Effects of the Distribution of Nuclear Magnetization on Hyperfine Structure

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I. INTRODUCTION

A FRUITFUL approach toward understanding the structure of the nucleus has been provided by measurement of various intrinsic nuclear properties such as spin, magnetic and electric multipole moments, charge density, etc. These empirical results are then compared with values calculated on the basis of an assumed model of the nucleus in attempts to develop a comprehensive theory of nuclear structure. This article presents the theory and experimental results which bear on the determination of a nuclear property which may be described as the *distribution of nuclear magnetization* (hereafter called DNM). This property is somewhat more difficult to discuss theoretically and to measure experimentally than, say, the nuclear magnetic dipole moment. However, the current nuclear models have reached a state of sophistication such that any theory which successfully predicts nuclear moments should also be capable of predicting the DNM. It is possible that in a few cases the experimental determination of the DNM will provide a sensitive test that will differentiate between various theories of magnetic moments.

We discuss the theory of the effects of the DNM with a view to distinguishing between the various nuclear parameters that enter into the theoretical evaluation of experimental results. Such results come primarily from accurate measurement of the hyperfine interaction in which the electronic density at the nuclear position is used to probe the DNM. Existing experimental results in the field are collected in tables and discussed from the point of view of the single particle model.

II. THEORY

1. General

The most readily observable effect of the DNM is its effect on the hyperfine interaction with an s electron. The electron density at the nucleus is in most cases negligible for p, d, $\cdots$ electrons so we confine this discussion to electronic s states, except where otherwise stated.

The hyperfine interaction energy was first calculated by Fermi and Segré (FE33)* by considering the nuclear magnetic moment as a point dipole. The inadequacy of the point dipole assumption for heavy nuclei was first suspected by Kopfermann (KO40) and was held responsible by Bitter (BI49) for the discrepancy between the ratios of nuclear moments and hyperfine interaction constants for the two stable rubidium isotopes. A quantitative theory to explain the effects of the DNM on hyperfine structure was developed by Bohr and Weisskopf (BO50) and enlarged upon by Bohr (BO51a, b). As the number of accurate measurements of this effect increased it became clear that, in keeping with the improved theories of nuclear moments it might be possible to calculate it in greater detail than had been considered justifiable by Bohr and Weisskopf in their original paper.† We therefore restate the essential parts of the B-W theory and enlarge upon those parts that may prove useful in nuclear structure interpretation.

2. Bohr-Weisskopf Theory

This theory is applicable to nuclei which are heavy enough so that the protons may be treated as a continuous charge distribution. The effects of the finite size of the nucleus on the hfs of an s electron are twofold. First, the effect of the finite size of the charge distribution is to make the electron potential non-Coulombic inside the nucleus. Since the magnitude and radial dependence of the hfs interaction density are proportional to the specific electron density, the nuclear charge size results in a reduction of the interaction relative to a "point" charge nucleus. To demonstrate this we plot in Fig. 1 the radial probability density for

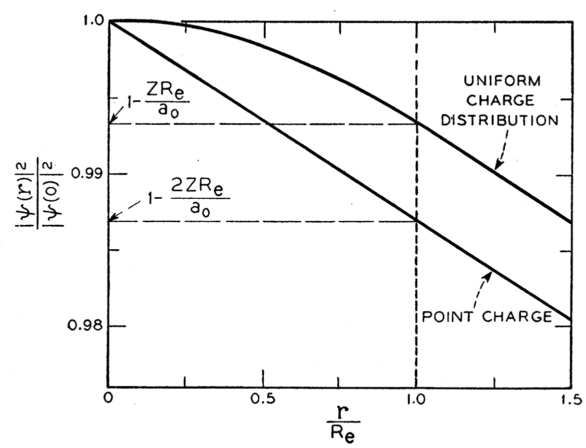

FIG. 1. Illustrating the effect of a uniformly charged nucleus of radius R_e on the nonrelativistic probability density distribution of an $s_{\frac{1}{2}}$ electron near the nucleus (Z).

* Refer to Bibliography at end of paper.

† Hereafter referred to as B-W.

an $s_{\frac{1}{2}}$ electron in the Schroedinger approximation for two cases: that of a point charge Ze and that of the same charge uniformly distributed throughout a sphere of radius R_e.

Let us estimate by how much the hfs of a finite nucleus with a uniform magnetization distribution differs from that of a point nucleus. The radial variation of the electron's density in the nuclear volume combined with the second effect of finite size—the distribution of nuclear magnetization—reduces the magnitude of the hfs interaction relative to that of a point magnetic dipole. To estimate the magnitude of the fractional hfs reduction, ϵ, we assume a uniform charge and uniform magnetization distribution throughout a sphere of radius R_e. The hfs reduction is then given by

$$\epsilon=\left\langle 1-\frac{|\psi(r)|^2}{|\psi(0)|^2}\right\rangle_{\mathrm{Av}}\approx\frac{ZR_e}{a_0}\left\langle\frac{R^2}{R_e^2}\right\rangle_{\mathrm{Av}}, \tag{1}$$

where $\psi(r)$ is the electronic wave function, a_0 is the first Bohr orbit radius, R is the radial nuclear coordinate from the center of the nucleus and the average is taken over the nuclear volume. For a relativistic electron ϵ is increased by approximately

$$\left(\frac{a_0}{2ZR_e}\right)^{2(1-\rho)},$$

where $\rho=(1-Z^2\alpha^2)^{\frac{1}{2}}$ and $\alpha=e^2/\hbar c$. For a uniform DNM the average value of R^2/R_e^2 is $\frac{3}{5}$ so that (1) leads to an ϵ which is less than 2% for $Z=50$. Thus even for the heavier nuclei the fractional hfs interaction reduction would only be a few percent which, unfortunately, is less than the accuracy to which we know electron wave functions for large Z.

(a) *Calculation of* Δ

Let us, however, compare the hfs interaction of two isotopes. The point dipole assumption predicts (FE33) that the hfs interaction energy for one isotope is

$$a_{pd}=\frac{16\pi}{3}g_I|\psi(0)|^2 \tag{2}$$

and that the ratio of a_{pd}'s for two isotopes should be equal to the ratio of their g_I's. For the DNM case we expect the hfs interaction energy of each isotope to be reduced by an amount ϵ and the ratio of the a's to differ from the ratio of the g_I's by an amount $\Delta\approx\epsilon_1-\epsilon_2$. The surprising experimental result is that in certain cases the observed Δ is comparable in magnitude to ϵ, indicating radically different DNM in the two isotopes. This result can be understood by considering the difference of orbital and spin magnetization distributions. For example a rotating uniformly charge sphere corresponds to an inwardly increasing magnetization and since this resembles more closely a point dipole, it would in its nuclear counterpart have a smaller reduction of the hfs interaction than a uniform (spin) magnetization.

Bohr and Weisskopf (BO50) calculated the interaction of a Dirac electron with the nuclear magnetic vector potentials associated with spin and orbital currents. Their theory may be summarized as follows. If F and G are the Dirac radial wave functions for an electron, the interaction energy for the spin and orbital cases are given by

$$a_s=\frac{16\pi}{3}g_s\int_{\mathrm{nucleus}}(d\tau_R)w(\mathbf{R})s_z \times\left[\int_R^\infty FGdr-\zeta^*(s_z,\theta,\phi)\int_0^R\frac{r^3}{R^3}FGdr\right] \tag{3}$$

$$a_l=\frac{16\pi}{3}g_l\int_{\mathrm{nucleus}}(d\tau_R)|\Phi(\mathbf{R})|^2l_z \times\left[\int_R^\infty FGdr+\int_0^R\frac{r^3}{R^3}FGdr\right], \tag{4}$$

where R and r refer to the radial nuclear and electron coordinates respectively, g_s and g_l are the nuclear spin and orbital g factors. $w(\mathbf{R})$ is the nuclear spin magnetization distribution and $\Phi(\mathbf{R})$ is the nuclear wave function. The factor ζ^* takes into account the angular asymmetry of the spin distribution (BO51b) and is given by $(1/s_z)[\boldsymbol{\sigma}-3(\boldsymbol{\sigma}\cdot\mathbf{R}_u)\mathbf{R}_u]_z$, where $\boldsymbol{\sigma}$ is the Pauli spin operator and $\mathbf{R}_u=\mathbf{R}/R$. ζ^* vanishes for a spherically symmetrical $w(\mathbf{R})$ and its value averaged over the nucleus, ζ, has been calculated for single particle states by Bohr (BO51b) and formulas for it appear in Appendix I. Note that (3) reduces to the Fermi-Segrè relation (2) as $R\rightarrow 0$.

Defining

$$\kappa_s\equiv\frac{\int_0^R\left(1+\zeta^*\frac{r^3}{R^3}\right)FGdr}{\int_0^\infty FGdr} \tag{5}$$

$$\kappa_l\equiv\frac{\int_0^R\left(1-\frac{r^3}{R^3}\right)FGdr}{\int_0^\infty FGdr} \tag{6}$$

the average values of κ_s and κ_l over the nucleus

$$\bar{\kappa}_s=\frac{a_{pd}-a_s}{a_{pd}},\qquad \bar{\kappa}_l=\frac{a_{pd}-a_l}{a_{pd}}$$

represent the fractional hfs reduction due to the finite extent of the spin and orbital magnetization distributions, respectively.

