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R-Matrix Theory of Nuclear Reactions

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I. INTRODUCTION

THIS paper describes advances made in the theoretical study of nuclear reactions during the last twenty years. We begin by surveying some of the main achievements in approximately chronological order.

In 1937 Bethe¹ gave a quantitative discussion of nuclear reactions in terms of the compound nucleus mechanism suggested just previously by Bohr.² The following few years saw publication of further papers by Weisskopf,³ Bethe and Placzek,⁴ and Weisskopf and Ewing,⁵ in which detailed consequences of Bohr's suggestion were thoroughly investigated.⁶

These early discussions were not based upon any general and rigorous theory of reactions but were constructed in an *ad hoc* fashion around the assumption that reactions proceed by the compound nucleus mechanism. When Bohr proposed this mechanism, there was no mathematical framework in quantum mechanics that could be adopted to give a quantitative description of it. The only type of theory that seemed

at all relevant was the Weisskopf-Wigner theory⁷ of resonance absorption and subsequent emission of optical radiation by an atomic system. This type of time-dependent perturbation theory was adapted to the treatment of nuclear reactions by Breit and Wigner⁸ and resulted in the formula that bears their name. Although nuclear forces can certainly *not* be realistically treated as perturbations, the expressions for cross sections derived by Breit and Wigner (and later by Bethe¹ with time-independent perturbation theory) were reasonable in *form*, i.e., in their energy dependence. For instance, the Breit-Wigner formula for cross sections of reactions proceeding through an isolated resonance gives excellent fits to observed cross sections. Absurdities appear only when attempts are made to interpret the values of the parameters (the matrix elements) obtained from such fitting. The reason for the correctness of the *form* is that this only depends on the condition that the reaction proceeds through an isolated (i.e., long-lived) intermediate state. Such a condition is satisfied both for atomic reactions of the kind described by the Weisskopf-Wigner theory and for nuclear reactions proceeding through a compound nucleus. However there are important differences between the two types of reaction. In the former case, the excitation energy is concentrated on a single particle (electron) and the long life is due to the *weakness* of the coupling of this particle to the radiation field. In the latter case, the *strength* of the nuclear forces leads to a sharing of the available energy by all particles (nucleons) and the long life is due to the small probability of the energy being concentrated in a mode that corresponds to disintegration by an "open" (i.e., energetically allowed) channel.

The unsatisfactory reliance on perturbation theory was removed by the presentation by Kapur and Peierls⁹ in 1938 of a rigorous theory of reactions which was not dependent upon any particular physical picture (such as the compound nucleus mechanism) or upon any dubious mathematical approximations (such as perturbation theory). This theory and its subsequent versions by other authors has provided a very satisfactory framework for discussing nuclear reactions. Its essential feature is the occurrence of a complete set of formal states (of all particles) defined in a volume of nuclear size by the imposition of some fixed boundary condition on the surface of this volume. Although the theory is rigorous and therefore capable of describing all types of reaction mechanisms, it is especially easily adapted for describing the compound nucleus mechanism. This is done by identifying the formal states with states of the compound nucleus. Neglect of mechanisms

¹ H. A. Bethe, *Revs. Modern Phys.* **9**, 69 (1937).

² N. Bohr, *Nature* **137**, 344 (1936); *Science* **86**, 161 (1937).

³ V. F. Weisskopf, *Phys. Rev.* **52**, 295 (1937).

⁴ H. A. Bethe and G. Placzek, *Phys. Rev.* **51**, 450 (1937).

⁵ V. F. Weisskopf and D. H. Ewing, *Phys. Rev.* **57**, 472 (1940).

⁶ F. L. Friedman and V. F. Weisskopf, *Niels Bohr and the Development of Physics* (Pergamon Press, London, 1955).

⁷ V. H. Weisskopf and E. P. Wigner, *Z. Physik* **63**, 54 (1930); **65**, 18 (1930).

⁸ G. Breit and E. P. Wigner, *Phys. Rev.* **49**, 519 (1936); **49**, 642 (1936); G. Breit, *ibid.* **40**, 127 (1932).

⁹ P. L. Kapur and R. E. Peierls, *Proc. Roy. Soc. (London)* **A166**, 277 (1938); R. E. Peierls, *Proc. Cambridge Phil. Soc.* **44**, 242 (1947).

other than the compound nucleus one is achieved by ignoring the presence of the formal states that are far away in energy.

Since the paper by Kapur and Peierls appeared in 1938, both Breit¹⁰ and Wigner¹¹ with co-workers have written papers on the same problem of providing a rigorous theory of nuclear reactions. Of course, the various approaches must be equivalent if they are all rigorous. They differ, in fact, in details and emphasis rather than in principle. Breit, for example, was concerned mainly with the formal problem of trying to remove the occurrence in the cross-section expressions of certain parameters (the interaction radii) whose values cannot affect the cross sections. Wigner,¹¹ in contrast, actually emphasized the occurrence of these parameters in the theory and exploited the fact that particular choices of these parameters have physical significance and therefore enable one to introduce physical information into the theory. This is especially true of the "R-matrix" theory, first expounded by Wigner and Eisenbud.¹²

The main achievement of Wigner's particular formulation has been to make the energy dependence of all expressions as explicit as possible. In contrast Kapur and Peierls⁹ were not so concerned with the questions of detailed energy dependence and worked with complete sets of states defined by energy-dependent (and complex) boundary conditions. Almost all of the quantities occurring in the Kapur-Peierls cross sections are energy-dependent, either explicitly or implicitly. In order to use their formulas, the energy dependences must be made explicit, and this means referring to a treatment such as Wigner's with energy-independent boundary conditions. For instance, in the case of an isolated resonance, the Kapur-Peierls resonance energy is implicitly energy-dependent. This dependence may be made explicit by referring to energy-independent boundary conditions. It is found that the dependence is just equivalent to the presence of the level shift factor which occurs explicitly in the Wigner formulation *from the beginning*. It is this sort of consideration which persuaded us to base the present review upon the R-matrix theory of Wigner and Eisenbud rather than upon the theory of Kapur and Peierls. However we emphasize again that the two formulations are absolutely equivalent; the only particular merit of the R-matrix theory is that it is more explicit.

The R-matrix theory is rigorous, and therefore its application is not confined to reactions that proceed by the compound nucleus mechanism; in principle, it can be used to describe all types of reaction phenomena. Since the theory was presented in 1947, a new type of phenomenon has been established and has been fitted into the framework of the theory. This is the possibility

that colliding nuclei can interpenetrate each other *without* necessarily forming a compound nucleus. This very significant fact was deduced by Feshbach,¹³ Porter, and Weisskopf in 1953 from the energy dependence of total neutron cross sections. Previously it had been assumed that any particle entering a nucleus must inevitably lose energy and thereby cause the formation of a compound nucleus. In the terminology invented by Bethe,¹⁴ this was expressed by saying that the nuclear "sticking probability" is equal to unity. Most earlier discussions of reactions were based upon this same assumption. It was embodied in the papers by Weisskopf and Ewing^{3,5} on the spectra of particles from compound states, and also in the papers by Feshbach, Peaslee, and Weisskopf¹⁵ and by Feshbach and Weisskopf¹⁶ on total cross sections.

For the sake of preciseness, we retain the term "model" for those phenomenological theories of nuclear collisions that prescribe probabilities of compound nucleus formation. The expression "strong absorption model" is identified with those theories in which the chance of collision without compound nucleus formation is very small; the expression "moderate absorption model" labels those theories in which the chance is appreciable. The R-matrix theory must be capable of describing *all* such models whether of the "strong absorption" or "moderate absorption" variety. According to Thomas,¹⁷ each model implies a special energy dependence of the so-called "strength function" which is the central quantity in the R-matrix theory; the moderate absorption model implies oscillatory dependence; in contrast the strong absorption model implies a smooth monotonic dependence. Recently, it has been shown¹⁸ that one may relate the form of the strength function to general dynamical properties of nuclear matter.

The validity of the moderate (as opposed to the strong) absorption model opens up the possibility of an entirely new class of nuclear reactions, namely "direct" reactions in which nuclei interpenetrate without forming a compound nucleus, and directly scatter each other with or without a change in their internal structures. Existence of such direct reactions was established by Butler¹⁹ in 1950 from the angular distributions of (*d,p*) reactions. The loose structure of the deuteron makes the chance of interaction without compound nucleus formation especially high for this class of reactions. Austern, Butler, and McManus²⁰ later proposed a

¹³ Feshbach, Porter, and Weisskopf, Phys. Rev. **90**, 166 (1953); **96**, 448 (1954).

¹⁴ H. A. Bethe, Phys. Rev. **57**, 1125 (1940).

¹⁵ Feshbach, Peaslee, and Weisskopf, Phys. Rev. **71**, 145 (1947).

¹⁶ H. Feshbach and V. F. Weisskopf, Phys. Rev. **76**, 1550 (1949).

¹⁷ R. G. Thomas, Phys. Rev. **97**, 224 (1955).

¹⁸ C. Bloch, J. phys. radium **17**, 510 (1956); Nuclear Phys. **3**, 137 (1957); G. E. Brown and C. T. de Dominicis, Proc. Phys. Soc. (London) **A70**, 668 (1957).

¹⁹ S. T. Butler, Phys. Rev. **80**, 1095 (1950); Proc. Roy. Soc. (London) **A208**, 559 (1951).

²⁰ Austern, Butler, and McManus, Phys. Rev. **92**, 350 (1953).

¹⁰ G. Breit, Phys. Rev. **58**, 506 (1940); Phys. Rev. **69**, 472 (1946).

¹¹ E. P. Wigner, Phys. Rev. **70**, 15 (1946); **70**, 606 (1946).

¹² E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947); E. P. Wigner, J. Am. Phys. Soc. **17**, 99 (1949).

similar interpretation for (n,p) reactions. In both treatments, direct processes were considered to occur only when colliding nuclei graze each other, i.e., they were considered to be "surface" phenomena. It has since become apparent from the work of Hayakawa, Kawai, and Kikuchi²¹ and of Brown and Muirhead,²² that this viewpoint may, at least in the (n,p) case, be too restrictive. Rather, it appears that direct reactions may originate in the volume of the target nucleus as well as at the edge. Although existence of direct mechanisms for reactions has been established beyond doubt, it seems that they account for only small fractions of most total cross sections, except when peculiar circumstances exaggerate their contribution relative to the compound nucleus one.

It is of considerable interest to see just how the **R**-matrix theory can describe the direct reaction mechanism; i.e., to see what particular assumptions must be made in order to eliminate other mechanisms such as the compound nucleus one. As mentioned above, **R**-matrix theory is easily specialized to the description of the compound nucleus mechanism. The means of making the specialization to the direct mechanism are not so evident. However it has recently been shown²³ how this can be done; one introduces assumptions into sections of just the "direct reaction" kind.

The present article is an account of only the existing *theory* of nuclear reactions, and does not describe the vast number of detailed applications that have been made to experimental data. For summarized accounts of such applications, the reader is referred to the book of Blatt and Weisskopf,²⁴ a review by Peaslee,²⁵ and especially to recent reviews by Kinsey²⁶ and Burcham.²⁷

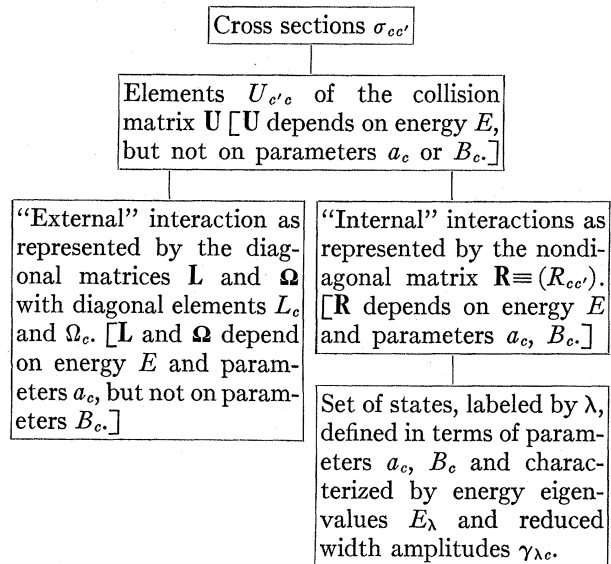
II. GENERAL DISCUSSION OF R-MATRIX THEORY

Although the essential basis of the **R**-matrix theory is quite straightforward, the full development is lengthy and involves considerable algebraic manipulation. Unless one is made familiar in advance with the essential ideas and objectives, one tends to lose track of the main arguments because of distraction by details. The present section is devoted to sketching an outline of the content and objectives of the theory.

1. Scheme of the Formal Theory

Any theory that can be specialized to the description of a nuclear reaction proceeding via formation of intermediate states of the compound nucleus, is expected to

be formulated in terms of some set of states that can be associated with those of the compound nucleus. The **R**-matrix theory¹² is such a theory. In it, a set of states of all nucleons is defined and the cross sections can ultimately be expressed in terms of these. In the general formulation, however, the algebraic relations connecting the cross section and the states are very complicated, so it is convenient to introduce intermediary quantities, one of which is the "collision matrix" and the others "the **L**, **Ω**, and **R** matrices." The general scheme is as follows:



Here $\sigma_{cc'}(E)$ is defined as the cross section for the production of the pair of nuclei of type c' when the two nuclei of the pair of type c are bombarded against each other with energy E . The element $U_{c'c}(E)$ of the so-called collision matrix **U** is defined as the amplitude of the outgoing waves of pair c' resulting from unit flux of bombardment with pair c . It follows that the cross section $\sigma_{cc'}$ must be proportional to $|U_{c'c}|^2$. The introduction of **U** with its rather trivial relation to the cross sections is, of course, not a peculiarity of the **R**-matrix theory but is the usual first step in any quantum-mechanical reaction theory. **U** is an especially convenient quantity because two very general physical principles, those of conservation of probability flux and time-reversibility, impose restrictions on any reaction theory which, as first shown by Breit,²⁸ can be simply stated in terms of the properties of **U**, namely, **U** must be unitary and symmetric (Sec. VI).

The special features of the **R**-matrix theory lie in the further steps, namely those in which **U** is expressed in terms of the matrices **L**, **Ω**, and **R**. The first two matrices are diagonal and take account of any long-range nonpolarizing interactions acting between any separated nuclei. The **R** matrix is nondiagonal and takes account of the effects of all other interactions, i.e., those

²¹ Hayakawa, Kawai, and Kikuchi, *Progr. Theoret. Phys.* **13**, 415 (1955).

²² G. Brown and H. Muirhead, *Phil. Mag.* **2**, Ser. 8, 473 (1957).

²³ C. Bloch, *Nuclear Phys.* **4**, 503 (1957); G. E. Brown and C. T. de Dominicis, *Proc. Phys. Soc. (London)* **A70**, 686 (1957).

²⁴ J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

²⁵ D. C. Peaslee, *Ann. Revs. Nuclear Sci.* **5**, 99 (1955).

²⁶ B. B. Kinsey, *Encyclopaedia of Physics* (Springer-Verlag, Berlin, 1957), Vol. 40.

²⁷ W. E. Burcham, *Encyclopaedia of Physics* (Springer-Verlag, Berlin, 1957), Vol. 40.

²⁸ G. Breit, *Phys. Rev.* **58**, 1068 (1940).

operating *inside* nuclei, both inside the compound nucleus and inside the nuclei of separated pairs. All three matrices depend on certain parameters a_c , one for each type of pair c . Given these parameters, the matrices \mathbf{L} and $\mathbf{\Omega}$ are fully determined. The unknown quantity is then the \mathbf{R} matrix. In addition to depending on E and the a_c , this matrix also depends on a set of boundary condition parameters B_c , one for each type of pair c . Even if all the parameters are specified, \mathbf{R} is still essentially unknown in general. In spite of this, Wigner and Eisenbud¹² have, shown that the energy dependence of any element of \mathbf{R} is expressible in a surprisingly simple form:

$$R_{cc'}(E) = \sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{E_{\lambda} - E},$$

where λ labels the members of a complete set of states and the $\gamma_{\lambda c}$, $\gamma_{\lambda c'}$, and E_{λ} are energy-independent quantities depending on a_c , B_c . The $\gamma_{\lambda c}$ are called "reduced width amplitudes." For each state λ , one $\gamma_{\lambda c}$ is defined for each pair c . The E_{λ} are the energy eigenvalues of the states λ .

Since few specific assumptions are made in the \mathbf{R} -matrix theory, which therefore has wide generality, one might guess that the simplicity of the analytic form of the elements of the \mathbf{R} matrix is the consequence of some third general physical principle in addition to those of conservation of probability and time-reversibility already mentioned. Following the initial suggestion by Schuster and Tiomno,²⁹ (Sec. IV, 8), it has been shown that, in the special case when only scattering of the initial pair is allowed to occur, this principle is the one of causality. (Roughly speaking, the causality principle says that the two nuclei cannot be scattered before they interact.) No proof has as yet been given in the general case that causality leads to the analytical form of \mathbf{R} , but there is no reason to doubt that such a proof could be given. Presumably the general form of the results of \mathbf{R} -matrix theory do not depend on special information or assumptions contained in the quantum-mechanical derivation of Wigner and Eisenbud but only on certain general physical principles.

For our purposes it is desirable, not only to know the analytic form of the \mathbf{R} matrix, but also to be able to interpret the parameters in this form such, as the $\gamma_{\lambda c}$. In the Wigner-Eisenbud derivation, these are given a precise meaning as a certain type of matrix element. This means that one can hope not merely to fit experimental cross sections in terms of the parameters of \mathbf{R} -matrix theory, but also to predict the values and properties of these parameters on the basis of some theory of nuclear forces and nuclear structure. For this reason, we do not base the present review on general derivation of the \mathbf{R} matrix from the causality principle, but use the Wigner-Eisenbud derivation.

2. Assumptions of the Theory

Certain broad assumptions are at the basis of the \mathbf{R} -matrix theory.

(i) *Applicability of nonrelativistic quantum mechanics* as implied in the assumption of the usual Hamiltonian equation defined in all space:

$$H\Psi = E\Psi,$$

where H is the sum of kinetic and potential energies. This equation must embody the three general physical principles (conservation of probability, time-reversibility, and causality) already mentioned. (For instance, conservation of probability demands the Hermiticity of the potential energy in H .) It follows that the collision matrix derived under the present assumption must automatically be unitary, symmetric, and have certain analytic properties. (This is verified in Sec. VII.)

Neglect of relativistic effects is just as appropriate in the usual \mathbf{R} -matrix theory as in other theories in low-energy nuclear physics. The justification is that nucleon kinetic energies inside nuclei are less than a few percent of the rest mass energy. Goertzel³⁰ has made an extension of the \mathbf{R} -matrix theory to Dirac particles, but we do not need to use his results.

(ii) *Absence or unimportance of all process in which more than two product nuclei are formed.*—This assumption is the most restrictive one. It implies that the theory is not strictly applicable to reactions where the bombarding energy is high enough to make three-body breakup energetically possible. However, we see later (Sec. XIII, 2) that approximate treatment of many-body decays can be given if these can be described as a succession of two-body decays.

(iii) *Absence or unimportance of all processes of creation or destruction.*—The main effect of this assumption is to exclude photons from discussion. We show in Sec. XIII, 3 how this restriction can be removed within the limits of a conventional perturbation treatment of the coupling of nuclear particles to the electromagnetic field.

(iv) *The existence, for any pair of nuclei c , of some finite radial distance of separation a_c , beyond which neither nucleus experiences any polarizing potential field from the other.*—We assume that, beyond separation a_c , any potential acting between the pair c can be written as a function of radial distance only. Given a value of a_c satisfying this assumption, any larger value will also satisfy it so that, in the absence of further conditions, the (a_c) are largely arbitrary. In \mathbf{R} -matrix theory no such further condition is mentioned. For the formal derivation, it is necessary only to be able to assume the existence of a set, *any* set, of the a_c that satisfies the above assumption. The theory is then framed in terms of this set. For instance the \mathbf{L} , $\mathbf{\Omega}$, and \mathbf{R} matrices depend explicitly and implicitly on the a_c . The collision matrix \mathbf{U} itself does not depend on any set of formal param-

²⁹ W. Schuster and J. Tiomno, Phys. Rev. **83**, 249 (1951).

³⁰ G. Goertzel, Phys. Rev. **73**, 1463 (1948).

eters such as the a_c , so the dependences of \mathbf{L} , $\mathbf{\Omega}$, and \mathbf{R} on a_c must be complementary in such a way so as to make \mathbf{U} independent of the a_c . It is clearly unsatisfactory and artificial to retain such a set of arbitrary formal parameters. For example, in order to analyze a cross-section curve and extract values of the $\gamma_{\lambda c}$ and E_λ , one would have to assume a set of values of the a_c , so that any extracted values would depend on the a_c . Then in making theoretical predictions about the $\gamma_{\lambda c}$ and E_λ , one would have to assume the same set of values. The whole comparison between experiment and theory would center around the set of a_c , a set of quite arbitrary formal parameters.

The point is that one could not, in general, extract values of $\gamma_{\lambda c}$ and E_λ from the experimental cross sections even if a set of the a_c were assumed. There are an infinite number of the $\gamma_{\lambda c}$ and E_λ in the theory, so it would never be possible to fit them from a given experimental cross section no matter how accurately measured. In order to be able to use \mathbf{R} -matrix theory at all, one must have some broad *a priori* knowledge of the role of the $\gamma_{\lambda c}$ and E_λ based on a physical picture. Since the $\gamma_{\lambda c}$ and E_λ depend on the a_c , this picture must be one in which one definite set of the a_c has a special physical significance. The idea that nuclear reactions can only occur when the closely packed colliding nuclei actually enter each other's volume, provides the kind of picture needed. It suggests choosing the a_c to be equal to the sum of the radii of the colliding nuclei, which is the minimum value allowed by condition (4). In future, we always understand the a_c to be defined as having such minimum values and refer to them as "interaction radii."

The idea that reactions can only originate at separation distances $< a_c$ does not imply that a reaction *must* take place if two nuclei approach closer than this. The actual situation may be anywhere between two following extremes.

(i) As two nuclei come together, there is sharp transition between the point where neither nucleus experiences any polarizing forces at all, and the point where both nuclei interact so strongly as to lose completely their individual identities. If the a_c are now defined as having their minimum values under assumption (4), they will be located fairly precisely in this sharp transition distance. This situation is that implied by the "strong absorption" model of compound nucleus formation in which two nuclei fuse together as soon as they touch. The ideal limit of an infinitely sharp transition region corresponds to all reactions proceeding via a compound nucleus and excludes the possibility of other types of reaction process (Sec. II, 4).

(ii) Nuclei may interpenetrate each other freely without appreciable disturbance of each other's internal structures. Thus the distance a_c marks the edge of a potential field which is smooth and refractive in character, and does not give rise to the violent interaction

needed to cause compound nucleus formation. Rather it either merely scatters the colliding nuclei or, occasionally, may cause a direct reaction. If this picture were true, nearly all reactions would be of the "direct" type not involving compound nucleus formation. The actual situation with nuclei lies somewhere between (i) and (ii). The evidence is that it is considerably closer to (i) than (ii), because, although in certain special reactions the direct process implied by (ii) predominates, most reactions are mainly governed by compound nucleus formation.

3. Specialization to the Compound Nucleus Mechanism

The qualitative picture of the compound nucleus enables one to guess the special properties that must be assigned to the $\gamma_{\lambda c}$ and E_λ in order to specialize \mathbf{R} -matrix theory to the compound nucleus mechanism.

First we consider the energy region in the compound nucleus just above the highest bound states, i.e., the region in which resonance levels are found in nuclear reactions. The strength of nuclear forces and the consequent complexity and compact nature of nuclear states leads us to believe that the spectrum of nuclear states will persist, in some fashion, for some way above the bound states. For instance, let us ensure such a continuation in the nuclear spectrum by enclosing the nucleus in a sphere of nuclear dimensions, and consider a state that would normally be above the bound states. On removing the sphere, the complexity of the state is such that a long time passes before the state "realizes that it is unbound" and disintegrates. Thus the state, although free, maintains much of the character of a bound state, the only difference being that it has a nonzero width Γ reflecting the fact that it does eventually disintegrate after a mean life \hbar/Γ . Defining the a_c as we have done introduces an effective enclosing sphere of nuclear dimensions, and this means that we can associate the E_λ , the formal energy eigenvalues of the eigenstates in the enclosure, with the energies of the physically observed quasi-bound "resonance" states. In particular, it is natural, in applying \mathbf{R} -matrix theory to a reaction proceeding through a certain resonance level, to drop all terms in the \mathbf{R} -matrix sums over λ except one. This leads to the famous "one-level" cross-section formula originally formulated by Breit and Wigner⁸ with a quite different theory.

At higher energies, as resonance levels become closer and broader, they overlap to form a continuum, so the one-level formula is no longer valid or useful. Instead we must use the many-level cross-section formula and try to average it over many levels. Thomas¹⁷ has shown, following the earlier considerations by Bethe,¹ that an averaging is possible if the signs of the (real) level amplitudes $\gamma_{\lambda c}$ can be assumed to be random (Sec. XI, 2). The extent of the randomness in the signs increases with the probability of compound nucleus

formation. In the idealized "strong absorption" limit when it is supposed that any interaction between two nuclei immediately leads to their merging into a compound nucleus, the randomness is expected to be complete. With this assumption, the many-level formula can be averaged to give the expected expressions for cross sections from compound nucleus formation in the continuum. As discussed in Sec. XI, 6, the idealized limit is not attained for actual nuclei, so that, in practice, the signs are not completely random. In particular there may be mild correlations in sign over large energy intervals. Nevertheless the randomness is sufficient to enable the averaging of the many-level formula to be performed to a good approximation.

4. Direct Reaction Mechanisms in R-Matrix Theory

Although several direct reaction mechanisms have been identified experimentally, the compound nucleus mechanism still accounts for the major part of the observed reaction cross sections. Only when one observes special individual products of bombardments, may one find that other mechanisms dominate. Experimentally a number of such special individual reactions have been established where the compound nucleus contribution is rivalled by those from other types of mechanism. The other contributions are usually incoherent with the compound nucleus contribution and their origins are believed to be understood, at least qualitatively. There are four main types of such reactions.

(i) *Rearrangement reactions producing individual low-lying states of the residual nuclei.*—Reactions in which "direct" contributions have been found are (N^{14}, N^{13}),³¹ (α, p) and (p, α),³² (p, t) and (p, He^3),³³ and others, but especially those involving deuterons like (d, p)¹⁹ and (p, d) reactions. The fact that deuteron reactions are especially liable to these special effects gives the clue to their origin. This lies in the diffuseness of the surfaces of nuclei; for the loosely bound deuteron this is especially pronounced. There is an appreciable probability of "partial" interaction between a deuteron and a target nucleus in which only the neutron interacts with the nucleus. The proton may be released in such "grazing" or "stripping" reactions without interacting with the target. Study¹⁹ of individual proton groups resulting from deuteron bombardment often shows that this type of mechanism may completely dominate other mechanisms. Only those deuterons that graze the edge of the target nucleus can lead to this process. Those deuterons with smaller impact parameters that hit the target head-on will form compound nuclei in the usual way. Consequently *total* deuteron cross sections

³¹ H. L. Reynolds and A. Zucker, Phys. Rev. **101**, 166 (1956).

³² Eisberg, Igo, and Wegner, Phys. Rev. **100**, 1309 (1955); J. G. Likely and F. P. Brady, *ibid.* **104**, 118 (1956).

³³ B. L. Cohen and T. H. Handley, Phys. Rev. **93**, 514 (1954); J. G. Likely and F. P. Brady, Phys. Rev. **104**, 118 (1956).

will not be seriously affected by the existence of stripping processes (except possibly at low energies where the Coulomb repulsion prevents head-on collisions or when the target nucleus is small, i.e., light).

(ii) *Inelastic and charge-exchange scattering producing individual low-lying states.*—There is evidence for "direct" contributions to (n, p),³⁴ (p, p'),³⁵ (d, d'),³⁶ and (α, α')³⁷ reactions. These contributions show up especially well in high energy bombardments because the compound nucleus cross sections are diminished as a consequence of the smallness of the compound nucleus competition factors. The origin of these effects is found partly in the surface diffuseness. An incident projectile may graze the edge of a target nucleus and just interact with a single nucleon in the surface. This may result in an energy exchange and inelastic scattering. Another source of the direct inelastic scattering is to be found in the possibility of interpenetration of nuclei without compound nucleus formation. This is especially possible in nucleon bombardment.^{22,23} From the "moderate absorption" model,¹³ one knows that a nucleon may occasionally enter quite far into a nucleus without losing energy or forming a compound nucleus. Eventually it may interact with a single nucleon and lose energy, but there is still a finite chance of either it or the struck nucleon reaching the surface of the nucleus without further interaction. This possibility implies that the direct mechanism for inelastic scattering is operative in the nuclear volume as well as at the surface. Although low-lying states tend to be produced by direct interaction, the compound nucleus mechanism generally dominates for all the higher states and so accounts for the majority of processes.

(iii) *Excitation of collective states (especially rotations) by inelastic scattering.*—This type of process has not yet been confirmed experimentally, but it is a reasonable theoretical expectation³⁸ that such processes can occur. This is particularly so when the target nucleus is appreciably nonspherical and has a rotational spectrum of states. The bombarding particle may interact just with the surface shape of the target and set it rotating, without actually forming a compound nucleus. From some points of view such events are more naturally regarded as part of the shape-elastic scattering since no change is induced in the internal state of the target nucleus.

(iv) *Inelastic scattering of charged particles producing individual excited states of the target nuclei.*—Assumption (4) is always, strictly speaking, violated when the bombarding particles are charged. To represent the

³⁴ D. L. Allan, Proc. Phys. Soc. (London) **A70**, 195 (1957).

³⁵ W. F. Hornyak and R. Sherr, Phys. Rev. **100**, 1409 (1955); Benevise, Finke, and Martinelli, Phys. Rev. **101**, 655 (1956).

³⁶ J. W. Haffner, Phys. Rev. **103**, 1398 (1956).

³⁷ H. J. Watters, Phys. Rev. **103**, 1763 (1956).

³⁸ D. M. Chase, Phys. Rev. **104**, 838 (1956); Hayakawa and Yoshida, Proc. Phys. Soc. (London) **A68**, 656 (1955); C. F. Wandel, thesis, (Copenhagen, 1953), (unpublished); M. Moshinsky, Rev. mex. fis. **V**, 1 (1956); D. M. Brink, Proc. Phys. Soc. (London) **A68**, 994 (1955).

Coulomb potential between the protons in the target nucleus and those in the projectile by the single term $Z_1 Z_2 e^2 / r_c$ is only an approximation. There are additional terms which, although smaller, can cause inelastic scattering. Such scattering by Coulomb forces is called "Coulomb excitation." It is well understood theoretically³⁹ and its existence has been thoroughly confirmed experimentally. Clearly the ratio of the cross section for excitation of a given level by Coulomb excitation to that by the compound nucleus mechanism may be increased to any desired amount by reducing the bombarding energy far enough below the Coulomb barrier. At these low energies, compound nucleus formation is exceedingly small since almost no projectiles penetrate the barrier and reach the target nucleus. Since the Coulomb excitation process does not require close collisions, it is not nearly as seriously inhibited by the low energy. It differs from the other three processes that have been mentioned because, at low enough energies, this special process will always account for a large fraction of the total reaction cross section. Nevertheless in general the cross sections for reactions other than inelastic scattering of charged particles will be very little influenced by Coulomb excitation. However there are cases where the presence of excitation processes (real and virtual) does affect other types of reaction. For instance, it has lately been shown⁴⁰ that the $N^{14}(N^{14}, N^{13})N^{15}$ transfer reaction is affected by virtual Coulomb excitation occurring before actual nuclear interactions come into play.

The fact that the Coulomb excitation mechanism is excluded by assumption (4) for any finite choice of a_c places this mechanism apart from the other three, which are consistent with (4) provided the a_c are taken large enough. Since **R**-matrix theory is exact, if (4) is satisfied, it must be possible to describe the three mechanisms (i), (ii), and (iii) with **R**-matrix theory. Although **R**-matrix theory must be capable of describing most reaction mechanisms, its most immediate application is to the compound nucleus mechanism. This follows from the fact that the theory is framed in terms of a set of states of all particles which are naturally associated with the compound states of the compound nucleus mechanism. It is not so easy to spot how the theory may be adapted to describe direct mechanisms. Nevertheless, this adaption can be made, as shown by Bloch²³ (and described in detail in Sec. XI, 6). The essential point is that the only factor in **R**-matrix theory that could possibly correspond to direct mechanisms are the sums over the far-away levels. In the adaption to the compound nucleus mechanism such sums are dropped as a result of the random-sign approximation. This approximation eliminates the direct mechanisms. Consequently, to describe these mechanisms one must not make the random-sign approximation, but must retain the far-

away levels and perform the sums over them by using closure relations (see Sec. XI, 6).

III. NUCLEAR CONFIGURATION SPACE AND FORM OF THE WAVE FUNCTIONS

We now establish the basic framework of **R**-matrix theory. First we discuss a number of concepts and definitions and introduce the idea of nuclear configuration space. Following this, we describe the various types of wave function that occur.

The theory of nuclear reactions, like any contemporary theories in physics, has been plagued by the use of a variety of conventions of notation. This is especially apparent on comparing the papers of Wigner^{11,12} and co-workers with the theory in the book of Blatt and Weisskopf.²⁴ (The symbol γ^2 is the reduced width in the former and the square root of the same quantity in the latter!) We follow Wigner's notation as far as possible with some minor changes. For instance, our reduced width γ^2 is equal to Wigner's reduced width divided by the distance a_c . Such a change seems necessary to make the dimensions of γ^2 such as to justify the term "width" (i.e., to give γ^2 the dimension of energy, instead of the confusing energy-times-length).

1. Definitions and Notation

Two of the basic concepts of **R**-matrix theory of a nuclear system are: (i) the total system being separated into various pairs of nuclei (sometimes these pairs are called "alternatives"); (ii) the associated interaction radii a_c . These concepts and their interpretation in the nuclear configuration space are now examined.

The Separated pairs or "alternatives" c .—As yet we have only loosely defined the symbol c as labeling pairs of nuclei. Now we specify the precise features of a given pair of nuclei that the symbol c implies. We consider a total system of A nucleons separated into two groups containing A_1 and A_2 nucleons. If these groups are bound, they will be two definite nuclei in certain quantum states, α_1 and α_2 (say) with spins I_1 and I_2 , whose components in some specified direction are i_1 and i_2 . We could label the pair of nuclei, including spin orientations, by the set of quantities $\{\alpha(I_1 I_2) i_1 i_2\}$, where α is written for the pair $\alpha_1 \alpha_2$. In practice it is convenient to use an alternative representation. Instead of specifying i_1 and i_2 , we prescribe the so-called "channel spin" s and its component ν , so that a state in the new representation is labeled by the set of quantities $\{\alpha(I_1 I_2), s \nu\}$. This channel spin s is formed by coupling I_1 and I_2 together: $\mathbf{s} = \mathbf{I}_1 + \mathbf{I}_2$, and can take values between $|I_1 - I_2|$ and $|I_1 + I_2|$.

In the definition of c , it is convenient, not only to include specification of the two nuclei themselves as given in this channel spin representation, but also features of their relative motion. For prescribing the latter, as in any reaction theory, we have the choice of two representations. One is the "angle representa-

³⁹ Alder, Bohr, Huus, Mottelson, and Winther, *Revs. Modern Phys.* **28**, 432 (1956).

⁴⁰ G. Breit and M. E. Ebel, *Phys. Rev.* **104**, 1030 (1956).

tion" in which the states are plane waves, each with a definite direction (usually specified by a wave-number vector \mathbf{k}) but containing all relative angular momenta. This is the more useful representation for high bombarding energies when a large number of relative angular momentum waves play a role in the reactions which involve mainly a small range of angles about the incident beam. The other representation is that in which a state has a definite relative orbital angular momentum l (and component m), but contains components of all angles. This representation is generally useful at lower bombarding energies where only a limited number of relative angular momenta can contribute to a reaction. Thus it should be the more suitable for discussing reactions through the compound nucleus. We adopt it here, so that the symbol c implies the total set of quantities $\{\alpha(I_1 I_2) s l m\}$.

Occasionally we find it useful to change this representation somewhat and combine the channel spin s , and the relative orbital angular momentum l into a "total spin" quantum number $\mathbf{J} = \mathbf{s} + \mathbf{l}$ and its component M . In this case, c implies the set $\{\alpha(I_1 I_2) s l, JM\}$.

The interaction radii.—For a given pair c , the "interaction radius" a_c is the minimum radial distance of separation of the pair at which neither nucleus experiences any polarizing force from the other. Since this definition implies a dependence only on α and not on $(s l m)$, it is just as appropriate to write a_α instead of a_c and we shall sometimes do this. Many suggestions have been made by various authors that a_α has a simple dependence on the mass-numbers A_1 and A_2 . A commonly used prescription has been

$$a_\alpha = r_0(A_1^{\frac{1}{3}} + A_2^{\frac{1}{3}}), \quad (1.1)$$

where r_0 is independent of A_1 and A_2 and has a numerical value between 1.40 and 1.50×10^{-13} cm.

Configuration space and channels.—It is useful to introduce the concept of the configuration space of all A nucleons. Corresponding to the three spatial degrees of freedom of each particle, this space has $3A$ dimensions. If we include the extra two degrees of freedom that each nucleon has by virtue of its intrinsic spin, this makes $6A$ dimensions altogether. In this space, there is a certain region, called the "internal region," by Wigner and Eisenbud, corresponding to all nucleons being close together in a volume of nuclear dimensions in physical space. Certain other regions, called channels by Breit,²⁸ correspond to the nucleons being separated into two groups, A_1 and A_2 , in physical space beyond distances a_α . All the rest of configuration space corresponds to situations in physical space which occur with negligible probability. The channel for each split $A = (A_1, A_2)$ is separated from the others by such regions of zero probability. This means that a given pair of nuclei in their channel cannot change directly into another pair (channel). Such a change can only take place indirectly via the internal region. This internal region is bounded by the "channel surfaces" $r_\alpha = a_\alpha$, r_α being taken as the relative radial coordinate of the pair α .

As yet we have only defined channels for different splits, (A_1, A_2) of the total number of nucleons. To a given split (A_1, A_2) , there corresponds in general many pairs c which have the same region of configuration space for their channel. For instance, all those pairs c that involve a definite pair of nuclei α but different values of (l, m) have the same channel in configuration space. Furthermore, all pairs c corresponding to different internal excitation states of the two nuclei in a given split (A_1, A_2) share the same channel. Nevertheless, it is possible to speak as though *any* given pair c has its own special channel that does not overlap any other channels without any contradictions or paradoxes arising. The reason is ultimately that, when the two members of a pair c are in the external region, they cannot disturb each other. In other words, the wave functions of the pairs cannot mix with each other because, from the definition of the interaction radii, there are no forces to cause any mixing in the external region.

Not only do many different pairs c share the same configuration space channel, but, conversely, there are several channels in configuration space corresponding to a given pair c . At least, this is true provided that we do not distinguish, in a given pair c , particular individual nucleons. If, for the pair c , the groups of A_1 and A_2 nucleons contain N_1 and N_2 neutrons, Z_1 and Z_2 protons, there are $\binom{N_1}{N} \binom{Z_1}{Z}$ configuration space channels corresponding to the pair c where N, Z are the total numbers of neutrons and protons ($N_1 + N_2 = N$, $Z_1 + Z_2 = Z$, $N + Z = A$). The indistinguishability of nucleons which, as for all Fermi-Dirac particles, is represented by the need to antisymmetrize all wave functions, makes it desirable *not* to distinguish individual nucleons in speaking of a pair c . Thus when we speak of the "channel of pair c " we mean implicitly the sum over all $\binom{N_1}{N} \binom{Z_1}{Z}$ configuration space channels.

The channel surfaces S_c .—The internal and external regions of configuration space are separated by the totality of channel surfaces $r_\alpha = a_\alpha$. These surfaces overlap but the regions of overlap correspond to physical situations that occur with negligible probability. For instance, consider two surfaces representing splits into a neutron and a residual nucleus and a proton and a residual nucleus. The overlap region between the surfaces corresponds to both a neutron and a proton being separated from the rest of the system. This physical situation will be relatively very improbable if the total energy of the system does not allow a real separation of this type, or if for any other reason three-body processes are unimportant. This is guaranteed by assumption (2).

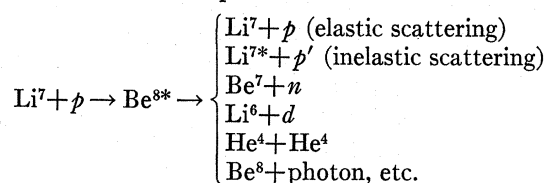
We define S_c as the channel surface for the pair c . All the channels (and so all the S_c) of the various pairs c arising from a given split (A_1, A_2) actually coincide, but we can think of the channels (and S_c) as distinct without contradictions arising. The totality of all

surfaces we write as $\mathcal{S} \equiv \sum_c \mathcal{S}_c$. Sachs⁴¹ has suggested that the surface \mathcal{S} may be visualized as a polyhedron, each hypersurface of which corresponds to a channel entrance; the channels are then cylinders normal to these planes. An element of the surface \mathcal{S}_c is

$$d\mathcal{S}_c = a_\alpha^2 d\Omega_c dq_\alpha, \quad (1.2)$$

where $d\Omega_c$ is the element of solid angle of the relative separation between the pair c and q_α represents the internal coordinates of the pair α . It is convenient not to specify individual nucleons in a given pair c . Thus \mathcal{S}_c is really a sum over $\binom{N_1}{N} \binom{Z_1}{Z}$ channel surfaces.

As an illustrative example, consider the nuclear reactions of Li^7 with protons:



The bombardment $\text{Li}^7 + p$ itself involves several channels c . Since the spin of Li^7 is $I_1 = \frac{3}{2}$ and that of the proton $I_2 = \frac{1}{2}$, the channel spin s is 1 or 2. Except at very low energies, several incoming orbital angular momentum waves l from an incident plane wave can contribute to the reaction. Thus the reaction is initiated by incident waves in several channels c . When the radial distance of separation in these channels falls below a_α , the proton enters the Li^7 nucleus. This "internal region" of configuration space corresponds to the compound nucleus, Be^{8*} . Decay of this nucleus leads to outgoing waves in all channels for which the relative energy of motion is positive. For instance, if decay into $\text{Be}^7 + n$ is energetically allowed, there will be outgoing waves in the several channels that can give this pair. (Since there are four neutrons that can be emitted, each of these channels is really a sum over four channels, one for each neutron.)

Finally we introduce the following additional channel characterizations:

$E_c \equiv E_\alpha$, the energy of relative motion of the particles of the pair c ;

$M_c \equiv M_\alpha = \frac{M_{\alpha_1} M_{\alpha_2}}{M_{\alpha_1} + M_{\alpha_2}}$, the reduced mass;

$k_c \equiv k_\alpha = \left(\frac{2M_\alpha |E_\alpha|}{\hbar^2} \right)^{\frac{1}{2}}$, the wave number;

$v_c \equiv v_\alpha = \hbar k_\alpha / M_\alpha$, the relative velocity;

$\eta_c \equiv \eta_{\alpha l} = \frac{Z_{1\alpha} Z_{2\alpha} e^2}{\hbar v_\alpha}$, the Coulomb field parameter;

$\sigma_c \equiv \sigma_{\alpha l} = \arg \Gamma(1 + l_c + i\eta_c)$, the Coulomb phase shift;

$\rho_c \equiv \rho_\alpha = k_\alpha r_\alpha$.

M_{α_1} , Z_{α_1} and M_{α_2} , Z_{α_2} are the mass, charge number of the two particles of the pair α , respectively. Occasionally, when no ambiguity can arise, these will be written M_1 , Z_1 , M_2 , Z_2 . The E_α are positive for those channels through which decay is energetically allowed and negative for those energetically forbidden; the latter channels may also be referred to as "negative-energy" or "virtual."

2. Wave Functions for the External Region

Throughout the whole of configuration space, the wave function of a given total system is assumed to satisfy the Hamiltonian equation

$$H\Psi = E\Psi,$$

where E is the total energy and H , the Hamiltonian operator, is the sum of T , the kinetic energy operator and V , the potential energy operator.

T has the form

$$\sum_i -\frac{\hbar^2}{2m_i} \nabla_{\mathbf{x}_i}^2,$$

where m_i is the mass of particle i and

$$\nabla_{\mathbf{x}_i} = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i} \right).$$

$\mathbf{X}_i = (x_i, y_i, z_i)$ is the position vector of particle i referred to a fixed arbitrary origin O . For any general system of A particles, the dependence of T on the centroid vector ($\mathbf{R} = (\sum_i m_i)^{-1} \sum_i m_i \mathbf{X}_i$) may be separated off by making an orthogonal transformation ($\mathbf{X}_i \rightarrow \mathbf{R}, q_i$), where the q_i are a set of $(3A-3)$ internal coordinates depending only on the relative positions of the particles. (The requirement that the transformation be orthogonal imposes the only restriction on the choice of q_i that concerns us here.) Under this transformation, T becomes

$$T = -\frac{\hbar^2}{2M} \nabla_{\mathbf{R}}^2 + T_{\text{int}}$$

where the first term is the kinetic energy of the motion of the centroid (M being $\sum_i m_i$, the total mass) and the second term is that of the internal motion. The latter has the form

$$\sum_i c_i \frac{\partial^2}{\partial q_i^2}$$

where the c_i are coefficients determined by the orthogonal transformation. Making this type of transformation separately on the kinetic energy operators of the subsystems α_1 and α_2 of the pair α leads to

$$T = T_{\alpha_1} + T_{\alpha_2} = -\frac{\hbar^2}{2M_{\alpha_1}} \nabla_{\mathbf{R}_1}^2 - \frac{\hbar^2}{2M_{\alpha_2}} \nabla_{\mathbf{R}_2}^2 + (T_{\text{int}})_{\alpha_1} + (T_{\text{int}})_{\alpha_2}$$

where \mathbf{R}_1 and \mathbf{R}_2 are the centroid position vectors of α_1

⁴¹ R. G. Sachs, *Nuclear Theory* (Addison-Wesley Press, Cambridge, 1953).

and α_2 . To obtain the final form of T suitable for a channel involving the pair α , it is necessary to make a second transformation, namely one from $(\mathbf{R}_1, \mathbf{R}_2)$ to the centroid position vector of the whole system \mathbf{R} , and the relative vector of α_1 and α_2 , \mathbf{r}_α :

$$\mathbf{R} = \frac{M_{\alpha_1}\mathbf{R}_1 + M_{\alpha_2}\mathbf{R}_2}{M_{\alpha_1} + M_{\alpha_2}}$$

$$\mathbf{r}_\alpha = \mathbf{R}_2 - \mathbf{R}_1.$$

This transformation leads to the kinetic energy:

$$T = -\frac{\hbar^2}{2M}\nabla_{\mathbf{R}}^2 - \frac{\hbar^2}{2M_\alpha}\nabla_{\mathbf{r}_\alpha}^2 + (T_{\text{int}})_{\alpha_1} + (T_{\text{int}})_{\alpha_2}.$$

From the definition of the external region and the channels, it follows that, in the channel c , V may be decomposed into the form $(V_{\text{int}})_{\alpha_1} + (V_{\text{int}})_{\alpha_2} + V_c(\mathbf{r}_\alpha)$, where the last term is composed of the Coulomb potential and any other long-range potential between α_1 and α_2 . Thus, in a given channel c , the Hamiltonian may be assumed to be the sum of four parts:

$$H = H_0 + H_c + H_{\alpha_1} + H_{\alpha_2}$$

where:

$$H_0 = -\frac{\hbar^2}{2M}\nabla_{\mathbf{R}}^2$$

$$H_c = -\frac{\hbar^2}{2M_\alpha}\nabla_{\mathbf{r}_\alpha}^2 + V_c(\mathbf{r}_\alpha)$$

$$H_{\alpha_1} = (T_{\text{int}})_{\alpha_1} + (V_{\text{int}})_{\alpha_1}$$

$$H_{\alpha_2} = (T_{\text{int}})_{\alpha_2} + (V_{\text{int}})_{\alpha_2}.$$

Correspondingly the wave function Ψ in the channel c may be taken as the *product* of four parts:

$$\Psi = \Phi(\mathbf{R})\chi(\mathbf{r}_\alpha)\psi_{\alpha_1}(q_{\alpha_1})\psi_{\alpha_2}(q_{\alpha_2})$$

where

$$H_0\Phi = \epsilon\Phi, \quad H_c\chi = \mathcal{E}\chi,$$

$$H_{\alpha_1}\psi_{\alpha_1} = E_{\alpha_1}\psi_{\alpha_1}, \quad H_{\alpha_2}\psi_{\alpha_2} = E_{\alpha_2}\psi_{\alpha_2}.$$

The wave functions Φ , χ , ψ_{α_1} , and ψ_{α_2} describe respectively the centroid motion, the relative motion of α_1 and α_2 and the internal states of α_1 and α_2 . ϵ , \mathcal{E} , E_{α_1} , and E_{α_2} are the associated energies.

From now on, we never need to mention the centroid motion explicitly. For convenience, we assume that the centroid is at rest, so that $\epsilon = 0$ and the total energy in a given channel is then only a sum of three parts: $E = \mathcal{E} + E_{\alpha_1} + E_{\alpha_2}$. We now give separate discussions of the wave functions of the internal states (the "channel spin wave functions") and the wave functions of relative motion.

a. Channel Spin Wave Functions

$V_c(\mathbf{r}_\alpha)$ does not depend on the relative orientation of the spins I_1 and I_2 of α_1 and α_2 , so that there is degeneracy with respect to these orientations. For this reason, we may work with a basic set of channel wave functions Ψ which are *linear combinations of products* (rather than single products) of the wave functions of the internal states of α_1 and α_2 . We will choose these combinations so that each Ψ has the previously defined channel spin s as a quantum number. To be more precise, the "channel spin wave function" $\psi_{\alpha s \nu}$ of the pair α is constructed by vector coupling the (normalized) wave functions of the individual fragments α_1 and α_2 :

$$\psi_{\alpha s \nu} = \sum_{i_1+i_2=\nu} (I_1 I_2 i_1 i_2 | s \nu) \psi_{\alpha_1 I_1 i_1} \psi_{\alpha_2 I_2 i_2}. \quad (2.1)$$

The coefficients $(I_1 I_2 i_1 i_2 | s \nu)$ are elements of the matrix of the orthogonal transformation from the $(I_1 i_1, I_2 i_2)$ scheme to the $(I_1 I_2, s \nu)$ scheme; they are called "vector-addition" coefficients as discussed by, for instance, Condon and Shortley.⁴²

The channel spin wave functions are mutually orthogonal and normalized:

$$\int \psi_{\alpha s \nu}^* \psi_{\alpha' s' \nu'} d\mathcal{S} = 4\pi \alpha^2 \delta_{\alpha s \nu, \alpha' s' \nu'} \quad (2.2)$$

where the integral is taken over the totality of channel surfaces $\mathcal{S} = \sum_c \mathcal{S}_c$ and where

$$\delta_{\alpha s \nu, \alpha' s' \nu'} = \delta_{\alpha \alpha'} \delta_{s s'} \delta_{\nu \nu'}.$$

The orthonormality with respect to s and ν is a consequence of the unitary property of the vector addition coefficients. When α and α' signify two different divisions of the total system, the orthonormality with respect to α is a consequence of the absence of spatial overlap between each ψ , whereas when they signify the same division but into different states of excitation, it is a consequence of the usual orthonormality of bound states of a given system. These functions are real in the sense that they have the time-reversal property

$$K\psi(s\nu) = (-1)^{s-\nu} \psi(s-\nu), \quad (2.3)$$

where we temporarily write $\psi(s\nu)$ for $\psi_{\alpha s \nu}$ and where K is the time-reversal operator.⁴² This property follows from (2.1) if $\psi(I_1 i_1)$ and $\psi(I_2 i_2)$ have similar properties. That is,

$$\begin{aligned} K\psi(s\nu) &= \sum_{i_1 i_2} (I_1 I_2 i_1 i_2 | s \nu) K\psi(I_1 i_1) K\psi(I_2 i_2) \\ &= \sum_{i_1 i_2} (I_1 I_2 i_1 i_2 | s \nu) (-1)^{I_1 - i_1 + I_2 - i_2} \psi(I_1 - i_1) \psi(I_2 - i_2) \\ &= \sum_{i_1 i_2} (I_1 I_2 - i_1 - i_2 | s - \nu) (-1)^{-s + \nu} \psi(I_1 - i_1) \psi(I_2 - i_2) \\ &= (-1)^{s-\nu} \psi(s-\nu), \end{aligned} \quad (2.4)$$

the third equality being a consequence of one of the symmetry properties of the vector addition coefficients

⁴² E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, New York, 1953).

and the fourth follows from (2.1) and the fact that $s-\nu$ is always an integer. As originally shown by Wigner,⁴³ the operator K implies the transition to the complex conjugate and multiplication with the spin operators $-i\sigma_y$ of all elementary particles in the nucleus. For more detailed accounts of K and the interpretation of its effect on wave functions the reader is referred to the literature.^{43,12,41} The particular behavior (2.3) was suggested by Biedenharn and Rose.⁴⁴

b. Wave Functions of Relative Motion

The wave function of relative motion has the form

$$\chi \sim r_\alpha^{-1} u_{\alpha s l}(\mathbf{r}_\alpha) (i^l Y_m^{(l)}(\Omega_\alpha)), \quad (2.5)$$

where r_α and Ω_α denote the length and the direction of the vector which goes from the particle 1 to particle 2. The $Y_m^{(l)}$ are the usual normalized spherical harmonics of Condon and Shortley⁴² with the property

$$Y_m^{(l)*} = (-1)^m Y_{-m}^{(l)}. \quad (2.6)$$

As pointed out by Huby,⁴⁵ it is convenient to work with spherical harmonic functions with the additional factor i^l because they satisfy the time-reversal condition of the type (2.3), i.e.,

$$K(i^l Y_m^{(l)}) = (-1)^{l-m} (i^l Y_{-m}^{(l)}), \quad (2.7)$$

and because the i^l factors which appear in plane-wave expansions can be absorbed, thus simplifying the formulas. The radial functions $u_{\alpha s l}(r_\alpha)$ are solutions to the radial Schroedinger equation

$$\left[\frac{d^2}{dr_\alpha^2} - \frac{l(l+1)}{r_\alpha^2} - \frac{2M_\alpha}{\hbar^2} (V_{\alpha s l} - E_\alpha) \right] u_{\alpha s l}(r_\alpha) = 0. \quad (2.8)$$

For the time being we consider only a Coulomb interaction $V_{\alpha s l} = Z_{\alpha 1} Z_{\alpha 2} e^2 / r_\alpha$ in the external region, in which case this equation is more conveniently put in the dimensionless form

$$u_{\alpha s l}''(\rho_\alpha) - [l(l+1)\rho_\alpha^{-2} + 2\eta_\alpha \rho_\alpha^{-1} \mp 1] u_{\alpha s l}(\rho_\alpha) = 0, \quad (2.9)$$

wherein a prime denotes differentiation with respect to ρ_α ; the upper sign applies to positive-energy channels and the lower to negative-energy ones.

For positive-energy channels, the two linearly independent solutions to (2.9) that occur most naturally in the theoretical development are those representing *incoming* (I) and *outgoing* (O) waves. It is convenient to work with solutions whose asymptotic forms for large ρ_α are

$$I_c^+ \equiv I_{\alpha l}^+ \sim \exp[-i(\rho_\alpha - \eta_\alpha \log 2\rho_\alpha - \frac{1}{2}l\pi + \sigma_{\alpha 0})], \quad (2.10a)$$

$$O_c^+ \equiv O_{\alpha l}^+ \sim \exp[i(\rho_\alpha - \eta_\alpha \log 2\rho_\alpha - \frac{1}{2}l\pi + \sigma_{\alpha 0})] \quad (2.10b)$$

where the superscripts $+$ signify positive energy. These

solutions are evidently complex conjugates of each other. For applications it is more convenient to have the formulas expressed in terms of two real, linearly independent solutions. It is customary to use those solutions which are *regular* (F) and *irregular* (G) at the origin and whose asymptotic forms for large ρ_α are

$$F_c \equiv F_{\alpha l} \sim \sin(\rho_\alpha - \eta_\alpha \log 2\rho_\alpha - \frac{1}{2}l\pi + \sigma_{\alpha l}), \quad (2.11a)$$

$$G_c \equiv G_{\alpha l} \sim \cos(\rho_\alpha - \eta_\alpha \log 2\rho_\alpha - \frac{1}{2}l\pi + \sigma_{\alpha l}). \quad (2.11b)$$

The Wronskian of this pair is

$$F_c' G_c - G_c' F_c = 1, \quad (2.12)$$

which holds for all values of ρ_α since Wronskians in general are independent of ρ_α . These functions have been extensively studied by Breit and his collaborators⁴⁶ and tables are available covering certain ranges of ρ , η , and l (see Appendix). The relations between the two sets of solutions are evidently

$$I_c^+ = (G_c - iF_c) \exp(i\omega_c), \quad (2.13a)$$

$$O_c^+ = (G_c + iF_c) \exp(-i\omega_c), \quad (2.13b)$$

where

$$\omega_c \equiv \omega_{\alpha l} = \sigma_{\alpha l} - \sigma_{\alpha 0} = \sum_{n=1}^l \tan^{-1}(\eta_\alpha/n). \quad (2.13c)$$

In the absence of a Coulomb field ($\eta_\alpha = 0$), the I and O functions are related to the Hankel functions according to

$$I_c^+ = -i(\pi\rho_\alpha/2)^{\frac{1}{2}} H_{l+\frac{1}{2}}^{(2)}(\rho_\alpha), \quad (2.14a)$$

$$O_c^+ = i(\pi\rho_\alpha/2)^{\frac{1}{2}} H_{l+\frac{1}{2}}^{(1)}(\rho_\alpha); \quad (2.14b)$$

and the relations between the F and G functions and the J -type Bessel functions are

$$F_c^+ = (\pi\rho_\alpha/2)^{\frac{1}{2}} J_{l+\frac{1}{2}}(\rho_\alpha), \quad (2.15a)$$

$$G_c^+ = (-1)^l (\pi\rho_\alpha/2)^{\frac{1}{2}} J_{-(l+\frac{1}{2})}(\rho_\alpha). \quad (2.15b)$$

For negative-energy channels, only the solution to (2.9) which vanishes at infinity occurs in the usual applications. We specify an O -type solution to have the asymptotic form

$$O_c^- \equiv O_{\alpha l}^- \sim \exp(-\rho_\alpha - \eta_\alpha \log 2\rho_\alpha), \quad (2.16)$$

so that, apart from a factor i^{-l} , it is just the analytical continuation from the positive real axis to the positive imaginary axis in the complex wave-number (k) plane of the O^+ -type solution which is determined by (2.10b). It is identical to the real, "exponentially decaying" Whittaker function described in the appendix

$$O_c^- = W(-\eta_\alpha, l+\frac{1}{2}; 2\rho_\alpha). \quad (2.17)$$

In the absence of a Coulomb field, it is related to the modified Bessel function of the second kind:

$$O_c^- = (2\rho_\alpha/\pi)^{\frac{1}{2}} K_{l+\frac{1}{2}}(\rho_\alpha). \quad (2.18)$$

⁴⁶ Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, Revs. Modern Phys. 23, 147 (1951).

⁴³ E. P. Wigner, Göttingen Nachr. 31, 546 (1932).

⁴⁴ L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. 25, 729 (1953).

⁴⁵ R. Huby, Proc. Phys. Soc. (London) A67, 1113 (1954).

An I^- -type solution could similarly be taken as the analytical continuation of the I^+ function which is determined by (2.10a). In view of its nonoccurrence in the usual applications, we need not be concerned with the precise specification even though it appears in the theoretical development.

c. Complete Channel Wave Functions

Complete channel wave functions can now be written down for positive energies corresponding to *incoming* and *outgoing* waves of unit flux crossing any sphere centered at the origin; in the $(\alpha slvm)$ channel scheme they are:

$$g_{\alpha slvm}^+ = (i^l Y_m^{(l)}) \frac{I_{\alpha l}^+}{v_{\alpha}^{\frac{1}{2}} r_{\alpha}} \psi_{\alpha sv} \quad (2.19a)$$

$$\Theta_{\alpha slvm}^+ = (i^l Y_m^{(l)}) \frac{O_{\alpha l}^+}{v_{\alpha}^{\frac{1}{2}} r_{\alpha}} \psi_{\alpha sv}. \quad (2.19b)$$

The corresponding functions in the $(\alpha slJM)$ scheme are

$$g_{\alpha slJM}^+ = \sum_{\nu+m=M} (slvm|JM) g_{\alpha slvm}^+, \quad (2.20a)$$

$$\Theta_{\alpha slJM}^+ = \sum_{\nu+m=M} (slvm|JM) \Theta_{\alpha slvm}^+. \quad (2.20b)$$

The functions of the latter set behave under time-reversal as

$$K g_{\alpha slJM}^+ = (-1)^{J-M} \Theta_{\alpha slJ-M}^+, \quad (2.21a)$$

$$K \Theta_{\alpha slJM}^+ = (-1)^{J-M} g_{\alpha slJ-M}^+ \quad (2.21b)$$

because of (2.3) and the fact that K acts on the radial parts of (2.19) as the complex-conjugation operator. Solutions analogous to (2.19) and (2.20) may be introduced for the negative-energy channels by simply replacing I^+ , O^+ by I^- , O^- . Because the radial parts of these solutions will be taken as real, the time-reversal equations are

$$K g_{\alpha slJM}^- = (-1)^{J-M} g_{\alpha slJ-M}^-, \quad (2.22a)$$

$$K \Theta_{\alpha slJM}^- = (-1)^{J-M} \Theta_{\alpha slJ-M}^-. \quad (2.22b)$$

It is convenient to introduce the surface functions

$$\varphi_{\alpha slvm} = r_{\alpha}^{-1} \psi_{\alpha sv} (i^l Y_m^{(l)}) \quad (2.23)$$

for the terms in (2.5) which multiply the radial function $u_{\alpha sl}$ to give the total channel wave function. These have the property of being mutually orthogonal and normalized on \mathcal{S} :

$$\int \varphi_{\alpha slvm}^* \varphi_{\alpha' s' l' v' m'} d\mathcal{S} = \delta_{\alpha slvm; \alpha' s' l' v' m'}. \quad (2.24)$$

The corresponding functions of the $(\alpha slJM)$ scheme,

$$\varphi_{\alpha slJM} = \sum_{\nu+m=M} (slvm|JM) \varphi_{\alpha slvm} \quad (2.25)$$

are also mutually orthogonal and normalized on \mathcal{S} :

$$\int \varphi_{\alpha slJM}^* \varphi_{\alpha' s' l' J' M'} d\mathcal{S} = \delta_{\alpha slJM; \alpha' s' l' J' M'}, \quad (2.26)$$

as a consequence of (2.24) and the unitary property of the vector addition coefficients. These respective functions are assumed to form complete sets of surface functions on \mathcal{S} . For a detailed discussion of this assumed completeness, the reader is referred to an account by Sachs.⁴¹

3. Wave Functions for the Internal Region

In the internal region the total wave function Ψ for any particular excitation energy E is composed of various wave functions Ψ_{JM} corresponding to definite angular momenta J and components M . These wave functions satisfy the wave equation,

$$H\Psi_{JM} = E\Psi_{JM}, \quad (3.1)$$

and can formally be expanded in the internal region in terms of certain mutually orthogonal eigenfunctions $X_{\lambda JM}$ of that region,

$$\Psi_{JM} = \sum_{\lambda} A_{\lambda J} X_{\lambda JM}, \quad (3.2)$$

these eigenfunctions being solutions to the wave equation at the real, energy eigenvalues $E_{\lambda J}$:

$$H X_{\lambda JM} = E_{\lambda J} X_{\lambda JM}. \quad (3.3)$$

We specify later (Sec. V, 2) how the $X_{\lambda JM}$ are defined by certain *eigenvalue* boundary conditions on the surface \mathcal{S} . The expansion coefficients $A_{\lambda J}$ will in general be functions of the excitation energy E and dependent on the *actual* boundary conditions on \mathcal{S} ; in the absence of a nonspherically symmetrical external field, such as a magnetic field, they are independent of M . The eigenfunctions are assumed to be real in the sense of (2.3),

$$K X_{\lambda JM} = (-1)^{J-M} X_{\lambda J-M}. \quad (3.4)$$

4. Wave Functions on the Boundary Surface \mathcal{S}

In the developments of Secs. V and VII, which involve matching of the external and external functions on \mathcal{S} , expressions are needed for the values and derivatives of the radial parts of the external and internal functions on \mathcal{S} that is, of the $u_{\alpha sl}(r_{\alpha})$ of (2.5). For this purpose it is convenient to introduce the *value quantity*,

$$V_c = (\hbar^2/2M_c a_c)^{\frac{1}{2}} u_c(a_c), \quad (4.1a)$$

and the *derivative quantity*

$$D_c = (a_c \hbar^2/2M_c)^{\frac{1}{2}} (du_c/dr_c)_{r_c=a_c}. \quad (4.1b)$$

The (dimensionless) ratio D_c/V_c is also important and referred to as the logarithmic derivative; it is actually the radius a_c times the logarithmic derivative of r_c .

times the radial part of the external or internal function on \mathcal{S} . These quantities may be expressed as surface integrals,

$$V_c = (\hbar^2/2M_c a_c)^{\frac{1}{2}} \int \varphi_c^* \Psi d\mathcal{S}, \quad (4.2a)$$

$$D_c = (\hbar^2/2M_c a_c)^{\frac{1}{2}} \int \varphi_c^* \text{grad}_n(r_c \Psi) d\mathcal{S} \\ = V_c + (a_c \hbar^2/2M_c)^{\frac{1}{2}} \int \varphi_c^* \text{grad}_n \Psi d\mathcal{S}, \quad (4.2b)$$

where Ψ is the complete wave function of the system and grad_n is the gradient normal to \mathcal{S} . The expansions on \mathcal{S} for Ψ and its normal gradient in terms of the assumed complete set of surface functions φ_c are therefore

$$\Psi = \sum_c (2M_c a_c / \hbar^2)^{\frac{1}{2}} V_c \varphi_c, \quad (4.3a)$$

$$\text{grad}_n(r_c \Psi) = \sum_c (2M_c a_c / \hbar^2)^{\frac{1}{2}} D_c \varphi_c \\ \text{grad}_n \Psi = \sum_c (2M_c / a_c \hbar^2)^{\frac{1}{2}} (D_c - V_c) \varphi_c. \quad (4.3b)$$

a. External Functions

Since the $u(r_\alpha)$ are expressed in the external region as a linear combination of I and O waves, it is useful to have some symbols for the frequently occurring combinations of the surface values and derivatives of these waves. The logarithmic derivative of the O -type wave function is designated as

$$L_c \equiv (\rho_c O_c' / O_c)_{r_c = a_c} = S_c + iP_c, \quad (4.4)$$

the real and imaginary parts of which are, according to (2.13b), (2.12), and (2.17), respectively,

$$S_c^+ = [\rho_c (F_c F_c' + G_c G_c') / (F_c^2 + G_c^2)]_{r_c = a_c}, \\ S_c^- = (\rho_c W_c' / W_c)_{r_c = a_c}; \quad (4.4a)$$

$$P_c^+ = [\rho_c / (F_c^2 + G_c^2)]_{r_c = a_c}, \\ P_c^- = \text{zero}. \quad (4.4b)$$

In the case of the positive-energy channels, the ratio

$$\Omega_c^+ = (I_c / O_c)_{r_c = a_c}^{\frac{1}{2}} \quad (4.5)$$

is a unit-modulus complex number which is expressible as

$$\Omega_c^+ \equiv \Omega_{\alpha l}^+ = \exp(i(\omega_c - \phi_c^+)), \quad (4.5a)$$

where

$$\phi_c^+ \equiv \phi_{\alpha l}^+ = \tan^{-1}(F_c / G_c). \quad (4.5b)$$

We also introduce

$$\mathfrak{L}_c = (\rho_c I_c' / I_c)_{r_c = a_c}, \quad (4.6a)$$

$$\mathfrak{P}_c = (\rho_c / I_c O_c)_{r_c = a_c} \quad (4.6b)$$

and the Wronskian

$$w = (O_c' I_c - I_c' O_c)_{r_c = a_c} \quad (4.6c)$$

which is not strictly a surface quantity because its value is independent of the position of \mathcal{S} . For the positive-energy channels

$$\mathfrak{L}_c^+ = L_c^{+*}, \quad (4.7a)$$

$$\mathfrak{P}_c^+ = P_c^+, \quad (4.7b)$$

$$w_c^+ = 2i. \quad (4.7c)$$

We need not be concerned with stating similar relations for the negative-energy channels.

There are four quantities involved in the specification of the external functions on \mathcal{S} : I, I', O, O' or F, F', G, G' . With specification of the Wronskian value (4.9c) or (2.12) for the positive-energy channels, only three independent quantities are needed. For these we may use S_c^+, P_c^+ , and ϕ_c^+ . S_c^+ (as well as S_c^-) is referred to as the *shift factor* because it appears as a factor in the level shift expression; P_c is referred to as the *penetration factor* because it appears as a factor in the level width expression; while $-\phi_c^+$ is called the *hard-sphere scattering phase shift* because it is the phase shift induced by an infinitely repulsive sphere of radius a_c , i.e., it corresponds to a node in the wave function at $r_c = a_c$. The vanishing of P_c^- is a reflection of the fact that the negative-energy channels do not transmit any particle fluxes. Some relations and approximations pertaining to these three quantities, which are useful for applications, are given in the appendix.

b. Internal Functions

The value and derivative quantities for the internal eigenfunctions are usually denoted by

$$\gamma_{\lambda c} \equiv V_{\lambda c} = (\hbar^2/2M_c a_c)^{\frac{1}{2}} \int \varphi_c^* X_{\lambda JM} d\mathcal{S}, \quad (4.8a)$$

$$\delta_{\lambda c} \equiv D_{\lambda c} = \gamma_{\lambda c} + (a_c \hbar^2/2M_c)^{\frac{1}{2}} \int \varphi_c^* \text{grad}_n X_{\lambda JM} d\mathcal{S} \quad (4.8b)$$

where c denotes $\alpha s l J M$. Important properties of these quantities are that they are real and independent of M . According to Wigner,^{43,12} these properties characterize any surface scalar product of functions which behaves in the same way under time reversal.

$$\int \varphi_{\alpha s l J M}^* X_{\lambda JM} d\mathcal{S} \equiv (\varphi_{JM}, X_{JM}) = (X_{JM}, \varphi_{JM})^* \\ = (K X_{JM}, K \varphi_{JM}) = (-1)^{2J-2M} (X_{J-M}, \varphi_{J-M}) \\ = (\varphi_{J-M}, X_{J-M})^*. \quad (4.9)$$

To obtain this result we have used (2.3) and the fact that $2J - 2M$ is always even. Since these scalar products are independent of M , one can conclude that

$$(\varphi_{JM}, X_{JM}) = (\varphi_{JM}, X_{JM})^*. \quad (4.10)$$

These scalar products vanish if the J values of the two functions differ, or if their M values differ.

The squared quantities $\gamma_{\lambda c}^2$ are referred to as the *reduced level widths*; from the definition (4.7b) they have the dimensions of energy. The unsquared quantities are referred to as reduced-width amplitudes. The ratio of the amplitudes

$$\delta_{\lambda c}/\gamma_{\lambda c} = D_{\lambda c}/V_{\lambda c} \equiv B_c \quad (4.11)$$

is a logarithmic derivative quantity which is involved in the specification of the (real) boundary conditions to be satisfied by the $X_{\lambda JM}$ on S (Sec. V, 2).

IV. ELASTIC SCATTERING OF SPINLESS PARTICLES BY A CENTRAL POTENTIAL

It is possible now to proceed directly with the complete formal derivation of **R**-matrix theory. However, as that includes considerations of spins, the possibility of reactions, Coulomb fields, and the use of arbitrary boundary conditions, the underlying physical principles are apt to be concealed by the consequent complexities. It is therefore desirable to begin with the simplest possible case which includes most of the principles. Such is the elastic scattering of a spinless particle by another spinless particle which results from a central force interaction.

1. Derivation of the Cross-Section Formula

Since the interacting particles are assumed to be spinless, the channel spin s and its component ν are zero; and since the incident plane wave beam has no angular momentum about the beam direction, the component m of the angular momentum quantum number l is also zero if we choose this direction for the axis of quantization. The channels symbols $s, \nu,$ and m therefore need not be used; there is also no need to use the symbol α because reactions are supposed not to occur. The quantum number l is then the only channel designation subscript that need be retained.

a. The *R* Function

The derivation of the general **R**-matrix relation of the next section is based on an application of a Green's theorem relation. For elastic scattering this may readily be obtained by manipulation of the Schroedinger equations for the radial parts $r^{-1}u_l(r)$ of the internal wave function for a particular l value at the two energies E_1 and E_2 :

$$\begin{aligned} (d^2u_1/dr^2) + (2M/\hbar^2)(E_1 - V)u_1 &= 0, \\ (d^2u_2/dr^2) + (2M/\hbar^2)(E_2 - V)u_2 &= 0. \end{aligned} \quad (1.1)$$

Here the interaction potential V includes the centrifugal "potential," and the subscripts 1 and 2 attached to the u refer to the energy rather than to the l value. The first of these is multiplied by u_2 and the second by u_1 , and the difference is then integrated from the origin to

the channel radius a to obtain

$$\int_0^a \left(u_2 \frac{d^2u_1}{dr^2} - u_1 \frac{d^2u_2}{dr^2} \right) dr + \frac{2M}{\hbar^2} (E_1 - E_2) \int_0^a u_1 u_2 dr = 0. \quad (1.2)$$

The *Green's theorem relation* is obtained by partial integration (Green's theorem) of the first integral,

$$\left(u_2 \frac{du_1}{dr} - u_1 \frac{du_2}{dr} \right)_{r=a} + \frac{2M}{\hbar^2} (E_1 - E_2) \int_0^a u_1 u_2 dr = 0, \quad (1.3)$$

the contribution from the origin vanishing because u must vanish there if the radial part $r^{-1}u(r)$ is to remain finite.

