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REVIEWS OF MODERN PHYSICS

ELECTRICAL DISCHARGES IN GASES

PART II. FUNDAMENTAL PHENOMENA IN ELECTRICAL DISCHARGES

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Chapter I. Introduction

PROGRESS in physics during the last 20 to 30 years has been characterized by the remarkable advance in our knowledge of electrons, ions, atoms and molecules, *as individuals*. A summary of this knowledge, in so far as it will aid in the understanding of electric discharges in vacuum and in gases, has been given in Part I, under the title of a "Survey of Fundamental Processes."

We now wish to consider how these *processes*, characteristic for the most part of individual electrons, ions, or atoms, cooperate to determine the *phenomena* of electric discharges. We shall have to deal primarily with the *collective behavior* of these charged and uncharged particles. This field of study in recent years has not received attention comparable to that devoted to the individual particle. The time seems now ripe to apply all this new knowledge in a systematic manner to a study of discharges.

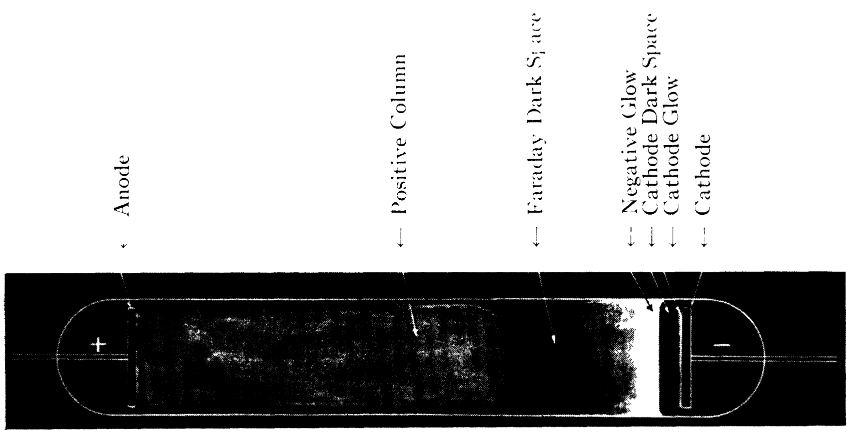

Fig. 34. Typical glow discharge in a gas at about 1 mm pressure.

Long prior to the beginning of the present century, certain types of electric discharge had been very extensively investigated. The typical phenomena that had been most frequently observed were those produced when a current was passed between two disk-shaped electrodes placed at some distance apart along the axis of a tube containing gas at a given pressure. The general effects of altering the pressure or the distance between the electrodes were well known.

Fig. 34 illustrates a typical discharge of this kind. Close to the surface of the cathode a glow, called the cathode glow, is observed. Beyond this is the cathode or Crookes' dark space. Then comes the negative glow which is usually of considerable intensity. Passing in the direction toward the anode, the intensity of this glow gradually decreases and becomes a second dark space, called the Faraday dark space, this usually being several times wider than the cathode dark space. Then comes the positive column which begins sharply at a definite position called the "head of the positive column." This surface of demarkation is convex on the side toward the cathode. In most cases the positive column is of uniform intensity all the way to the anode. Sometimes, however, it is broken up into striations, which appear to consist of alternations of Faraday dark spaces and short sections of positive column. Close to the anode, especially if this is of small size, there may be an anode glow.

Typical phenomena such as those illustrated in Fig. 34 are usually observed most readily at gas pressures in the neighborhood of one millimeter of mercury. At any given pressure the positions of the negative glow, the Faraday dark space, and the head of the positive column are fixed with reference to the cathode. Thus, for example, if the anode is moved, these positions do not change, whereas, if the cathode is moved, these boundaries move with it. As the distance between anode and cathode decreases, the anode may reach the head of the positive column so that the positive column disappears. In a similar way, the anode can be moved through the Faraday dark space and even into the cathode dark space. If the pressure is lowered, these distances from the cathode all increase approximately inversely proportional to the pressure. Thus, with fixed distances between the electrodes, on lowering the pressure, the cathode dark space expands until it reaches the anode. The discharge then becomes one of a type studied particularly by Sir William Crookes. It was the study of such Crookes' tubes by Roentgen in 1895 that led to the discovery of x-rays.