TABLE I. Values of coefficients entering into the fractional reduction of the hfs for $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ electrons due to the finite size and distribution of the nuclear spin and orbital magnetization are tabulated for various Z. These are required in Eqs. (7) and (8). The b's for the $s_{\frac{1}{2}}$ state are obtained directly from the modified B-W theory but the $p_{\frac{1}{2}}$ coefficients have been corrected as indicated in Appendix I.

	$s_{\frac{1}{2}}$ (percent)						$p_{\frac{1}{2}}$ (percent)					
Z	b_{s2}	b_{s2}'	b_{l2}	b_{s4}	b_{s4}'	b_{l4}	b_{s2}	b_{s2}'	b_{l2}	b_{s4}	b_{s4}'	b_{l4}
10	0.065	0.026	0.039	0.007	0.004	0.003						
15	0.11	0.046	0.069	0.012	0.007	0.005	0.001					
20	0.18	0.072	0.11	0.019	0.010	0.008	0.002	0.001	0.001			
25	0.26	0.10	0.15	0.028	0.016	0.012	0.005	0.002	0.003	0.001		
30	0.35	0.14	0.21	0.041	0.023	0.017	0.010	0.004	0.006	0.001	0.001	
35	0.47	0.19	0.28	0.057	0.032	0.024	0.020	0.007	0.012	0.002	0.001	
40	0.61	0.25	0.37	0.077	0.044	0.033	0.034	0.014	0.020	0.004	0.003	0.002
45	0.79	0.32	0.47	0.10	0.060	0.045	0.058	0.023	0.035	0.007	0.004	0.003
50	1.01	0.40	0.60	0.14	0.080	0.060	0.093	0.038	0.056	0.012	0.007	0.005
55	1.27	0.51	0.76	0.19	0.11	0.076	0.15	0.060	0.090	0.021	0.012	0.009
60	1.58	0.63	0.95	0.25	0.14	0.095	0.22	0.091	0.14	0.034	0.020	0.015
65	1.98	0.79	1.19	0.33	0.19	0.14	0.35	0.14	0.21	0.055	0.032	0.024
70	2.44	0.98	1.46	0.43	0.25	0.19	0.51	0.21	0.31	0.088	0.050	0.038
75	2.98	1.19	1.79	0.56	0.32	0.24	0.75	0.30	0.45	0.14	0.077	0.058
80	3.62	1.45	2.17	0.73	0.42	0.31	1.06	0.42	0.64	0.21	0.12	0.089
85	4.33	1.73	2.59	0.93	0.53	0.40	1.48	0.59	0.89	0.31	0.18	0.13
90	5.07	2.03	3.04	1.16	0.66	0.50	2.04	0.81	1.22	0.45	0.26	0.19

Assuming the potential of a uniform spherical charge distribution, Bohr and Weisskopf obtain the power series solutions to the Dirac equation which are required for evaluating the numerator integrals in Eqs. (5) and (6). In Appendix I we reproduce these solutions. We evaluated κ_s and κ_l keeping all terms necessary in the series expansion to limit the error to a few percent. In this way we obtain for the average values of the κ's

$$\bar{\kappa}_s=[b_{s2}(s_{\frac{1}{2}})+\zeta b_{s2}'(s_{\frac{1}{2}})]\mathcal{R}_{e2}-[b_{s4}(s_{\frac{1}{2}})+\zeta b_{s4}'(s_{\frac{1}{2}})]\mathcal{R}_{e4} \quad (7)$$

$$\bar{\kappa}_l=b_{l2}(s_{\frac{1}{2}})\mathcal{R}_{l2}-b_{l4}(s_{\frac{1}{2}})\mathcal{R}_{e4} \quad (8)$$

with similar expressions for the $p_{\frac{1}{2}}$ state. $\mathcal{R}_{e2}$ and $\mathcal{R}_{e4}$ are the average values of R^2/R_e^2 and R^4/R_e^4 over the nucleus. Writing R_V for the nuclear potential radius we can define $\mathcal{R}_{V2}$ and $\mathcal{R}_{V4}$ in an analogous manner. The $\mathcal{R}_{Vi}$ $(i=2, 4)$ can be calculated on the basis of the shell model; such calculations are illustrated in Appendix II. $\mathcal{R}_{ei}$ is then obtained from the relation

$$\mathcal{R}_{ei}=\left(\frac{R_V}{R_e}\right)^i\mathcal{R}_{Vi}. \quad (9)$$

In B-W the $\mathcal{R}_{e4}$ dependence of κ_s and κ_l is lumped with $\mathcal{R}_{e2}$ by making the approximation that $\mathcal{R}_{e4}\simeq(6/7)\mathcal{R}_{e2}\simeq(6/7)\mathcal{R}_{V2}$. We have chosen the somewhat more elaborate presentation of Eqs. (7)–(9) on the basis of the following considerations. First, recent work (AR54, BL53, BL54) has shown that nuclear magnetic moments may in many cases be accurately predicted using a single particle plus configuration mixing model. Secondly, it is clear that the hyperfine interaction reduction depends on the moments of both the single particle radius and the nuclear charge radius, R_e, where the latter is considered to be different from the potential radius (RO56).

The coefficients b needed for Eq. (7) and Eq. (8) and their equivalents for the $p_{\frac{1}{2}}$ state are given in Table I; the procedure used in evaluating them appears in Appendix I. In calculating the b's, $R_e=r_0A^{\frac{1}{3}}$ with $r_0=1.20\times10^{-13}$ cm, was used. They reduce to the b's given in B-W when suitable approximations are made and B-W's $r_0=1.5\times10^{-13}$ cm is substituted.

The fractional hyperfine interaction reduction for a given isotope is then

$$\epsilon=-(\bar{\kappa}_s\alpha_s+\bar{\kappa}_l\alpha_l), \quad (10)$$

where α_s and α_l are the fractional contributions of spin and orbital moment to the magnetic moment $(\alpha_s+\alpha_l=1)$. When expressed in terms of the nuclear g factors α_s is given by

$$\alpha_s=\frac{g_s}{g_I}\frac{g_I-g_l}{g_s-g_l}. \quad (11)$$

We have now, a formalism for evaluating the *fractional* reduction of the hfs interaction, but because of the uncertainty in the normalization of the entire electronic wave function we do not have, with a few exceptions discussed below, accurate knowledge of the absolute magnitude of the interaction for the point-dipole case. However, by comparing the ratio of the hfs interaction for two isotopes this uncertainty largely disappears. Denoting the two isotopes by 1 and 2, and defining the so-called hfs anomaly as

$$\Delta\equiv\epsilon_1-\epsilon_2,$$

then

$$\Delta=\frac{a_1}{a_2}\frac{g_2}{g_1}-1, \quad (12)$$

in which the squares of quantities of order ϵ are neglected.‡ Since the b's show little isotopic variation, Δ

‡ See footnote y of Table III for a consequence of this approximation.

measures the *difference* in the distribution of magnetization for the two isotopes. Various factors can cause Δ to be anywhere between zero and of the same order of magnitude as ϵ. Neglecting the spin asymmetry for the moment, b_{li} is less than b_{si} so that spin and orbital magnetization differences in two isotopes will be weighted differently in their contribution to Δ. In addition, large asymmetries in the spin magnetization of certain nuclei weight the spin contribution to Δ of these isotopes even more strongly.

The theory as thus outlined only suffices to deal with a single $s_{\frac{1}{2}}$ or $p_{\frac{1}{2}}$ electron. In cases in which the atomic ground state of the atom involves the coupling of two or more electrons the fractional $s_{\frac{1}{2}}(p_{\frac{1}{2}})$ character of the electronic wave function must be determined.

(*b*) *Additional Effects Contributing to* Δ_{exp}

Since one wishes to extract from the measured Δ_{exp} for a pair of isotopes, information concerning the difference in the DNM, one must estimate the nature and magnitude of those contributions to Δ_{exp} which arise from isotopic variation of the electron's wave functions and of the dipole interaction (SC57).

Isotope shift.—The reduced mass correction to the electrons wave function is $(1+m/AM)^{-3}$, which, for two isotopes differing by two neutrons, will give a correction to Δ_{exp} of order $3m/M(2/A^2)$, which is appreciable for the lightest isotopes but of the order of 10^{-6} for $A=50$.

Breit-Rosenthal effect.—The charge distribution and hence the non-Coulombic part of the potential which affects the electron will, in general, be different for two isotopes. This affects the value of the hfs "a" and therefore contributes to Δ_{exp} (Breit-Rosenthal effect). Theoretical estimates (RO32, CR49) indicate that its contribution to Δ_{exp} is always less than one part in 10^4. In some cases (cf. $Tl^{203,205}$) it may, however, exceed the Bohr-Weisskopf effect. Considerable uncertainty in calculating the Breit-Rosenthal effect [mainly due to lack of knowledge of the difference in the nuclear radii of the two isotopes (WI53)] exists, thereby placing an upper limit on the precision with which experimental values of Δ may at present be compared with theory.