At certain real energies E_λ , the solutions u_λ to the Schroedinger equation (111.3.1) will have zero derivatives at the surface:

$$(du_\lambda/dr)_{r=a} = 0. \quad (1.4)$$

These energies are the energy *eigenvalues* and the corresponding solutions are the *eigenfunctions* which satisfy the *boundary conditions* (1.4) at $r=a$. By applying (1.3) to two eigenfunctions u_λ and $u_{\lambda'}$ belonging to two different eigenvalues E_λ and $E_{\lambda'}$, one immediately finds that the eigenfunctions are orthogonal in the internal region; they are also considered to be normalized:

$$\int_0^a u_\lambda u_{\lambda'} dr = \delta_{\lambda\lambda'}. \quad (1.5)$$

The solution to (1.1) at *any* energy E may now be expanded in the internal region in terms of the u_λ :

$$u_E(r) = \sum_\lambda A_\lambda u_\lambda(r), \quad 0 \leq r \leq a, \quad (1.6)$$

where

$$A_\lambda = \int_0^a u_\lambda u_E dr.$$

For the determination of the expansion coefficients A_λ , the Green's theorem relation may be used again:

$$-u_\lambda(a)(du_E/dr)_a + (2M/\hbar^2)(E_\lambda - E) \int_0^a u_E u_\lambda dr = 0$$

or

$$A_\lambda = \frac{\hbar^2}{2M} \frac{u_\lambda(a)}{E_\lambda - E} \left(\frac{du_E}{dr} \right)_a. \quad (1.7)$$

One thereby obtains the relation

$$u_E(r) = G(r, a) (adu_E/dr)_a, \quad (1.8)$$

where

$$G(r, a) = \frac{\hbar^2}{2Ma} \sum_\lambda \frac{u_\lambda(r)u_\lambda(a)}{E_\lambda - E} \quad (1.9)$$

is the *Green's function* which relates the value of the wave function in the internal region to its derivative on the surface.² The *R function* is defined as

$$R = G(a, a) = \sum_{\lambda} \gamma_{\lambda}^2 / (E_{\lambda} - E), \quad (1.10)$$

where

$$\gamma_{\lambda} = (\hbar^2/2Ma)^{1/2} u_{\lambda}(a) \quad (1.10a)$$

is the reduced-width amplitude which was introduced in (111.4.8a). According to (1.8) R is equal to the reciprocal of a times the logarithmic derivative of u at a :

$$R = u_E(a) / a (du_E/dr)_a. \quad (1.11)$$

b. The Collision Function U

A general solution Ψ_l may always be expressed in the external region as a linear combination of the linearly independent g_l and θ_l waves of (2.19):

$$\Psi_l \sim g_l - U_l \theta_l. \quad (1.12)$$

The coefficient U_l is thus the amplitude of the unit-flux outgoing wave θ_l which is associated with a unit-flux incoming wave g_l ; it is called the *collision* or *scattering function*.

c. The Relation between the R Function and the Collision Function

The collision function may be expressed in terms of the R function by equating the logarithmic derivatives of the internal and external wave functions at $r = a$:

$$\left(\frac{u_l}{\rho u_l'} \right)_a = R_l = \frac{(I_l - U_l O_l)}{\rho (I_l' - U_l O_l')}. \quad (1.13)$$

It follows that:

$$U_l = \frac{I_l - L_l^* R_l}{O_l - L_l R_l}, \quad (1.14)$$

where L_l is the logarithmic derivative quantity of (III.4.4). Since R_l is real, U_l may be expressed in terms of a *phase shift* δ_l as

$$U_l = \exp(2i\delta_l), \quad (1.15)$$

where

$$\delta_l = \tan^{-1} [R_l P_l / (1 - R_l S_l)] - \phi_l + \omega_l, \quad (1.15a)$$

the quantities S_l , P_l , ϕ_l , ω_l being given by (III.4.4 and 5) and (III.2.13c). Since the Coulomb field is ignored, ω_l is zero. In general there will be different $\gamma_{\lambda l}^2$, $E_{\lambda l}$, R_l , δ_l for each partial wave l that is effective. Evidently the resonance contribution to δ_l from the first term increases by π as E goes from level to level, and between resonances where $R_l = 0$, δ_l is equal to $-\phi_l$, the so-called *hard sphere scattering phase shift*, to within an integral multiple of π .

If the energy E is sufficiently close to one of the level positions E_{λ} , it is justified to neglect all but the contribution from that level in the R function expansion

(1.10):

$$R_l \approx \gamma_{\lambda l}^2 / (E_{\lambda} - E). \quad (1.16)$$

One thereby obtains the one-level approximation to the phase shift,

$$\delta_l = \tan^{-1} \left(\frac{\frac{1}{2} \Gamma_{\lambda l}}{E_{\lambda l} + \Delta_{\lambda l} - E} \right) - \phi_l, \quad (1.17a)$$

where the *level width*

$$\Gamma_{\lambda l} = 2\gamma_{\lambda l}^2 P_l \quad (1.17b)$$

determines how fast the phase changes when E passes through the *resonance energy* $E_{r_l} = E_{\lambda l} + \Delta_{\lambda l}$, and the *level shift*

$$\Delta_{\lambda l} = -\gamma_{\lambda l}^2 S_l \quad (1.17c)$$

is the amount by which the resonance energy is shifted from the eigenvalue position $E_{\lambda l}$. The first term on the right side of (1.17a) is referred to as the resonance contribution and the second as the potential scattering contribution.

The extreme one-level approximation of (1.16) can be relaxed to some extent without much complication by retaining the contribution to R_l from all the other levels as a contribution R_l^0 :

$$R_l = R_l^0 + \gamma_{\lambda l}^2 / (E_{\lambda l} - E). \quad (1.18)$$

The phase shift may then be expressed as

$$\delta_l = \tan^{-1} \left(\frac{\frac{1}{2} \Gamma_{\lambda l}'}{E_{\lambda l} + \Delta_{\lambda l}' - E} \right) - \phi_l', \quad (1.19)$$

where

$$\begin{aligned} \frac{1}{2} \Gamma_{\lambda l}' &= \gamma_{\lambda l}^2 P_l', & \Delta_{\lambda l}' &= -\gamma_{\lambda l}^2 S_l', \\ \phi_l' &= \phi_l - \tan^{-1} [R_l^0 P_l / (1 - R_l^0 S_l)], \\ S_l' &= [S_l (1 - R_l^0 S_l) - R_l^0 P_l^2] / d_l, & P_l' &= P_l / d_l, \\ d_l &= (1 - R_l^0 S_l)^2 + (R_l^0 P_l)^2. \end{aligned}$$

If R_l^0 is chosen to make (1.18) exact, then (1.19) is also exact like (1.15a). Although (1.19) has the same form as (1.17a), the level width, $\Gamma_{\lambda l}'$, the level shift $\Delta_{\lambda l}'$, and the potential scattering phase ϕ_l' are now modified by the contribution from the other levels. It is shown later (Sec. XI, 4) that the contribution to ϕ_l' from the other levels is of the same order of magnitude as the hard sphere contribution ϕ_l . Therefore, there is little justification to the interpretation of the observed potential scattering as that caused by a hard sphere.

d. The Relation between the Collision Function and the Cross Section

In order to obtain an expression for the differential elastic scattering cross section in terms of the U_l , one forms the following linear combinations of solutions to the wave equation:

$$i\pi^{1/2} k^{-1} \sum_l (2l+1)^{1/2} (g_l - U_l \theta_l), \quad (1.20)$$

This wave function remains unchanged if we add to and subtract from it the following wave function,

$$v^{-\frac{1}{2}} \exp(ikz)\psi = v^{-\frac{1}{2}} k^{-1} \sum_l i^l (2l+1) r^{-1} F_l(\rho) P_l(\cos\theta) \psi \\ = i\pi^{\frac{1}{2}} k^{-1} \sum_l (2l+1)^{\frac{1}{2}} (g_l - \theta_l), \quad (1.21)$$

which is the function representing a unit-flux beam of particles directed along the z axis; here $P_l(\cos\theta)$ is the Legendre polynomial and the sum over l is from $l=0$ to $l=\infty$. One thereby obtains the solution

$$v^{-\frac{1}{2}} \exp(ikz)\psi + i\pi^{\frac{1}{2}} k^{-1} \sum_l (2l+1)^{\frac{1}{2}} (1-U_l) \theta_l, \quad (1.22)$$

the asymptotic form of which is (noting that $Y_0^{(l)} = (2l+1/4\pi)^{\frac{1}{2}} P_l$)

$$v^{-\frac{1}{2}} \left[\exp(ikz) + \frac{\exp(ikr)}{r} A(\theta) \right] \psi, \quad (1.23)$$

where

$$A(\theta) = \frac{1}{2} i k^{-1} \sum_l (2l+1) (1-U_l) P_l(\cos\theta)$$

is the complex *scattering amplitude*. The solution (1.22) is evidently the sum of a unit-flux plane wave, which represents the incident beam, and outgoing waves, which represent the scattered particles. The differential elastic scattering cross section per unit solid angle is therefore

$$\sigma(\theta) = |A(\theta)|^2 \\ = \frac{1}{4} k^{-2} \left| \sum_l (2l+1) (1-U_l) P_l(\cos\theta) \right|^2. \quad (1.24)$$

By forming the absolute square of (1.24) and integrating over all solid angles, one obtains the total cross-section expression

$$\sigma_t = \int \sigma(\theta) d\Omega = \pi k^{-2} \sum_l (2l+1) |1-U_l|^2. \quad (1.25)$$

In the review article by Blatt and Biedenharn⁴⁷ an expansion is given for (1.24) as a series of Legendre polynomials with real coefficients. Further remarks concerning these expansions may be found there.

2. Dependence of the R Function on the Boundary Condition and Interaction Radius

The derivation of the last subsection involved the use of a particular boundary condition on the eigenfunctions, namely the zero-derivative condition (1.4). As far as the formal theory is concerned one can just as well use a more general boundary condition on the

⁴⁷ J. M. Blatt and L. C. Biedenharn, *Revs. Modern Phys.* **24**, 258 (1952); Biedenharn, Blatt, and Rose, *Revs. Modern Phys.* **24**, 249 (1952).

logarithmic derivative, like

$$\left(\frac{r}{u_\lambda} \frac{du_\lambda}{dr} \right)_a = B \quad (2.1)$$

where B is a fixed (that is, independent of λ) real number. The orthonormalization condition (1.5) is also valid with such a boundary condition, and the other relations are modified as follows. For the expansion coefficients of (1.6), one finds

$$A_\lambda = \frac{\hbar^2}{2M} \left(u_\lambda \frac{du_E}{dr} - u_E \frac{du_\lambda}{dr} \right)_a / (E_\lambda - E) \quad (2.2)$$

so that

$$u_E(r) = \frac{\hbar^2}{2Ma} \left(a \frac{du_E}{dr} - B u_E \right)_a \sum_\lambda \frac{u_\lambda(r) u_\lambda(a)}{E_\lambda - E}. \quad (2.3)$$

Putting $r=a$ in this equation and using the definition (1.10) of the R function, we have

$$R(B) = (f - B)^{-1}, \quad (2.4)$$

where, following Feshbach, Peaslee, and Weisskopf,¹⁵ we have put

$$f = \left(\frac{r}{u_E} \frac{du_E}{dr} \right)_a,$$

i.e., f is the logarithmic derivative quantity of the radial wave function times a . Since f does not depend on B , the relation between $R(B)$ and the R function for zero slope boundary condition, i.e., $R(0)$, is

$$BR(B) = \frac{BR(0)}{1 - BR(0)}. \quad (2.5a)$$

Alternatively the dependence of R on B can be expressed as a differential equation:

$$\frac{\partial R}{\partial B} = R^2. \quad (2.5b)$$

The formulas from (1.13) through (1.19) are adapted to this boundary condition by simply replacing the shift factor S by S^0 , where

$$S^0 = S - B. \quad (2.6)$$

Although the expression (1.15a) for the phase shift would then contain a dependence on B through the factor S^0 , this dependence is compensated by the dependence of R on B , so the phase shift is actually independent of B , as it must be. Selection of the most appropriate boundary condition is discussed in Sec. XII.

From knowledge of the reduced widths and level positions for a particular boundary condition, say $B=0$, one can in principle obtain the corresponding quantities for any other boundary condition. From (2.5) it is

apparent that the energies $E_\lambda(B)$ are the solutions to the equation

$$R(0) = \frac{1}{B}. \quad (2.7)$$

The reduced widths $\gamma_\lambda^2(B)$ may be determined by consideration of the quantity $R^2/(dR/dE)$. This quantity, which we write as $\gamma^2(E)$, is independent of the choice of boundary condition B . [This can be seen from (2.4), which shows that the quantity equals $-(df/dE)^{-1}$, and the fact that f is independent of B .] Thus,

$$R^2(B) \left(\frac{\partial R(B)}{\partial E} \right)^{-1} = \gamma^2(E) = R^2(0) \left(\frac{\partial R(0)}{\partial E} \right)^{-1}. \quad (2.8)$$

Taken with the choice of energy $E = E_\lambda(B)$, this equation gives, on using (2.7),

$$\gamma_\lambda^2(B) = \gamma^2(E_\lambda(B)) = \left[B^2 \left(\frac{\partial R(0)}{\partial E} \right)_{E=E_\lambda(B)} \right]^{-1}. \quad (2.9)$$

Just as the *structure* of the formal theory is the same for any particular choice of boundary condition (i.e., of B), so it is also the same for any particular choice of interaction radius (i.e., of a), provided only that it is taken greater than the minimum value allowed by assumption (4) of Sec. II.2. In fact, as far as the formal theory is concerned, a and B are just parameters. This is not true in applications where, as will be seen later (e.g., Sec. XII), one must use particular values of a and B in order to derive most benefit from the theory. At present, however, we are only considering formal aspects of the theory.

The collision function and cross section cannot depend on a and B , of course. Consequently, in the expressions for these quantities, although a and B occur, they always occur in compensating fashion. For instance, we have just seen how, in the case of B , the combination $f = B + 1/R(B)$ must be independent of B . In the case of a , the compensation is somewhat more involved because a occurs in both the "internal" function R and the "external" functions I , O , and L .

Teichmann⁴⁸ examined the formal dependence of the quantities E_λ and γ_λ^2 on choice of a and B , and exhibited this dependence in the form of simple first-order differential equations. The dependence on B follows from (2.5b) by considering the limit $E \rightarrow E_\lambda$. This gives

$$\frac{\partial E_\lambda}{\partial B} = -\gamma_\lambda^2$$

$$\frac{\partial(\gamma_\lambda^2)}{\partial B} = 2\gamma_\lambda^2 \sum_{\mu(\neq\lambda)} \frac{\gamma_\mu^2}{E_\mu - E_\lambda}.$$

The corresponding equations for the dependence on a

⁴⁸ T. Teichmann, Phys. Rev. **77**, 506 (1950); thesis, Princeton (unpublished, 1949).

are somewhat more complicated since, as expected, they involve the external potentials. For these equations, and for the corresponding equations for the general many-channel theory of the following sections, we refer to Teichman's original work.

3. Special Case of a Square Potential Well

The special case of a square well for the internal interaction illustrates some of the properties of the R -function expansion. In this case the interaction potential V of (1.1) contains a constant contribution V_0 in the internal region where $r < a$; there is also the centrifugal potential contribution $l(l+1)(\hbar^2/2Mr^2)$ for angular momentum l .

a. Zero Angular Momentum ($l=0$)

The solution to (1.1) for arbitrary energy E is in the internal region

$$u_E(r) = \sin Kr, \quad (3.1)$$

where

$$K = [2M(E - V_0)/\hbar^2]^{1/2},$$

and according to (1.11) the R function for a zero boundary condition is therefore

$$R = \mu^{-1} \tan \mu, \quad (3.2)$$

where $\mu = Ka$. The energy eigenvalues are those energies at which (3.1) has zero derivative at the surface, that is, at which

$$\mu_\lambda = K_\lambda a = \pi(\lambda - \frac{1}{2}), \quad \lambda = 1, 2, 3 \dots, \quad (3.3a)$$

or

$$E_\lambda = V_0 + (\pi^2 \hbar^2 / 2M a^2) (\lambda - \frac{1}{2})^2. \quad (3.3b)$$

The reduced-width amplitudes may be obtained using (1.10a); after noting that (3.1) needs to be normalized, one obtains

$$\gamma_\lambda = (-1)^{\lambda+1} (\hbar^2 / M a^2)^{1/2}. \quad (3.4)$$

The R -function expansion is then

$$R = \sum_\lambda \frac{\gamma_\lambda^2}{E_\lambda - E} = \sum_{\lambda=1}^{\infty} \frac{2}{\pi^2 (\lambda - \frac{1}{2})^2 - \mu^2}, \quad (3.5)$$

which is recognized as the partial fraction expansion for $\mu^{-1} \tan \mu$ of (3.2).

It is possible to derive a simple expression for the contribution to (3.5) at a particular energy level E_λ from the other levels, that is for the quantity $R^0(E_\lambda)$ of (1.18). It follows from (2.8) that

$$d\gamma^2/dE = 2R - R^2 \frac{d^2 R}{dE^2} / \left(\frac{dR}{dE} \right)^2. \quad (3.6)$$

By substituting (1.18) into the above and evaluating in the limit $E = E_\lambda$, one obtains

$$R^0(E_\lambda) = -\frac{1}{2} (d\gamma^2/dE)_{E_\lambda}, \quad (3.7)$$

valid whether or not R^0 is treated as a constant. Expression (3.2) is then used to evaluate the right side of (3.7), with the result that

$$[R^0(E_\lambda)]^{-1} = 2\mu(E_\lambda)^2 = 2\pi^2(\lambda - \frac{1}{2})^2. \quad (3.8)$$

The contribution from the other levels clearly diminishes as λ increases. This tendency reflects the stronger cancellation of positive and negative terms in the R^0 function sum as λ increases from $\lambda=1$ (where all terms are positive).

b. Arbitrary Angular Momentum

The solution to (1.1) for arbitrary angular momentum is the function F_l of (III.2.15a) with $\mu=Ka$. The R function for zero boundary condition

$$R_l = f_l^{-1} = F_l(\mu)/\mu F_l'(\mu), \quad (3.9)$$

where the prime denotes differentiation with respect to μ . It is convenient to introduce a dimensionless quantity

$$\theta_l^2 = (Ma^2/\hbar^2)\gamma_l^2 \quad (3.10)$$

then, from (2.4) and (2.8):

$$-\theta_l^{-2} = (\hbar^2/Ma^2)(df_l/dE) = \mu^{-1}f_l'(\mu). \quad (3.11)$$

Using the equation for $F_l(\mu)$

$$F_l'' + [1 - l(l+1)\mu^{-2}]F_l = 0, \quad (3.12)$$

it follows that

$$\theta_l^{-2} = [(f_l + l)(f_l - l - 1) + \mu^2]/\mu^2. \quad (3.13)$$

Now choosing

$$B_l = l \quad (3.14)$$

as the boundary condition, it is evident that $\theta_l^{-2}(E_\lambda) = 1$. From (2.9) it follows that $\theta_\lambda^2 = 1$ and therefore the reduced-width amplitude expression (3.4) holds for all values of l . To find the energy eigenvalues $E_{\lambda l}$, one determines the corresponding values $\mu_{\lambda l}$ for which the right-hand side of (3.9) equals $-1/l$. Using the recursion formula

$$F_l' = F_{l-1} - (l/\mu)F_l, \quad (3.15)$$

this equality becomes

$$F_{l-1}(\mu_{\lambda l}) = 0. \quad (3.16a)$$

so that

$$E_{\lambda l} = V_0 + (\hbar^2/2Ma^2)\mu_{\lambda l}^2, \quad (3.16b)$$

where $\mu_{\lambda l}$ is the λ th root of (3.16a). The quantity $R^0(E_\lambda)$ is determined as in the case of zero angular momentum. Using (3.13) together with (3.6) and (3.7), one obtains

$$R_l^0(E_{\lambda l}) = (2l+1)/2\mu_{\lambda l}^2. \quad (3.17)$$

The effect of the other levels thus becomes more important as l increases.

4. Definition of a Resonance Level

It is appropriate to discuss at this place our definition of a resonance, or virtual, energy level. According to

(1.17c), the resonance levels occur at those energies E_r at which

$$E_\lambda + \Delta_\lambda(E_r) - E_r = 0. \quad (4.1)$$

They are thus the energies at which the resonance contribution to the phase shift formula [(1.17a), or, preferably, (1.19)] is equal to odd integral multiples of $\frac{1}{2}\pi$. This is the definition which is used throughout this review. Since the level shift $\Delta_\lambda(E)$ is a bounded monotonic function of the energy E , there will be a one-to-one correspondence of the resonance levels and the energy eigenvalues. As there is an infinite number of the latter, there will also be an infinite number of the former.

The definition may seem unsatisfactory for several reasons. First, the precise values that one obtains for the E_r depend upon which formula one uses for the level shift in the interpretation of the resonance data, that is, (1.17c) or the corresponding formula of (1.19). There may also be a slight dependence on the boundary condition when (1.19) is used for the potential scattering contribution to the phase shift. The main reason, however, is that this definition implies that even in field-free space, there will be an infinite spectrum of resonance levels, as indicated by (3.3b) with V_0 set equal to zero. In this case the resonance and potential scattering contributions to the phase shift are equal in magnitude but opposite in sign so that the phase shift vanishes. This last reason is an extreme example of the practical difficulty that one may encounter when trying to separate the resonance from the potential scattering contribution to the observed phase shift. For these reasons resonance energies are sometimes defined in the literature as those energies at which the phase shift equals an odd-integral multiple of $\pi/2$. Although this definition involves no ambiguities, there is then no longer a one-to-one correspondence between the E_r and the E_λ , and indeed, a nuclear system could have only a few or even no such resonance levels.

In spite of these difficulties, we adhere to the definition (4.1) to avoid confusion, and because of its acceptance for the interpretation of the type of "resonance" data with which we are primarily concerned. This problem of establishing the best definition of resonance is a rather academic one. In practice, at a sharp peak in a cross section most definitions give an energy inside the width of the peak. Only for very broad peaks do different definitions give very different energies. When this occurs, it is a warning that the interpretation in terms of a resonance is not a suitable one.

5. Expansion of the Derivative of the Wave Function

If one tries to obtain the derivative of $u_E(r)$ at $r=a$ by differentiating the individual terms of the sum for $G(r,a)$ on the right side of (1.8), one obtains a null result, because derivatives of the individual terms are all zero according to (1.4). The explanation for this paradox is that the derivative series thus obtained is not uniformly convergent in the vicinity of the surface. That it does converge, albeit nonuniformly, was shown

in the following way by Jackson⁴⁹ for the case where the interaction potential is bounded.

The derivative series for (1.8) is

$$u'(r) = \frac{\hbar^2}{2M} \sum_{\lambda} \frac{u_{\lambda}'(r) u_{\lambda}(a)}{E_{\lambda} - E} u'(a), \quad (5.1)$$

where a prime now denotes differentiation with respect to r . By integrating the Schroedinger equation (1.1) for the $u_{\lambda}(r)$ from r to a and by noting the condition (1.4), one obtains

$$u_{\lambda}'(r) = (2M/\hbar^2) \int_r^a (E_{\lambda} - V) u_{\lambda} dr. \quad (5.2)$$

The derivative series can therefore be expressed as

$$u'(r)/u'(a) = \sum_{\lambda} u_{\lambda}(a) \int_r^a u_{\lambda} dr + \sum_{\lambda} (E_{\lambda} - E)^{-1} u_{\lambda}(a) \int_r^a (E - V) u_{\lambda} dr. \quad (5.3)$$

According to Courant's minimax considerations, if V is bounded, no E_{λ} differs from the corresponding E_{λ} for free space by more than the bound, and consequently the second sum of (5.3) converges in general and vanishes as $r \rightarrow a$, because the latter $E_{\lambda} \sim \lambda^2$ for large λ as indicated by (3.3b). When r is near a , the first sum is just the expansion of a function which is unity from r to a and zero elsewhere, so that $u'(r) \rightarrow u'(a)$ as $r \rightarrow a$.

It is not possible to develop a significant \mathbf{R} -matrix theory in terms of X_{λ} which are zero on S (i.e. with $B = \infty$), because of the above difficulty with regard to interchange of order of summation and differentiation. With such X_{λ} one might presume that the development of subsection 1 could be carried out in an analogous manner to give a relation

$$"R" = \left(\frac{r}{u_E} \frac{du_E}{dr} \right)$$

corresponding to (1.11) but with the "R" function of the form

$$\sum_{\lambda} \frac{\delta_{\lambda}^2}{E_{\lambda} - E},$$

where the δ_{λ}^2 are the quantities defined in (III.4.7b). However, the above-mentioned difficulty of interchange is encountered in the last step leading to this relation, namely matching of internal and external normal gradients on \mathcal{S} . Nevertheless, a procedure which is described in Sec. V. 3a and does not involve this matching may be used to obtain an "R" function of the form

$$"R" = R^{\infty} + \sum_{\lambda} \sum (\Delta + E) / (E_{\lambda} + \Delta)] \delta_{\lambda}^2 / (E_{\lambda} - E),$$

where Δ is an arbitrary constant energy and R^{∞} a real symmetrical matrix. The constant energy Δ arises from the identity

$$\frac{(E_2 - E_1)}{(E_{\lambda} - E_1)(E_{\lambda} - E_2)} = \frac{\Delta + E_2}{(E_{\lambda} + \Delta)(E_{\lambda} - E_2)} - \frac{(\Delta + E_1)}{(E_{\lambda} + \Delta)(E_{\lambda} - E_1)}$$

which can be used instead of that of Sec. V. 3a. If Δ were infinite, as in the ordinary \mathbf{R} -matrix expansion, the above sum would in general not converge because the δ_{λ}^2 tend to increase with E_{λ} , rather than being essentially constant or decreasing, as in the case of the γ_{λ}^2 . The Mittag-Leffler expansion for the logarithmic derivative of the square well wave function corresponds to such an "R" function expansion with $\Delta = 0$ and $R^{\infty} = -1$.

6. Properties of R Functions

a. General Mathematical R Functions

The "physical" R function defined by (1.10) may be considered as belonging to the broader class of "mathematical" R functions. The latter functions are defined as meromorphic

⁴⁹ J. L. Jackson, thesis, New York University (unpublished, 1950).

functions of a complex variable, the imaginary parts of which functions are non-negative in the upper half-plane and non-positive in the lower half. The R of (1.10) is evidently such a function when considered as a function of the complex variable $\varepsilon = E + iP$,

$$R(\varepsilon) = \sum_{\lambda} \gamma_{\lambda}^2 / (E_{\lambda} - \varepsilon), \quad (6.1)$$

provided that the poles E_{λ} are real and their residues $-\gamma_{\lambda}^2$ are negative, as in physical applications. Although the theory of elastic scattering, just described, is only concerned with the evaluation of R along the real axis where ε is identified with the real energy E of the nuclear system, later (Sec. XI) we show that, when absorption is considered together with scattering, evaluations are needed for complex energies, the imaginary parts of which are one-half of the total absorption widths. The properties of the mathematical R functions in general and certain types of them of special physical interest in the resonance theory have been studied extensively by Wigner.⁵⁰⁻⁵⁴ Some of his results which have applications in physical problems are stated in the following.

(i) An R function is real on the real axis and assumes real values only on the real axis.—It follows from this theorem and the definition that

$$R(\varepsilon^*) = R(\varepsilon)^*. \quad (6.2)$$

(ii) The derivative of an R function is positive at every regular point of the real axis.—For the physical R functions given by (6.1) this is obvious because

$$\frac{dR(E)}{dE} = \sum_{\lambda} \frac{\gamma_{\lambda}^2}{(E_{\lambda} - E)^2} > 0, \quad (6.3)$$

except at the singularities E_{λ} . This property is particularly important and one of its consequences in scattering theory is developed in the next section. Excepting again the E_{λ} , all of the odd derivatives of (6.1) are positive on the real axis, and all of the even derivatives satisfy there the inequality⁵¹

$$(R^{(n+m)} / (n+m)!)^2 \leq R^{(2n+1)} R^{(2m-1)} / (2n+1)!(2m-1)! \quad (6.4)$$

(iii) All poles of an R function must lie on the real axis, and they are simple ones and have negative residues.—These properties are manifest in the physical R functions of (6.1).

(iv) Every R function can be expanded in an absolutely convergent Mittag-Leffler series.—

$$R = \alpha \varepsilon + \beta + \sum_{\mu} \left(\frac{\gamma_{\mu}^2}{E_{\mu} - \varepsilon} - \frac{\gamma_{\mu}^2}{E_{\mu}} \right) \quad (6.5)$$

with α and γ_{μ}^2 positive, β and E_{μ} real.—For the physical R functions (6.1), $\alpha = 0$ and $\beta = \sum_{\mu} \gamma_{\mu}^2 / E_{\mu}$.

(v) The linear fractional function

$$Q = (a_1 R + b_1) / (a_2 R + b_2) \quad (6.6)$$

of an R function with positive determinant ($a_1 b_2 - a_2 b_1 > 0$) and real coefficients is an R function.—A special case of this theorem has already been encountered in the consideration of the change induced in R by a change in B (subsection 2). A more significant application of the theorem concerns (1.15a) where it shows that if the quantities P and S of (1.15a) can be treated as constants, the linear fractional function

$$Q = RP / (1 - RS) \quad (6.6a)$$

of that equation is an R function because P is always positive. Therefore, Q has a series expansion of the form (6.5) and, moreover, it can be shown that the absence of a linear term $\alpha \varepsilon$ in the R of (1.15a) entails the absence of a similar term in Q . Hence, the phase shift can be expressed as

$$\delta = \tan^{-1} \left(Q_0 + \sum_{\mu} \frac{\theta_{\mu}^2}{H_{\mu} - E} \right) - \phi, \quad (6.7a)$$

⁵⁰ E. P. Wigner, Proc. Cambridge Phil. Soc. 47, 790 (1951).

⁵¹ E. P. Wigner, Ann. Math. 53, 36 (1951).

⁵² E. P. Wigner, Ann. Math. 55, 7 (1952).

⁵³ E. P. Wigner, Ann. Math. Monthly 59, 669 (1952).

⁵⁴ E. P. Wigner and J. V. Neumann, Ann. Math. 59, 418 (1954).

where the new constants H_μ and Q_0 are real, θ_μ^2 positive, these being related to the P , S , $\gamma\lambda^2$, and E_λ , although in a complicated manner. In practice P and S are not constants, and P in particular will have a rather violent energy dependence when there is a strong Coulomb barrier in the external region. S , on the other hand, can usually be well approximated over a fairly wide energy range on the real axis by a linear function $S=S_0+S_1E$ with $S_1 \geq 0$. The linear fractional function $R/[1-R(S_0+S_1E)]$ is also an R function if S_1 is positive, and the linear term of its series (6.5) is again absent. It is therefore preferable to expand the phase shift as

$$\delta = \tan^{-1} P \left(Q_0 + \sum_{\mu} \frac{\theta_{\mu}^2}{H_{\mu} - E} \right) - \phi, \quad (6.7b)$$

which involves no restriction on P and allows S to vary in the indicated manner; the constants here are not the same as those of (6.7a).

(vi) Every R function can be expressed as:—

$$R = b + a \tan g(\varepsilon) \quad (6.8)$$

with any real b , positive a , and a convergent series representation for $g(\varepsilon)$ of the form,

$$g(\varepsilon) = \eta \varepsilon + E_0 + \sum_{\nu} \left(\tan^{-1} \frac{\varepsilon - E_{\nu}}{\kappa_{\nu}} + \tan^{-1} \frac{E_{\nu}}{\kappa_{\nu}} \right) \quad (6.8a)$$

with positive η , κ_{ν} , and real E_0 , E_{ν} . The constants η , E_0 , E_{ν} , κ_{ν} in the expansion for a particular R will depend on the choice of a and b . Owing to the violent energy dependence and the discontinuous nature of R functions, it is sometimes convenient when interpreting experimental data to plot the function $g(E) = \tan^{-1} \times (R - b/a)$, which is continuous, rather than R itself.

(vii) Every R function can be expanded in an absolutely convergent product of the form,

$$R(\varepsilon) = c \frac{x_0 - \varepsilon}{Z_0 - \varepsilon} \prod_{\nu} \frac{1 - \varepsilon/x_{\nu}}{1 - \varepsilon/Z_{\nu}} \prod_{\nu} \frac{1 - \varepsilon/x_{-\nu}}{1 - \varepsilon/Z_{-\nu}}, \quad (6.9)$$

the order of the numbers x_{ν} , Z_{ν} being

$$\dots Z_{-2} < x_{-2} < Z_{-1} < x_{-1} < Z_0 < x_0 < Z_1 < x_1 \dots$$

(viii) The solution $R(r, \varepsilon)$ to Riccati's equation

$$dR(r, \varepsilon)/dr = R^2 + \mathcal{U}(r)R - \mathcal{V}(r) + \varepsilon \quad (6.10)$$

with the initial condition $R(0, \varepsilon) = R_0$, with real R_0 , is an R function of ε for every value of r in the interval $(0, a)$ where $\mathcal{U}(r)$ and $\mathcal{V}(r)$ are real continuous functions. With the substitution $R = -(du/dr)/u$ in (1.1), one obtains (6.10) with $\mathcal{U} = 0$, $\mathcal{V} = (2M/\hbar^2)V(r)$, $\varepsilon = 2ME/\hbar^2$. Thus the negatives of the logarithmic derivatives of Bessel functions and of many other functions, including the Γ functions, are R functions and permit partial fraction and product representations of the form (6.5) and (6.9), respectively. Although the physical R function is the reciprocal of the logarithmic derivative of the wave function, theorem (v) shows that the negative of this derivative is also an R function.

b. Uniform R Functions

An R function is said to be uniform if it has both a pole density ρ and a definite strength s , and if there is no linear term $\alpha\varepsilon$ in its expansion (6.5). By "definite strength," it is meant that for every ε there is an $L(\varepsilon)$ such that the sum of the residues of all poles within any interval of length $L > L(\varepsilon)$ on the real axis is between $-(1-\varepsilon)sL$ and $-(1+\varepsilon)sL$ with $s > 0$. The term "pole density" has a similar meaning. The utility of these functions will be apparent after the statements of some theorems pertaining to them and after the introduction below of a special class of the uniform R functions called "statistical R functions."

A trivial example of a uniform R function is $\pi s \tan \pi \rho \varepsilon$. An example of a nonuniform R function is (3.5), the reciprocal of the logarithmic derivative of the square-well wave function.

(ix) If an R function has a pole density, that density is invariant under the linear fractional transformation (6.6).—This theorem

shows that if the R in (1.15a) is uniform, then the transformation $R/(1-RS)$ therein, S being considered as constant, will have the same pole (or level) density.

(x) If an R function has a definite strength s , and if there is no linear term $\alpha\varepsilon$ in its expansion (6.5), the imaginary part of R will converge uniformly (in E) to πs as the imaginary part F of $\varepsilon = E + iF$ goes to infinity.—This property may be made evident by replacing the sum over levels of a uniform R function of the form (6.1) by an integration for the evaluation of the imaginary part when F is large compared with the mean spacing. The above example has this property since $\tan i\infty = i$. A uniform R function also has the property that its real part approaches a value which is independent of E when F goes to infinity, provided that the convergence of average strength to the asymptotic value is sufficiently rapid, specifically, that the $L(\varepsilon)$ which is involved in the definition of s satisfies the subsidiary condition $L(\varepsilon) < A/\varepsilon^{\nu}$, with positive A and ν , when $\varepsilon < 1$.

Since the linear fractional transformation Q of (6.6) of a uniform R function which satisfies such a subsidiary condition will also be uniform and satisfy that condition, one can obtain the strength s_Q and the corresponding real part Q_0 by evaluating the right side of (6.6) at $\varepsilon = i\infty$. For the transformation $Q = R/(1-RS)$ the result is that, just as $E(\varepsilon) \rightarrow R_0 + i\pi s_R$, so $Q(\varepsilon) \rightarrow Q_0 + i\pi s_Q$ as $\varepsilon \rightarrow E$, where

$$\begin{aligned} Q_0 &= R_0(1-R_0S)/[(1-R_0S)^2 + (\pi s_R S)^2] \\ s_Q &= s_R/[(1-R_0S)^2 + (\pi s_R S)^2]. \end{aligned} \quad (6.11)$$

The last of these equations shows that the θ_{μ}^2 of a series expansion such as that in (6.7a) for Q will have an average which will in general differ from the average of the $\gamma\lambda^2$, in spite of the fact that the level density will remain the same.

(xi) For every uniform R function, the distance of two closest poles never can exceed a definite limit and there is an upper limit for the residue of any pole.—This follows immediately from the definition of the strength and the corresponding definition of the pole density.

c. Statistical R Functions

Uniform R functions having definite distributions of spacings and of strengths which are independent of the interval involved in obtaining the distributions are called statistical R functions. The condition for the existence of a distribution $s(r)$ for strengths is that for any positive $\varepsilon < 1$ there is an $L(\varepsilon)$ such that for any integral $L > L(\varepsilon)$ on the real axis the integral

$$\int_0^{\varepsilon} |s(r) - s_L(r)/L| dr < \varepsilon,$$

where $s_L(r)$ is the number of residues which are smaller than $-r$ and belong to the poles of the interval L ; the strength $s = \int s(r) dr$. The quantities $s_L(r)$ and $s(r)$ are monotonously decreasing functions of their argument. A similar condition applies to the distance D of subsequent poles from each other which leads to the specification of a distribution function $P(D)$ for the number of distances larger than D ; here the pole density is $\rho = P(0) = s(0)$. An example of a statistical R function is

$$R = \tan \left[\eta \varepsilon + \vartheta + \sum_k \tan^{-1} \left\{ \tanh(\lambda_k \kappa_k) \tan(\lambda_k(\varepsilon - x_k)) \right\} \right] \quad (6.12)$$

in which the sum over k is a finite one and all Greek symbols with the exception of ϑ are positive. Although Wigner³¹ was able to show that the distributions for spacing and strength of R functions of the form (6.12) are invariant under an orthogonal fractional transformation [$a_1^2 + b_1^2 = 1$, $a_2 = -b_1$, $b_2 = -a_1$, in (6.6)], provided that λ_k are incommensurable with η , only a plausibility argument could be given for the invariance of the transformation of "most" statistical R functions.

The general linear fractional transformation (6.6) of a uniform R function which satisfies the subsidiary condition can be considered as a succession of three linear fractional transformations.

The first of these is just the linear one,

$$R_n = (R - R_0)/\pi s, \quad (6.13)$$

where R_0 is the quantity defined after theorem (x). The R function so obtained has the property that $R_n(E+i\infty) = i$, and it is referred to as a normalized uniform R function. The second is an "orthogonal" one ($a^2 + b^2 = 1$),

$$\begin{aligned} Q_n &= (aR_n + b)/(-bR_n + a) \\ &= (R_n \cos\phi + \sin\phi)/(-R_n \sin\phi + \cos\phi) \\ &= (R_n + \tan\phi)/(1 - R_n \tan\phi). \end{aligned} \quad (6.14)$$

The third is another linear one,

$$Q = cQ_n + c'. \quad (6.15)$$

In the case of the transformation (6.6a) $\tan\phi = \pi s S/(1 - RS)$, $c = \pi s/q$, $c' = [R_0(1 - R_0 S) - (\pi s)^2 S]/q$, where $q = (\pi s S)^2 + (1 - R_0 S)^2$. The first and third transformations do not change the positions of the poles, and merely multiply all of the residues by the same number. Hence they affect neither the distribution of the distances of successive poles nor the statistics of the ratios of the residues. The second does not affect the density of poles, and transforms normalized R functions into normalized R functions. However, it may influence the frequency with which certain distances of poles occur and may influence the statistics of the residues without affecting the average value of either quantity.

To study the second transformation [i.e., (6.14)], one represents both R and Q_n by expressions (6.8) with $a=1$, $b=0$;

$$R_n = \tan g_R(\varepsilon); \quad Q_n = \tan g_Q(\varepsilon). \quad (6.16)$$

In view of the orthogonal nature of the transformation, it follows that

$$\dot{g}_Q = \dot{Q}_n/1 + Q_n^2 = \dot{R}_n/1 + R_n^2 = \dot{g}_R, \quad (6.17)$$

where a dot denotes differentiation with respect to energy, so that the derivative function

$$\dot{g}(\varepsilon) = \eta + \sum_\nu \kappa_\nu / [\kappa_\nu^2 + (\varepsilon - E_\nu)^2] \quad (6.18)$$

is the same for both representations and is for that reason referred to as the "invariant derivative." Both R_n and Q_n may therefore be expressed as

$$\tan \int_c^\varepsilon \dot{g}(\varepsilon) d\varepsilon \quad (6.19)$$

but with different lower integration limits c . Indeed, (6.14) reveals that

$$g_Q = g_R + \phi, \quad (6.20)$$

where $\phi = \tan^{-1}(b/a)$. By looking at Fig. 1 where the invariant derivative is plotted, one sees that starting at a particular pole, say E_1 , the next pole E_2 is determined as the abscissa point at which the area under \dot{g} between E_1 and E_2 is just π . By inserting the series expansion (6.1) for R_n into (6.17), one finds that the reduced widths are simply

$$\gamma_\lambda^2 = 1/\dot{g}(E_\lambda). \quad (6.21)$$

The levels and corresponding widths of Q_n can be obtained from the same construction but starting, however, at a different abscissa. Unless the κ_ν and E_ν of (6.18) have special values or unless $\eta=0$, the chance of hitting on a definite value of \dot{g} will be in the long run the same no matter at which abscissa point one started originally. It therefore seems plausible that the distributions for the widths and spacings of Q_n will be the same as those for R_n , i.e. the *distributions of pole strengths and spacings are invariant under orthogonal fractional transformations* (6.14). Since the same is true for the transformations (6.13) and (6.15), it follows that the *distributions are invariant under any fractional transformation* (6.6).

When applied to the transformation (2.5) corresponding to a change in boundary condition B , this tells us what we might suspect on physical grounds, *viz.*, the distributions of γ_λ^2 and D do not depend on the choice of B . Another application is to the transformation (6.6a). In this case, one can conclude that the statistical distributions of the θ_μ^2 and H_μ of (6.7a) are probably

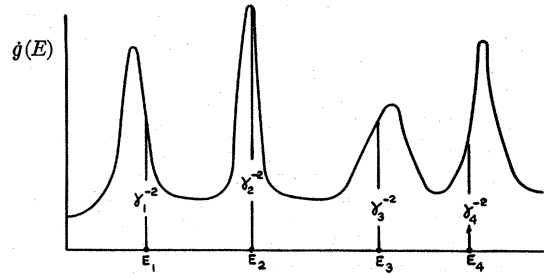


FIG. 1. Construction of eigenvalues E_ν and reduced widths γ_ν^2 from the invariant derivative $\dot{g}(E)$ of (6.18). If the eigenvalue E_1 is given, the position of E_2 is found by the condition that the integrated area under the curve between E_1 and E_2 is equal to π . The positions of E_3 , E_4 and all the other poles may be found similarly. The figure has been drawn for the situation $\eta\Delta \ll \pi$, $\kappa_\nu \ll \Delta$, where Δ is the mean spacing of the E_ν . This corresponds to separated peaks and small background. The area under a completely isolated peak equals π so that, for isolated peaks there is a one-one correspondence between the peaks and the E_ν . In actuality the peaks are not isolated so that, if E_1 is near a maximum of g , the succeeding E_ν are increasingly to the left to the maxima and eventually begin to approach maxima from the right. Thus, at least for $\eta > 0$, the values $\dot{g}(E_\nu)$, which are the reciprocals of the reduced widths, wander through maxima and minima and their distribution is independent of the position of E_1 .

not affected by the interactions in the external region which are responsible for the nonvanishing factor S in the transformation (6.6a), provided this factor can be considered as essentially constant and that the κ_ν , E_ν , and η , of (6.18), do not have "special" values.

Invariance of the distribution laws does not reveal anything about the forms of the distribution laws themselves. To deduce these forms requires considerable extra information such as specification of the distributions of pole strengths and spacings of $\dot{g}(\varepsilon)$ of (6.18). As an illustrative example, Wigner⁵² has shown that if the E_ν of (6.18) are distributed at random and if the κ_ν have some arbitrary distribution, one can write down expressions for the distributions of γ_λ^2 and of differences of subsequent E_λ .

As yet, we have said nothing of the possibility of correlations between the pole strengths and spacings of levels. Certainly the foregoing arguments about the invariance of distributions are expected to apply to such correlations. Again, however, the invariance of the correlations reveals nothing about their importance or specific forms.

If, for instance, the plot of \dot{g} against energy consists of a series of well-separated sharp spikes (i.e. if $\kappa_\nu \ll$ pole spacing) then, since the area under each spike is $\sim \pi$, it follows that any correlation between the values of γ_λ^2 and the spacings of subsequent E_λ of the R function is directly determined by the correlation (if any) between values of κ_ν and spacings of subsequent E_ν of the \dot{g} function. On the other hand, if the function \dot{g} is smoothed out, there will emerge an additional superposed, correlation between the values of γ_λ^2 and spacings of subsequent E_λ *viz.*, large widths will be associated with large spacings, and small widths with small spacings. Such a correlation was first suggested by Feshbach, Peaslee, and Weisskopf¹⁵ on the basis of more qualitative arguments. It has also been suggested by Teichmann and Wigner⁵⁵ on the basis of Eq. (2.9). The trouble with these speculations is that they are based on implicitly attributing qualitative properties to functions that are really quite unknown. For instance, (2.9) in itself gives no information about correlations [as can be seen by considering $B=0$ in which case (2.9) reduces to the identity: $\gamma_\lambda^2(0) = \gamma_\lambda^2(0)$]. Thus, some *ad hoc* assumption must be made about the behavior of the function $\gamma_\lambda^2(B)$. The speculations^{15,55} appear to assume that widths have a fairly narrow distribution. The

⁵⁵ T. Teichmann and E. P. Wigner, Phys. Rev. **87**, 123 (1952).

experimental evidence, however, is that the distribution is very broad.⁵⁶

d. Physical R Functions

The "physical" R functions of nuclear reaction theory are not "uniform" in the strict sense of the definition given. It is a well-known experimental fact that the mean level width and the pole strength function have long-range variations. Nevertheless one can assert that the physical R functions are uniform in a restricted sense, namely in restricted energy intervals. In other words, it is possible to find an interval I containing a large number of levels wherein the pole strength function $s(E)$ and level density $\rho(E)$ are well defined and do not vary significantly from one end of the interval to the other. (Of course it also follows that, if the distributions do not vary in the interval I , the physical R functions are statistical R functions in the same restricted sense.) Thus we can write the actual (physical) R function as the sum of an R function to be written R' which is uniform *everywhere* and a term $-R_0$ which will take into account the deviation of the actual R function from uniformity outside of I

$$R(\varepsilon) = R'(\varepsilon) - R_0(\varepsilon). \quad (6.22)$$

The uniform R function R' is constructed so that its poles in I coincide with those of R , the respective residues being equal. Outside of I the poles and residues of R' need be specified only to the extent that the uniformity property is realized, and these poles and residues will in general differ from those of R . As a consequence, it is reasonable for the approximate evaluation of R_0 to replace the sums in the combination $R' - R$ by an integration, whereupon one obtains for complex $\varepsilon = E + iF$ with positive F :

$$R_0(\varepsilon) \equiv R' - R \approx i\pi s(E) + \bar{R}(\varepsilon), \quad (6.23a)$$

where

$$\bar{R}(\varepsilon) = \int \frac{s(E') dE'}{E' - \varepsilon} \quad (6.23b)$$

is just the Stieltjes transform of $s(E)$. If a strength $s(E)$ can be defined and if it is slowly varying with respect to energy intervals containing a large number of levels, then R_0 is also slowly varying, that is, it is essentially constant in an interval of type I .

The inverse of the transform (6.23b) is

$$s(E) = \lim_{F \rightarrow 0+} \left[\frac{1}{\pi} \text{Im} \bar{R}(\varepsilon) \right]. \quad (6.23c)$$

For evaluation of R on the real axis, one may therefore consider the relation

$$R(E) = R'(E) + \lim_{F \rightarrow 0+} \text{Re} \bar{R}(\varepsilon), \quad (6.24)$$

the limit term being recognized as just the principal value of $\int s(E') dE' / (E' - E)$. On the other hand, when the imaginary part F is large compared with the level spacing, $R'(\varepsilon)$ approaches $i\pi s(E)$ according to the property (x), and hence

$$R(\varepsilon) \sim \bar{R}(\varepsilon). \quad (6.25)$$

As the transform $\bar{R}(\varepsilon)$ is presumed to be only a slowly varying function of E , the violent energy dependence or "resonance structure" of R , which was manifest for real values of the energy is no longer evident. This limiting property of $R(\varepsilon)$ plays an important role in considerations of compound nucleus formation in the region of overlapping levels (Sec. XI.2).

7. Consideration of Bound States

Although R-matrix theory was developed for treatment of nuclear reactions and scattering, it is of interest to apply it to bound states.

In the case of simple two-body systems, bound states appear at those energies at which the internal logarithmic derivative

⁵⁶ C. E. Porter and R. G. Thomas, Phys. Rev. **104**, 483 (1956).

matches the external one for the wave function which vanishes at infinity. The latter logarithmic derivative is the quantity S_c^- of (III, 4.4a) and the former one is, from (2.4), equal to $(R-B)^{-1}$. Since, for uncharged particles, the quantity S_c^- is monotonic increasing, as is the quantity R , there will be either a bound or a resonant state associated with each of the levels E (although shifted in energy from E_λ).

The amplitude of the external wave function depends upon the magnitude of the invariant quantity γ^2 of (2.8) or its dimensionless equivalent θ^2 of (3.10). When the complete wave function is normalized to unity over the entire configuration space, internal plus external,

$$\int_0^a u^2 dr + \int_a^\infty u^2 dr = 1. \quad (7.1)$$

The external part of this integral may be transformed by the relation

$$\int_a^\infty u^2 dr = -au^2(a) \left(\frac{\partial S}{\partial (\rho^2)} \right)_a \quad (7.2)$$

which is derivable from a Green's theorem relation similar to (1.3) but taken in the limit as E_1 goes to E_2 and with the integration from the channel radius to infinity. The amplitude of the external wave function is expressed in terms of a normalization constant N by which the O -type wave function of (III, 2.18) is to be multiplied to give the normalization (7.1)

$$u(r) = \theta \left(\frac{2N}{a} \right)^{\frac{1}{2}} \left(\frac{O(r)}{O(a)} \right) \quad (r \geq a). \quad (7.3)$$

One finds that, from the definition of θ in (3.10) and (1.10a):

$$N^{-1} = 1 - 2\theta^2 \left(\frac{\partial S}{\partial (\rho^2)} \right)_a \quad (7.4)$$

The quantity $(\partial S / \partial \rho^2)_a$ is negative for bound channels; it is tabulated in the appendix for a few values of the angular momentum. The amplitude factor is directly proportional to θ for small values of θ , as expected since θ is just proportional to the amplitude of the wave function at the surface, when the internal function is normalized to unity.

8. Analytical Properties of the Collision Function in the Complex Plane: the Causality Condition

Considerable attention has recently been given to the restrictions that are imposed by the causality principle on the analytical form of the collision function $U(k)$ considered as a function of wave-number k . In considerations of this sort it is convenient to begin by establishing the analytic continuation of $U(k)$ from the positive real axis into the complex k plane. In particular we will now describe some relations that hold between the functions $U(k)$, $U(k^*)$, and $U(-k)$ as a result of general physical principles other than causality.

The collision function $U(k)$ is defined as minus the coefficient of the outgoing wave e^{ikr} at infinity corresponding to incoming wave e^{-ikr} . In other words, the asymptotic form of the wave function ψ_k is

$$r\psi_k(r) \sim e^{-ikr} - U(k)e^{ikr}. \quad (8.1)$$

Here ψ_k is the solution of the equation $H\psi_k = E\psi_k$ where the energy E is related to k by $E = (\hbar^2 k^2 / 2m)$. Assuming only the H is self-adjoint, it follows that, for two solutions ψ_1, ψ_2 corresponding to energies E_1 and E_2 ,

$$(E_1^* - E_2) \int_{r < a} \psi_1^* \psi_2 d\tau \sim \left[(r\psi_1)^* \frac{d}{dr} (r\psi_2) - (r\psi_2) \frac{d}{dr} (r\psi_1^*) \right]_a \quad (8.2)$$

where $r = a$ is any large sphere. On putting $k_1 = k_2$ the left-hand side vanishes and the insertion of (8.1) into the right-hand side gives

$$U(k)[U(k^*)]^* = 1. \quad (8.3)$$

This condition is the generalization for the complex plane of the

usual unitary condition on the real positive k axis:

$$U(k)U^*(k) \equiv |U(k)|^2 = 1.$$

The relation between $U(k)$ and $U(-k)$ follows from putting $-k$ for k in (8.1) and multiplying by $-U(k)$:

$$-U(k)r_{l,-k}(r) \sim [U(k)U(-k)]e^{-ikr} - U(k)e^{ikr}.$$

Comparing with (8.1) and assuming that, for a given outgoing wave, there is only one possible incoming wave it follows that

$$U(k)U(-k) = 1. \quad (8.4)$$

The specific form (1.14) for U from R function theory satisfies the relations (8.3) and (8.4). (The fact the $U(k)$ can be written as $\exp[2i\delta(k)]$, where $\delta(k)$ is a real odd function of k is sufficient for this purpose.) It is expected that (8.3) and (8.4) should hold for forms of U derived from specific theories since these relations are supposed to follow from general physical principles which must be embodied in any specific theory. When taken on the real axis, (8.3) is a consequence of the conservation condition expressed in the self-adjointness of H . According to Van Kampen,⁵⁷ the relation (8.4) on the real axis follows provided that the total energy has a lower bound.

We turn to the causality condition. Van Kampen⁵⁸ has formulated this condition quantitatively and has shown from it and the conservation condition that $U(k)$ is regular in the first quadrant of the k plane (except for possible poles on the positive imaginary axis) and can be assigned the analytic form:

$$U(k) = \prod_{\lambda} \left(\frac{1 - k^2/k_{\lambda}^2}{1 - k^2/k_{\lambda}^{*2}} \right) \exp \left(ibk^2 - i \int_0^{\infty} \frac{s^2 k^2 - 1}{s^2 + k^2} d\alpha(s) \right),$$

where b is a non-negative constant and $\alpha(s)$ is a function of bounded variation. On using the condition (8.4), it follows that $U(k)$ is regular in the second quadrant and, the form of $U(k)$ can be stated more specifically as

$$U(k) = e^{-2i\alpha k} \prod_{\lambda} \left(\frac{1 - k/k_{\lambda}}{1 - k/k_{\lambda}^*} \cdot \frac{1 + k/k_{\lambda}^*}{1 + k/k_{\lambda}} \right) \prod_{\mu} \left(\frac{1 - k/iP_{\mu}}{1 + k/iP_{\mu}} \right) \quad (8.5)$$

where α is a constant $\geq a$, the inner radius of the field-free region, and the iP_{μ} run over all zeros of U on the imaginary axis.

This form (8.5) for U is equivalent to a dispersion-type relation, namely:

$$\frac{U(k)e^{2i\alpha k} - 1}{k^2} = \frac{2}{\pi} \int_0^{\infty} \frac{\text{Im}[U(k')e^{2i\alpha k'}]}{k'(k'^2 - k^2)} dk' - \sum_n \frac{2b_n}{K_n^2(K_n^2 + k^2)} \quad (8.6)$$

where $h^2 K_n^2 / 2M = -E_n$ are singularities associated with bound states and the b_n are corresponding residues.

The original motivation for these investigations into the analytical form of $U(k)$ was provided by remarks made by Wigner and the paper of Schuster and Tiomno.²⁹ These latter authors suggested (although they were not able to give a rigorous proof) that the analytical form of $U(k)$ given by the R -function theory is a consequence of the causality condition. Although it is not evident from inspection of (8.5), it is possible to show⁵⁹ that this form for $U(k)$ is equivalent to the specific form (1.14) derived from the R -function theory for a bounded interaction. In other words, the form (8.5) for $U(k)$ implies that the function $R(E)$ defined by

$$R(E) = \frac{1}{ik\alpha} \frac{Ue^{2i\alpha k} - 1}{Ue^{2i\alpha k} + 1}, \quad (8.7)$$

where $E = (\hbar^2 k^2 / 2M)$ has the essential properties of our previously established R function, i.e., R is real and meromorphic and holomorphic in the upper half-energy plane.

Any angular momentum barrier or long-range potential is allowed in the derivation from causality only if it is assumed to be cut off and is treated as a part of the internal interactions. In other words, as might be expected, the derivation of $U(k)$ from

the causality condition alone does not distinguish between "internal" interactions and "external" barriers. Only the more specific derivation from the R -function theory can take account of physical information of this type.

9. Some Features of the Energy Dependence of the Phase Shifts δ_l

In subsection 1, the phase shifts δ_l wave defined in terms of the collision function U_l by the relation (1.15):

$$U_l = \exp(2i\delta_l).$$

This implies the relation (1.15a) between δ_l and the R function,

$$\delta_l = \tan^{-1} \left(\frac{R_l P_l}{1 - R_l S_l} \right) - \phi_l + \omega_l,$$

and the following relation between δ_l and the cross section for the l th partial wave:

$$\sigma_l = \frac{4\pi}{k^2} (2l+1) \sin^2 \delta_l = \frac{4\pi (2l+1)}{k^2 + (k \cot \delta_l)^2}. \quad (9.1)$$

a. Wigner's Limit⁶⁰ on the Energy Derivative

Wigner noticed that a consequence of the positive definite nature of

$$\frac{dR_l}{dE} = \sum_{\lambda} \frac{\gamma_{\lambda}^2}{(E_{\lambda} - E)^2}$$

is the existence of a lower limit on the values of $(d\delta_l/dE)$. On differentiating (1.15a) with respect to E , the coefficient of dR_l/dE is $P_l[(1 - R_l S_l)^2 + (R_l P_l)^2]^{-1}$, which is positive. Thus omission of the term in dR_l/dE from the differentiated equation results in a lower limit expression for $d\delta_l/dE$. As an example, consider the case of s -wave neutrons where $S=0$, $P=\rho$, $\omega=0$, $\phi=\rho$:

$$\frac{d\delta}{dE} = \frac{\rho}{1 + (\rho R)^2} \frac{dR}{dE} + \left(\frac{R}{1 + (\rho R)^2} - 1 \right) \frac{d\rho}{dE} > \left(\frac{\sin 2(\delta + \rho)}{2\rho} - 1 \right) \frac{d\rho}{dE}. \quad (9.2)$$

Given the value of δ at any energy, this equation gives a lower limit on the value of $d\delta/dE$ at that energy. For low energies where $\rho \ll 1$, $d\delta/dE$ must be positive. At very low energies, $(d\rho/dE) \gg (dR/dE)$ and so the inequality tends to an equality. In this limit (9.2) only expresses the known fact that, at very low energies, δ is proportional to ρ .

b. Effective Range Expansion

The work of Blatt and Jackson⁶¹ and Bethe⁶² has shown that, in certain situations, it is useful to consider the quantity $f(k) \cot \delta_l$ as a function expanded in powers of k^2 . If $f(k)$ is chosen appropriately, the power series can converge rapidly so that the energy dependence of the cross section (9.1) is expressed in terms of only two or three parameters instead of the infinite number implied by using the R function relation (1.15a). These parameters correspond to the coefficients $R^{(0)} \equiv R(E=0)$, $R^{(1)} \equiv (dR/dE)_{E=0}$, ... in the Taylor series expansion of $R(E)$ about zero bombarding energy $E=0$:

$$R(E) = R^{(0)} + ER^{(1)} + \frac{E^2}{2!} R^{(2)} + \dots \quad (9.3)$$

This correspondence shows that the useful application of the effective range expansion is limited to energy regions in which R is not strongly varying. For instance, it is not possible to treat energy regions containing whole resonances [i.e., infinities of $R(E)$]. The condition that R varies slowly is, of course, well satisfied in cases to which the effective range theory is normally applied, such as nucleon-nucleon scattering. In cases where there

⁵⁷ N. G. Van Kampen, *Physica* 20, 115 (1954).

⁵⁸ N. G. Van Kampen, *Phys. Rev.* 91, 1267 (1953).

⁵⁹ N. G. Van Kampen, *Rev. mex. fis.* 2, 233 (1953).

⁶⁰ E. P. Wigner, *Phys. Rev.* 98, 145 (1955).

⁶¹ J. M. Blatt and J. D. Jackson, *Phys. Rev.* 86, 18 (1949).

⁶² H. A. Bethe, *Phys. Rev.* 76, 38 (1949).

are enough nucleons to form many compound states, the condition is not satisfied. In fact, there are only a few instances in light nuclei where the effective range theory is of any possible use. In other words, there is very little overlap between the situations of practical applicability of the R -function expansion and the effective range expansion. The relationship between the two theories has been examined by Teichmann⁶³ whose analysis begins with a rearranged form of (1.15a),

$$\cot(\delta_l - \omega_l) = -\left(\frac{G_l - \rho G_l' R_l}{F_l - \rho F_l' R_l}\right), \quad (9.4)$$

where F and G are the functions defined in Sec. III (evaluated at $r=a$), and the prime represents $d/d\rho$.

For s -wave neutrons, this can be immediately reduced to the form of the standard effective range expansion,

$$k \cot \delta = -\alpha + \frac{1}{2} r_0 k^2 - p r_0^3 k^4 \dots \quad (9.5)$$

with the following expressions for the coefficients:

$$\begin{aligned} \text{"scattering length"} &= 1/\alpha = a - aR^{(0)} \\ \text{"effective range"} &= r_0 = 2 \left[a - a^2 \alpha + \frac{\alpha^2 a^3}{3} - \frac{\hbar^2 \alpha a^2}{2M} R^{(1)} \right]. \end{aligned} \quad (9.6)$$

Since $k \cot \delta$ cannot depend on the choice of a , neither can the coefficients α , r_0 , etc. Therefore Eqs. (9.6) are really equations for the dependence of $R^{(0)}$, $R^{(1)}$, \dots on the choice of a .

In terms of the logarithmic derivative $f=R^{-1}$, Eqs. (9.6) become

$$\begin{aligned} [f]_{E=0}^{-1} &= R^{(0)} = 1 - 1/\beta, \\ \gamma^2(E=0) &= -\left(\frac{df}{dE}\right)_{E=0}^{-1} = -\frac{R^{(1)}}{[R^{(0)}]^2} = \frac{\hbar^2}{2M a^2} \left(\frac{(1-\beta)^2}{1-\beta + \frac{1}{3}\beta^2 - r_0/2a} \right), \end{aligned} \quad (9.7)$$

where $\beta = a\alpha$.

Some physical interpretation of the coefficients $1/\alpha$ and r_0 can be obtained by considering the case of a square potential well of radius a and depth V in the following situations:

$$(i) \text{ resonance at } E=0; R^{(0)} = \infty, \frac{\hbar^2}{2M a^2} \frac{R^{(1)}}{[R^{(0)}]^2} = 1:$$

$$1/\alpha = \infty, \quad r_0 = a.$$

$$(ii) \text{ "antiresonance" at } E=0; R^{(0)} = 0$$

$$1/\alpha = a, \quad r_0 = \frac{2a}{3} \left[1 - \frac{3}{8V} \left(\frac{\hbar^2}{M a^2} \right) \right].$$

These relations illustrate the facts that the value of α depends sensitively on the position of the nearest resonance to $E=0$, whereas the value of r_0 depends much less on this and is always near to the range of the well.

For s -wave protons, the effective range expansion can be deduced from (9.4), (9.3) and expansions of F and G to be:

$$\frac{2\pi\eta(k \cot \eta)}{e^{2\pi\eta} - 1} + \eta k g(\eta) = -\alpha_c + \frac{1}{2} r_{0c} k^2 - p r_{0c}^3 k^4, \quad (9.8)$$

where

$$g(\eta) = -2 \ln \eta + 2\eta^2 \sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu^2 + \eta^2)}. \quad (9.9)$$

The subscript c indicates that these quantities depend on the presence of the Coulomb field. This is to be expected since the addition of one potential to another potential changes the phase shift in a nonsimple fashion in general. If the added potential is smooth, we expect the value of the effective range r_0 should not be much changed, even though the scattering length may be appreciably altered. The extent to which the latter alters depends on the magnitude of the added potential. In the case of the addition of an external Coulomb potential, Teichmann⁶³ determines an approximation for the difference $\alpha_c - \alpha$ to be

$$\alpha_c - \alpha \approx 2\eta k [\ln 2\eta a + \gamma], \quad (9.10)$$

where γ is Euler's constant and a is put equal to the radius of the

short range (nuclear) interaction, i.e., essentially r_0 . Jackson and Blatt⁶⁴ show that there should be an additional term -0.824 in the square brackets when the Coulomb field also acts in the internal region.

The importance of the difference is determined by the magnitude of $2\eta k$ relative to α . For proton-proton bombardment, $2\eta k$ is only a few percent of 10^{13} cm^{-1} and so, except for anomalously small α , the effect of the Coulomb field on α is small. For a general nuclear bombardment, however, $2\eta k$ is much larger and there is no simple relation between α and α_c which may be quite different.

V. THE R MATRIX IN THE GENERAL (MANY-CHANNEL) CASE

The considerations in Sec. IV can be immediately generalized to take account of the occurrence of spins and reaction channels. We now establish the generalized version of the fundamental R -matrix relation. This can be derived in a number of ways. As in Sec. IV, we emphasize the derivation using a Green's theorem relation.

1. Use of Green's Theorem

The general derivation, which considers the possibility of reactions as well as scattering, is based on an application of the many-channel analog to the one-channel Green's-theorem relation given by (IV, 1.3). The derivation of the analog relation is similar to that of (IV, 1.3), and, as one might expect, in the many-channel case the "surface" term of (IV, 1.3) is replaced by the sum of such terms for each of the participating channels.

As in (IV, 1.1), the wave equation for a system of A nucleons is written for general solutions at two energies,

$$H\Psi_1 = E_1\Psi_1, \quad H\Psi_2 = E_2\Psi_2 \quad (1.1)$$

where H is the Hamiltonian operator. The first of these is multiplied by Ψ_2^* and the complex-conjugate of the second by Ψ_1 ; the difference between the resulting equations is integrated over the internal region τ :

$$\begin{aligned} (E_2 - E_1) \int_{\tau} \Psi_2^* \Psi_1 d\tau \\ = \int_{\tau} [(H\Psi_2)^* \Psi_1 - \Psi_2^* H\Psi_1] d\tau. \end{aligned} \quad (1.2)$$

The interaction term V in H is assumed to be self-adjoint so that

$$\int_{\tau} [(V\Psi_2)^* \Psi_1 - \Psi_2^* V\Psi_1] d\tau = 0, \quad (1.3)$$

and the kinetic energy terms which remain are integrated by Green's theorem (in $3A$ -dimensional space)

⁶⁴ J. D. Jackson and J. M. Blatt, *Revs. Modern Phys.* **22**, 77 (1950).

⁶³ T. Teichmann, *Phys. Rev.* **83**, 141 (1951).

to give

$$(E_2 - E_1) \int_{\tau} \Psi_2^* \Psi_1 d\tau = \int_S (\hbar^2/2M_c) (\Psi_2^* \text{grad}_n \Psi_1 - \Psi_1 \text{grad}_n \Psi_2^*) dS = \sum_c (V_{2c}^* D_{1c} - V_{1c} D_{2c}^*). \quad (1.4)$$

The last equality follows by substitution of the surface representations (III, 4.2) for Ψ and $\text{grad}_n \Psi$. One noteworthy feature in (1.4) is the appearance of the reduced channel mass M_c , a consequence of the form of the kinetic energy operator at the channel surfaces (Sec. III, 2). The surface contribution to (1.4) takes account only of the possibility of the breakup of Ψ into pairs of fragments, breakup into triplets, quadruplets, etc., having been neglected. Therefore, the formula cannot be applied to an $(n,2n)$ reaction, for example, except insofar as the emission of neutrons may be treated as a succession of two processes (Sec. XIII, 2).

A useful special case of (1.4) occurs when $\Psi_1 = \Psi_2 = \Psi$ and $E_1 = E_2 = E$; for this case, by taking the real and imaginary parts of (1.4), we have:

$$\int_{\tau} |\Psi|^2 d\tau = \sum_c \left\{ -2 \text{Im}(f_c) \text{Im} \left(V_c \frac{dV_c^*}{dE} \right) - \frac{d}{dE} [\text{Re}(f_c)] |V_c|^2 \right\} \quad (1.5)$$

and

$$O = \sum_c |V_c|^2 \text{Im}(f_c), \quad (1.6)$$

where $\text{Re}(f_c)$ and $\text{Im}(f_c)$ are defined as the real and imaginary parts of the logarithmic derivatives

$$f_c = D_c/V_c. \quad (1.7)$$

A similar type of relation also holds for the overlap integral in the volume between two surfaces S_1 and S_2 in a given channel c . Applying (1.4) to such a volume,

$$(E_2 - E_1) \int_{S_1}^{S_2} \Psi_2^* \Psi_1 d\tau = \left(V_{2c}^* D_{1c} - V_{1c} D_{2c}^* \right)_{S_1}^{S_2}. \quad (1.8)$$

Corresponding to (1.5), i.e., putting $\Psi_1 = \Psi_2 = \Psi$ and $E_1 = E_2$:

$$\int_{S_1}^{S_2} |\Psi|^2 d\tau = \left[-2 \text{Im}(f_c) \text{Im} \left(V_c \frac{dV_c^*}{dE} \right) - \frac{d}{dE} [\text{Re}(f_c)] |V_c|^2 \right]_{S_1}^{S_2}. \quad (1.9)$$

2. The Fundamental R-Matrix Relation

The fundamental relation of the R-matrix theory is the many-channel generalization of (IV, 1.10). Such a

relation is used to determine the collision matrix \mathbf{U} just as (IV, 1.10) was used to determine the collision function U in the one-channel case. This relation may be derived by means of the Green's-theorem relation (1.4).

By analogy with (IV, 2.1) for the one-channel case, we now specify the general boundary conditions to be satisfied by the complete set of states $X_{\lambda JM}$ of (III, 3.3) on the surfaces S_c . These conditions are taken to have the form:

$$\frac{D_{\lambda c}}{V_{\lambda c}} \left(\equiv \frac{\delta_{\lambda c}}{\gamma_{\lambda c}} \right) = B_c, \quad (2.1)$$

where the B_c are independent of the λ . (The labels JM will be assumed understood and dropped.) By applying (1.4) to any two proper solutions X_{λ} , $X_{\lambda'}$ belonging to energy values E_{λ} , $E_{\lambda'}$, it is evident that the set of proper solutions are mutually orthogonal with such boundary conditions; they are normalized

$$\int_{\tau} X_{\lambda'}^* X_{\lambda} d\tau = \delta_{\lambda\lambda'}. \quad (2.2)$$

A wave function Ψ may be expanded in τ in terms of them,

$$\Psi = \sum_{\lambda} A_{\lambda} X_{\lambda}, \quad (2.3)$$

the energy-dependent coefficients A_{λ} being given by

$$A_{\lambda} = \int_{\tau} X_{\lambda}^* \Psi d\tau. \quad (2.4)$$

As before these coefficients may be determined by applying (1.4) to the solution Ψ of energy E and the X_{λ} of energy E_{λ} ; the expansions (III.4.3) are used for Ψ and $\text{grad}_n \Psi$ on S . One finds immediately that

$$A_{\lambda} = (E_{\lambda} - E)^{-1} \sum_c D_c^0 \gamma_{\lambda c}, \quad (2.5)$$

$$D_c^0 = D_c - B_c V_c$$

and the $\gamma_{\lambda c}$ are put for $V_{\lambda c}$ using definition (III, 4.8a). Expansion (2.3) now becomes:

$$\Psi = \sum_c \left[\sum_{\lambda} \frac{X_{\lambda} \gamma_{\lambda c}}{E_{\lambda} - E} \right] D_c^0. \quad (2.6)$$

This equation relates the value of Ψ at any point in τ to its "derivatives" D_c^0 on S ; it is the analog to the relation (IV, 1.8), the quantity in the square bracket of (2.6) being considered as the Green's function. By operating on (2.6) with $\int_S \varphi_{c'}^* dS$ and using (III, 4.8a), one obtains the fundamental R-matrix relation

$$V_{c'} = \sum_c R_{c'c} D_c^0, \quad (2.7a)$$

where

$$R_{c'c} = \sum_{\lambda} \gamma_{\lambda c'} \gamma_{\lambda c} / (E_{\lambda} - E). \quad (2.8a)$$

Equations (2.7) and (2.8) are more conveniently treated in matrix notation:

$$\mathbf{V} = \mathbf{R}\mathbf{D}^0, \quad (2.7b)$$

$$\mathbf{R} = \sum_{\lambda} (\boldsymbol{\gamma}_{\lambda} \times \boldsymbol{\gamma}_{\lambda}) / (E_{\lambda} - E). \quad (2.8b)$$

\mathbf{V} and \mathbf{D}^0 are considered as column vectors, the components of which are the V_c and D_c^0 . The matrix $(\boldsymbol{\gamma}_{\lambda} \times \boldsymbol{\gamma}_{\lambda})$ is the matrix square of the vector $\boldsymbol{\gamma}_{\lambda}$, the components of which are the $\gamma_{\lambda c}$. In addition to being symmetrical, the \mathbf{R} matrix is also real because the $\gamma_{\lambda c}$ are real.

It is convenient at this stage to consider the so-called "square integral" of the internal region which is a measure of the probability of all particles being together in a "compound system" (not necessarily a compound nucleus) of nuclear dimensions. This is calculated from (2.6) to be

$$\begin{aligned} \int_{\tau} |\Psi|^2 d\tau &= \sum_{\lambda} \left| \sum_c (E_{\lambda} - E)^{-1} D_c^0 \gamma_{\lambda c} \right|^2 \\ &= \sum_{c,c'} D_c^0 \frac{d}{dE} (R_{cc'}) D_{c'}^{0*}. \end{aligned} \quad (2.9a)$$

This integral may be expressed in matrix notation as the scalar product

$$\int_{\tau} |\Psi|^2 d\tau = \left(\mathbf{D}^{0*}, \frac{d}{dE} (\mathbf{R}) \mathbf{D}^0 \right). \quad (2.9b)$$

The notation (\mathbf{x}, \mathbf{y}) for the scalar product of two column vectors \mathbf{x} and \mathbf{y} stands for the transpose of \mathbf{x} (from a column to a row) times \mathbf{y} . (No complex conjugate of \mathbf{x} is taken.) The overlap integral for wave functions $\Psi_1(E_1)$ and $\Psi_2(E_2)$ with surface derivative quantities D_{1c}^0 and D_{2c}^0 may be expressed similarly:

$$\begin{aligned} \int_{\tau} \Psi_1^* \Psi_2 d\tau &= \sum_{\lambda} \left[\sum_c (E_{\lambda} - E_1)^{-1} \gamma_{\lambda c} D_{1c}^{0*} \right] \left[\sum_c (E_{\lambda} - E_2)^{-1} \gamma_{\lambda c} D_{2c}^0 \right] \\ &= (\mathbf{D}_1^{0*}, \mathbf{R}_{12}(E_1, E_2) \mathbf{D}_2^0) \end{aligned} \quad (2.10)$$

where:

$$\mathbf{R}_{12}(E_1, E_2) = \frac{\mathbf{R}(E_1) - \mathbf{R}(E_2)}{E_1 - E_2} = \sum_{\lambda} \frac{\boldsymbol{\gamma}_{\lambda} \times \boldsymbol{\gamma}_{\lambda}}{(E_{\lambda} - E_1)(E_{\lambda} - E_2)}.$$

With the aid of (2.7) we see that the expressions for the two overlaps in (2.9) and (2.10) are equivalent to those given previously, namely (1.5) and (1.4).

3. Alternative Derivations of the \mathbf{R} Matrix

Three alternative derivations of the \mathbf{R} -matrix relation (2.7) have been given in the literature. The first two are fairly closely related to the preceding derivation. The other one is different, being based on the causality condition.

a. Wigner's Original Derivation

In Wigner's first paper¹¹ on \mathbf{R} -matrix theory, the wave function was not assumed to be continuous across \mathcal{S} , as in the deduction of (2.7) from (2.6). Instead, the Green's theorem relation (1.4) was applied again to two solutions of the form (2.6) belonging to different energies. The result of this procedure is that the \mathbf{R} matrix of (2.7) is given by

$$\mathbf{R} = \mathbf{R}^{\infty} + \sum_{\lambda} \frac{\boldsymbol{\gamma}_{\lambda} \times \boldsymbol{\gamma}_{\lambda}}{E_{\lambda} - E}, \quad (3.1)$$

where \mathbf{R}^{∞} is an arbitrary, real-symmetric matrix which is independent of energy. The continuity requirement is thus equivalent to the vanishing of \mathbf{R}^{∞} . This derivation of (3.1) is as follows.

If the expansions (2.6) for any two solutions Ψ_1, Ψ_2 of energies E_1, E_2 are substituted into the Green's-theorem relation (1.4) one obtains

$$\begin{aligned} \sum_c (V_{2c}^* D_{1c}^0 - V_{1c} D_{2c}^{0*}) \\ = (E_2 - E_1) \sum_{c', c''} \left(\sum_{\lambda} \frac{\gamma_{\lambda c'} \gamma_{\lambda c''}}{(E_{\lambda} - E_1)(E_{\lambda} - E_2)} \right) D_{1c'}^0 D_{2c''}^{0*}. \end{aligned} \quad (3.2)$$

Assuming solutions of the form (2.7) for the V_{2c}^* and V_{1c} of the left side of (3.2), one has upon rearranging

$$\begin{aligned} \sum_{c,c'} \left[(R_{2cc'}^* - R_{1cc'}) \right. \\ \left. - \left(\sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{E_{\lambda} - E_2} - \sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{E_{\lambda} - E_1} \right) \right] D_{1c}^0 D_{2c'}^{0*} = 0. \end{aligned} \quad (3.3)$$

By setting, first a particular $D_{1c}^0 = 1$ and a particular $D_{2c}^{0*} = 1$, all others being zero and, secondly, $D_{1c}^0 = 1, D_{2c}^{0*} = 1$, all others being zero, one observes that the solution for \mathbf{R} is as given by (3.1) with the properties specified there for \mathbf{R}^{∞} . A symmetric constant \mathbf{R}^{∞} may formally be replaced by a sum $\sum_{\lambda} (\boldsymbol{\gamma}_{\lambda} \times \boldsymbol{\gamma}_{\lambda}) / (E_{\lambda} - E)$ in which the E_{λ} and $\boldsymbol{\gamma}_{\lambda}$ are infinite in such a way as to give a finite contribution.

b. Derivation from a Variation Principle

This type of derivation is discussed for the one-channel case. Subsequent generalization to the many channel case is quite straightforward.⁶⁵

The usual variational principle for the energy of a bound state is not suitable here. A related principle is needed which for a given energy leads to a stationary value of the logarithmic derivative $f_E(a)$ of the wave function at some "external" point $r = a$. Kohn⁶⁵ has written down such a principle, *viz.*,

$$f_E(a) = \frac{2Ma \int_{\delta}^a u_E (H - E) u_E dr}{\hbar^2 \frac{du_E}{dr}(a)} - \frac{a}{u_E} \frac{du_E}{dr}. \quad (3.4)$$

For small variation in the function $u_E(r)$ about the exact wave function, the right-hand side is stationary so $\delta f_E = 0$. In the spirit of the derivation of the R -function relation in IV, 1 and V, 2, we consider a trial wave function of the form:

$$u_E = \sum_{\lambda} A_{\lambda} u_{\lambda}, \quad (3.5)$$

where the u_{λ} are members of an orthonormal complete set of states that are exact solutions of

$$H u_{\lambda} = E_{\lambda} u_{\lambda} \quad (3.6)$$

and satisfy the boundary condition $u_{\lambda}'(a) = 0$. For the present, it is assumed that there are only a finite number of terms in the trial function (3.5).

On putting (3.5) in (3.4) and using (3.6),

$$f_E(a) = \sum_{\lambda} A_{\lambda}^2 (E_{\lambda} - E) / (\sum A_{\lambda} \gamma_{\lambda})^2, \quad (3.7)$$

where the γ_{λ} are the reduced width amplitudes of (IV, 1.10a).

⁶⁵ W. Kohn, Phys. Rev. **74**, 1763 (1948); J. L. Jackson, Phys. Rev. **83**, 301 (1951).

From the stationary property of f

$$\partial f_E / \partial A_\lambda = 0, \quad (3.8)$$

it follows that

$$A_\lambda (E_\lambda - E) / \gamma_\lambda = \sum_\mu A_\mu^2 (E_\mu - E) / \sum_\mu A_\mu \gamma_\mu = C \text{ (say)} \quad (3.9)$$

where C is independent of λ . Insertion in (3.7) immediately gives

$$f_E(a) = \left(\sum_\lambda \frac{\gamma_\lambda^2}{E_\lambda - E} \right)^{-1}. \quad (3.10)$$

On allowing the number of elements in the trial function expansion (3.5) to go to infinity, this expression for f_E becomes equivalent to the R function relation (IV, 1.10).

c. Derivation from Causality Condition

In Sec. IV, 8, we described how, in the case of one channel with the assumption of no channel barrier beyond some finite point $r=a$, the analytic form of the R function is a consequence of the causality condition. This condition implies a certain analytic form for the collision function $U(k)$ which implies a definite analytic form for $R(E)$, that deduced from the R -function theory. At present only a preliminary investigation has been made of the problem of extending this result to the case of many channels.⁶⁶

VI. COLLISION MATRIX \mathbf{U}

A precise definition of the \mathbf{U} matrix will now be given and some of its general properties listed. The unitarity and symmetry properties were not brought out in Sec. IV because, in the case of one channel (elastic scattering), they are trivial.

1. Definition

As mentioned in Sec. III, there are two possible channel designation schemes $\{\alpha slvm\}$ and $\{\alpha slJM\}$. Accordingly \mathbf{U} can be defined in either scheme. The form of the definition is quite unchanged by the scheme we choose.

Let us consider a completely general solution to the wave equation in the external region. Following the definitions of incoming and outgoing waves in Sec. III, 2, this may be written

$$\Psi(\text{general}) = \sum_c (x_c \Theta_c + y_c \mathcal{G}_c). \quad (1.1)$$

The numbers y_c are the amplitudes of the incoming waves \mathcal{G}_c in the various channels c , while the x_c are the amplitudes of the outgoing waves Θ_c . For a given total system, when the y_c are given, the numbers x_c are determined by the nature of the system. The role of the collision matrix is simply to give an expression for the x_c in terms of the y_c as follows:

$$x_{c'} = - \sum_c U_{c'c} y_c \quad (1.2a)$$

or, in matrix notation:

$$\mathbf{x} = - \mathbf{U} \mathbf{y}. \quad (1.2b)$$

A wave \mathcal{G}_c or Θ_c in one scheme is simply transformed to the other scheme by rearranging the vector coupling, so it is easy to transform \mathbf{U} in one scheme to \mathbf{U} in the other scheme (subsection 3).

⁶⁶ N. G. Van Kampen (private communication).

An often-used special case of the general solution (1.1) corresponds to incident waves in one channel only. Suppose this channel is c and put $y_c=1$ for this channel and zero for all others, then

$$\Psi(\text{incident in } c) = \mathcal{G}_c - \sum_{c'} U_{c'c} \Theta_{c'}. \quad (1.3)$$

2. Some General Properties of \mathbf{U}

As originally shown by Breit,²⁸ the collision matrix has certain general properties which do not depend on specific reaction theories (such as \mathbf{R} -matrix theory) but only on broad physical principles. We now establish the unitarity and symmetry of \mathbf{U} .

a. Unitary Property (from Conservation)

Consider two particular solutions like (1.3) for the wave equation which asymptotically are of the form

$$\begin{aligned} \Psi_{1p} &= \sum_c (\delta_{pc} \mathcal{G}_{1c} - U_{1cp} \Theta_{1c}), \\ \Psi_{2m} &= \sum_c (\delta_{mc} \mathcal{G}_{2c} - U_{2cm} \Theta_{2c}). \end{aligned} \quad (2.1)$$

Ψ_{1p} is a solution at the energy E_1 with an incoming wave only in channel p , and Ψ_{2m} is a solution at E_2 with an incoming wave only in channel m . By substituting these into the Green's-theorem relation (V, 1.4) with the surface \mathcal{S} taken to be very large, one obtains

$$\begin{aligned} (E_1 - E_2) \int_{\mathcal{S}} \Psi_{2m}^* \Psi_{1p} d\tau &= \frac{1}{2} \sum_{c^+} [(\delta_{mc} O_{2c'} - U_{2cm}^* I_{2c'}) (\delta_{pc} I_{1c} - U_{1cp} O_{1c}) \\ &\quad - (\delta_{pc} I_{1c'} - U_{1cp} O_{1c'}) (\delta_{mc} O_{2c} - U_{2cm}^* I_{2c})] \\ &= \frac{1}{2} \hbar \{ \delta_{pm} (O_{2p}' I_{1p} - I_{1p}' O_{2p}) \\ &\quad + \sum_{c^+} U_{2cm}^* U_{1cp} (I_{2c}' O_{1c} - O_{1c}' I_{2c}) \\ &\quad - U_{2pm}^* I_{2p}' I_{1p} - U_{1mp} O_{2m}' O_{1m} \\ &\quad + U_{1mp} O_{1m}' O_{2m} + U_{2pm}^* I_{1p}' I_{2p} \} \end{aligned} \quad (2.2)$$

(provided that the surface \mathcal{S} is put far enough out so that the contributions from the negative-energy channels c^- can be ignored). If $E_1 = E_2$, the left side of (2.2) vanishes so that, with the Wronskian (III, 4.7c) we have

$$\sum_{c^+} U_{cm}^* U_{cp} = \delta_{cp} \quad (2.3a)$$

or, in matrix notation,

$$(\mathbf{U}^{++})^\dagger (\mathbf{U}^{++}) = \mathbf{1}, \quad (2.3b)$$

where \dagger means "complex conjugate transpose" and the superscript $++$ signifies that only elements for positive energy channels are concerned. Equation (2.3b) is equivalent to the statement that \mathbf{U}^{++} is unitary. One

can say that this property is a consequence of conservation of probability flux since it is this conservation that is expressed in the Green's theorem relation (V, 1.4) that is the basis of the present proof.

Unlike the symmetry property discussed in the following, the unitary property is established without special reference to either of the two-channel designation schemes.

b. The Symmetry Property (from Time Reversal)

We consider again solutions which are asymptotically of the form (2.1) in the $\{\alpha s l J M\}$ channel designation scheme:

$$\Psi_m = \sum_{c^+} (\delta_{mc} g_c - U_{cm} \theta_c) - \sum_{c^+} U_{cm} \theta_c, \quad (2.4)$$

the negative-energy channels being considered separately. Since the time-reversal operator K of Sec. III, 2 commutes with the Hamiltonian, the following is also a solution

$$\begin{aligned} & (-1)^{J-M} K \Psi_{-m} \\ &= \sum_{c^+} (\delta_{-mc} \theta_{-c} - U_{c-m}^* g_{-c}) - \sum_{c^+} U_{c-m}^* \theta_{-c} \\ &= \sum_{c^+} (\delta_{mc} \theta_c - U_{cm}^* g_c) - \sum_{c^+} U_{cm}^* \theta_c, \end{aligned} \quad (2.5)$$

because of the time-reversal properties of the g_c and θ_c and the fact that \mathbf{U} is independent of M ; a negative sign preceding the channel subscript indicates that the negative of the total angular momentum component M is to be used. If Ψ_r is a solution of the form (2.4) with an incident wave in only channel r , then a linear combination of such solutions is also a solution; in particular, the following combination is a solution:

$$\begin{aligned} & - \sum_{r^+} U_{rm}^* \Psi_r \\ &= - \sum_{c^+ r^+} (\delta_{rc} U_{rm}^* g_c - U_{rm}^* U_{cr} \theta_c) \\ & \quad + \sum_{c^+ r^+} U_{rm}^* U_{cr} \theta_c, \quad (2.6) \\ &= \sum_{c^+} \left[\sum_{r^+} (U_{rm}^* U_{cr} \theta_c) - U_{cm}^* g_c \right] \\ & \quad + \sum_{c^+ r^+} U_{rm}^* U_{cr} \theta_c. \end{aligned}$$

The particular solutions (2.5) and (2.6) have identical incoming waves, and hence the coefficients of the outgoing waves must be the same in each channel:

$$\sum_{r^+} U_{rm}^* U_{c^+ r} = \delta_{mc^+}, \quad (2.7a)$$

$$\sum_{r^+} U_{rm}^* U_{c^- r} = -U_{c^- m}^*. \quad (2.8a)$$

In matrix form these are expressed as

$$(\mathbf{U}^{++})(\mathbf{U}^{++})^* = \mathbf{1}, \quad (2.7b)$$

$$(\mathbf{U}^{-+})(\mathbf{U}^{++})^* + (\mathbf{U}^{-+})^* = 0. \quad (2.8b)$$

Comparing (2.7b) with (2.3b), it is evident that the

positive-energy submatrix \mathbf{U}^{++} is symmetric:

$$\mathbf{U}^{++} = \text{transpose}(\mathbf{U}^{++}). \quad (2.9)$$

The time-reversed solution of a solution of type Ψ_m can be expressed as a linear combination of solutions having incident waves in various channels,

$$K \Psi_m = (-1)^{J-M-1} \sum_{r^+} U_{r-m}^* \Psi_r. \quad (2.10)$$

The time-reversed solutions are therefore not independent of the original set of solutions.

These considerations apply to the $\{\alpha s l J M\}$ channel designation scheme. It is straightforward to show that in the $\{\alpha s l m\}$ channel designation scheme

$$U_{rp}^{++} = (-1)^{r+p} U_{-p-r}^{++} = U_{pr}^{++}, \quad (2.11)$$

where $(-1)^r$ means $(-1)^{s-r+l-m}$ and $-r$ designates channel r with all components reversed in direction. These equalities are applicable to the $\{\alpha s l J M\}$ scheme as well because in that scheme $(-1)^{r+p} = 1$, which corresponds to the fact that \mathbf{U} is independent of M .

c. Analytical Properties of $\mathbf{U}(k)$

It is of formal interest to consider the collision matrix as an analytical function of its channel wave number variables k_c . The present discussion is a close parallel of Sec. IV, 8 which dealt with the special case of the collision function for elastic scattering. The only special feature of the general (many-channel) case to which attention need be drawn is the presence of negative energy or bound channels. These can be included in the discussion just like positive energy channels provided that the usual relation $E_c = \hbar^2 k_c^2 / 2M$ is assumed to hold (i.e., the wave numbers k_c are pure imaginary for the bound channels), and provided that ingoing and outgoing waves are chosen to have the same Wronskian (III, 4.6c and 7c) as for positive energy channels.

As in Sec. IV, 8, we first establish the generalizations for the complex plane of the usual unitarity and symmetry properties of \mathbf{U} that apply on the positive real k axes. From the asymptotic behaviors of the outgoing and incoming wave functions of (III, 2.10), it is evident that, if the Coulomb field is neglected or cut off, the following symmetry relations exist for their radial parts:

$$\begin{aligned} O(k^*)^* &= I(k), & I(k^*)^* &= O(k). \\ O(-k) &= I(k), & I(-k) &= O(k). \end{aligned} \quad (2.12)$$

For the generalization of the unitarity property, we assume the self-adjointness of the Hamiltonian as expressed in the Green's-theorem relation (1.2). On putting

$$\Psi_{1p} = \Psi_{1p}(k), \quad \Psi_{2m} = \Psi_{2m}(k^*),$$

the difference of the energies on the left-hand side vanishes and the right-hand side yields the matrix

relation:

$$\mathbf{U}(k)[\mathbf{U}(k^*)]^\dagger = \mathbf{1}, \quad (2.13)$$

which is to be compared with the unitary relation (2.3) applying on the real energy axis. For the generalization of the symmetry property (2.9), we follow the same procedure of (2.5) and (2.6). The relation,

$$\mathbf{U}(k)[\mathbf{U}(k^*)]^* = \mathbf{1}, \quad (2.14)$$

follows if the quantities in (2.4) and (2.5) are considered as functions of k^* rather than k . Equations (2.13) and (2.14) together give the symmetry property

$$\mathbf{U}(k) = \text{transpose } \mathbf{U}(k). \quad (2.15)$$

A second symmetry property which involves $\mathbf{U}(-k)$ (and therefore has no analog on the positive real axis) is obtained by consideration of a wave function of the form

$$\begin{aligned} \Psi_p(-k) &= g_p(-k) - \sum_c U_{cp}(-k) \Theta_c(-k) \\ &= (-i) \Theta_p(k) - \sum_c U_{cp}(-k) (-i) g_c(k). \end{aligned} \quad (2.16)$$

The factor $(-i)$ of the second equality arises from the normalization factor in the running waves. A linear combination of solutions such as (2.16) is also a solution; in particular the following combination is a solution:

$$\begin{aligned} \Psi_r(k) &= -i \sum_p U_{pr}(k) \Psi_p(-k) \\ &= \sum_{cp} U_{pr}(k) U_{cp}(-k) g_c(k) - \sum_p U_{pr}(k) \Theta_p(k). \end{aligned} \quad (2.17)$$

The outgoing waves of (2.17) are the same as those of the standard expansion

$$\Psi_r(k) = g_r(k) - \sum_p U_{pr}(k) \Theta_p(k), \quad (2.18)$$

so that the incoming waves must be the same, indicating that

$$\mathbf{U}(-k) \mathbf{U}(k) = \mathbf{1}. \quad (2.19)$$

By comparing (2.19) with (2.14) one obtains

$$\mathbf{U}(-k) = [\mathbf{U}(k^*)]^*. \quad (2.20)$$

By considering the \mathbf{R} matrix expression for the collision matrix \mathbf{U} in the form (1.6) of the next section with the external functions satisfying the symmetry properties (2.12) and the associated Wronskian, it is evident that this special form of \mathbf{U} satisfies the above symmetry properties (2.13), (2.15), and (2.20) in the complex planes.

To make a complete parallel with the special case of elastic scattering in Sec. IV, 8, one should describe the analytic properties of \mathbf{U} that are a consequence of causality. However, as far as we are aware, the only attention that has been given to this matter consists in certain unpublished observations by Van Kampen.⁶⁶

3. Connection between the Collision Matrices of the $\{\alpha s I J M\}$ and $\{\alpha s l v m\}$ Channel Designation Schemes

The collision matrix that emerges from the theory of Secs. V and VII refers to the $\{\alpha s I J M\}$ channel designation scheme. For the determination of the differential cross sections in Sec. VIII, we need the collision matrix referring to the $\{\alpha s l v m\}$ scheme. To obtain the connection between the two, we consider the solution having an incoming wave in only the $\{\alpha s I J M\}$ channel. In the external region, this has the form:

$$\Psi_{JM} = g_{\alpha s I J M} - \sum_{\alpha' s' l' v'} U_{\alpha' s' l' v', \alpha s I J} \Theta_{\alpha' s' l' v' J M} \quad (3.1a)$$

which, by substituting the expansions (III.2.20), may also be written as

$$\begin{aligned} \Psi_{JM} &= \sum_{v'' m''} (s l v'' m'' | J M) g_{\alpha s l v'' m''} - \sum_{\alpha' s' l' v'} \\ &\quad \times U_{\alpha' s' l' v', \alpha s I J} \sum_{v' m'} (s' l' v' m' | J M) \Theta_{\alpha' s' l' v' m'}. \end{aligned} \quad (3.1b)$$

A linear combination of such solutions is also a solution, and, in particular, the following combination is a solution:

$$\begin{aligned} \Psi_{v m} &= \sum_{J M} (s l v m | J M) \Psi_{J M} \\ &= \sum_{J M v'' m''} (s l v m | J M) (s l v'' m'' | J M) g_{\alpha s l v m} \\ &\quad - \sum_{J M \alpha' s' l' v' m'} (s l v m | J M) \\ &\quad \times U_{\alpha' s' l' v', \alpha s I J} (s' l' v' m' | J M) \Theta_{\alpha' s' l' v' m'}. \end{aligned} \quad (3.2)$$

By utilizing the unitary property

$$\sum_{J M} (s l v m | J M) (s l v'' m'' | J M) = \delta_{v m, v'' m''},$$

this solution reduces to

$$\Psi_{v m} = g_{\alpha s l v m} - \sum_{\alpha' s' l' v' m'} U_{\alpha' s' l' v' m', \alpha s l v m} \Theta_{\alpha' s' l' v' m'} \quad (3.3a)$$

where

$$\begin{aligned} U_{\alpha' s' l' v' m', \alpha s l v m} &= \sum_{J M} (s l v m | J M) \\ &\quad \times U_{\alpha' s' l' v', \alpha s I J} (s' l' v' m' | J M) \end{aligned} \quad (3.3b)$$

is the "ordinary" collision matrix for the $\{\alpha s l v m\}$ scheme.

4. Generalized Collision Matrices

The collision matrix of subsection 1 gives, by definition, the coefficients of the outgoing wave functions Θ in terms of the coefficients of the incoming wave functions g . These two types of wave functions are independent as their Wronskian (III, 4.6c) is not zero. The derivation of subsection 1 can be carried out using any arbitrary pairs of independent wave functions, the Wronskians of which are not zero. These wave functions will be referred to as generalized incoming and outgoing waves, g^\dagger and Θ^\dagger , respectively. The matrix which gives the coefficients of the Θ^\dagger waves in terms of the coefficients of the g^\dagger waves is referred to as the generalized collision matrix \mathbf{U}^\dagger . The radial parts I^\dagger , O^\dagger of the generalized waves are related to the original ones by a set of four quantities:

$$I_c^\dagger = \xi_c I_c + \eta_c O_c, \quad O_c^\dagger = \zeta_c I_c + \chi_c O_c, \quad (4.1)$$

and the generalized Wronskian is

$$w_c^\dagger = O_c^{\dagger\dagger} I_c^\dagger - I_c^{\dagger\dagger} O_c^\dagger = 2i(\eta_c \zeta_c - \xi_c \chi_c). \quad (4.2)$$

In the external region the particular solutions are then of the

generalized form

$$\Psi_m = \sum_c (\delta_{cm} \mathcal{G}_c^\dagger - U_{cm}^\dagger \mathcal{O}_c^\dagger). \quad (4.3)$$

By replacing the daggered waves by the undaggered ones using the relations (4.1), one finds that

$$\Psi_m = \sum_c [(\xi_c \delta_{cm} - U_{cm}^\dagger \zeta_c) \mathcal{G}_c - (U_{cm}^\dagger \chi_c - \eta_c \delta_{cm}) \mathcal{O}_c]. \quad (4.4)$$

Defining ξ , η , ζ as diagonal matrices with elements ξ_c , η_c , ζ_c respectively, the linear combination

$$\begin{aligned} \Psi_t &= \sum_m (\xi - \zeta \mathbf{U}^\dagger)_{mt}^{-1} \Psi_m \\ &= \sum_c (\delta_{ct} \mathcal{G}_c - U_{ct} \mathcal{O}_c) \end{aligned}$$

is also a solution, where

$$\mathbf{U} = (\chi \mathbf{U}^\dagger - \eta) (\xi - \zeta \mathbf{U}^\dagger)^{-1} \quad (4.5)$$

is the relation between the generalized \mathbf{U}^\dagger and the \mathbf{U} of subsection 1. From (4.5), one has the inverse relation

$$\mathbf{U}^\dagger = (\chi + \mathbf{U} \zeta)^{-1} (\mathbf{U} \xi + \eta). \quad (4.5a)$$

Although \mathbf{U} is symmetrical with respect to channels having the same Wronskian values, it is evident from the arguments of subsection 2a that it is only unitary if the Wronskian (4.2) is pure imaginary and the same for all channels.

a. The \mathbf{Q} Matrix

An example of a generalized collision matrix is the \mathbf{Q} matrix (or "reactance matrix"), introduced by Teichmann and Wigner.⁵⁶ The radial external wave functions associated with this matrix are $I^\dagger = F$ and $O^\dagger = -G$ so that the quantities of (4.1) and (4.2) are $\xi = \frac{1}{2}i$, $\eta = -\frac{1}{2}i$, $\zeta = -\frac{1}{2}$, $\chi = -\frac{1}{2}$, $w^\dagger = 1$. According to (4.5)

$$\mathbf{U} = (\mathbf{1} + i\mathbf{Q})(\mathbf{1} - i\mathbf{Q})^{-1} \quad (4.6)$$

and by substituting F , $-G$, F' , $-G'$, 1 for I , O , I' , O' , w in (1.6a) of the next section, one obtains

$$\begin{aligned} \mathbf{Q} &= -\mathbf{F}\mathbf{G}^{-1} + \rho^{\frac{1}{2}}\mathbf{G}^{-1}(\mathbf{1} - \mathbf{R}\mathbf{L}^0)^{-1}\mathbf{R}\rho^{\frac{1}{2}}\mathbf{G}^{-1} \\ &= -\mathbf{F}'\mathbf{G}' + \rho^{-\frac{1}{2}}(\mathbf{G}')^{-1}[(\mathbf{L}^0)^{-1} - \mathbf{R}]^{-1}\rho^{-\frac{1}{2}}(\mathbf{G}')^{-1} \end{aligned} \quad (4.7)$$

where

$$\mathbf{L}^0 = \rho\mathbf{G}'\mathbf{G}^{-1};$$

the second equation is obtained by rearranging and applying the Wronskian relation (III, 2.12) to the first; it is the formula which was considered by Teichmann and Wigner. The procedures of Sec. IX can be used to obtain level expansions for the \mathbf{Q} matrix; the details and discussions are analogous to those of Sec. IX. The \mathbf{Q} matrix is evidently real and symmetric. In the one-channel case (4.6) shows that $Q = \tan\delta$ where δ is the phase shift of Sec. IV, 9.

The \mathbf{Q} matrix gives the coefficients of the external functions with radial parts equal to the irregular functions G in terms of those with radial parts equal to the F functions which are regular at the origin. Its components therefore provide a measure of the extent to which the internal nuclear interactions generate the irregular functions in the external region. In the absence of such interactions, the \mathbf{Q} matrix vanishes, as may be verified by substituting into (4.7) the diagonal \mathbf{R} matrix of Sec. IV, 3 (with $V_0 = 0$) which describes free space. The components of the \mathbf{Q} matrix are also related to the components of the interaction matrix elements which generate the G functions in terms of the F , in the familiar Green's function method.

b. The \mathbf{R} Matrix

The \mathbf{R} matrix itself may be considered as a generalized collision matrix. The appropriate transformation (4.1) has the coefficients

$$\begin{aligned} \xi &= \frac{1}{2}i(G + iF)\rho^{-\frac{1}{2}}, \\ \eta &= -\frac{1}{2}i(G - iF)\rho^{-\frac{1}{2}}, \\ \zeta &= -\frac{1}{2}i(G' + iF')\rho^{\frac{1}{2}}, \\ \chi &= \frac{1}{2}i(G' - iF')\rho^{\frac{1}{2}}, \end{aligned} \quad (4.8)$$

the quantities of the external region being evaluated at the

surface \mathcal{S} . The values and derivatives of the radial parts of the \mathcal{G}^\dagger and \mathcal{O}^\dagger wave functions on \mathcal{S} are therefore, from (4.1),

$$I^\dagger = 0, \quad I^{\dagger'} = \rho^{-\frac{1}{2}}; \quad O^\dagger = \rho^{\frac{1}{2}}, \quad O^{\dagger'} = 0, \quad (4.9)$$

and

$$w^\dagger = 1.$$

By substituting these into (1.6a) of the next section one finds that

$$\mathbf{U}^\dagger = -\mathbf{R}. \quad (4.10)$$

In other words, when \mathcal{G}^\dagger and \mathcal{O}^\dagger are defined by (4.1) and (4.8), the particular solutions having only a single \mathcal{G}_m^\dagger function are then in the external region of the form

$$\Psi_m \sim \sum_c (\delta_{cm} \mathcal{G}_m^\dagger + R_{cm} \mathcal{O}_c^\dagger). \quad (4.11)$$

By considering this equation and its derivative on the surface \mathcal{S} and making use of (4.9) it is immediately evident that (4.11) is consistent with the \mathbf{R} -matrix relation (V, 2.7) and constitutes its continuation into the external region.

VII. RELATION BETWEEN THE \mathbf{R} MATRIX AND THE COLLISION MATRIX

Equations (V, 2.7) and (V, 2.8) give the connection between the derivative and the value of the wave function on the surface \mathcal{S} which must hold if the wave function is to be continuable into the internal region. The connection between the \mathbf{R} matrix and collision matrix \mathbf{U} will now be derived. Since the \mathbf{R} matrix specifies the form of the wave function on the surface \mathcal{S} and the \mathbf{U} matrix specifies the form at infinity, the connection between the two is established by joining these regions. This process of joining introduces into the theory reference to the "external" wave functions like the F and G of Sec. III.

1. Derivation of the Relation

In the *external region* any solution Ψ can be expressed as a sum of the incoming and outgoing waves of (III, 2.19 and 20), the coefficients of which are y_c and x_c , respectively, from (VI, 1.1):

$$\Psi = \sum_c (x_c \mathcal{O}_c + y_c \mathcal{G}_c). \quad (1.1)$$

For this expression the surface "value" and "derivative" quantities of (III, 4.2) are

$$\begin{aligned} V_c &= (\hbar^2/2M_c a_c)^{\frac{1}{2}} (v_c^{-\frac{1}{2}} O_c x_c + v_c^{-\frac{1}{2}} I_c y_c), \\ D_c &= (\rho_c^{\frac{1}{2}} \hbar^2/2M_c a_c)^{\frac{1}{2}} (v_c^{-\frac{1}{2}} O_c' x_c + v_c^{-\frac{1}{2}} I_c' y_c), \end{aligned} \quad (1.2a)$$

or

$$\begin{aligned} V_c &= (\frac{1}{2}\hbar)^{\frac{1}{2}} (\rho_c^{-\frac{1}{2}} O_c x_c + \rho_c^{-\frac{1}{2}} I_c y_c), \\ D_c &= (\frac{1}{2}\hbar)^{\frac{1}{2}} (\rho_c^{\frac{1}{2}} O_c' x_c + \rho_c^{\frac{1}{2}} I_c' y_c). \end{aligned} \quad (1.2b)$$

Since these must satisfy the fundamental \mathbf{R} -matrix relation (V, 2.7), one finds, in matrix notation, that

$$(\rho^{-\frac{1}{2}} \mathbf{O} \mathbf{x} + \rho^{-\frac{1}{2}} \mathbf{I} \mathbf{y}) = \mathbf{R} [(\rho^{\frac{1}{2}} \mathbf{O}' \mathbf{x} + \rho^{\frac{1}{2}} \mathbf{I}' \mathbf{y}) - \mathbf{B}(\rho^{-\frac{1}{2}} \mathbf{O} \mathbf{x} + \rho^{-\frac{1}{2}} \mathbf{I} \mathbf{y})]$$

or

$$(\rho^{-\frac{1}{2}} \mathbf{O} - \mathbf{R} \rho^{\frac{1}{2}} \mathbf{O}') \mathbf{x} = -(\rho^{-\frac{1}{2}} \mathbf{I} - \mathbf{R} \rho^{\frac{1}{2}} \mathbf{I}') \mathbf{y}, \quad (1.3)$$

where

$$\rho \mathbf{O}' = \rho \mathbf{O}' - \mathbf{B} \mathbf{O}, \quad \rho \mathbf{I}' = \rho \mathbf{I}' - \mathbf{B} \mathbf{I}.$$

The collision matrix \mathbf{U} gives, by definition (VI, 1.2), the outgoing wave coefficients in terms of the incoming

wave coefficients:

$$x_{c'} = -\sum_c U_{c'e} y_c$$

or, in matrix notation,

$$\mathbf{x} = -\mathbf{U}\mathbf{y}. \quad (1.4)$$

By comparing with (1.3), one obtains

$$\begin{aligned} \mathbf{U}^J &= (\mathbf{O}\mathbf{O}^{-\frac{1}{2}} - \mathbf{R}^J \mathbf{O}' \mathbf{O}^{\frac{1}{2}})^{-1} (\mathbf{I}\mathbf{O}^{-\frac{1}{2}} - \mathbf{R}^J \mathbf{I}' \mathbf{O}^{\frac{1}{2}}) \\ &= \mathbf{O}^{\frac{1}{2}} \mathbf{O}^{-1} (\mathbf{1} - \mathbf{R}^J \mathbf{L}^0)^{-1} (\mathbf{1} - \mathbf{R}^J \mathbf{Q}^0) \mathbf{I}\mathbf{O}^{-\frac{1}{2}} \\ &= \mathbf{O}\mathbf{W}^J \mathbf{O} \quad (\text{say}). \end{aligned} \quad (1.5)$$

In these expressions $\mathbf{1}$ is written for the diagonal unit matrix and the following three matrices have been introduced:

$$\mathbf{L}^0 = \mathbf{L} - \mathbf{B}; \quad \mathbf{Q}^0 = \mathbf{Q} - \mathbf{B}$$

$$\begin{aligned} \mathbf{W}^J &= \mathfrak{P}^{\frac{1}{2}} (\mathbf{1} - \mathbf{R}^J \mathbf{L}^0)^{-1} (\mathbf{1} - \mathbf{R}^J \mathbf{Q}^0) \mathfrak{P}^{-\frac{1}{2}} \\ &= \mathbf{1} + \mathfrak{P}^{\frac{1}{2}} (\mathbf{1} - \mathbf{R}^J \mathbf{L}^0)^{-1} \mathbf{R}^J \mathfrak{P}^{\frac{1}{2}} \mathbf{w} \end{aligned} \quad (1.6a)$$

$$\begin{aligned} &= [\mathbf{1} - \mathfrak{P}^{\frac{1}{2}} (\mathbf{L}^0)^{-1} \mathbf{w}] + [\mathfrak{P}^{\frac{1}{2}} (\mathbf{L}^0)^{-1}] \\ &\quad \times [(\mathbf{L}^0)^{-1} - \mathbf{R}^J]^{-1} [\mathfrak{P}^{\frac{1}{2}} (\mathbf{Q}^0)^{-1}] \mathbf{w}. \end{aligned} \quad (1.6b)$$

The matrix \mathbf{W}^J is a new quantity. It is often more convenient to use than \mathbf{U}^J itself. The components of the "surface" diagonal matrices $\mathfrak{P}^{\frac{1}{2}}$, \mathbf{L} , \mathbf{Q} , \mathbf{O} , \mathbf{w} were specified in III, 4; \mathbf{B} is the real, diagonal boundary condition matrix whose components are given by (V, 2.1). The dependence of \mathbf{U} and \mathbf{R} on total spin and parity has been indicated in (1.5) and (1.6) by the superscript J ; this superscript will be omitted except when relevant. Equations (1.5) and (V, 2.8) give the collision matrix in terms of the real quantities $\gamma_{\lambda c}$, E_{λ} , σ_c , ϕ_c , S_c , P_c . Although these quantities depend on the parameters a_c and B_c , the \mathbf{U} matrix, which is a property of the physical system, must itself be independent of these parameters.

2. Verification of the Symmetry and Unitarity Properties of the Collision Matrix

In Sec. VI, 2 we saw that, from very general physical principles, the collision matrix \mathbf{U} must be symmetric and unitary. The specific form (1.5) satisfies these requirements. Clearly the symmetry and unitarity natures of \mathbf{W} are equivalent to the same properties of \mathbf{U} .

From the symmetry of \mathbf{R} and the fact that

$$\mathbf{R}(\mathbf{1} - \mathbf{L}^0 \mathbf{R})^{-1} = [\mathbf{1} - \mathbf{R}(\mathbf{L}^0)^{-1}] \mathbf{R},$$

it follows that

$$\text{transpose}(\mathbf{W}) = \mathbf{1} + \mathbf{w} \mathfrak{P}^{\frac{1}{2}} (\mathbf{1} - \mathbf{R}\mathbf{L}^0)^{-1} \mathbf{R} \mathfrak{P}^{\frac{1}{2}}. \quad (2.1)$$

Therefore, \mathbf{W} (and \mathbf{U}) is symmetrical with respect to channels having the same values of the Wronskians w_c . In particular, the submatrix \mathbf{W}^{++} corresponding to positive energy channels is symmetrical since, from (III, 4.7c), $w_c^+ = 2_i$ for all such channels.

As to the unitarity of \mathbf{U} , it is convenient to examine the role of negative energy channels a little more closely.

$-\mathbf{U}$ is the matrix that, multiplying the vector \mathbf{y} which is composed of the amplitudes of incoming waves, it gives the corresponding vector \mathbf{x} that represents outgoing waves. The \mathbf{U} matrix gives the amplitudes of the positive-energy outgoing wave functions O_c^+ and, the negative-energy exponentially decaying O_c^- in terms of the incoming waves I_c^+ and the negative-energy unbounded functions I_c^- . As there are no physical situations in which the I_c^- occur, the components of the submatrix \mathbf{W}^{+-} are not physically significant and one might as well set them equal to zero as can be seen from (1.6b). This may be accomplished without affecting the components of \mathbf{W}^{++} and \mathbf{W}^{-+} by setting the negative-energy components of the Wronskian matrix equal to zero; $w_c^- = 0$. (This means that the O_c^- and I_c^- are not linearly independent.) Another consequence is that

$$\mathbf{Q}_c^0 = L_c^{0*} \quad (2.2)$$

for the negative- as well as the positive-energy channels, so that one can eliminate the quantities \mathbf{Q}_c completely by replacing them by L_c^* . For instance, (1.6a) becomes

$$\mathbf{W} = \mathbf{P}^{\frac{1}{2}} (\mathbf{1} - \mathbf{R}\mathbf{L}^0)^{-1} (\mathbf{1} - \mathbf{R}\mathbf{L}^{0*}) \mathbf{P}^{-\frac{1}{2}}. \quad (2.3)$$

This matrix is of the form

$$\mathbf{W} = \left\| \begin{array}{c|c} \mathbf{W}^{++} & \mathbf{0} \\ \hline \mathbf{W}^{-+} & \mathbf{1} \end{array} \right\|. \quad (2.4)$$

No significance is attached to the unit value of the \mathbf{W}^{--} components, which like the \mathbf{W}^{+-} components are of no physical interest.

From (2.3) it is evident that

$$\mathbf{W}\mathbf{W}^* = \mathbf{1}. \quad (2.5)$$

By substituting the "supermatrix" (2.4) into (2.5), one finds that

$$\mathbf{W}^{++}(\mathbf{W}^{++})^* = \mathbf{1}, \quad (2.6)$$

and that

$$(\mathbf{W}^{-+})(\mathbf{W}^{++})^* + (\mathbf{W}^{-+})^* = \mathbf{0}. \quad (2.7)$$

Since \mathbf{W}^{++} is symmetrical, (2.6) indicates that it is also unitary; $\mathbf{U}^{++} \equiv (\mathbf{O}\mathbf{W}\mathbf{O})^{++}$ is also unitary because the unitary diagonal matrix appears on each side of \mathbf{W} :

$$(\mathbf{W}^{++})(\mathbf{W}^{++})^\dagger = (\mathbf{U}^{++})(\mathbf{U}^{++})^\dagger = \mathbf{1}. \quad (2.8)$$

Likewise (2.7) leads to

$$(\mathbf{U}^{-+})(\mathbf{U}^{++})^* + (\mathbf{U}^{-+})^* = \mathbf{0}. \quad (2.9)$$

3. Two-Channel Collision Matrix

Although the expressions (1.6b) give formally the components of the collision matrix in terms of those of the \mathbf{R} matrix and the diagonal surface matrices, it is unsuited for applications involving many channels because of the difficulty of inverting the matrix $(\mathbf{1} - \mathbf{R}\mathbf{L}^0)$. This difficulty reflects the difficulty of solving many equations in many unknowns that is implied in

the fundamental \mathbf{R} -matrix relation (V, 2.7). Conversion of (1.6b) to more practical though equally general forms is carried out in Secs. IX and X.

For only one or two participating channels, the matrix inversion is easily effected. In the one-channel case the inversion is just ordinary division and (1.6b) in conjunction with (1.5) gives after some rearranging the collision function (IV, 1.14). In the case of two positive-energy channels (1 and 2), the components of the collision matrix are

$$\begin{aligned} W_{11} &= 1 + 2iP_1[R_{11} - L_2^0(R_{11}R_{22} - R_{12}^2)]d^{-1}, \\ W_{22} &= 1 + 2iP_2[R_{22} - L_1^0(R_{11}R_{22} - R_{12}^2)]d^{-1}, \\ W_{12} &= W_{21} = 2iP_1^{\frac{1}{2}}R_{12}P_2^{\frac{1}{2}}d^{-1}, \end{aligned} \quad (3.1)$$

where d is the determinant

$$d = (1 - R_{11}L_1^0)(1 - R_{22}L_2^0) - L_1^0R_{12}^2L_2^0.$$

An exceptional feature of the reaction components W_{12} of (1.28) is that it vanishes between levels for which the products $\gamma_{\lambda_1}\gamma_{\lambda_2}$ have the same sign. The associated cross section, being proportional to $|W_{12}|^2$, also has this feature. Wigner⁶⁷ has shown that (i) the elastic scattering cross section does not vanish in general for any value of the energy, except in the one-channel case; and (ii) no reaction cross sections in the case of three or more channels will vanish in general for any value of the energy. These remarks apply to the partial cross section for an individual J value. Even in the two-channel case the *total* reaction cross section also does not vanish in general for any value of the energy, because the individual contributions would each have to vanish for the same E and every J .

4. Inclusion of Nonpolarizing Potentials

It is possible to include in the formalism the presence of any nonpolarizing potentials in the various channels. By assumption (4) of Sec. II such potentials must not be capable of significantly exciting or "polarizing" the two separated subsystems.

If there were no nuclear forces between the two systems, the radial wave function of relative motion would just be the function F_l defined by (III, 2.11). This function is regular at $r=0$ and takes account of the phase shift induced by a Coulomb potential. In the presence of an extra potential, let us assume that the radial wave function is f_l corresponding to an extra phase shift $2\theta_l$ then:

$$f_l = F_l \cos\theta_l + G_l \sin\theta_l = \sin(\rho - \eta \log 2\rho - \frac{1}{2}l\pi + \sigma_l + \theta_l). \quad (4.1)$$

The notation f has been selected in analogy with the use of F for the regular solution in the absence of a nuclear potential. The irregular solution g_l is defined so that it has the asymptotic behavior

$$g_l = -F_l \sin\theta_l + G_l \cos\theta_l = \cos(\rho - \eta \log 2\rho - \frac{1}{2}l\pi + \sigma_l + \theta_l). \quad (4.2)$$

Similarly one can define incoming waves i_l and outgoing waves o_l , analogous to (III, 2.10) and (III, 2.13), which have radial parts

$$\begin{aligned} i_l &= (G_l - iF_l) \exp[i(\omega_l - \theta_l)] = I_l \exp(-i\theta_l), \\ o_l &= (G_l + iF_l) \exp[-i(\omega_l - \theta_l)] = O_l \exp(i\theta_l), \end{aligned} \quad (4.3)$$

respectively. The Wronskians for the sets $\{f, g\}$ and $\{i, o\}$ are the same as those of $\{F, G\}$ and $\{I, O\}$, respectively. Let \mathbf{u} denote the

collision matrix, analogous to \mathbf{U} of (1.1) and (1.4), which refers to the system of o and i waves; it may be expressed in terms of the \mathbf{R} matrix by means of (1.5b) and (1.6) by merely substituting f and g for F and G , respectively. The actual collision matrix \mathbf{U} may be found by multiplying the solution in the external region of the form

$$\Psi_c \sim i_c - \sum_{c'} u_{cc'} o_{c'} \quad (4.4)$$

by $\exp i\theta_c$ and comparing the result with (1.1) and (1.4) in which only a single $y_c (=1)$ is nonvanishing: the result is, in matrix notation, that

$$\mathbf{U} = \mathbf{e}^{i\theta} \mathbf{u} \mathbf{e}^{i\theta}. \quad (4.5)$$

\mathbf{U} is thus given by (1.5) and (1.6) with the f and g substitutions and with the additional phase θ in the potential scattering matrix Ω :

$$\begin{aligned} \Omega &= \exp i(\omega + \theta - \phi), \\ \phi &= \tan^{-1}(f/g), \\ S &= \rho(f'f' + gg')/(f^2 + g^2), \\ P &= \rho/(f^2 + g^2). \end{aligned} \quad (4.6)$$

Thus the procedure for including the effect of the external potentials is the same as that of including the Coulomb potential.

As to the evaluation of f and g , these can be determined by solving the Schroedinger equation directly. When the external potential is not too large, two integral equation methods may be used.

The first method applies the Green's-theorem relation to the functions g and G of the same energy and with the surface integrations performed at the radii r and infinity. This relation is

$$(gG' - Gg')_r = \int_r^\infty P(r'')g(r'')G(r'')dr'', \quad (4.7)$$

where $\hbar^2P/2M$ is the additional potential to which g is subject, and the surface contribution at infinity vanishes because g and G have the same asymptotic form; the prime denotes differentiation with respect to r . Both sides of (4.7) are then divided by $G^2(r')$ and integrated from r to infinity; by noting that

$$(gG' - Gg')dr = -G^2d(g/G),$$

the following result is obtained:

$$g(r) = G(r) \left[1 + \int_r^\infty \frac{dr'}{G^2(r')} \int_{r'}^\infty \bar{P}G^2 dr'' \right], \quad (4.8)$$

where

$$\bar{P} = (g/G)P.$$

This result is used in Sec. XIII, 1 in the determination of the behavior of cross sections near thresholds. It also applies to the other three functions f , i , and o if the appropriate substitutions are made.

The second method uses a Green's function, and although it requires knowledge of both F and G for the determination of either f or g , only a single integration is involved. For the determination of g the Green's theorem relation with surface integrations at r and ∞ is applied twice, once with the pair g, G and once with the pair g, F . Considering that $g=G$ and $g'=G'$ at infinity, and that $F'G - G'F = k$, these relations reduce to

$$-k - (Fg' - gF')_r = \int_r^\infty P(r'')g(r'')F(r'')dr'', \quad (4.9)$$

$$- (Gg' - gG')_r = \int_r^\infty P g G dr'. \quad (4.10)$$

The first equation is multiplied through by $G(r)$ and the second by $F(r)$; the difference of the resulting equations is the integral equation for $g(r)$,

$$g(r) = G(r) + k^{-1} \int_r^\infty G(r'')P(r'')g(r'')dr'' \quad (4.11)$$

with the Green's function

$$G(r, r') = G(r)F(r') - F(r)G(r').$$

⁶⁷ E. P. Wigner, Proc. Natl. Acad. Sci. U. S. 32, 302 (1946).

In a similar manner, the integral equation for $f(r)$ is found to be

$$f(r) = F(r) - k^{-1} \int_r^\infty G(r, r') P(r') f(r') dr'. \quad (4.12)$$

VIII. RELATIONS BETWEEN THE CROSS SECTIONS AND THE ELEMENTS OF THE COLLISION MATRIX

We now complete the last link in the basic scheme of the \mathbf{R} -matrix theory as described in Sec. II, 1. In Sec. V we obtained the relation between the \mathbf{R} matrix and the eigenstates X_λ . In Sec. VII we derived the collision matrix \mathbf{U} in terms of the \mathbf{R} matrix and the matrices \mathbf{L} and $\mathbf{\Omega}$ representing the external interactions. Now we express the cross sections in terms of the elements of the collision matrix.

1. Scattering Amplitudes

By substituting (VII, 1.4) into (VII, 1.1), the general solution of the wave equation can be expressed in the external region in terms of the unspecified coefficients y_c of the incoming waves:

$$\Psi(\text{general}) = \sum_{cc'} (\delta_{cc'} \mathcal{G}_c - U_{c'c} \mathcal{O}_{c'}) y_c. \quad (1.1)$$

Consider now the wave function with the same y_c , the radial parts of which are proportional to the regular function F_c in each channel:

$$\Psi' = \sum_{cc'} (\delta_{cc'} \mathcal{G}_c - e^{2i\omega c} \delta_{cc'} \mathcal{O}_{c'}) y_c. \quad (1.2)$$

Adding Ψ' to and subtracting it from the right side of (1.1) gives

$$\Psi(\text{general}) = \Psi' + \sum_{cc'} (e^{2i\omega c} \delta_{cc'} - U_{c'c}) \mathcal{O}_{c'} y_c, \quad (1.3)$$

in which explicit reference to the incoming waves has been eliminated. This solution is now particularized by an appropriate choice of the y_c so that Ψ' represents an incident wave of particles of type α , channel spins and component ν , moving along the z axis and disturbed only by the Coulomb field (if any), this choice is

$$y_c \equiv y_{\alpha s l \nu 0} = \frac{i\pi^{\frac{1}{2}}}{k_\alpha} (2l+1)^{\frac{1}{2}}, \quad (1.4)$$

all others being zero. From the definitions (III, 2.13) and (III, 2.19) and the fact that

$$Y_0^{(l)}(\Omega) = [(2l+1)/4\pi]^{\frac{1}{2}} P_l(\cos\theta)$$

it follows that

$$\Psi_{\alpha s \nu}' = v_\alpha^{-\frac{1}{2}} k_\alpha^{-1} \sum_l i^l (2l+1) \times e^{i\omega_\alpha l} \left(\frac{F_{\alpha l}}{r_\alpha} \right) P_l(\cos\theta_\alpha) \psi_{\alpha s \nu}. \quad (1.5)$$

According to Schiff⁶⁸ (see pp. 116–119), (1.5) is an alternative form of the function which is asymptotically†

$$\Psi_{\alpha s \nu}' \sim v_\alpha^{-\frac{1}{2}} \psi_{\alpha s \nu} \left[\left(1 - \frac{\eta_\alpha^2}{ik_\alpha(r_\alpha - z_\alpha)} \right) \times \exp i \{ k_\alpha z_\alpha - \eta_\alpha \log k_\alpha (r_\alpha - z_\alpha) - \sigma_{\alpha 0} \} - \frac{\pi^{\frac{1}{2}}}{r_\alpha k_\alpha} C_\alpha(\theta_\alpha) \exp i \{ \rho_\alpha - \eta_\alpha \log 2\rho_\alpha + \sigma_{\alpha 0} \} \right] \quad (1.6)$$

where

$$C_\alpha(\theta_\alpha) = (4\pi)^{-\frac{1}{2}} \eta_\alpha \operatorname{cosec}^2 \left(\frac{\theta_\alpha}{2} \right) \exp \left\{ -2i\eta_\alpha \log \sin \left(\frac{\theta_\alpha}{2} \right) \right\}.$$

This function represents an incident plane wave in the z direction of the type $\alpha s \nu$ in a Coulomb field, together with a Coulomb scattered wave, the coefficient of which is $C_\alpha(\theta_\alpha)$. With the choice (1.4), the sum on the right side of (1.3) may also be evaluated, and the asymptotic form of the particular solution is obtained:

$$\Psi(\text{particular}) \sim \Psi_{\alpha s \nu}' + \frac{i\pi^{\frac{1}{2}}}{k_\alpha} \sum_{\alpha' s' l' \nu' m'} (2l'+1)^{\frac{1}{2}} \times [e^{2i\omega_{\alpha'} l'} \delta_{\alpha' s' l' \nu' m', \alpha s l \nu 0} - U_{\alpha' s' l' \nu' m', \alpha s l \nu 0}] \times \frac{\exp i \{ \rho_{\alpha'} - \eta_{\alpha'} \log 2\rho_{\alpha'} + \sigma_{\alpha' 0} \}}{v_{\alpha'}^{\frac{1}{2}} r_{\alpha'}} Y_{m'}^{(l')}(\Omega_{\alpha'}) \psi_{\alpha' s' l' \nu'}. \quad (1.7)$$

The amplitudes $A_{\alpha' s' l' \nu', \alpha s \nu}(\Omega_{\alpha'})$ of the outgoing waves of type $\alpha' s' l' \nu'$ at infinity, which are associated with the unit-flux incident plane wave of type $\alpha s \nu$ are defined by the scalar product

$$A_{\alpha' s' l' \nu', \alpha s \nu}(\Omega_{\alpha'}) = r_{\alpha'} v_{\alpha'}^{\frac{1}{2}} \exp i \{ \rho_{\alpha'} - \eta_{\alpha'} \log 2\rho_{\alpha'} + \sigma_{\alpha' 0} \} \times \lim_{r_{\alpha'} \rightarrow \infty} \int \psi_{\alpha' s' l' \nu'}^* \Psi d\Omega_{\alpha'}. \quad (1.8)$$

The differential cross sections are then by definition given by

$$d\sigma_{\alpha s \nu, \alpha' s' l' \nu'} = |A_{\alpha' s' l' \nu', \alpha s \nu}(\Omega_{\alpha'})|^2 d\Omega_{\alpha'} \quad (1.9)$$

with the scattering and reaction amplitudes of (1.7) as

$$A_{\alpha' s' l' \nu', \alpha s \nu}(\Omega_{\alpha'}) = \frac{\pi^{\frac{1}{2}}}{k_\alpha} \left\{ -C_{\alpha'}(\theta_{\alpha'}) \delta_{\alpha' s' l' \nu', \alpha s \nu} + i \sum_{l' m' l} (2l'+1)^{\frac{1}{2}} \times [e^{2i\omega_{\alpha'} l'} \delta_{\alpha' s' l' \nu' m', \alpha s l \nu 0} - U_{\alpha' s' l' \nu' m', \alpha s l \nu 0}] Y_{m'}^{(l')}(\Omega_{\alpha'}) \right\}. \quad (1.10)$$

⁶⁸ L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949).

† If the colliding particles are identical, then either the symmetric or antisymmetric "plane waves" of the form

$$\exp ikz + \exp(-ikz) \quad \text{and} \quad \exp(ikz) - \exp(-ikz)$$

must be used, depending upon the nature of the particles and the symmetry of the $\psi_{\alpha s \nu}$.

2. Differential Cross Sections

We omit discussion of cross sections involving either polarized incident particles or measurement of polarization in the reaction products. (For discussion of polarization cross sections and references see Simon and Welton⁶⁹ and Biedenharn and Rose⁴⁴ and Satchler.⁷⁰ For unpolarized incident and target particles, the cross section (1.9) may be summed over the ν' and averaged with respect to the ν to obtain the differential cross sections for the processes $\alpha s \rightarrow \alpha' s'$,

$$d\sigma_{\alpha s, \alpha' s'} = (2s+1)^{-1} \sum_{\nu\nu'} |A_{\alpha' s' \nu', \alpha s \nu}(\Omega_{\alpha'})|^2 d\Omega_{\alpha'}. \quad (2.1)$$

Likewise, (2.1) may be summed over the s' and averaged with respect to the s to obtain the differential cross section for the processes $\alpha \rightarrow \alpha'$:

$$d\sigma_{\alpha\alpha'} = [(2I_1+1)(2I_2+1)]^{-1} \times \sum_{s's'\nu\nu'} |A_{\alpha' s' \nu', \alpha s \nu}(\Omega_{\alpha'})|^2 d\Omega_{\alpha'}, \quad (2.2)$$

where I_1 and I_2 are the spins of the particles of the pair α .

To proceed further the representation of \mathbf{U} will be changed from the $\{\alpha s l \nu m\}$ scheme to the $\{\alpha s l J M\}$ scheme. Substituting the transformation (VI, 3.3b) into the amplitude expression (1.10) gives:

$$A_{\alpha' s' \nu', \alpha s \nu}(\Omega_{\alpha'}) = \frac{\pi^{\frac{1}{2}}}{k_{\alpha}} \left[-C_{\alpha'}(\theta_{\alpha'}) \delta_{\alpha' s' \nu', \alpha s \nu} + i \sum_{J M l l' m' m'} (2l+1)^{\frac{1}{2}} (s l \nu 0 | J M) (s' l' \nu' m' | J M) \times T_{\alpha' s' \nu', \alpha s l' J} Y_{m'}^{(l')}(\Omega_{\alpha'}) \right], \quad (2.3)$$

where

$$T_{\alpha' s' \nu', \alpha s l' J} = e^{2i\omega_{\alpha' l'}} \delta_{\alpha' s' \nu', \alpha s l} - U_{\alpha' s' \nu', \alpha s l' J}.$$

In performing the absolute squaring operation, one introduces the two sets of summing integers

$$\{J_1 M_1 l_1 l_1' m_1'\} \quad \text{and} \quad \{J_2 M_2 l_2 l_2' m_2'\}$$

for the single set of (2.3), and thereby obtains for (2.1)

$$(2s+1) \frac{k_{\alpha}^2}{\pi} d\sigma_{\alpha s, \alpha' s'} d\Omega_{\alpha'} = (2s+1) |C_{\alpha'}(\theta_{\alpha'})|^2 \delta_{\alpha' s', \alpha s} + \sum_{\substack{J_1 J_2 M_1 M_2 \\ l_1 l_1' l_2 l_2' \\ \nu \nu' m_1' m_2'}} (2l_1+1)^{\frac{1}{2}} (2l_2+1)^{\frac{1}{2}} (s l_1 \nu 0 | J_1 M_1) \times (s l_2 \nu 0 | J_2 M_2) (s' l_1' \nu' m_1' | J_1 M_1) (s' l_2' \nu' m_2' | J_2 M_2) \times (T_{\alpha' s' l_1', \alpha s l_1}^{J_1} Y_{m_1'}^{(l_1')}(\Omega_{\alpha'})) \times (T_{\alpha' s' l_2', \alpha s l_2}^{J_2} Y_{m_2'}^{(l_2')}(\Omega_{\alpha'}))^* + \sum_{\substack{J M l \\ m' \nu \nu'}} (2l+1)^{\frac{1}{2}} (s l \nu 0 | J M) (s' l' \nu' m' | J M) \times \delta_{\alpha' s' \nu', \alpha s \nu} 2 \operatorname{Re} [i T_{\alpha' s' \nu', \alpha s l' J} Y_{m'}^{(l')}(\Omega_{\alpha'}) C_{\alpha'}(\theta_{\alpha'})]. \quad (2.4)$$

The first term may be identified as pure Coulomb scattering, the second as the resonance scattering and reaction, and the last as the interference term from the first two. Sums in the resonance term may be evaluated by the procedure described by Blatt and Biedenharn⁴⁷ and sums in the interference terms by means of the relation

$$\sum_{M\nu} (s l \nu 0 | J M) (s' l' \nu m' | J M) = \frac{2J+1}{2l+1} \delta_{l'm'}, \quad (2.5)$$

The result is that

$$(2s+1) \frac{k_{\alpha}^2}{\pi} d\sigma_{\alpha s, \alpha' s'} d\Omega_{\alpha'} = (2s+1) |C_{\alpha'}(\theta_{\alpha'})|^2 \delta_{\alpha' s', \alpha s} + \frac{1}{\pi} \sum_L B_L(\alpha' s', \alpha s) P_L(\cos\theta_{\alpha'}) + (4\pi)^{-\frac{1}{2}} \sum_{Jl} (2J+1) \times 2 \operatorname{Re} [i T_{\alpha' s' \nu', \alpha s l' J} C_{\alpha'}(\theta_{\alpha'}) P_l(\cos\theta_{\alpha'})], \quad (2.6)$$

where

$$B_L(\alpha' s', \alpha s) = \frac{1}{4} (-)^{s-s'} \sum_{J_1 J_2 l_1 l_2 l_1' l_2'} \bar{Z}(l_1 J_1 l_2 J_2, s L) \times \bar{Z}(l_1' J_1 l_2' J_2, s' L) (T_{\alpha' s' l_1', \alpha s l_1}^{J_1}) (T_{\alpha' s' l_2', \alpha s l_2}^{J_2})^*. \quad (2.7)$$

The \bar{Z} coefficients are related to the Racah coefficients W according to

$$\bar{Z}(l_1 J_1 l_2 J_2, s L) = (2l_1+1)^{\frac{1}{2}} (2l_2+1)^{\frac{1}{2}} (2J_1+1)^{\frac{1}{2}} \times (2J_2+1)^{\frac{1}{2}} (l_1 l_2 0 0 | L 0) W(l_1 J_1 l_2 J_2, s L) \quad (2.8a)$$

and to the Z coefficients of Blatt and Biedenharn,

$$\bar{Z}(l_1 J_1 l_2 J_2, s L) = i^{l_1-l_2-L} Z(l_1 J_1 l_2 J_2, s L). \quad (2.8b)$$

The present result for B_L thus differs by a factor $i^{l_1-l_2+l_1'-l_2'-2L}$ from that of Blatt and Biedenharn.⁴⁷ This is due to the difference of the time-reversal conventions.

In some applications⁴⁷ it is desirable to separate from \mathbf{U} a diagonal part \mathbf{D} , leaving a generally nondiagonal remainder \mathbf{X} :

$$\mathbf{U} = \mathbf{D} + \mathbf{X}. \quad (2.8)$$

The formulas are especially simple if \mathbf{D} , like $C_{\alpha}(\theta_{\alpha})$, is not dependent upon J ,

$$D_{\alpha' s' \nu', \alpha s l' J} = D_{\alpha s l}. \quad (2.9)$$

Then

$$D_{\alpha' s' \nu' \nu' m', \alpha s l \nu 0} = \delta_{\alpha' s' \nu' \nu' m', \alpha s l \nu 0} D_{\alpha s l}. \quad (2.10)$$

By defining a potential scattering quantity as

$$p_{\alpha s} = -i \sum_l (2l+1)^{\frac{1}{2}} (e^{2i\omega_{\alpha l}} - D_{\alpha s l}) Y_0^{(l)}(\Omega_{\alpha}) = -i (4\pi)^{-\frac{1}{2}} \sum_l (2l+1) (e^{2i\omega_{\alpha l}} - D_{\alpha s l}) P_l(\cos\theta_{\alpha}), \quad (2.11)$$

the scattering and reaction amplitudes reduce to

$$A_{\alpha' s' \nu', \alpha s \nu}(\Omega_{\alpha'}) = -\frac{\pi^{\frac{1}{2}}}{k_{\alpha}} \{ (C_{\alpha'}(\theta_{\alpha'}) + p_{\alpha' s'}) \delta_{\alpha' s' \nu', \alpha s \nu} + i \sum_{l' m'} (2l'+1)^{\frac{1}{2}} X_{\alpha' s' \nu' \nu' m', \alpha s l' \nu 0} Y_{m'}^{(l')}(\Omega_{\alpha'}) \}. \quad (2.12)$$

⁶⁹ A. Simon and T. A. Welton, Phys. Rev. **90**, 1036 (1953); **94**, 943 (1954); A. Simon, *ibid.* **92**, 1050 (1953).

⁷⁰ G. R. Satchler, Proc. Phys. Soc. (London) **A68**, 1041 (1955).

Evidently the differential cross sections of (2.6) are modified by the substitutions

$$C \rightarrow C + \rho, \quad \mathbf{T} \rightarrow -\mathbf{X}. \quad (2.13)$$

3. Integrated and Total Cross Sections

By the "integrated" cross section for the reactions $\alpha s \rightarrow \alpha' s'$, we mean the cross section obtained by integrating the differential cross sections (2.6) over the solid angle $\Omega_{\alpha'}$. In this integration only the B_0 contribution is nonvanishing. Using the relation

$$\bar{Z}(l_1 J_1 l_2 J_2, s_0) = \delta_{l_1 J_1, l_2 J_2} (-)^{J_1 - s} (2J_1 + 1)^{\frac{1}{2}} \quad (3.1)$$

one obtains the well-known result

$$\sigma_{\alpha s, \alpha' s'} = \frac{\pi}{k_{\alpha}^2 (2s + 1)} \sum_{J l l'} (2J + 1) |T_{\alpha' s' l', \alpha s l}^J|^2. \quad (3.2a)$$

The infinite contribution to the integrated elastic scattering from the Coulomb field has been omitted from (3.2). The observed cross sections, which do not distinguish the various s' and which are obtained with unpolarized incident and target nuclei, are obtained by adding over s' and averaging over s :

$$\sigma_{\alpha \alpha'} = \frac{\pi}{k_{\alpha}^2} \sum_{J l l' s s'} g_J |T_{\alpha' s' l', \alpha s l}^J|^2 \quad (3.2b)$$

where g_J is the spin statistical factor defined as

$$g_J = \frac{2J + 1}{(2I_1 + 1)(2I_2 + 1)}. \quad (3.2c)$$

The observed total cross section is obtained by summing (3.2b) with respect to all possible α' (including $\alpha' = \alpha$). From the unitary nature of U , this cross section is

$$\sigma_{\alpha}(\text{tot}) = \frac{\pi}{k_{\alpha}^2} \sum_J 2g_J \sum_{sl} (1 - \text{Re}(U_{\alpha sl, \alpha s l}^J)). \quad (3.3)$$

4. Relation between Total Cross Section and Imaginary Part of the Forward Scattering Amplitude

According to (3.3) the total cross section is related to a sum of the diagonal components of the collision matrix and their complex conjugates. Since a similar sum appears in the expression for the imaginary part of the forward elastic scattering amplitude, a relation between these quantities is expected when there is no Coulomb field.

According to (2.3) the forward amplitude for *pure* elastic scattering (that is, $s = s'$ as well as $\alpha = \alpha'$) is given by

$$A_{\alpha s \nu, \alpha s \nu}(0) = \frac{1}{2} i k_{\alpha}^{-1} \sum_{J M l l'} (2l + 1)^{\frac{1}{2}} (2l' + 1)^{\frac{1}{2}} \times (s l \nu 0 | J M) (s l' \nu 0 | J M) T_{\alpha s l, \alpha s l'}^J \quad (4.1)$$

since only the $Y_0^{(l)}(0)$ are nonvanishing, there are no spin-flip terms with $\nu' \neq \nu$. The average of this amplitude with respect to s and ν may readily be obtained by using (2.5) for the ν sum, and

one finds for the imaginary part

$$\begin{aligned} \langle \text{Im} A_{\alpha s \nu, \alpha s \nu}(0) \rangle_{s, \nu} &= [(2I_1 + 1)(2I_2 + 1)]^{-1} \sum_{s \nu} \text{Im} A_{\alpha s \nu, \alpha s \nu}(0) \\ &= (4k_{\alpha})^{-1} \sum_{J s l} 2g_J (1 - \text{Re}(U_{\alpha s l, \alpha s l}^J)). \end{aligned} \quad (4.2)$$

By comparing (4.2) with (3.3), one obtains the expected relation

$$\langle \text{Im} A_{\alpha s \nu, \alpha s \nu}(0) \rangle_{s, \nu} = (k_{\alpha}/4\pi) \sigma_{\alpha}(\text{tot}). \quad (4.3)$$

An inequality relating $\sigma_{\alpha}(\text{tot})$ to the forward elastic differential scattering cross section may be deduced from (4.3). This differential cross section is

$$d\sigma_{\alpha \alpha}(0)/d\Omega_{\alpha} = [(2I_1 + 1)(2I_2 + 1)]^{-1} \times [\sum_{s \nu} |A_{\alpha s \nu, \alpha s \nu}(0)|^2 + \sum_{s' (\neq s) \nu} |A_{\alpha s' \nu, \alpha s \nu}(0)|^2]. \quad (4.4)$$

By means of Schwartz' inequality

$$\left(\sum_{i=1}^N a_i b_i \right)^2 \leq \left(\sum_{i=1}^N a_i^2 \right) \left(\sum_{i=1}^N b_i^2 \right)$$

with $a_i = \text{Im}(A_{\alpha s \nu, \alpha s \nu}(0))$ and $b_i = 1$, it follows that

$$\begin{aligned} [(2I_1 + 1)(2I_2 + 1)]^{-1} \sum_{s \nu} (\text{Im} A_{\alpha s \nu, \alpha s \nu}(0))^2 \\ \geq [(2I_1 + 1)^{-1}(2I_2 + 1)^{-1} \sum_{s \nu} \text{Im} A_{\alpha s \nu, \alpha s \nu}(0)]^2. \end{aligned} \quad (4.5)$$

It is evident then that

$$d\sigma_{\alpha \alpha}(0)/d\Omega_{\alpha} \geq (k_{\alpha} \sigma_{\alpha}(\text{tot})/4\pi)^2. \quad (4.6)$$

This is sometimes referred to as Wick's inequality.⁷¹ It is useful for setting a lower limit to the forward differential elastic scattering cross section from the known value of the total cross section as it is usually difficult, if not impossible, to measure this differential cross section.

IX. FURTHER DEVELOPMENT OF THE RELATION BETWEEN THE U AND R MATRICES: EIGENVALUE EXPANSIONS FOR U

In practical applications of the theory, some approximations have to be made. These are not conveniently introduced into the theory as it stands at present. For this reason we outline some alternative presentations of the theory.

1. Real Eigenvalue Expansion for U

The expression for U so far derived, [Eq. (VII, 1.5)] involves inversion of the matrix $(\mathbf{1} - \mathbf{R}\mathbf{L}^0)$ whose dimensions equal the number of channels. Although this inversion is trivial if there is only one channel (Sec. IV, 1) and relatively easy in the two-channel case (Sec. VII, 3), it is in general impossible to carry out unless some assumptions are introduced. These inversion and expansion difficulties reflect the difficulty of solving many equations in many unknowns as required by the fundamental matrix relation (V, 2.7). It is often convenient to replace the problem of inverting the matrix $(\mathbf{1} - \mathbf{R}\mathbf{L}^0)$ by the equivalent problem of inverting another matrix. As a result of the special form of the \mathbf{R} matrix (V, 2.8), it is possible to transform the matrix (VII, 1.6) for \mathbf{W} from its present form involving the inversion of a "channel" matrix, the components of which refer to channels c , to one involving the inversion of a "level" matrix, the components of which refer to the proper levels λ of the system. In the one-level

⁷¹ G. C. Wick, Phys. Rev. 75, 1459 (1949).

case where only one level is explicitly considered, inversion of the level matrix is trivial no matter how many channels are involved. Even when many levels are involved, some circumstances of practical interest (Sec. XI) permit a diagonal level approximation to be made, and the inversion is then trivial. In the absence of such approximations the level form is not simpler and may actually be more difficult.

a. Inversion of $(\mathbf{1}-\mathbf{R}\mathbf{L}^0)$

The problem is to transform the problem of inverting the channel matrix $(\mathbf{1}-\mathbf{R}\mathbf{L}^0)$ in the expression (VII, 1.6) for \mathbf{W} :

$$\mathbf{W} = \mathbf{1} + \mathfrak{P}^\dagger (\mathbf{1}-\mathbf{R}\mathbf{L}^0)^{-1} \mathbf{R} \mathfrak{P}^\dagger \mathbf{w}$$

into that of inverting a level matrix. We begin by splitting the \mathbf{R} matrix into two parts corresponding to the division of the levels into two groups:

$$\mathbf{R} = \mathbf{R}^0 + \mathbf{R}' \quad (1.1)$$

The matrix product $(\mathbf{1}-\mathbf{R}\mathbf{L}^0)^{-1}\mathbf{R}$ becomes

$$(\mathbf{1}-\mathbf{R}\mathbf{L}^0)^{-1}\mathbf{R} = (\mathbf{1}-\mathbf{R}^0\mathbf{L}^0)^{-1}\mathbf{R}^0 + (\mathbf{1}-\mathbf{R}^0\mathbf{L}^0)^{-1}(\mathbf{1}-\mathbf{R}'\mathbf{L}^0)^{-1}\mathbf{R}'(\mathbf{1}-\mathbf{L}^0\mathbf{R}^0)^{-1}, \quad (1.2)$$

where \mathbf{L}' is defined:

$$\mathbf{L}' = \mathbf{L}^0(\mathbf{1}-\mathbf{R}^0\mathbf{L}^0)^{-1}. \quad (1.3)$$

The reason for making the split (1.1) is that it may be possible to invert $(\mathbf{1}-\mathbf{R}^0\mathbf{L}^0)$ where \mathbf{R}^0 is a part of \mathbf{R} . Then from (1.2) the problem reduces to inverting $(\mathbf{1}-\mathbf{R}'\mathbf{L}^0)$ where \mathbf{R}' is the remaining part of \mathbf{R} .

The inversion procedure is to assume an expansion

$$(\mathbf{1}-\mathbf{R}'\mathbf{L}^0)^{-1} = \mathbf{1} + \sum_{\mu\nu} (\gamma_\mu \times \beta_\nu) A_{\mu\nu}, \quad (1.4)$$

where the vector

$$\beta_\nu = \mathbf{L}'\gamma_\nu = \mathbf{L}^0(\mathbf{1}-\mathbf{R}^0\mathbf{L}^0)^{-1}\gamma_\nu$$

and the level quantities $A_{\mu\nu}$ are presumably to be functions of the energy. If (1.4) is actually permissible, the equations for the $A_{\mu\nu}$ can be found by multiplying both sides by

$$(\mathbf{1}-\mathbf{R}'\mathbf{L}^0) = \mathbf{1} - \sum_\lambda (\gamma_\lambda \times \beta_\lambda) / (E_\lambda - E), \quad (1.5)$$

obtaining

$$\left[\mathbf{1} - \sum_\lambda \frac{(\gamma_\lambda \times \beta_\lambda)}{E_\lambda - E} \right] \left[\mathbf{1} + \sum_{\mu\nu} (\gamma_\mu \times \beta_\nu) A_{\mu\nu} \right] = \mathbf{1} \quad (1.6)$$

or

$$-\sum_\lambda \frac{(\gamma_\lambda \times \beta_\lambda)}{E_\lambda - E} + \sum_{\mu\nu} (\gamma_\mu \times \beta_\nu) A_{\mu\nu} - \sum_{\lambda\mu\nu} \frac{(\gamma_\lambda \times \beta_\nu)}{E_\lambda - E} \xi_{\lambda\mu} A_{\mu\nu} = \mathbf{0} \quad (1.7)$$

where

$$\xi_{\lambda\mu} = (\beta_\lambda, \gamma_\mu) \quad (1.8)$$

is a scalar product with respect to channels. The $\xi_{\lambda\mu}$ originates through the application of the theorem for

column vectors: $(\mathbf{x} \times \mathbf{y})(\mathbf{z} \times \omega) = (\mathbf{y}, \mathbf{z})(\mathbf{x}, \omega)$. According to the theorem $(\mathbf{M}\mathbf{x}, \mathbf{N}\mathbf{y}) = (\mathbf{x}, (\mathbf{M}, \mathbf{N}\mathbf{y}))$ where \mathbf{M} , \mathbf{N} are matrices, the $\xi_{\lambda\mu}$ are symmetric in the $\lambda\mu$ because the matrix \mathbf{L}' is symmetrical. Eq. (1.7) may be rearranged to read

$$\sum_{\lambda\nu} (\gamma_\lambda \times \beta_\nu) \left[-\frac{\delta_{\lambda\nu}}{E_\lambda - E} + A_{\lambda\nu} - \sum_\mu \frac{\xi_{\lambda\mu} A_{\mu\nu}}{E_\lambda - E} \right] = \mathbf{0}, \quad (1.9)$$

which is satisfied if for all λ, ν

$$(E_\lambda - E)A_{\lambda\nu} - \sum_\mu \xi_{\lambda\mu} A_{\mu\nu} = \delta_{\lambda\nu}. \quad (1.10)$$

By introducing the level matrix \mathbf{A} with components $A_{\lambda\nu}$, the symmetrical matrix ξ , the real diagonal matrix, \mathbf{e} with components E_λ , and the unit matrix $\mathbf{1}$ with components $\delta_{\lambda\mu}$, the set of Eqs. (1.10) may be expressed as a single level matrix equation

$$\mathbf{A} = (\mathbf{e} - \mathbf{E} - \xi)^{-1}, \quad (1.11)$$

where the matrix \mathbf{E} is the energy E times the unit matrix. The problem of inverting the channel matrix $(\mathbf{1}-\mathbf{R}'\mathbf{L}^0)$ is thus that of inverting the level matrix $(\mathbf{e} - \mathbf{E} - \xi)$.