Orbital corrections to the electron's orbital g factor.—Schwartz points out (SC57) that reduced mass corrections to the electron's orbital g factor give an additional contribution to Δ_{exp} for non-s electrons. It is difficult to estimate this, in general, but again the effect will be important only for very light nuclei.

Perturbations resulting from neighboring electronic levels.—For single $p_{\frac{1}{2}}$ electrons for example, the presence of the neighboring fine structure level $(p_{\frac{3}{2}})$ will, in second order, perturb levels of the same $F(\mathbf{F}=\mathbf{I}+\mathbf{J})$ and modify a. The resulting contributions to Δ_{exp} will be important for light atoms where the fine structure doublet spacing is smallest, and the DNM contribution to Δ_{exp} for light nuclei is unfortunately small. This same perturbation enters into measurements of the ratios of g_I's in the $p_{\frac{1}{2}}$ state (FO50, CL54) (see Sec. III-3).

These considerations indicate that measurements of Δ for atoms in $s_{\frac{1}{2}}$ states for which the difference in the DNM represent major contributions are subject to the fewest uncertainties in theoretical interpretation.

3. Other Effects of the Distribution of Magnetization

(*a*) *Internal Conversion Coefficients*

Information about the DNM may also be obtainable through accurate determination of certain internal conversion coefficients. No such experiments have been published to date and no attempt will be made here to give the complete relevant theory.

Following the calculation of internal conversion coefficients by Rose *et al.* (RO49, 51) who made the point charge nucleus assumption, Sliv *et al.* (SL51, SL52) pointed out that important corrections are necessary due to the finite size of the nucleus, which affects the shape of the radial wave functions of the initial and final state of the conversion electron. Existence of a further correction, which is or particular interest from the point of view of DNM, was pointed out by Church and Weneser (CH56a, b). It arises from the fact that penetration of the electron inside the charge and current distribution of the nucleus gives rise to additional internal conversion matrix elements, which have an essentially different form from the leading matrix elements and are sensitive to the details of the nuclear magnetization (cf. EW57). These structure-sensitive matrix elements are particularly important when the normal gamma-ray emission matrix elements vanish for one reason or another. In such cases measurement of the conversion coefficient may lead to important conclusions about the DNM. Several instances of forbidden or inhibited gamma-ray emission are discussed by Church and Weneser (CH56b).

(*b*) *Magnetic Octupole Moments*

Just as the electric quadrupole moment gives a measure of the second radial moment of the electric charge distribution in a nucleus so does the nuclear magnetic octupole moment, Ω, give a measure of the nuclear magnetization§ distribution. Following the observation (JA54) of the effects of a nuclear magnetic octupole moment in the hfs of I^{127}, a detailed theory of hfs (SC55) was given in which, among other things, the character of Ω was discussed. If one defines

$$\Omega \equiv \langle \tfrac{1}{2}(5Z^3-3ZR^2)\,\mathrm{div}\mathbf{M}\rangle_{I,I}, \tag{13}$$

where $\mathbf{M}$ is the volume density of nuclear dipole mo-

§ Unlike the magnetic dipole moment the existence of a Ω implies a finite DNM.

ment, one can show for single particle orbits

$$\Omega=\mu_n\frac{3}{2}\frac{2I-1}{(2I+4)(2I+2)}f(I,g_l,g_s)\mathcal{R}_{V^2}R_V{}^2, \qquad (14)$$

where

$$f(I,g_lg_s)=\begin{cases}(I+2)(I-\frac{3}{2})(g_l+g_s); & I=l+\frac{1}{2}\\ (I-1)(I+\frac{5}{2})(g_l-g_s); & I=l-\frac{1}{2}.\end{cases}$$

The octupole moment manifests itself principally in its hfs interaction with atomic $p_{\frac{3}{2}}$ electrons. (The existence of a multipole moment of this order and its observation through a study of hfs require that $I\geq\frac{3}{2}$ and $J\geq\frac{3}{2}$.) The magnitude of Ω calculated from the observed hfs interaction is subject to uncertainty, principally, because of lack of knowledge of the radial electron wave functions. The several moments that have been measured seem to confirm the single particle character associated with nuclear magnetism, especially if interpreted (SC55) in terms of a particle-core coupling reduction effect predicted by the collective model (BO53).

In view of the success of the configuration mixing model (AR54, BL53, BL54) in accurately predicting odd-even nuclear magnetic dipole moments, it might be interesting to calculate Ω from the same point of view. If one compares the ratio of the magnetic octupole hfs interaction for two isotopes (after each has been corrected for the dipole-quadrupole pseudo-octupole terms which are easily evaluated) the difference in the distribution of nuclear magnetization of the two isotopes could be obtained, and the principal uncertainty associated with the lack of knowledge of electron wave functions would be removed. As with nuclear magnetic dipole moments, one might expect different predictions for Ω from the collective model and the configuration mixing model.

References to octupole moment measurements have been collected in (SC57).

(c) *Scattering Experiments*

It was recently pointed out by Newton (NE56) that the DNM of aligned nuclei can be measured in scattering experiments using polarized electron beams. Unfortunately such experiments are not feasible at the present time.

III. EXPERIMENTAL TECHNIQUES

1. Measurement of Δ

The most successful method of exploring the DNM has been the precise determination of the hyperfine interaction constants and the nuclear g factors for two (or more) isotopes of the same element. The so-called "hyperfine structure anomaly" Δ is then a function of the DNM of isotopes 1 and 2 and is given by Eq. (11). Because Δ is in general a quantity of order of magnitude of 10^{-3} or less one needs at least an over-all accuracy of about 10^{-4} in all experimental quantities to obtain values accurate to 10% for Δ. Several experimental methods have been used to obtain accurate values of (a_1/a_2) and (g_1/g_2). The necessary precision is attained by employing resonance techniques in which the crucial measurement is that of the ratio of two frequencies.

2. Measurement of a_1/a_2

(a) *Atomic Beam Magnetic Resonance*

The requisite accuracy in the measurement of a's has been achieved, principally, by use of atomic beam magnetic resonance measurements (ABMR) of the ground states of free atoms. The ABMR methods have been reviewed in (RA56) and (KI56). The requirement of having an $S_{\frac{1}{2}}$ or $P_{\frac{1}{2}}$ electronic ground state has with a few exceptions limited experiments, to Group I and Group III elements, all of whose stable isotopes have been measured by ABMR. Recently (NI56, HU56) spin measurements have been made with as few as 10^9 atoms by using radioactive detection techniques in conjunction with the ABMR method. This seems to be a promising method for measurement of the ratio of a's for several isotopes in a given odd-proton species, especially for the short-lived isotopes. Measurements of the hfs interaction in isomeric states (GO55, GI55) of certain nuclei have been made in this way, so that one may obtain information in the near future about the difference in DNM in two nuclear states of a given isotope.

(b) *Paramagnetic Resonance*

Accurate measurements of the hfs interaction have been made using PR techniques in free atom gases (RO46, WI56) but, as yet, not on isotopic pairs. In general, PR measurements require upward of 10^{12} atoms.

It has recently become possible to measure a_1/a_2 in solids to considerable accuracy by means of paramagnetic resonance (PR). The precision is greatly increased through the use of electron-nuclear double-resonance techniques (FE56) which are capable of reducing the usual line widths in solids by several orders of magnitude. By applying this method to donor atoms in silicon (EI58) it was possible to measure hfs interactions in the $S_{\frac{1}{2}}$ state for atoms whose free atom ground state hfs may not be amenable to a simple analysis (SM57). For example the free Sb atom is in a $^4S_{\frac{3}{2}}$ ground state (which involves the coupling of 3 $p_{\frac{1}{2}}$ electrons) whereas Sb as a donor in a silicon crystal is in a $^2S_{\frac{1}{2}}$ state. The absolute value of "a" for the donor atom is different from the hfs interaction of the free atom but a_1/a_2 should be unchanged (EI58).

3. Measurement of g_1/g_2

Where isotopic abundance is not a problem the now standard Bloch-Purcell techniques of nuclear magnetic

TABLE II. Experimental results of measurement of the hfs and magnetic moment ratios of some light elements. Numbers in parentheses indicate the uncertainties in the last quoted figures.

Atom	I	$\Delta\nu$ (Mc/sec)	Reference	μ ratios	Reference
$H^1(^2S_{\frac{1}{2}})$	$\frac{1}{2}$	1 420.405 73(5)	a,b	$\mu_s/\mu_p = 658.228\ 8(6)$	c
$H^2(^2S_{\frac{1}{2}})$	1	327.384 302(2)	a	$\mu_{H^1}/\mu_{H^2} = 3.257\ 199\ 86(45)$	d
$H^3(^2S_{\frac{1}{2}})$	$\frac{1}{2}$	1 516.701 70(7)	e	$\mu_{H^3}/\mu_{H^1} = 1.066\ 636(10)$	f
$He^3(^3S_1)$	$\frac{1}{2}$	6 739.71(5)	g		
$He^{3+}(^2S_{\frac{1}{2}})$	$\frac{1}{2}$	1 083.354 99(20)	h,i	$\mu_{He^3}/\mu_{H^1} = 0.761\ 812(1)$	j,k
$Li^6(^2S_{\frac{1}{2}})$	1	228.208(5)	l		
$Li^7(^2S_{\frac{1}{2}})$	$\frac{1}{2}$	803.512(15)	l	$g_{Li^7}/g_{Li^6} = 2.640\ 91(1)$	m

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resonance (NMR) have been employed to measure the ratio of the g values of two isotopes embedded in a nonparamagnetic material. A precision of one part in 10^5 is not uncommon but at least 10^{17} nuclei are required for NMR. The NMR method has been reviewed in (AN55).