The quantity appearing in (1.2) to be evaluated is

$$\begin{aligned} (\mathbf{1}-\mathbf{R}'\mathbf{L}^0)^{-1}\mathbf{R}' &= [\mathbf{1} + \sum_{\mu\nu} (\gamma_\mu \times \beta_\nu) A_{\mu\nu}] [\sum_\lambda (\gamma_\lambda \times \gamma_\lambda) / (E_\lambda - E)] \\ &= \sum_\lambda \frac{(\gamma_\lambda \times \gamma_\lambda)}{E_\lambda - E} + \sum_{\mu\nu\lambda} \frac{(\gamma_\mu \times \gamma_\lambda) \xi_{\nu\lambda} A_{\mu\nu}}{E_\lambda - E}. \end{aligned} \quad (1.12)$$

By considering the symmetry of ξ and \mathbf{A} , and applying (1.10) to the ν sum, (1.12) immediately reduces to the desired result,

$$(\mathbf{1}-\mathbf{R}'\mathbf{L}^0)^{-1}\mathbf{R}' = \sum_{\lambda\mu} (\gamma_\lambda \times \gamma_\mu) A_{\lambda\mu}. \quad (1.13)$$

By substituting this into (1.2), and into (VII, 1.6), the real eigenvalue expansion is obtained for the collision matrix,

$$\mathbf{W} = \mathbf{1} + \mathfrak{P}^\dagger [(\mathbf{1}-\mathbf{R}^0\mathbf{L}^0)^{-1}\mathbf{R}^0 + \sum_{\lambda\mu} (\alpha_\lambda \times \alpha_\mu) A_{\lambda\mu}] \mathfrak{P}^\dagger \mathbf{w}, \quad (1.14)$$

where

$$\alpha_\lambda = (\mathbf{1}-\mathbf{R}^0\mathbf{L}^0)^{-1}\gamma_\lambda = (\mathbf{L}^0)^{-1}\beta_\lambda. \quad (1.15)$$

It is convenient to separate the matrix ξ into real and imaginary parts,

$$\xi = -\mathbf{\Delta} + \frac{1}{2}i\mathbf{\Gamma}, \quad (1.16)$$

the matrix $\mathbf{\Delta}$ of the real part being referred to as the *shift matrix* and the $\mathbf{\Gamma}$ of the imaginary part as the *width matrix*. They may be determined as follows: According to (1.15),

$$\gamma_\lambda = (\mathbf{1}-\mathbf{R}^0\mathbf{L}^0)\alpha_\lambda,$$

and, since the γ_λ are real,

$$\gamma_\lambda = (\mathbf{1}-\mathbf{R}^0\mathbf{L}^{0*})\alpha_\lambda^*.$$

Therefore,

$$\begin{aligned} \xi_{\lambda\mu} &= (\gamma_\lambda, \beta_\mu) = ((1 - \mathbf{R}^0 \mathbf{L}^{0*}) \alpha_\lambda^*, \mathbf{L}^0 \alpha_\mu) \\ &= (\alpha_\lambda^*, \mathbf{L}^0 \alpha_\mu) - (\mathbf{R}^0 \mathbf{L}^{0*} \alpha_\lambda^*, \mathbf{L}^0 \alpha_\mu) \\ &= (\alpha_\lambda^*, \mathbf{L}^0 \alpha_\mu) - (\alpha_\lambda^*, \mathbf{L}^{0*} \mathbf{R}^0 \mathbf{L}^0 \alpha_\mu) \end{aligned} \quad (1.17)$$

using the theorem mentioned below (1.8) and the fact that \mathbf{R}^0 is symmetrical. The second term of (1.17) is real so, substituting $\mathbf{L}^0 = \mathbf{S}^0 + i\mathbf{P}$ of (III, 4.4) in the first where $\mathbf{L}^0 = \mathbf{L} - \mathbf{B}$, $\mathbf{S}^0 = \mathbf{S} - \mathbf{B}$, one has

$$\frac{1}{2} \Gamma_{\lambda\mu} = (\alpha_\lambda^*, \mathbf{P} \alpha_\mu) = \frac{1}{2} \sum_c \Gamma_{\lambda\mu, c} \quad (1.18a)$$

$$\Gamma_{\lambda\mu, c} = 2P_c \alpha_{\lambda c}^* \alpha_{\mu c} \quad (1.18b)$$

$$\Delta_{\lambda\mu} = (\alpha_\lambda^*, (\mathbf{L}^{0*} \mathbf{R}^0 \mathbf{L}^0 - \mathbf{S}^0) \alpha_\mu). \quad (1.18c)$$

If \mathbf{R}^0 is diagonal, these simplify to

$$\frac{1}{2} \Gamma_{\lambda\mu} = \frac{1}{2} \sum_c \Gamma_{\lambda\mu, c} \quad (1.19a)$$

with

$$\Gamma_{\lambda\mu, c} = 2P'_c \gamma_{\lambda c} \gamma_{\mu c}$$

and

$$\Delta_{\lambda\mu} = \sum_c \Delta_{\lambda\mu, c} \quad (1.19b)$$

with

$$\Delta_{\lambda\mu, c} = -S'_c \gamma_{\lambda c} \gamma_{\mu c},$$

where S' and P' are defined by $L' = S' + iP'$. Although the total width matrix components of both (1.18a) and (1.19a) are expressed as the sum of partial widths, only in the case of diagonal \mathbf{R}^0 of (1.19b) can the total shift matrix components be expressed as the sum of partial level shifts.

b. One-, Two-, and Three-Level Expansions

Inversion of (1.11) can be carried out without difficulty by means of the usual procedures in the cases where one, two, and three levels are assigned to the \mathbf{R}' matrix of (VII, 1.6). For this purpose, we introduce the diagonal level matrix

$$\epsilon = \mathbf{e} - \mathbf{E} - \xi + \xi', \quad (1.20)$$

where ξ' is the nondiagonal part of ξ so that $\xi - \xi'$ is the diagonal part. The results are:

One level

$$A = 1/\epsilon. \quad (1.21)$$

Two levels (1 and 2)

$$\begin{aligned} A_{11} &= \epsilon_2/D, & A_{22} &= \epsilon_1/D, \\ A_{12} &= A_{21} = \xi_{12}/D, \\ D &= \epsilon_1 \epsilon_2 - \xi_{12}^2. \end{aligned} \quad (1.22)$$

Three levels (1, 2, and 3)

$$\begin{aligned} DA_{11} &= \epsilon_2 \epsilon_3 - \xi_{23}^2, \quad \text{etc.} \\ DA_{12} &= -\xi_{12} \epsilon_3 - \xi_{13} \xi_{23}, \quad \text{etc.}, \\ D &= \epsilon_1 \epsilon_2 \epsilon_3 - \epsilon_1 \xi_{23}^2 - \epsilon_2 \xi_{13}^2 - \epsilon_3 \xi_{12}^2 - 2\xi_{12} \xi_{13} \xi_{23}. \end{aligned} \quad (1.23)$$

c. Probability Integral for the Compound System

With the expansion (1.4) it is possible to give a simple expression for the absolute square integral (V, 2.9b) of the wave function in the internal region when there is a unit flux incoming wave in only a single channel. If this channel is designated as e (for entrance), the coefficients of the derivative quantities of (VII, 1.2b) which are involved in (V, 2.9b) are

$$y_e = \delta_{ce}, \quad x_e = -U_{ce}. \quad (1.24)$$

By substituting these in (VII, 1.2b), and using the collision matrix expression (VII, 1.6b) with $w_e = 2i$, one obtains

$$\mathbf{V} = -i(2\hbar)^{\frac{1}{2}} (\mathbf{1} - \mathbf{R}\mathbf{L}^0)^{-1} \mathbf{R} \mathfrak{P}^{\frac{1}{2}} \Omega, \quad (1.25a)$$

$$\mathbf{D}^0 = -i(2\hbar)^{\frac{1}{2}} [\mathbf{1} + \mathbf{L}^0 (\mathbf{1} - \mathbf{R}\mathbf{L}^0)^{-1} \mathbf{R}] \mathfrak{P}^{\frac{1}{2}} \Omega. \quad (1.25b)$$

The \mathbf{V} and \mathbf{D}^0 quantities are now considered as matrices, rather than as column vectors, with each column referring to a different entrance channel.

Upon substituting the expansions (1.2) and (1.4) into (1.25b), the derivative matrix becomes

$$\mathbf{D}^0 = -i(2\hbar)^{\frac{1}{2}} [(\mathbf{1} - \mathbf{L}^0 \mathbf{R}^0)^{-1} + \sum_{\nu\mu} A_{\nu\mu} (\beta_\nu \times \alpha_\mu)] \mathfrak{P}^{\frac{1}{2}} \Omega, \quad (1.26)$$

the identity $[\mathbf{1} + \mathbf{L}^0 (\mathbf{1} - \mathbf{R}^0 \mathbf{L}^0)^{-1} \mathbf{R}^0] = (\mathbf{1} - \mathbf{L}^0 \mathbf{R}^0)^{-1}$ having been used. By considering (1.10) which is satisfied by the components of the \mathbf{A} matrix and the matrix theorems mentioned in subsection 1a, the scalar product of the vector γ_λ and the e th column of the derivative matrix is found to be

$$(\gamma_\lambda, \mathbf{D}^0 e) = -i(2\hbar)^{\frac{1}{2}} (E_\lambda - E) \sum_\mu A_{\lambda\mu} \alpha_{\mu e} \mathfrak{P}^{\frac{1}{2}} \Omega_e. \quad (1.27)$$

In view of this relation and the symmetry of the \mathbf{A} matrix, and if we consider \mathbf{R}^0 as energy independent, the absolute square integral of (V, 2.9b) may be written as the trace (Tr) of a level matrix,

$$\begin{aligned} \int |\Psi_e|^2 d\tau &= \hbar \text{Tr} (\mathbf{A} \mathbf{A}^* \Gamma_e) \\ &= \hbar \text{Tr} \left[\frac{\Gamma_e}{[(\mathbf{e} + \mathbf{\Delta} - \mathbf{E})^2 + \frac{1}{4} \mathbf{\Gamma}^2 + \frac{1}{2} i \mathbf{\Gamma} (\mathbf{e} + \mathbf{\Delta}) - \frac{1}{2} i (\mathbf{e} + \mathbf{\Delta}) \mathbf{\Gamma}]} \right], \end{aligned} \quad (1.28)$$

the components of the level matrices $\mathbf{\Gamma}$, $\mathbf{\Delta}$, and $\mathbf{\Gamma}_e$ being given by (1.18a), (1.18c), and (1.18b), respectively. For the one-level expansion, this integral is particularly simple,

$$\int |\Psi_e|^2 d\tau = \hbar \Gamma_{\lambda\lambda, e} / [(E_\lambda + \Delta_{\lambda\lambda} - E)^2 + \frac{1}{4} \Gamma_{\lambda\lambda}^2], \quad (1.29)$$

which is valid with the contribution \mathbf{R}^0 from the other levels included, provided it can be considered as constant.

An interesting consequence of (1.27) that will be needed later is the expansion of the wave function Ψ_e

for unit incoming flux in any given channel in the internal region. From (V, 2.3 and 5) and (1.27) it follows that

$$\Psi_e = -i(2\hbar)^{\frac{1}{2}} \sum_{\lambda\mu} (A_{\lambda\mu} \alpha_{\mu e} P_e^{\frac{1}{2}} \Omega_e) X_\lambda. \quad (1.30)$$

If we set $\mathbf{R}^0 = \mathbf{0}$ so $\mathbf{R}' = \mathbf{R}$ and $\alpha_{\mu e} = \gamma_{\mu e}$, the expansion becomes

$$\begin{aligned} \Psi_e &= -i(2\hbar)^{\frac{1}{2}} \sum_{\lambda\mu} (A_{\lambda\mu} \gamma_{\mu e} P_e^{\frac{1}{2}} \Omega_e) X_\lambda \\ &= -i\hbar^{\frac{1}{2}} \Omega_e \sum_{\lambda\mu} (A_{\lambda\mu} \Gamma_{\mu e}^{\frac{1}{2}}) X_\lambda. \end{aligned} \quad (1.31)$$

The corresponding elements of the collision matrix are found by evaluating the scalar product of Ψ_e with ψ_c at infinity:

$$U_{ce} \equiv U_{ec} = i\Omega_e \Omega_c \sum_{\lambda\mu} A_{\lambda\mu} \Gamma_{\mu e}^{\frac{1}{2}} \Gamma_{\lambda c}^{\frac{1}{2}}. \quad (1.32)$$

This agrees with (1.14).

2. Complex Eigenvalue Expansion for U

The expansion (1.14) for the collision matrix is formally expressed in terms of the *real* eigenvalues E_λ , reduced width amplitudes $\gamma_{\lambda c}$, and boundary conditions B_c , as well as the real parameters S_c , P_c , ϕ_c characterizing the external functions on the surface \mathcal{S} . The expansion involves a *double* sum. It is also possible to make a *single* sum, level expansion of the collision matrix. The E_λ and $\gamma_{\lambda c}$ of this expansion are, however, no longer real quantities, and this is the price that is paid for having a single sum instead of a double one.

We now describe a procedure for the single sum expansion which starts with the results involving the real E_λ and $\gamma_{\lambda c}$. Then we describe a simpler and more direct derivation that introduces at the outset the complex eigenvalues and complex eigenfunctions. This procedure was adopted by Kapur and Peierls⁹ who gave the first rigorous derivation of the general resonance formulas. Both of these derivations give a single sum eigenvalue expansion for U with eigenvalues H_λ and reduced width amplitudes $\omega_{\lambda c}$ that depend on energy E . This dependence is unfortunate from some points of view and, in subsection 2c, we describe an expansion with energy independent parameters that was introduced by Siegert⁷² and developed by Humblet.⁷³

a. Indirect Derivation of the Kapur-Peierls Expansion from the Real Eigenvalue Expansion

It is well known that, if a symmetric matrix like $(\mathbf{e} - \xi)$ has no double characteristic values, the matrix \mathbf{T} which transforms it to diagonal form \mathbf{H} is complex orthogonal, so that

$$\mathbf{H} = \mathbf{F} - \frac{1}{2}i\mathbf{\Gamma} = \mathbf{T}(\mathbf{e} - \xi)\mathbf{T}^{-1} = \mathbf{T}(\mathbf{e} - \xi)\mathbf{T}^{\dagger*}, \quad (2.1)$$

where \mathbf{F} and $\mathbf{\Gamma}$ are real diagonal matrices and where $\mathbf{T}^{\dagger*}$ is the transpose of \mathbf{T} . Therefore,

$$\mathbf{A} = \mathbf{T}^{\dagger*}(\mathbf{H} - \mathbf{E})^{-1}\mathbf{T} \quad (2.2)$$

and from (1.13):

$$(\mathbf{1} - \mathbf{R}'\mathbf{L}')^{-1}\mathbf{R}' = \sum_{\lambda} (\omega_{\lambda} \times \omega_{\lambda}) / (H_{\lambda} - E) \quad (2.3)$$

where

$$\omega_{\lambda c} = \sum_{\nu} T_{\lambda\nu} \gamma_{\nu c}, \quad (2.4)$$

the $\omega_{\lambda c}$ being complex. By substituting (2.3) into (1.2), and (1.2) into (VII, 1.6b), using the theorem mentioned below (1.15), one

⁷² A. J. F. Siegert, Phys. Rev. **56**, 750 (1939).

⁷³ J. Humblet, thesis, Roy. soc. des sci. de Liège Ser. 4, **7**, No. 4 (1952).

obtains for the collision matrix the single sum expansion:

$$\mathbf{W} = \mathbf{1} + \mathfrak{P}^{\frac{1}{2}} \left[(\mathbf{1} - \mathbf{R}^0 \mathbf{L}^0)^{-1} \mathbf{R}^0 + \sum_{\lambda} \frac{(\theta_{\lambda} \times \theta_{\lambda})}{F_{\lambda} - E - \frac{1}{2}i\Gamma_{\lambda}} \right] \mathfrak{P}^{\frac{1}{2}} \mathbf{w} \quad (2.5)$$

where

$$\theta_{\lambda} = (\mathbf{1} - \mathbf{R}^0 \mathbf{L}^0)^{-1} \omega_{\lambda} = \sum_{\nu} T_{\lambda\nu} \alpha_{\nu}. \quad (2.6)$$

Since the \mathbf{L}^0 are energy dependent, the $T_{\lambda\nu}$ and F_{λ} as well as the θ_{λ} will be energy dependent.

If the sum in (2.5) were a normal resonance sum, the imaginary part of H_{λ} , i.e., $-\frac{1}{2}i\Gamma_{\lambda}$, would be related to the quantities $(2P)^{\frac{1}{2}}\theta_{\lambda c}$ in the numerators by: $\frac{1}{2}i\Gamma_{\lambda} = \sum_c P_c |\theta_{\lambda c}|^2$. It is of interest to see what the actual relation is. From (2.1) it follows that

$$\mathbf{T} \mathbf{e} \mathbf{T}^{\dagger} - \mathbf{T} \xi \mathbf{T}^{\dagger} = \mathbf{H} \mathbf{T} \mathbf{T}^{\dagger}, \quad (2.7a)$$

the complex conjugate of which is

$$\mathbf{T}^* \mathbf{e} \mathbf{T}^{\dagger*} - \mathbf{T}^* \xi^* \mathbf{T}^{\dagger*} = \mathbf{H}^* \mathbf{T}^* \mathbf{T}^{\dagger*}. \quad (2.7b)$$

The diagonal components of the first term on the left-hand and the right-hand side of (2.7a) are, respectively,

$$(\mathbf{T} \mathbf{e} \mathbf{T}^{\dagger})_{\lambda\lambda} = \sum_{\nu} E_{\nu} |T_{\lambda\nu}|^2, \quad (2.8)$$

$$(\mathbf{H} \mathbf{T} \mathbf{T}^{\dagger})_{\lambda\lambda} = (F_{\lambda} - \frac{1}{2}i\Gamma_{\lambda}) N_{\lambda}, \quad (2.9)$$

where

$$N_{\lambda} = \sum_{\nu} |T_{\lambda\nu}|^2. \quad (2.10)$$

By means of (1.17) and (2.6), the diagonal components of the second term of the left-hand side of (2.7a) is found to be

$$(\mathbf{T} \xi \mathbf{T}^{\dagger})_{\lambda\lambda} = i \sum_c P_c |\theta_{\lambda c}|^2 - (\theta_{\lambda}^*, (\mathbf{L}^0 \mathbf{R}^0 \mathbf{L}^0 - \mathbf{S}^0) \theta_{\lambda}), \quad (2.11)$$

the sum over channels and the scalar product being both real. The difference between (2.7a) and (2.7b) gives therefore

$$\frac{1}{2}i\Gamma_{\lambda} = N_{\lambda}^{-1} \sum_c P_c |\theta_{\lambda c}|^2, \quad (2.12)$$

while the sum corresponds to

$$F_{\lambda} = N_{\lambda}^{-1} [\sum_{\nu} E_{\nu} |T_{\lambda\nu}|^2 + (\theta_{\lambda}^*, (\mathbf{L}^0 \mathbf{R}^0 \mathbf{L}^0 - \mathbf{S}^0) \theta_{\lambda})]. \quad (2.13)$$

Since

$$(\mathbf{T} \mathbf{T}^{\dagger})_{\lambda\lambda} = \sum_{\nu} T_{\lambda\nu}^2 = 1,$$

and since \mathbf{T} is in general complex, it is evident that $N_{\lambda} \geq 1$. From (2.12), it follows that the imaginary parts $\frac{1}{2}i\Gamma_{\lambda}$ of the eigenvalues H_{λ} satisfy the inequalities

$$\Gamma_{\lambda} \leq \sum_c \Gamma_{\lambda c}, \quad (2.14)$$

where

$$\Gamma_{\lambda c} = 2P_c |\theta_{\lambda c}|^2.$$

Since the $\Gamma_{\lambda c}$ are equal to the absolute squares of the vectors $(2P_c)^{\frac{1}{2}}\theta_{\lambda c}$ of the numerators of (2.5) (the factor $2^{\frac{1}{2}}$ arising from the Wronskian), the above inequality asserts that the total widths of the denominators are less than or equal to the sums of the partial widths of the numerators.

b. Direct Derivation of the Kapur-Peierls Expansion

These results actually constitute a generalization of the Kapur-Peierls theory in the sense that an arbitrary part of the \mathbf{R} matrix (*viz.* \mathbf{R}^0) is not transformed. To make a comparison with the Kapur-Peierls results, we put $\mathbf{R}^0 = \mathbf{0}$. This immediately implies $\mathbf{R}' = \mathbf{R}$, $\mathbf{L}' = \mathbf{L}^0$, and $\omega_{\lambda} = \theta_{\lambda}$. The collision matrix of (2.5), then becomes

$$\mathbf{W} = \mathbf{1} + \mathfrak{P}^{\frac{1}{2}} \left[\sum_{\lambda} \frac{\omega_{\lambda} \times \omega_{\lambda}}{H_{\lambda} - E} \right] \mathfrak{P}^{\frac{1}{2}} \mathbf{w}. \quad (2.15)$$

We now establish the plausible fact that the same expression for the collision matrix results from carrying through the \mathbf{R} -matrix theory using complex eigenfunctions and eigenvalues from the outset. This procedure is the one used by Kapur and Peierls.⁹ The $\omega_{\lambda c}$ and H_{λ} of (2.3) correspond to the reduced width amplitudes $\gamma_{\lambda c}$ and eigenvalues E_{λ} of and \mathbf{R} -matrix theory with complex states. Kapur and Peierls introduced complex X_{λ} , E_{λ} , and solutions X_{λ}^* (more precisely, the time reversed solutions KX_{λ}) belonging to the eigenvalue E_{λ}^* . These solutions have boundary conditions B_c and B_c^* , respectively, and their reduced widths are complex. The Green's-theorem relation when applied to X_{λ}

and its complex conjugate gives then an orthonormality condition which is $\int X_\lambda X_\lambda d\tau = \delta_{\lambda\lambda'}$, rather than the condition $\int X_\lambda^* X_\lambda d\tau = \delta_{\lambda\lambda'}$ of (V, 2.2). An application of a Green's-theorem relation with $\Psi_1 = \Psi_2 = \Psi$ shows that the imaginary parts Γ_λ of the H_λ satisfy a condition of the form (2.12), in which the θ_λ are the complex γ_λ and with $N_\lambda = \int |X_\lambda|^2 d\tau$. Since $\int |X_\lambda|^2 d\tau \geq \int X_\lambda^2 d\tau$ the inequality (2.14) is satisfied.

Kapur and Peierls choose the complex boundary conditions parameters B_c to be the energy-dependent logarithmic derivatives L_c corresponding to outgoing waves. As a result, the $L^0 (= L - B)$ matrix of (1.2) and (1.4) vanishes identically and the double sum of (1.4), as well as the ξ matrix of (1.11), also vanishes. The A matrix is therefore diagonal and the collision matrix of (1.14) reduces to

$$W = 1 + \mathfrak{P} \left[\sum_\lambda \frac{\gamma_\lambda \times \gamma_\lambda}{E_\lambda - E} \right] \mathfrak{P}^\dagger w. \quad (2.15')$$

Comparison with (2.15) immediately confirms the expected interpretation of the $\omega_{\lambda c}$ and H_λ as the reduced width amplitudes and eigenvalues of an R -matrix theory with complex states.

Although the expansion (2.15') is much simpler than (1.14), the difficulty of the matrix inversion of (1.11), or of (1.12), is now replaced by the problem of handling the $\gamma_{\lambda c}$ and E_λ with their implicit energy dependence and their complex nature. Any attempt to remove the energy dependences and make the $\gamma_{\lambda c}$ and E_λ real leads back to the real eigenvalue expansion and the difficulty of matrix inversion. For these reasons the real eigenvalue expansion is preferred, and used as the basis of this review.

c. Expansion of Siegert and Humlet

A feature of the expansion of Kapur and Peierls is that the width amplitudes $\omega_{\lambda c}$ and the eigenvalues H_λ are energy dependent, as seen by inspection of either of the direct (2b) or indirect (2a) derivations. In the former case at a given energy E , the complete set of states X_λ is defined as satisfying the boundary conditions for outgoing waves in all channels at the energy E (not at the state energies H_λ). Thus at two different energies E_1 and E_2 , the sets of values $\omega_{\lambda c}$ and H_λ are different. Except in certain limiting physical situations such as that of discrete resonances, the dependence of $\omega_{\lambda c}$ and H_λ upon energy is not at all simple. One way of avoiding this dependence is to use real energy-independent boundary conditions (as done here following Wigner and Eisenbud).¹² Another way was suggested by Siegert⁷² and investigated by Humlet.⁷³ According to Siegert one should work with a set of states X_λ which have outgoing waves in all channels, i.e., the boundary conditions are those of outgoing waves at the state energies H_λ , not at some prescribed energy E . These are usually called "radioactive states." These states seem to combine the best of two worlds that they lead to a single sum expansion for U and, at the same time, the parameters $\omega_{\lambda c}$ and H_λ are energy-independent. However, these states have certain disadvantages. They are complex and so the $\omega_{\lambda c}$ and H_λ are complex. Furthermore they are not orthogonal in the internal region, and so the usual R -matrix method for deriving the U matrix is not applicable. Nevertheless, one can use the fact that the energies H_λ must be poles of the collision matrix to develop a Mittag-Leffler series for the latter. Alternatively we may start by using the results of the subsection 1. Adopting the latter procedure, we assert that, in the neighborhood of some real energy E of interest, the matrix $(1 - R'L)^{-1}R'$ of (1.2) may be assumed to have an expansion of the form

$$(1 - R'L)^{-1}R' = \sum_\lambda (\omega_\lambda \times \omega_\lambda) / (H_\lambda - E) \quad (2.16)$$

in terms of complex eigenvalues H_λ and complex "reduced widths" ω_λ . By multiplying (2.16) on the left by the matrix $(1 - R'L)$, it becomes

$$R' = \sum_\lambda ((1 - R'L)\omega_\lambda \times \omega_\lambda) / (H_\lambda - E). \quad (2.17)$$

At the complex poles H_λ , the left side is bounded so that the set of linear homogeneous equations for the ω_λ ,

$$(1 - R'(H_\lambda)L)\omega_\lambda = 0 \quad (2.18)$$

will have nontrivial solutions, the H_λ being the roots of the associated determination equation

$$|1 - R'(H_\lambda)L| = 0. \quad (2.19)$$

As the L' in (2.18) and (2.19) are generally energy-dependent, they must be assumed to be analytically continuable from E to the complex H_λ where they are to be evaluated (this assumption will be qualified in subsection 2f).

The evaluation of (2.18) in the case where R' has only one level is of particular interest. By writing it as

$$[(H_\lambda - E_\lambda) + (\gamma_\lambda \times \beta_\lambda)]\omega_\lambda = 0, \quad (2.20)$$

it is evident that it is satisfied with

$$\gamma_\lambda = \omega_\lambda, \quad (2.21a)$$

$$H_\lambda = E_\lambda - \xi_{\lambda\lambda}. \quad (2.21b)$$

By expanding the determinant corresponding to (2.20) in descending powers of $(H_\lambda - E_\lambda)$, the coefficients of which are the ascending orders of the principal determinants of the rank-one matrix $(\gamma_\lambda \times \beta_\lambda)$, it is readily verified that (2.19) is also satisfied because all of these determinants of order greater than one vanish.

d. Poles of the Siegert-Humlet Expansion

As described in Sec. IV, 8, the work of Schuster and Tiomno⁸⁰ (and later Van Kampen⁸⁷⁻⁸⁹) has shown by imposition of a "causality" condition that the poles of the (one channel) collision function $U(k)$, where k is the complex wave number, must lie either on the imaginary k axis or in the lower half-plane. Those on the positive imaginary axis correspond to the bound states. In view of the general symmetry condition $U(k) = U(-k)^*$ described in Sec. IV, 8, those in the lower half-plane occur in pairs symmetrically situated on each side of the negative imaginary axis; they correspond to the "resonance" or "radioactively decaying" states of the system. In the many-channel case also the poles of the collision matrix of the R -matrix theory lie either on the imaginary k axis or in the lower half-plane. This was first demonstrated by Moshinsky.⁷⁴

In discussing the analytical continuation of the collision function $U(k)$ or $U(E)$ it is necessary to consider that $U(E)$ is a two-valued function because in general $U(k) \neq U(-k)$. Two energy planes, or Riemannian sheets, are employed, the first corresponding to the upper-half k plane and the second to the lower-half k plane. In discussing the analytical properties of the collision matrix there is the additional complication that a separate k plane for each channel is involved. For simplicity the channels may be enumerated according to their binding energies b_c and the energy scale referred to that of the first channel. The relations between the complex E and k variables are then

$$E = (\hbar^2/2M_1)k_1^2 = (\hbar^2/2Mc)k_c^2 + b_c$$

with $0 = b_1 < b_2 < \dots < b_c \dots$.

To show that the poles of the collision matrix (i.e., the poles H_λ of the expansion (2.16) with $R^0 = 0$, $R' = R$, $L' = L^0$) satisfy the "causality" requirement, one proceeds by forming the scalar product of $(L^0\omega_\lambda)^*$ with expression (2.18) for the homogeneous system of equations satisfied by the ω_λ at H_λ , obtaining

$$\begin{aligned} (\omega_\lambda^*, L^0\omega_\lambda) &= (\omega_\lambda^*, L^0 R L^0 \omega_\lambda) \\ &= \sum_\lambda (E_\lambda - H_\lambda)^{-1} |\sum_c \omega_{\lambda c} L_c^0 \gamma_{\lambda c}|^2. \end{aligned} \quad (2.22)$$

Both sides of (2.22) are multiplied by the imaginary part of H_λ ($\text{Im}H_\lambda$) and the signs (Sgn) of the resulting equations are considered. The sign of the right side of (2.22) is equal to $\text{Sgn}(\text{Im}H_\lambda)$ so that

$$\text{Sgn}(\text{Im}H_\lambda) \cdot \text{Sgn}(\text{Im}(\omega_\lambda^*, L^0\omega_\lambda)) = (\text{Sgn}(\text{Im}H_\lambda))^2 = +. \quad (2.23)$$

If the components $L_c^0 = S_c^0 - iP_c$ of the L^0 matrix are considered as constants everywhere, then it is evident that $\text{Sgn}(\text{Im}H_\lambda) = -$

⁷⁴M. Moshinsky, Phys. Rev. **91**, 984 (1953); Acad. Bras. de cien **25**, No. 4 (1953).

or that $\text{Im}(H_\lambda) = 0$ since $\text{Sgn}(P_c) = +$ for positive energy channels and $P_c = 0$ for negative energy channels. In this approximation the "causality" condition is satisfied, and the poles are either all on the imaginary k_1 axis or all in the lower-half k_1 plane.

Proceeding without approximation, but neglecting the possibility of a Coulomb field, the Green's-theorem relation (V, 1.4) is applied to an outgoing wave $O_c(k_c)$ for the complex k_c corresponding to an assumed pole at H_λ and to the complex conjugate of $O_c(k_c)$,

$$[O^*(dO/dr) - O(dO^*/dr)]_a^r + (k^2 - k^{*2}) \int_a^r |O|^2 dr = 0. \quad (2.24)$$

Since $O_c \sim \exp i(kr - \frac{1}{2}\pi)$ when r is large, it is evident that if $\text{Im} k > 0$ then $|O(r)|^2 \rightarrow 0$ and the integral converges as $r \rightarrow \infty$, in which case

$$\text{Im} L^0(a) = (\partial M a / \hbar^2) \text{Im} H_\lambda |O(a)|^{-2} \int_a^\infty |O|^2 dr$$

and

$$\text{Sgn}(\text{Im} L^{0*}) = -\text{Sgn} \text{Im} H_\lambda. \quad (2.25)$$

The sign equation (2.23) is therefore not satisfied for H_λ corresponding to $\text{Im} k_1 > 0$ (and likewise to $\text{Im} k_c > 0$) with the exception of H_λ corresponding to points on the positive imaginary k_1 axis where $\text{Im} H_\lambda = 0$. The poles are therefore confined to this semiaxis (bound states) and to the lower-half k_1 plane (radioactive states).

e. Reduced Widths of the Siegert-Humblert Expansion

Still considering the case $\mathbf{R}^0 = 0$ (i.e., $\mathbf{R}' = \mathbf{R}$, $\mathbf{L}' = \mathbf{L}^0$), we now wish to give an interpretation of the quantities $\omega_{\lambda c}$ to parallel that given for the poles H_λ . Equation (2.18) for the $\omega_{\lambda c}$ does not specify any normalization of the $\omega_{\lambda c}$. On multiplying both sides of (2.17) on the left with $(\omega_\lambda \times \omega_\lambda) \mathbf{L}^0$, we have

$$\begin{aligned} (\omega_\lambda \times \omega_\lambda) [(\mathbf{L}' \mathbf{R}' - 1) + 1] \\ = \sum_\mu (\omega_\lambda \times \omega_\lambda) ((\mathbf{L}' - \mathbf{L}' \mathbf{R}' \mathbf{L}') \omega_\mu \times \omega_\mu) / (H_\mu - E) \\ = \sum_\mu (\mathbf{L}' - \mathbf{L}' \mathbf{R}' \mathbf{L}') \omega_\lambda \cdot \omega_\mu (\omega_\lambda \times \omega_\mu) / (H_\mu - E) \end{aligned} \quad (2.26)$$

after using the theorems mentioned below (1.8) and (1.15), and the symmetry property of \mathbf{R}' . In the vicinity of H_λ one can argue, using (2.18), that

$$(\omega_\lambda \times \omega_\lambda) (\mathbf{L}' \mathbf{R}' (H_\lambda) - 1) = (\omega_\lambda \times (\mathbf{R}' (H_\lambda) \mathbf{L}' - 1) \omega_\lambda) = 0, \quad (2.27)$$

and (2.26) therefore reduces to

$$1 = (\omega_\lambda \cdot (\mathbf{L}' - \mathbf{L}' \mathbf{R}' (H_\lambda) \mathbf{L}') \omega_\lambda) / (H_\lambda - E). \quad (2.28)$$

The normalization is thus obtained from the derivative of the matrix in the numerator of (2.28):

$$1 = \left[\omega_\lambda \cdot \left(-\frac{d}{dE} (\mathbf{L}') + \mathbf{L}' \mathbf{R}' \frac{d}{dE} (\mathbf{L}') + \frac{d}{dE} (\mathbf{L}') \mathbf{R}' \mathbf{L}' + \mathbf{L}' \frac{d}{dE} (\mathbf{R}' \mathbf{L}') \right) \omega_\lambda \right]. \quad (2.29)$$

\mathbf{R}' , \mathbf{L}' , and $(d/dE) (\mathbf{L}')$ being evaluated at H_λ . By applying (2.18) once again, the above condition simplifies to

$$\left[\omega_\lambda \cdot \left(\frac{d}{dE} (\mathbf{L}') + \mathbf{L}' \frac{d}{dE} (\mathbf{R}' (H_\lambda) \mathbf{L}') \right) \omega_\lambda \right] = 1. \quad (2.30)$$

With this normalization condition on the $\omega_{\lambda c}$ we may now interpret these quantities. As mentioned in the last subsection, the poles of the collision matrix correspond to radioactive states, i.e., states Ψ with outgoing waves in all channels. For such a state, we have an equation like (V, 2.9b) but with Ψ^2 instead of $|\Psi|^2$ thus:

$$\int \Psi^2 d\tau = \left(\mathbf{D}^0 \cdot \frac{d}{dE} (\mathbf{R} (H_\lambda)) \mathbf{D}^0 \right),$$

where D_c^0 are the quantities defined for (V, 2.6) as $D_c - B_c V_c$, V_c and D_c being the value and derivative quantities for the state Ψ . Since L_c is the logarithmic derivative for outgoing waves, it equals D_c/V_c when evaluated at the pole energy H_λ . It follows that $D_c^0 = L_c^0 V_c$, so by putting $V_c = \omega_{\lambda c}$, the above volume

integral reduces to the second term of the normalization condition (2.30). Ignoring the other term in (2.30) for the moment, we can interpret the $\omega_{\lambda c}$ as the surface values $V_{\lambda c}$ (which are equal to the reduced width amplitudes $\gamma_{\lambda c}$) for the wave function Ψ . The first term in (2.30) simply corresponds to extending the usual normalization of states over the internal region to normalization over all space including the channels. This is evident from the special case of the Green's-theorem relation with squared, rather than absolute-squared, wave functions. The integration is extended from the radius a to a large r , and there are consequently two terms for (V, 1.5), one from the "surface" at a and the other from r . For positive-energy channels and wave functions of the form $O \sim G + iF$, although the integral does not converge, it is effectively "cutoff" by the surface contribution at r . If the Coulomb asymptotic phase $\eta \log 2\rho$ of O is ignored and if φ represents the remaining constant phase, then by adding and subtracting

$$\int_a^r e^{2i(kr' + \varphi)} dr' = \frac{1}{2ik} [e^{2i(kr + \varphi)} - e^{2i(ka + \varphi)}]$$

the individual channel contributions to (2.30) may be written as

$$\omega_{\lambda c}^2 \frac{d}{dE} (L^0) = \frac{k a \omega_{\lambda c}^2}{O^2(a) H_\lambda} \left[\int_{ka}^\infty \{O^2(r) - e^{2i(\rho + \varphi)}\} d\rho + \frac{1}{2} i e^{2i(ka + \varphi)} \right] \quad (2.31)$$

the convergence of which is evident since, as $r \rightarrow \infty$, $G + iF \rightarrow \exp 2i(\rho + \varphi)$.

f. Relations between the Expansions of Kapur-Peierls and Siegert-Humblert

By substituting (2.16) into (1.2), and (1.2) in turn into (VII, 1.6), an expansion is obtained for the collision matrix which is identical in form to the Kapur-Peierls expansion (2.5). These respective expansions are, however, not the same because the $\omega_{\lambda c}$ and H_λ of (2.5) differ from those of (2.16). In the former case, on account of the energy dependence of the matrix ξ in (2.1), it is necessary to perform a diagonalization at every energy and, as a result, the $\omega_{\lambda c}$ of (2.4), and the H_λ of (2.1) will be energy-dependent. On the other hand, the H_λ of (2.10) and $\omega_{\lambda c}$ of (2.18) are energy-independent, at least if the \mathbf{L}' matrix is considered as analytic. [Actually this matrix is not wholly analytic because of the existence of branch points at the thresholds where the negative-energy channels become positive-energy channels. Nevertheless the expansion (2.16) should be valid over a limited energy range where the \mathbf{L}' can be considered as analytic; it is certainly valid to the extent that the components of \mathbf{L}' can be considered as constants plus linear functions of the energy.]

If the dependence of \mathbf{L}' on E in the expansion (2.16) is considered as a parametric one, rather than as a complex-variable one, then the expansion parameters $\omega_{\lambda c}$, H_λ of (2.3) and (2.16) must, of course, be the same. This parametric dependence on the real energy E means that at each real energy E a separate determination is made of the $\omega_{\lambda c}$, H_λ by means of (2.18), (2.19), (2.20), and that the $\omega_{\lambda c}$, H_λ are therefore functions of the real parameter E . It is worthwhile to demonstrate this equivalence in detail and to obtain an expression for the components of the complex-orthogonal matrix \mathbf{T} .

By comparing (2.18)

$$\omega_\mu = \mathbf{R} (H_\mu) \mathbf{L}' \omega_\mu = \sum_\nu \frac{\gamma_\nu (\mathcal{G}_\nu, \omega_\mu)}{E_\nu - H_\mu}$$

with (2.4), it is evident that

$$T_{\mu\nu} = \frac{(\mathcal{G}_\nu, \omega_\mu)}{E_\nu - H_\mu}. \quad (2.32)$$

It is also evident that if $(d/dE) (\mathbf{L}') = 0$, Eq. (2.26) can be written as

$$\begin{aligned} (\omega_\lambda \times \omega_\lambda) = \sum_\mu (\omega_\lambda \times \omega_\mu) \frac{(\mathbf{L}' \omega_\lambda \cdot (\mathbf{R} (H_\mu) - \mathbf{R} (H_\lambda)) \mathbf{L}' \omega_\mu)}{H_\mu - H_\lambda} \\ = \sum_\mu (\omega_\lambda \times \omega_\mu) \sum_\nu T_{\lambda\nu} \tilde{T}_{\nu\mu} \end{aligned} \quad (2.33)$$

since

$$\mathbf{R}(H_\mu) - \mathbf{R}(H_\lambda) = (H_\mu - H_\lambda) \sum_\nu \frac{(\boldsymbol{\gamma}_\nu \times \boldsymbol{\gamma}_\nu)}{(E_\nu - H_\mu)(E_\nu - H_\lambda)}$$

From this result the components given by (2.32) are those of a complex orthogonal matrix,

$$\mathbf{T}\mathbf{T}^* = \mathbf{1}. \quad (2.34)$$

By forming the scalar product of (2.18) with $\boldsymbol{\beta}_\nu$, one obtains

$$(\boldsymbol{\beta}_\nu, \boldsymbol{\omega}_\lambda) - \sum_\mu \xi_{\nu\mu} \frac{(\boldsymbol{\beta}_\mu, \boldsymbol{\omega}_\lambda)}{E_\mu - H_\lambda} = 0$$

or

$$T_{\lambda\nu}(E_\nu - H_\lambda) - \sum_\mu \xi_{\nu\mu} T_{\mu\lambda}^\dagger = 0$$

or

$$\mathbf{T}(\mathbf{e} - \boldsymbol{\xi})\mathbf{T}^{\dagger*} = \mathbf{H},$$

indicating that the matrix \mathbf{T} with components given by (2.32) does indeed diagonalize the level matrix $(\mathbf{e} - \boldsymbol{\xi})$ of (2.1).

X. CHANNEL ELIMINATION METHOD OF TEICHMANN AND WIGNER

In the general theory, \mathbf{R} and \mathbf{U} are square matrices whose dimensions equal the total number of channels, whether of positive or negative energy. The only restriction is that no channels corresponding to breakup into three or more parts are included. It is assumed that such breakups are not energetically possible, so explicit reference to such processes can be eliminated by including their channels in the "internal" region. The general formulation is hardly of practical use as it stands, since there may be hundreds or thousands of channels. This implies a correspondingly large number of parameters, i.e., the elements of \mathbf{R} which are involved in the relation between \mathbf{U} and \mathbf{R} , and most of which can never be determined because of limitations on the accuracy of data. Consequently one tries to reduce the number of parameters in the theory of the number that can be reasonably determined from the data. Two examples are the following.

(i) *Measurement of all cross sections in the energy region about an isolated resonance.*—Each single cross section corresponds to a positive-energy channel so one may extract all the widths for these channels. However, one cannot hope to extract any information about the negative-energy channels, (except possibly if the threshold for such a channel is near). Thus one tries to remove explicit reference to such channels. The simplest way to do this is to include them in the internal region, i.e., to draw their channel surfaces at infinity so that $\mathbf{R}^{--} = \mathbf{R}^{+-} = \mathbf{R}^{-+} = 0$. Formally there is no objection to this. An alternative method is that of Teichmann and Wigner⁵⁵ in which one exploits the fact that, at a given energy, the boundary conditions for negative-energy channels are known. This latter method is preferable when one is examining the energy range about the threshold of a given channel because it leads to a continuous treatment through the threshold region. (The former method must be dropped at the threshold and a switch made to regarding the channel as a normal free channel with a finite channel surface.)

(ii) *Measurement of the total and the elastic scattering cross sections at a fixed energy for one given channel, i.e.,*

given $c \equiv \alpha sl$.—These measurements can determine only two parameters. In the first instance, these are simply the real and imaginary parts of U_{cc} . As shown by Feshbach, Peaslee, and Weisskopf these can be translated into terms of the real and imaginary parts of the logarithmic derivative of the radial wave function at the channel surface: $r_c = a_c$. The usual \mathbf{R} matrix relates the values and derivatives on *all* channel surfaces. Therefore we wish to find a "reduced" \mathbf{R} matrix that refers to only one channel. This can be done by using the method of Teichmann and Wigner, i.e., the so-called "channel elimination" method. This enables one to replace the actual \mathbf{R} matrix by a suitable "reduced" \mathbf{R} matrix, whose dimensions are smaller and comparable with the number of parameters one can hope to determine from any given data. The step of introducing the reduced \mathbf{R} matrix is one of convenience. It does not introduce any new information in the theory. A theoretical analysis of a set of empirical values of the elements of the reduced \mathbf{R} matrix, in terms of level widths and energies $\gamma_{\lambda c}$ and E_λ can only be made by referring back to the actual \mathbf{R} matrix. In other words, "the channel elimination" referred to in the Teichmann-Wigner method is not a complete elimination, but only one made for convenience. It is clear that, in the present example, the presence or absence of certain open channels will affect the data in general so that in any complete theoretical interpretation of the data, some reference must be made to such channels. Nevertheless, as we see in Sec. XI, 2, under the "random sign" approximation the relation between the reduced \mathbf{R} matrix and the actual \mathbf{R} matrix becomes very simple so that a given reduced \mathbf{R} matrix may be directly interpreted in terms of the corresponding $\gamma_{\lambda c}$ and E_λ .

1. Reduced \mathbf{R} Matrix (\mathfrak{R})

a. Definition

We are given the general matrix relation (VII, 1.6) between the collision matrix and the \mathbf{R} matrix, and are interested in a particular nondiagonal element $U_{cc'}$ of the collision matrix. This element is conveniently regarded as one element of the 2×2 matrix

$$\begin{pmatrix} U_{cc} & U_{cc'} \\ U_{c'c} & U_{c'c'} \end{pmatrix}$$

and so, from the relation (VII, 1.6), we wish to extract an expression for this little submatrix which refers only to the two channels of interest, c and c' . If we had been interested in a diagonal element U_{cc} , the corresponding submatrix would be just U_{cc} itself. It is possible, in a formal way, to extract an expression directly from (VII, 1.6), for a required submatrix. This direct procedure is less useful than an indirect one which introduces a "reduced \mathbf{R} matrix" having the same dimensions as the submatrix of interest. The "reduced \mathbf{R} matrix" (\mathfrak{R}) is essentially just a convenient intermediary between the general relation (VII, 1.6) and

the small submatrix. One can define its role in a purely formal mathematical way [see below, where (1.5) is the relation between \mathfrak{R} and the submatrix, and (1.4) is the relation between \mathfrak{R} and \mathbf{R} . Eliminating \mathfrak{R} between these two leads back to the general relation (VII, 1.6) for the elements of the submatrix]. However, it is possible to introduce the reduced \mathbf{R} matrix in a more physical way.

To be somewhat more general, suppose we are interested in an arbitrary number of channels, instead of just one or two. These channels will be called "retained" channels, labeled by r , in contrast to the other channels which are to be eliminated and will be labeled by e . We now suppose that we have a state of affairs with incoming waves in the r channels but none in the e channels. (Note: This need not correspond to any experimental situation.) The e channels then contain only outgoing waves, whereas the other channels contain mixtures of incoming and outgoing waves.

With this separation the fundamental \mathbf{R} -matrix relation (V, 2.7) reduces to two submatrix relations,

$$\begin{aligned} \mathbf{V}_e &= \mathbf{R}_{ee}\mathbf{D}_e^0 + \mathbf{R}_{er}\mathbf{D}_r^0, \\ \mathbf{V}_r &= \mathbf{R}_{rr}\mathbf{D}_r^0 + \mathbf{R}_{re}\mathbf{D}_e^0. \end{aligned} \quad (1.1)$$

The subscripts e and r refer respectively to the groups of e and r channels, and not to particular channels; the \mathbf{R}_{ee} , \mathbf{R}_{rr} , and \mathbf{R}_{er} are matrices, the last being in general nonsquare. Since the e channels have only \mathcal{O} waves, their logarithmic derivatives at the surface S are known to be those of the outgoing functions O , so that

$$\mathbf{L}_e^0 = \mathbf{D}_e^0 \mathbf{V}_e^{-1} = \mathbf{D}_e \mathbf{V}_e^{-1} - \mathbf{B}_e = \mathbf{L}_e - \mathbf{B}_e. \quad (1.2)$$

By substituting (1.2) into (1.1) and solving the resulting matrix expressions, one obtains

$$\mathbf{V}_r = \mathfrak{R}_{rr}\mathbf{D}_r^0, \quad \mathbf{V}_e = \mathfrak{R}_{er}\mathbf{D}_r^0 \quad (1.3)$$

where

$$\begin{aligned} \mathfrak{R}_{rr} &= \mathbf{R}_{rr} + \mathbf{R}_{re}\mathbf{L}_e^0(\mathbf{1} - \mathbf{R}_{ee}\mathbf{L}_e^0)^{-1}\mathbf{R}_{er}, \\ \mathfrak{R}_{er} &= (\mathbf{1} - \mathbf{R}_{ee}\mathbf{L}_e^0)^{-1}\mathbf{R}_{er}. \end{aligned} \quad (1.4)$$

\mathfrak{R}_{rr} is the symmetrical reduced \mathbf{R} matrix; it is real, only if the e group contains no positive-energy channels.

b. Relation to Collision Matrix

By the procedure used to arrive at (VII, 1.6b), it is possible to obtain directly from (1.1) the general expressions for the collision submatrices; they are

$$\mathbf{W}_{rr} = \mathbf{1} + \mathfrak{P}_r^{\frac{1}{2}}(\mathbf{1} - \mathfrak{R}_{rr}\mathbf{L}_r^0)^{-1}\mathfrak{R}_{rr}\mathfrak{P}_r^{\frac{1}{2}}\mathbf{w}_r, \quad (1.5a)$$

$$\mathbf{W}_{er} = \mathfrak{P}_e^{\frac{1}{2}}\mathfrak{R}_{er}(\mathbf{1} - \mathbf{L}_r^0\mathfrak{R}_{rr})^{-1}\mathfrak{P}_r^{\frac{1}{2}}\mathbf{w}_r. \quad (1.5b)$$

Although the positive-energy part \mathbf{W}_{rr}^{++} of \mathbf{W}_{rr} is symmetric, it is unitary only if there are no eliminated positive-energy channels.

c. Transformation of Reduced \mathbf{R} Matrix to a Level Matrix

As in the deduction of (IX, 1.4), the inverse of the channel matrix appearing in (1.4) can be expressed in

terms of the inverse of a level matrix:

$$\mathfrak{R}_{rr} = \mathbf{R}_{rr}^0 + \mathbf{R}_{re}^0\mathbf{L}_{ee}^0\mathbf{R}_{er}^0 + \sum_{\lambda\mu}(\omega_{\lambda r} \times \omega_{\mu r})A_{\lambda\mu} \quad (1.6)$$

where

$$\omega_{\lambda r} = \gamma_{\lambda r} + \mathbf{R}_{re}^0\mathbf{L}_{ee}^0\gamma_{\lambda e}$$

and

$$\mathfrak{R}_{er} = \sum_{\lambda\mu}(\alpha_{\lambda e} \times \gamma_{\mu r})A_{\lambda\mu}$$

where

$$\alpha_{\lambda e} = (\mathbf{1} - \mathbf{R}_{ee}^0\mathbf{L}_e^0)^{-1}\gamma_{\lambda e},$$

$$\mathbf{L}_{ee}^0 = \mathbf{L}_e^0(\mathbf{1} - \mathbf{R}_{ee}^0\mathbf{L}_e^0)^{-1}.$$

As in Sec. IX, 1, the components of the symmetrical level matrix \mathbf{A} are obtained from the matrix relation

$$\mathbf{A} = (\mathbf{e} - \mathbf{E} - \xi)^{-1} \quad (1.7)$$

in which the components of the complex, symmetrical level matrix $\xi = -\mathbf{\Delta} + \frac{1}{2}i\mathbf{\Gamma}$ are given by the scalar products with respect to *eliminated* channels only,

$$\xi_{\lambda\mu} = (\gamma_{\lambda e}, \mathbf{L}_{ee}^0\gamma_{\mu e}) \quad (1.8)$$

in contrast with (IX, 1.8) where the r channels are included as well. Again the components of the real, diagonal matrix \mathbf{e} are the proper values E_λ and the real, diagonal matrix \mathbf{E} is the energy E times the unit matrix. As in Sec. IX, 1, the definition (1.8) can be developed to give expressions like (IX, 1.18) for the total width ($\mathbf{\Gamma}$) and shift ($\mathbf{\Delta}$) matrices.

With the matrix $\mathbf{R}^0 = 0$, Eqs. (1.6) simplify to

$$\mathfrak{R}_{rr} = \sum_{\lambda\mu}(\gamma_{\lambda r} \times \gamma_{\mu r})A_{\lambda\mu}, \quad (1.9a)$$

$$\mathfrak{R}_{er} = \sum_{\lambda\mu}(\gamma_{\lambda e} \times \gamma_{\mu r})A_{\lambda\mu}. \quad (1.9b)$$

2. Elimination of Negative-Energy Channels

It is often desirable to eliminate explicit reference to the negative-energy channels as there are apt to be many of them and they usually have only a slight effect on the behavior of the reaction and scattering cross sections. If this is done, and if there are no positive-energy e channels, then the reduced \mathbf{R} matrix of (1.4) is real, as is the ordinary \mathbf{R} matrix. However, it is in a much more complicated form. If the components of the logarithmic derivative matrix \mathbf{L}_e^0 of (1.4) can be replaced by constants or linear functions of E (the coefficients of the E term being always positive), and approximate \mathfrak{R}_{rr} will be obtained which is expansible in the form (V, 2.8), the γ_λ and E_λ of which are independent of E though different in general from those which occur in (V, 2.8). This assertion may be verified by means of the expansion of (1.9a) and may readily be interpreted in terms of a modification of the normalization used for the γ_λ of (V, 2.8).

If the boundary conditions parameters B_e for the negative-energy channels are chosen to be very close to the S_e , so that the $\mathbf{L}_e^0 (= \mathbf{S}_e^0 = \mathbf{S}_e - \mathbf{B}_e)$ are very small, the off-diagonal part ξ' of the real matrix ξ of (1.7) ($\mathbf{P}_e = 0$) may be considered as small compared with the diagonal part $\mathbf{e} = \mathbf{e} - \mathbf{E} - \xi + \xi'$ and all but the first term of the

series expansion for (1.7)

$$\mathbf{A} = \boldsymbol{\varepsilon}^{-1} + \boldsymbol{\varepsilon}^{-1} \boldsymbol{\xi}' \boldsymbol{\varepsilon}^{-1} + \dots \quad (2.1)$$

may be disregarded; (1.9a) reduces to the single sum

$$\mathfrak{R}_{rr} \approx \sum_{\lambda} (\gamma_{\lambda r} \times \gamma_{\lambda r}) / (E_{\lambda} + \Delta_{\lambda\lambda} - E) \quad (2.2)$$

where the $\Delta_{\lambda\lambda}$ are the diagonal components of the level shift matrix for the e channels. When $\mathbf{S}_e^0 = 0$, $\Delta_{\lambda\lambda}$ vanishes and (1.11) reduces to the submatrix \mathbf{R}_{rr} of (V, 2.8). However, the energy dependences of the \mathbf{S}_e must be considered. By expanding the \mathbf{S}_e linearly with respect to energy about the energy E of interest, one obtains

$$\mathfrak{R}_{rr} \approx \sum_{\lambda} \frac{(\gamma_{\lambda r}^0 \times \gamma_{\lambda r}^0)}{E_{\lambda} - E}, \quad (2.3)$$

where

$$\gamma_{\lambda r}^0 = \frac{\gamma_{\lambda r}}{\left(1 - \frac{d\Delta_{\lambda\lambda}}{dE}\right)^{\frac{1}{2}}}, \quad (2.4)$$

$$\frac{d\Delta_{\lambda\lambda}}{dE} = - \sum_e \frac{dS_e}{dE} \gamma_{\lambda e}^2.$$

According to results in the appendix and Sec. XII, 3 and the definition of the $\gamma_{\lambda e}$, $-d\Delta_{\lambda\lambda}/dE$ gives the volume integration of the negative-energy channel wave functions of the external region which join on to the X_{λ} of the internal region. When $-d\Delta_{\lambda\lambda}/dE$ is added to the unity term in the denominator of (2.4), a modified normalization of the $\gamma_{\lambda e}$ is obtained which is effective with respect to the internal region plus the negative-energy part of the external region. This could have been inferred immediately from the derivation of Sec. V by allowing the channel radii of the negative-energy channels to approach infinity where the wave functions are vanishingly small. (2.3) is correct to the extent that the S_e can be approximated by linear functions of the energy. This approximation, and likewise (2.3), is therefore invalid through thresholds where S_e has anomalous behavior (see appendix).

3. Elimination of Reaction Channels

According to Eqs. (VIII, 3.2 and 3), the experimental cross sections for bombardment by pairs of unpolarized particles α are (with $c = \alpha sl$, $c' = \alpha' s'l'$):

$$\text{total: } \sigma_{\alpha}(\text{tot}) = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} 2g_J (1 - \text{Re}(U_{cc}^J)) \quad (3.1)$$

$$\text{elastic: } \sigma_{\alpha}(\text{el}) = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} g_J \times \{ |1 - U_{cc}^J|^2 + \sum_{\substack{(\alpha' = \alpha) \\ s'l' (\neq sl)}} |U_{cc'}^J|^2 \} \quad (3.2)$$

$$\begin{aligned} \text{absorption: } \sigma_{\alpha}(\text{abs}) &= \sigma_{\alpha}(\text{tot}) - \sigma_{\alpha}(\text{el}) = \sum_{\alpha' (\neq \alpha)} \sigma_{\alpha\alpha'} \\ &= \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} g_J \{ 1 - |U_{cc}^J|^2 - \sum_{\substack{(\alpha' = \alpha) \\ s'l' (\neq sl)}} |U_{cc'}^J|^2 \} \end{aligned} \quad (3.3)$$

$$\text{particular reaction: } \sigma_{\alpha\alpha'} = \frac{\pi}{k_{\alpha}^2} \sum_{\substack{Jsl \\ s'l'}} g_J |U_{cc'}^J|^2, \quad (3.4)$$

where g_J is the spin statistical factor:

$$g_J = \frac{2J+1}{(2I_1+1)(2I_2+1)}. \quad (3.5)$$

Of these cross sections, the total involves U_{cc} only, the particular reaction involves $U_{cc'}$ only and the other two involve, in general, both U_{cc} and $U_{cc'}$.

a. Elimination of All Reaction Channels

As noted by Feshbach, Peaslee, and Weisskopf,¹⁵ the diagonal element U_{cc}^J is uniquely determined by f_c^J , the logarithmic derivative at $r_c = a_c$ of the wave function in the channel c when there are incoming waves in this channel only (whether this corresponds to an actual experimental situation is irrelevant). By definition of U_{cc}^J , the radial wave function is

$$u_{cc}^J \sim I_c - U_{cc}^J O_c \quad (3.6)$$

with the consequence that

$$f_c^J = \rho_c \left(\frac{I_c' - U_{cc}^J O_c'}{I_c - U_{cc}^J O_c} \right) \quad (3.7)$$

or, equivalently, using the formulas of Sec. III, 4,

$$U_{cc}^J = e^{2i(\omega_c - \phi_c)} \left(\frac{f_c - L_c^*}{f_c - L_c} \right). \quad (3.8)$$

From the definition of f_c^J and the definition of \mathfrak{R}_{rr}^J in (1.3) in the case when all reaction channels are eliminated, it follows that

$$f_r^J = (\mathfrak{R}_{rr}^J)^{-1} + B. \quad (3.9)$$

From (1.6) with $R^0 = 0$, \mathfrak{R}_{rr} is the quadratic expression:

$$\mathfrak{R}_{rr} = \sum_{\lambda\mu} \gamma_{\lambda r} A_{\lambda\mu} \gamma_{\mu r}. \quad (3.10)$$

Equation (3.9) establishes the connection between our reduced \mathbf{R} matrix and the f function of Feshbach, Peaslee, and Weisskopf which was used in their discussion of total cross sections. In the special case when there are no open reaction channels, $A_{\lambda\mu} \rightarrow (E_{\lambda} - E)^{-1} \delta_{\lambda\mu}$, $\mathfrak{R} \rightarrow R$ and f becomes real.

b. Elimination of All Reaction Channels but One

Let us denote a typical pair of such channels by c and c' . Equation (3.6) is now replaced by the pair of

equations corresponding to incoming waves in both channels c and c' :

$$\begin{aligned} u_c^J &= I_c - (U_{cc}^J O_c + U_{cc'}^J O_{c'}), \\ u_{c'}^J &= I_{c'} - (U_{c'c}^J O_c + U_{c'c'}^J O_{c'}). \end{aligned} \quad (3.11)$$

Clearly there is no longer the simple relation (3.7) between U_{cc}^J and the logarithmic derivative f_c^J in the channel c . Rather one has the matrix relation (1.5a) for \mathbf{U} in terms of \mathfrak{R} where \mathbf{U} and \mathfrak{R} are 2×2 matrices and \mathfrak{R} satisfies the value-derivative matrix relation (1.3): $\mathbf{V} = \mathfrak{R}\mathbf{D}^0$. From (1.5a) we have

$$\begin{aligned} W_{cc} &= 1 + 2iP_c [\mathfrak{R}_{cc} - L_{c'}^0 (\mathfrak{R}_{cc} \mathfrak{R}_{c'c'} - \mathfrak{R}_{cc'}^2)] d^{-1}, \\ W_{c'c} &= 2iP_{c'} \frac{1}{2} P_c \frac{1}{2} \mathfrak{R}_{cc'} d^{-1} \end{aligned} \quad (3.12)$$

where

$$d = (1 - L_c^0 \mathfrak{R}_{cc}) (1 - L_{c'}^0 \mathfrak{R}_{c'c'}) - L_c^0 L_{c'}^0 \mathfrak{R}_{cc'}^2. \quad (3.13)$$

If only the two channels c and c' are open, the \mathfrak{R} matrix is equal to the usual real \mathbf{R} matrix and (3.12) and (3.13) become equivalent to (VII, 3.1).

XI. THE R-MATRIX THEORY APPLIED TO TREATMENT OF AVERAGE CROSS SECTIONS

The considerations so far have been of an exact and formal nature. Now we begin to make approximations that are relevant for treating various practical situations. It is a well-known experimental fact that, with increasing excitation energy of the total (compound) system, resonance levels become broader and closer together. There is a continuous transition from sharp, well-isolated levels in the region just above the lowest threshold to the so-called "continuum" where levels overlap so that no structure remains in the cross sections. Both of the extreme situations of complete isolation and complete overlap can be described within the framework of \mathbf{R} -matrix theory by appropriate approximations. This section discusses average cross sections, with emphasis on the case of overlapping levels; the next section discusses isolated levels. The intermediate case of partial overlap of levels, will not be discussed. Attention is drawn to certain formulas (2.18) that appear to hold in both extreme situations and can therefore may be plausibly assumed to hold in the intermediate case.

Two types of approximation have been suggested for treating large numbers of levels. Both lead to cross-section expressions having the form of a product of two factors, one for the bombarding channel, the other for the breakup channel. Such expressions evidently imply the property of independence of formation and decay that is the essential feature of the Bohr picture of the compound nucleus. The difference between the two approximations is that one (due to Newton and Teichmann) implies that this property holds at *all* discrete energies whereas the other (due to Bethe) implies that the property only holds *on the average* over energy intervals containing large numbers of levels.

The former approximation involves some very strong correlations between elements of the \mathbf{R} matrix, i.e., between the $\gamma_{\lambda c}$. For instance, if we think of a cross section containing contributions from two isolated levels represented by Breit-Wigner formulas, the demand of independence of formation and decay at all energies implies that the ratios of the widths for the various channels must be the same for both levels. In fact, the requirement of independence of formation and decay at *all* energies is far too stringent and leads to strange predictions.

The other approximation (due to Bethe) is based on an assumption that is almost the opposite of Newton's. Bethe's assumption is that the signs of the width amplitudes $\gamma_{\lambda c}$ are random. This assumption results in formulas for average cross sections which exhibit features expected of reactions proceeding by the Bohr compound nucleus mechanism. The cross-section expression for a given reaction $\alpha \rightarrow \alpha'$ proceeding through states of given spin and parity has the form of a product of two factors, actually the so-called "transmission factors" one for α and one for α' , thereby expressing independence of formation and decay. When one sums over spins and parities, this independence no longer applies, a point which does not appear to be generally appreciated. The independence of formation and decay, and also the isotropy of outgoing groups for many final states, are only expected to be exhibited by observed cross sections if a very specific property of level densities is assumed.

In subsection 3, we show that the "transmission factors" are expressible in terms of the "strength function" whose form is not given by \mathbf{R} -matrix theory and is, to that extent, arbitrary. There is a degree of freedom in the cross-section formulas. As mentioned in Sec. I, this freedom corresponds to the various "models" for the probability of compound nucleus formation. In most published applications of the theory, as reviewed by Blatt and Weisskopf,²⁴ this freedom was not appreciated and the transmission factors and strength function were fixed as those of the "strong absorption" model. This model is based on the assumption of inevitable compound nucleus formation in collisions and was used without question until the exposure of its failings by Feshbach, Porter, and Weisskopf¹³ who replaced it by their "moderate absorption" model. This does not automatically mean that all analyses previously made with the strong coupling model are useless. This would indeed be a pity because of the great labor that has gone into such work. For instance, all the fits of continuous spectra from reactions in terms of Maxwellian distributions and nuclear temperatures presupposed the strong absorption model. Fortunately it appears that such fits are not *directly* impaired by the change to the "moderate coupling" model because the main factor in such fits is the level density law (as a function of excitation energy), the transmission factors playing only a secondary role. However, as is now well known,

the fits to spectra may be seriously disturbed by the presence of "direct reactions" in certain circumstances. In subsection 6, we see that the direct reaction mechanism owes its existence ultimately to the "moderate absorption" model (which implies a mild breakdown in the assumption of random signs for the $\gamma_{\lambda c}$). To this extent, we may say that the change to the moderate absorption model may *indirectly* impair some of the existing fits of spectra made with the strong absorption model.

1. Consequences of Newton's Assumption⁷⁵ of Strong Correlations between the $\gamma_{\lambda c}$

If the \mathbf{R} matrix is of low rank, say, one, two, or three, Teichmann⁴⁸ has shown that it is straightforward to obtain the exact expressions for the collision matrix. That the \mathbf{R} matrix has a low rank means essentially that it can be put in the form

$$\mathbf{R} = \sum_i \mathbf{x}_i \times \mathbf{x}_i / t_i, \quad (1.1)$$

where the sum extends to $i=1, 2, 3$ if the rank is 1, 2, 3, respectively. These forms are similar to those of the 1, 2, 3 level \mathbf{R} -matrix expansions of Sec. IX, 1. The collision matrices for the 1, 2, 3 level expansions of Sec. IX, 1 give the collision matrix for (1.1) if the γ_{λ} are replaced by the \mathbf{x}_i and the $(E-E_{\lambda})$ by the t_i . The forms (1.1) also apply if the $\gamma_{\lambda c}$ are factorizable into groups of levels i in the manner

$$\gamma_{\lambda ic} = a_{\lambda i} b_{ic}. \quad (1.2)$$

In each group i the arrangement of the relative signs and magnitudes of the various channels c is the same for all levels λ_i of the group (to within an over-all factor which is determined by the level factor $a_{\lambda i}$). The $\gamma_{\lambda ic}$ of one group are, however, not correlated in any way with those of the other groups. Thus the lower the rank of \mathbf{R} , the more highly correlated are the $\gamma_{\lambda c}$. With this form, one has $x_{ic} = b_{ic}$ and

$$t_i^{-1} = \sum_{\lambda_i} \frac{a_{\lambda_i}^2}{E_{\lambda_i} - E}. \quad (1.3)$$

The cases of rank 2 and rank 3 and the corresponding cross sections have been discussed by Teichmann. We confine ourselves to the rank 1 case for which the collision matrix has the one-level form,

$$\mathbf{W} = \mathbf{1} + \frac{2i(\mathbf{P}^{\dagger} \mathbf{b} \times \mathbf{P}^{\dagger} \mathbf{b})}{t + \Delta - \frac{i}{2}\Gamma}, \quad (1.4a)$$

and the reaction cross sections are proportional to

$$|W_{cc'}|^2 = \frac{\Gamma_c \Gamma_{c'}}{(t + \Delta)^2 + \frac{1}{4}\Gamma^2}, \quad (1.4b)$$

⁷⁵ T. D. Newton, Can. J. Phys. **30**, 53 (1952).

where

$$\Gamma = \sum_c \Gamma_c$$

$$\Gamma_c = 2P_c b_c^2.$$

This form of reaction cross section is unusual because it vanishes between each pair of resonances where $t(E)$ becomes infinite. (As mentioned in Sec. VII, 3 Wigner showed that when there are three or more open channels no reaction cross section vanishes *in general* for any value of the energy.) Moreover, the reaction cross sections all have the same type of energy variation and differ only in their relative amplitudes. Thus the relative decay probabilities are independent of the energy and of the formation mode; in fact, Newton⁷⁵ arrived at the form (1.4a) by requiring that the cross sections have this characteristic. Teichmann showed that these cross sections may readily be averaged if $a_{\lambda}^2 = 1$ for all λ and if the levels are uniformly spaced by an amount D , in which case

$$t = -\frac{D}{\pi} \tan\left(\frac{\pi E}{D}\right). \quad (1.5)$$

With this expression for t , the average cross sections are easily found from (VIII, 3.2) to be

$$\langle \sigma_{\alpha\alpha'} \rangle = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl's'l'} g_J \left(\frac{4\Gamma_c \Gamma_{c'}}{\Gamma^2} \right)$$

$$\times \frac{\left(\frac{\pi\Gamma}{2D}\right) \left(1 + \frac{\pi\Gamma}{2D}\right)}{\left(\frac{\pi\Delta}{D}\right)^2 + \left(1 + \frac{\pi\Gamma}{2D}\right)^2} \quad (1.6)$$

where $c \equiv \alpha sl$, $c' \equiv \alpha' s'l'$. By choosing the B_c equal to S_c evaluated at the particular energy of interest, the level shift will vanish at that energy and will be negligible in the vicinity of it. Therefore, in the region of overlapping levels where $(\pi\Gamma/2D) \gg 1$, (1.6) simplifies to

$$\langle \sigma_{\alpha\alpha'} \rangle = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl's'l'} g_J \left(\frac{4\Gamma_c \Gamma_{c'}}{\Gamma^2} \right). \quad (1.7)$$

The total absorption cross section is obtained by summing over $\alpha' (\neq \alpha)$:

$$\langle \sigma_{\alpha}(\text{abs}) \rangle \equiv \sum_{\alpha' (\neq \alpha)} \langle \sigma_{\alpha\alpha'} \rangle = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} g_J \left(\frac{4\Gamma_c}{\Gamma} \right),$$

where we have dropped the term in $(\Gamma_c/\Gamma)^2$ as small. Let us now consider raising the bombarding energy to the point where many partial l waves contribute. Since the cutoff in l occurs at $l \sim ka \gg 1$, if we assume that (Γ_c/Γ) is independent of s, l and J

$$\langle \sigma_{\alpha}(\text{abs}) \rangle \approx \pi a^2 \left(\frac{4\Gamma_c}{\Gamma} \right). \quad (1.8)$$

If there are many channels for absorption so that $\Gamma \gg \Gamma_c$ the absorption cross section becomes much less than the geometrical cross section πa^2 found experimentally and it does not seem likely that this is a consequence of the special assumptions made in connection with (1.5). For this reason the form (1.4) does not provide an adequate description of the compound nucleus even though it does satisfy the requirement of independence of formation and decay. This tendency for the absorptions to vanish is evidently due to destructive interference of the various contributing levels caused by the extreme correlations of the signs of the $\gamma_{\lambda c}$. Such destructive interference is also evident in the results obtained by Kalckar, Oppenheimer, and Serber⁷⁶ as well as Bohr, Peierls, and Placzek.⁷⁷ In the following subsection a more suitable form of the collision matrix for the compound nucleus is derived. This does not lead to destructive interference in the absorption cross section, and implies independence of formation and decay *on the average*.

2. Consequences of Bethe's Assumption of Random Signs of the $\gamma_{\lambda c}$

a. **R**-Matrix Description of the Original Treatment

Bethe¹ and, later Bethe and Placzek⁴ derived an expression for the averages of reaction cross sections which is valid even if the levels overlap, provided that each of the average partial level widths $\langle \Gamma_{\lambda c} \rangle$ is smaller than the mean spacing D . This derivation is based on the assumption that the signs of matrix elements for the formation and decay of the various intermediate nuclear states are uncorrelated, and the result thereby obtained shows that the excited nucleus decays on the average with probabilities which are independent of the formation mode, and that the average total absorption cross sections will tend to be of the order of magnitude of the nuclear area, in conformity with experiment. Although the concept of a collision matrix had not been introduced at the time of Bethe's work, the form of this matrix can be inferred from his equations. The purpose of this section is to derive Bethe's result using **R**-matrix theory and to show that, in this theory, the signs of the $\gamma_{\lambda c}$ for the states of the compound nucleus must be considered as uncorrelated.

The set of Eqs. (IX, 2.18) and (IX, 2.19) for the determination of the parameters of the collision matrix can be solved approximately in the case of uncorrelated signs and overlapping levels by the standard perturbation theory procedures such as those reviewed by Morse and Feshbach.⁷⁸ However, for simplicity we use a less rigorous procedure.

With the assumption of random signs for the $\gamma_{\lambda c}$ the nondiagonal components of the ξ matrix given by

(IX, 1.8) will be smaller in absolute magnitude than the diagonal components and one can make an expansion of the matrix expression for **A** given by (IX, 1.11) in a power series about the diagonal part $\epsilon = \mathbf{e} - \mathbf{E} - \xi + \xi'$ whose components are $\epsilon_{\lambda} = E_{\lambda} + \Delta_{\lambda} - E - (i/2)\Gamma_{\lambda}$ where ξ' is the nondiagonal part of ξ . Thus

$$\mathbf{A} = \epsilon^{-1} + \epsilon^{-1} \xi' \epsilon^{-1} + \dots, \quad (2.1)$$

and therefore from (IX, 1.7) with $\mathbf{R}^0 = 0$

$$(\mathbf{I} - \mathbf{R}\mathbf{L}^0)^{-1}\mathbf{R} = \sum_{\lambda} \frac{\gamma_{\lambda} \times \gamma_{\lambda}}{\epsilon_{\lambda}} + \sum_{\lambda \mu} \frac{\gamma_{\lambda} \times \gamma_{\mu}}{\epsilon_{\lambda} \epsilon_{\mu}} \xi_{\lambda \mu}' + \dots, \quad (2.2a)$$

the components of which are

$$[(\mathbf{I} - \mathbf{R}\mathbf{L}^0)^{-1}\mathbf{R}]_{cc'} = \sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{\epsilon_{\lambda}} + i \sum_{c''} P_{c''} \sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c''}}{\epsilon_{\lambda}} \sum_{\mu (\neq \lambda)} \frac{\gamma_{\mu c''} \gamma_{\mu c'}}{\epsilon_{\mu}} + \dots \quad (2.2b)$$

For simplicity we have set $B_c = S_c$ so that the level shift matrix may (temporarily) be disregarded, and the expressions given by (IX, 1.16) and (IX, 1.18) for the resulting components of ξ' have been used to arrive at (2.2b) and (2.2a). We now determine the conditions for the validity of the approximation (2.3) below for the collision matrix which is obtained by neglecting all but the first term of (2.2a or b). The sum over the channels c'' of (2.2b) is considered in two parts: (1) the contributions from the channels c and c' ; (2) the contributions from the remaining channels. If the $\mu = \lambda$ terms are added to the μ sum of (2.2b), which is permissible if the total level widths are much larger than the spacings, the contribution from the channel c' is observed to be

$$i P_{c'} \sum_{\mu} \frac{\gamma_{\mu c'}^2}{\epsilon_{\mu}}$$

times the contribution from the first sum of (2.2b). By replacing this μ sum by an integration, it may readily be estimated as

$$\frac{\langle \gamma_{\mu c'}^2 \rangle}{\pi D}.$$

The contribution from the channel c may be determined in a similar manner. The magnitude of the contribution from the c and c' channels together is thus $(\pi/D)\langle \Gamma_{\lambda c} + \Gamma_{\lambda c'} \rangle$ times that from the first sum of (2.2a) or (2.2b). In the consideration of the contribution (2) from the remaining channels, no two of the same $\gamma_{\lambda c}$ appear multiplied together. Since the signs of these are presumed to be uncorrelated, the most-probable value of this contribution is zero; its root-mean-square magnitude may be estimated as follows. The rms magnitude of the contribution to the components of ξ' from the remaining channels may be estimated as equal to $\frac{1}{2}\langle \Gamma_{\lambda} \rangle$,

⁷⁶ Kalckar, Oppenheimer, and Serber, Phys. Rev. **52**, 273 (1937).

⁷⁷ Bohr, Peierls, and Placzek, Nature **144**, 200 (1939).

⁷⁸ P. M. Morse and H. Feshbach *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953).

the average magnitude of the diagonal components of ξ , divided by the square root of the number of open channels $c'' \neq c, c'$, this number being approximately equal to $(\Gamma_\lambda)/\langle\Gamma_{\lambda c''}\rangle$, the average being with respect to both λ and c'' . The magnitude of the contribution from the μ sum of $1/\epsilon_\mu$, individual terms having random signs, may be estimated as $(2\pi/D\langle\Gamma_\lambda\rangle)^{1/2}$. The rms magnitude of the contribution to the second sum of (2.2b) from these channels is thus estimated as the channel average of $(\pi\langle\Gamma_{\lambda c''}\rangle/D)^{1/2}$ times the first sum. A necessary condition for the validity of the approximation

$$\mathbf{W} = \mathbf{1} + 2i\mathbf{P}^{\frac{1}{2}} \left[\sum_{\lambda} \frac{\gamma_{\lambda} \times \gamma_{\lambda}}{E_{\lambda} - E - \frac{i}{2}\Gamma_{\lambda}} \right] \mathbf{P}^{\frac{1}{2}} \quad (2.3)$$

is therefore that all of the ratios $\pi\langle\Gamma_{\lambda c''}\rangle/2D$ must be much less than unity, as imposed by the contribution (1). The third and higher terms of the expansion (2.2b) are proportional to the square and higher powers of these ratios. If the level shifts Δ_{λ} are included in the denominators of (2.3), the ratios $\pi\langle\Delta_{\lambda c''}\rangle/D$ must also be less than unity. Equation (2.3) is the many-level approximation to the collision matrix which is valid in general when the total level widths are much less than their spacings. This result corresponds to the equality condition of the relation (IX, 2.14).

With $c \equiv \alpha s l$, $c' \equiv \alpha' s' l'$, the reaction cross sections obtained from (2.3) are

$$\sigma_{\alpha\alpha'} = \frac{4\pi}{k_{\alpha}^2} \sum_{J s l s' l'} g_J \times \sum_{\lambda\mu} \frac{P_c \gamma_{\lambda c} \gamma_{\mu c} \gamma_{\lambda c'} \gamma_{\mu c'} P_{c'}}{\left(E_{\lambda} - E - \frac{i}{2}\Gamma_{\lambda}\right) \left(E_{\mu} - E + \frac{i}{2}\Gamma_{\mu}\right)}. \quad (2.4)$$

Following the procedure indicated by Bethe,¹ averages of these with respect to an energy interval δ , which is large compared with the total widths, are found to be

$$\langle\sigma_{\alpha\alpha'}\rangle = \frac{4\pi}{k_{\alpha}^2} \sum_{J s l s' l'} g_J \times \frac{\pi}{\delta} \sum_{\lambda\mu(\text{in } \delta)} \frac{P_c P_{c'} \gamma_{\lambda c} \gamma_{\mu c} \gamma_{\lambda c'} \gamma_{\mu c'} (\Gamma_{\lambda} + \Gamma_{\mu})}{(E_{\lambda} - E_{\mu})^2 + \frac{1}{4}(\Gamma_{\lambda} + \Gamma_{\mu})^2}, \quad (2.5)$$

the sums being restricted to levels within the interval δ . For each level in the λ sum there will be about $\langle\Gamma_{\lambda}\rangle/D$ levels contributing from the μ sum and thus about $\delta\langle\Gamma_{\lambda}\rangle/D^2$ pairs altogether. In view of the random nature of the $\gamma_{\lambda c}$ signs, the contribution from the terms with $\lambda \neq \mu$ will be proportional to the square root of this number of pairs in contrast with the number δ/D of contributions from the positive definite terms with $\lambda = \mu$. The cross-product terms may therefore be ne-

glected so that

$$\langle\sigma_{\alpha\alpha'}\rangle = \frac{\pi}{k_{\alpha}^2} \sum_{J s l s' l'} g_J \left(\frac{T_c^J T_{c'}^J}{\sum_{c''} T_{c'', J}} \right), \quad (2.6)$$

where

$$T_c^J = \frac{2\pi\langle\Gamma_{\lambda c}^J\rangle}{D^J} \quad (2.7)$$

are quantities similar to Weisskopf's transmission factors.²⁴ From the form of (2.6) the decay of the compound nucleus is on the average independent of the formation mode, for each pair given spin J and parity (but not when summed over J). By summing over decay channels α' , the total absorption cross section is obtained.

$$\langle\sigma_{\alpha}(\text{abs})\rangle = \frac{\pi}{k_{\alpha}^2} \sum_{J s l} g_J T_c^J. \quad (2.8)$$

When the bombarding energy is greater than the barrier height, the penetration factors $P_l \sim ka$, and the strong-coupling theory (see subsection 4) suggests that the ratios $(\langle\gamma_{\lambda c}^2\rangle/D) \sim (\pi Ka)^{-1}$ where $K \sim 1 \times 10^{-13}$ cm is a wave number characteristic of nucleon motion within the nucleus. Under these conditions the transmission factor is $T_c = (4k/K)$, which is of the order of magnitude unity at moderate and high energies, and the summing procedure leading to (1.8) indicates that the total absorption is of the order of magnitude of the nuclear area. However, when T_c is of the order unity, the ratio $\pi\langle\Gamma_{\lambda c}\rangle/2D$ is also of this order, and the expansion (2.3) is not valid. Transmission factors calculated from (2.6) can exceed unity for high energies (that is if $4k$ exceeds K) in violation of the requirement that the collision matrix be unitary. Subsection 2b is devoted to an alternative procedure which avoids these difficulties by dealing with an expansion similar to (2.2b) except for the nonappearance of the objectionable contributions from the c and c' channels in the ξ' matrix. This avoidance is accomplished by means of the channel elimination procedure of Sec. X. The result obtained is similar to (2.6) though valid under the less restrictive condition, imposed by the channel contribution (2) above, that the mean with respect to channels c of the $\langle\Gamma_{\lambda c}\rangle$ be less than D ; the transmission factors that are obtained cannot exceed unity and are of the form proposed by Weisskopf.²⁴

b. Improved Treatment Using the Reduced R Matrix

If the signs of the $\gamma_{\lambda c}$ are uncorrelated, the arguments used to justify the approximation (2.3) may also be used¹⁷ to justify the approximation

$$\mathfrak{R}_{rr} = \sum_{\lambda} \frac{\gamma_{\lambda} \times \gamma_{\lambda}}{E_{\lambda} + \Delta_{\lambda}' - E - \frac{i}{2}\Gamma_{\lambda}'} \quad (2.9)$$

to (X, 1.9a), the widths Γ_λ' and shifts Δ_λ' of this expression being the diagonal components of the total width and shift matrices for the eliminated channels. However, (2.9) is valid under the less restrictive condition that the means with respect to λ and c of the partial level widths and shifts for the eliminated channels be less than the spacings, because the contributions to the second sum from the channels c, c' have been eliminated in the expansion corresponding to (2.2b) for the components of \mathfrak{R}_{rr} .

When the Γ_λ are much larger than the spacings D , it is permissible to replace the individual widths and shifts in (2.9) by suitable averages with respect to an energy interval of size comparable to the $\langle \Gamma_\lambda \rangle$

$$\Gamma_\lambda' \approx \langle \Gamma_\lambda \rangle \equiv \Gamma(E), \quad \Delta_\lambda' \approx \langle \Delta_\lambda \rangle \equiv \Delta(E). \quad (2.10)$$

Both Γ and D are expected to be mildly energy-dependent because of the presence of the energy-dependent factors P and S in the individual contributing terms and of possible systematic long-range variations of the reduced widths of the eliminated channels. This approximation is reasonable because it is not expected that the individual level widths and shifts will deviate significantly from one another, and the results obtained are usually not sensitive to the actual values of Γ and Δ . We also apply (2.10) in those circumstances where the Γ_λ are narrow or comparable to D by allowing the Γ and Δ to vary in an appropriate manner from level to level.

Approximations (2.9) and (2.10) together lead to the interesting result that the various components of the reduced \mathbf{R} matrix at the energy E can be obtained approximately by simply evaluating the components of the ordinary \mathbf{R} matrix at the complex energy $\mathcal{E} = E - \Delta + (i/2)\Gamma$:

$$\mathfrak{R}_{rr}(E) = \mathbf{R}_{rr}(\mathcal{E}). \quad (2.11)$$

This result is particularly useful because the latter components are analytical functions of the complex energy \mathcal{E} . As discussed in Sec. IV, 6 the diagonal components of the \mathbf{R} matrix are meromorphic functions, the imaginary parts of which are non-negative on the upper half-plane and nonpositive on the lower; their poles E_λ are confined to the real axis and have negative residues $-\gamma_\lambda^2$. In this connection frequent reference is made to the *pole strength function* s_c for channel c , defined as the sum of the $\gamma_{\lambda c}^2$ per unit energy interval of the E_λ , averaged with respect to an interval of appropriate length.

For the determination of a particular diagonal component of the collision matrix, one eliminates explicit reference to all but the channel referred to by that component, while for the determination of a particular nondiagonal component, one eliminates explicit reference to all but the two channels referred to by that component.

Diagonal Components.—If there is only one r channel,

the collision matrix given by (X, 1.5a) becomes the collision function (X, 3.8), which, on using (X, 3.9) has the form

$$U = e^{2i(\omega - \phi)} \left(\frac{1 - \mathfrak{R}(E)L^{0*}}{1 - \mathfrak{R}(E)L^0} \right). \quad (2.12)$$

By introduction of Wigner's statistical R function, R' (see Sec. IV, 4)

$$R'(\mathcal{E}) = \sum_{\lambda'} \frac{\gamma_{\lambda'}^2}{E_{\lambda'} - \mathcal{E}} \quad (2.13)$$

it is possible to develop a useful representation for the approximation (2.11) which is expected to be rather accurate whether the level width is small or large compared with the mean level spacing D . The R' function has the following property. In an energy interval I containing the energy E of interest and of a length such that $D \ll I \ll s(ds/dE)^{-1}$ where $s(E)$ is the strength of R in the vicinity of E , the poles $E_{\lambda'}$, and residues $-\gamma_{\lambda'}^2$ of R' are equal to those of R , whereas outside of I they maintain the same statistical distributions as within and thus will in general differ from those of R which may evidence long-range fluctuations. The strength s of R' is thus everywhere equal to the strength $s(E)$ of R at the energy E . It has therefore the useful property of approaching $i\pi s(E)$ when evaluated at a complex energy whose imaginary part is large compared with D , as is evident by replacing the sum over levels in (2.13) by an integration for the evaluation. It is convenient to introduce the "complex smoothed R function," denoted by \bar{R} and defined by

$$\bar{R}(\mathcal{E}) = R(\mathcal{E}) - R'(\mathcal{E}) + i\pi s(E). \quad (2.14)$$

Since the terms λ of R and R' that occur inside I are the same, only those terms from outside of I will contribute to \bar{R} . The sums over these outside terms can be replaced by integrals:

$$\bar{R}(\mathcal{E}) = \int_{(\infty-I)} \frac{s(E')}{E' - \mathcal{E}} dE' - \int_{(\infty-I)} \frac{s'(E')dE'}{E' - \mathcal{E}} + i\pi s(E)$$

where $(\infty - I)$ denotes the range of all energies excluding the interval I , and where $s'(E')$ is the strength function for R' which, by the definition of R' , is a constant everywhere equal to $s(E)$. From this last fact, we may add

$$\int_{(I)} \frac{s(E')dE'}{E' - \mathcal{E}} - \int_{(I)} \frac{s'(E')dE'}{E' - \mathcal{E}} \equiv 0$$

to the last equation to obtain

$$\bar{R}(\mathcal{E}) = \int_{(\infty)} \frac{s(E')dE'}{E' - \mathcal{E}} - \int_{(\infty)} \frac{s'(E')dE'}{E' - \mathcal{E}} + i\pi s(E).$$

Since $s'(E')$ is a constant, the second term equals

$i\pi s(E)$ and cancels the third so that

$$\bar{R}(\mathcal{E}) = \int_{(\infty)} \frac{s(E')dE'}{E' - \mathcal{E}} \quad (2.15a)$$

In other words, the complex smoothed R function \bar{R} is the Stieltjes transform of s ; clearly we have that, in the limit when $\Delta, \Gamma \ll s(ds/dE)^{-1}$, i.e., when $|\mathcal{E} - E| \ll s(ds/dE)^{-1}$

$$\text{Re}[\bar{R}(\mathcal{E})] \rightarrow \text{Pr} \int \frac{s(E')dE'}{E' - E} = \bar{R}(E) \quad (2.15b)$$

$$\text{Im}[\bar{R}(\mathcal{E})] \rightarrow \pi s(E) \quad (2.15c)$$

where "Pr" signifies "principal part of." The $\bar{R}(E)$ defined by (2.15b) is called the "real smoothed" R function. Here $\bar{R}(E)$ is not equal to the limiting value of $\bar{R}(\mathcal{E})$ when $\mathcal{E} \rightarrow E$, but is only the real part of this limit. We now apply this discussion to the special situations $\Gamma \gg D$ and $\Gamma \ll D$.

(i) $\Gamma \gg D$. In this case, as mentioned above (2.14) we may use the property of R' that, for $\Gamma \gg D$:

$$R'(\mathcal{E}) \approx i\pi s(E)$$

to deduce from (2.11) and (2.14) that

$$\mathfrak{R}(E) = R(\mathcal{E}) = \bar{R}(\mathcal{E}).$$

From (2.12), the collision function is then

$$\bar{U}(E) = e^{2i(\omega - \phi)} \left(\frac{1 - \bar{R}(\mathcal{E})L^{0*}}{1 - \bar{R}(\mathcal{E})L^0} \right), \quad (2.16a)$$

the bar having been placed over U to indicate that $\mathfrak{R}(E)$ has been replaced by $\bar{R}(\mathcal{E})$. This function manifests no resonances and is essentially constant in the interval I .

(ii) $\Gamma \ll D$. In this case we have from (2.15c): $\text{Im}[\bar{R}(\mathcal{E})] = \pi s(E)$, so from (2.14) $R(\mathcal{E}) = \bar{R}(E) + R'(\mathcal{E})$, the term $\bar{R}(E)$ representing the net contribution to $R(\mathcal{E})$ from the levels outside of I . If \mathfrak{R}' is defined as having the same relation to \mathfrak{R} as R' has to R :

$$\mathfrak{R}'(E) = \sum_{\lambda'} \frac{\gamma_{\lambda'}^2}{E_{\lambda'} + \Delta_{\lambda'} - E - \frac{i}{2}\Gamma_{\lambda'}},$$

then $\mathfrak{R}(E) - \mathfrak{R}'(E) = R(\mathcal{E}) - R'(\mathcal{E}) = \bar{R}(E)$, and the collision function (2.12) for this case is conveniently rewritten

$$U = e^{2i(\omega - \phi')} \left(\frac{1 - \mathfrak{R}'(E)L'^*}{1 - \mathfrak{R}'(E)L'} \right) \quad (2.16b)$$

where

$$L' = L^0(1 - \bar{R}(E)L^0)^{-1}$$

and

$$\phi' = \phi - \tan^{-1} \left(\frac{\bar{R}(E)P}{1 - \bar{R}(E)S} \right)$$

is the negative of the actual potential scattering phase shift. For example, in the collisions of slow neutrons with nuclei, the potential term appearing in elastic scattering is found to be modified by a factor $[1 - \bar{R}(E)]$ by which the radius is multiplied.

For convenience, we list the various R functions and their properties:

$$R \text{ function} \equiv R(\mathcal{E}) = \sum_{\lambda} \frac{\gamma_{\lambda}^2}{E_{\lambda} - \mathcal{E}}$$

Reduced R function $\equiv \mathfrak{R}(E)$; from (2.9):

$$\mathfrak{R}(E) = \sum_{\lambda} \frac{\gamma_{\lambda}^2}{E_{\lambda} + \Delta_{\lambda} - E - \frac{i}{2}\Gamma_{\lambda}}$$

and, from (2.11), when $\Gamma \gg D$:

$$\mathfrak{R}(E) \approx R(\mathcal{E}) = \sum_{\lambda} \frac{\gamma_{\lambda}^2}{E_{\lambda} - \mathcal{E}}$$

Statistical R function $\equiv R'(\mathcal{E})$ is defined to have the same γ_{λ}^2 and E_{λ} as R inside an interval I , and to have the same statistical distributions of γ_{λ}^2 and E_{λ} outside I . R' has the property, for $\Gamma \gg D$:

$$R'(\mathcal{E}) = i\pi s(E).$$

Reduced statistical R function $\equiv \mathfrak{R}'[E]$ has the same form as $\mathfrak{R}(E)$ above, except that the terms λ outside the interval I are replaced by terms λ' such that the $\gamma_{\lambda'}^2$ and $E_{\lambda'}$ have the same statistical distribution as those of terms λ inside I .

Complex smoothed R function $\equiv \bar{R}(\mathcal{E})$ is defined as (2.14)

$$\bar{R}(\mathcal{E}) = R(\mathcal{E}) - R'(\mathcal{E}) + i\pi s(E),$$

and this can be shown to equal (2.15a):

$$\bar{R}(\mathcal{E}) = \int_{(\infty)} \frac{s(E')dE'}{E' - \mathcal{E}}$$

Real smoothed R function $\equiv \bar{R}(E)$ is defined as

$$\int_{(\infty)} \frac{s(E')dE'}{E' - E}$$

As yet we have assumed nothing about the relative magnitudes of Γ and $s(ds/dE)^{-1}$. If we assume that $\Gamma \ll s(ds/dE)^{-1}$ (which is reasonable for most situations since the latter magnitude is expected to be $\gtrsim 1$ Mev—see subsection 4), then we may use (2.15b) and (2.15c) to deduce from (2.11) and (2.14) that

$$\mathfrak{R}(E) = R'(\mathcal{E}) + \bar{R}(E).$$

In the case (i) where $\Gamma \gg D$, this becomes

$$\mathfrak{R}(E) = i\pi s(E) + \bar{R}(E).$$

Nondiagonal components.—For evaluation of the cc' nondiagonal component of \mathbf{U} , one can eliminate explicit reference to all but the c and c' channels by introducing the reduced \mathbf{R} matrix with components \mathfrak{R}_{cc} , $\mathfrak{R}_{cc'}$, $\mathfrak{R}_{c'c'}$ into the expression (X, 1.5) for the collision matrix. By inverting the two-channel matrix $(1 - \mathfrak{R}_{rr}L_r^0)$ as in (X, 3.12) one finds the nondiagonal component of the W_{rr} matrix to be

$$W_{cc'} = 2iP_e^{\frac{1}{2}}\mathfrak{R}_{cc'}P_{c'}^{\frac{1}{2}}d^{-1}, \quad (2.17)$$

where

$$d = (1 - L_c^0\mathfrak{R}_{cc})(1 - L_{c'}^0\mathfrak{R}_{c'c'}) - L_c^0\mathfrak{R}_{cc'}L_{c'}^0;$$

the approximations (2.9) may be used for the various \mathfrak{R} components, the Γ_λ and Δ_λ now including contributions from the partial widths and shifts of all but the c and c' channels.

(i) When $\Gamma \gg D$, we may use (2.11). It can be shown by means of Rademacher's theorem that $|\mathfrak{R}_{cc'}|^2$ averaged with respect to all possible choices of the random signs of the $\gamma_{\lambda c}$ is equal to $s_c s_{c'}(2\pi D/\Gamma)$, which is negligible compared with the product

$$\mathfrak{R}_{cc}\mathfrak{R}_{c'c'} = \bar{R}_c(\mathcal{E})\bar{R}_{c'}(\mathcal{E})$$

therefore may be replaced by

$$W_{cc'} = 2iP_e^{\frac{1}{2}} \sum_{\lambda} \frac{\bar{\alpha}_{\lambda c}\bar{\alpha}_{\lambda c'}}{E_{\lambda} + \Delta_{\lambda}' - E - \frac{i}{2}\Gamma_{\lambda}'} P_{c'}^{\frac{1}{2}} \quad (2.18a)$$

where

$$\bar{\alpha}_{\lambda c} = (1 - L_c^0\bar{R}_c(\mathcal{E}))^{-1}\gamma_{\lambda c} \quad (2.19)$$

$\bar{\alpha}_{\lambda c}$ is defined by this equation like the $\alpha_{\lambda c}$ of (IX, 1.15) except that R^0 in the latter is replaced here by $\bar{R}(\mathcal{E})$.

(ii) When $\Gamma \ll D$, at a given level λ , we may put

$$\mathfrak{R}_{cc}(E) = \frac{\gamma_{\lambda c}\gamma_{\lambda c}}{E_{\lambda} + \Delta_{\lambda}' - E - \frac{i}{2}\Gamma_{\lambda}'} + \bar{R}_{cc}(\mathcal{E})$$

(and similarly for $\mathfrak{R}_{c'c'}$) and

$$\mathfrak{R}_{cc'}(E) = \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_{\lambda} + \Delta_{\lambda}' - E - \frac{i}{2}\Gamma_{\lambda}'},$$

thereby assuming as in (i) that the contribution from other levels to $\mathfrak{R}_{cc'}$ is much smaller than the contribution $\bar{R}_{cc}(\mathcal{E})$ to \mathfrak{R}_{cc} . Insertion of these expressions in (2.17) gives, after a little rearrangement,

$$W_{cc'} = 2iP_e^{\frac{1}{2}} \sum_{\lambda} \frac{\bar{\alpha}_{\lambda c}\bar{\alpha}_{\lambda c'}}{E_{\lambda} + \Delta_{\lambda} - E - \frac{i}{2}\Gamma_{\lambda}} P_{c'}^{\frac{1}{2}}, \quad (2.18b)$$

where the Δ_λ and Γ_λ now *include* the contributions from channels c and c' , and the $\bar{\alpha}_{\lambda c}$ are the same as defined

by (2.19) except that the $\bar{R}_c(\mathcal{E})$ of (2.19) is replaced by $\bar{R}_c(E)$. In fact, neither of these differences between (2.18a) and (2.18b) is significant; clearly the replacement of Δ_{λ}' and Γ_{λ}' by Δ_λ and Γ_λ in (2.18a) is allowed by the condition $\Gamma \gg D$. Furthermore, the replacement of the $\bar{\alpha}_{\lambda c}$ of (2.18b) by those of (2.19) is permissible since $1 - L_c^0\bar{R}_c(\mathcal{E}) \sim 1 - L_c^0\bar{R}_c(E)$ when $\Gamma \ll D$. Thus we conclude that (2.18b) with $\bar{\alpha}_{\lambda c}$ given by (2.19) is valid when $\Gamma \ll D$ and when $\Gamma \gg D$. One would also presume it to be reasonably accurate when $\Gamma \sim D$ although in that case one may refer directly to (2.17).

From consideration of the term $L_c^0\bar{R}_c(\mathcal{E})$ in $\alpha_{\lambda c}$, it is apparent that, as expected, (2.18) differs from (2.3) when the magnitudes $\langle \Gamma_{\lambda c} \rangle / D$ and $\langle \Delta_{\lambda c} \rangle / D$ approach unity.

This form of the collision matrix may violate the relation (IX, 2.14), although to order of the channel mean of the ratios $\langle \Gamma_{\lambda c} \rangle / D$, which necessarily must be small. This slight defect is readily amended by considering the partial widths and shifts which contribute to the respective totals to be

$$\Gamma_{\lambda c} = 2P_c |\bar{\alpha}_{\lambda c}|^2, \quad \Delta_{\lambda c} = -S_c^0 |\bar{\alpha}_{\lambda c}|^2 \quad (2.20)$$

rather than $2P_c\gamma_{\lambda c}^2$ and $-S_c^0\gamma_{\lambda c}^2$. The equality alternative of (IX, 2.14) is then satisfied. Finally although the nondiagonal components of W have been expressed in the expansion form (IX, 2.5) the diagonal components have not. No particular significance is attributed to this departure other than that it does appear to make \mathbf{W} unitary.

3. Average Cross Sections

(a) Average Total Cross Section

From (X, 3.1), the average total cross section for bombardment with unpolarized particles α is (with $c \equiv \alpha sl$)

$$\langle \sigma_{\alpha}(\text{tot}) \rangle = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} 2g_J (1 - \text{Re} \langle U_{cc}^J \rangle). \quad (3.1)$$

As in (X, 3.2 and 3), this cross section is composed of two parts, the experimental average elastic scattering and absorption cross sections:

$$\langle \sigma_{\alpha}(\text{el}) \rangle = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} g_J \times \{ \langle |1 - U_{cc}^J|^2 \rangle + \sum_{\substack{s'l' (\neq sl) \\ (\alpha' = \alpha)}} \langle |U_{c'c}^J|^2 \rangle \} \quad (3.2)$$

$$\langle \sigma_{\alpha}(\text{abs}) \rangle = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} g_J \times \{ 1 - \langle |U_{cc}^J|^2 \rangle - \sum_{\substack{s'l' (\neq sl) \\ (\alpha' = \alpha)}} \langle |U_{c'c}^J|^2 \rangle \}. \quad (3.3)$$

Evidently the average total cross section is determined by the average of the diagonal elements of the collision

matrix. In contrast, the averages of the experimental elastic scattering and absorption cross sections (3.2) and (3.3) involve other quantities like $\langle |U|^2 \rangle$.

Feshbach, Porter, and Weisskopf¹³ pointed out that there are two ways of splitting the average total cross section (3.1) into two parts. One is the split corresponding to experiment described by (3.2) and (3.3). The second way is more meaningful theoretically and is the split into the so-called "shape elastic" and "compound nucleus formation" cross sections which are defined by

$$\sigma_{\alpha}(\text{s.e.}) = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} g_J |1 - \langle U_{cc}^J \rangle|^2 \quad (3.4)$$

$$\sigma_{\alpha}(\text{c.n.}) = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} g_J (1 - |\langle U_{cc}^J \rangle|^2). \quad (3.5)$$

Clearly the sum of these two identically equals the average total cross section:

$$\sigma_{\alpha}(\text{s.e.}) + \sigma_{\alpha}(\text{c.n.}) = \langle \sigma_{\alpha}(\text{tot}) \rangle = \langle \sigma_{\alpha}(\text{el}) \rangle + \langle \sigma_{\alpha}(\text{abs}) \rangle. \quad (3.6)$$

The two splits are, in general, different and one speaks of the difference between the average experimental elastic and the shape elastic cross sections as the "compound-elastic" cross section:

$$\begin{aligned} \sigma_{\alpha}(\text{c.e.}) &= \langle \sigma_{\alpha}(\text{el}) \rangle - \sigma_{\alpha}(\text{s.e.}) = \sigma_{\alpha}(\text{c.n.}) - \langle \sigma_{\alpha}(\text{abs}) \rangle \\ &= \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} g_J \{ \langle |U_{cc}^J|^2 \rangle - |\langle U_{cc}^J \rangle|^2 + \sum_{\substack{J' \neq J \\ (\alpha' = \alpha)}} |U_{c'c'}^J|^2 \}. \end{aligned} \quad (3.7)$$

Freidman and Weisskopf⁹ have discussed the interpretation of (3.4) and (3.5) from the point of view of the temporal development of a reaction. They point out that (3.4) is just the cross section for the elastic pulse that appears as soon as the incident pulse reaches the target nucleus. The cross section (3.5) corresponds to the delayed pulse arising from the decay of the compound nucleus. This latter cross section may contain an elastic component, *viz.* $\sigma_{\alpha}(\text{c.e.})$ of (3.7), but this is incoherent with the former cross section as a result of the time delay.

Since all of the poles of the collision function are situated in the lower half of the complex energy plane, with the exception of those on the real axis associated with bound states, the path of integration involved in the averaging of U may be displaced, without crossing poles, far enough upward so that R' becomes essentially equal to $i\pi s(E)$, and therefore $\mathfrak{R} = \bar{R}$ according to (2.14). If the average is with respect to an energy interval I such that $D \ll I \ll s(ds/dE)^{-1}$, it may be presumed that the contributions from the connecting sides of the contour effectively cancel and that \bar{R} is nearly constant on the displaced path, in which case

$$\langle U \rangle = \bar{U} \quad (3.8)$$

the quantity \bar{U} being given by (2.15a). The interval I must also be small enough so that the external functions S , P , and ϕ may be considered as constants. It is then evident that $\langle \sigma_{\alpha}(\text{tot}) \rangle$, $\langle \sigma_{\alpha}(\text{s.e.}) \rangle$, and $\langle \sigma_{\alpha}(\text{c.n.}) \rangle$ are obtained by simply substituting \bar{U} for $\langle U \rangle$ in (3.1), (3.4), and (3.5).

No such simple result of a general nature can be obtained for the quantity $\langle |U|^2 \rangle$ which also appears in (3.2), (3.3), and (3.7), because its poles are situated in both halves of the \mathcal{E} plane. However, by assuming that the γ_{λ}^2 of R' are all equal and the levels E_{λ} uniformly spaced by an amount D so that

$$R' = \pi s \tan(\pi \mathcal{E}/D), \quad (3.9)$$

then it is not difficult to include the residues of the poles of $|U|^2$ in the contour used to derive (3.8). The result is that

$$(1 - \langle |U|^2 \rangle)^{-1} = (1 - \langle |U|^2 \rangle)^{-1} + (1 - \omega^{-1})^{-1} \quad (3.10)$$

where

$$\omega = e^{-2\pi\Gamma/D},$$

Γ being the total width for absorption. If we ignore the nondiagonal terms in $U_{c'c}$, it follows that

$$\langle \sigma_{\alpha}(\text{abs}) \rangle = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} g_J \left(\frac{1 - \omega^J}{1 - \omega^J + \omega^J T_c^J} \right) \quad (3.11a)$$

$$\langle \sigma_{\alpha}(\text{c.e.}) \rangle = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} g_J \left(\frac{\omega^J T_c^J}{1 - \omega^J + \omega^J T_c^J} \right) \quad (3.11b)$$

where

$$T_c^J = 1 - |\bar{U}_{cc}^J|^2 \quad (3.12)$$

is the transmission factor in the form proposed by Weisskopf. This result was obtained in a somewhat different manner by Snowden and Whitehead.⁷⁹

According to (3.11) the relative probabilities of compound elastic scattering and absorption are ωT and $(1 - \omega)$, respectively. The following interpretation of these probabilities was suggested by Porter.⁸⁰ According to Weisskopf⁸¹ a period $(2\pi\hbar/D)$ may be attributed to the compound nucleus. Since the decay rate for absorption is (Γ/\hbar) , the probability for absorption in this period is $(1 - \omega)$ while that of no absorption is ω . At the end of each period the system is presumed to be in a configuration for decay into the entrance channel from which it was formed, and the probability for penetration through this channel is the transmission factor T , a quantity which by definition cannot exceed unity. Therefore, the chance of the occurrence of compound elastic scattering in one period is ωT . The periodic motion of the compound nucleus is repeated until there is decay either one way or the other.

⁷⁹ S. C. Snowden and W. D. Whitehead, Phys. Rev. **94**, 1267 (1954).

⁸⁰ C. E. Porter (private communication).

⁸¹ V. F. Weisskopf, Helv. Phys. Acta **23**, 187 (1950); Science **113**, 101 (1951).

Both the detailed calculation by means of (3.9) and the interpretation involving the attribution of a period to the compound nucleus depend upon the assumption of a uniform level spacing. This assumption may be questioned because the alternative view that the behavior of the compound nucleus is "chaotic" would imply that as in the familiar one-level resonance formula the relative probabilities for these decays are Γ for the entrance channel and the corresponding transmission factor $2\pi\Gamma/D$ for absorption. In this case, the probability for compound elastic scattering is relatively larger than ωT , especially when $\omega \ll 1$ or equivalently $2\pi\Gamma/D \gg 1$. This alternative view may correspond to something like a random distribution for the level spacings. Unfortunately there is scant experimental information regarding this distribution, and the true behavior may lie anywhere between these extremes.

(b) *Average Cross Sections for Particular Reactions*

The average cross sections are obtained from (2.18) in the same manner as (2.6) was obtained from (2.3). We have, from (2.18) with (2.20):

$$\langle \sigma_{\alpha\alpha'} \rangle = \frac{\pi}{k_{\alpha}^2} \sum_{J s l s' l'} g_J \left(\frac{T_c^J T_{c'}^J}{\sum_{c''} T_{c''}^J} \right) \quad (3.13)$$

where the transmission factors T_c^J are defined by

$$T_c^J \equiv T_{\alpha s l}^J = \frac{2\pi \langle \Gamma_{\lambda c}^J \rangle}{D^J} \quad (3.14)$$

and $\langle \Gamma_{\lambda c} \rangle$ is determined from (2.20) as

$$\langle \Gamma_{\lambda c} \rangle = 2P_c |\bar{\alpha}_{\lambda c}|^2 = \frac{2P_c \langle \gamma_{\lambda c}^2 \rangle}{|1 - \bar{R}_c(\mathcal{E}) L_c^0|^2} \quad (3.15)$$

It follows from the definition of the strength function s_c below (2.11) that the transmission factor T_c^J may be written

$$T_c^J = \frac{4\pi s_c^J P_c}{|1 - \bar{R}_c(\mathcal{E}) L_c^0|^2} \quad (3.16)$$

On using (2.15b) and (2.16a), it is apparent that this factor is identical with the one defined above in (3.12). In other words $[(\pi/k_{\alpha}^2) g_J] T_c^J$ is the contribution from spin J and channel c to the compound nucleus formation cross section by unpolarized pairs α as defined by (3.5):

$$T_c^J = \left(\frac{\pi}{k_{\alpha}^2} g_J \right)^{-1} \sigma_c^J(\text{c.n.}) \quad (3.17)$$

where σ_c^J denotes the part of (3.5) from spin J and channel c . This simple but important result that the factors T_c^J occurring in the reaction cross section (3.13) are proportional to the cross sections for compound

nucleus formation was first shown by Weisskopf³ on the basis of the principle of detailed balance.

The evaluation of the transmission factors has been surveyed by Blatt and Weisskopf²⁴ for the case when the strength function s_c is chosen in accord with the "strong absorption" model (see subsection 4). For the "moderate absorption" (complex potential) model, some evaluations have been made by Oleska.⁸²

We now discuss angular distributions. The cross section (3.13) does *not* exhibit independence of formation and decay. In other words, it is not simply a product of two factors, one for the formation mode and one for the decay mode. The presence of the sum over J prevents one writing (3.13) as such a product, and this reflects the fact that the compound nucleus cannot quite lose memory of the formation mode, i.e., it must "remember" the total spin J and parity. Of course, each separate term of (3.13) *does* exhibit independence of formation and decay.

In spite of this, there is a certain special situation when (3.13) can be resolved into a product. Let us suppose that the cross section $\sigma_{\alpha\alpha'}$ is averaged not only over compound states but also over residual states α' corresponding to the energy of emission being in a small region $dE_{\alpha'}$ about $E_{\alpha'}$. Let us further suppose that for *all* residual pairs α'' one particle of the pairs α'' (say α_1'') has no excited states and is therefore always in the ground state with spin $I_{\alpha_1''}$; in contrast the nuclei α_2'' , the "residual" nuclei, may be left in any of a very large number of spin states $I_{\alpha_2''}$. This average cross section may be written, from (3.13) as

$$\langle \sigma_{\alpha\alpha'} \rangle dE_{\alpha'} = \frac{\pi}{k_{\alpha}^2} \sum_{J s l s' l'} g_J \times \left[\frac{T_{\alpha s l}^J (\sum_{I'} T_{\alpha' E' s' l'}^J \rho_{\alpha' I'}(\epsilon_{\alpha'}))}{\sum_{\alpha'' s'' I'' l''} \int_0^{\epsilon_{\alpha'} m} T_{\alpha'' E'' s'' l''}^J \rho_{\alpha'' I''}(\epsilon_{\alpha'}) d\epsilon_{\alpha'}} \right] dE_{\alpha'} \quad (3.18)$$

where we now understand α' and α'' to label just the residual nuclei and not particular states of them. The spins $I_{\alpha_2''}$ and $I_{\alpha_2'}$ have been abbreviated to I'' and I' , and the new quantities $\epsilon_{\alpha''}$, $\epsilon_{\alpha'}$ are excitation energies of the residual nuclei α'' , α' . For instance, $\epsilon_{\alpha'}$ is such that its sum with $E_{\alpha'}$, the emission kinetic energy, and the binding energy $b_{\alpha'}$ of the pair α' in the ground state, makes the excitation E^* of the compound nucleus:

$$\epsilon_{\alpha'} + E_{\alpha'} + b_{\alpha'} = E^* \quad (3.19)$$

The maximum value of $\epsilon_{\alpha'}$ is evidently achieved when $E_{\alpha'} = 0$:

$$\epsilon_{\alpha'}^m = E^* - b_{\alpha'} \quad (3.20)$$

We now make a very specific assumption, that the

⁸² S. Oleska, Phys. Rev. **101**, 1034 (1956).

dependence of the level density $\rho_{\alpha''}^{I''}$ of residual nucleus α'' on I'' is the following:

$$\rho_{\alpha''}^{I''}(\epsilon_{\alpha''}) = (2I''+1)\rho_{\alpha''}(\epsilon_{\alpha''}) \quad (3.21)$$

where $\rho_{\alpha''}(\epsilon_{\alpha''})$, without the superscript I'' , is independent of I'' . It is a pity that this assumption has to be made to derive the results below because the various theories of level densities suggest that this form is only appropriate for low values of I'' . (In particular, a number of authors have shown that the dependence on I'' is of the form $(2I''+1)\exp[-(I''+\frac{1}{2})^2/2\sigma^2]$, where σ is a dispersion coefficient. Bloch⁸⁸ has shown that, in a light nucleus, the second factor is significantly less than unity for I'' as low as ~ 3 and excitations less than 12 Mev.) Nevertheless it appears that (3.21) is essential as can be seen on trying to perform the sum over I'' in the denominator of (3.18). For fixed $I_{\alpha_1''}$ and s'' we have

$$\sum_{I''} (2I''+1) = (2s''+1)(2I_{\alpha_1''}+1). \quad (3.22)$$

Where the sum is over all I'' that can satisfy the vector relation: $I'' + \mathbf{I}_{\alpha_1''} = \mathbf{s}''$. For summands other than $(2I''+1)$ one cannot, in general, write the sum in a closed algebraic form and this is why (3.21) is so vital.

Assuming that $T_{\alpha''E_{\alpha''}s''l''J}$ is the same (say $T_{\alpha''E_{\alpha''}l''J}$) for all s'' satisfying the vector relation $\mathbf{s}'' + \mathbf{I}'' = \mathbf{J}$ and zero otherwise, we have

$$\begin{aligned} \sum_{s''} (2s''+1)T_{\alpha''E_{\alpha''}s''l''J} \\ = T_{\alpha''E_{\alpha''}l''J}(2l''+1)(2J+1). \end{aligned} \quad (3.23)$$

This assumption is expected to be approximately correct but not exactly. (For instance in the case of nucleon channels, the known existence of a nucleon spin-orbit force means that the strength function s_c for such channels depends on s and J to some extent. This implies that the transmission factors also depend on s and J .) With (3.23), Eq. (3.18) now reads

$$\begin{aligned} \langle \sigma_{\alpha\alpha'} \rangle dE_{\alpha'} = \frac{\pi}{k_{\alpha}^2} \sum_{Jsl} g_J \\ \times \frac{(2I_{\alpha_1'}+1)T_{\alpha sl}^J (\sum_{l'} (2l'+1)T_{\alpha'E_{\alpha'}l'J})\rho_{\alpha'}(\epsilon_{\alpha'})}{\sum_{\alpha''l''} (2I_{\alpha_1''}+1)(2l''+1) \int_0^{\epsilon_{\alpha''m}} T_{\alpha''E_{\alpha''}l''J}\rho_{\alpha''}(\epsilon_{\alpha''})d\epsilon_{\alpha''}} \\ \times dE_{\alpha'}. \end{aligned} \quad (3.24)$$

When dealing with specific final states α' , the sums over l and l' are not independent but rather are connected by the need to preserve spin J and parity. In the present case however, the fact that residual states of all spins I' and both parities are assumed to participate means that the sum over l' is not restricted by fixing l and J . Thus, if the transmission factors are assumed not to depend on J if J satisfies $\mathbf{s} + \mathbf{I} = \mathbf{J}$, and to be zero other-

wise, one can perform the sums over the values of J and s allowed by $\mathbf{s} + \mathbf{I} = \mathbf{J}$, $\mathbf{I}_{\alpha_1} + \mathbf{I}_{\alpha_2} = \mathbf{s}$:

$$\sum_s \sum_J g_J = 1 \quad (3.25)$$

to obtain

$$\begin{aligned} \langle \sigma_{\alpha\alpha'} \rangle dE_{\alpha'} = \sigma_{\alpha}(\text{c.n.}) \\ \times \frac{(2I_{\alpha_1'}+1)\sigma_{\alpha'}(\text{c.n.})k_{\alpha'}^2\rho_{\alpha'}(\epsilon_{\alpha'})dE_{\alpha'}}{\sum_{\alpha''} (2I_{\alpha_1''}+1) \int_0^{\epsilon_{\alpha''m}} \sigma_{\alpha''}(\text{c.n.})k_{\alpha''}^2\rho_{\alpha''}(\epsilon_{\alpha''})d\epsilon_{\alpha''}} \end{aligned} \quad (3.26)$$

where $\sigma_{\alpha}(\text{c.n.})$ is the form taken on by (3.5) under the assumption that is independent of s and J :

$$\sigma_{\alpha}(\text{c.n.}) = \frac{\pi}{k_{\alpha}^2} \sum_l (2l+1)T_{\alpha l}. \quad (3.27)$$

The form (3.26) for the average reaction cross section is often quoted in the literature. It has the form of a product, and so implies independence of formation and decay. However (3.26) is based on two assumptions, one of which (transmission factors do not depend on s , J) is only approximately true, while the other ($\rho^I \propto (2I+1)$) may be badly violated in practice. The previous form (3.18) is not based on these assumptions so that it is desirable to use this in preference to (3.26) if circumstances and information permit.

The general expression for the angular distribution of the products of a nuclear reaction is given by Eq. (VIII, 2.6):

$$\begin{aligned} d^2\sigma_{\alpha\alpha'} = [k_{\alpha}^2(2I_1+1)(2I_2+1)]^{-1} \\ \times \sum_{Lss'} B_L(\alpha'E_{\alpha'}s',\alpha s)P_L(\cos\theta_{\alpha'})dE_{\alpha'}d\Omega_{\alpha'} \end{aligned} \quad (3.28)$$

where B_L is given by the expression (VIII, 2.7).

This expression contains a sum the individual terms of which are proportional to the absolute squares of components of the collision matrix, in addition to sums of terms which are proportional to the real part of the product of one component and the complex conjugate of another, that is, cross-product terms. The random sign approximation corresponds to the omission of all but the sum of terms proportional to the absolute squares of the collision matrix components. Thus, as indicated by (2.18) in the statistical-theory approximation, the retained terms are proportional to factors such as $|R_{c'e}^J|^2$ whereas the omitted ones are proportional to factors such as $R_{c_1c_1'}^{J_1}R_{c_2c_2'}^{J_2*}$ which are effectively zero when averaged over an energy interval much larger than the level spacing. In this situation B_L becomes

$$\begin{aligned} B_L(\alpha'E_{\alpha'}s',\alpha s) = \frac{1}{4}(-)^{s-s'} \sum_{Jl'l'} Z(JlJl',sL) \\ \times Z(l'Jl'J',s'L) |U_{\alpha'E_{\alpha'}s'l',\alpha sl'}|^2. \end{aligned} \quad (3.29)$$

⁸⁸ C. Bloch, Phys. Rev. 93, 1094 (1954).

The often-quoted result that the distributions of reactions through the compound nucleus have back-and-front symmetry is already apparent at this stage. It is a consequence of the fact that the Z functions of (3.29) vanish unless L is even (since, from (VIII, 2.8a), the Z function contains the factor $(U00|L0)$ which has this property).

To derive the result that the distributions are isotropic, it appears necessary to make both of the assumptions made in connection with (3.26). As said before, this is a pity because assumption (3.21) in particular, is not thought to have a wide range of validity. With (3.21) the differential cross section per unit solid angle and unit interval of excitation energies $E_{\alpha'}$ may be expressed as

$$d^2\sigma_{\alpha\alpha'} = [k_{\alpha'}^2(2I_1+1)(2I_2+1)]^{-1} \times \sum_{ss'I'} \left(\sum_L B_L(\alpha'E_{\alpha'}s'l, \alpha sl) P_L(\cos\theta_{\alpha'}) \right) \times (2I'+1)\rho_{\alpha'}(E_{\alpha'}) dE_{\alpha'} d\Omega_{\alpha'}. \quad (3.30)$$

Evaluation of the sums is performed as follows. For the product of the coefficients, we may use

$$Z(IJl, sL)Z(I'J'l, s'L) = (-)^{(2l+1)(2l'+1)(2J+1)^2} \times (U00|L0)(I'J'00|L0) \sum_{\lambda} (2\lambda+1)(-)^{s-s'-\lambda-L} \times W^2(IJ\lambda s, s'l)W(I'J'\lambda l, \lambda L). \quad (3.31)$$

The sum of the right side of (3.31) can be simplified if it is assumed that $|U_{\alpha'e}^J|^2$ is independent of I' and s' . Then we can use the relations

$$\sum_{I'} (2I'+1) = (2I_{\alpha'}+1)(2s'+1) \quad (3.32)$$

$$\sum_{s'} (2s'+1)(2l+1)W^2(IJ\lambda s, s'l) = 1. \quad (3.33)$$

Finally, on using

$$(-)^{\lambda}(U00|00)(I'J'00|00) = W(I'J'\lambda 0), \quad (3.34)$$

the sum over λ may then be performed by multiplying (3.31) by

$$1 = (-)^{\lambda}W(I'J'\lambda 0)/(U00|00)(I'J'00|00)$$

which gives

$$\sum_{\lambda} (2\lambda+1)W(I'J'\lambda 0)W(I'J'\lambda L) = \delta_{L0}. \quad (3.35)$$

It follows that $L=0$ and that the product distributions are isotropic, as originally shown by Wolfenstein and by Hauser and Feshbach⁸⁴ for the case of inelastic neutron scattering. The reaction cross section, when integrated over angle, agrees with (3.26).

⁸⁴ W. Hauser and H. Feshbach, Phys. Rev. **87**, 366 (1952); L. Wolfenstein, *ibid.* **82**, 690 (1951).

4. Phenomenological Forms of the Pole Strength Function

From Eqs. (3.8) and (2.16a) with (2.15), the vital quantity determining average total cross sections is the pole strength function s . Similarly, from (3.16) and (3.13) this quantity is also decisive for evaluating average reaction cross sections.

Discussions of average total cross sections that have been given in the literature are not framed in terms of the pole strength function but, rather, are based on certain phenomenological models. These theories give prescriptions for an "effective" logarithmic derivative (\bar{f} , say) of the wave function at the nuclear surface as a function of energy. This \bar{f} is defined to be such that its use in the relation (X, 3.8) gives the average collision function:

$$\langle U \rangle = e^{2i(\omega-\phi)} \left(\frac{\bar{f}-L^*}{\bar{f}-L} \right). \quad (4.1)$$

By comparing this equation with (2.16a) and using the relation (3.8) it follows that

$$\bar{f}(E) - B = [\bar{R}(E)]^{-1}. \quad (4.2)$$

This is an important equation which enables us to relate any phenomenological model to our R -matrix (resonance level) theory on the basis of the condition that the two give the same average total cross section. From (2.15b and c) we deduce that, provided

$$\Gamma \ll s(ds/dE)^{-1}$$

(which is satisfied in most situations of interest):

$$\bar{f}(E) - B = \{\bar{R}(E) + i\pi s(E)\}^{-1} \quad (4.3)$$

so that the strength function is related to \bar{f} by

$$s(E) = \frac{1}{\pi} \text{Im}[\bar{f}(E) - B]^{-1} \quad (4.4)$$

and the real smoothed R function is related to \bar{f} by

$$\bar{R}(E) = \text{Re}[\bar{f}(E) - B]^{-1}. \quad (4.5)$$

If a model is sufficiently specific to predict $\bar{f}(E)$ as an analytic function for all E , then such a function must obey, for any B , the relation:

$$\text{Re}[\bar{f}(E) - B]^{-1} = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{\text{Im}[\bar{f}(E') - B]^{-1}}{E' - E} dE' \quad (4.6)$$

that is implied by the relation (2.15b) between $\bar{R}(E)$ and $s(E)$. Usually however a proposed model only specifies $\bar{f}(E)$ is a limited energy range, in which case (4.6) can be satisfied by assuming appropriate behavior for \bar{f} outside the range.

a. The "Strong Absorption" Model of Feshbach, Peaslee, and Weisskopf¹⁵

This phenomenological theory for average cross sections is based on the idea that *all* incident particles crossing the surface of a target nucleus lose energy in collisions and are thereby "absorbed." If we suppose that, just inside the target, the incident particles have wave number K then for any given orbital angular momentum $l < Ka$, we have, dropping subscripts l

$$\bar{f} \equiv \left(\frac{\rho u'}{u} \right)_{r=a} \sim -iKa \quad (4.7)$$

which is the boundary condition for incoming waves. Choosing $B=0$, it follows from (4.4) and (4.5) that

$$s(E) = \frac{1}{\pi Ka}; \quad \bar{R}(E) = 0. \quad (4.8)$$

The strong absorption model is applied to the energy region $0 < E \lesssim 30$ Mev where E signifies incident energy. In other words, the relations (4.7) and (4.8) are only to be applied in this limited energy region, and nothing is said of the form of $s(E)$ in the regions $E < 0$, $E \gtrsim 30$ Mev. This means that the expressions (4.8) for s and $\bar{R}(E)$ can be quite consistent with the relation (2.15b).

Although the strong absorption model makes no specification about $s(E)$ in the region $E < 0$, one can argue that $s = (\pi Ka)^{-1}$ in this region also from the fact that $s(E)$ is a function that is smooth and continuous through the region $E \sim 0$ (since the quantities $\langle \gamma \lambda^2 \rangle$ and D are not affected by the states λ being bound or free). Furthermore, if b is the binding energy of the ground state of the compound nucleus, we can argue that $s(E)$ is zero beyond $E < -b$. If we accept these assertions, and provided from K is either constant or not very strongly energy dependent, it follows that, as a result of (2.15b) we cannot consistently retain the relation $\bar{R}(E) = 0$ in the region $0 < E < 30$ Mev. For instance near $E=0$, the contribution to the integral form of \bar{R}_0 in (2.15b) from the range $-b < E < 0$ is over-compensated by that from the range $0 < E < 30$ Mev, leaving a net positive value for $\bar{R}(E)$. Whatever contribution comes from the region $E > 30$ Mev can only make $\bar{R}(E)$ larger [since such a contribution must be positive as a result of the positive definite nature of $s(E)$].

Thus, although the relation (4.7) is free from objection as long as it and (4.8) are only held to apply in a restricted range, this is no longer true when one makes plausible extrapolations of $s(E)$. Consequently it is of interest to enquire what form emerges for \bar{f} if one starts from a form for $s(E)$ that is specified for most energies.

For instance, let us consider the form

$$s(E) = \begin{cases} (\pi Ka)^{-1} & (E > -b) \\ 0 & (E < -b) \end{cases} \quad (4.9)$$

where

$$K^2 = \frac{2M}{\hbar^2}(E+V). \quad (4.10)$$

Here V is a mean potential energy of order 30 Mev, and K is the wave number corresponding to motion of the incident particle in this potential. (This form (4.9) evidently satisfies the Wigner-Teichmann rule⁵⁵: $\sum_{\lambda} \gamma_{\lambda} e^2 = \infty$ i.e., $\int_{-\infty}^{\infty} s(E) dE = \infty$.) The Stieltjes transform (2.15a), of (4.9) is

$$\begin{aligned} [\bar{f}(E)]^{-1} &= \bar{R}(\mathcal{E}) \\ &= \left(\frac{\hbar^2}{2Ma^2} \right) \frac{x-iy}{(V-b)^{\frac{1}{2}}} \left\{ \frac{1}{2\pi} \ln \frac{(1+x)^2 + y^2}{(1-x)^2 + y^2} \right. \\ &\quad \left. + i \left[1 - \frac{1}{\pi} \tan^{-1} \left(\frac{y}{1+x} \right) - \frac{1}{\pi} \tan^{-1} \left(\frac{y}{1-x} \right) \right] \right\} \end{aligned}$$

where

$$\begin{aligned} x &= \left(\frac{\frac{1}{2}(V-b)}{H^2 + \frac{1}{4}\Gamma^2} \right)^{\frac{1}{2}} \left[(H^2 + \frac{1}{4}\Gamma^2)^{\frac{1}{2}} + H \right]^{\frac{1}{2}} \\ y &= \left(\frac{\frac{1}{2}(V-b)}{H^2 + \frac{1}{4}\Gamma^2} \right)^{\frac{1}{2}} \left[(H^2 + \frac{1}{4}\Gamma^2)^{\frac{1}{2}} - H \right]^{\frac{1}{2}} \\ H &= E - \Delta + V, \end{aligned} \quad (4.11)$$

the inverse tangents being evaluated in the first quadrant. The absorption width Γ is generally much less than the characteristic potential energy V , so that the above expression may be accurately approximated by

$$\begin{aligned} [\bar{f}(E)]^{-1} &= \underset{\mathcal{E} \rightarrow E}{L} \bar{R}(\mathcal{E}) \\ &= (\pi Ka)^{-1} \log \left\{ \frac{H^{\frac{1}{2}} + (V-b)^{\frac{1}{2}}}{H^{\frac{1}{2}} - (V-b)^{\frac{1}{2}}} \right\} + \frac{i}{Ka}. \end{aligned} \quad (4.12)$$

In typical cases, the real and imaginary parts of this transform are of about the same magnitude. In this theory the potential scattering is not given by the hard-sphere formula but by that which is obtained from the phase shift $-\phi'$ of (2.16b). Just as the usual strong coupling theory corresponds to a surface wave function $u \sim e^{-iKr}$ as implied by (4.7), so the revised theory corresponds to the form $u \sim e^{(K'-iK)r}$ with the damping coefficient of about the same magnitude as K .

b. "Moderate Absorption" ("Complex Potential") Model of Feshbach, Porter, and Weisskopf¹³

For the phenomenological treatment of average nucleon cross sections, the strong absorption model has been superseded by the more successful "moderate absorption" or complex potential model. This is based

on the idea that incident particles inside a target nucleus effectively move in a complex potential well $-(V(r)+iW(r))$ with $V, W > 0$. The real attractive potential $-V$ simply refracts incident nucleons while the presence of the imaginary term $-iW$ implies "absorption" of nucleons. According to the physical interpretation of the model (which does not concern us here so is not discussed in detail), the absorption is not regarded as loss of flux from the entrance channel, but as absorption of the incident particle into a compound nucleus, which may eventually re-emit the particle into the entrance channel. Put otherwise, the absorption potential is not associated necessarily with the reaction cross sections; in fact, one can still apply the model, without modification, to the energy region where only elastic scattering is energetically possible.

For simplicity, we consider that V and W are rectangular wells of equal radius a (but different depth). The wave function inside such a square complex well is

$$u = F(K_r) \sim \sin\left(K_r r - \frac{l\pi}{2}\right), \quad (4.13)$$

where

$$K^2 = \frac{2M}{\hbar^2}(E + V + iW),$$

the approximation holding for $l < Ka$. The logarithmic derivative corresponding to (4.13) is, setting $\mu = Ka$

$$\bar{f} = \frac{\mu F'(\mu)}{F(\mu)} \sim \mu \cot\left(\mu - \frac{l\pi}{2}\right). \quad (4.14)$$

From the discussion of Sec. IV, 3, this function can be written, on replacing E by $E + iW$ as

$$(\bar{f} + l)^{-1} = \sum_p \frac{(\hbar^2/Ma^2)}{E_p - E - iW} \quad (4.15)$$

where the E_p are the energies of single particle levels of the real potential V that are determined by the boundary condition $B = -l$.

From (4.2) it follows that the corresponding \bar{R} is

$$\bar{R}(\mathcal{E}) = \sum_p \frac{\hbar^2/Ma^2}{E_p - E - iW} \equiv \sum_p \frac{\hbar^2/Ma^2}{E_p' - \mathcal{E} - iW'} \quad (4.16)$$

where $E_p' = E_p + \Delta$, $W' = W - \Gamma/2$ and, from (4.4) and (4.5), we have

$$\pi s(E) = \sum_p \frac{W'(\hbar^2/Ma^2)}{(E_p' - E)^2 + W'^2} \quad (4.17)$$

$$\bar{R}(E) = \sum_p \frac{(E_p' - E)(\hbar^2/Ma^2)}{(E_p' - E)^2 + W'^2}.$$

Of course, Δ and Γ are small compared with the other magnitudes like W and so they can be put equal to zero in (4.17) and (4.18), i.e., $E_p' = E_p$, $W' = W$.

If V and W are constant, (4.15) expresses the logarithmic derivative at the edge of the potential well for all energies from $-\infty$ to ∞ . It is automatic that (2.15b) and (4.6) are satisfied since the function \bar{R} of (4.16) is an R function (*viz.*, that of a potential well). Alternatively, if Eqs. (4.14) to (4.17) are only to be applied in a limited range like $0 < E \lesssim 30$ Mev, there is also no problem of consistency with (2.15b) even if V and W are allowed to vary with energy because behavior outside the range can be found to satisfy (2.15b).

A little study of (4.16) and (4.17a and b) shows that these relations between the complex potential model and \mathbf{R} -matrix theory are not quite as satisfactory as they at first appear. In fact, these relations are really not valid, as exemplified by the fact that $s(E)$, as given by (4.17a) does not vanish for energies $E < -b$. To understand this paradoxical situation, we must investigate more carefully the connection between the function (4.15) of the square potential well model and the \bar{R} function of \mathbf{R} -matrix theory.

We consider the latter, and recall from Secs. II and III that one of the basic assumptions of \mathbf{R} -matrix theory is that the surfaces S_e at which the strength functions are defined must be large enough so that there is negligible overlap between the two separating systems. For instance, in the case of a neutron channel, when a neutron and the residual nucleus are separated by distance $r = a$, there must be negligible probability of any neutrons of the residual nucleus also being at $r = a$; otherwise the basic orthogonality relation (III, 2.2) is not correct. Now let us suppose that the actual nuclear system is describable as a system of neutrons (and protons) moving in a square potential well. Since many neutrons may be found simultaneously at the edge of the well, it follows that the interaction radius of the \mathbf{R} -matrix theory must be taken some way outside the edge of the well. In other words, it is not possible to even define the strength function in terms of a surface taken at the edge of the well. It can only be defined in terms of a surface taken at least as far outside the well edge as the reciprocal of the external wave number for the lightest bound nucleons, i.e., the order of 1 to 2×10^{-13} cm. Consequently, to relate the potential well model to the strength function of \mathbf{R} -matrix theory, we must form the function $(\bar{f} + 1)^{-1}$, not at the edge of the well as in (4.16), but further out. We may always assume an expansion of $(\bar{f} + 1)^{-1}$ at any given radial distance $r = a$:

$$(\bar{f} + l)^{-1} = \sum_p \frac{\gamma_p^2}{E_p - E - iW_p} \quad (4.18)$$

where \bar{f} , γ_p^2 , E_p , and W_p all depend upon a . The difference between (4.15) and (4.18) is that, in the latter, γ_p^2 and W_p depend on p . In particular, if $r = a$ is chosen far enough outside the edge of the well, the values of γ_p^2 for the lowest orbits p are very small, so these orbits contribute very little to the sum. In this case, the

imaginary part of (4.18) drops essentially to zero for energies less than $E \sim -b$. This is, of course, just the proper feature of the strength function. Thus in contrast to (4.16), the $(\bar{f}+l)$ function of (4.18) can sensibly be identified with the function $\bar{R}(\mathcal{E})$ characterizing the **R**-matrix (fine structure) approach.

The paradox associated with (4.16) and (4.17) would not have appeared so strongly if the well had been chosen to have sloping sides. If \bar{f} is evaluated at the edge of such a well, the γ_p^2 for the lower p orbits are already small, as required, because of the exponential attenuation of wave functions in the sloping region.

Finally we comment on the effect on the strength function of including a spin-orbit coupling term $U(r)\mathbf{l}\cdot\boldsymbol{\sigma}$ with the potential $-(V+iW)$. In the absence of such a term, the strength function for any channel $c \equiv \alpha s l$, and total spin J is given by the $s(E)$ of (4.17) and is the same for all s , J consistent with $\mathbf{s}+\mathbf{l}=\mathbf{J}$. In the presence of a spin-orbit term, it is convenient to define new reduced width amplitudes $\gamma_{\lambda(\alpha j I \alpha_2)}^J$ in which the coupling sequence: $\mathbf{I}_{\alpha_1}+\mathbf{I}_{\alpha_2}=\mathbf{s}$, $\mathbf{l}+\mathbf{s}=\mathbf{J}$, is replaced by $\mathbf{I}_{\alpha_1}+\mathbf{l}=\mathbf{j}$, $\mathbf{I}_{\alpha_2}+\mathbf{j}=\mathbf{J}$ (where we assume α_1 to be a nucleon so that $\mathbf{I}_{\alpha_1}=2\boldsymbol{\sigma}$). With the aid of a little Racah algebra, it can be shown that

$$\gamma_{\lambda(\alpha s l)}^J = \sum_{j=l\pm\frac{1}{2}} U(\frac{1}{2}I_{\alpha_2}Jl, sj)\gamma_{\lambda(\alpha j I \alpha_2)}^J. \quad (4.19)$$

From the properties of the U function [which is defined as $(2s+1)^{\frac{1}{2}}(2j+1)^{\frac{1}{2}}$ times the W function] it follows that, as expected,

$$\sum_s [\gamma_{\lambda(\alpha s l)}^J]^2 = \sum_j [\gamma_{\lambda(\alpha j I \alpha_2)}^J]^2. \quad (4.20)$$

In the presence of a spin-orbit force, the strength functions $s_{\alpha j I \alpha_2}^J$ in the $(\alpha j I \alpha_2)$ scheme depend on j , but are the same for all J consistent with $\mathbf{I}_{\alpha_2}+\mathbf{j}=\mathbf{J}$. The strength function for the two j values, $j=l\pm\frac{1}{2}$, have the same shape as (4.17) but are split in energy. Assuming that the $\gamma_{\lambda(\alpha j I \alpha_2)}^J$ have random signs, it follows from (4.19) that

$$s_{\alpha s l}^J = \sum_{j=l\pm\frac{1}{2}} U^2(\frac{1}{2}I_{\alpha_2}Jl, sj)s_{\alpha j I \alpha_2}^J. \quad (4.21)$$

This shows that, in general, for given channel spin, $s_{\alpha s l}^J$ depends on J , and vice versa.

Equation (4.20) shows that $\sum_s s_{\alpha s l}^J$ may take one of three values depending on whether J is consistent with $\mathbf{J}=\mathbf{I}_{\alpha_2}+\mathbf{j}$ for

- (i) $j=l+\frac{1}{2}$ but not for $j=l-\frac{1}{2}$
- (ii) $j=l-\frac{1}{2}$ but not for $j=l+\frac{1}{2}$
- (iii) both.

5. Theoretical Form of the Pole Strength Function for Nucleon Channels

Two types of attack have been made upon the problem of accounting for the complex potential model and

its successes. One^{85,21} accepts the broad presumption of the complex potential model that the nucleus affects incident particles like a classical optical sphere affects light. With this presumption one can estimate values for V and W in terms of observed properties of nucleon-nucleon scattering, the essential relation being, for $V \gg W, E$:

$$V+iW = \hbar v \rho \left\langle \frac{2\pi}{k_r} f(0) \right\rangle \quad (5.1)$$

where the average is over all relative wave numbers k_r of the incident nucleon and the target nucleons, and $f(0)$ is the forward scattering amplitude for nucleon-nucleon scattering, v is the velocity of the incident particles inside the target and ρ is the density of target nucleons. A second approach, due to Bloch,¹⁸ is altogether more fundamental in that it does not use the optical analogy. It begins with the fact that the empirical success of the complex potential model provides one with a certain empirical form for the smoothed \bar{R} function, namely the form (4.16). Then it attempts to derive this form from a theory of nuclear structure.

From (2.14), since $R'(\mathcal{E}) \sim i\pi s(E)$ when the imaginary part of $\mathcal{E} \gg D$, it follows that, for such an \mathcal{E} :

$$\bar{R}(\mathcal{E}) = R(\mathcal{E}) = \sum_{\lambda} \frac{\gamma_{\lambda}^2}{E_{\lambda} - \mathcal{E}}. \quad (5.2)$$

Now let us expand the states X_{λ} in terms of a complete set of states of the form:

$$X_{c p} = \phi_c u_p(r_A) \quad (5.3)$$

where $\phi_c \equiv \phi_{\alpha s l}$ are the channel wave functions defined in (III, 2.25) and $u_p(r_A)$ are the radial wave functions of a set of single particle states of the A th nucleon defined for some real potential $-V$ with the same boundary condition as the X_{λ} . The expansion of X_{λ} is

$$X_{\lambda} = \sum_{c p} C_{\lambda, c p} X_{c p}. \quad (5.4)$$

From the definition of reduced width amplitude in (III, 4.7a), it follows that

$$\gamma_{\lambda c} = \left(\frac{\hbar^2}{2M_c a_c} \right)^{\frac{1}{2}} \sum_p C_{\lambda, c p} u_p(a_c). \quad (5.5)$$

Inserting this expansion in (5.2) gives

$$\bar{R}_{cc}(\mathcal{E}) = \left(\frac{\hbar^2}{2M_c a_c} \right) \sum_{p p'} u_p(a_c) u_{p'}(a_c) \times \left(X_{c p} \left| \frac{1}{H - \mathcal{E}} \right| X_{c p'} \right) \quad (5.6)$$

⁸⁵ M. Goldberger, Phys. Rev. 74, 1269 (1948); A. M. Lane and C. F. Wandel, Phys. Rev. 98, 693 (1955); E. Clement and C. Villi, Nuovo cimento 10, 176 (1955); Morrison, Muirhead, and Murdock, Phil. Mag. 46, 475 (1955).

where we have used closure to sum over λ and introduced H , the total Hamiltonian operator. This equation is an important one. It relates the vital quantity $\bar{R}(\mathcal{E})$ that determines average cross sections to a set of nuclear matrix elements involving only a particular state of the target nucleus (the ground state in practice). All reference to the compound states λ has been eliminated.

It may appear that the discussion so far is general and exact. This is not true however because of the rather subtle role of antisymmetrization. We have implicitly assumed that the states X_λ have the same symmetry as the χ_{cp} , which are antisymmetric in particles $1 \dots (A-1)$ but not in the A th particle. We now enquire what happens when X_λ is made antisymmetric in all nucleons. An expansion of the type (5.4), where the χ_{cp} are *not* antisymmetric in the A th particle, is still possible. (It is, in fact, a "fractional parentage" expansion.) The contribution of the channel surface for the A th particle to the reduced width amplitude then equals $A^{-\frac{1}{2}}$ times (5.5). Since X_λ is antisymmetric in all nucleons, each channel surface must contribute the same amount so the final reduced width amplitude is $A^{\frac{1}{2}}$ times (5.5). Thus it might seem that the effect of antisymmetrization upon (5.6) is to multiply the right-hand side by a factor A . This is not correct because, in deriving (6.6), a closure relation has been assumed and this is now invalid. The essential point is that, when X_λ is completely antisymmetric, the expansion (5.4) is not reversible, since one can expand a state in terms of states of lower symmetry (as in (5.4)), but not vice versa. It follows that there is no basis for the relation

$$\sum_{\lambda} C_{\lambda, cp}^* C_{\lambda, c'p'} = \delta_{cp, c'p'} \quad (5.7)$$

or for the closure relation used in (5.6). To avoid this difficulty one can now try to frame our discussion on a new set χ_{cp} , namely a set which is completely antisymmetrized. This means that the states X_λ and χ_{cp} have the same degrees of symmetry. Unfortunately a new snag appears. Although one can now expand the χ_{cp} in terms of the X_λ (with the consequence that closure is valid), the expansion (5.4) itself is no longer correct and so (5.5) is not correct. The cause of this breakdown in (5.4) is that the fully antisymmetric χ_{cp} are no longer orthonormal, i.e.,

$$\int \chi_{cp}^* \chi_{c'p'} \equiv \langle \chi_{cp} | \chi_{c'p'} \rangle \neq \delta_{cp, c'p'}. \quad (5.8)$$

However, it seems likely that they are approximately orthonormal. According to the "intermediate model" that we discuss below, the most important target states in the sum (5.4) are bound states and the important single particle states u_p are free states. In this case, since the particle states do not occur in the target states, u_p is orthogonal to the target states and the χ_{cp}

are orthonormal. Accepting this point of view, when the χ_{cp} and X_λ are antisymmetric, (5.4), (5.5), and (5.6) are still valid (without any factor of $A^{\frac{1}{2}}$), subject to only a small correction from the lack of orthonormality of the χ_{cp} .

Returning to (5.6), we now introduce the so-called "intermediate" assumption of Lane, Thomas, and Wigner.⁸⁶ One argues that, in the reverse expansion to (5.4):

$$\chi_{cp} = \sum_{\lambda} C_{\lambda, cp}^* X_{\lambda} \quad (5.9)$$

the states λ that account for most of the sum are confined to a restricted energy band about the energy of the state χ_{cp} say E_{cp} . (This is the sum of the energy E_c of the target state of the channel c and the energy E_p of the single particle state.) In other words (see Fig. 2), the situation is intermediate between the case when only one term contributes (i.e., no coupling at all between an incident nucleon and the target nucleons) and the case when terms from all energies contribute (i.e., strong coupling in which the single particle states p are completely spread out or "dissolved"). The implication of this picture for (5.5) and (5.6) is that, for a given energy E , the sums are dominated by the nearest single particle level p so that we may drop cross terms $p \neq p'$ and put

$$\gamma_{\lambda c} \approx \left(\frac{\hbar^2}{2M_c a_c} \right) \sum_p u_p^2(a_c) C_{\lambda, cp}^2 \quad (5.10)$$

$$\bar{R}_{cc}(\mathcal{E}) \approx \left(\frac{\hbar^2}{2M_c a_c} \right) \sum_p u_p^2(a_c) \left(\chi_{cp} \left| \frac{1}{H - \mathcal{E}} \right| \chi_{cp} \right). \quad (5.11)$$

We note three things in connection with these equations:

(i) Since $\sum_{\lambda} C_{\lambda, cp}^2 = 1$ (5.10) demonstrates that the *total* reduced width of the original single particle level is preserved.

(ii) For a square well with boundary condition $B = -l$, $u_p^2(a) = (2/a)$ (see Sec. IV, 3).

(iii) The assumption of random signs for contributions of different p would also justify (5.10) and (5.11) but this assumption is probably not correct (see the following).

Bloch attempted the evaluation of the matrix element in (5.11). For the A th term in the antisymmetrized χ on the left, we make the expansion of $1/H - \mathcal{E}$ in powers of $H' \equiv H - H_0$:

$$\begin{aligned} \frac{1}{H - \mathcal{E}} &= \frac{1}{H_0 - \mathcal{E}} + \frac{1}{H_0 - \mathcal{E}} \frac{H'}{H_0 - \mathcal{E}} \\ &+ \frac{1}{H_0 - \mathcal{E}} \frac{H'}{H_0 - \mathcal{E}} \frac{H'}{H_0 - \mathcal{E}} \frac{1}{H_0 - \mathcal{E}} \dots \quad (5.12) \end{aligned}$$

⁸⁶ Lane, Thomas, and Wigner, Phys. Rev. **98**, 693 (1955).