The ABMR method has been employed to measure the ratio of the nuclear g values of radioactive isotopes of relatively low abundance (EI53, ST57) and the electron-nuclear double-resonance method can also be used to determine g_1/g_2. Since these methods of measurement of g values are less familiar we summarize them briefly. Considering a nucleus whose spin I is coupled to an electron with $J=\frac{1}{2}$ in an external field H_0, the eigenvalues of the "high-field" Hamiltonian, E, are given by

$$E(m_I, m_J) = a m_I m_J + m_J g_J \mu_0 H_0 + m_I g_I \mu_n H_0, \quad (15)$$

where terms quadratic in the m's and the field have been omitted. If one observes a transition which corresponds to a reorientation of the nuclear moment with respect to the external field but leaving the electron-external field coupling unchanged ($\Delta m_I = \pm 1$, $\Delta m_J = 0$) in each of the two electron states ($m_J = \pm\frac{1}{2}$), then the difference in energy of the two transitions is equal to $2g_I\mu_n H_0$. The precision is comparable with NMR techniques wherever applicable and is very useful for isotopes of low abundance.

IV. DISCUSSION OF RESULTS

To date all experimental results bearing on the DNM have been obtained through a study of the hyperfine interactions and the nuclear g factors by the methods outlined above. These results are collected below and discussed in some detail. We treat separately the measurements involving light nuclei ($A<20$) and those of medium and heavy nuclei, since the theoretical treatment of the results is essentially different for these groups.

1. Experimental Results: $A<20$

The nuclei in this group contain too few nucleons to permit consideration from the point of view of a continuous nuclear charge distribution. It is possible, on the other hand, to set up detailed nuclear wave functions for the lightest nuclei whose predictions for the distribution of nuclear magnetization can be compared with the measured values of the hyperfine structure.

Table II shows the results of the measurements. Since the pure $s_{\frac{1}{2}}$ hydrogen wave function is still the only one which can be calculated to sufficient accuracy, the three hydrogen isotopes (and the He^{3+} ion) offer so far, the only opportunity, of obtaining an absolute value of ϵ.

The contribution of nuclear structure to the hyperfine splitting ($\Delta\nu$) of very light nuclei was first considered by Bohr (BO48). He pointed out that the electron wave function in H^2 will have a characteristic motional frequency much higher than the corresponding nuclear frequencies so that the electron orbits will adjust to the instantaneous positions of the nucleons. In calculating $\Delta\nu$ from the magnetic moment, the spatial distributions of nuclear currents and moments must therefore be taken into account. The deuterium calculations were subsequently refined by Low (LO50) and relativistic effects were taken into account by Salpeter and Newcomb (SA52). Similar analyses for H^3 and He^3 were undertaken by Adams (AD51) and Sessler and Foley (SE54, SE55). Recently the effect of *nucleon* structure on the hyperfine structure of H^1 was pointed out (MO55) and calculated (ZE56) and corresponding corrections for H^2, H^3, and He^3 obtained (SE58a).

A comparison of calculated and observed values of $\Delta\nu(H^1)$ may be made, following the work of Zemach (ZE56). Using the values of $\Delta\nu$ and μ in Table II and the size of the proton [with $\langle r^2\rangle_e^{\frac{1}{2}} = \langle r^2\rangle_m^{\frac{1}{2}} = 0.8\times10^{-13}$ cm (GO58) the proton size correction is of the order of 40 ppm] and all other known corrections (ZE56) one finds a discrepancy of 20 ppm between theory and experiment for the hyperfine structure of H^1. The recent modification of the theoretical value of the fourth-order contribution to the magnetic moment of the electron (SO57) does not affect the calculated value of $\Delta\nu$ since there is a compensating change in the fine structure constant.

The hyperfine structure of H^2 was recently discussed by Sessler and Mills (SE58a). After allowing for proton and neutron structure effects (128 and 17 ppm, respectively, in H^2), and other corrections, and comparing the calculated and experimental values of $[\Delta\nu(H^1)]/[\Delta\nu(H^2)]$ (Table II), a hyperfine structure anomaly of 208 ± 32 ppm remains. They observe that this is consistent with 3.9% D state and no relativistic or interaction moment contribution to the magnetic moment. The experiments are also in agreement with the (3 ± 1)% D state obtained by Sugawara (SU55a, b) using field-theoretic arguments.

The hfs of H^3 was analyzed by Adams (AD51) and Sessler and Foley (SE55). Since the two neutrons couple to zero spin they are not expected to contribute to the hyperfine structure. The proton size contribution will be almost the same as for H^1 so that the hfs anomaly, Δ, will be small. Using the experimental data of Table I, one finds that $|\Delta| = |\epsilon(H^1) - \epsilon(H^3)| < 12$ ppm.

The hfs of He^3 in the $^3S_1$ ground state has been measured (WE54) and many of the corrections required for a theoretical value of $\Delta\nu$ have been calculated (SE55). The triplet electronic wave function is not well enough known for a comparison with the experimental value in the table. Novick and Commins (NO56, CO57) have recently measured $\Delta\nu$ of $He^{3+}(S_{\frac{1}{2}})$ to great precision and since the electronic wave function is hydrogenic, $\Delta\nu$ can also be computed. Novick (NO58) reports a hyperfine discrepancy $\epsilon(He^3) = 1 - (\Delta\nu_{obs}/\Delta\nu_{pt})$ of 186.5 $+9.2$ ppm. Several proposed three-body nuclear wave functions predict values of $\epsilon(He^3)$ which are all too small by 20 to 40 ppm. This can probably be accounted for by nucleon structure and interaction current effects. Some forms of the latter have been considered by Sessler and Foley (SE55) and it appears that velocity-dependent nucleon interactions can be excluded on the basis of experimental results for H^3 and He^3, since they would lead to interaction current terms of about 230 ppm. None of the calculations to date has taken into account effects of the spin-orbit contribution of nuclear forces and some of the above conclusions may have to be modified if this effect should turn out to be appreciable.

The experimental values of $\Delta\nu$ and μ for the Li isotopes lead to a hfs anomaly of 70 ± 30 ppm. No detailed comparison of this value with one calculated on the basis of nuclear wave functions has been published. At the same time Li is too light to be treated by the Bohr-Weisskopf theory.

2. Experimental Results: $A>20$

Table III lists measured ratios of hyperfine interaction constants "a" and nuclear g factors for pairs of isotopes and values of Δ obtained from them. Since Δ contains only information about the difference of the effect of the DNM for the two isotopes it is clear that such measurements would be most useful if performed for a series of isotopes of the same element. This has been done so far only for four Cs isotopes and three K isotopes.

In many cases, the experimental Δ was compared to a computed value by those who made the measurement. These theoretical values are usually based on one, or several, of a variety of nuclear models. It is desirable that the chosen models in addition to reproducing the correct Δ, be also capable of predicting the correct magnetic dipole moments. This requirement has sometimes been met by a fairly arbitrary procedure such as ascribing to the odd nucleon an intrinsic moment different from the value for the *free* nucleon, chosen to make the single particle moment agree with the experimental value rather than the Schmidt value. While these calculations served their original purpose of confirming the single particle model as opposed to, say, a model of a nucleus in which all protons contribute to the orbital moment ($g_l \cong Z/A$), they are inadequate for gaining detailed information about the DNM. If we require of a theory (BL56) of nuclear moments that it explain the deviations from the Schmidt lines of all or nearly all nuclei in a systematic way there seem to be only two theories which need to be considered at present: the collective model of Bohr and Mottelson (BO53) in which the odd nucleon is coupled to an asymmetric nuclear core, and the configuration mixing model which has been described by Arima and Horie (AR54) and Blin-Stoyle and Perks (BL53, BL54). These two theories attempt to account for the observed magnetic moments by ascribing the deviations from the Schmidt limits to angular momentum sharing between the core and external nucleon, and to admixtures of near lying "valence" nucleon states, respectively. The collective model should be most satisfactory in regions of large nuclear distortions ($150<A<190$ and $A>225$) where no measurements of Δ have yet been made. The configuration mixing model has been very successful for odd-A nuclei in reproducing almost all observed magnetic moments. A systematic calculation of Δ based on this theory, for the measured isotope pairs, is being undertaken (ST58). In the following we confine ourselves to qualitative comparison between theory and experiment and illustrate the methods of calculating Δ by a few examples (cf. Sec. 3).