In the split $H=H_0+H'$, one has in mind that H_0 should be the sum of the Hamiltonian for the target and the single particle Hamiltonian $T+(-V)$, where $-V$ is a suitable mean potential for the i th particle. H' is the difference between the actual potential $\sum_j v_{ij}$ felt by the i th particle and the mean potential:

$$H' = \sum_j v_{ij} - (-V). \quad (5.13)$$

In evaluating (5.11), there will be a direct and an exchange term for each i . The exchange term will be ignored so that the matrix element will now be equal to that with nonantisymmetric χ_{cp} with a definite particle (say, the A th) in the single particle orbit p .

Although (5.12) has the form of a perturbation series, Bloch does not evaluate it as such. Rather he assumes that the matrix elements $(cp|H'|c'p')$ have random signs. For any given term in the series (5.12), this enables him to pick out a "leading part." This is the part corresponding to each alternate operation by H' leading back to cp . Choosing V such that $(cp|H'|cp)=0$ then gives the following form for the matrix element of (5.11):

$$\begin{aligned} \left(\chi_{cp} \left| \frac{1}{H-\mathcal{E}} \right| \chi_{cp} \right) &= -\sum_{n=0}^{\infty} \frac{1}{(\mathcal{E}-E_{cp})^n} \left[\frac{\varphi_{cp}(\mathcal{E})}{\mathcal{E}-E_{cp}} \right]^n \\ &= [E_{cp} - \mathcal{E} + \varphi_{cp}(\mathcal{E})]^{-1} \end{aligned} \quad (5.14)$$

where $\varphi_{cp}(\mathcal{E})$ is defined as

$$\begin{aligned} \varphi_{cp}(\mathcal{E}) &= \sum_{c'p'} \frac{|(cp|H'|c'p')|^2}{\mathcal{E}-E_{c'p'}} \\ &= \left(cp \left| H' \frac{1}{\mathcal{E}-H_0} H' \right| cp \right). \end{aligned} \quad (5.15)$$

On changing the sum of (5.15) into an integral, and then letting $\mathcal{E} \rightarrow E$:

$$\varphi_{cp}(E) = \Delta_{cp}(E) - iW_{cp}(E) \quad (5.16)$$

where

$$\Delta_{cp}(E) = \text{Pr} \int \frac{|(cp|H'|c'p')|^2}{E-E_{c'p'}} \rho(E_{c'p'}) dE_{c'p'} \quad (5.17)$$

$$\begin{aligned} W_{cp}(E) &= \pi \rho_{cp}(E) \int |(cp|H'|c'p')|^2 \\ &\quad \times \delta(E-E_{c'p'}) dE_{c'p'}. \end{aligned} \quad (5.18)$$

In the integral for Δ , the principal value is to be taken. By combining (5.16), (5.14) and (5.11) the final expression for $\bar{R}(\mathcal{E})$ is,

$$\bar{R}_{cc}(\mathcal{E}) = \sum_p \frac{\frac{a_c}{2} u_p^2(a_c) \left(\frac{\hbar^2}{M_c a_c^2} \right)}{E_{cp} + \Delta_{cp}(E) - \mathcal{E} - iW_{cp}(E)}. \quad (5.19)$$

This expression is very similar to the empirical form (4.16) implied by the complex potential model. In fact,

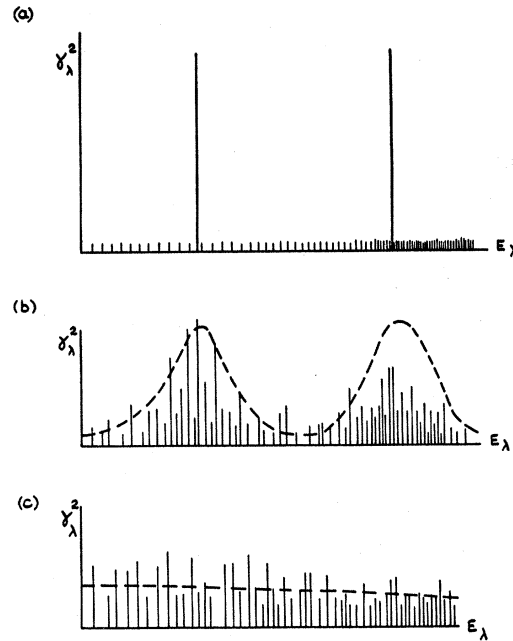


Fig. 2. Schematic plots of γ_{λ}^2 against E_{λ} , each γ_{λ}^2 being denoted by the height of the vertical line put at the corresponding energy E_{λ} . (a) Illustrates the situation when there is almost no coupling between an incident particle and the target nucleons so that no reactions are possible and only shape elastic scattering occurs. In this case each eigenstate λ is a simple product state of an internal wave function of the colliding particles and a wave function of relative motion. The only appreciable values of γ_{λ}^2 are those at the states λ corresponding to the colliding particles being in their ground states and the energy being at single particle resonances. Thus the plot of γ_{λ}^2 consists of isolated spikes $\gamma_{\lambda}^2 \sim (\hbar^2/Ma^2)$. (c) illustrates the opposite extreme situation of very strong coupling. In this case the single particle (i.e., the simple product) states are entirely "dissolved" amongst the myriads of fine structure states. The broken line represents the strength function $s = \langle \gamma_{\lambda}^2 \rangle / D$ which is smooth and monotonic in this situation. (b) represents the intermediate situation where the single particle states are dissolved amongst the fine structure states but the sharing is confined to restricted energy regions around the single particle energy levels. The strength function s is shown to have a corresponding oscillatory behavior. One noteworthy feature of (b) and (c) is that the sharing should be such as to conserve the total strength \hbar^2/Ma^2 within the region of each original single particle level, i.e., $\sum_{\lambda} \gamma_{\lambda}^2 (= \int s dE) \approx (\hbar^2/Ma^2)$.

if Δ_{cp} and W_{cp} do not depend on p , it is evident from comparison with the R -function expansion (IV, 1.10) for the potential well $(-V)$:

$$R_V(\mathcal{E}) = \sum_p \frac{\frac{a_c}{2} u_p(a_c) \left(\frac{\hbar^2}{M_c a_c^2} \right)}{E_p - \mathcal{E}} \quad (5.20)$$

that $\bar{R}_{cc}(\mathcal{E})$ is simply the same thing evaluated at the energy $E - \Delta_{cp} - E_c - iW_{cp}$:

$$\bar{R}_{cc}(\mathcal{E}) = R_V(\mathcal{E} - \Delta_{cp} - E_c + iW_{cp}). \quad (5.21)$$

Put otherwise, R_{cc} is the R function for the potential well $-V_c'$ equal to $(-V)$ modified by the addition of the complex part: $E_c + \Delta_{cp} - iW_{cp}$. Remembering that

$-V$ has been previously defined to make $(c\mathcal{P}|H'|c\mathcal{P})=0$, it follows that the real part of the effective potential $-V_{c'}$ is

$$\text{Re}(-V_{c'}) = (c\mathcal{P}|\sum_j v_{ij}|c\mathcal{P}) + \int \frac{|(c\mathcal{P}|v_{ij}|c'\mathcal{P}')|^2}{E-E_{c'\mathcal{P}'}} \rho(E_{c'\mathcal{P}'}) dE_{c'\mathcal{P}'}. \quad (5.22)$$

There are corrections to (5.19) and (5.21) due to the assumptions made in the derivation. Especially there is the neglect of exchange terms, the selection of "leading parts" to arrive at (5.14) and the neglect of cross terms $\mathcal{P} \neq \mathcal{P}'$ in passing from (5.6) to (5.11). Using the results of the next subsection, the leading parts of the terms $\mathcal{P} \neq \mathcal{P}'$ can be summed to give a contribution to \bar{R}_{cc} of $(cu_E|H'|cu_E)$ where u_E is the single particle wave function in the complex potential $V_{c'}$.

One noteworthy feature of (5.17), (5.18), and (5.20) is that, although the forms are very similar to familiar expressions from second-order perturbation theory, there is a detailed difference, namely that the expressions are energy dependent as a consequence of the appearance of E instead of $E_{c\mathcal{P}}$.

Cini and Fubini⁸⁷ have used perturbation theory to arrive at results similar to (5.17) and (5.18). They have made numerical estimates of (5.18) on the basis of a Fermi gas model for the target nucleus and conclude that there is qualitative agreement with the empirical values of W . In particular the strong increase of W with E is derived and found, as expected,⁸¹ to result from the action of the Pauli principle applied to the target. An interesting point in connection with using a Fermi gas to evaluate (5.18) is that this procedure can be shown to be equivalent to the computation of W from the optical formula (5.1), except that the latter contains the *observed* total cross section [i.e., the imaginary part of $f(0)$], whereas the former involves the Born approximation to this same quantity.

Lane, Thomas, and Wigner⁸⁶ tried to estimate W as the square root of the second moment of the distribution of $C_{\lambda, c\mathcal{P}^2}$ in (5.10):

$$M_2 \equiv \sum_{\lambda} (E_{c\mathcal{P}} - E_{\lambda})^2 C_{\lambda, c\mathcal{P}^2} = (c\mathcal{P}|H'^2|c\mathcal{P}). \quad (5.23)$$

The trouble with this approach is that it gives too much weight to the distant components with the consequence that, if the distribution of $C_{\lambda, c\mathcal{P}^2}$ has a long tail, W will be badly overestimated. In fact, the numerical value (~ 20 Mev) obtained for $M_2^{\frac{1}{2}}$ when a Fermi gas is used for the target nucleus is considerably larger than the observed values for W . Some improvement is found if correlations are introduced between the incident nucleon and the target nucleons in the

⁸⁷ M. Cini and S. Fubini, *Nuovo cimento* **10**, 75 (1955).

wave function $\chi_{c\mathcal{P}}$ according to work by Vogt⁸⁸ and by Lane and Verlet.⁸⁹

Bloch¹⁸ pointed out that M_2 of (5.23) may be written as

$$M_2 = \int |(c\mathcal{P}|H'|c'\mathcal{P}')|^2 \rho(E_{c'\mathcal{P}'}) dE_{c'\mathcal{P}'}. \quad (5.24)$$

If the distribution of matrix elements $(c\mathcal{P}|H'|c'\mathcal{P}')$ is assumed to be fairly smooth up to some cutoff energy separation $(E_{c\mathcal{P}} - E_{c'\mathcal{P}'}) = \epsilon$, then

$$(W/M_2^{\frac{1}{2}}) \approx (\pi W/2\epsilon)^{\frac{1}{2}}. \quad (5.25)$$

This equation corroborates our surmise above that if the distribution of $C_{\lambda, c\mathcal{P}^2}$ is well spread out, i.e., $\epsilon \gg W$, then $W \ll M_2^{\frac{1}{2}}$.

Wigner⁹⁰ obtained very much the same results as Bloch for the form of the strength function (i.e., π^{-1} times the imaginary part of (5.19)) by making much the same assumptions but using a quite different mathematical method. He starts from the Hamiltonian equation in matrix form

$$\mathbf{H}\mathbf{C}_{\lambda} = \mathbf{E}_{\lambda}\mathbf{C}_{\lambda} \quad (5.26)$$

where \mathbf{H} is in the $\chi_{c\mathcal{P}}$ representation and \mathbf{C}_{λ} is a column vector with components $C_{\lambda, c\mathcal{P}}$ of (5.4). The problem of finding the strength function s_c near $E\mathcal{P}$ is, from (5.10), equivalent to determining $\langle C_{\lambda, c\mathcal{P}^2} \rangle_I / D_I$, where D_I is the mean spacing of eigenvalues in the interval I . Thus the problem is reduced to a purely mathematical one requiring considerable familiarity with matrix algebra. Wigner makes the assumption that the non-diagonal elements of H have random signs but equal magnitudes as far as a certain number of terms off the diagonal. Beyond that point (corresponding to energy ϵ), the elements are supposed to be zero. The diagonal elements are assumed to be equally spaced. With these assumptions, in the case when $|(c\mathcal{P}|H'|c'\mathcal{P}')|^2 \rho \ll \epsilon$, Wigner's result for the strength function agrees with Bloch's.

6. Implications of a Partial Breakdown in the Assumption of Random Signs

Let us consider the element of the collision matrix for a particular inelastic scattering process $c \rightarrow c'$. The essential part, as given by (2.3) or (2.18) is the sum:

$$\sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{E_{\lambda} + \Delta_{\lambda} - E - \frac{i}{2} \Gamma_{\lambda}}. \quad (6.1)$$

For the evaluation of the cross sections (i.e., essentially the square of this quantity) in subsection 2 [see

⁸⁸ E. Vogt, *Phys. Rev.* **101**, 1792 (1956); E. Vogt and J. Lascoux, *ibid.* **107**, 1028 (1957).

⁸⁹ A. M. Lane and L. Verlet, *Phys. Rev.* **100**, 956 (1955).

⁹⁰ E. P. Wigner, *Ann. Math.* **62**, 548 (1955).

(2.4), (2.5), and (2.6)], we made the assumption that the signs of the $\gamma_{\lambda c}$ and $\gamma_{\lambda c'}$ are random. This implies that the contributions from states λ outside a certain limited region about E can be ignored, and leads to the conventional compound nucleus cross sections, (2.6) or (3.13), exhibiting independence of formation and decay. It is now of interest to examine the assumption of random signs, and, in particular, to enquire how the far-away terms contribute to (6.1) when the assumption is not strictly valid.

With the strong coupling model discussed in subsection 4, the random sign approximation is expected to be valid over almost arbitrarily large energy intervals (except, ultimately, over the infinite energy range where the Wigner-Teichmann rule⁵⁵ $\sum_{\lambda} \gamma_{\lambda c} \gamma_{\lambda c'} = 0$ for $c \neq c'$ imposes a mild restraint). We base this statement on the usual idea of the "strong absorption" theory that nuclear motion is completely chaotic, with no vestige of any single particle or other coherent motion. Following on from this notion, it is plausible that, in cases where single particle motion exists, it should be associated with a partial breakdown in the random sign approximation. This would not apply over arbitrarily small energy intervals, but would only be expected to appear as a long-range correlation on summing over all levels within single particle resonance (broadened by any absorption potential W). Clearly if W is increased the correlation should become more and more mild until ultimately the strong-absorption situation is achieved.

We may arrive at the same conclusions by considering the recently established existence of the so-called "direct interaction" processes which contribute to nuclear cross sections, specially for inelastic scattering. Experimentally these processes often dominate the compound nucleus process in the cross sections for producing definite low-lying states. They are clearly distinguished by the strong forward peaking in the angular distributions. From the discussion in subsection 3, absence of back-and-front symmetry in an angular distribution implies a breakdown in the random sign assumption. The direct processes have been subjected to theoretical analysis in terms of a phenomenological model that is closely related to the complex potential model. The basic physical idea is that, as a consequence of the fairly long mean free path against collision implied by the complex potential (with small W), it is possible for an incident nucleon to enter a nucleus, exchange energy with a target nucleon, and for one or the other particle to escape directly without further energy loss or formation of a compound nucleus. The success of this idea in applications means that the breakdown in the random signs assumption is associated with the existence of single particle motion—just as deduced in the preceding paragraph. Also if W is considered to increase (i.e., the mean free path to decrease) then the single particle motion is eventually eliminated so that no direct processes are possible, and the only process is the compound nucleus one.

Now we put these ideas into more quantitative form. The problem is that of showing that when a sum is made over many long-lived states, it is possible to have certain correlations between the phases such that the lifetime associated with the sum is a very short one. We consider that the term $E - \Delta_{\lambda} + (i/2)\Gamma_{\lambda}$ is independent of λ (which will be so effectively if levels are overlapping or if the cross section if being averaged). As in subsection 5, we denote this term by the complex energy \mathcal{E} and then apply closure to obtain

$$R_{cc'}(\mathcal{E}) = \left(\frac{\hbar^2}{2M_c a_c} \right) \sum_{pp'} u_p(a_c) u_{p'}(a_c) \times \left(\chi_{cp} \left| \frac{1}{H - \mathcal{E}} \right| \chi_{c'p'} \right). \quad (6.2)$$

The remarks about antisymmetrization in the last subsection also apply here.

Bloch²³ attempted the evaluation of (6.2) along just the same lines that he used for the diagonal elements R_{ccc} . On making the expansion (5.12), neglecting the exchange integrals, and selecting leading terms on the basis of random signs for the $(cp|H'|c'p')$, (6.2) can be reduced, just like (5.11), to the form

$$R_{cc'}(\mathcal{E}) = \left(\frac{\hbar^2}{2M_c a_c} \right) \times \sum_{pp'} \frac{u_p(a_c) u_{p'}(a_c) (cp|H'|c'p')}{(E_{cp} - \mathcal{E} + \varphi_{cp}(E))(E_{c'p'} - \mathcal{E} + \varphi_{c'p'}(E))}, \quad (6.3)$$

where the quantities φ_{cp} are defined in (5.15). We can now take the limit $\mathcal{E} \rightarrow E$ and, by defining $u_{Ec}(r)$ as

$$u_{Ec}(r) = \left(\frac{\hbar^2}{2M_c a_c} \right) \sum_p \frac{u_p(a_c) u_p(r_c)}{E_{cp} + \Delta_{cp} - E - iW_{cp}}, \quad (6.4)$$

it follows that

$$R_{cc'} = (cu_{Ec}|H'|c'u_{E'c'}). \quad (6.5)$$

The definition (6.4) for u_{Ec} is just the exact expansion (IV, 1.8) of the eigenfunction u_E for the real potential well $-V$ at energy E except that $(E_c + \Delta_{cp} - E - iW_{cp})$ replaces $-E$. If Δ_{cp} and W_{cp} do not depend on p then clearly the function (6.4) is just wave-function solution for the well $-V_c'$ introduced in subsection 5. This implies that the quantity, (6.5) corresponds precisely to the matrix elements of the so-called "distorted wave" theory that one would use to compute cross sections with the phenomenological theory of direct processes; i.e., matrix elements of the "perturbing" potential H taken between wave functions each one of which is a product of the appropriate target state wave function and a distorted single particle wave function appropriate to scattering by the complex potential well.†

† For simplicity, we have ignored the factor $(1 - L_c^0 R_c)^{-1}$ in (2.18a). This ≈ 1 only at low energies in general. If it is included in (6.4), the normalization of u_{Ec} is that of fixed incoming flux, i.e., that required in the matrix elements of distorted wave theory.

It is very satisfying to have the phenomenological theory of direct processes related to \mathbf{R} -matrix theory in this simple way. It provides an explicit demonstration of the fact that in \mathbf{R} -matrix theory the direct processes correspond to long-range correlations in the signs of the $\gamma_{\lambda c}$ over the giant resonances.

In the course of the above developments of $R_{cc'}$, we have "lost" the compound nucleus contribution. Since the only essential approximation has been Bloch's selection of leading terms in passing from (6.2) to (6.3), we must conclude that this is the stage at which the compound nucleus contributions is implicitly dropped. A deficiency of the present theory is that, in concentrating on the contribution from far away terms in (6.1), we have lost the contribution from nearby terms. Nevertheless one can accept on grounds of plausibility the fact that a good approximation to (6.1) is

$$(c u_{E_c} | H' | c' u_{E' c'}) + \sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{E_{\lambda} + \Delta_{\lambda} - E - \frac{i}{2} \Gamma_{\lambda}} \quad (6.6)$$

where the sum in the last term is to be taken over an energy region $\gg \langle \Gamma_{\lambda} \rangle$, but \ll the breadth of single particle resonances. This last requirement implies that the signs of the $\gamma_{\lambda c}$ can be regarded as random with the consequence that, if we take the square modulus of (6.6) and average over many levels to obtain the average cross section, all cross interference terms cancel, and the final cross section is an incoherent sum of a cross section for direct processes containing

$$|(c u_{E_c} | H' | c' u_{E' c'})|^2$$

plus a cross section of the type (3.13) from compound nucleus formation.

Although we have dealt specifically with the case of direct inelastic scattering, the same methods and approach could be developed in order to treat other types of direct reaction such as deuteron stripping. Again one can argue that \mathbf{R} -matrix theory is exact when the interaction radius is taken large enough to include all interaction. Since the familiar angular distributions of stripping are not consistent with predictions of the theory based on the random sign approximation, it follows that stripping is describable in terms of \mathbf{R} -matrix theory by allowing for correlations in sign. One detailed difference with the case of incident nucleons is that, for these, one assumes that the absorption potential is fairly constant over the nucleus so that single particle motion exists to the same extent over the nucleus and direct processes may take place in the volume as well as at the surface; in contrast, for incident deuterons, one imagines that W is small for a certain region just inside $r_c = a_c$ but that, inside the bulk of the nucleus, W is very large. Thus the single particle motion responsible for the correlation in phases is to be found only at the edge of the nucleus. If one defined a

set of the $\gamma_{\lambda c}$ for deuterons on an *inner* surface, one would expect these to have random signs. Only when the surface region right up to $r_c = a_c$ is included do the correlations between signs of $\gamma_{\lambda c}$ emerge.

XII. \mathbf{R} -MATRIX THEORY APPLIED TO TREATMENT OF ISOLATED LEVELS

The occurrence of sharp resonance peaks in the excitation curves of low-energy nuclear reactions is one of the most striking phenomena encountered in this field of study. Since the time that the first resonances were found experimentally over twenty years ago, the number of observed resonances has increased to the order of thousands. Many of these levels have been fitted with the famous Breit-Wigner⁸ one-level resonance formula. The fits vary in quality from those in which the experimental curve is duplicated completely over the region of the peak to those in which the fit is in only qualitative agreement with experiment.

For some years, the one-level formula was applied to resonances more or less by analogy with similar applications in atomic radiation problems. Since the resonance formula in these problems was derived on the basis of a perturbation theory⁷ which had no justifiable counterpart in the nuclear case, this situation was unsatisfactory. Amongst other things, it meant that the interpretation of the widths (perturbation matrix elements) obtained in fitting peaks was obscure, and the determination of corrections to the one-level formula was not possible in any quantitative fashion. With the advent of the rigorous general theory of nuclear reactions of Kapur and Peierls⁹ in 1938 and, later, the \mathbf{R} -matrix theory,¹² these difficulties were largely resolved. In such theories, the widths are given explicit quantitative form in terms of the nuclear wave functions, and the corrections to the one-level formula appear explicitly when the general theory is approximated to the one-level case. The familiar condition for the validity of the one-level formula is that the level width Γ should be much less than the level spacings, D . In deducing the one-level formula from the general reaction theory, it can be seen that this condition will justify use of the formula and this is not surprising. The advantage of having \mathbf{R} -matrix theory comes when the condition is not well satisfied so that other levels may influence the cross section near a peak. In such cases the \mathbf{R} -matrix theory can give explicit modifications to the one-level formula in terms of parameters representing the presence of the other levels. With these extra parameters, one has more freedom in fitting so that a poor fit obtained with a one-level formula may be improved. For instance, if the condition $\Gamma \ll D$ fails because of the anomalous proximity of just one other level, one can use the "two-level formula" instead of the one-level formula.

Another advantage of the general \mathbf{R} -matrix theory arises from the fact that, even when only one level is significant, the simplest type of one level formula may still give poor fits on occasion. This is because not all

of the energy dependence in the cross section may be expressible in terms of a resonance denominator and phase-space factors. In certain circumstances, the width of a level and the resonance energy itself vary significantly with energy through the width of the resonance. When the one-level formula is derived from \mathbf{R} -matrix theory, these energy variations are given explicit form in the penetration (P) and shift (S) factors. This type of situation has been studied extensively by Breit⁹¹ and co-workers, especially for cases where the reduced widths are large, i.e. the "one-body" or "single-particle" resonances.

In the first three of the following subsections, we develop the general theory in a form suitable for specialization to the one-level case and indicate various approximations that can lead to a one-level formula. We then discuss the physical significance and interpretation of the one-level formula and finally we discuss its applications in practice. In the last two subsections the two-level formula is presented and remarks are made on the manifestations of level interference in the regions between resonances.

1. The One-Level Approximation

Referring back to Sec. IX, it follows from (IX, 1.14) that, if we make the split

$$\mathbf{R} = \mathbf{R}^0 + \mathbf{R}' \quad (1.1a)$$

with the particular choice:

$$\mathbf{R}' = \frac{\gamma_\lambda \times \gamma_\lambda}{E_\lambda - E}, \quad (1.1b)$$

then the collision matrix is expressible as

$$\mathbf{U} = \mathbf{U}^0 + 2i\Omega \mathfrak{P}^\dagger \left[\frac{\alpha_\lambda \times \alpha_\lambda}{E_\lambda + \Delta_\lambda - E - \frac{i}{2}\Gamma_\lambda} \right] \mathfrak{P}^\dagger \Omega \quad (1.2)$$

with the "background matrix" \mathbf{U}^0 given by

$$\mathbf{U}^0 = \Omega [\mathbf{1} + 2i\mathfrak{P}^\dagger (\mathbf{1} - \mathbf{R}^0 \mathbf{L}^0)^{-1} \mathbf{R}^0 \mathfrak{P}^\dagger] \Omega \quad (1.3a)$$

$$= \Omega \mathfrak{P}^\dagger (\mathbf{1} - \mathbf{R}^0 \mathbf{L}^0)^{-1} (\mathbf{1} - \mathbf{R}^0 \mathbf{L}^{0*}) \mathfrak{P}^{-\dagger} \Omega \quad (1.3b)$$

and with the quantities α_λ , Δ_λ , and Γ_λ given, from (IX, 1.18 and 20), by

$$\alpha_\lambda = (\mathbf{1} - \mathbf{R}^0 \mathbf{L}^0)^{-1} \gamma_\lambda \quad (1.4)$$

$$\Delta_\lambda \equiv \Delta_{\lambda\lambda} = (\alpha_\lambda^*, (\mathbf{L}^{0*} \mathbf{R}^0 \mathbf{L}^0 - \mathbf{S}^0) \alpha_\lambda) = \sum_c \Delta_{\lambda\lambda c} \quad (1.5)$$

$$\frac{1}{2}\Gamma_\lambda \equiv \frac{1}{2}\Gamma_{\lambda\lambda} = (\alpha_\lambda^*, \mathfrak{P} \alpha_\lambda) = \frac{1}{2} \sum_c \Gamma_{\lambda\lambda c}. \quad (1.6)$$

Before introducing the one-level approximation it is of interest to consider some formal properties of (1.2).

⁹¹ Ofstrosky, Breit, and Johnson, *Phys. Rev.* **49**, 22 (1936); B. E. Freeman and J. L. McHale, *ibid.* **89**, 223 (1953); J. L. Johnson and J. L. McHale *ibid.* **91**, 87 (1953); J. L. Johnson and H. M. Jones, *ibid.* **93**, 1286 (1954).

For this purpose, it is convenient to write (1.2) in the form

$$\mathbf{U} = \mathbf{U}^0 + 2i \frac{\tau_\lambda \times \tau_\lambda}{\epsilon_\lambda} \quad (1.7)$$

where τ_λ and ϵ_λ are defined by

$$\begin{aligned} \tau_\lambda &= \Omega \mathfrak{P}^\dagger \alpha_\lambda \\ \epsilon_\lambda &= E_\lambda + \Delta_\lambda - E - \frac{i}{2}\Gamma_\lambda. \end{aligned}$$

The form (1.7) satisfies the conditions:

$$\mathbf{U}^0 = \mathbf{U}^{0\dagger}, \quad (1.8a)$$

$$\mathbf{U}^0 (\mathbf{U}^0)^\dagger = \mathbf{1}, \quad (1.8b)$$

$$\mathbf{U}^0 \tau_\lambda^* = \tau_\lambda, \quad (1.8c)$$

$$\text{Im}(\epsilon_\lambda) = -(\tau_\lambda^*, \tau_\lambda), \quad (1.8d)$$

where \dagger denotes the complex conjugate transpose.

It may be made plausible from more general considerations than \mathbf{R} -matrix theory that (1.7) has the natural form of a one-level approximation and that the conditions (1.8) are expected to be satisfied in general. The deduction proceeds from the expansion of an N -dimensional unitary symmetric matrix \mathbf{U} in terms of its real orthonormal eigenvectors \mathbf{g}_λ belonging to the unit-modulus eigenvalues $\exp(2i\delta_\lambda)$,

$$\mathbf{U} = \sum_{\lambda=1}^N e^{2i\delta_\lambda} (\mathbf{g}_\lambda \times \mathbf{g}_\lambda). \quad (1.9)$$

From considerations of causality it has been shown in Sec. IV, that in the one-channel case the collision function $\exp(2i\delta)$ may be expanded in terms of its poles and zeros, and that in the vicinity of a particular pole H (in the lower half of the energy plane where $\text{Im}(H) < 0$) this function has the form

$$\begin{aligned} e^{2i\delta} &= \exp(2i\delta^0) \left(\frac{H^* - E}{H - E} \right) \\ &= \exp(2i\delta^0) \left(1 + \frac{2i \text{Im}(H)}{H - E} \right). \end{aligned}$$

The quantity $\exp(2i\delta^0)$ contains the singularities at the remaining poles and other factors; if the assumption is made that the individual eigenvalues of the many-channel expansion (1.9) also have this form in the vicinity of a pole H and if the singularity near E corresponds to the first eigenvalue $\lambda = 1$, then it can be written in the form (1.7) with

$$\begin{aligned} \mathbf{U}^0 &= \exp(2i\delta_1^0) (\mathbf{g}_1 \times \mathbf{g}_1) + \sum_{\lambda=2}^N e^{2i\delta_\lambda} (\mathbf{g}_\lambda \times \mathbf{g}_\lambda) \\ \tau_1 &= [-\text{Im}(H_1)]^\dagger \exp(i\delta_1^0) \mathbf{g}_1. \end{aligned}$$

The conditions (1.8) which are satisfied by the \mathbf{R} -matrix expansion are also satisfied by (1.9). We note that the \mathbf{U} of (1.9) also satisfies the general condition $\mathbf{U}(E^*)^* \mathbf{U}(E) = \mathbf{1}$ of the collision matrix theory. Alternative derivations of (1.8) and additional discussions have been given by Breit.¹⁰

A number of special features of \mathbf{R} -matrix theory not possessed by the general theory should be noted. In the former theory the nuclear system is assumed to have a rather well defined boundary corresponding to the surface S . As a result of such an assumption the partial widths $\Gamma_{\lambda c}$ are found to be proportional to the penetration factors P_c and the partial level shifts are proportional to the shift factors S_c^0 ; both P_c and S_c^0 are calculable and their dependences on energy are known. The levels E_λ may be associated with eigenfunctions for certain boundary conditions B_c ; the shift Δ_λ of the actual resonances from the E_c may be expressed

in terms of the S_c^0 and the reduced widths of the system. In some cases it is also possible to make reasonably significant estimates of the magnitude of the background matrix. Since the general theory involves no assumption of a well-defined region of interaction, it does not involve any of this information.

The various cross sections are given by the absolute squares of the various components of \mathbf{U} (see Sec. VIII). The squares of the elements of the second matrix of (1.2) represent the familiar "resonance" part of scattering or reaction; the (cc') component is

$$\frac{\Gamma_{\lambda c}\Gamma_{\lambda c'}}{(E_{\lambda}+\Delta_{\lambda}-E)^2+\frac{1}{4}\Gamma_{\lambda}^2}$$

Notice that a typical partial width $\Gamma_{\lambda c}$ of (1.6) contains the penetration factor P , but also depends on energy through the occurrence of the \mathbf{L}^0 matrix and of \mathbf{R}^0 . The elements of the matrix \mathbf{U}^0 give rise to what may be called "potential" scattering or reaction. There also occur in the total cross-section cross-product terms which represent the interference between the "resonance" and "potential" terms.

All expressions given so far are exact, but it is apparent that they are not useful as such because, amongst other things, the inversion of matrices is implied. The usefulness of such expressions is restricted to situations in which approximations can be made about the nature of the matrix \mathbf{R}^0 , which represents the presence of levels other than the particular one λ in which interest centers. Notice that \mathbf{R}^0 occurs in combination with the penetration matrix \mathfrak{P} or the \mathbf{L}^0 matrix. If at a given energy we choose appropriate boundary conditions, the \mathbf{L}^0 matrix effectively becomes the penetration matrix, so it is the magnitudes of the combinations of \mathbf{R}^0 with \mathbf{P} that we are concerned with. This implies that when \mathbf{P} is very small, the one-level formula is applicable no matter what the value of \mathbf{R}^0 as long as it is not $\gg 1$. We expect this situation because, in the limit as $\mathbf{P} \rightarrow 0$, the states become bound and any effect of other states at a given state must fall to zero.

There are three types of approximation involving $\mathbf{R}^0\mathbf{L}^0$ that, broadly speaking, may be called "one-level" approximations.

a. First Approximation: $\mathbf{R}^0\mathbf{L}^0=0$

This is the strongest approximation we can make in trying to apply the general formulas, and it is the one usually used in practice. The consequences of this assumption are simple and can be read off immediately on specializing the general formulas (1.2) to (1.6). The background matrix \mathbf{U}^0 is diagonal with diagonal values

$$U_{cc}^0 = e^{2i(\omega_c - \phi_c)} \quad (1.10)$$

and the widths and shifts are given by

$$\Gamma_{\lambda c} = 2P_c\gamma_{\lambda c}^2, \quad (1.11a)$$

$$\Delta_{\lambda c} = -S_c^0\gamma_{\lambda c}^2 = -(S_c - B_c)\gamma_{\lambda c}^2 \quad (1.11b)$$

and a general element of the \mathbf{U} matrix is, from (1.2) and (1.10):

$$U_{cc'} = (U_{cc}^0 U_{c'c'}^0)^{\frac{1}{2}} \times \left[\delta_{cc'} + \frac{i\Gamma_{\lambda c}\Gamma_{\lambda c'}^{\frac{1}{2}}}{E_{\lambda} + \Delta_{\lambda} - E - \frac{i}{2}\Gamma_{\lambda}} + O\left(\frac{\langle \Delta - \frac{i}{2}\Gamma \rangle}{D}\right) \right] \quad (1.12)$$

[This could have been obtained from (IX, 1.14 and 20).] On inserting this expression in Eq. (VIII, 3.2), we have the famous Breit-Wigner one-level formula for a reaction cross section ($\alpha' \neq \alpha$) proceeding through an isolated resonance λ of spin J and definite parity:

$$\sigma_{\alpha\alpha'} = \frac{\pi}{k_{\alpha}^2} g_J \frac{(\sum_{ls} \Gamma_{\lambda c})(\sum_{l's'} \Gamma_{\lambda c'})}{(E_{\lambda} + \Delta_{\lambda} - E)^2 + \frac{1}{4}\Gamma_{\lambda}^2} \quad (1.13)$$

where primed sums over $ls, l's'$ are such that the relations $\mathbf{l} + \mathbf{s} = \mathbf{J}$, $\mathbf{l}' + \mathbf{s}' = \mathbf{J}$ are satisfied and such that parity is conserved. The corresponding formula for the elastic scattering cross section ($\alpha = \alpha'$) is

$$\sigma_{\alpha\alpha} = \frac{\pi}{k_{\alpha}^2} \times \left\{ \begin{aligned} & \sum_{Jsl} 4g_J \sin^2\phi_c - g_J (\sum_{sl} \Gamma_{\lambda c}) \\ & \times \left[\frac{2(E_{\lambda} + \Delta_{\lambda} - E) \sin 2\phi_c + \Gamma_{\lambda}(1 - \cos 2\phi_c)}{(E_{\lambda} + \Delta_{\lambda} - E)^2 + \frac{1}{4}\Gamma_{\lambda}^2} \right] \\ & + g_J \frac{(\sum_{sl} \Gamma_{\lambda c})(\sum_{s'l'} \Gamma_{\lambda c'})}{(E_{\lambda} + \Delta_{\lambda} - E)^2 + \frac{1}{4}\Gamma_{\lambda}^2} \end{aligned} \right\} \quad (1.14)$$

The last term is the so-called "resonance term" the second is the "interference term" and the first is the potential scattering term. Since ϕ_c the hard sphere phase, does not depend on s or J , this potential scattering term is simply

$$\frac{4\pi}{k_{\alpha}^2} \sum_l (2l+1) \sin^2\phi_{\alpha l} \quad (1.15)$$

which is just the hard-sphere scattering cross section from the definition of $\phi_{\alpha l}$ in Sec. III, 4. (If a long-range Coulomb interaction is present, this term diverges.)

In (1.12), the last term signifies the order of the correction due to the effects of nearby levels (but not far-away levels). This represents the presence of the $\mathbf{R}^0\mathbf{L}^0$ terms and, in the one-channel case, it is evident that $\mathbf{R}^0\mathbf{L}^0 \sim O(\langle \Delta - (i/2)\Gamma \rangle / D)$ where $\langle \rangle$ signifies an average taken over nearby levels.

As yet we have said nothing about the choice of the boundary condition parameters B_c . The criterion for

making the choice is that the one-level approximation be as accurate as possible, i.e., that the error $O(\langle\Delta - (i/2)\Gamma\rangle/D)$ be as small as possible. Now the quantities $\langle\Gamma\rangle$ and D are relatively insensitive to the choice of B_c , but Δ_c is very sensitive, as is evident from (1.11b). Thus, the one-level approximation is most accurate when B_c is such that $\langle\Delta\rangle \lesssim \langle\Gamma\rangle$, which is equivalent to saying that B should be chosen so that the eigenvalue E_λ lies within the width of the observed resonance. This is, of course, a very reasonable requirement. One can make the choice $B_c = +S_c(E_r)$ where E_r is the energy at the center of the resonance. This makes Δ_λ zero at the energy E_r and so this choice certainly satisfies the above criterion. There are other choices that also satisfy the criterion, and these are no better or worse than the particular choice $B_c = +S_c(E_r)$. The point here is that one can never reduce the error in the one-level formula below the "intrinsic" error $O(\langle\Gamma\rangle/2D)$ and so, all choices of B_c that reduce $O(\langle\Delta - (i/2)\Gamma\rangle/D)$ to this intrinsic error are equally suitable.

Use of a one-level formula (1.13) to fit an experimental resonance curve *implies* that the boundary conditions satisfy this criterion. Since the conditions do not have to be specified explicitly at any stage in the fitting, there is no need to draw attention to any specific boundary conditions in presenting the results of such a fitting. However, it should be understood by the reader of such results that the extracted values of the parameters (reduced widths, eigenvalues) correspond to boundary conditions that satisfy the above criterion and are therefore approximately: $B_c = +S_c(E_r)$.

We have not drawn attention in the above discussion to the fact that $\Delta_{\lambda c}$ is energy-dependent (through S_c) so that setting $\Delta_{\lambda c} = 0$ at the center of a resonance does not imply that $\Delta_{\lambda c}$ is everywhere zero. In most cases, one can neglect the energy dependence of $\Delta_{\lambda c}$ through the width of a resonance. This is fortunate because, generally speaking, the problem of fitting an experimental curve is made much more complicated if $\Delta_{\lambda c}$ is varying rapidly. As discussed in subsection 3 below, one can probably represent the energy dependence of $\Delta_{\lambda c}$ by a Taylor series expansion:

$$\Delta_{\lambda c}(E) = \Delta_{\lambda c}(E_r) + (E - E_r) \left(\frac{d\Delta_{\lambda c}}{dE} \right)_{E=E_r} + \dots \quad (1.16)$$

With this expansion, the magnitude of $d\Delta_{\lambda c}/dE$ is the important quantity in determining how sharply $\Delta_{\lambda c}(E)$ varies with energy. In the present approximation, we have, from (1.11b):

$$\frac{d\Delta_{\lambda c}}{dE} = -\gamma_{\lambda c}^2 \frac{dS_c}{dE}. \quad (1.17)$$

b. Second Approximation: $\mathbf{R}^0\mathbf{L}^0$ is Small

This approximation is not especially useful but we briefly discuss it for the sake of completeness. It implies

that we can expand:

$$(\mathbf{1} - \mathbf{R}^0\mathbf{L}^0)^{-1} = \mathbf{1} + \mathbf{R}^0\mathbf{L}^0 + (\mathbf{R}^0\mathbf{L}^0)^2 + \dots \quad (1.18)$$

and take only two terms. On taking two terms, expressions (1.5) and (1.6) for the partial widths and shifts become

$$\frac{1}{2}\Gamma_{\lambda c} = P_c \{ \gamma_{\lambda c}^2 + 2 \sum_{c'} \gamma_{\lambda c} R_{cc'}^0 S_{c'}^0 \gamma_{\lambda c'} \} \quad (1.19a)$$

$$\Delta_{\lambda c} = -S_c^0 \gamma_{\lambda c}^2 - \sum_{c'} S_c^0 \gamma_{\lambda c} R_{cc'}^0 S_{c'}^0 \gamma_{\lambda c'} + \sum_{c'} P_c \gamma_{\lambda c} R_{cc'}^0 P_{c'} \gamma_{\lambda c'}. \quad (1.19b)$$

Choosing the B_c to make S_c^0 vanish at resonance $E = E_r$ the widths reduce to $2P_c \gamma_{\lambda c}^2$ at resonance. Thus $\Gamma_{\lambda c} = 2P_c \gamma_{\lambda c}^2$ is true at resonance to second order in R^0P . With the same choice of B_c , the second term in $\Delta_{\lambda c}$ vanishes (besides the first), and the term it fosters in $d\Delta_{\lambda c}/dE$ is second order in $(E - E_r)$ and so is probably small compared to the contribution from the first term. The third term in (1.19b) and its energy derivative are more difficult to estimate since the signs of individual terms with $c \neq c'$ will fluctuate. The contribution from terms with $c = c'$ is given below in (1.34) and (1.35) and it might be expected that the sum of terms with $c \neq c'$ is small compared to the sum of terms with $c = c'$. If there are only two channels, the contribution to the third term in (1.19b) from $c \neq c'$ is roughly

$$\pm \frac{1}{2} \frac{\langle\Gamma_{\lambda c}\rangle \langle\Gamma_{\lambda c'}\rangle}{D^2}.$$

The background matrix \mathbf{U}^0 is not diagonal in this approximation so that it contributes to the reaction cross sections.

c. Third Approximation: $\mathbf{R}^0\mathbf{L}^0$ is Diagonal

From the theoretical and experimental study of average total cross sections (see Secs. XI, 4 and 5), the matrix \mathbf{R}^0 has strong diagonal matrix elements so that, unless the factor \mathbf{L}^0 is small, it is not satisfactory to use the first approximation above, *viz.*, $\mathbf{R}^0\mathbf{L}^0$ is zero. A natural extension of this approximation is to assume that $\mathbf{R}^0\mathbf{L}^0$ is diagonal so that matrix inversion becomes simply ordinary division and we can write the above formulas without any implied matrix inversion. This assumption is not unreasonable because the individual level contributions to the nondiagonal components of \mathbf{R}^0 can be positive as well as negative and so tend to cancel, whereas the diagonal components are all proportional to the positive reduced widths. With $\mathbf{R}^0\mathbf{L}^0$ diagonal, the widths and shifts become, from (1.5) and (1.6):

$$\Gamma_{\lambda c} = \frac{2P_c \gamma_{\lambda c}^2}{d_c} \quad (1.20a)$$

$$\Delta_{\lambda c} = \frac{P_c (R_{cc}^0 P_c) - S_c^0 (1 - R_{cc}^0 S_c^0)}{d_c} \gamma_{\lambda c}^2 \quad (1.20b)$$

where

$$d_c = (1 - R_{cc}^0 S_c^0)^2 + (R_{cc}^0 P_c)^2, \quad (1.21)$$

and the elements of the diagonal background matrix are now

$$U_{cc}^0 = e^{2i(\omega_c - \phi_c')}, \quad (1.22)$$

where

$$\begin{aligned} \phi_c' &= \phi_c + \arg(1 - R_{cc}^0 L_c^0) \\ &= \phi_c - \tan^{-1} \left(\frac{R_{cc}^0 P_c}{1 - R_{cc}^0 S_c^0} \right). \end{aligned} \quad (1.23)$$

With the revised definitions of $\Gamma_{\lambda c}$, $\Delta_{\lambda c}$, and U_{cc}^0 , and putting ϕ_c' for ϕ_c the previous formulas (1.12), (1.13), and (1.14) for the collision matrix elements and the

cross sections can be taken over as they stand. In the case of only one channel, the present treatment is exact. The above formulas are equivalent to those previously considered for the one-channel case in Sec. IV, 1. The present approximation has been used by Krotkov⁹² to analyze the resonance scattering and capture of neutrons by Mn⁵⁵ for energies up to the kilovolt region.

If R^0 and Δ_λ are not very energy-dependent, they can be regarded as constant in the vicinity of a single resonance. If this is not so, the fitting of cross sections becomes complicated and one tries to represent the energy dependence of $\Delta_{\lambda c}$ by (1.16). In the present approximation, we have, from (1.20b),

$$\begin{aligned} & \left(\frac{d\Delta_{\lambda c}}{dE} \right)_{E_r} \\ &= \gamma_{\lambda c}^2 \left\{ \frac{(2x - 2xy + 4xy^2) \frac{dP_c}{dE} + (x^2 - 3y^2 + 2y^3 + 2x^2y + y^4 - x^4) \left(\frac{1}{R_{cc}^0} \right)^2 \frac{dR_{cc}^0}{dE} + (-1 - 4x^2y + x^2 - 2y + 3y^2 + y^3) \frac{dS_c}{dE}}{[(1-y)^2 + x^2]^2} \right\}, \end{aligned} \quad (1.24)$$

where $x = (P_c R_{cc}^0)_{E=E_r}$, $y = (S_c^0 R_{cc}^0)_{E=E_r}$.

The evaluation of $\Delta_{\lambda c}$ and $(d\Delta_{\lambda c}/dE)$ depends on the choice of boundary condition B_c . Following the same reasoning as in the discussion of the first approximation above, one chooses B_c to make the one-level approximation as good as possible. This implies making $\Delta_{\lambda c} \lesssim \Gamma_{\lambda c}$ inside the resonance. We mention two choices of B_c that satisfy this requirement

(i) We may define B_c to be such that

$$\Delta_{\lambda c}(E_r) = 0 \quad (1.25)$$

for each channel c .

Introducing an angle A_c by the definition

$$\sin A_c = 2(P_c R_{cc}^0)_{E=E_r} = 2x \quad (1.26)$$

then the choice of B_c implied by (1.25) is from (1.20b):

$$B_c = S_c(E_r) - P_c(E_r) \tan \frac{A_c}{2}. \quad (1.27)$$

This choice of B implies that, at $E = E_r$:

$$\Gamma_{\lambda c} = 2P_c \gamma_{\lambda c}^2 \sec^2 \frac{A_c}{2}. \quad (1.28)$$

(ii) An alternative choice of boundary condition would be that which makes $d_c = 1$ at $E = E_r$ so that, at $E = E_r$,

$$\Gamma_{\lambda c} = 2P_c \gamma_{\lambda c}^2. \quad (1.29)$$

Such a condition can be stated in the form (1.29) with an angle A_c' in place of A_c , where A_c' is given by

$$\sin A_c' = (P_c R_{cc}^0)_{E=E_r} = x. \quad (1.30)$$

The level shifts in this case are, at resonance $E = E_r$,

$$\Delta_{\lambda c} = P_c \gamma_{\lambda c}^2 \tan \frac{A_c'}{2} = \gamma_{\lambda c}^2 (S_c - B_c). \quad (1.31)$$

If x is small, both of the boundary conditions (i) and (ii) reduce to $y=0$, and so are the same to first order, as are the expressions for the widths and shifts. To first order in x :

$$\frac{d\Delta_{\lambda c}}{dE} = \gamma_{\lambda c}^2 \left(2x \frac{dP_c}{dE} - \frac{dS_c}{dE} \right). \quad (1.32)$$

To second order in x ,

$$\begin{aligned} \frac{d\Delta_{\lambda c}}{dE} &= \gamma_{\lambda c}^2 \\ &\times \left\{ \frac{2x \frac{dP_c}{dE} + P_c^2 \frac{dR_{cc}^0}{dE} + \frac{dS_c}{dE} (x^2 - 2y - 1)}{1 - 4y + 2x^2} \right\}. \end{aligned} \quad (1.33)$$

With the first and second boundary conditions ($y = x^2$, $y = x^2/2$ to second order), this becomes, respectively,

$$\frac{d\Delta_{\lambda c}}{dE} = \gamma_{\lambda c}^2 \left\{ \frac{\frac{d}{dE} (R_{cc}^0 P_c)^2 - \frac{dS_c}{dE} (1 + x^2)}{1 - 2x^2} \right\} \quad (1.34)$$

$$\frac{d\Delta_{\lambda c}}{dE} = \gamma_{\lambda c}^2 \left\{ \frac{\frac{d}{dE} (R_{cc}^0 P_c)^2 - \frac{dS_c}{dE}}{1 - 2x^2} \right\}. \quad (1.35)$$

⁹² R. Krotkov, Can. J. Phys. 33, 622 (1955).

The magnitude of dS_e/dE is usually $\lesssim (Ma^2/\hbar^2)$ except for S -wave neutrons for which it is zero. In the latter instance, we can estimate R_{cc}^0 to be roughly constant at $(Ka)^{-1}$ where $K \sim 10^{13} \text{ cm}^{-1}$ (see Sec. XI, 4) so that

$$(d/dE)(R_{cc}^0 P_c)^2 \sim (Ma^2/\hbar^2) \cdot (2/aK) \sim \frac{1}{3}(Ma^2/\hbar^2).$$

It thus appears that the quantity in brackets in (1.34) or (1.35) is always $\sim (Ma^2/\hbar^2)$ since, in just those situations when the second term is smallest, the first term is largest. Consequently $d\Delta_{\lambda c}/dE$ is only small (and the Taylor series expansion of $\Delta_{\lambda c}$ only useful) when $\gamma_{\lambda c}^2 \ll \hbar^2/Ma^2$.

2. Interpretation of the One-Level Approximation

Although the one-level theory furnishes formulas for analysis of resonances along with expressions for correction terms, it is not readily associated with a physical picture of a reaction. In particular, use of a stationary treatment conceals the time-development of a reaction and does not obviously complement the Bohr idea² of a compound state decaying by various competing modes of decay. In contrast, earlier theories of reactions actually assumed aspects of the supposed physical picture to begin with. This made them phenomenological and meant that they lacked rigor, although they were more easily grasped in physical content. Now we try to complete the present one-level theory by giving it some physical color.

In Sec. IX, 1c, we derived an expression for the integral of the probability $|\Psi_e|^2$ over the internal region, where the subscript e indicates that the internal wave function Ψ_e is formed with unit incoming (*spherical wave*) flux in the channel e . Such an integral is naturally interpreted as a measure of the probability that the two nuclei in pair e penetrate into each other to form a compound nucleus. From (IX, 1.29) the integral is

$$\int_{\tau} |\Psi_e|^2 d\tau = \frac{\hbar\Gamma_{\lambda e}}{(E_{\lambda} + \Delta_{\lambda} - E)^2 + \frac{1}{4}\Gamma_{\lambda}^2}. \quad (2.1)$$

This is not only true when \mathbf{R}^0 of (1.1) is ignored, but also when \mathbf{R}^0 is included provided that it can be taken as constant with energy. (From this point of view, the most logical definition of the one-level approximation would be that in which we assume only that \mathbf{R}^0 is constant. However, as seen in subsection 1, further assumptions usually have to be made in order to derive useful expressions for cross sections.)

In connection with (2.1), if the function Ψ_e arises from a unit-flux *plane wave* beam, the right side of (V, 2.1) is to be multiplied by the actual flux intensity $\pi/k_e^2[(2J+1)/(2s_e+1)]$. With this multiplying factor the integral has the dimensions of area-times-time, and the interpretation is again straightforward. By dividing it by the resonance part of the total cross section one obtains the expected value \hbar/Γ_{λ} as the mean lifetime of the compound nucleus.

The physical nature of the one-level approximation is also evidenced in the evaluation of the overlap integral

$$I_{12} = \int_{\tau} \Psi_1^* \Psi_2 d\tau / \left[\int_{\tau} |\Psi_1|^2 d\tau \right]^{\frac{1}{2}} \left[\int_{\tau} |\Psi_2|^2 d\tau \right]^{\frac{1}{2}} \quad (2.2)$$

for internal wave functions Ψ_1 and Ψ_2 of two different energies, E_1 and E_2 , but which both correspond to an \mathbf{R} matrix with one level λ and a constant contribution \mathbf{R}^0 from the remaining levels. By means of the expressions (V, 2.9 and 10) for the integrals in (2.2), one can show that

$$\begin{aligned} |I_{12}|^2 &= \frac{|(\mathbf{D}_1^{0*}, (\gamma_{\lambda} \times \gamma_{\lambda}) \mathbf{D}_2^0)|^2}{(\mathbf{D}_1^{0*}, (\gamma_{\lambda} \times \gamma_{\lambda}) \mathbf{D}_1^0)(\mathbf{D}_2^{0*}, (\gamma_{\lambda} \times \gamma_{\lambda}) \mathbf{D}_2^0)} = 1. \end{aligned} \quad (2.3)$$

By considering the Schwarz identity,

$$\begin{aligned} |(\mathbf{f}^*, \mathbf{g})|^2 &= (\mathbf{f}^*, \mathbf{f})(\mathbf{g}^*, \mathbf{g}) \\ &\quad - \frac{1}{2} \int \int |\mathbf{f}(x)\mathbf{g}(\xi) - \mathbf{f}(\xi)\mathbf{g}(x)|^2 dx d\xi, \end{aligned} \quad (2.4)$$

one can assert that in the internal region Ψ_1 is equal to a constant, complex multiplicative factor times Ψ_2 . Notice that I_{12} is independent of how the compound systems described by Ψ_1 and Ψ_2 are formed. Thus, the shape of the internal wave function describing the general one-level system as defined here is independent of energy and of the channel by which it is formed; its amplitude and phase may, however, depend upon them. By starting with the assumption of this independence of the shape, Wigner¹¹ gave the first derivation of the generalized one-level formula with the energy-independent \mathbf{R}^0 included.

In the particular one-level approximation of assuming the \mathbf{R}^0 is zero, the observations of the last paragraph are perhaps trivial because in that case Ψ is assumed to be expandible in terms of a single X_{λ} . This brings up the matter of the physical significance of the \mathbf{R}^0 matrix and how it happens that in the restricted one-level approximation the actual Ψ with running waves can be described by a single standing-wave type function X_{λ} . The answer to the second question is that the Green's theorem relation used in Sec. V serves as an approximate means of integrating X_{λ} in a manner which enables it to join onto the external functions with their running waves. This integration changes the slope of the wave function at the surface but not its shape in the internal region, as evidenced by applying the Green's theorem relation (V, 1.4) directly to Ψ and X_{λ} and assuming that $\Psi = X_{\lambda}$ in the internal region. The effect of the inclusion of the constant \mathbf{R}^0 matrix is to change the absolute value of the wave function on S . That is, the integration by means of the Green's theorem relation introduces on S a discontinuity in the value as well as

the slope. We consider the ratio of the total probability of finding all nucleons on a channel surface S_c (in which case they will be formed into the pair c separated by the interaction radius) to the total probability of finding all nucleons inside the internal region (in which case they are in the compound nucleus):

$$\int_{S_c} |\Psi_e|^2 dS / \int_{\tau} |\Psi_e|^2 d\tau. \quad (2.5)$$

With (III, 4.3a) the surface integral can be expressed as

$$\int_{S_c} |\Psi_e|^2 dS = \frac{2Ma_c}{\hbar^2} |V_c|^2, \quad (2.6)$$

the value quantities V_c being given by (IX, 1.25a) for a system formed from a particular channel e . By inserting the one-level expansion (1.1) in the matrix $(\mathbf{1} - \mathbf{R}\mathbf{L}^0)^{-1}$ of \mathbf{V} , and using (2.1) for the volume integral, our ratio (2.5) becomes

$$\frac{2Ma_c}{\hbar^2} | \{ (\mathbf{1} - \mathbf{R}^0\mathbf{L}^0)^{-1} \mathbf{R}^0 \}_{ce\epsilon_\lambda + \alpha_{\lambda c} \alpha_{\lambda e}} |^2 \frac{P_e}{\hbar\Gamma_{\lambda e}} \quad (2.7)$$

where α_λ is given by (IX, 1.15) as

$$\alpha_\lambda = (\mathbf{1} - \mathbf{R}^0\mathbf{L}^0)^{-1} \gamma_\lambda.$$

If \mathbf{R}^0 is assumed to be zero, then the ratio becomes

$$\frac{2Ma_c}{\hbar^2} \gamma_{\lambda c}^2, \quad (2.8)$$

and is independent of the energy and of the formation channel. However, if \mathbf{R}^0 is not zero, even though perhaps constant, no cancellation of the energy-dependent ϵ_λ of (2.7) with that of (2.1) occurs and as a result the ratio is in general dependent on energy and of the formation mode. The ratio given by (2.8) is recognized as that for the eigenfunction X_λ . Since the shape of the one-level wave function Ψ_e in τ is the same as that of X_λ , the departure of the ratio from (2.8) indicates that the effect of a constant \mathbf{R}^0 term is to change discontinuously the value of Ψ_e on S_c .

To conclude we make a few remarks about a time-dependent interpretation of the one-level approximation. In general, to determine time-dependent features of some process from knowledge of its stationary characteristics, we construct wave packets by superposing stationary wave functions of the various energies in some energy range, and then examine the behavior of these wave packets.

In the one-level approximation, from (IX, 1.31 and 21), the wave function in the internal region is given by

$$\Psi_e(E) = \frac{\exp i(\omega_e - \phi_e) \Gamma_{\lambda e} \frac{1}{2} \hbar^{\frac{1}{2}}}{E_\lambda + \Delta_\lambda - E - \frac{i}{2} \Gamma_\lambda} X_\lambda \exp(iEt/\hbar) \quad (2.9)$$

where we have included the time dependence [notice that (2.9) implies (2.1)]. Let us superpose stationary states of this type with some energy-dependent amplitude $A(E)$ that is constant over the region of the resonance and goes smoothly to zero on either side of the resonance. We then have a wave packet

$$\begin{aligned} & \int A(E) \Psi_e(E) dE \\ &= \Gamma_{\lambda e} \frac{1}{2} \hbar^{\frac{1}{2}} \exp i(\omega_e - \phi_e) X_\lambda \int \frac{A(E) \exp(iEt/\hbar) dE}{E_\lambda + \Delta_\lambda - E - \frac{i}{2} \Gamma_\lambda} \\ &\sim \Gamma_{\lambda e} \frac{1}{2} \hbar^{\frac{1}{2}} \exp i(\omega_e - \phi_e) X_\lambda \exp[i(E_\lambda + \Delta_\lambda) - \Gamma_\lambda] \frac{t}{\hbar}. \quad (2.10) \end{aligned}$$

This wave packet can be seen to decay with lifetime \hbar/Γ_λ and can be said to represent the compound nucleus and its time decay. The theories of Kapur-Peierls⁹ and Seigert-Humblet^{72,73} were formulated with wave-packet states X_λ that decayed in time in just this manner. In the Kapur-Peierls paper, instead of choosing real boundary conditions to define the states X_λ , the condition of outgoing waves in all channels was selected. Such a condition directly complements the physical notion of the decay of a compound nucleus in contrast to the arbitrary real conditions used here. Although basically the two types of formulation are quite equivalent (Sec. IX, 2), the older one is more physically suggestive.

As a more concrete illustration, let us consider the simple example of a pure scattering process near an isolated resonance. The following treatment is due to Van Kampen.⁹³ Generally, at an energy E , we have at asymptotic distances:

$$\Psi(E, \mathbf{r}, t) \sim \frac{1}{r} (e^{-ikr} - U e^{ikr}) \exp(iEt/\hbar). \quad (2.11)$$

Let us form wave packets from these stationary states as in the foregoing

$$\begin{aligned} \Psi(\mathbf{r}, t) &\sim \int A(E) \Psi(E, \mathbf{r}, t) dE \\ &= \int \frac{A(E)}{r} (e^{-ikr} - U e^{ikr}) \exp(iEt/\hbar) dE, \quad (2.12) \end{aligned}$$

where $A(E)$ varies smoothly inside some suitably chosen energy region and goes to zero outside this region. The method of stationary phase shows that the incoming wave packet is localized at time t at the point $\mathbf{r} = -t/v$ where v is the mean velocity. By the same

⁹³ N. G. Van Kampen (private communication).

method the *outgoing* wave packet is found where

$$r = vt - 2 \frac{d\delta}{dk} \quad (2.13)$$

where U has been expressed as $\exp(2i\delta)$. Near an isolated resonance:

$$\delta = (\omega - \phi) + \tan^{-1} \left\{ P \left(\frac{\gamma_\lambda^2}{E_\lambda - E} + R^0 \right) \right\}. \quad (2.14)$$

If R^0 is ignored,

$$\frac{d\delta}{dk} = \frac{(\hbar v/2)\Gamma_\lambda}{(E - E_\lambda)^2 + \frac{1}{4}\Gamma_\lambda^2} + O(a).$$

Thus the time delay $(2/v)(d\delta/dk)$ of the packet is $\sim 2\hbar/\Gamma$ near the resonance, and is very much smaller off resonance. If R^0 is retained but taken as a constant, this result is unchanged provided that PR^0 is not large.

3. Practical Application of the One-Level Approximation with $R^0 L^0 = 0$

We now discuss problems associated with fitting experimental resonances with the one-level formula (1.13) based on the neglect of R^0 . The only serious complication is that the energy dependences of the width and shift factors P and S in (1.11a and b) must be considered in some cases.^{91,94} There are two approximate procedures for taking these dependences into account: the first is to expand the shift linearly with respect to energy while treating the widths exactly; the second procedure is to expand the quantity ϵ_λ of (1.7) linearly with respect to energy while treating the widths in the numerator of (1.13) exactly. There are cases, such as resonances near thresholds, where it is undesirable to make either approximation.

a. First Approximate Method: Linear Expansion of Δ_λ

It is usually a good approximation⁹⁴ to treat Δ_λ as a linear function of E over a fairly wide range of energies that will often include the width of a resonance. It is then convenient to rewrite Eq. (1.13) in terms of the resonance energy E_r which we will now define as the solution to

$$E_\lambda + \Delta_\lambda(E_r) - E_r = 0 \quad (3.1)$$

so that

$$E_\lambda + \Delta_\lambda - E = (E_r - E) \left(1 - \frac{d\Delta_\lambda}{dE} \right)_{E_r} + \dots,$$

the higher order terms being neglected. Equation (1.13) then becomes

$$\sigma_{cc'} = \frac{\pi}{k_c^2} \frac{g_j \Gamma_{\lambda c^0} \Gamma_{\lambda c'^0}}{(E - E_r)^2 + (\frac{1}{2}\Gamma_{\lambda c^0})^2} \quad (3.2)$$

⁹⁴ R. G. Thomas, Phys. Rev. 81, 148 (1951).

where

$$\Gamma_{\lambda c^0} = \frac{\Gamma_{\lambda c}}{\left(1 - \frac{d\Delta_\lambda}{dE} \right)_{E_r}} = \frac{\Gamma_{\lambda c}}{1 + \sum_c \gamma_{\lambda c}^2 \frac{dS_c}{dE}} \quad (3.3)$$

$$\Gamma_\lambda^0 = \sum_c \Gamma_{\lambda c^0}. \quad (3.4)$$

The $\Gamma_{\lambda c^0}$ may be interpreted as the "observed" partial widths in contrast with the "formal" partial widths $\Gamma_{\lambda c}$, the latter being dependent on the values chosen for the channel radii. In the barrier region the penetration factors P_c are very sensitive to the energy and no approximate procedure for taking this fact into consideration can be recommended. Let us define new reduced widths in terms of the "observed" partial widths by the usual relation,

$$\Gamma_{\lambda c^0} = 2P_c(\gamma_{\lambda c}^0)^2; \quad (3.5)$$

then, from (3),

$$(\gamma_{\lambda c}^0)^2 = \frac{\gamma_{\lambda c}^2}{1 + \sum_c \gamma_{\lambda c}^2 \frac{dS_c}{dE}}. \quad (3.6)$$

The $(\gamma_{\lambda c}^0)^2$, which are obtained by dividing the *observed* partial reduced widths by twice the penetration factors, may be called the "observed" partial reduced widths. From resonance data it is generally not possible to measure the $(\gamma_{\lambda c}^0)^2$ for the negative-energy channels. It is therefore convenient to separate the sum in the denominator of (3.6) into two parts,

$$\sum_c \gamma_{\lambda c}^2 \frac{dS_c}{dE} = N + \sum_{c^+} \gamma_{\lambda c}^2 \frac{dS_c}{dE}, \quad (3.7)$$

where

$$N = \sum_{c^-} \gamma_{\lambda c}^2 \frac{dS_c}{dE} \quad (3.8)$$

and the additional channel designation $+$ or $-$ indicates whether the channel has positive or negative energy of relative motion. By substituting into (3.6) one finds that

$$\gamma_{\lambda c^0}^2 / (\gamma_{\lambda c}^0)^2 = \frac{1 + N}{1 - \sum_{c^+} \gamma_{\lambda c}^2 \frac{dS_c}{dE}}. \quad (3.9)$$

In applications dS_c/dE has been found to be positive so that this ratio $> (1 + N)$. Because the $(\gamma_{\lambda c}^0)^2$ are positive, the following inequality must be satisfied:

$$\sum_{c^+} (\gamma_{\lambda c}^0)^2 \left(\frac{dS_c}{dE} \right)_{E=E_r} < 1. \quad (3.10)$$

If the separation 3.7 had not been made, then $N=0$ and the sums in (3.10) and in the denominator of (3.9) would include negative as well as positive energy chan-

nels. From results given in the appendix it can be shown that N is the volume integral of the probability density in the part of the external region with negative-energy channels, that of the internal region being normalized to unity. Except near a threshold for a new positive-energy channel, N is expected to be small compared with one; however, as an s -neutron threshold is approached, it becomes indefinitely large because the external probability integral becomes indefinitely large.

A few qualitative remarks can be made concerning the distinction between the observed $\Gamma_{\lambda c}^0$ and the formal $\Gamma_{\lambda c}$. According to (3.6) the distinction between these quantities may be large when the $\gamma_{\lambda c}^2$ are large, in which case the wave function in the c channel will be large in the external region provided that it is not attenuated too rapidly by the barrier. Since the rate of attenuation of the external wave function is inversely proportional to dS_c/dE (see appendix), the distinction reflects the relative extent to which the nuclear wave function extends into the external region and the consequent uncertainty of the size of the actual nuclear system. The distinction is expected to be large when the barrier is weak although extending a large distance from the nuclear surface. It is therefore expected to increase with decreasing l_c and to be most important for weak Coulomb barriers. In the case of alpha-particle decay, for example, the difference between $\Gamma_{\lambda c}^0$ and $\Gamma_{\lambda c}$ is small, less than a few percent, because the wave function is attenuated very rapidly just beyond the nuclear surface, thus making it possible to define rather unambiguously an actual size for the nuclear system. Further examples are given in papers by Breit and co-workers.⁹¹ There is an important exception which must be made in the foregoing interpretation. The difference would be expected to be most pronounced in the case of an s -wave neutron channel in which the wave function extends essentially unimpeded into the external region. However, for such channels $dS_c/dE=0$ so that they do not contribute to making a difference between $\Gamma_{\lambda c}$ and $\Gamma_{\lambda c}^0$, at least in the present approximation where the contributions from the other levels are neglected. (See the remarks made in the discussion of the "third approximation" in subsection 1.)

Finally the idea of a "dimensionless reduced width" $\theta_{\lambda c}^2$ will be introduced. This quantity is defined as the ratio of the reduced width $\gamma_{\lambda c}^2$ to $(\hbar^2/M_c a_c^2)$:

$$(\theta_{\lambda c}^0)^2 = \left(\frac{\hbar^2}{M_c a_c^2} \right)^{-1} (\gamma_{\lambda c}^0)^2. \quad (3.11)$$

Similarly, corresponding to the *observed* reduced width $(\gamma_{\lambda c}^0)^2$:

$$(\theta_{\lambda c}^2)^2 = \left(\frac{\hbar^2}{M_c a_c^2} \right)^{-1} (\gamma_{\lambda c}^2)^2. \quad (3.12)$$

From the discussion of Sec. XI, 4, the following rule is implied for the $\theta_{\lambda c}^2$:

$$\sum_{\lambda} \theta_{\lambda c}^2 \sim 1 \quad (3.13)$$

where the sum is over all levels λ in any energy interval equal to the spacing between single particle levels. The rule (3.10) is, for comparison,

$$\sum_{c^+} (\theta_{\lambda c}^0)^2 \left[\frac{dS_c}{dE} \frac{\hbar^2}{M_c a_c^2} \right] < 1. \quad (3.14)$$

The correction to a reduced width (summed over channel spins) for a given channel arising from the variation of the partial level shift for *that channel only* is, from (3.6):

$$\left[\sum_s (\theta_{\lambda c}^0)^2 \right]^{-1} - \left(\sum_s \theta_{\lambda c}^2 \right)^{-1} = \frac{dS_c}{dE} \frac{\hbar^2}{M_c a_c^2}. \quad (3.15)$$

The quantity $(dS_c/dE)(\hbar^2/M_c a_c^2)_c$ is of order unity or less (see the appendix). For instance, in the case of l -wave neutrons ($l \neq 0$), the maximum value of this quantity, which is equal to $2(dS_c/d\rho_c^2)$ for neutrons, is $2/(2l-1)$ and is achieved at zero energy. Taking $(2dS_c/d\rho_c^2)$ as unity, the fractional corrections to the corrected reduced widths are equal to the uncorrected values. Since other uncertainties usually prevent the determination of reduced width values to better than 10% one may say that the correction is only significant if the uncorrected value of $\sum_s (\theta_{\lambda c}^0)^2$ is $\gtrsim 0.1$.

b. Second Approximate Method: Linear Expansion of $\Delta_{\lambda} - (i/2)\Gamma_{\lambda}$

The approximate result obtained after expanding the quantity $\Delta_{\lambda} - \frac{1}{2}i\Gamma_{\lambda}$ in the denominator of the resonance amplitude linearly with respect to energy has somewhat more physical significance than the previous approximation of simply expanding Δ_{λ} , although it is probably somewhat less useful in applications. Although in the barrier region a linear expansion of the width would appear to be a very poor one, it is actually satisfactory in many applications because the width term in the denominator is effective only in the immediate vicinity of the resonance; the partial widths of the numerator should always be treated exactly.

The quantity ϵ_{λ} of (1.7) is expanded linearly with respect to energy about the real energy E_0 which satisfies the equation

$$E_{\lambda} + \Delta_{\lambda}(E_0) - E_0 = 0, \quad (3.16)$$

so that

$$\epsilon_{\lambda} \approx E_{\lambda} - E - \xi_{\lambda}(E_0) - (E - E_0)(d\xi_{\lambda}/dE)_{E=E_0} \quad (3.17)$$

where

$$\xi_{\lambda} = -\Delta_{\lambda} + \frac{1}{2}i\Gamma_{\lambda}.$$

By substituting (3.16) into (3.17) one eventually obtains

$$\epsilon_{\lambda} = [1 + (d\xi_{\lambda}/dE)_{E=E_0}][E_r - E - \frac{1}{2}i\Gamma_{\lambda}^{\dagger}], \quad (3.18)$$

where

$$E_r = E_0 - \frac{1}{2}\Gamma_{\lambda}(E_0) \frac{\text{Im}(d\xi_{\lambda}/dE)}{[1 + \text{Re}(d\xi_{\lambda}/dE)]^2 + [\text{Im}(d\xi_{\lambda}/dE)]^2} \quad (3.19)$$

is the resonance energy, and

$$\Gamma_\lambda^\dagger = \sum_c \Gamma_{\lambda c}^\dagger, \\ \Gamma_{\lambda c}^\dagger / \Gamma_{\lambda c} = 1 + \operatorname{Re} \frac{d\xi_\lambda}{dE} / \left(1 + \operatorname{Re} \frac{d\xi_\lambda}{dE} \right)^2 + \left(\operatorname{Im} \frac{d\xi_\lambda}{dE} \right)^2. \quad (3.20)$$

The resonance contribution to the cross sections may then be considered proportional to

$$\frac{\Gamma_{\lambda c}^* \Gamma_{\lambda c}^*}{(E_r - E)^2 + \frac{1}{4} \Gamma_\lambda^{\dagger 2}}, \quad (3.21)$$

where

$$\Gamma_{\lambda c}^* = 2P_c |\omega_{\lambda c}|^2, \\ \omega_{\lambda c} = \frac{\gamma_{\lambda c}}{\{ [1 + \operatorname{Re}(d\xi_\lambda/dE)]^2 + [\operatorname{Im}(d\xi_\lambda/dE)]^2 \}^{\frac{1}{2}}} \\ \times \exp \frac{1}{2} i \left\{ \tan^{-1} \left(\frac{\operatorname{Im}(d\xi_\lambda/dE)}{1 + \operatorname{Re}(d\xi_\lambda/dE)} \right) \right\}.$$

The total width Γ_λ^\dagger in the denominator of (3.21) is in general less than the sum of the partial widths $\Gamma_{\lambda c}^*$ of the numerator, in accordance with the inequality (IX, 2.14). In many applications $\operatorname{Im}(d\xi_\lambda/dE)$ is small and this difference is likewise small.

The results of this approximation differs from the previous one by the inclusion of the effect of $\operatorname{Im}(d\xi_\lambda/dE)$ in the widths of (3.20) and (3.21). The quantities $\omega_{\lambda c}$ of (3.21) correspond to those of (IX, 2.4), and the "renormalization" factor $[1 + (d\xi_\lambda/dE)]^{-\frac{1}{2}}$ of (3.21) corresponds to the contribution from the term dL/dE in the normalized condition (IX, 2.30). As mentioned in Sec. IX, 2, this contribution is proportional to the volume integral in the external channels for the complex eigenfunctions with energy $H_\lambda = E_r - \frac{1}{2} i \Gamma_\lambda^\dagger$.

c. Displacements in Mirror Levels

Some remarks concerning the positions of corresponding levels in mirror nuclei are relevant here. It is well known that various pairs of corresponding levels show different displacements. If the spectra of two given mirror nuclei like C^{13} and N^{13} are drawn along side each other with the ground states on the same horizontal line, some of the pairs of excited states are only displaced by a kev or so, whereas others are displaced by 100's of kev. The maximum observed displacement is that of the first excited states of C^{13} and N^{13} where it is over 700 kev. Ehrman⁹⁵ and Thomas⁹⁶ show that this displacement may be explained qualitatively in terms of the different boundary conditions at the channel entrance for separation of C^{13} into $C^{12} + n$ and N^{13} into $C^{12} + p$. Referring back to (3.1) we have

$$E_\lambda + (B - S(E)) \gamma_\lambda^2 - E = 0, \quad (3.22)$$

where we have assumed that the only contribution to Δ_λ that need be considered is that from the channel for separation into C^{12} (in the ground state) and a nucleon. Since $S(E)$ is different according to whether the nucleon is a neutron or a proton, so the solution $E = E_r$ of this

equation is different for the two cases. Denoting the neutron and proton cases by N and P , and choosing the same value of B for the two cases:

$$E_{rN} - E_{rP} = (E_{\lambda N} - E_{\lambda P}) \\ + (S_P(E_{rP}) - S_N(E_{rN})) \gamma_\lambda^2, \quad (3.23)$$

where we have taken γ_λ^2 to be the same for the two cases. This will be correct if the two internal wave functions X_λ may be assumed to be the same. The predicted difference in the displacements of the first excited states and the ground states follows on taking the difference between $E_{rN} - E_{rP}$ evaluated for the two sets of states. Assuming that the internal Coulomb energy difference $E_{\lambda N} - E_{\lambda P}$ is the same for the ground state and first excited states the difference in displacements is predicted to equal the difference in the quantity

$$[S_P(E_{rP}) - S_N(E_{rN})] \gamma_\lambda^2 \quad (3.24)$$

when evaluated for the two pairs of states, $\lambda=0$ and 1.