(a) $Cu^{63,65}$, $Cs^{133,135}$, $Cs^{135,137}$

Each of these three pairs of isotopes has the same spin and similar magnetic moments. All are even N-odd Z nuclei. According to the single particle model (SP)

TABLE III. Experimental results of measurement of the hfs and nuclear g-factor ratio for some medium and heavy atoms. Numbers in parentheses indicate uncertainties in the last quoted figures. The experimental techniques are identified by ABMR, NMR, and PR. All atomic beam measurements of the hyperfine interaction constant "a" were done with the atom in the $S_{\frac{1}{2}}$ state unless otherwise indicated. The last column shows the hfs anomaly as defined in Eq. (12).

Isotopes (1,2) and their spins (I)		g_1/g_2	Method	a_1/a_2	Method	$\Delta=(a_1/a_2)(g_2/g_1)-1$ (%)
${}_{17}Cl^{35}(\frac{3}{2})$,	$Cl^{37}(\frac{3}{2})$	1.20135(8)	NMR[a,b]	1.20136(1)	ABMR($P_{\frac{1}{2}}$)[c]	0.000(8)
${}_{19}K^{39}(\frac{3}{2})$,	$K^{40}(4)$	0.800421(16)	ABMR[d]	0.8007962(4)	ABMR[d,e]	0.467(19)
${}_{19}K^{39}(\frac{3}{2})$,	$K^{41}(\frac{3}{2})$	1.82185(16)	NMR[f,g]	1.81768(1)	ABMR[e,g]	0.229(9)
${}_{29}Cu^{63}(\frac{3}{2})$,	$Cu^{65}(\frac{3}{2})$	0.933424(19)	NMR[h]	0.933567(2)	ABMR[i]	0.015(2)
${}_{31}Ga^{69}(\frac{3}{2})$,	$Ga^{71}(\frac{3}{2})$	0.7870147(12)	NMR[j]	0.7870196(6)	ABMR($P_{\frac{1}{2}}$)[k]	0.00062(23)
${}_{37}Rb^{85}(\frac{5}{2})$,	$Rb^{87}(\frac{3}{2})$	0.2950737(11)	NMR[l,m]	0.2961104(6)	ABMR[n,o]	0.3513(6)[y]
${}_{47}Ag^{107}(\frac{1}{2})$,	$Ag^{109}(\frac{1}{2})$	0.86985(1)	NMR[p]	0.86627(3)	ABMR[q]	−0.412(4)
${}_{49}In^{113}(9/2)$,	$In^{115}(9/2)$	0.9978609(12)	NMR[r]	0.99786841(28)	ABMR($P_{\frac{1}{2}}$)[s]	0.00075(13)
${}_{51}Sb^{121}(\frac{5}{2})$,	$Sb^{123}(\frac{7}{2})$	1.84661(1)	NMR[t]	1.84012(9)	PR($S_{\frac{1}{2}}$)[t]	−0.352(5)
				1.84076(5)	ABMR($^4S_{\frac{3}{2}}$)[u]	−0.317(3)
${}_{55}Cs^{133}(\frac{7}{2})$,	$Cs^{134}(4)$	0.98574(29)	ABMR[v]	0.987405(2)	ABMR[v,w]	0.169(30)
${}_{55}Cs^{133}(\frac{7}{2})$,	$Cs^{135}(\frac{7}{2})$	0.945001(8)	ABMR[v]	0.9453527(15)	ABMR[v,w]	0.037(9)
${}_{55}Cs^{135}(\frac{7}{2})$,	$Cs^{137}(\frac{7}{2})$	0.961492(8)	ABMR[v]	0.9612967(15)	ABMR[v]	−0.020(9)
${}_{81}Tl^{203}(\frac{1}{2})$,	$Tl^{205}(\frac{1}{2})$	0.9902578(10)	NMR[x]	0.9903622(5)	ARMR($P_{\frac{1}{2}}$)[k]	0.0105(15)

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the moment is therefore wholly due to the proton and the addition of two neutrons to the lighter isotope should not affect the DNM. The foregoing pairs do indeed exhibit small Δ's in spite of the fact that the atomic ground states have $S_{\frac{1}{2}}$ character.

The experimental Δ for $Cs^{135,137}$ is negative (ST57). If in the B-W theory one assumes the appropriate κ's to be the same for both isotopes, 1 and 2, then

$$\Delta=[\bar{\kappa}_s-\bar{\kappa}_l]\frac{g_s g_l}{g_s-g_l}\left(\frac{1}{g(1)}-\frac{1}{g(2)}\right), \tag{16}$$

where $\bar{\kappa}_s$, $\bar{\kappa}_l$, g_s and g_l are always positive quantities with $\bar{\kappa}_s>\bar{\kappa}_l$ and $g_s>g_l$. It follows that if

$$g(1)<g(2) \quad \text{with} \quad g(1)\approx g(2)>0$$

or

$$g(1)>g(2) \quad \text{with} \quad g(1)\approx g(2)<0$$

then $\Delta\geq 0$. From Table III, $Cs^{135,137}$ appears to be the only measured exception to this rule, which apparently can only be explained by the configuration mixing model. Since ${}_{55}Cs_{82}{}^{137}$ is neutron "magic," one might think the anomalous sign could be explained by assuming the charge radius to be smaller for this isotope than for Cs^{135}. This would require a difference in the two radii of the order of 10% which seems most unlikely.

(b) $K^{39,41}$, $Ag^{107,109}$

These two pairs again have odd protons in the same state for both isotopes. Here the discrepancy between observed moments and between the differences of their distributions with the predictions of the SP model are more striking. The measured Δ's are appreciable and can be explained approximately, using both the collective model (BO51b, WE53) and the configuration mixing model (ST58).

(c) $Cl^{35,37}$, $Ga^{69,71}$, $In^{113,115}$, $Tl^{203,205}$

These pairs of isotopes are similar to groups a and b as far as their nuclear properties are concerned but differ from them in that the hfs measurements were performed on the atoms in the $P_{\frac{1}{2}}$ states. The reduced electron density at the nucleus precludes a large Δ except in the heaviest elements and it is difficult to draw conclusions from the measured small Δ's about the DNM except that they do not contradict the theory.

(d) $Rb^{85,87}$, $Sb^{121,123}$

These two pairs have different spins so that the addition of two neutrons to the lighter isotope seems to have affected the nuclear structure profoundly. Here the strict SP model is not applicable at all. The spin change points to the closeness of two incomplete shell levels resulting from strong spin-orbit coupling. This leads to different DNM for the two isotopes and since the hfs was measured in an $S_{\frac{1}{2}}$ state, Δ is expected to be large (BO51b, EI58). The calculation of Δ for $Rb^{85,87}$ according to the two models is given as an example in the following section.

Smith has recently measured "a" for Sb^{121} and Sb^{123} in the $^4S_{\frac{3}{2}}$ ground state by an atomic beam method (SM57). His value of Δ is of course not comparable to $\Delta(^2S_{\frac{1}{2}})$ (EI58) but it may be possible to obtain a theoretical value for $\Delta(^4S_{\frac{3}{2}})$ in the future.

(e) $K^{39,40}$, $Cs^{133,134}$

These are comparisons between an even-odd and an odd-odd isotope. The spin and moment of the latter depend on the coupling of the unpaired neutron and proton and the difference in DNM should be considerable. The experimental results bear this out. Theoretical values (EI53, ST57) of Δ based on the collective model are in approximate agreement with the observations. A configuration mixing theory of nuclear moments of odd-odd nuclei has not been developed yet.

(f) *Odd Neutron-Even Proton Isotopes*

It is unfortunate that there have been no measurements to date of odd neutron-even proton isotopic pairs. Since the effect on the hfs of the difference in DNM arises, mainly from the different weighting given to spin and orbital magnetization, the prediction of the B-W theory using the extreme SP model ($g_l=0$) is that Δ would be very small. In fact, even if the two isotopes had different spins, we would expect Δ to reflect principally the difference in the spin magnetization asymmetry for the two isotopes. In the CM model there may also be contributions to the orbital moment distribution difference arising from virtual proton excitations.

3. Comparison with Theory

As an illustrative example we compare the predictions of the two most promising theories of nuclear magnetic moments as applied to the two stable isotopes of rubidium, $Rb^{85,87}$.

(a) *Collective or Asymmetric Core Model*

The collective model (C) was suggested by Bohr and Mottelson (BO53). It has been successful in explaining rotational level spacings, electric quadrupole moments and other features of nuclear structure. In its application to magnetic moments one considers the odd nucleon coupled to an asymmetric nuclear core whose angular momentum operator and g factor are $\mathbf{R}$ and $g_R \approx Z/A$ respectively. Writing g_s and g_l for the spin and orbital g factors of the odd particle and $\mathbf{S}$ and $\mathbf{L}$ for the corresponding angular momentum operators we have for the nuclear moment,

$$\mu_C = \langle \Psi | g_s\mathbf{S} + g_l\mathbf{L} + g_R\mathbf{R} | \Psi \rangle_{I,I}. \tag{17}$$

Since in this model the nuclear wave functions are not uniquely determined by I and m_I, (15) can only be evaluated by making an assumption about the coupling strengths. Denoting the spin-orbit coupling strength by λ, the coupling of particle to core ($\mathbf{L}\cdot\mathbf{R}$) by β and the core rotational level spacing parameter by ϵ_R one can discuss the moments by using certain "intermediate" coupling schemes (since the observed moments do not, in general, correspond to any of three extreme coupling cases). Having chosen the coupling scheme to fit the observed moments, the observed DNM becomes a test of the theory.

Rb^{85}.—Nuclear properties are given in Table III. Bohr (BO51b) treats this nucleus as a single $f_{\frac{5}{2}}$ proton coupled to an asymmetric core. For the intermediate coupling scheme that fits the moments, $\lambda=\beta\gg\epsilon_R$. The fractional spin and orbital contributions to the magnetic moment α_s and α_l are more difficult to establish than in the SP model, since the coupling of l and s to the nuclear axis for the case of $I=l-\frac{1}{2}$ depends on the relative admixture of spin "up" and spin "down" wave functions, which is not uniquely determined as it is in the SP model. A similar difficulty arises in calculation of the spin asymmetry parameter. Bohr finds that $\zeta=1.25$ and $\alpha_s=-0.71$. (The negative value of α_s corresponds to spin and total angular momentum being in opposite directions.) From Table I we have then

$$b_{s2}=0.53\% \quad b_{s2}=0.21\% \quad b_{s4}=0.065\%$$
$$b_{s4}=0.037\% \quad b_{l2}=0.32\% \quad b_{l4}=0.028\%.$$

Using these values in Eq. (7) with $\mathcal{R}_{e2}=0.73$, and $\mathcal{R}_{e4}=0.69$ obtained by methods outlined in Appendix II, we find ϵ (Rb^{85}) $=-0.01\%$.

Rb^{87}.—The coupling scheme capable of fitting the moment requires $\beta>\lambda\gg\epsilon_R$ with the odd proton in a $p_{\frac{3}{2}}$ state. The parameters are $\zeta=0.2$, $\alpha_s=0.6$, and $\alpha_l=0.4$. With the b's given above and $\mathcal{R}_{e2}=0.58$ and $\mathcal{R}_{e4}=0.56$ obtained in Appendix II, we obtain $\epsilon_C(Rb^{87})=-0.24\%$. We have finally $\Delta_C=\epsilon_C(Rb^{85})-\epsilon_C(Rb^{87})=0.23\%$ which is in only approximate agreement with the experimental value. The lack of agreement may be ascribed, in part, to failure to consider the motion of the several equivalent odd particles.

(b) *Configuration Mixing Model* (*CM*)

Blin-Stoyle (BL53) and Arima and Horie (AR54) have accounted for the deviation of the magnetic moments of odd-even nuclei from the Schmidt limits, in the majority of cases, by configuration admixing of SP model states with the ground state. The interconfiguration mixing is presumed to result from a short-range internucleon force, for which previous evidence existed in "pairing-energy" effects. The wave function in the presence of configuration interaction may be described by

$$\Psi_j = \chi_j + \sum_i \alpha_i \Phi_i^j, \tag{18}$$

where χ_j and Φ_i^j are the ground state and excited SP state wave functions, respectively, and α_i is the augmentation parameter. The magnetic moment is given by

$$\mu_{\mathrm{CM}} = \langle \Psi | \sum_n (g_s{}^n \mathbf{S} + g_l{}^n \mathbf{L}) | \Psi \rangle_{I,I}. \tag{19}$$

The smallness of $\alpha_i (\alpha_i \leq 0.1)$ enables one to limit the contributions to the magnetic moment to terms linear in α_i. Taken in conjunction with the single particle character of the magnetic moment operator, this further restricts one to those states for which χ_j and Φ_i^j differ by one particle and for which $\Delta l = 0$. For such states a single particle is transferred from a $(j = l + \frac{1}{2})$ state to a $(j = l - \frac{1}{2})$ state. For $I = l + \frac{1}{2}$ and $I = l - \frac{1}{2}$ nuclei this requires that there is more than one nucleon or more than one hole in the respective odd particle ground states. There also exists the possibility in a closed $j' = l' + \frac{1}{2}$ subshell of exciting a nucleon to an incomplete $j' = l' - \frac{1}{2}$ subshell. The sole problem is to determine the strength of the short-range interaction and its A dependence. Having done this Arima-Horie predict for the μ's of interest $\mu_{\mathrm{CM}}(\mathrm{Rb}^{85}) = 1.32\mu_n$ $(\mu_{\mathrm{exp}} = 1.35\mu_n)$ and $\mu_{\mathrm{CM}}(\mathrm{Rb}^{87}) = 2.79\mu_n (\mu_{\mathrm{exp}} = 2.75\mu_n)$. Stroke and Jaccarino (ST58) have used a CM model to predict the DNM. The calculation of $\Delta(\mathrm{Rb}^{85,87})$ based on their procedure is given below.

Rb^{85}.—The proton and neutron configurations are $(p_{\frac{3}{2}})^4 (f_{\frac{5}{2}})^5$ and $(g_{9/2})^8$, respectively, where only the possible admixable valence states are listed. The fractional contributions of spin and orbit to the magnetic moment $\alpha_s{}^i$ and $\alpha_l{}^i$ for each of the states (i) involved may now be calculated, where now, $\sum_i \alpha_s{}^i + \alpha_l{}^i = 1$. Each excited state is characterized by a definite l so that the corresponding $\mathcal{R}_{e2}$ and $\mathcal{R}_{e4}$ for each may be found (see Appendix II). These, together with the appropriate values of the b's (Table I), yield $\epsilon_{\mathrm{CM}}(\mathrm{Rb}^{85}) = 0.13\%$.

This procedure neglects the fact that the spin asymmetry operator will admix states of a different l, $\Delta l = 2$, and that there also exist off-diagonal elements connecting states of different n and j (SC58).

Rb^{87}.—The proton configuration is $(p_{\frac{3}{2}})^3 (f_{\frac{5}{2}})^6$ and the neutron configuration is $(g_{9/2})^{10}$. Proceeding as before we find $\epsilon_{\mathrm{CM}}(\mathrm{Rb}^{87}) = -0.21\%$ so that $\Delta_{\mathrm{CM}} = 0.34\%$, which is to be compared to an experimental value of $\Delta = (0.3513 \pm 0.0006)\%$.

The agreement between theory and experiment is quite satisfactory. To demonstrate the sensitivity of the theory to the model used, one may compute a Δ based on nuclear model in which all protons contribute to the orbital moment, i.e., $g_l = Z/A$. This leads to $\Delta = 0.10\%$ for $\mathrm{Rb}^{85,87}$ in disagreement with the observed Δ.

The values of $\epsilon_{\mathrm{C}}(\mathrm{Rb}^{85})$ and $\epsilon_{\mathrm{CM}}(\mathrm{Rb}^{85})$ and those of $\epsilon_{\mathrm{C}}(\mathrm{Rb}^{87})$ and $\epsilon_{\mathrm{CM}}(\mathrm{Rb}^{87})$ differ considerably more than the values of Δ_{C} and Δ_{CM}. The measurement of another Rb isotope might therefore provide a crucial test of the C and CM theories.

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APPENDIX I

Assuming the proton charge distribution to be that of a uniformly charged sphere, of radius R_e, then the "interior" potential is

$$V_e = \left(\tfrac{3}{2} - \tfrac{1}{2}x^2\right)\frac{Ze}{R_e}, \quad x \equiv \frac{r}{R_e} < 1. \tag{I-1}$$

A series solution to the radial part of the Dirac equation yields in that case

$$\left.\begin{matrix} F(s_{\frac{1}{2}}) \\ -G(p_{\frac{1}{2}}) \end{matrix}\right\} = \tfrac{1}{2}k\gamma x\left[1 + L - \left(\frac{1}{5} + \frac{9}{40}\gamma^2\right)x^2 + \tfrac{1}{10}\gamma^2 x^4 \cdots\right] \tag{I-2}$$

and

$$\left.\begin{matrix} G(s_{\frac{1}{2}}) \\ F(p_{\frac{1}{2}}) \end{matrix}\right\} = k\left[1 - \tfrac{3}{8}\gamma^2 x^2 + \tfrac{1}{10}\gamma^2 x^4 \cdots\right], \tag{I-3}$$

where

$$L = \begin{cases} 0 & \text{for } s_{\frac{1}{2}} \text{ electrons} \\ \dfrac{4}{3}\dfrac{R_e}{\gamma}\dfrac{mc}{\hbar} & \text{for } p_{\frac{1}{2}} \text{ electrons} \end{cases}$$

and $\gamma = \alpha Z$. The R-dependent integrals in Eqs. (5) and (6) are

$$\int_0^R FGdr = (\pm)\tfrac{1}{4}k^2\gamma R_e\left\{(1+L)\left(\frac{R^2}{R_e^2}\right) - \left[\frac{1}{10} + \left(\frac{3}{10} + \frac{3L}{16}\right)\gamma^2\right]\left(\frac{R^4}{R_e^4}\right)\right\} \tag{I-4}$$