Numerically the term for the excited states dominates the other. This is because of the nature of the quantity S , which varies most strongly with energy at energies near threshold and l values near zero. (The excited state energies in the first term are ~ 2.5 Mev nearer threshold than the ground state energies in the second; and the excited states are $l=0$, whereas the ground states are $l=1$.) Ehrman⁹⁵ and Thomas⁹⁶ found that this first term in (3.24) was qualitatively equal to the 700 kev relative displacement. The main uncertainty was due to lack of knowledge of the value of γ_λ^2 to better than a factor of ~ 2 .

4. The Two-Level Approximation

Occasionally one may find experimental instances of two levels (of the same spin and parity) occurring fortuitously close together in a region of normally well-isolated levels. In such a case, the two levels should be analyzed together as a single anomaly. For this purpose, one may try to use the two-level formula that is the counterpart of (1.13) for the one-level case. Such a formula follows from Eqs. (IX, 1.14 and 22) and (VIII, 3.2). For simplicity we will set $\mathbf{R}^0 = 0$, so that the \mathbf{R} matrix is considered to be

$$\mathbf{R} = \frac{\gamma_1 \times \gamma_1}{E_1 - E} + \frac{\gamma_2 \times \gamma_2}{E_2 - E}. \quad (4.1)$$

A certain amount of algebraic manipulation leads to the required formula,

$$\sigma_{\alpha\alpha'} = \frac{\pi g_J}{k^2} \sum_{s's'l'l'} \frac{[\bar{E}_2 - E] \Gamma_{1c}^{\frac{1}{2}} \Gamma_{1c'}^{\frac{1}{2}} + (\bar{E}_1 - E) \Gamma_{2c}^{\frac{1}{2}} \Gamma_{2c'}^{\frac{1}{2}} - \Delta_{12} (\Gamma_{1c}^{\frac{1}{2}} \Gamma_{2c'}^{\frac{1}{2}} + \Gamma_{2c}^{\frac{1}{2}} \Gamma_{1c'}^{\frac{1}{2}})]^2 + \frac{1}{4} [\sum_{c''c'} \Pi_{c''c} \Pi_{c''c'}]^2}{[(\bar{E}_1 - E)(\bar{E}_2 - E) + \frac{1}{4} (\Gamma_{12}^2 - \Gamma_1 \Gamma_2) - \Delta_{12}^2] + \frac{1}{4} [\Gamma_1 (\bar{E}_2 - E) + \Gamma_2 (\bar{E}_1 - E) - 2 \Delta_{12} \Gamma_{12}]^2}, \quad (4.2)$$

⁹⁵ J. B. Ehrman, Phys. Rev. **81**, 412 (1951).

⁹⁶ R. G. Thomas, Phys. Rev. **88**, 1109 (1952).

where

$$\begin{aligned}\Pi_{c''c} &= \Gamma_{1c''} \Gamma_{2c}^{\frac{1}{2}} - \Gamma_{2c''} \Gamma_{1c}^{\frac{1}{2}}; \quad \Gamma_1 \equiv \Gamma_{11} \\ \bar{E}_1 &= E_1 + \Delta_1 \\ \Gamma_{12} &= \sum_{c''} 2P_{c''} \gamma_{1c''} \gamma_{2c''}.\end{aligned}\tag{4.3}$$

Selecting the boundary condition $B_c = S_c$ and provided that S_c is reasonably constant in the energy range of interest (so that $\Delta_1 = \Delta_2 = \Delta_{12} = 0$), (4.2) simplifies somewhat to

$$\sigma_{\alpha\alpha'} = \frac{\pi g_J}{k_\alpha^2} \sum_{s, s'} \frac{[(E_2 - E)\Gamma_{1c} \Gamma_{2c'}^{\frac{1}{2}} + (E_1 - E)\Gamma_{2c} \Gamma_{1c'}^{\frac{1}{2}}]^2 + \frac{1}{4} [\sum_{c''} \Pi_{c''c} \Pi_{c''c'}]^2}{[(E_1 - E)(E_2 - E) + \frac{1}{4}(\Gamma_{12}^2 - \Gamma_1 \Gamma_2)]^2 + \frac{1}{4} [\Gamma_1(E_2 - E) + \Gamma_2(E_1 - E)]^2}.\tag{4.4}$$

In the special case when there are only 2 channels, the second term in the numerator is zero. From a study of the remaining term it is clear that $\sigma_{\alpha\alpha'}$ will vanish between resonances if $\gamma_{1c} \gamma_{1c'}$ and $\gamma_{2c} \gamma_{2c'}$ have the same sign, but not otherwise. In the general many-channel case, the second term in the numerator means that $\sigma_{\alpha\alpha'}$ never vanishes. These findings are in keeping with those deduced from general arguments mentioned in Sec. VII, 3.

5. Interference between Isolated Levels

a. Cross-Section Minima between Levels

In the preceding subsection it was shown that the one-level expansion provides a useful description of the behavior of the cross section maxima when the ratio $\Gamma/D \ll 1$. Teichmann⁴⁸ has shown that, in the same region, it is possible to make estimates of magnitudes and behaviors of the cross-section minima between resonances. Unlike the maxima, these minima may range in magnitudes over several powers of Γ/D , and the estimates therefore have only qualitative value.

The estimates are made by expanding in a power series in the vicinity of the minima the matrix $(\mathbf{1} - \mathbf{R}\mathbf{L}^0)^{-1}$, or equivalently the matrix $(\mathbf{L}^0)^{-\frac{1}{2}}(\mathbf{1} - \mathbf{L}^0 \mathbf{R}\mathbf{L}^0)(\mathbf{L}^0)^{\frac{1}{2}}$. A sufficient condition for the permissibility of such an expansion is that the norm

$$\|\mathbf{L}^0 \mathbf{R}\mathbf{L}^0\| = \sum_{cc'} |L_c^0 R_{cc'} L_{c'}^0|^2\tag{5.1}$$

be smaller than one, and the usefulness of the expansion will depend upon the norm being much less than one. This condition is most likely to be satisfied if the boundary conditions parameters B_c , are set equal to the shift factors, S_c , in which case $L_c = iP_c$. Now, in between the resonances, the order of magnitude of the \mathbf{R} components may be estimated as $\langle \gamma_{\lambda c} \gamma_{\lambda c'} \rangle / D$, although it is quite possible that some of the nondiagonal components as well as all of the diagonal components may actually vanish there. (On the other hand, as pointed out in Sec. XI, 6, there may be a cumulative contribution to $R_{cc'}$ from far-away levels. Following

Teichmann,⁴⁸ we ignore this possibility without real justification.) Thus, if the signs of the $\gamma_{\lambda c}$ are assumed to be as likely positive as negative, their magnitudes independent of λ , and the levels uniformly spaced by an amount D , the average of $R_{cc'}^2$ with respect to all possible choices of signs is found to be

$$\langle R_{cc'}^2 \rangle_{\pm} = \sum_{\lambda} \gamma_{\lambda c}^2 \gamma_{\lambda c'}^2 / (E_{\lambda} - E)^2 \approx \gamma_{\lambda c}^2 \gamma_{\lambda c'}^2 (\pi/D)^2.$$

Likewise, if the signs of the $\gamma_{\lambda c} \gamma_{\lambda c'}$ are alternately positive and negative, the same numerical factor of π is arrived at for the magnitude of $R_{cc'}$. In view of the fact that some of the $R_{cc'}$ components may vanish, it seems reasonable to assert that

$$\begin{aligned}\|L^0 \mathbf{R}\mathbf{L}^0\| &\lesssim (\pi/D)^2 \sum_{cc'} P_c P_{c'} \gamma_{\lambda c}^2 \gamma_{\lambda c'}^2 \\ &= (\pi\Gamma/2D)^2,\end{aligned}\tag{5.2}$$

where Γ and D are appropriate mean total level width and spacing. The following considerations are therefore likely to be applicable when $\Gamma \ll D$.

The power series expansions for the nondiagonal reaction components of the collision matrix are

$$W_{cc'} = 2iP_c \frac{1}{2} [R_{cc'} + \sum_{c''} R_{cc''} L_{c''}^0 R_{c''c'} + \dots] P_{c'}.\tag{5.3}$$

In the consideration of this expansion it is necessary to distinguish (Fig. 3) three possibilities. (1) $R_{cc'}$ has an extremum between resonances but does not pass through zero. (2) $R_{cc'}$ vanishes between resonances but has no extremum. (3) $R_{cc'}$ goes through zero twice and has an extremum between these zeros. In the third case, which is somewhat unusual, the reaction cross section has between resonances two minima and one maxima which is itself not a resonance. For the first possibility to occur between resonances at E_{λ} , $E_{\lambda+1}$ it is necessary that the products $\gamma_{\lambda c} \gamma_{\lambda c'}$ and $\gamma_{\lambda+1, c} \gamma_{\lambda+1, c'}$ have opposite signs. Consideration of the first term of the expansion (5.3) then suffices for an estimate of the magnitude and shape of the minimum. The third possibility is a special case of the first and could occur if the contribution to $R_{cc'}$ from the totality of other levels on both sides is the same sign as $\gamma_{\lambda c} \gamma_{\lambda c'}$ and of sufficient magnitude to annul the contribution from the levels E_{λ} and $E_{\lambda+1}$.

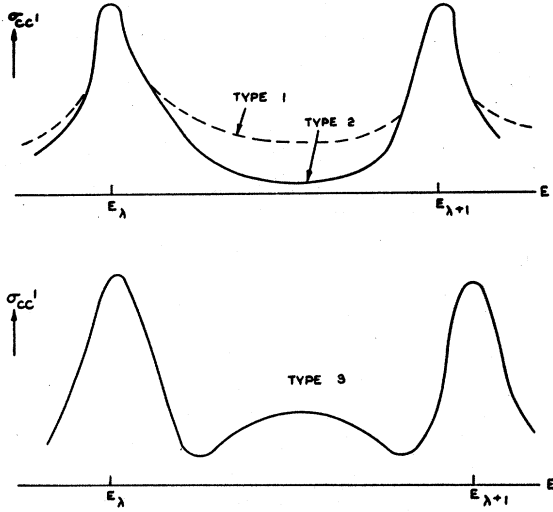


FIG. 3. Different types of behavior for cross sections in between resonances. The origin of the three types is described in the text.

The extremum of $R_{cc'}$, which occurs between the two energies of annulment, leads to a cross-section minimum with the same shape as the minimum which occurs when $R_{cc'} \neq 0$. When Γ/D is small, such an occurrence probably requires an anomalously high value of $\gamma_{\mu c}^2$ at a nearby level, or a rather sudden decrease in the level spacing. For the occurrence of the second possibility it is generally necessary that the products $\gamma_{\lambda c} \gamma_{\lambda c'}$ and $\gamma_{\lambda+1, c} \gamma_{\lambda+1, c'}$ have the same sign. The behavior of the cross-section minimum in this case will depend largely upon the second term of the expansion (5.3).

Minima of Type (i)

If $R_{cc'} \neq 0$ in the vicinity of the minimum, the cc' component of the collision matrix may be considered as approximately given by

$$\begin{aligned} |W_{cc'}|^2 &\approx 4P_c P_{c'} [R_{cc'}^{(\min)} + (E - E_{\min})^2 (d^2/dE^2)(R_{cc'}^{(\min)})]^2 \\ &\approx 4P_c P_{c'} [(R_{cc'}^{(\min)})^2 \\ &\quad + 2R_{cc'}^{(\min)}(E - E_{\min})^2 (d^2/dE^2)(R_{cc'})]. \end{aligned} \quad (5.4)$$

The shape of the cross-section minimum is evidently parabolic. As before it is estimated that $(R_{cc'})^2 = \gamma_{\lambda c}^2 \gamma_{\lambda c'}^2 \times (\pi/D)^2$. Similarly, for the second term, it is estimated that

$$\begin{aligned} R_{cc'}^{(\min)} \frac{d^2}{dE^2} (R_{cc'}) &= \sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{E_{\lambda} - E} \sum_{\mu} \frac{\gamma_{\mu c} \gamma_{\mu c'}}{(E_{\mu} - E)^3} \\ &\approx \gamma_{\lambda c}^2 \gamma_{\lambda c'}^2 \sum_{\lambda} (E_{\lambda} - E)^{-4} \\ &\approx \gamma_{\lambda c}^2 \gamma_{\lambda c'}^2 (32/D^4) \sum_{n=0}^{\infty} (2n+1)^{-4} \\ &= \frac{1}{3} \gamma_{\lambda c}^2 \gamma_{\lambda c'}^2 (\pi/D)^4 \end{aligned} \quad (5.5)$$

and hence

$$|W_{cc'}|^2 \approx \Gamma_{\lambda c} \Gamma_{\lambda c'} (\pi/D)^2 [1 + \frac{2}{3} (\pi/D)^2 (E - E_{\min})^2]. \quad (5.6)$$

The ratio of the cross-section minimum to the resonance maximum is

$$\sigma_{cc'}^{(\min)} / \sigma_{cc'}^{(\max)} \approx (\pi \Gamma / 2D)^2. \quad (5.7)$$

If a width W of the minimum is defined as the interval between the energies at which $\sigma_{cc'}$ exceeds $\sigma_{cc'}^{(\min)}$ by a factor of four, then from (5.4)

$$W \approx 1.4D.$$

Minima of Type (ii)

When $R_{cc'} = 0$ in the vicinity of the minimum, the absolute square of the cc' component of the collision matrix may be approximated by

$$|W_{cc'}|^2 \approx 4P_c P_{c'} \left[\left(\frac{dR_{cc'}}{dE} \right)^2 (E - E_{\min}) + (Q_{cc'})^2 \right] \quad (5.8)$$

where

$$Q_{cc'} = \sum_{c''} R_{cc''} P_{c''} R_{c''c'} = \sum_{c''} \sum_{\lambda \mu} \frac{\gamma_{\lambda c} \gamma_{\lambda c''}}{E_{\lambda} - E} P_{c''} \frac{\gamma_{\mu c''} \gamma_{\mu c'}}{E_{\mu} - E}$$

since the first term of (5.3) is real whereas the second nonvanishing term is imaginary. The quantity $Q_{cc'}$ may be estimated as follows: the sum over channels c is of the order of or less than $\frac{1}{2}\Gamma$; as before, the magnitude of the sum and the μ sum may be considered as $\langle \gamma_{\lambda c} \rangle \times (\pi/D)$ and $\langle \gamma_{\mu c'} \rangle (\pi/D)$, respectively; therefore, $Q_{cc'} \lesssim \frac{1}{4} (\gamma_{\lambda c}^2 \gamma_{\lambda c'}^2) \Gamma (\pi/D)^2$. By substituting this estimate into (5.8), one obtains

$$|W_{cc'}|^2 \lesssim \Gamma_{\lambda c} \Gamma_{\lambda c'} (\pi/D)^4 [(E - E_{\min})^2 + \frac{1}{4} \Gamma^2]. \quad (5.9)$$

The ratio of the cross-section minimum to the resonance maximum is evidently

$$\sigma_{cc'}^{(\min)} / \sigma_{cc'}^{(\max)} \approx (\pi \Gamma / 2D)^4 \quad (5.10)$$

which is two orders of (Γ/D) smaller than the previous one. The width of the parabolic minimum is

$$W \sim \Gamma. \quad (5.11)$$

b. Collision Matrix for Regions Containing Isolated Levels

When $\Gamma/D \ll 1$, it is possible to obtain an approximate expansion for \mathbf{U} in terms of the levels of the system. The form of this expansion was first deduced by Bethe¹ and recently by Wigner¹¹ using the more rigorous procedure of \mathbf{R} -matrix theory.

If the widths are small and good boundary conditions employed, then the components of the matrix ξ of (IX, 1.9) are small and one can attempt to expand the

matrix $\mathbf{A} = (\boldsymbol{\varepsilon} - \boldsymbol{\xi}')^{-1}$ about the diagonal part $\boldsymbol{\varepsilon} = \mathbf{e} - \mathbf{E} - \boldsymbol{\xi} + \boldsymbol{\xi}'$ whose components are $\varepsilon_\lambda = E_\lambda - E - \frac{1}{2}i\Gamma$, where $\boldsymbol{\xi}'$ is the nondiagonal part of $\boldsymbol{\xi}$. Thus

$$\mathbf{A} = \boldsymbol{\varepsilon}^{-1} + \boldsymbol{\varepsilon}^{-1}\boldsymbol{\xi}'\boldsymbol{\varepsilon}^{-1} + \dots, \quad (5.12)$$

and therefore, according to (IX, 1.13)

$$\begin{aligned} (\mathbf{1} - \mathbf{R}\mathbf{L}^0)^{-1}\mathbf{R} &= \sum_\lambda (\boldsymbol{\gamma}_\lambda \times \boldsymbol{\gamma}_\lambda) \varepsilon_\lambda^{-1} \\ &+ \sum_{\lambda\mu} \varepsilon_\lambda^{-1} (\boldsymbol{\gamma}_\lambda \times \boldsymbol{\gamma}_\mu) \xi_{\lambda\mu}' \varepsilon_\mu^{-1} + \dots, \end{aligned} \quad (5.13)$$

the components of which are

$$\begin{aligned} [(\mathbf{1} - \mathbf{R}\mathbf{L}^0)^{-1}\mathbf{R}]_{cc'} &= \sum_\lambda (\boldsymbol{\gamma}_{\lambda c} \boldsymbol{\gamma}_{\lambda c'} / \varepsilon_\lambda) + i \sum_{c''} P_{c''} \sum_\lambda \\ &\times (\boldsymbol{\gamma}_{\lambda c} \boldsymbol{\gamma}_{\lambda c''} / \varepsilon_\lambda) \sum_{\mu (\neq \lambda)} (\boldsymbol{\gamma}_{\mu c''} \boldsymbol{\gamma}_{\mu c'} / \varepsilon_\mu). \end{aligned} \quad (5.13a)$$

The components of $|\xi_{\lambda\mu}| < \sum_{c''} P_{c''} |\boldsymbol{\gamma}_{\lambda c''} \boldsymbol{\gamma}_{\mu c''}|$ are less than or of the order $\frac{1}{2}\Gamma$. If a particular λ is near resonance, then the terms of the μ sum are not and the magnitude of their sum may be estimated as $\langle \boldsymbol{\gamma}_{\mu c'} \rangle \times (\pi/D)$. On the other hand, if a μ term is near resonance, the terms of the λ sum are not and the magnitude of their sum is similarly $\langle \boldsymbol{\gamma}_{\lambda c} \rangle (\pi/D)$. Clearly then if $\Gamma/D \ll 1$, the second and higher terms of (5.13) may be neglected, and the approximation to W is obtained by retaining only the first sum of (5.13):

$$\mathbf{W} = \mathbf{1} + 2i\mathbf{P}^{\frac{1}{2}} \left[\sum_\lambda \frac{(\boldsymbol{\gamma}_\lambda \times \boldsymbol{\gamma}_\lambda)}{\varepsilon_\lambda} \right] \mathbf{P}^{\frac{1}{2}}. \quad (5.14)$$

Although (5.14) has the form of the more exact expansion (IX, 2.5), its $\boldsymbol{\gamma}_{\lambda c}$ are real and the imaginary parts $\frac{1}{2}\Gamma_\lambda$ of the denominators satisfy the condition (IX, 2.14) in the exceptional case of equality. If the Γ_λ are of the same order or larger than the energy differences between resonances, (5.14) may become grossly inaccurate. Nevertheless, it is shown in Sec. XI, 2 that if the signs of the $\boldsymbol{\gamma}_{\lambda c}$ may be considered as likely to be positive as negative, (13.14) is valid for any value of Γ/D provided only that the individual ratios $\langle 2\pi\Gamma_{\lambda c}/D \rangle$ are all much less than unity.

The reactions cross sections corresponding to (5.14) are, from (VIII, 3.3),

$$\sigma_{\alpha\alpha'} = \frac{\pi}{k_\alpha^2} \sum_{J s l s' l'} g_J \left| \sum_\lambda \frac{\Gamma_{\lambda c}^{\frac{1}{2}} \Gamma_{\lambda c'}^{\frac{1}{2}}}{E_\lambda + \Delta_\lambda - E - \frac{i}{2}\Gamma_\lambda} \right|^2, \quad (5.15)$$

where the sum over λ is over levels of given spin J and parity. For this formula, if the signs of $\boldsymbol{\gamma}_{\lambda c'}$ are random with respect to c' , the interference terms will be very small in cross sections that imply sums over large numbers of channels. For this reason experimental slow neutron capture cross sections (involving the hundreds of photon channels) can be analyzed into superpositions of Breit-Wigner resonance terms without interference.

The fission cross sections, in contrast, exhibit strong level interference effects in keeping with the suggestion⁵⁶ that only 2 or 3 channels are involved.

XIII. SPECIAL TOPICS

The present section deals with miscellaneous special topics. The discussion of three-body disintegration and photon processes will partly remove restrictions hitherto imposed on the theory.

1. Cross Sections near Thresholds

Although the main results reviewed here on behavior of cross sections near thresholds have been known since the introduction of the Born approximation into nuclear theory, not until recently was a derivation given by Wigner⁹⁷ in which the rigor was such as to leave no doubt as to the validity of these well-known results. Wigner's derivation goes somewhat beyond the earlier ones in that it shows that cusps can occur in reaction and scattering cross sections at the thresholds. This derivation does not make use of the one-level approximation but does require that the range of nuclear excitation energies of the compound nucleus being considered be much less than the distance to the next level. It also requires that, in general, $\rho \ll 1$ and, in the case of charged particles, that in addition $\eta \gg 1$.

From (VII, 1.6), a typical nondiagonal element of W may be written

$$W_{cc'} = 2i[P_c^{\frac{1}{2}}(L_c^0)^{-1}] \{ [(\mathbf{L}^0)^{-1} - \mathbf{R}]^{-1} \}_{cc'} [P_{c'}^{\frac{1}{2}}(L_{c'}^0)^{-1}]. \quad (1.1)$$

Near the threshold for either channel c or c' , the vital quantities determining the energy dependence of the cross section are $P_c |L_c^0|^{-2}$ and $P_{c'} |L_{c'}^0|^{-2}$, respectively. In terms of the Coulomb wave functions introduced in Sec. III, we have, dropping subscripts c ,

$$\begin{aligned} P^{-1} |L^0|^2 &= \frac{1}{P} \{ P^2 + (B - S)^2 \} \\ &= \frac{F^2 + G^2}{\rho} \left\{ \left(\frac{\rho}{F^2 + G^2} \right)^2 + \left(B - \rho \frac{FF' + GG'}{F^2 + G^2} \right)^2 \right\}. \end{aligned} \quad (1.2)$$

At energies near the threshold for a channel c with $l \neq 0$, $F \ll G$, $F' \ll G'$, $G \gg 1$ and

$$P^{-1} |L^0|^2 = \frac{1}{\rho} [GB - \rho G']^2. \quad (1.3)$$

In the absence of a Coulomb barrier, we may use the limiting forms for G and G' which are valid at very low

⁹⁷ E. P. Wigner, Phys. Rev. **73**, 1002 (1948); also G. Breit, *ibid.* **107**, 1612 (1957).

energies (which, in this case, means for $\rho \ll 1$),

$$G_l = \frac{(2l-1)!!}{\rho^l}; \quad G_l' = \frac{-l(2l-1)!!}{\rho^{l+1}} \quad (1.4)$$

so that

$$P|L^0|^{-2} = \rho^{l+1}[(B+l)(2l-1)!!]^{-2}. \quad (1.5)$$

The case $l=0$ is exceptional when there is no barrier; since $P=\rho$, $S=0$, when $l=0$, Eq. (1.2) gives, for this case,

$$P|L^0|^{-2} = \rho/(\rho^2+B^2). \quad (1.6)$$

When a Coulomb barrier is present, the limiting forms for low energies are (see appendix),

$$\begin{aligned} G_l &= (4\rho/\pi)^{\frac{1}{2}} e^{\pi\eta} K_{2l+1}(\beta), \\ G_l' &= -(1/\pi\rho)^{\frac{1}{2}} e^{\pi\eta} \{\beta K_{2l}(\beta) + 2lK_{2l+1}(\beta)\}, \end{aligned} \quad (1.7)$$

where $\beta = (8\rho\eta)^{\frac{1}{2}}$ and the K 's are modified Bessel functions of the second kind. (We follow Wigner⁹⁷ and use K 's that are $(-1)^{2l+1}$ times those defined by Watson.) The combination $\rho\eta$ does not depend on energy so that β remains finite as $E \rightarrow 0$. For this case, from (1.3),

$$P|L^0|^{-2} = \pi\rho e^{-2\pi\eta} [2\rho^{\frac{1}{2}}(B+l)K_{2l+1}(\beta) + \rho^{\frac{1}{2}}BK_{2l}(\beta)]. \quad (1.8)$$

The quantity $[(L^0)^{-1} - \mathbf{R}]$ in the expression (1.1) for the collision matrix is essentially constant for the nonthreshold channels in the range of energies near a threshold. For the threshold channel the imaginary part of $(L^0)^{-1}$ is very small and its real part is finite and essentially constant. The quantity $[(L^0)^{-1} - \mathbf{R}]^{-1}$ may thus be regarded as constant in first approximation and the energy dependences of the reaction cross sections are dominated by $P_c|L_c^0|^{-2}$. The cross section for the production of a new particle type α' with a negative Q will thus behave as

$$\sigma \sim k_{\alpha'}^{2l+1} \quad \text{when } \eta = 0$$

and

$$\sigma \sim e^{-2\pi\eta\alpha'} \quad \text{when } \eta > 0.$$

If the Q of a reaction is positive, the cross section for the reaction will depend on the energy of the bombarding particle α for small values of E_{α} as

$$\sigma \sim k_{\alpha}^{2l-1} \quad \text{when } \eta = 0$$

and

$$\sigma \sim k_{\alpha}^{-2} e^{-2\pi\eta\alpha} \quad \text{when } \eta > 0.$$

The first of the last two dependences gives the well-known $1/v$ law for s -wave neutron capture. The second gives the proper relation to use for extrapolating charged-particle reaction cross sections to low bombarding energies; in this case the effect of the centrifugal

field is completely washed out by the Coulomb field as regards the asymptotic behavior, although the absolute value of the cross section will depend to some extent on the value of angular momentum. We also observe the well-known paradox that while the ratios of the cross sections for various angular momenta remain finite when there is a Coulomb field, they do not in the absence of such a field.

For reactions with negative Q 's, the next approximation in the consideration of the threshold behavior is to include the energy dependence of $[(L^0)^{-1} - \mathbf{R}]$. If the new particle type is uncharged, then it will emerge at threshold with zero angular momentum, the bombarding particle bringing in the balance required to conserve momentum, so that, for positive energy channels: $L_c^0 = (i\rho_c - B_c)$ and for negative energy channels: $L_c^0 = \rho_c - B_c$.

Treating the two types of channels together: $L_c^0 = (-2M_c E_c k_{\alpha}^2 / \hbar^2)^{\frac{1}{2}} - B_c$. Considered as a complex function of E , L^0 has a branch point at $E=0$, resulting in the introduction of singular energy dependences (cusp or "S-on-its-side") into the various cross sections. Such branch points do not occur when there is a barrier for the new particle type.

These results may be illustrated by the special case where the cross section near the threshold is dominated by a single resonance level, λ . In this case, from Sec. XII, 1a

$$|W_{cc'}|^2 = \frac{\Gamma_{\lambda c} \Gamma_{\lambda c'}}{(E_{\lambda} + \Delta_{\lambda} - E)^2 + \frac{1}{4}\Gamma_{\lambda}^2}. \quad (1.9)$$

The threshold energy dependences of (1.5) and (1.8) are now contained in the widths of the numerator in the form of the penetration factors P . Just above the threshold for either c or c' , the denominator may be regarded as constant relative to the energy dependence of the numerator. In the case when the energy passes through the region of a third channel c'' however, the situation may be reversed. The numerator is essentially constant but the denominator may vary sharply because of the terms $\Delta_{c''}$ and $\Gamma_{\lambda c''}$. This variation is most pronounced for an s -wave channel without barrier. In this case choosing $B_c = 0$, $\Delta_{\lambda c''} = \rho_{c''} \gamma_{\lambda c''}^2$, $\Gamma_{\lambda c''} = 0$ below threshold and $\Gamma_{\lambda c''} = 2\rho_{c''} \gamma_{\lambda c''}^2$, $\Delta_{\lambda c''} = 0$ above, so there are actually discontinuities in the energy derivatives of $\Delta_{\lambda c''}$ and $\Gamma_{\lambda c''}$ when the threshold is crossed. It is the resulting discontinuity in the slope of the denominator (1.9) that produces the cusp or "S-on-its-side" anomaly. For all other types of channels there is no actual discontinuity in $\Delta_{\lambda c''}$ or $\Gamma_{\lambda c''}$ but they may still vary sharply through the threshold region when the l value or the channel barrier is small.

Results obtained for the behavior of cross sections near thresholds for two-body decays can also be applied to three-body decay thresholds if these decays can be treated with sufficient accuracy as two-stage processes

as will be done in subsection 2. An important example is the $(n,2n)$ reaction in which the probability for the emission of the first neutron with an energy E_1 is proportional to $E_1^{\frac{1}{2}}dE_1$ and the probability for the emission of the second neutron with an energy E_2 is proportional to $E_2^{\frac{1}{2}}$ so that $\sigma(n,2n) \sim (E_1E_2)^{\frac{1}{2}}dE_1$. If the total energy in excess of the threshold is $E = E_1 + E_2$, then by integrating over the neutron distribution one finds that $\sigma(n,2n) \sim E^2$. These dependences were derived in a complicated though more rigorous manner by Snow. As emphasized by him, the energy range of their validity is so limited that they are of no use for analyzing any presently existing data. As a criterion for their validity, Snow⁹⁸ gives the permissible range of E as the smaller of $D_A/2500$ and $D_{A+1}/100$ where D_A denotes the mean energy spacing between levels of the bombarded nucleus A at the threshold excitation and D_{A+1} is the mean level spacing of the compound nucleus. Since $(n,2n)$ reactions generally take place at high excitation energies, these ranges of E are usually of the order of an electron volt which is much smaller than any attainable beam resolution. On the other hand, Weisskopf's statistical theory^{3,5,24} gives an expression for this energy dependence which is valid in the opposite extreme where the ranges of energies E involved are large compared to both level spacings. This expression is

$$\sigma(n,2n) \sim \pi a^2 \left[1 - \left(1 + \frac{E}{T} \right) e^{-E/T} \right],$$

where T is the temperature of the bombarded nucleus at the threshold excitation. Although this statistical expression also indicates a $\sigma \sim E^2$ dependence when $E \ll T$, this dependence is unrelated to that just obtained from the resonance theory; there is no reason that these two quadratic relations should join smoothly onto one another in the intermediate ranges of energies E which are comparable to the level spacings as their absolute coefficients are not expected to be the same.

Wigner⁹⁷ studied also the effect of perturbing potentials on the asymptotic behaviors. His investigation shows that if the small additional potential is twice integrable to infinity, i.e., if it drops as fast as r^{-n} with $n > 2$, the asymptotic behavior is unaffected although the magnitude of the barrier and the range of validity of the asymptotic expressions will be affected. This shows then that the centrifugal potential is just about the weakest potential to change the asymptotic behavior but that its effect can even be washed out by the more influential Coulomb field. The condition that the potentials be insignificant and do not affect the region of validity of the asymptotic laws is that the second integral P_2 of the quantity P in the perturbing potential $\hbar^2 P/2M$ be small compared with unity (see Sec. VII, 4). In practice this is equivalent with the condition that

⁹⁸ G. Snow, thesis (Princeton University, 1950), (unpublished).

the WKB integral be small. As an example, the penetration of the magnetic moment barrier is considered. In this case $P \sim (2M/\hbar^2)(e\hbar/2Mc)^2(\mu_1\mu_2/r^3)$ for which the second integral is

$$P_2 = \int_a^\infty \int_{r'}^\infty P(r) dr dr' = \frac{e^2 \mu_1 \mu_2}{4Mc^2 a},$$

which is of the order 10^{-3} thus showing that the magnetic interaction should be without appreciable effect on slow neutron processes even though the height of this barrier at the radius a may be several kev.

2. Three-Body Disintegration Treated as a Succession of Two Two-Body Disintegrations

Hitherto it has been assumed that the excitation energy of the nuclear system was below the thresholds for its disintegration into three or more bodies. We now consider the possibility that one of the two bodies, α_1 or α_2 , of the disintegration pair α may have sufficient excitation energy to disintegrate in one or more different ways into additional pairs of bodies, resulting in a three-body decay for the compound system. The modifications of some of the formulas of previous sections to include this possibility are given below, and the validity and usefulness of the results are discussed.

We can best begin our discussion by reference back to the relation (V, 1.4),

$$(E_2 - E_1) \int \Psi_2^* \Psi_1 d\tau = \sum_c (V_{2c}^* D_{1c} - V_{1c} D_{2c}^*). \quad (2.1)$$

The equation is the fundamental one in **R**-matrix theory. The term on the right results from using Green's theorem to transform the integral on the left over the internal region into an integral over the surface S . In the case when only two-body breakups are involved, this surface S is defined to be drawn far enough out in configuration space so that it is equivalent to a sum over nonoverlapping channel surfaces S_c —hence the appearance of a sum over c on the right-hand side. A further consequence of the definition of S is that, from (III, 2.1), all the channel surface wave functions ψ_c are orthogonal on S ,

$$\int \psi_c^* \psi_{c'} dS = 4\pi a_c^2 \delta_{cc'}. \quad (2.2)$$

The essential complication introduced by the presence of three-body decay is that, in general, one cannot define corresponding channel surfaces S_c . According to definition, S_c is equivalent to the spherical surface $r_c = a_c$ which is such that there is no polarizing interaction between the two members of c when separated by a distance $> a_c$. As long as both members of c are

bound, a_c can be assigned some finite value. However if one (or both) of the members is unbound (i.e., its wave function spreads over all space), then no finite value can, in general, be assigned to the a_c . It is clear from these remarks that the theory can be applied to three body breakup only when the unbound member has a wave function that is small in all the unbound channels so that a_c may be chosen at a finite value approximately in accordance with its usual definition. This is equivalent to demanding that the unbound member be long lived so that the three-body breakup has the character of two successive two-body breakups. Of course, if the a_c are chosen large enough, these conditions will be satisfied by any type of three-body decay, but the theory is no longer useful. Given such approximate channel surfaces for three-body decay, the only modification of the **R**-matrix theory that are needed to cope with three-body decay are those implied by the occurrence of continuum wave functions. These must be chosen to satisfy the orthogonality condition (2.2), and, once this is done, the three-body decay processes may be included in the sum in (2.1) and the **R**-matrix theory can be carried through in the usual way.

The only matter to be settled is the enumeration of a complete set of breakup states for three-body decay to include in the sum of (2.1). For a given type of three-body decay consisting, for definiteness, of a bound system α_1 and an unbound system α_2 , we first label states by the kinetic energy T_1 of α_1 . Fixing T_1 fixes the amount of energy E_2 left with the unbound system α_2 . For a given E_2 and spin I_2 there will be several linearly independent states of α_2 , the precise number being determined by the number of open channels of α_2 (counting channel spins and l values as well as different breakup modes). We can conveniently take these states to be those with *outgoing* waves in single channels [i.e., the time-reversed versions of (VI, 1.3)]. Labeling open channels by r ,

$$\psi_{E_2 r} = \hbar^{-\frac{1}{2}} [\Theta_r(E_2) - \sum_{r'} U_{rr'}^*(E_2) \mathcal{G}_{r'}(E_2)], \quad (2.3)$$

where Θ and \mathcal{G} are defined in Sec. III, 2. We have inserted the factor $\hbar^{-\frac{1}{2}}$ so that the $\psi_{E_2 r}$ satisfy the usual continuum normalization condition:

$$\int_{E_2-\Delta}^{E_2+\Delta} dE_2 \int_{\infty} \psi_{E_2 r}^* \psi_{E_2' r'} d\tau = \begin{cases} \delta_{rr'} & \text{if } |E_2 - E_2'| < \Delta \\ 0 & \text{otherwise,} \end{cases} \quad (2.4)$$

where the volume integral is over all space and Δ is any small number. [Equation (2.4) follows straightforwardly on inserting (2.3) on the left-hand side and integrating over all space, then over energy.] Put otherwise (2.4) reads

$$\int_{\infty} \psi_{E_2 r}^* \psi_{E_2' r'} d\tau = \delta_{rr'} \delta(E_2 - E_2') \quad (2.5)$$

from the definition of the δ function. This implies that the orthogonality condition (2.2) may be extended to three-body decay,

$$\int \psi_{c(E_2 r)}^* \psi_{c'(E_2' r')} dS = 4\pi a_c^2 \delta_{cc'} \delta_{rr'} \delta(E_2 - E_2'), \quad (2.6)$$

where c (and likewise c') specifies the composition of the subsystems α_1 and α_2 along with their spins and the channel spin s and the relative angular momentum l , but not the energy E_2 or the state r of α_2 .

We may now carry through the **R**-matrix theory with the simple modification that \sum_c is everywhere replaced by $\sum_c \sum_{r'} \int dE_2$, the first replacement being on the right-hand side of (2.1). The collision matrix elements $U_{c'(E_2' r'), c}$ for unbound channels c are defined such that the total asymptotic wave functions have the form

$$\Psi_c \sim \mathcal{G}_c - \sum_{c'} U_{c'c} \Theta_{c'} - \sum_{c' r'} \int U_{c'(E_2' r'), c} \Theta_{c'} dE_2',$$

where the first sum is over two-body channels. This definition gives $U_{c'(E_2' r'), c}$ the dimension (energy) $^{-\frac{1}{2}}$ and leads to the following cross section for the initial two-body breakup:

$$\frac{d\sigma_{\alpha, \alpha'(E_2' r')}}{dE_2'} = \frac{\pi}{k_{\alpha}^2} \sum_{J s l s' l'} g_J |U_{c'(E_2' r'), c^J}|^2. \quad (2.7)$$

The cross section for producing three definite end products, irrespective of their kinetic energies follows on integrating over E_2' and summing over all r' consistent with the three products, (i.e., over channel spins and l values).

In the particular case of the one-level formula, (2.7) becomes

$$d\sigma_{\alpha, \alpha'(E_2' r')} = \frac{\pi}{k_{\alpha}^2} \sum_{s l s' l'} g_J \frac{\Gamma_{\lambda c} \Gamma_{\lambda c'(E_2' r')}}{(E_{\lambda} + \Delta_{\lambda} - E)^2 + \frac{1}{4} \Gamma_{\lambda}^2} dE_2', \quad (2.8)$$

where the quantity $\Gamma_{\lambda c'(E_2' r')}$ is dimensionless and where the total width and shift in the denominator include integrations of $\Gamma_{\lambda c'(E_2' r')}$ over dE_2' besides the usual summations over discrete c' .

The reduced widths $\gamma_{\lambda c'(E_2' r')}^2$ for three-body decay are unknown functions of E_2' so that (2.8) is generally not as useful as the one-level formula for two-body decay. However, if the continuum systems $c'r'$ can also be described over a range of the energies E_2' by a one-level formula, the following useful approximation can be justified:

$$\gamma_{\lambda c'(E_2' r')} \approx \gamma_{\lambda c'(\lambda')}^2 \int_{\tau_{\alpha_2}} |\psi_{c'(E_2' r')}|^2 d\tau_{\alpha_2}, \quad (2.9)$$

in which the constant reduced width factor $\gamma_{\lambda c'(\lambda')}^2$ is independent of r' and E_2' but depends upon α_2' and the energy level λ' of the system α_2' . The basis for this factorization is (Sec. XII, 2), that in the one-level approximation (even with a constant \mathbf{R}^0 term included) the shape of the internal wave function for a particular α_2' is independent of both r' and E_2' ; the $\gamma_{\lambda c(E_2 r')}^2$ is expected to be proportional to the absolute square of this wave function in the internal region $\tau_{\alpha_2'}$ of the system α_2' and thus also to the integral over $\tau_{\alpha_2'}$ of the absolute square of this wave function. For the probability integral in (2.9) we can apply (XII, 2.1) obtaining

$$\gamma_{\lambda c'(E_2 r')}^2 \approx \frac{1}{2\pi} \gamma_{\lambda c'(\lambda')}^2 \frac{\Gamma_{\lambda' r'}}{(E_{\lambda'} + \Delta_{\lambda'} - E)^2 + \frac{1}{4}\Gamma_{\lambda'}^2}, \quad (2.10)$$

in which a factor \hbar^{-1} has been included as indicated by (2.3) for the energy normalization. If $\Gamma_{\lambda' r'}$ and $\Delta_{\lambda'}$ do not vary with E_2' over the resonance λ' (we could consider $\Delta_{\lambda'}$ as a linear function of E_2'), Eq. (2.10) may be integrated over the λ' resonance with the result that

$$\int_{\lambda'} \gamma_{\lambda c'(E_2 r')}^2 dE_2' \approx \gamma_{\lambda c'(\lambda')}^2 \frac{\Gamma_{\lambda' r'}}{\Gamma_{\lambda'}}. \quad (2.11)$$

Finally, we may sum (2.11) over the various possible r' with the result that

$$\sum_{r'} \int_{\lambda'} \gamma_{\lambda c'(E_2 r')}^2 dE_2' \approx \gamma_{\lambda c'(\lambda')}^2. \quad (2.12)$$

Thus we have obtained the result that the ordinary one-level formula

$$\sigma_{\alpha, \alpha'(\lambda')} = \frac{\pi}{k_{\alpha}^2} \sum_{s s' l' l'} g_J \frac{\Gamma_{\lambda c} \Gamma_{\lambda c'(\lambda')}}{(E_{\lambda} + \Delta_{\lambda} - E)^2 + \frac{1}{4}\Gamma_{\lambda}^2} \quad (2.13)$$

is valid for the cross section for the production of a given continuum state λ' of α_2' provided that this state may be described as an isolated level. From (2.11) it follows that the cross section for producing a definite trio of end products is obtained by multiplying (2.13) by the ratio $(\sum_{r'} \Gamma_{\lambda' r'} / \Gamma_{\lambda'})$ where the sum over r' is over all spins consistent with the three products.

These considerations provide some background for the usual treatments of three-body disintegration given in the literature.²⁴ Almost all of these are based on assumption of successive two-body decays with the consequence that the cross section for definite end products becomes a product of two parts in the manner just described. Although we have only discussed the case of isolated levels, these considerations can be extended to the case where the (first) compound states are overlapping by using the considerations of the statistical picture described in Sec. XI, 3. In applications, this type of treatment based on successive decay has been successful not only for three-body disintegra-

tions, but for reactions giving as many as seven simultaneous end products.⁹⁹

3. Inclusion of Photon Channels

The derivation of the theory as given in previous sections certainly does not apply to the electromagnetic field particles or "photons" with their special properties. These properties include the possibilities of production and annihilation and the nonlocalizability of photons in space which prevents their description by orthodox spatial wave functions. We can extend the heavy particle theory to include photon processes, by exploiting the fact that the coupling of nucleons to the electromagnetic field is relatively small, adopting the usual perturbation approach to electromagnetic phenomena and considering processes in which at most one photon, virtual or real, exists in the system at any given time. Because of the smallness of the coupling, the probability of two photons being simultaneously present is much less than that for one photon and so the "one-photon" theory should be quite adequate except in those anomalous situations in which one-photon transitions are forbidden by selection rules. (For example, transitions between two states of spin zero.)

Although the treatment of photons is different from the treatment of heavy particles, the qualitative role played by photons is similar in nuclear reactions to that of heavy particles. This belief is manifested by the fact that photon processes are usually included in theories of nuclear reactions by *analogy with* particle processes.²⁴ For example, the dispersion formula for reactions through a compound nucleus state is extended to photons by merely including a damping width for photon processes in the total width. This is made plausible by the Bohr picture of the compound nucleus which decays by competition through the various channels in a manner independent of the mode of formation. On this picture, photon emission is just one particular competing mode of decay so it is expected to damp the formation of the compound state just like any other mode.

For most nuclear reactions, the widths for photon processes are so small compared with the particle widths that their damping effect is quite negligible. The well-known exception to this rule is provided by thermal neutron reactions where photon widths may be even a thousand times larger than the particle widths and so damping must be taken into account.

a. One-Photon Theory without Damping

We denote wave functions with and without photon-particle coupling by Φ and Ψ , respectively, so that

$$H\Phi = E\Phi, \text{ where } H = (H_{\text{part}} + H_{\text{phot}}) + (H' + H'^*), \quad (3.1)$$

$$\bar{H}\Psi = E\Psi, \text{ where } \bar{H} = H_{\text{part}} + H_{\text{phot}}, \quad (3.2)$$

⁹⁹ J. D. Jackson, Can. J. Phys. 34, 767 (1956).

where H' creates single photons and H'^* absorbs them.

Consider the state Φ_e corresponding to unit incoming flux in the channel $c=e$. Allowing for at most one photon to be present we may write this state as

$$\Phi_e = \Phi_e^{(0)} + \sum_p A_p \Psi_p^{(1)}. \quad (3.3)$$

Superscripts signify the number of photons present. $\Psi_p^{(1)}$ has the form of a product of a nuclear wave function and a state vector of a single photon, p , which, for definiteness, we assume to be in the $(k; \mathcal{L}\mathfrak{M}\pi)$ representation (k =photon wave number, \mathcal{L} =angular momentum, \mathfrak{M} =component, π =parity) and normalized to one photon in a large volume V .

Substituting (3.3) in (3.1), and multiplying through by $\Phi_e^{(0)*}$ and integrating over all coordinates

$$\int \Phi_e^{(0)*} (\bar{H} - E) \Phi_e^{(0)} + \sum_p A_p \int \Phi_e^{(0)*} H' \Psi_p^{(1)} = 0. \quad (3.4)$$

Multiplying by $\Psi_p^{(1)*}$ and integrating gives

$$(E_p - E) A_p + \int \Psi_p^{(1)*} H' \Phi_e^{(0)} = 0. \quad (3.5)$$

Utilizing the well-known Dirac procedure,¹⁰⁰ we assert that the amplitude U_{ep} of the actual outgoing photons at large distances is

$$U_{ep} = \left\{ \left(\frac{2\pi\rho}{\hbar} \right)^{\frac{1}{2}} (E_p - E) A_p \right\}_{E_p = E} \\ = \left\{ - \left(\frac{2\pi\rho}{\hbar} \right)^{\frac{1}{2}} \int \Psi_p^{(1)*} H' \Phi_e^{(0)} \right\}_{E_p = E}, \quad (3.6)$$

where ρ is the energy density of photon states and $E = E_p$ expresses the energy balance for real photons. (Notice that $\rho^{\frac{1}{2}}$ contains a factor $V^{\frac{1}{2}}$ which compensates the $V^{-\frac{1}{2}}$ in the photon state vector.) The amplitudes U_{ep} defined by (3.6) are elements of the collision matrix appropriate for reactions involving photon states p . Cross sections are related to collision matrix elements in the same way as for particles (Sec. VIII). The only difference is that, for the differential cross sections, the photon wave $(\mathcal{L}\mathfrak{M}\pi)$ is not to be associated with the angular function $Y_{\mathfrak{M}}^{(\mathcal{L})}(\Omega)$ but with the special photon vector function $\mathbf{X}_{\mathfrak{M}}^{(\mathcal{L})}(\Omega)$ that is defined in (3.40) below.

The discussion may now be specialized to the case of the first-order theory without damping by replacing $\Phi_e^{(0)}$ in (3.6) by $\Psi_e^{(0)}$, the nuclear wave function in the absence of photon coupling. From (IX, 1.31), this wave

function has the form in the internal region,

$$\Psi_e^{(0)} = -i\hbar^{\frac{1}{2}} \Omega_e \sum_{\lambda\mu} (A_{\lambda\mu} \Gamma_{\mu e}^{\frac{1}{2}}) X_{\lambda} \quad (3.7)$$

and corresponds to the collision matrix for particle channels,

$$U_{ec} = i\Omega_e \Omega_c \sum_{\lambda\mu} A_{\lambda\mu} \Gamma_{\mu e}^{\frac{1}{2}} \Gamma_{\lambda c}^{\frac{1}{2}}. \quad (3.8)$$

Thus, assuming that the internal region is large enough to include all significant contributions to the matrix element in (3.6), we may assert that

$$U_{ep} = i\Omega_e \Omega_p \sum_{\lambda\mu} A_{\lambda\mu} \Gamma_{\mu e}^{\frac{1}{2}} \Gamma_{\lambda p}^{\frac{1}{2}}, \quad (3.9)$$

where the product $(\Omega_p \Gamma_{\lambda p}^{\frac{1}{2}})$ of the photon phase factor and width amplitude is defined as

$$\Omega_p \Gamma_{\lambda p}^{\frac{1}{2}} = \left\{ (2\pi\rho)^{\frac{1}{2}} \int \Psi_p^{(1)*} H' X_{\lambda} \right\}_{E_p = E}. \quad (3.10)$$

In fact, since the matrix element in (3.10) must be real we may set $\Omega_p = 1$ without loss in generality. Comparing (3.9) and (3.8), the expressions for collision matrix elements for particles and photons have almost identical forms, suggesting that, once photon processes are assigned widths in accordance with (3.10), they may be treated on the same footing as particles. Unfortunately this is not quite correct because the $A_{\lambda\mu}$ occurring in (3.8) and (3.9) are for the unperturbed problem, so that they involve no reference to photons and are therefore not symmetrical in photon and particle widths. As an illustration, let us consider a particle-photon cross section proceeding through an isolated resonance. In this case, from (IX, 1.21), all $A_{\lambda\mu}$ are zero except

$$A_{\lambda\lambda} = [E_{\lambda} + \Delta_{\lambda} - E - (i/2)\Gamma_{\lambda}] \quad (3.11)$$

so, from (3.8) or (3.9), for both particle and photon outgoing channels,

$$U_{cc} = \frac{i\Omega_e \Omega_c \Gamma_{\lambda e}^{\frac{1}{2}} \Gamma_{\lambda c}^{\frac{1}{2}}}{E_{\lambda} + \Delta_{\lambda} - E - \frac{i}{2}\Gamma_{\lambda}} \quad (3.12)$$

and, from (VIII, 3.3),

$$\sigma_{ec} = \frac{\pi}{k_e^2} \sum_{st} g^J \frac{\Gamma_{\lambda e} \Gamma_{\lambda c}}{(E_{\lambda} + \Delta_{\lambda} - E)^2 + \frac{1}{4}\Gamma_{\lambda}^2}. \quad (3.13)$$

The level shift Δ_{λ} and width Γ_{λ} in these three equations only contain contributions from the *particle* channels. In general, since photon widths are \ll particle widths, this lack of symmetry in the theory between photons and particles is of no practical importance. However there is the special case of slow neutron resonance capture where often $\sum_p \Gamma_{\lambda p} \gg \Gamma_{\lambda n}$. In such cases, Eq.

¹⁰⁰ P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, New York, 1947), third edition, pp. 188-204.

(3.13) is known to be quite wrong if one does not include the photon contribution in the total width and shift in the denominator. In other words, it is vital to include the damping effect of photon production. This means that for insertion in (3.6) one must find a better approximation to $\Phi_e^{(0)}$ than that given by $\Psi_e^{(0)}$ of (3.7). If photons are to occupy a similar role to particles, this better approximation must be such that the coefficients $A_{\lambda\mu}$ of (3.7) are modified to include photon widths symmetrically with the particle widths. There are two arguments to show that this modification ought to be made.

b. Effects of Damping

(i) *Plausibility Derivation Based on the Unitarity Requirement.*—The quantities $A_{\lambda\mu}$ in the U_{ep} of (3.9) are, at present, to be evaluated for the unperturbed system, i.e., ignoring the presence of photon channels. This means the U is not unitary, as is evident in the special case of an isolated level (3.12). In this special case, imposition of the condition of unitarity implies that the photon width should be included in the total width in the denominator of (3.12), which is equivalent to its being included symmetrically in $A_{\lambda\lambda}$. We will now show that, in the general case the imaginary part of the scalar product defining the ξ -matrix components, which components are involved in the determination of the A -matrix components of (3.9), must include an additional contribution from the photon channels for each component $\lambda\mu$, each of the two factors of this contribution being specified by (3.10). (The contribution to the real part of the ξ components from the additional channel remains unspecified by this argument.)

To show in detail the condition for unitarity, one evaluates the components of the matrix product

$$UU^* = \Omega \left[1 + 2i \sum_{\lambda\mu} (\mathbf{P}^{\frac{1}{2}} \gamma_\lambda \times \mathbf{P}^{\frac{1}{2}} \gamma_\mu) A_{\lambda\mu} \right] \times \left[1 - 2i \sum_{\lambda\mu} (\mathbf{P}^{\frac{1}{2}} \gamma_\lambda \times \mathbf{P}^{\frac{1}{2}} \gamma_\mu) A_{\lambda\mu}^* \right] \Omega^*. \quad (3.14)$$

This product reduces to

$$UU^* = \Omega \left\{ 1 + 2i \mathbf{P}^{\frac{1}{2}} \sum_{\lambda\mu} (\gamma_\lambda \times \gamma_\mu) [A_{\lambda\mu} - A_{\lambda\mu}^* - \sum_{\sigma\nu} A_{\lambda\sigma} (\xi_{\sigma\nu} - \xi_{\sigma\nu}^*) A_{\nu\mu}^*] \mathbf{P}^{\frac{1}{2}} \right\} \Omega^*, \quad (3.15)$$

where

$$\xi_{\sigma\nu} = (\gamma_\sigma, \mathbf{L} \gamma_\nu),$$

the scalar product including contributions from all channels. One notes that only the imaginary part of the ξ components are involved in (3.15). In order that U be unitary, it is necessary that

$$\mathbf{A} - \mathbf{A}^* - \mathbf{A}(\xi - \xi^*)\mathbf{A}^* = 0. \quad (3.16)$$

If $\mathbf{A} = (\mathbf{e} - \mathbf{E} - \xi)^{-1}$, with the ξ components as in (3.15), then $\mathbf{A}(\mathbf{e} - \mathbf{E} - \xi)\mathbf{A}^* = \mathbf{A}^*$ and $\mathbf{A}(\mathbf{e} - \mathbf{E} - \xi^*)\mathbf{A}^* = \mathbf{A}$ and

by subtraction

$$\mathbf{A} - \mathbf{A}^* = \mathbf{A}\xi\mathbf{A}^* - \mathbf{A}\xi^*\mathbf{A}^*,$$

thus satisfying (3.16).

(ii) *Derivation from Proper Damping Theory.*—If (3.3) is substituted in (3.1), and the resulting equation is multiplied by X_λ^* and integrated, we have, instead of (3.4):

$$\int X_\lambda^* (\bar{H} - E) \Phi_e^{(0)} + \sum_p A_p \int X_\lambda^* H'^* \Psi_p^{(1)} = 0. \quad (3.17)$$

Let us now assume the expansion

$$\Phi_e^{(0)} = \sum_\mu A_\mu X_\mu; \quad (3.18)$$

then (3.5) becomes

$$(E_p - E) A_p + \sum_\mu A_\mu \int \Psi_p^{(1)*} H' X_\mu = 0. \quad (3.19)$$

Substituting A_p from (3.19) into (3.17) and remembering the condition of outgoing waves:

$$\int X_\lambda^* (\bar{H} - E) \Phi_e^{(0)} = \sum_\mu \sum_p \left\{ -\frac{1}{E - E_p} + i\pi\delta(E - E_p) \right\} \times \int (\Psi_p^{(1)*} H' X_\mu) \left(\int X_\lambda^* H'^* \Psi_p^{(1)} \right). \quad (3.20)$$

On extracting from the summation over p the summation over E_p and replacing it by an integration, we have

$$\int X_\lambda^* (\bar{H} - E) \Phi_e^{(0)} = -\sum_{\mu p} A_\mu \left(\Delta_{\lambda\mu p} - \frac{i}{2} \Gamma_{\lambda\mu p} \right), \quad (3.21)$$

where the photon shift $\Delta_{\lambda\mu p}$ and width $\Gamma_{\lambda\mu p}$ are defined by

$$\Delta_{\lambda\mu p} = \int \rho_{E_p} dE_p$$

$$\times \frac{\left(\int \Psi_p^{(1)*} H' X_\mu \right) \left(\int X_\lambda^* H'^* \Psi_p^{(1)} \right)}{E - E_p}, \quad (3.22)$$

$$\Gamma_{\lambda\mu p} = \Gamma_{\lambda p}^{\frac{1}{2}} \Gamma_{\mu p}^{\frac{1}{2}}, \quad (3.23)$$

where $\Gamma_{\lambda p}^{\frac{1}{2}}$ is defined by (3.10).

Equation (3.21) is the basic one of the present considerations. If $\Phi_e^{(0)}$ were replaced by $\Psi_e^{(0)}$, the unperturbed state, the left-hand side would vanish as demanded by the conservation theorem. It follows that the right-hand side of (3.21) accounts for the leaking of probability from the nucleon configuration space due to photon processes. Applying Green's theorem to the

left-hand side of (3.21) gives

$$\begin{aligned} \int X_\lambda^*(\bar{H}-E)\Phi_e^{(0)} &= (E_\lambda-E)A_\lambda + \sum_c (V_e D_{\lambda c} - D_e V_{\lambda c}) \\ &= \sum_\mu A_\mu \left\{ (E_\lambda-E)\delta_{\lambda\mu} + \sum_{c(\neq e)} \left(\Delta_{\lambda\mu c} - \frac{i}{2}\Gamma_{\lambda\mu c} \right) \right\} \\ &\quad + (V_e D_{\lambda e} - D_e V_{\lambda e}). \end{aligned} \quad (3.24)$$

Combining (3.21) and (3.24), the presence of photon channels manifests itself in just the same way as all other outgoing channels. We have

$$\begin{aligned} \sum_\mu A_\mu \left\{ (E_\lambda-E)\delta_{\lambda\mu} + \sum_{c(\neq e)} \left(\Delta_{\lambda\mu c} - \frac{i}{2}\Gamma_{\lambda\mu c} \right) \right\} \\ = V_e D_{\lambda e} - D_e V_{\lambda e}, \end{aligned} \quad (3.25)$$

where the sum over c now includes photon channels. Evaluating the wave function at the entrance channel surface from inside and outside gives the relations

$$\sum_\mu A_\mu \gamma_{\mu e} = V_e, \quad (3.26a)$$

$$D_e - V_e L_e = i(2\hbar)^{\frac{1}{2}} P_e^{\frac{1}{2}} \Omega_e, \quad (3.26b)$$

so the right-hand side of (3.25) is

$$\begin{aligned} V_e D_{\lambda e} - D_e V_{\lambda e} &= -\sum_\mu A_\mu \left(\Delta_{\lambda\mu e} - \frac{i}{2}\Gamma_{\lambda\mu e} \right) \\ &\quad + i(2\hbar)^{\frac{1}{2}} P_e^{\frac{1}{2}} \Omega_e \gamma_{\lambda e}. \end{aligned} \quad (3.27)$$

Defining the matrix $\mathbf{A} = [A_{\lambda\mu}]$ in the usual way,

$$\mathbf{A} = \mathbf{B}^{-1},$$

$$B_{\lambda\mu} = (E_\lambda - E)\delta_{\lambda\mu} + \sum_c \left(\Delta_{\lambda\mu c} - \frac{i}{2}\Gamma_{\lambda\mu c} \right), \quad (3.28)$$

where the sum over c includes photon channels, we may extract from (3.25) and (3.27) the solution for the expansion coefficients A_λ ,

$$A_\lambda = -i(2\hbar)^{\frac{1}{2}} P_e^{\frac{1}{2}} \Omega_e \sum_\mu A_{\lambda\mu} \gamma_{\mu e}. \quad (3.29)$$

Substituting (3.29) in (3.18) gives the expansion (3.7). Continuing the solution (3.7) through the channel surfaces to infinity gives the collision matrix elements (3.8) for particle channels. Substituting (3.7) in (3.6) gives (3.9) for the collision matrix elements for photon channels. This completes the demonstration of the fact that photon channels and particle channels occur completely symmetrically in the collision matrix when damping is taken into account.

c. Transitions in the External Region

If the particle channel surfaces are chosen in the normal way the external region of configuration space may contribute to the electromagnetic transition matrix elements.¹⁰¹ This possibility may be removed by enlarging the internal region, or may be taken into account by the method we now discuss. Usually the contribution of the external region will be relatively very small. For instance when the bombarding energy is on a strong resonance, the external contribution can be ignored. The situation in which the external contribution may be appreciable is a combination of rather special circumstances that seems to be occasionally realized in light nuclei.

The wave function of the final nuclear state, $X_{pJ'M'}$, say, takes the following form in the channel $c = \alpha sl'$:

$$\begin{aligned} X_{pJ'M'} &\rightarrow \left(\frac{2}{a_c} \right)^{\frac{1}{2}} \theta_{\lambda c} N_p \frac{E_c(r_c)}{r_c} \\ &\times \left\{ \sum_{\nu'+m'=M'} (sl'\nu'm'|J'M') \psi_{\alpha s\nu'}(i^\nu Y_{m'}^{(\nu')}) \right\}. \end{aligned} \quad (3.30)$$

Here it is assumed that X_p is a bound state, normalized over all space and the normalizing factor N_p is defined by

$$N_p^{-1} = 1 + \sum_c \frac{2\theta_{pc}^2}{a_c} \int_{a_c}^{\infty} E_c^2(r_c) dr_c. \quad (3.31)$$

θ_{pc} is the dimensionless reduced width amplitude and $E_c(r_c)$ is the external radial wave function fixed by the condition: $E_c(a_c) = 1$ so that, in terms of the O function of Sec. III:

$$E_c(r_c) = O_c(r_c)/O_c(a_c). \quad (3.32)$$

Making the reasonable assumption that external transitions are only significant in channels α in which there are incident waves, the external contribution to the collision matrix U_{ep} of (3.6) is

$$\begin{aligned} U_{ep}^J(\text{ext}) &= N_p \left(\frac{2\pi\rho}{\hbar} \right)^{\frac{1}{2}} \sum_{\substack{\nu+m=M \\ \nu'+m'=M'}} (sl\nu m|JM)(sl'\nu'm'|J'M') \\ &\times \theta_{pc} \left(\frac{2}{a_e v_e} \right)^{\frac{1}{2}} \left(i^\nu Y_{m'}^{(\nu)} \frac{E_c}{r_c} \left| H_{3\pi}(\mathcal{E}\pi) \right. \right. \\ &\quad \left. \left. \times i^{\nu'} Y_{m'}^{(\nu')} \frac{(I_e - U_{ee} O_e)}{r_e} \right) \right), \end{aligned} \quad (3.33)$$

where $e = \alpha sl$, the entrance channel. This term must be added to N_p times the internal contribution (3.9) to obtain the total collision matrix element. The operator $H_{3\pi}(\mathcal{E}\pi)$ in (3.33) is the part of H' that remains after

¹⁰¹ R. G. Thomas, Phys. Rev. 84, 1061 (1951).

the operation of creation of the photon ($k; \mathcal{L}\mathcal{M}\mathcal{N}\pi$) specified by p has been made (it only depends on the nucleon coordinates). As is evident from (3.33), the channel transitions are assumed to be electric multipole in nature. (In fact, electric dipole transitions are the only ones likely to ever have appreciable external contributions.) Using some Racah algebra (3.33) can be developed a little to give

$$U_{ep}^J(\text{ext}) = (-)^{2s} \left(\frac{4\pi\rho}{\hbar a_e v_e} \right)^{\frac{1}{2}} (J\mathcal{L}M\mathcal{N}\pi | J'M') \\ \times \sum_{l'} \frac{U(\mathcal{L}lJ's, l'J)}{(2l'+1)^{\frac{1}{2}}} \theta_{p_e} \\ \times \left(i^{l'} Y^{(l')} \frac{E_c}{r_e} \parallel \mathbf{H}^{(\mathcal{L}\pi)} \parallel i^{l'} Y^{(l)} \frac{I_e - U_{ee} O_e}{r_e} \right), \quad (3.34)$$

where the reduced matrix element has been introduced.

d. Widths for Photons

The procedure for determining the expressions for the widths of the photon channels involves the multipole expansions which have been described by Blatt and Weisskopf,²⁴ Rose,¹⁰² MacDonald,¹⁰³ and others.

In field-free space, one can consider incoming and outgoing vector potentials of the form

$$\mathfrak{S}_{lm1t} = (k/2\pi\hbar)^{-\frac{1}{2}} r^{-1} I_l(i^{l'} Y_m^{(l)}) \mathbf{e}_t, \\ \mathfrak{D}_{lm1t} = (k/2\pi\hbar)^{-\frac{1}{2}} r^{-1} O_l(i^{l'} Y_m^{(l)}) \mathbf{e}_t, \quad (3.35)$$

which are similar to the functions (III, 2.25) for particle wave functions; they are normalized to unit flux and the unit vectors \mathbf{e}_t are

$$\mathbf{e}_1 = -2^{-\frac{1}{2}} (\mathbf{e}_x + i\mathbf{e}_y), \\ \mathbf{e}_0 = \mathbf{e}_z \\ \mathbf{e}_{-1} = 2^{-\frac{1}{2}} (\mathbf{e}_x - i\mathbf{e}_y)$$

with the properties

$$(\mathbf{e}_t^* \cdot \mathbf{e}_{t'}) = \delta_{tt'}.$$

The vector potential multipole fields having a definite total angular momentum \mathcal{L} , component \mathcal{M} , are linear combinations of the form

$$\begin{cases} \mathfrak{S}_{\mathcal{L}\mathcal{M}\mathcal{N}\pi} \\ \mathfrak{D}_{\mathcal{L}\mathcal{M}\mathcal{N}\pi} \end{cases} = -2^{\frac{1}{2}} \sum_{mt} (L1mt | \mathcal{L}\mathcal{M}\mathcal{N}) (L1-11 | L0) \\ \times \begin{cases} \mathfrak{S}_{\mathcal{L}\mathcal{M}\mathcal{N}1t} \\ \mathfrak{D}_{\mathcal{L}\mathcal{M}\mathcal{N}1t} \end{cases} \quad (3.36a)$$

¹⁰² M. E. Rose, *Multipole Fields* (John Wiley & Sons, Inc., New York, 1955).

¹⁰³ W. M. MacDonald, *Phys. Rev.* **98**, 60 (1955).

for the magnetic multipole fields \mathfrak{C} , and

$$\begin{cases} \mathfrak{S}_{\mathcal{L}\mathcal{M}\mathcal{N}\pi} \\ \mathfrak{D}_{\mathcal{L}\mathcal{M}\mathcal{N}\pi} \end{cases} = -2^{\frac{1}{2}} \sum_{mt} \sum_{l(\neq \mathcal{L})} (l1mt | \mathcal{L}\mathcal{M}\mathcal{N}) \\ \times (L1-11 | l0) \times \begin{cases} \mathfrak{S}_{lm1t} \\ \mathfrak{D}_{lm1t} \end{cases} \quad (3.36b)$$

for the electric multipole fields \mathcal{E} ; these fields are also normalized to unit flux. The respective incoming and outgoing multipole fields are related according to

$$k^{-1} \left(\nabla \times \begin{cases} \mathfrak{S}_{\mathcal{L}\mathcal{E}\mathcal{N}\pi} \\ \mathfrak{D}_{\mathcal{L}\mathcal{E}\mathcal{N}\pi} \end{cases} \right) = \begin{cases} \mathfrak{S}_{\mathcal{L}\mathcal{E}\mathcal{N}\pi} \\ \mathfrak{D}_{\mathcal{L}\mathcal{E}\mathcal{N}\pi} \end{cases}. \quad (3.37)$$

We can deal with the fields (3.36) in the same way as the channel functions of Sec. III.2 in (VIII,1.5). The vector potential \mathbf{A} for a unit-flux plane wave of wave vector \mathbf{k} , polarization state ϵ , may be expanded in terms of these fields as

$$\mathbf{A}(\mathbf{k}, \epsilon) = -i\pi^{\frac{1}{2}} k^{-1} 2^{-\frac{1}{2}} \sum_{\mathcal{L}\mathcal{M}\mathcal{N}\pi} (2\mathcal{L}+1)^{\frac{1}{2}} \\ \times [(\mathfrak{S}_{\mathcal{L}\mathcal{E}\mathcal{N}\pi} - \mathfrak{D}_{\mathcal{L}\mathcal{E}\mathcal{N}\pi}) + \epsilon(\mathfrak{S}_{\mathcal{L}\mathcal{C}\mathcal{E}\mathcal{N}\pi} - \mathfrak{D}_{\mathcal{L}\mathcal{C}\mathcal{E}\mathcal{N}\pi})] D_{\mathcal{M}\pi}^{(\mathcal{L})}, \quad (3.38)$$

where the quantities $D_{\mathcal{M}\pi}^{(\mathcal{L})}$ are the components of the irreducible matrices representing the rotation from the axis of quantization to the direction of \mathbf{k} .

A semiclassical procedure for determining the collision matrix components U_{ep} of the above outgoing multipole fields is given in Appendix B, Sec. 4, of Blatt and Weisskopf.²⁴ In terms of the nucleon current sources \mathbf{j}_{ep} and magnetization sources \mathbf{M}_{ep} , one finds from the Maxwell equations that, for magnetic and electric \mathcal{L} -pole radiation, respectively,

$$U_{ep} = -(2\pi\hbar k)^{-\frac{1}{2}} (4\pi/c) \\ \times \int r^{-1} F_{\mathcal{L}}(i^{\mathcal{L}} \mathbf{X}_{\mathcal{M}\pi}^{(\mathcal{L})})^* \cdot (\mathbf{j}_{ep} + c\nabla \times \mathbf{M}_{ep}) dV, \quad (3.39a)$$

$$U_{ep} = -(2\pi\hbar k)^{-\frac{1}{2}} (4\pi/c) \\ \times \int r^{-1} F_{\mathcal{L}}(i^{\mathcal{L}} \mathbf{X}_{\mathcal{M}\pi}^{(\mathcal{L})}) \\ \cdot (k^{-1} \nabla \times \mathbf{j}_{ep} + ck\mathbf{M}_{ep}) dV, \quad (3.39b)$$

where the $\mathbf{X}_{\mathcal{M}\pi}^{(\mathcal{L})}$ are the vector angular functions for photons defined by Blatt and Weisskopf:

$$\mathbf{X}_{\mathcal{M}\pi}^{(\mathcal{L})} = [\mathcal{L}(\mathcal{L}+1)]^{-\frac{1}{2}} \mathbf{L} Y_{\mathcal{M}\pi}^{(\mathcal{L})}, \quad (3.40)$$

These functions are normalized:

$$\int [\mathbf{X}_{\mathcal{M}\pi}^{(\mathcal{L})}]^* \cdot [\mathbf{X}_{\mathcal{M}'\pi'}^{(\mathcal{L}')}] d\Omega = \delta_{\mathcal{L}\mathcal{M}\pi, \mathcal{L}'\mathcal{M}'\pi'}.$$

\mathbf{L} is the differential operator $-i\mathbf{r}\times\nabla$. These amplitudes differ from the corresponding amplitudes \mathcal{G}_ε and \mathcal{H}_ε of Blatt and Weisskopf in the normalization and the phase. In the case of a transition from an initial state Ψ_e to a final (bound) state X_p , the quantum-mechanical current and magnetization expressions are

$$\mathbf{j}_{ep} = \sum \frac{1}{2} [X_p^* (\mathbf{j}\Psi_e) + (\mathbf{j}X_p)^* \Psi_e], \quad (3.41a)$$

$$\mathbf{M}_{ep} = \sum (e\hbar/2Mc)\mu [X_p^* \boldsymbol{\sigma}\Psi_e], \quad (3.41b)$$

where \sum indicates summation over all nucleons, \mathbf{j} is the current operator for protons $-ie\hbar\nabla$, $\boldsymbol{\sigma}$ is the Pauli spin operator, and μ is magnetic moment expressed in units of the Bohr magneton.

From (3.9), (3.10), and (3.39), it follows that

$$\Gamma_{\lambda p}^{\frac{1}{2}} = \left(\frac{8\pi k}{c^2}\right)^{\frac{1}{2}} \int dV j_\varepsilon(kr) (i^\varepsilon \mathbf{X}_{\mathfrak{L}\mathfrak{M}}^{(\varepsilon)})^* \begin{cases} \mathbf{j}_{\lambda p} + c(\nabla \times \mathbf{M}_{\lambda p}) \\ k^{-1}(\nabla \times \mathbf{j}_{\lambda p}) + ck\mathbf{M}_{\lambda p} \end{cases}, \quad (3.42)$$

where we have put the free space function $j_\varepsilon(kr)$ for $r^{-1}F_\varepsilon(kr)$ and where the source expressions in $\{$ are

$$\mathbf{j}_{\lambda p} = \frac{e}{2M} [X_p^* (\mathbf{j}X_\lambda) + (\mathbf{j}X_p)^* X_\lambda] \quad (3.43a)$$

$$\mathbf{M}_{\lambda p} = \left(\frac{e\hbar}{2Mc}\right)\mu [X_p^* \boldsymbol{\sigma}X_\lambda], \quad (3.43b)$$

the upper expression in $\{$ applies to magnetic radiation and the lower to electric.