$$\int_0^R \frac{r^3}{R^3}FGdr = (\pm)\tfrac{1}{4}k^2\gamma R_e\left\{\tfrac{2}{5}(1+L)\left(\frac{R^2}{R_e^2}\right) - \left[\frac{2}{35} + \left(\frac{6}{35} + \frac{3L}{28}\right)\gamma^2\right]\left(\frac{R^4}{R_e^4}\right)\right\}, \tag{I-5}$$

where the upper and lower sign refer to the $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ states, respectively. The error introduced by neglecting terms containing powers higher than R^4/R_e^4 is less than 2%. The denominator integral $\int_0^\infty FGdr$ was evaluated by matching the solutions for an unscreened Coulomb field to the interior solutions at $r=R_e$. Bohr and Weisskopf (BO50) point out the reliability of this procedure for $s_{\frac{1}{2}}$ states and for $p_{\frac{1}{2}}$ states for large Z. In this way one obtains

$$\int_0^\infty FGdr=\tfrac{1}{4}k^2\frac{\hbar}{mc}\left(\frac{2R_eZ}{a_0}\right)^{2(1-\rho)}\frac{3}{\rho(4\rho^2-1)}[(2\rho-1)!]^2 \times\begin{cases}(1-\tfrac{1}{2}\gamma^2)^2 & (s_{\frac{1}{2}})\\ (-1)\dfrac{4}{3\gamma^2}(1-0.72\gamma^2)^2 & (p_{\frac{1}{2}}).\end{cases} \qquad \text{(I-6)}$$

Schwartz (SC57) recently calculated a correction to this expression for the $p_{\frac{1}{2}}$ state which is quite important at low Z—due to the fact that a $p_{\frac{1}{2}}$ function has an appreciable density at large r where the electron is subject to considerable screening. This correction is included in the numerical computations that follow.

The integrals in Eqs. (5) and (6) can now be obtained from (I-4), (I-5), and (I-6) and lead to Eqs. (7) and (8) and similar expressions for the $p_{\frac{1}{2}}$ state.

The b's are functions of Z and R_e and using $R_e=1.20A^{\frac{1}{3}}\times10^{-13}$ cm, R_e may be expressed as a function of Z by choosing equivalent "stable valley" values of A. The values of the b's computed in this manner are given in Table I.

The calculation of $\mathcal{R}_{e2}$ and $\mathcal{R}_{e4}$ required for Eq. (7) and Eq. (8) is discussed in Appendix II.

The spin asymmetry parameter ζ^* must be averaged over the nuclear polar angles. A procedure for doing this is given by Bohr (BO51b). For the extreme single particle model, writing ζ for the average value of ζ^* one finds

$$\zeta=\frac{2I-1}{4(I+1)}, \qquad I=l+\tfrac{1}{2} \qquad \text{(I-7)}$$

$$\zeta=\frac{2I+3}{4I}, \qquad I=l-\tfrac{1}{2}. \qquad \text{(I-8)}$$

APPENDIX II. CALCULATION OF $\mathcal{R}_{ei}$

The calculation of the effect of the DNM requires a knowledge of $\mathcal{R}_{e2}$ and $\mathcal{R}_{e4}$. $\mathcal{R}_{ei}$ is defined as the average value over the nucleus of $(R^i/R_e{}^i)$ where R is the radial coordinate of the nucleon under consideration and R_e is the nuclear charge radius. These parameters may be obtained from the single particle model of the nucleus in which a nucleon is considered to move in a spherically symmetric potential well. We assume, for simplicity, that the well is square, of depth V and radius R_V, but diffuse wells may be considered in a similar fashion. Square wells have been shown to reproduce the observed shell structure of nuclei ($A>20$) quite accurately when strong spin-orbit coupling is included (MA55).

Our procedure is to obtain a well-behaved normalized radial wave function for the odd nucleon using reasonable values for its separation energy‖ S and the well parameters V and R_V. The angular momentum of the nucleon is presumed known from the measured nuclear spin and the shell-model level scheme. From the radial wave functions we calculate the average value of $(R^i/R_V{}^i)$ which we write $\mathcal{R}_{Vi}$. Thus, $\mathcal{R}_{ei}$ is then given by

$$\mathcal{R}_{ei}=\left(\frac{R_V}{R_e}\right)^i\mathcal{R}_{Vi}. \qquad \text{(II-1)}$$

The value of (R_V/R_e) is somewhat greater than unity; calculations for Au^{197} (FO56) indicate that it is approximately 1.1. R_e, the charge radius is assumed as $R_e=r_0A^{\frac{1}{3}}$ with $r_0=1.20\times10^{-13}$ cm. Here we assume that $R_V/R_e=1.1$ and independent of A. It should be kept in mind that this is considerably less certain than the values of $\mathcal{R}_{Vi}$ and R_e. This value of r_0 is based on the recent high-energy electron scattering experiments of Hofstadter

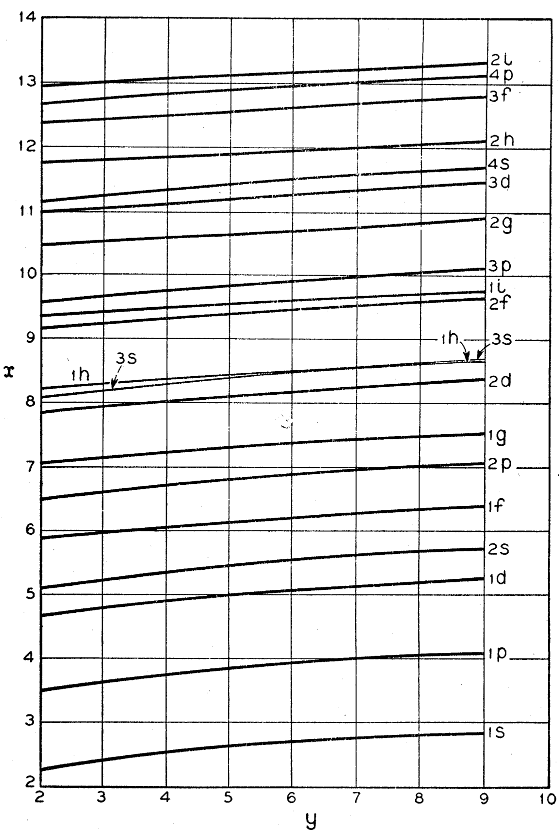

FIG. 2. Solutions of the eigenvalue equation (II-2) which match the inside and outside solutions of single particle wave equations. The nuclear potential is considered to be a square well and the parameters x and y depend on the size of the nucleus and the binding energy of the odd nucleon.

‖ The separation energy S is defined as the work that must be done to move the last unpaired nucleon from inside the nucleus to infinity. It is approximately the same for protons and neutrons (BL52) and can be obtained from the binding energies, B, of the nuclei or (γ,n) thresholds. $S(A,Z)=B(A,Z)-B(A-1,Z-1)$ for odd protons and $S(A,Z)=B(A,Z)-B(A-1,Z)$ for odd neutrons.