These width expressions can be put into more convenient forms for numerical evaluations by performing a number of partial integrations. One thereby obtains for the width amplitude of the photon channel p with multipolarity ε :

$$\Gamma_{\lambda p}^{\frac{1}{2}} = \frac{(8\pi)^{\frac{1}{2}} k^{\varepsilon+\frac{1}{2}} (\varepsilon+1)^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}} (2\varepsilon+1)!!} \begin{cases} \mathfrak{H}_{\varepsilon\mathfrak{M}} + \mathfrak{H}'_{\varepsilon\mathfrak{M}} \\ \mathcal{E}_{\varepsilon\mathfrak{M}} + \mathcal{E}'_{\varepsilon\mathfrak{M}} \end{cases}, \quad (3.44)$$

where

$$\mathfrak{H}_{\varepsilon\mathfrak{M}} = -2i \frac{(2\varepsilon+1)!!}{(\varepsilon+1)k^\varepsilon} \left(\frac{e\hbar}{2Mc}\right) \int dV \times \{j_\varepsilon(i^\varepsilon Y_{\mathfrak{M}}^{(\varepsilon)})^* \nabla \cdot (X_p^* \mathbf{L}X_\lambda)\} \quad (3.45a)$$

$$\mathfrak{H}'_{\varepsilon\mathfrak{M}} = -i \frac{(2\varepsilon+1)!!}{(\varepsilon+1)k^\varepsilon} \left(\frac{e\hbar}{2Mc}\right) \mu \times \int dV \left\{ \frac{\partial}{\partial r} (rj_\varepsilon)(i^\varepsilon Y_{\mathfrak{M}}^{(\varepsilon)})^* \nabla \cdot (X_p^* \boldsymbol{\sigma}X_\lambda) - j_\varepsilon(i^\varepsilon Y_{\mathfrak{M}}^{(\varepsilon)})^* (X_p^* \boldsymbol{\sigma} \cdot \mathbf{r}X_\lambda) \right\} \quad (3.45b)$$

$$\mathcal{E}_{\varepsilon\mathfrak{M}} = e \frac{(2\varepsilon+1)!!}{(\varepsilon+1)k^\varepsilon} \int dV \left\{ \frac{\partial}{\partial r} (rj_\varepsilon)(i^\varepsilon Y_{\mathfrak{M}}^{(\varepsilon)})^* (X_p^* X_\lambda) + \frac{\hbar k}{Mc} \left[\frac{1}{2} \frac{\partial}{\partial r} (rj_\varepsilon)(i^\varepsilon Y_{\mathfrak{M}}^{(\varepsilon)})^* (X_p^* X_\lambda) + j_\varepsilon(i^\varepsilon Y_{\mathfrak{M}}^{(\varepsilon)})^* X_p^* X_\lambda + j_\varepsilon(i^\varepsilon Y_{\mathfrak{M}}^{(\varepsilon)})^* X_p \left(\frac{\partial}{\partial r}\right) X_\lambda \right] \right\} \quad (3.46a)$$

$$\mathcal{E}'_{\varepsilon\mathfrak{M}} = -i \left(\frac{e\hbar}{2Mc}\right) \frac{(2\varepsilon+1)!!}{(\varepsilon+1)k^\varepsilon} \mu k \int dV \times \{j_\varepsilon(i^\varepsilon Y_{\mathfrak{M}}^{(\varepsilon)})^* \nabla \cdot (X_p^* (\mathbf{r} \times \boldsymbol{\sigma}) X_\lambda)\}. \quad (3.46b)$$

These expressions for the moments $\mathfrak{H}_{\varepsilon\mathfrak{M}}$ and $\mathcal{E}_{\varepsilon\mathfrak{M}}$ are similar to those given by MacDonald.¹⁰³ In the long wavelength approximation which assumes that $ka \ll 1$, one obtains

$$\mathfrak{H}_{\varepsilon\mathfrak{M}} = \frac{-i}{\varepsilon+1} 2\mu \left(\frac{e\hbar}{2Mc}\right) \int dV \times \{r^\varepsilon (i^\varepsilon Y_{\mathfrak{M}}^{(\varepsilon)})^* \nabla \cdot (X_p^* \mathbf{L}X_\lambda)\} \quad (3.47a)$$

$$\mathfrak{H}'_{\varepsilon\mathfrak{M}} = -i\mu \left(\frac{e\hbar}{2Mc}\right) \int dV \times \{r^\varepsilon (i^\varepsilon Y_{\mathfrak{M}}^{(\varepsilon)})^* \nabla \cdot (X_p^* \boldsymbol{\sigma}X_\lambda)\} \quad (3.47b)$$

$$\mathcal{E}_{\varepsilon\mathfrak{M}} = e \int dV \{r^\varepsilon (i^\varepsilon Y_{\mathfrak{M}}^{(\varepsilon)})^* (X_p^* X_\lambda)\} \quad (3.48a)$$

$$\mathcal{E}'_{\varepsilon\mathfrak{M}} = \frac{-i\mu k}{\varepsilon+1} \left(\frac{e\hbar}{2Mc}\right) \int dV \times \{r^\varepsilon (i^\varepsilon Y_{\mathfrak{M}}^{(\varepsilon)})^* \nabla \cdot (X_p^* (\mathbf{r} \times \boldsymbol{\sigma}) X_\lambda)\}. \quad (3.48b)$$

These expressions are similar to those derived by Blatt and Weisskopf.

The above width amplitudes $\Gamma_{\lambda p}^{\frac{1}{2}}$ and associated moment expressions actually refer to the emission of a multipole $\varepsilon\mathfrak{M}$ in a transition from an initial nuclear state $X_{\lambda JM}$ having a definite spin J and component M to a final state $X_{pJ'M'}$ having a definite spin J' and component M' . Reduced matrix elements may be introduced in the usual way:

$$\Gamma_{\lambda p(\varepsilon\mathfrak{M}J'M'JM)}^{\frac{1}{2}} = (\varepsilon J' \mathfrak{M} M' | JM) \Gamma_{\lambda p(\varepsilon J' J')}^{\frac{1}{2}}. \quad (3.49)$$

4. Alpha Decay

It is of interest to enquire to what extent we can describe alpha decay within the framework of \mathbf{R} -matrix theory. First we treat the alpha-emitting state as a resonance level that could in principle (but not in practice) be excited by bombarding a daughter product

with alpha particles. This treatment gives expected relations between widths, etc., but, being a time-independent treatment, it does not mention the quantities that are actually observed, namely the decay lifetimes. One knows that the lifetime of an alpha-emitting state (in a time-dependent description) is equal to \hbar/Γ , where Γ is the width of the state (in the time-independent description). This relation is based on the uncertainty relation between energy and time and is almost self-evident. Nevertheless it is instructive to investigate a time-dependent description in detail and to see just how the relation between width and lifetime enters, and how an actual decaying state spreads out in time and space.

a. Time-Independent Description

Following Thomas¹⁰⁴ (in whose paper further references are given) the wave function in the internal region corresponding to unit flux in the entrance channel e is, from (IX, 1.31),

$$\Psi_e = -i\hbar^2\Omega_e \sum_{\lambda\mu} A_{\lambda\mu}\Gamma_{\mu e}^{\frac{1}{2}}X_\lambda \quad (4.1)$$

and the corresponding reaction elements of the collision matrix are, from (IX, 1.32),

$$U_{ec} = i\Omega_e\Omega_c \sum_{\lambda\mu} A_{\lambda\mu}\Gamma_{\lambda e}^{\frac{1}{2}}\Gamma_{\mu c}^{\frac{1}{2}}. \quad (4.2)$$

From the discussion of the one-level approximation in (XII, 1a), we know that, if $\mathbf{R}^0\mathbf{L}^0$ is put equal to zero (where $\mathbf{R}^0 = \mathbf{R} - [(\boldsymbol{\gamma}_\lambda \times \boldsymbol{\gamma}_\lambda)/(E_\lambda - E)]$), then

$$A_{\lambda\lambda} = \left(E_\lambda + \Delta_\lambda - E - \frac{i}{2}\Gamma_\lambda \right)^{-1}, \quad (4.3)$$

and all other $A_{\lambda\mu} = 0$. In the case when the entrance channel is one of incident alphas bombarding the daughter product of an alpha-decay process, $\mathbf{R}^0\mathbf{L}^0$ may certainly be made very small by appropriate choice of boundary condition. By choosing $B = S(E_r)$, where E_r is the energy at the center of resonance, $L^0(E) = (E - E_r)(dS/dE)_{E=E_r} + iP(E)$ which is very small within the width of the resonance because of the huge channel barriers. This same choice sets $E_\lambda = E_r$, $\Delta_\lambda(E_r) = 0$ so (4.3) is

$$A_{\lambda\lambda}^{-1} = (E_\lambda - E)(1 - d\Delta_\lambda/dE) - \frac{i}{2}\Gamma_\lambda \\ = (1 - d\Delta_\lambda/dE) \left(E_\lambda - E - \frac{i}{2} \frac{\Gamma_\lambda}{1 - d\Delta_\lambda/dE} \right). \quad (4.4)$$

As discussed in Sec. XII, 3, the term Δ_λ essentially renormalizes the wave function X_λ to unity within the classical turning points of the channel barriers (instead of in the internal region). Because of the size of the

barriers, the wave function decays very sharply inside the barriers so that this difference is very small. In fact, $d\Delta_\lambda/dE$ was estimated¹⁰⁴ for alpha decay as $\lesssim 10^{-2}$. Another consequence of the size of the barrier is that $S(E_r)$ is very large so that the boundary condition parameter B is very large.

From (4.1) and the ensuing discussion, we can approximate the running wave Ψ_e by the single standing wave X_λ when the channel barriers are large. If the barriers are imagined to increase without limit, then $P \rightarrow 0$, $\Gamma_\lambda \rightarrow 0$, and the relation $\Psi_e \sim X_\lambda$ becomes exact.

This is about as far as we can go towards giving a picture of the alpha-decay process by restriction to a time-independent theory with real energies.

b. Time-Dependent Description

In the consideration of alpha-particle decay, one needs to know the relation between the decay rate and the resonance parameters. By a continuation of the solutions VI.3, 1a into the complex wave-number plane solutions with only outgoing waves occur at the poles of \mathbf{U} . All of these poles lie in the lower halves of the k planes with the exception of those on the positive imaginary axis which correspond to the bound states of the system (Sec. IV, 8). If the λ th pole in the lower-half of one of the k_c planes lies at $k_{\lambda c} = \Delta_\lambda = K_{\lambda c} - iG_{\lambda c}$, it lies in the energy plane at $E = H_\lambda = F_\lambda - \frac{1}{2}i\Gamma_\lambda$ where $F_\lambda = (\hbar^2/2M_c)(K_{\lambda c}^2 - G_{\lambda c}^2) + b_c$, $\frac{1}{2}\Gamma_\lambda = (\hbar^2/M_c)K_{\lambda c}G_{\lambda c}$, and b_c is the binding energy for channel c . At such a pole the time-dependent solutions are therefore asymptotically of the form

$$\Psi(r, t) \sim \sum_c U_c(H_\lambda) \exp i[k_{\lambda c}r - H_\lambda t \hbar^{-1}] \\ = \sum_c U_c(H_\lambda) \exp i[K_{\lambda c}r - F_\lambda t \hbar^{-1}] \\ \times \exp(-\frac{1}{2}\lambda_\lambda t) \exp(G_{\lambda c} t), \quad (4.5)$$

where c labels the various alpha-particle channels and

$$\lambda_\lambda = \Gamma_\lambda/\hbar. \quad (4.6)$$

The absolute square of such solutions exhibit the familiar exponential time decay characteristic of radioactive states; (4.6) gives then the desired relation between the decay constant λ_λ for the λ th level and the resonance width Γ_λ . This solution also exhibits the expected exponential increase with radius of the amplitude at a particular time. Thus, the second exponential of (4.5) can be rewritten $\exp(\frac{1}{2}\lambda_\lambda r_c/v_{\lambda c})$ where $v_{\lambda c} = \hbar K_{\lambda c}/M_c$ is the velocity of emission corresponding to $K_{\lambda c}$.

Although (4.5) may give accurately the asymptotic form of a decaying system with a long lifetime, it is subject to the objection of being the solution of a system with complex energy, which actually does not exist. In other words, (4.5) represents only part of an actual solution which consists of a superposition, or wave packet, of time-dependent solutions with real energies. It is possible to show that a term of the form (4.5)

¹⁰⁴ R. G. Thomas, Progr. Theoret. Phys. 12, 253 (1954).

appears in a wave packet representing the emission from a nucleus of a particle whose energy lies in the neighborhood of the real energy F_λ of one of the poles. It is proportional to the residue of the pole $U_e(H_\lambda)$ which contributes when it is permissible to displace the wave packet integration path across the pole to a path at infinity in the lower half k plane. In the following considerations suggested by N. G. Van Kampen¹⁰⁵ the subscripts are omitted and the analysis is confined to the one-channel case and to the $l=0$ interaction.

In the one-channel, s -wave case the time-dependent solutions are of the general form

$$r\psi(r,t) = \int_0^\infty A(k) [\exp(-ikr) - U(k) \exp(ikr)] \\ \times \exp(-ihk^2t/2M) dk \\ + \sum_n A_n \exp(-K_n r) \exp(-ihK_n^2t/2M), \quad (4.7)$$

when $r > a$, the nuclear radius; the sum is over the bound levels n with energies $\hbar^2 K_n^2/2M$ while the integration is over the continuum levels with energies $\hbar^2 k^2/2M$; the A_n , $A(k)$ are arbitrary square integrable amplitudes. A wave packet is now constructed with amplitudes

$$A_n = 1/(iK_n + \Lambda) U'(-iK_n), \\ A(k) = \frac{i}{2\pi} \left\{ \frac{\exp[-i(\delta + ka)]}{k - \Lambda} + \frac{\exp[i(\delta + ka)]}{k + \Lambda} \right\} \\ \times \exp(-i\delta); \quad (4.8)$$

$\Lambda = K - iG$ is the wave number of a pole of U in the fourth quadrant ($K > 0$, $G > 0$) and Λ^* is the corresponding zero of U in the first quadrant; iK_n are the poles of U on the positive imaginary axis associated with bound states and $-iK_n$ are the corresponding zeros of U ; $\delta(k)$ is the real phase shift of the collision function $U(k) = \exp(2i\delta)$; and a prime denotes differentiation with respect to k . By means of the symmetry property $U(k)U(-k) = 1$, the wave function (4.7), with the amplitudes (4.8) may be written as

$$r\psi(r,t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{(k - \Lambda)U(k) \exp(ika)} \\ \times [\exp(-ikr) - U(k) \exp(ikr)] \exp(-ihk^2t/2M) \\ + \sum_n \frac{\exp(-K_n r - ihK_n^2t/2M)}{(iK_n + \Lambda)U'(-iK_n)}. \quad (4.9)$$

It is now shown that $\psi(r,0) = 0$ when $r > a$, so that $\psi(r,t)$ represents the emission of a particle by the nucleus. Thus, when $t=0$, the path of integration for the second

term of the integration of (4.9) can be displaced upwards into the complex k plane where the integrand eventually vanishes because $r > a$. For the evaluation of the first term the path of integration is displaced downwards; the residue of each zero of $U(k)$ at $-iK_n$ contributes a term to the sum of (4.9), and according to Van Kampen the path at infinity does not contribute. The sum term of (4.9) can be disregarded henceforth because it tends to vanish when r is sufficiently large and does not give rise to any emission.

By evaluating the integrals of (4.9) after a long time and at a large distance, it can also be shown that this packet describes a particle whose energy lies in the neighborhood of the real part F_λ of the pole at $E = H_\lambda$. The outgoing part of the integrand of (4.9) may be integrated approximately by the method of stationary phase. The exponential is expressed as $\exp if(k)$ where $f(k) = k(r-a) - (\hbar^2 k^2/2M)$ and $f(k)$ is expanded about its stationary value $k_0 = (M/\hbar)(r-a)t^{-1}$ where $f'(k_0) = 0$: $f(k) \approx f(k_0) - \frac{1}{2}(k-k_0)^2(\hbar t/M)$. With this approximation to $f(k)$ and assuming that $(k-\Lambda)$ is constant, the integration is immediately found to give

$$-i(M/2\pi\hbar t)^{1/2} e^{-i\pi} (k_0 - \Lambda)^{-1} \\ \times \exp i[k_0(r-a) - (\hbar k_0^2 t/2M)]. \quad (4.10)$$

At each point (r,t) one observes a wave with a velocity $v_0 = \hbar k_0/M = (r-a)t^{-1}$ indicating that t is so large that all Fourier components in the original packet are separated owing to their different velocities. Indeed the condition for the validity of this method is found to be $(\hbar/M)iG^2 \gg 1$ by requiring that the second derivative of the argument of $\exp[-\frac{1}{2}i(k-k_0)^2(\hbar t/m)]$ be large compared with that of

$$(k-\Lambda) = [(k-K)^2 + G^2]^{1/2} \exp[i \cot^{-1}(k-K/G)].$$

Introducing a "velocity-uncertainty" $w = \hbar G/M$ corresponding to the imaginary part G of Λ , similar to the velocity $v = \hbar K/M$ for the real part, this condition may be put more perspicuously as $(wt)G \gg 1$ indicating that the distance spread wt must be large compared with the wavelength $1/G$. The particle density corresponding to (4.10) is

$$\frac{(\hbar t/2\pi M)}{((r-a) - vt)^2 + (wt)^2}, \quad (4.11)$$

which is concentrated about the point $r = a + vt$ with a spread wt . The incoming part of the integration of (4.10) may similarly be evaluated about its stationary point $k_1 = -(M/\hbar)(r-a) = -k_0$. One finds it to be

$$i(M/2\pi\hbar t)^{1/2} e^{-i\pi} \exp[-ik_0(r-a) - ik_0^2 \hbar t/2M] \\ \times [(-k_0 - \Lambda)U(-k_0)]^{-1}, \quad (4.12)$$

which is always small compared with (4.10) because the magnitude of $(k_0 + \Lambda)$ is always large compared with G .

A different path of integration gives information

¹⁰⁵ N. G. Van Kampen (private communication).

about the shape of the wave packet at distances which are not so great that the packet is decomposed into its Fourier components. For this purpose a new complex variable $\zeta = \xi + i\eta = k - k_0$ is introduced, in terms of which the incoming part of the integrand of (4.10) is

$$-\frac{i}{2\pi} \exp\left[\frac{iM(r-a)^2}{2\hbar t}\right] \int_{-\infty}^{\infty} \frac{\exp(-i\hbar t \zeta^2/2M) d\zeta}{k_0 - \Lambda + \zeta}. \quad (4.13)$$

The path of integration is now turned clockwise in the ζ plane onto the imaginary axis. It is essential that it be turned in this direction because in the first and third quadrant the exponential factor becomes infinitely large at infinity; it is possible to show that the contributions from the arcs at infinity in the second and fourth quadrants vanish. By the stationary phase method one finds that the contribution to the integral of (4.13) from the integration along the imaginary axis is

$$-\frac{i(2\pi\hbar t/M)^{1/2} \exp(\frac{1}{4}i\pi)}{((r-a) - vt) + iG}. \quad (4.14)$$

The condition for the validity of this method is now that

$$((r-a) - vt)^2 \gg (vt/K); \quad (4.15)$$

in other words, the difference $(r-a) - vt$ must be large compared with the geometric mean of vt and the particle wavelength $1/K$, indicating that (4.14) gives only the tail of the wave packet which is centered about vt . However, if $vt > r-a$, the integrand of (4.13) has a pole at $\zeta = \Lambda - k_0$ in the fourth quadrant; the contribution from the residue, which is to be added to (4.14), is then

$$-\exp[-i(M/2\hbar t)(r-a)^2] \exp[i\Lambda(r-a)] \times \exp(-i\Lambda^2\hbar t/M). \quad (4.16)$$

This term is the one that is usually considered as representing the emitted wave of a radioactive state. The second and third exponentials of (4.16) correspond to those of (4.5), and considering the discussion in connection with (14.1), the real part of the arguments of these may combine to $\frac{1}{2}(\lambda/v)[(r-a) - vt]$. Since the pole appears in the fourth quadrant only when $vt > r-a$, it is apparent that the magnitude of (4.16) is always less than unity.

The result is that at a large distance behind the wave front at $r-a = vt$ the wave packet is determined by an asymptotic expansion whose first terms are (4.14) and (4.16). At large distances before the front the first nonvanishing term is (4.14). However, there is also the incoming part of (4.10) to be considered. For this part it is not possible to move the integration path through the second quadrant so that only one-half is moved into the lower half-plane. Putting

$$k = k_1 + \zeta = \zeta - (r-a)/(\hbar t/M)$$

one obtains

$$\frac{i}{2\pi} \exp[i(M/2\hbar t)(r-a)^2] \times \left[\int_{-\infty}^0 \frac{\exp(-\frac{1}{2}i\ell\xi^2) d\xi}{(k_1 - K + iG + \xi) U_a(k_1 + \xi)} + \int_0^{-\infty} \frac{\exp(\frac{1}{2}i\ell\eta^2) i d\eta}{(k_1 - K + iG + i\eta) U_a(-k_1 + i\eta)} \right], \quad (4.17)$$

where

$$U_a = \exp(2ika)U.$$

Both integrals can again be evaluated by means of the stationary phase method with the same condition (4.15), yielding together

$$\frac{2U_a(k_0)(2\pi\hbar t/M)^{1/2} \exp(-\frac{1}{4}i\pi)}{(r-a) + vt - i\omega t}, \quad (4.18)$$

which is always small compared with (4.14), even in the head of the packet, and it does not affect the over-all shape of the packet.

5. Isotopic Spin Selection Rule for Reactions with Light Nuclei

Several authors^{106,107} have proposed and discussed a selection rule for nuclear reactions which can forbid certain reactions involving deuterons and alphas as bombarding particles on self-conjugate nuclei. This selection rule is the so-called "isotopic spin selection rule" which says that "total isotopic spin is conserved in nuclear reactions." If, in particular, the two initial nuclei in a reaction have pure isotopic spins T_1 and T_2 and the two final nuclei have pure isotopic spins T_1' and T_2' , then, according to the rule, the reaction is forbidden unless the isotopic spins satisfy the vector equation:

$$\mathbf{T}_1 + \mathbf{T}_2 = \mathbf{T}_1' + \mathbf{T}_2'. \quad (5.1)$$

If the bombarding and emitted particles are alphas or deuterons (which have $T=0$), the rule forbids reactions to all those residual states with isotopic spin different from that of the target nucleus. Although the rule has had considerable experimental corroboration, it does not have an exact theoretical basis, even where the separated initial and final nuclei have a high degree of isotopic spin purity. For example, let us imagine a reaction proceeding through a compound nucleus with an appreciable time delay. If the excitation of the compound nucleus is to a region where states are closely spaced, the Coulomb forces acting between protons, although relatively much weaker than nuclear forces, may easily be strong enough to mix the isotopic spins of neighboring states. Then although the initial

¹⁰⁶ R. K. Adair, Phys. Rev. **86**, 155 (1952).

¹⁰⁷ R. F. Christy, Phys. Rev. **89**, 839 (1953).

separated system may have isotopic spin purity, this purity can be destroyed in the compound system with the result that isotopic spin need *not* be conserved in the reaction.

There is recent experimental evidence that under certain circumstances, isotopic spin need not be conserved in nuclear reactions. We now try to formulate the theoretical conditions under which we expect isotopic spin to be conserved, for reactions proceeding through a compound nucleus and for those "direct" reactions that do not proceed in this way.

a. Conditions for Conservation of Isotopic Spin for Reactions Proceeding through a Compound Nucleus

Whether isotopic spin is conserved in a given compound nucleus reaction depends on the structure of the compound nucleus. It depends on the relative magnitudes of the three following quantities that characterize the compound nucleus:

- (1) the average level spacing D^J between levels of the same spin, J , and parity
- (2) the average total width $\langle\Gamma^J\rangle$ of such states
- (3) the average matrix element $\langle H_c \rangle$ of the Coulomb forces taken between such states. Presumably, in contrast to D and $\langle\Gamma\rangle$, this is not especially dependent on J and parity.

The most familiar criterion for conservation of isotopic spin for states of given J and parity is:

$$\langle H_c \rangle \ll D^J. \quad (\text{"static" criterion}) \quad (5.2)$$

This says that the Coulomb forces are so weak that they are not able to mix appreciably the isotopic spins of neighboring states. Thus the states of the compound nucleus have pure isotopic spin, so isotopic spin must be conserved in reactions via the compound nucleus. We call this the "static criterion" because it is usually applied to estimate the isotopic spin purity of bound stationary states. A second criterion ensures conservation of isotopic spin in reactions has no relevance for bound states, and we call it the "dynamic criterion." It arises from the fact that there is a time $\sim \hbar/\langle H_c \rangle$ associated with the action of the Coulomb forces.¹⁰⁸ If the width of the compound state is large enough so that the state decays before the Coulomb forces have time to act, the isotopic spin purity of the initial separated system will be preserved. Since the lifetime of a typical compound state is $\hbar/\langle\Gamma^J\rangle$, the dynamic criterion can be written as

$$\langle H_c \rangle \ll \langle\Gamma^J\rangle. \quad (\text{"dynamic" criterion}) \quad (5.3)$$

Considering both criteria, it follows that if the mean Coulomb matrix element is less than *either* the mean level spacing *or* the mean width, then isotopic spin should be conserved. Only when the matrix element is

of the order of or greater than *both* the width and level spacing can conservation be violated.

Let us now consider the two criteria in more detail for the situations $\langle\Gamma^J\rangle < D^J$, and $\langle\Gamma^J\rangle > D^J$ (we drop the superscript J from now on and consider it as understood).

$\langle\Gamma\rangle < D$.—This condition implies that only one isolated resonance can be excited for a given bombarding energy. The weaker of conditions (5.2) and (5.3) is $\langle H_c \rangle \ll D$. This, along with $\langle\Gamma\rangle < D$, certainly ensures conservation of isotopic spin because the isotopic spin of an isolated compound state must be a good quantum number under these conditions. Thus the stronger condition $\langle H_c \rangle \ll \langle\Gamma\rangle$ automatically ensures isotopic spin conservation.

$\langle\Gamma\rangle > D$.—This condition implies that many compound states are excited by a given bombarding energy. The weaker of the two conditions (5.1) and (5.2) is now $\langle H \rangle \ll \langle\Gamma\rangle$ and so we want to see in detail why isotopic spin should be conserved under the combined conditions $\langle\Gamma\rangle \gg \langle H \rangle$, $\langle\Gamma\rangle > D$. From Sec. XI, 2, we have the result that the cross section $\sigma_{cc'}$ for a reaction through compound states λ in the continuum is determined essentially by the absolute square of the expression:

$$\sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{E_{\lambda} - E - \frac{i}{2} \langle\Gamma\rangle}. \quad (5.4)$$

This was derived by the assumption of random signs for the reduced width amplitudes $\gamma_{\lambda c}$, and leads to independence of formation and decay processes expected for reactions through the compound nucleus.

As suggested by C. Schwartz,¹⁰⁹ we consider the usual set of states X_{λ} of the total system, and also another set $X_{\mu T^0}$ which are defined similarly except for the fact that Coulomb forces are ignored so that isotopic spin T is a good quantum number. We can make the expansion,

$$X_{\lambda} = \sum_{\mu} a_{\lambda\mu} X_{\mu T^0} \quad (5.5)$$

and its dual

$$X_{\mu T^0} = \sum_{\lambda} a_{\lambda\mu}^* X_{\lambda}, \quad (5.6)$$

where

$$\sum_{\lambda} a_{\lambda\mu} a_{\lambda\mu'}^* = \delta_{\mu\mu'} \quad (5.7)$$

$$\sum_{\mu} a_{\lambda\mu} a_{\lambda'\mu}^* = \delta_{\lambda\lambda'}. \quad (5.8)$$

From the first expansion, we have the relations

$$\gamma_{\lambda c} = \sum_{\mu} a_{\lambda\mu} \gamma_{\mu c^0}, \quad (5.9)$$

$$\gamma_{\lambda c'} = \sum_{\mu} a_{\lambda\mu} \gamma_{\mu c'^0}, \quad (5.10)$$

¹⁰⁸ H. Morinaga, Phys. Rev. **97**, 444 (1954).

¹⁰⁹ C. Schwartz (private communication).

where $\gamma_{\mu c}^0$ and $\gamma_{\mu c'}^0$ are reduced width amplitudes defined for the states $X_{\mu T^0}$. From (5.5), (5.7), (5.9), and (5.10) it follows that

$$\sum_{\lambda} \gamma_{\lambda c} \gamma_{\lambda c'} = \sum_{\mu} \gamma_{\mu c}^0 \gamma_{\mu c'}^0. \quad (5.11)$$

Let us assume that the isotopic spins of the channels c and c' are $T_c, T_{c'}$. Since the states μ have pure isotopic spin T , $\gamma_{\mu c}$, and $\gamma_{\mu c'}$ contain $\delta_{T T_c}$ and $\delta_{T T_{c'}}$ respectively, so that

$$\sum_{\lambda} \gamma_{\lambda c} \gamma_{\lambda c'} = (\text{factor}) \times \delta_{T_c T_{c'}}. \quad (5.12)$$

The states in the sum over λ may be restricted to those within an energy distance $\Delta \ll \langle \Gamma \rangle$ but $\gg \langle H_c \rangle$ of a given energy. By the condition $\langle H_c \rangle \ll \langle \Gamma \rangle$ and the fact that a state λ only contains appreciable components of states μ from within a distance $\sim \langle H \rangle$, it follows that the states λ or μ within such a distance $\sim \Delta$ form a complete set to a good approximation. Thus the above expansions and, in particular, Eq. (5.11) are still valid when the states in the sums are restricted this way.

From the condition $\langle \Gamma \rangle > D$, we can say that for groups of states λ in (5.4) within energy intervals Δ , the denominator in (5.4) can be approximately replaced by a constant outside the sum. From (5.12) it follows that expression (5.4) contains a factor $\delta_{T_c T_{c'}}$, so the validity of the isotopic spin selection rule under the conditions $\langle \Gamma \rangle > D$, $\langle \Gamma \rangle \gg \langle H_c \rangle$ follows.

b. Conservation of Isotopic Spin in Other Types of Reaction

For completeness we mention a few aspects of isotopic spin conservation in certain "direct" types of reaction. In inelastic scattering processes that proceed by the mechanism known as Coulomb excitation,³⁹ the essential part of the theoretical cross sections is the electromagnetic matrix element between initial and final states of the target nucleus. In general this element is nonzero between pairs of target states such that the change in the target isotopic spin ΔT is $-1, 0$ or $+1$. A well-known special selection rule^{103, 110} occurs for electric dipole transitions which are zero in self-conjugate nuclei when $\Delta T = 0$.

There are also the direct reactions such as $(\alpha\alpha')$ (dd') (pp') (pn) that are caused by bombarding particles with large impact parameters interacting with nucleons on the edge of the target nucleus. Since such reactions involve no time delay, the Coulomb forces between the target and bombarding particle have no time to operate so we expect total isotopic spin to be conserved in such reactions. This does not always imply that the isotopic spin of the target nucleus cannot be changed. Clearly, for alpha and deuteron bombardment, it cannot change because these projectiles have $T=0$ so that

conservation of total isotopic spin implies conservation of target isotopic spin. For nucleon projectiles²⁰ the situation is different. An example, a $T=0$ target can be raised to a $T'=1$ state by a nucleon collision without changing the *total* isotopic spin of $\frac{1}{2}$. Whether the target isotopic spin changes depends on:

- (1) whether the outgoing nucleon is the incident nucleon or a target nucleon; that is, whether it is a direct or an exchange nucleon.
- (2) the existence of charge exchange forces

$$[1 + (\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)] v(r_{ij})$$

acting between the incident nucleon and the target nucleons.

When the outgoing nucleon is an exchange nucleon, the isotopic spin of the target can change in general, so there is no selection rule on the target isotopic spin. When the outgoing nucleon is the direct (incident) nucleon, the isotopic spin of the target can only be changed by action of charge exchange forces.

Consider the isotopic spin matrix elements in the direct and exchange amplitudes.

$$\text{Direct: } M = \langle \alpha T M, \beta \frac{1}{2} m | V_{on} | \alpha' T' M', \beta' \frac{1}{2} m' \rangle \quad (5.13)$$

where $\alpha T M, \beta \frac{1}{2} m$ are the initial states of the target nucleus and projectile respectively labeled by isotopic spin and component and with α, β accounting for all other quantum numbers. The final states are labeled with primes. V_{on} is the interaction of the incident nucleon (0, say) with a target nucleon (n , say) and has one of the following two forms:

$$V_{on} = v_{on}(\mathbf{r}_{0n}) \begin{cases} 1 & \text{ordinary} \\ \frac{1}{2} [1 + (\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_n)] & \text{charge exchange} \end{cases} \quad (5.14)$$

where v_{on} does not depend on isotopic spin.

We can assume a parentage expansion of the initial target state

$$\langle \alpha T M | = \sum \langle \alpha T | \rangle_{\alpha_1 T_1 \alpha_2} \times \langle \alpha_1 T_1 M_1 | \langle \alpha_2 \frac{1}{2} m_2 | {}_n (T_1 \frac{1}{2} M_1 m_2 | T M) \quad (5.16)$$

(and a similar one for the final state), where the coefficient of fractional parentage is taken to include the vector coupling of mechanical angular momenta up to total J_i and where $\alpha_1 T_1 M_1$ and $\alpha_2 \frac{1}{2} m_2$ label parent states and states of the n th nucleon. Upon insertion in (5.13), corresponding to forms (5.14), (5.15), we have:

$$\begin{aligned} M &= \delta_{T M m, T' M' m'} \sum_{T_1} \langle \alpha \beta | v | \alpha' \beta' \rangle_{T_1, \text{dir}} \\ M &= \frac{3}{2} \sum_{T_1} \{ \delta_{T M m, T' M' m'} + (-)^{M' - M + T - T'} \\ &\quad \times (1 \frac{1}{2} (m - m') m' | \frac{1}{2} m) (1 T' (M - M') M' | T M) \\ &\quad \times U(1 \frac{1}{2} T T_1, \frac{1}{2} T') \} \langle \alpha \beta | v | \alpha' \beta' \rangle_{T_1, \text{dir}} \end{aligned} \quad (5.17)$$

¹¹⁰ M. Gell-Mann and V. L. Telegdi, Phys. Rev. **91**, 169 (1953); L. A. Radicati, *ibid.* **87**, 521 (1952).

where we define the matrix element $\langle \alpha\beta | v | \alpha'\beta' \rangle_{T_1, \text{dir}}$ as:

$$\sum_{\alpha_1\alpha_1'\alpha_2\alpha_2'} \langle \alpha T | \alpha_1 T_1 \alpha_2 \rangle \langle \alpha' T' | \alpha_1' T_1 \alpha_2' \rangle \times \langle \alpha_2 \beta | v_{0n} | \alpha_2' \beta' \rangle_{\text{dir}}. \quad (5.18)$$

As expected, an ordinary force cannot change the isotopic spin but a charge exchange force can. For the latter force, the matrix element is zero when the target and residual states are in the same self-conjugate nucleus and $T=T'$. This is on account of the vanishing of the vector coupling coefficient $(T'100|T0)$ when $T=T'$.

Exchange: When the outgoing nucleon is the target nucleon and the incident nucleon is left behind in the residual nucleus, instead of the direct matrix elements (5.17) we have the exchange ones

$$M = \sum_{T_1} (T_1 \frac{1}{2} M_1 m' | TM) (T_1 \frac{1}{2} M_1 m | T' M') \times \langle \alpha\beta | v_{0n} | \alpha'\beta' \rangle_{T_1, \text{ex}}$$

$$M = \delta_{T M m, T' M' m'} \sum_{T_1} \langle \alpha\beta | v_{0n} | \alpha'\beta' \rangle_{T_1, \text{ex}}.$$

The charge exchange force has zero exchange matrix element when $m=-m'$. This is expected since this force exchanges charge and so any target (exchange) nucleon ejected by this force must have the charge of the incident nucleon, i.e., $m=m'$.

It thus seems very doubtful that the surface nucleon-nucleon reactions will reveal any selection rule on the isotopic spin of the final nucleus. In general, all four types of reaction amplitudes should contribute to (pp') and (nn') reactions, and two of them should contribute to (np) and (pn) reactions. Even if the nucleon-nucleon interactions contain no charge exchange force, there are still two and one contributing amplitudes respectively. Furthermore, since the angular distributions of the four amplitudes should be qualitatively similar (forward peaks for medium-high incident energies), there is probably no possibility of separating out the various amplitudes from the observed angular distributions in order to study each one individually.

c. Discussion of the Experimental Situation

The experimental material has been reviewed by Burcham.¹¹¹ First let us consider the reactions $(\alpha\alpha')$, (dd') , (αd) and $(d\alpha)$. The last two types are presumably compound nucleus reactions since a severe rearrangement of nucleons is implied. The first two can proceed by the surface or compound nucleus mechanisms with probably the former predominating in general for the production of definite final states. Since surface reactions should conserve isotopic spin, breakdown of the selection rule can only arise through the compound nucleus contribution. For (αd) and $(d\alpha)$ reactions, the ratio of the intensity of a forbidden group to those of allowed groups should provide a measure of the breakdown in the rule for the compound nucleus mechanism. For $(\alpha\alpha')$ and (dd') reactions such a ratio may underestimate this breakdown in view of the possibly large surface reaction contributions to the allowed

groups. Angular distribution measurements would reveal whether these contributions are important.

Any detailed theoretical analysis of data is inhibited by the fact that states of different spin and parity may have completely different isotopic spin characteristics. For instance, in a given nucleus at a given excitation, one can imagine that states of low spin are dense with $D \ll \langle H_c \rangle$ but broad with $\langle \Gamma \rangle \gg \langle H_c \rangle$, whereas states of high spin are rare and sharp ($\langle \Gamma \rangle \ll D \ll \langle H_c \rangle$). In both cases the selection rule should work but for opposite reasons [criterion (5.3) for the low spin states, (5.2) for the high spin states]. Breakdown of the rule may only occur for states of intermediate spin, but these states usually constitute only a fraction of those excited so that, in general, the rule should appear to work experimentally.

Now consider states of a given spin at different excitations. At low excitations the rule should work by (5.2). At high excitations it should also work, but by (5.3). A breakdown will occur if the two regions do not overlap, i.e., if with increasing excitation, D falls below $\langle H_c \rangle$, before $\langle \Gamma \rangle$ increases above $\langle H_c \rangle$. Such an intermediate region can occur in practice, at least in one special case (see "Mass 14" below).

Detailed analysis of data is also rendered difficult by the sporadic nature of the existing data. Often measurements are made at a fixed bombarding energy and a fixed angle of observation. It is significant that one of the first attempts to observe a reaction over energy and angular ranges discovered the most serious breakdown of the isotopic spin selection rule yet found (see "Mass 18" below).

We now discuss the existing experimental data from nuclear reactions that is significant from the point of view of isotopic spin conservation according to the compound nuclei involved. We do not mention inelastic and charge exchange nucleon-nucleon reactions because they are not expected to reveal any selection rules on isotopic spin. However, we note that there is a curious experimental absence of a proton group to the first excited $T=1$ state at 2.31 Mev in N^{14} from the $N^{14}(p, p')$ reaction at bombarding energies of 9.5¹¹² and 17 Mev.¹¹³ This is in contrast to the $N^{14}(p, n)-O^{14}(E_p=17 \text{ Mev})$ reaction,¹¹⁴ which seems to yield the corresponding isobaric state easily enough. The $N^{14}(n, p)C^{14}$ reaction has been examined only at low energies. A similar situation exists in Li^6 where the $T=1$ state at 3.6 Mev is only weakly excited in the $Li^6(p, p')$ reaction at bombarding energies of 15 and 19 Mev.¹¹⁵ In the case of the $N^{14}(p, p')$ reaction, the absence of the group has been recently interpreted in a convincing way by Levinson and Bannerjee¹¹⁶ who have shown that the same fortuitous cancellation in matrix element that gives C^{14} its long life against β decay also operates to reduce the cross section for production of the group.

Where no references are cited, in the following they may be found in the published compilations of data.¹¹⁷

Mass 8.—The reaction $Li^6(dd')Li^{6*}$ at $E_d=7.4$ Mev and $\theta=90^\circ$ does not reveal the first $T=1$ state at 3.57 Mev in Li^{6*} although the 2.19 ($T=0$) state is readily excited. The excitation of the compound nucleus, Be^8 , is 28 Mev, where levels of low spins would seem to be broad, dense, and overlapping if one extrapolates from the situation as observed near 20 Mev.

Mass 10.—The reaction $Li^6(\alpha\alpha')Li^{6*}$ at $E_\alpha=30$ Mev and all angles excites the 2.19 ($T=0$) state but not the 3.57 ($T=1$) one. The 4.52 ($T=0$) state is found as well, and an upper limit of 5%⁸⁷ can be put on the relative intensity of a group to the 3.57 state. The angular distributions of the observed groups are strongly

¹¹² Freemantle, Prowse, and Rotblat, Phys. Rev. **96**, 1268 (1954).

¹¹³ R. Sherr (private communication).

¹¹⁴ F. Ajzenberg and W. Franzen, Phys. Rev. **94**, 409 (1954).

¹¹⁵ Sherr, Hornyak, and Yoshiki, Bull. Am. Phys. Soc. Ser. II, **1**, 231 (1956); Levinson, Banerjee, Albright, and Toboeman, *ibid.* Ser. II, **1**, 194 (1956).

¹¹⁶ C. Levinson and M. K. Banerjee (to be published).

¹¹⁷ F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **24**, 321 (1952); **27**, 77 (1955); P. M. Endt and J. C. Kluyver, *ibid.* **26**, 95 (1954).

¹¹¹ W. E. Burcham, *Progress in Nuclear Physics* (Pergamon Press, London, 1955).

peaked forward showing that these reactions are mainly surface ones. Since such reactions cannot produce a $T=1$ state, the observed limit on the relative intensity is not a sound guide for estimating a possible breakdown of isotopic spin conservation for the compound nucleus contribution. The excitation of the compound nucleus B^{10} is about 22 Mev where, from the observed situation at 7 Mev, levels of low spins are dense and broad.

Mass 12.—The reaction $B^{10}(dd')B^{10*}$ has been observed at 6, 7, 7.6, and 14 Mev. States of $T=0$ at 0.72, 2.15, and 3.58 Mev excitation are found in the first three cases at special angles, with the 1.74 state unobserved. An upper limit to the relative intensity of a group to the 1.74 state compared to those observed is about 2%. At 14 Mev, however, a group to the 1.74-Mev state is observed with a relative intensity of ~5%.¹¹⁸ For the reasons we have mentioned above, this can be regarded as a lower limit on the relative intensity of the compound nucleus contributions to the groups, which is probably several times larger. Since the excitation of the compound nucleus, C^{12} , is about 37 Mev where most important spin states should satisfy criterion (5.3), such a breakdown is surprising. It is conceivable that in general in a situation like this some process akin to Coulomb excitation contributes to the forbidden group. However, this would be hard to maintain in the present case because the transition must be $M3$ which implies a very small cross section. Theoretical investigation of such processes for above-barrier energies would be interesting.

Mass 14.—Although the reaction $B^{10}(\alpha\alpha')B^{10*}$ has not been used to look for the 1.74 ($T=1$) state in B^{10} , the $B^{10}+\alpha$ reactions have revealed a very interesting fact about the isotopic spin situation in the compound nucleus N^{14} at excitations ~13 Mev. The reactions $B^{10}(\alpha p)C^{13}$, $B^{10}(\alpha n)N^{13}$ have been studied for a number of isolated resonances and the results have been analyzed with one level resonance formulas.¹¹⁹ If isotopic spin were a good quantum number for the compound nuclear states, the reduced widths for neutron and proton emission from any given state should be equal. However, it is found that these widths for the 12.69 (3-) and 12.79 (4+) states differ by a factor of five or so, and this cannot be accounted for by any variation of the nuclear interaction radii. So presumably isotopic spin is not a good quantum number. A second 44 state is observed at 12.92 indicating that the level spacing of states of the same spin and parity is of the order of hundreds of kev. (The levels widths are of the order of tens of kev.) Coulomb mixing matrix elements of the order of hundreds of kev would thus be sufficient to give rise to strong mixing of states of different isotopic spins. From estimates that have been made in special cases, such values are quite reasonable.^{103,110} It seems clear then that for states of spin 3- and 4+ in N^{14} near 13 Mev, there exists a region which is intermediate between those regions satisfying criteria (5.2) and (5.3).

Mass 16.—The reactions $N^{14}(dd')N^{14*}$ have been studied at a bombarding energy of 6.98 Mev. The 3.95 ($T=0$) state has been seen, but the 2.31 ($T=1$) state has not. The upper limit on the relative intensity of a possible deuteron group to the 2.31 state is about 5% of the observed group to the 3.95 state. The excitation of the compound nucleus O^{16} is about 27 Mev where one imagines all low spin states to satisfy criterion (5.3).

Mass 18.—The reactions $O^{16}(d\alpha)N^{14}$ have been studied at bombarding energies between 6 and 7.5 Mev and at several angles.¹²⁰ The reactions have also been studied at 19 Mev over all angles greater than 15°. In the latter case, the 2.31 ($T=1$) state in N^{14} is not observed and an upper limit of about 4% on the corresponding intensity relative to the intensity of the observed group to the ground state can be imposed. At the lower energies,¹²⁰ in contrast, an alpha group is observed to the 2.31 state with an

intensity of the order of 5 to 10% of the groups to the 3.95 ($T=0$) and ground states. The energy dependence of the yield of the 2.31 state shows resonance-like anomalies at 6.2, 6.7, and 7.0 Mev with widths of the order of 0.25 Mev. At the 7.0-Mev peak, the intensity at 30° is about one-half of that for the production of the 1.95 state and one-seventh of that for the ground state; the excitation curve for yielding the ground state group also exhibits peaks but not at the same energies as the group to the 2.31-Mev state.

The excitation of the compound nucleus for the bombarding energies 19 and 6.5 Mev are about 24 and 13 Mev, respectively. Although the former excitation should be in the region when the most important states satisfy (5.3), the latter is not. According to the observed spectrum of F^{18} , many levels are still fairly well resolved at 11 Mev or so, although dense. It thus seems to be quite possible that, at 13 Mev, the situation is much like the situation at the same energy in N^{14} that we have discussed, namely intermediate between the regions where (5.2) and (5.3) are satisfied. If this is correct, the observed breakdown in isotopic spin conservation is quite understandable. Because of the shape of the excitation curve for the "forbidden" group, isolated resonances are presumably mainly responsible.

The reaction $N^{14}(\alpha\alpha')N^{14}$ has been observed at $E_{\alpha}=21.2$ Mev¹²² and an angle of 35°. The 2.31 ($T=1$) excited state is not observed, and neither are the believed $T=1$ states at 8.06 and 8.70 Mev. However the state at 9.17 Mev is found although its strong $E1$ radiation to the ground state suggests that it is $T=1$. (Possibly two levels are involved.)^{122,123} Most other states, presumably $T=0$, are found, and so, on the whole, isotopic spin appears to be conserved. The excitation of the compound nucleus is ~21 Mev. The same reaction has been studied at $E_{\alpha}=31$ Mev and at all angles out to 70°. An upper limit of 5% can be placed on the intensity of a group to the 2.31 state relative to the observed groups.^{122,124}

Mass 22.—The reaction $Ne^{20}(d,\alpha)F^{18}$ for $E_d=7.8$ Mev has been observed to go to ten states in F^{18} . An interesting feature of the α spectrum is that the strongest group is centered on an energy corresponding to the first $T=1$ state in F^{18} at 1.05 Mev excitation.

There are two possible explanations (1) the group corresponds to $T=0.8$ states (those at 0.94 and 1.25 Mev perhaps) (2) isotopic spin conservation breaks down very seriously. It may indeed happen that the latter is correct because the situation is quite like that in the previous $O^{16}(d,\alpha)N^{14}$ reaction where we have seen that isotopic spin conservation fails badly. The excitation energy of the compound nucleus, Na^{22} is about 18 Mev which is rather higher than the 13 Mev in the $O^{16}(d,\alpha)N^{14}$ reaction. Although the level spectrum of Na^{22} is little known, one can infer that at 18 Mev, level densities will be higher than in the previous reaction because of the extra energy and larger number of nucleons.

Mass 26.—The reaction $Mg^{24}(d,\alpha)Na^{22}$ has been examined at bombarding energies of 5, 6, and 7 Mev and at an angle of 90°. A group is observed going to the 0.59 state in Na^{22} which has been established in the $F^{19}(\alpha,n\gamma)$ reaction¹²⁵ and has about the right energy to be the first $T=1$ state. At $E_d=6$ Mev, the yield to this state is about the same as that to the ground state and a breakdown in isotopic spin conservation may be implied. The excitation of the compound nucleus Al^{26} is about 17 Mev.

Mass 30.—The $Si^{28}(d,\alpha)Al^{26}$ reaction has been investigated at $E_d=7$ Mev and $\theta_{\alpha}=90^\circ$.¹²⁸ The first excited state in Al^{26} at 0.22

¹¹⁸ J. M. McGruer (private communication).

¹¹⁹ Shire, Wormald, Lindsay-Jones, Lunden, and Stanley, Phil. Mag. 44, 1197 (1953); E. S. Shire and R. Edge, *ibid.* 46, 640 (1955).

¹²⁰ C. P. Browne, Phys. Rev. 100, 1252(A) (1955).

¹²¹ Freemantle, Gibson, Prowse, and Rotblat, Phys. Rev. 92, 1268 (1953).

¹²² Carmichael, Sampson, and Johnson, Phys. Rev. 91, 473(A) (1953); Miller, Gupta, Rasmussen, and Sampson, Phys. Rev. 98, 1184(A) (1955); Miller, Carmichael, Gupta, Rasmussen, and Sampson, Phys. Rev. 101, 740 (1956).

¹²³ V. K. Rasmussen (private communication).

¹²⁴ H. Watters (private communication).

¹²⁵ C. P. Browne and W. C. Cobb, Phys. Rev. 99, 644(A) (1955).

¹²⁶ N. P. Heydenberg and G. M. Temmer, Phys. Rev. 94, 1252 (1954).

¹²⁷ S. A. Moszkowski and D. C. Peaslee, Phys. Rev. 93, 455 (1954); P. Stahelin, Helv. Phys. Acta 26, 691 (1953); Phys. Rev. 92, 1076 (L) (1953).

¹²⁸ C. P. Browne, Phys. Rev. 95, 860 (1954).

Mev is not observed and neither is a state at 2.074 Mev. The excitation of the compound nucleus P^{30} is about 19 Mev. The states at 0.22 and 2.074 have been established in the $Mg^{26}(p,\gamma)Al^{26}$ reaction¹²⁹ and the 0.22 state has about the right energy to be the first $T=1$ state. It seems then that conservation of isotopic spin holds in the $Si^{28}(d,\alpha)Al^{26}$ reaction at $E_d=7$ Mev and $\theta=90^\circ$. However a more thorough investigation¹³⁰ carried out over energies from 5.5 to 7.5 Mev and angles from 15° to 130° has revealed the presence of an alpha group leading to the 0.22-Mev state. The intensity of this group varies by a factor of about 10. Relative to the ground state group, it varies from 0.65 to 2. At 7.03 Mev the ratio of intensities decreases from 0.62 at 15° to 0.12 at 60° , and the group is not seen at larger angles. This accounts for its absence in the previous investigation.

Mass 34.—The $S^{32}(d,\alpha)P^{30}$ reaction has been examined at bombarding energies between 1.6 to 2.9 Mev, corresponding to an excitation of the compound nucleus of about 13.6 Mev.¹³¹ Observations at 63° and 90° found that the levels at 0.71 Mev in P^{30} is excited just as strongly as the ground state but that the level at 0.69 Mev is only weakly excited. From a study of the $Si^{29}(p,\gamma)P^{30}$ reaction¹³² it has been suggested that the 0.69-Mev state is the lowest $T=1$ state of P^{30} . Consideration of the Coulomb energy displacement¹³¹ puts the state rather lower at ~ 0.33 Mev. If the 0.69 state is $T=1$, and the 0.71 Mev state has $T=0$, the (d,α) results imply isotopic spin conservation.

APPENDIX. COULOMB WAVE FUNCTIONS AND NEUTRON WAVE FUNCTIONS

This Appendix is concerned with the solutions of the equation (III, 2.9):

$$u''(\rho) - [l(l+1)\rho^{-2} + 2\eta\rho^{-1} \mp 1]u(\rho) = 0, \quad (A.1)$$

where the prime denotes differentiation with respect to ρ ; the upper and lower signs apply to positive and negative energy channels, respectively. ρ and η are defined by

$$\rho = kr, \\ \eta = \frac{Z_1 Z_2 e^2 M}{\hbar^2 k},$$

where k is the channel wave number defined in terms of the channel energy of relative motion by:

$$k = \left(\frac{2M|E|}{\hbar^2} \right)^{\frac{1}{2}}.$$

M is the *reduced* channel mass and Z_1, Z_2 are the charges on the two fragments. r is the radial distance of separation of the fragments.

Convenient solutions of (A.1) are

(i) *Positive energies.*— $u_1 = F_l, G_l$, the solutions⁴⁶ that are regular and irregular at $r=0$ with asymptotic forms

$$F_l \sim \sin(\rho - \eta \log 2\rho - \frac{1}{2}l\pi + \sigma_l), \quad (A.2a)$$

$$G_l \sim \sin(\rho - \eta \log 2\rho - \frac{1}{2}l\pi + \sigma_l), \quad (A.2b)$$

¹²⁹ Sherr, Kavanagh, and Mills, Phys. Rev. **98**, 1185 (1954).
¹³⁰ C. P. Browne, Bull. Am. Phys. Soc. Ser. II, **1**, 212 (1956).
¹³¹ L. L. Lee and F. P. Mooring, Phys. Rev. **104**, 1342 (1956).
¹³² Broude, Green, Singh, and Wilmott, Phys. Rev. **101**, 1052 (1956).

where σ_l is the phase $\arg\Gamma(1+l+i\eta)$. For the case $\eta=0$ (neutron channels), F_l and G_l become simply related to spherical Bessel functions of half-integral order,

$$F_l = \left(\frac{\pi\rho}{2} \right)^{\frac{1}{2}} J_{l+\frac{1}{2}}(\rho) = \rho j_l(\rho), \quad (A.3a)$$

$$G_l = (-)^l \left(\frac{\pi\rho}{2} \right)^{\frac{1}{2}} J_{-(l+\frac{1}{2})}(\rho) = (-)^l \rho j_{-l}(\rho). \quad (A.3b)$$

(ii) *Negative energies.*—The only physically significant solution is the exponentially decaying one,

$$u_l = W(-\eta, l+\frac{1}{2}, 2\rho)$$

with the asymptotic form

$$W \sim \exp(-\rho - \eta \log 2\rho). \quad (A.4)$$

This W function is the "Whittaker function" which has the integral representation

$$W(-\eta, l+\frac{1}{2}, 2\rho) = \frac{\exp(-\rho - \eta \log 2\rho)}{\Gamma(1+l+\eta)} \int_0^\infty t^{l+\eta} e^{-t} \left(1 + \frac{t}{2\rho} \right)^{l-\eta} dt. \quad (A.5)$$

For the case $\eta=0$ (neutron channels), W becomes simply related to a modified Bessel function of the second kind:

$$W(0, l+\frac{1}{2}, 2\rho) = \left(\frac{2\rho}{\pi} \right)^{\frac{1}{2}} K_{l+\frac{1}{2}}(\rho). \quad (A.6)$$

a. General Formulas

The quantities involving F, G , and W that occur in R-matrix theory are S, P , and ϕ . It is often convenient in computation to work directly with these quantities—or, at least, closely related ones, namely the so-called amplitude A and phase ϕ . For positive energies

$$F = A \sin\phi, \quad G = A \cos\phi \quad (A.7)$$

$$A = +(F^2 + G^2)^{\frac{1}{2}} = (\rho/P)^{\frac{1}{2}} \quad (A.8)$$

$$\frac{\rho A'}{A} = S. \quad (A.9)$$

For negative energies, the only relevant quantity is S :

$$\frac{\rho W'}{W} = S. \quad (A.10)$$

The description of the F and G functions in terms of a phase ϕ and amplitude A forms the basis of the so-called phase-amplitude methods, which are useful for obtaining numerical evaluations when the argument ρ is large. According to (A.8) the penetration factor P is inversely proportional to A^2 and the shift factor S is

ρ times the logarithmic derivative of A . In terms of A and ϕ , the Wronskian (III, 2.12) is

$$A^2\phi' = 1. \quad (\text{A.11})$$

From Powell's recursion formulas¹³³ for F and G and their derivatives, one of us (R.G.T.) has obtained recursion formulas for S , P , and ϕ . They are:

$$P_l = a_l P_{l-1} / [(b_l - S_{l-1})^2 + P_{l-1}^2], \quad (\text{A.12})$$

$$S_l + b_l = a_l (b_l - S_{l-1}) / [(b_l - S_{l-1})^2 + P_{l-1}^2] \quad (\text{A.13})$$

$$\phi_l = \phi_{l-1} - \tan^{-1} [P_{l-1} / (b_l - S_{l-1})], \quad (\text{A.14})$$

where

$$a_l = \rho^2 + (\rho\eta/l)^2, \\ b_l = l + (\rho\eta/l).$$

Since the negative-energy Whittaker function is obtained to within a constant factor by making the substitutions $\rho \rightarrow i\rho$, $\eta \rightarrow -i\eta$ in the expressions for the positive-energy O function, the recursion formula (A.13) for S holds in this case if ρ^2 in the expression for a_l is replaced by $-\rho^2$. Errors will accumulate from the repeated use of (A.14) in the barrier region where ϕ and P are small. In this region $G \gg F$ and the approximation

$$F_l G_l \approx \rho / [(2l+1)^2 + x^2]^{\frac{1}{2}} \quad (\text{A.15})$$

may be valid, so that as an alternative to (A.14) one may use the approximation

$$\phi_l \approx F_l / G_l \approx P_l / [(2l+1)^2 + x^2]^{\frac{1}{2}}, \quad (\text{A.16})$$

where

$$x = (8\rho\eta)^{\frac{1}{2}}.$$

The above formulas and considerations also apply in the absence of a Coulomb field, when $\eta = 0$.

The quantity P is always > 0 and $\rightarrow 0$ as $\rho \ll 2\eta$ (energy $E \ll$ barrier height B) and $\rightarrow \rho$ when $\rho \gg 2\eta$ (i.e., $E \gg B$).

In contrast, the quantity S is always < 0 and $\rightarrow 0$ at $\rho \gg 2\eta$ (i.e., $E \gg B$) but remains finite at $\rho \ll 2\eta$ (see (A.24) below). It is not evident that S is always < 0 but one of us (R.G.T.) has proved this by using the relation,

$$\frac{1}{2P} = \frac{F^2 + G^2}{2\rho} \int_0^\infty dz e^{-2\rho z} Q(z),$$

where

$$Q(z) = \exp(2\eta \tan^{-1} z) (1+z^2)^{i\eta} \\ \times {}_2F_1(l+1+i\eta, -l+i\eta, 1; -z^2).$$

This leads to,

$$S \left(\frac{F^2 + G^2}{\rho} \right) = FF' + GG' = - \int_0^\infty z e^{-2\rho z} Q'(z) dz.$$

The negative nature of S follows from the positive nature of $Q'(z)$ which can be seen from the power series in z^2 of the ${}_2F_1$ function.

¹³³ J. L. Powell, *72*, 626 (1947).

We have not been able to prove that dS/dE is always positive or zero, but this appears to be true in most cases of practical interest [see (A.30) below]. In particular, it is true for negative energy channels [see (A.27)].

b. The JWKB Approximation for Coulomb Functions in the Barrier Region

In the barrier region the JWKB approximation may be considered, provided that the quantity $[\frac{1}{4}x^2 + l(l+1)]$ is large compared with both ρ^2 and 1. It has been found to be very accurate in alpha-particle decay applications. The JWKB penetration factor is given by

$$P_l \approx \rho / G_l^2 = \zeta_l \exp(-2C_l), \quad (\text{A.17})$$

where

$$\zeta_l = (L^2 + \frac{1}{4}x^2 - \rho^2)^{\frac{1}{2}}; \quad L = l + \frac{1}{2},$$

$$C_l = \eta \left[\frac{\pi}{2} + \sin^{-1} \frac{(\eta - \rho)}{(\eta^2 + L^2)^{\frac{1}{2}}} \right] - \zeta_l + L \log \frac{L\zeta_l + L^2 + \frac{1}{8}x^2}{\rho(\eta^2 + L^2)^{\frac{1}{2}}};$$

the \sin^{-1} lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. The Langer modification of¹³⁴ replacing l by $l + \frac{1}{2}$ has been used. The corresponding shift factor is given by

$$S_l \approx \rho G_l' / G_l = -\zeta_l + \frac{1}{2}\zeta_l^{-2} (\frac{1}{8}x^2 + L^2), \quad (\text{A.18})$$

and its rate of change with respect to energy can be obtained from

$$\left(\frac{\partial S}{\partial E} \right)_a \left(\frac{\hbar^2}{2Ma^2} \right) = (\partial S_l / \partial \rho^2)_x \\ \approx \frac{1}{2}\zeta_l^{-4} (\zeta_l^3 + \frac{1}{8}x^2 + L^2). \quad (\text{A.19})$$

In alpha-particle decay applications, the second term on the right side of (A.18) is small compared with the first so that

$$S_l \approx -\zeta_l. \quad (\text{A.20})$$

Our penetration factor P_l differs by the factor ζ_l from Bethe's¹ barrier penetrability P_l , as given by his Eq. (632a). The JWKB approximations to the phase ϕ_l and to the product $F_l G_l$ are given by (A.16) and (A.15), respectively, with the denominators in each case replaced by $2\zeta_l$. If the conditions for the validity of the JWKB approximation are satisfied, these distinctions are, of course, negligible, and (A.15) and (A.16) may themselves be considered as the JWKB formulas.

c. Expansions of Coulomb Functions in Terms of Bessel Functions of Imaginary Argument

The expansions of the Coulomb functions in terms of the Bessel functions of imaginary argument are useful for applications in light nuclei at low energies.^{135,97} Such expansions converge rapidly if the parameter η is large compared with both unity and the parameter x . We list here only a few special cases of these expansions which

¹³⁴ R. E. Langer, *Phys. Rev.* **51**, 669 (1937).

¹³⁵ J. G. Beckerley, *Phys. Rev.* **67**, 11 (1945).

TABLE A. 1. Quantities characterizing positive energy neutron wave functions in terms of the argument $z = \rho^2$.

| l | $U_l \rho^l$ | $V_l \rho^{l-1}$ | $A_l z^l$ | $A_l S_l z^l$ | $(A_l z^l)^2 \frac{dS}{dz} = (A_l z^l)^2 \left(\frac{\hbar^2}{2M a^2} \right) \left(\frac{dS}{dE} \right)$ |
|-----|--------------|------------------|--------------------|--------------------|--|
| 0 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | $1+z$ | -1 | 1 |
| 2 | $3-z$ | 3 | $9+3z+z^2$ | $-3(6+z^2)$ | $3(z^2+12z+9)$ |
| 3 | $15-6z$ | $15-z$ | $225+45z+6z^2+z^3$ | $-3(225+30z+2z^2)$ | $3(2z^4+60z^3+765z^2+1800z+3375)$ |

are useful in the consideration of nuclear reactions near thresholds.

The irregular function G is expansible as

$$G_l = -x B_l(\eta) C_0^{-1} K_{2l+1}(x) Q_l(x, \eta), \quad (A.21)$$

where

$$\begin{aligned}
 B_l &= [1 + \eta^{-2}] [1 + \eta^{-2}(l-1)^2] \dots [1 + \eta^{-2}]^{\frac{1}{2}} \\
 &= 1 + \frac{1}{12} \eta^{-2} l(l+1)(2l+1) + \dots \\
 C_0 &= [2\pi\eta / (\exp(2\pi\eta) - 1)]^{\frac{1}{2}}, \\
 Q_l(x, \eta) &= 1 + \left(\frac{x}{2}\right)^4 \frac{\eta - 2}{24(l+1)} \\
 &\quad \times \left\{ \frac{K_{2l+3}(x)}{K_{2l+1}(x)} \left[1 - 2 \left(\frac{2}{x}\right)^2 l(l+1) \right] - 1 \right\} + \dots
 \end{aligned} \quad (A.22)$$

so that in the barrier region where $G \gg F$, the penetration factor is

$$P_l \approx \rho / G_l^2 = \left(\frac{1}{4}\pi\right) \exp(-2\pi\eta) / K_{2l+1}(x)^2 B_l(\eta)^2 Q_l(x, \eta)^2. \quad (A.23)$$

One can also obtain for $S_l \approx (\rho G'_l / G_l)$,

$$\lim_{\eta \rightarrow \infty} (\rho G'_l / G_l) = -\frac{x K_{2l}(x)}{2 K_{2l+1}(x)} - l \quad (A.24)$$

as well as the correction term for nonzero energies, which we list only for $l=0$,

$$\begin{aligned}
 \lim_{\eta \rightarrow \infty} \left(\frac{\partial(\rho G'/G)}{\partial \eta^{-2}} \right)_x &= \frac{1}{24} \left(\frac{x}{2}\right)^4 \\
 &\quad \times \left[\frac{K_3(x) - 3K_1(x) - \frac{1}{2}xK_0(x)}{K_1(x)} \right. \\
 &\quad \left. + \left(\frac{x}{2}\right) \frac{K_3(x)K_0(x)}{K_1(x)^2} \right]. \quad (A.25)
 \end{aligned}$$

$K_\nu(x)$ is the "irregular" Bessel function of imaginary argument, as defined by Watson.¹³⁶ The leading term in the regular function expansion is

$$F_l = (\pi x)^{\frac{1}{2}} \exp(-\pi\eta) I_{2l+1}(\beta), \quad (A.26)$$

¹³⁶ G. N. Watson, *The Theory of Bessel Functions* (Cambridge University Press, New York, 1948).

and the leading term in the expansion for the product (FG) is the approximation (A.15), where I_ν is the "regular" Bessel function of imaginary argument, as defined by Watson.

d. Neutron Functions

For neutrons, the positive energy solutions F and G reduce to Bessel functions as exhibited by (A.3a, b). It is convenient to write F and G as

$$\begin{aligned}
 \rho^{-l} F_l &= U_l \sin \rho - \rho V_l \cos \rho \\
 \rho^{-l} G_l &= U_l \cos \rho + \rho V_l \sin \rho,
 \end{aligned} \quad (A.27)$$

where U_l and V_l are polynomials in $\rho^2 (= z, \text{ say})$. In Table A.1, we list the quantities $U, V, A_l z^l, A_l z^l S_l$, and $(A_l z^l)^2 (dS_l/dz)$ —all of which are polynomials in z . The phase ϕ_l of (A.7) is such that

$$\tan(\rho - \phi_l) = V_l / U_l. \quad (A.28)$$

In Table A.2, the quantities W, S , and $[dS/d(\rho^2)]$ are listed for negative energies. The expressions for S in Table A.2 are obtainable by substituting $\rho \rightarrow i\rho$ in the expressions for $L = S + iP$ taken from Table A.1.

From Tables A.1 and A.2, one sees that dS/dE is discontinuous at $E=0$ in the case $l=0$ but not otherwise.

e. Energy Derivative of S , the Shift Factor

Let us first consider negative energy channels for which, by (A.10), $S = \rho W' / W$. From the special case (V, 1.5) of the Green's theorem relation we have

$$\begin{aligned}
 \left(\frac{\partial S}{\partial E} \right)_a \left(\frac{\hbar^2}{2M a^2} \right) &= - \left(\frac{\partial S}{\partial(\rho^2)} \right)_{\rho, \eta} \\
 &= \frac{1}{\rho} [W(\rho)]^{-2} \int_{\rho}^{\infty} [W(\rho')]^2 d\rho', \quad (A.29)
 \end{aligned}$$

where we have abbreviated $W(-\eta, l + \frac{1}{2}, 2\rho)$ to $W(\rho)$. It follows from this and the definitions (III, 4.7a) of reduced width γ^2 and (IV, 1.17c) of level shift Δ that:

$$\begin{aligned}
 \frac{d\Delta}{dE} &= -\gamma^2 \left(\frac{dS}{dE} \right)_a \\
 &= +2\theta^2 \left(\frac{\partial S}{\partial(\rho^2)} \right)_{\rho, \eta} = - \int_{\rho}^{\infty} |\Psi|^2 d\tau \quad (A.30)
 \end{aligned}$$

TABLE A. 2. Quantities characterizing the negative-energy wave functions in terms of the argument $\rho = kr$.

| l | $W(0, l+\frac{1}{2}, 2\rho)$ | S | $\frac{dS}{d(\rho^2)} = \frac{\hbar^2}{2Ma^2} \frac{dS}{dE}$ |
|-----|---|--|---|
| 0 | $e^{-\rho}$ | $-\rho$ | $1/2\rho$ |
| 1 | $(1+1/\rho)e^{-\rho}$ | $-\rho - \frac{1}{1+\rho}$ | $\frac{2+\rho}{2(1+\rho)^2}$ |
| 2 | $(1+3/\rho+3/\rho^2)e^{-\rho}$ | $-\rho - \frac{3\rho+6}{\rho^2+3\rho+3}$ | $\frac{\rho^3+6\rho^2+12\rho+6}{2(\rho^2+3\rho+3)^2}$ |
| 3 | $(1+6/\rho+15/\rho^2+15/\rho^3)e^{-\rho}$ | $-\rho - \frac{6\rho^2+30\rho+45}{\rho^3+6\rho^2+15\rho+15}$ | $\frac{\rho^5+12\rho^4+60\rho^3+150\rho^2+180\rho+90}{2(\rho^3+6\rho^2+15\rho+15)^2}$ |

i.e., $-(d\Delta/dE)$ is the square integral in the bound channel of the wave function Ψ that is normalized to unity in the internal region. Consequently if Γ is any quantity depending upon the square of the wave function that is normalized in the internal region, $\Gamma[1-(d\Delta/dE)]^{-1}$ is equal to the corresponding quantity defined for a "renormalized" wave function whose normalization region is extended to include the bound channel.

On grounds of continuity, we expect that the same result is approximately true for positive energies and large channel barriers where the channel integral is extended to the classical turning-point. This may be seen by applying Green's theorem (V, 1.9) to the solution $(G+iF)$

$$\int_{\rho_1}^{\rho_2} (G^2+F^2)d\rho = \left[-\frac{1}{P} \left(E \frac{\partial S}{\partial E} \right) - \frac{2P}{\rho} \left\{ F \left(E \frac{\partial G}{\partial E} \right) - G \left(E \frac{\partial F}{\partial E} \right) \right\} \right]_{\rho_1}^{\rho_2} \quad (\text{A.31})$$

On choosing ρ_2 somewhat beyond the turning point ρ_0 , and using the fact that, as F and G become sinusoidal, the first term on the right $\rightarrow 0$, and the second term $\rightarrow \rho$, we find

$$\frac{\hbar^2}{2Ma^2} \frac{\partial S}{\partial E} \approx \frac{1}{\rho G^2(\rho)} \int_{\rho}^{\rho_0} G^2(\rho') d\rho', \quad (\text{A.32})$$

where we have used the fact that $F \ll 1 \ll G$ for $\rho \ll \rho_0$. It follows that the quantity $1-(d\Delta/dE)$, where Δ is for an open channel with a barrier, changes the normalization to the extent of including the channel (as far as the turning point) in the normalization volume.

f. Numerical Formulas and References to Graphs and Tables

The following numerical formulas are useful for the consideration of channel functions:

$$\begin{aligned} \rho &= ka = 0.21954M^{\frac{1}{2}}E^{\frac{1}{2}}a \\ \rho^2 &= 0.048196MEa^2, \\ \eta &= Z_1Z_2e^2M/\hbar k = 0.158052Z_1Z_2M^{\frac{1}{2}}E^{-\frac{1}{2}}, \\ 2\pi\eta &= 0.99307Z_1Z_2M^{\frac{1}{2}}E^{-\frac{1}{2}}, \\ x^2 &= 8\rho\eta = 4g^2 = 0.069396MZ_1Z_2a, \\ B &= Z_1Z_2e^2/a = 1.43987Z_1Z_2a^{-1} \text{ Mev}, \\ Z &= E/B = (\rho^2/2\rho\eta) = 0.69451EaZ_1^{-1}Z_2^{-1}, \\ k^2 &= 4.8196ME \text{ (barns)}^{-1}, \\ (4\pi/k^2) &= 2.6073M^{-1}E^{-1} \text{ barns}, \\ \hbar &= 0.65817 \times 10^{-21} \text{ Mev sec}, \\ (2Ma^2/\hbar) &= 3.17216 \times 10^{-23} Ma^2 \text{ sec}, \\ (\hbar^2/2Ma^2) &= 20.748(Ma^2)^{-1} \text{ Mev}, \end{aligned}$$

where

E is the energy in Mev of relative motion in the gravicentric system,
 a is the channel radius in units of 10^{-13} cm
 $M = M_1M_2/M_1+M_2$ is the reduced mass in units of the proton mass,
 B is the Coulomb barrier height.

Fairly extensive tables and graphs of Coulomb functions have been compiled. The ones that are most useful for nuclear reaction studies include:

Graphs

(1) R. F. Christy and R. Latter, *Revs. Modern Phys.* **20**, 185 (1948).

These graphs are for proton bombardment of individual nuclei and plot a quantity Γ which we write Γ_{CL} ; this is such that

$$\Gamma_{CL} = P \left(\frac{\hbar^2}{2Ma^2} \right)^{\frac{1}{2}}$$

$$\theta^2 \equiv \gamma^2 \left(\frac{\hbar^2}{Ma^2} \right)^{-1} = \frac{\Gamma}{\Gamma_{CL}} \left(\frac{Ma^2}{8\hbar^2} \right)^{\frac{1}{2}} = 0.0549 \left(\frac{M^{\frac{1}{2}}a\Gamma}{\Gamma_{CL}} \right),$$

where Γ is the observed partial width of a level in Mev and a, M are as in the foregoing.

(2) Sharp, Gove, and Paul, Chalk River Report, TPI-70 (1953) unpublished.

These graphs are of the functions A_i, ϕ_i , and other functions in terms of ρ and η .

Tables

(1) Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, *Revs. Modern Phys.* **23**, 147 (1951).

These tables give A, ϕ, F, G , etc., for values of $\rho/2\eta$

from unity down to the point where the JWKB approximation becomes usable.

(2) National Bureau of Standards Tables of Coulomb Wave Functions, Vol. 1. Applied Mathematics Series, No. 17 (1952).

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Errata: Study of Nuclear Structure by Electromagnetic Excitation with Accelerated Ions

[*Revs. Modern Phys.* **28**, 432-542 (1956)]

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IT has been pointed out by Mr. J. Bro-Jørgensen that the estimates of the cross sections for double excitation in Sec. II D.3 contain a numerical error which has led to an overestimate of these cross sections.

Thus, the coefficient in (II D.18) should be reduced from 0.240 to 0.0218, while the coefficient in the third term of (II D.20) should read 0.12 instead of 0.4. As a consequence of these changes, the numerical factor in (II D.19) is reduced from 2.1 to 0.19, and the cross section quoted in the third line of the first column of page 532 is reduced from 50 to 5 mb.

The qualitative estimate (II D.17) now considerably exceeds that obtained by quantitative evaluation, but the difference can be understood in terms of the dependence of the excitation probability on the scattering angle, which was neglected in the order of magnitude estimate (II D.17).

We also wish to point out that in the last column of Table II.11, the factors $1/(1+\delta^2)$ in the $A_4(2)$ coefficients should be replaced by $\delta^2/(1+\delta^2)$.