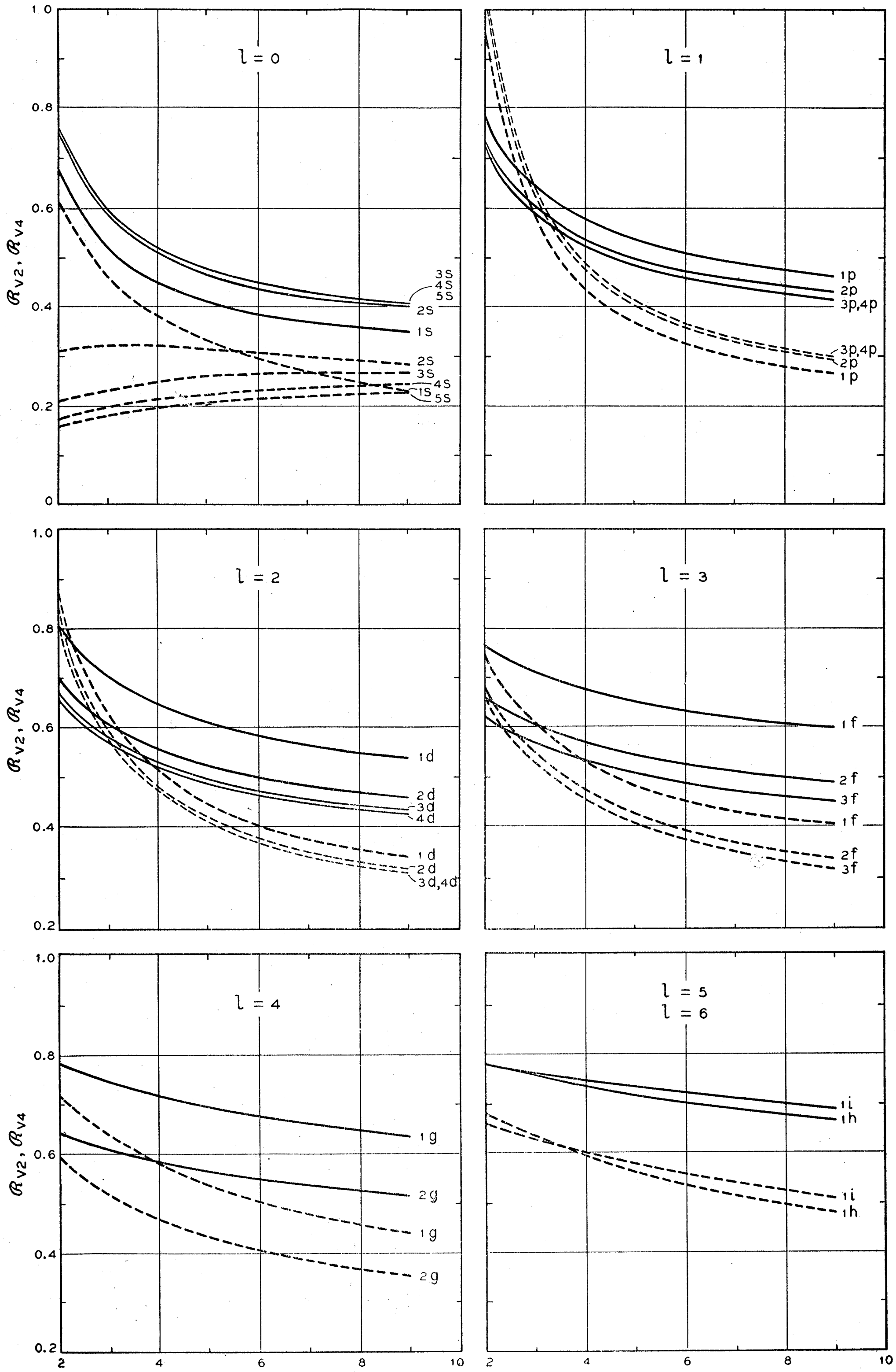

FIG. 3. Values of $\mathcal{R}_{V2}$ (solid curves) and $\mathcal{R}_{V4}$ (broken curves) which are the 2nd and 4th moments of the SP radial wave function divided by R_V^2 and R_V^4, respectively. The parameter y is defined in Eq. (II-2). It is larger the heavier the nucleus and the more tightly bound the nucleon.

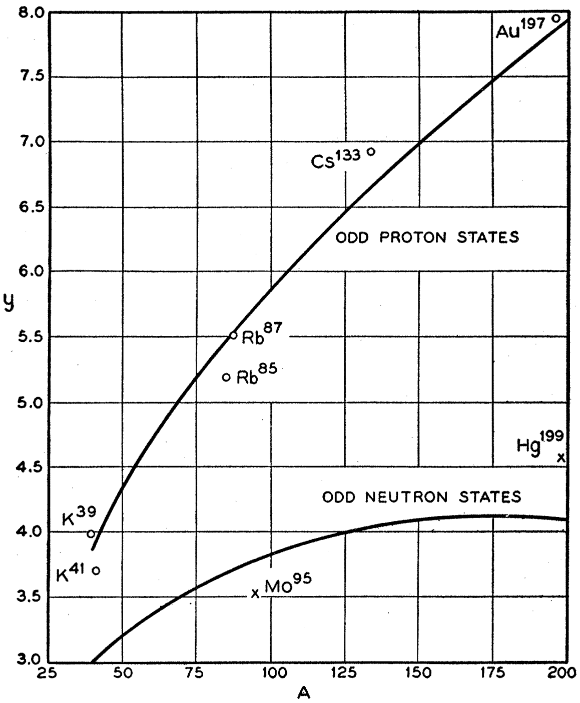

FIG. 4. The value of the parameter y for the unpaired nucleon in even-odd nuclei [see Eq. (II-4)] as a function of A. The curves for neutrons and protons are obtained by assuming a smooth relationship between the separation energy S and A as given by the empirical masses of the nuclei. The Coulomb barrier is $(Z-1)e^2/R_e$, $R_e=r_0A^{\frac{1}{3}}$. $r_0=1.20$ fermis, $R_V=1.1R_e$ and an empirical "stable valley" relationship between Z and A is used. More accurate values of y for ${}_{19}K^{39,41}$, ${}_{37}Rb^{85,87}$, ${}_{79}Au^{197}$, ${}_{42}Mo^{95}$, and ${}_{80}Hg^{199}$ computed from the measured binding energies of these nuclei are indicated.

and collaborators (HO53, HO54, HA55) and is in agreement with the μ-mesonic charge radius. The experiments and methods of analysis used to obtain r_0 have recently been reviewed by Ford and Hill (FO55).

Solutions of the radial wave equation of a particle in a square well potential are well known (SC49), and can be expressed as spherical Bessel functions for $R<R_V$ and spherical Hankel functions for $R>R_V$. The eigenfunctions are obtained by making the value and the first derivative continuous at $R=R_V$. This condition can be written as

$$x\left[\frac{j_l'(\rho)}{j_l(\rho)}\right]_{\rho=x}=iy\left[\frac{h_l'(i\rho)}{h_l(i\rho)}\right]_{\rho=y}, \tag{II-2}$$

where l is the angular momentum of the odd nucleon, j_l and h_l are the spherical Bessel and Hankel functions, respectively, and

$$x=\alpha R_V; \qquad \alpha=\hbar^{-1}[2M(V-S)]^{\frac{1}{2}} \tag{II-3}$$

$$y=\beta R_V; \qquad \beta=\hbar^{-1}[2ME]^{\frac{1}{2}}, \tag{II-4}$$

where M is the nucleon mass. The binding energy E for odd neutrons is simply the separation energy S. For odd protons the binding energy is increased by a Coulomb potential barrier which we approximate by a flat potential of height $E_c=(Z-1)e^2/R_e$. Since the proton is inside the nucleus most of the time this approximation introduces a negligible error. For odd protons then $E=S+E_c$, and for odd neutrons $E=S$.

For any given nucleus one may obtain y from Eq. (II-4) above by using the appropriate values of E and R_V. For this purpose (II-4) may be conveniently written as $y=0.219R_VE^{\frac{1}{2}}$ where R_V is in fermis (10^{-13} cm) and E in Mev. From (II-2) one can now derive the appropriate eigenvalue x to make the wave function well behaved over its entire range of R. These solutions are shown in Fig. 2. The potential well depth corresponding to any given x can be determined from Eq. (II-3). It is approximately constant (≈ 60 Mev) for all nuclei (FO56).

$\mathcal{R}_{Vi}$ is given by the expressions $\mathcal{R}_{Vi}=\langle R^i\rangle_{\text{Av}}/R_V{}^i$,

$$\langle R^i\rangle_{\text{Av}}=\frac{\int_0^{R_V}[j_l(\alpha R)]^2R^{i+2}dR+\int_{R_V}^{\infty}[h_l(\beta R)]^2R^{i+2}dR}{\int_0^{R_V}[j_l(\alpha R)]^2R^2dR+\int_{R_V}^{\infty}[h_l(\beta R)]^2R^2dR}. \tag{II-5}$$

$\mathcal{R}_{Vi}$ is a function of x, y, l, and n where n is the principal quantum number and $n=1$, 2, etc. refers to the first, second, etc. root of Eq. (II-2). $\mathcal{R}_{Vi}$ was computed with the aid of an IBM-650 computer by using power series expansions for j_l and h_l. Solutions were obtained¶ for $l=0, 1\cdots 6$ with $2\leq y\leq 9$ and values of n which would be of interest in real nuclei. These are displayed in Fig. 3.

Using the empirical mass formula to calculate S (BL52), it is found to vary between 9 Mev and 4.5 Mev when A goes from 40 to 220. With this and the usual formulas for $R_V(=1.10R_e)$ and E_c one can derive, with the aid of Eq. (II-4), an approximate value of y for any given A. This relation is given in Fig. 4 and with it and Fig. 3 one can obtain values of $\mathcal{R}_{Vi}$ that are good to a few percent for most odd-proton nuclei. For odd-neutron nuclei y is more sensitive to variations in S since $E_c=0$ and y should be obtained by the more precise procedure outlined above.

We illustrate this method by determining $\mathcal{R}_{Vi}$ for Rb^{87}. The odd proton is in a $2p_{\frac{3}{2}}$ state so that $l=1$, $n=2$. From the binding energies of Rb^{87} and Kr^{86} tabulated by Wapstra (WA55) we find $S=8.8$ Mev. The Coulomb barrier is $E_c=9.8$ Mev so that $E=18.6$ Mev. Using $R_V=5.8$ fermis one obtains $y=0.219R_VE^{\frac{1}{2}}=5.5$. Finally, from the appropriate curve of Fig. 3, we find $\mathcal{R}_{V2}=0.48$ and $\mathcal{R}_{V4}=0.38$. If $R_V/R_e=1.10$, this leads to $\mathcal{R}_{e2}=0.58$ and $\mathcal{R}_{e4}=0.56$.

¶ We are indebted to Dr. M. C. Gray for suggesting the recursion formulas and expansions required for the evaluation of the integrals of Eq. (II-5).

The use of diffuse wells (RO56) will lead to somewhat larger values of $\mathcal{R}_{Vi}$.

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