At high pressures, the cathode dark space and Faraday dark space move so close to the cathode that they become practically invisible and the whole tube is thus filled with the positive column. Gradually, with increasing pressure, the positive column detaches itself from the walls of the tube and becomes arc-like in character.

Discharges of the kind that we have just studied are usually referred to as glow discharges. Many other types of discharge have been observed, for example, spark discharges, arcs between carbon or metallic electrodes at atmospheric pressure, corona discharges and the low current discharges observed when gases are rendered conducting by x-rays or radio-active materials.

Instead of trying to explain all these apparently complicated phenomena of electric discharges in gases, we plan to approach these problems through a consideration of the simplest types of phenomena involved in these discharges. We propose to postulate the existence of certain simple conditions and draw conclusions regarding the types of phenomena that should result. In other words, we shall construct "models" of simplified types of discharge that might conceivably exist and later shall attempt to explain the more complicated phenomena commonly observed in gaseous discharges in terms of the elementary phenomena with which we have then become familiar.

For example, we propose to deal first with electric discharges in very high vacuum where the current is carried by particles of one sign only (unipolar discharges) and where the carriers of the electric current pass across the vacuous space from one electrode (emitter) to another electrode (collector) without suffering loss of energy or change in momentum by collisions with gas molecules. We shall therefore not need to consider the generation of ions and electrons by collisions with gas molecules, nor the recombination of ions and electrons.

In analyzing these high vacuum discharges, we shall first deal with current densities so low that the number of charged particles present at any time in the space between the electrodes is so small that the electric field produced by them is negligible, and the potential distribution is practically the same as if no space charges were present, involving only a solution of Laplace's equation. With higher current densities, the number of charged particles which carry the current becomes so great that the field produced by them can no longer be ignored and the potential distribution is then to be determined by a solution of Poisson's equation. We shall see that currents that flow under such conditions depend essentially on the presence of space charge and the various "space-charge-equations" that we shall obtain will prove to be of fundamental importance in the understanding of discharges of many types.

After dealing with the phenomena in high vacuum, we shall then proceed to a consideration of the fundamental phenomena occurring in the presence of very low pressures of gas, pressures sufficient to cause the generation of ions and electrons in space, but yet so low that the motions of the resulting carriers are not appreciably interfered with by the presence of gas. We shall see that under these conditions the electrons and ions which are generated in the space by electron impacts recombine on the walls of the tube and at the electrodes (but not in the space).

Further consideration of the effects produced by the generation of ions and electrons in space will show that the potential distribution becomes such that a *potential maximum* develops in which low speed electrons are trapped. The accumulation of the trapped electrons causes a region to appear in which the space charge of the ions is neutralized by the electrons. We have named this part of the discharge the *plasma*. Near the electrodes and near the walls there are still regions where there are large space charges and where the conditions are still essentially those of a unipolar discharge in high vacuum. These regions of large space charge and intense electric fields are called the *sheaths*. They usually surround the electrodes and cover the glass walls. We shall then study in considerable detail the properties of the plasma and of the sheaths.

At still higher pressures, collisions of the electrons and ions with gas molecules profoundly modify their movements so that alterations are needed in the space charge equations and in the equations which determine the distribution of potential within the plasma. Recombination of ions and electrons may then also occur in the body of the gas and lead to important changes in the conditions.

In experimental studies of gaseous discharges at low pressures, it is important to make measurements of the concentrations of electrons and ions and of their velocity distribution, etc. For this purpose various types of *collectors* can be used and considerable space will be devoted to the theory of such collectors and to the ways in which they can be used in the studies of gaseous discharges.

After these investigations of the fundamental phenomena, we shall then be in a position to study their applications in explaining the ordinary types of electric discharge. For example, we shall attempt to explain the properties of the positive column, the Faraday dark space, the cathode glow, and the cathode dark space, and various phenomena near the anode. Some of the concepts developed may be of use in understanding other types of discharge, such as corona discharges, high pressure arcs and low current discharges at high pressures, but we shall not attempt to give any systematic treatment of these applications.

Chapter II. The Electrostatic Fields Due to Electrodes and to Uniform Space Charges

The electric discharges of most interest to us are usually those taking place in vessels of glass or metal which contain two or more metallic electrodes maintained at definite potentials or between which definite currents are passed. The phenomena of the discharge by which the flow of current is determined usually depend on the distribution of potential in the space between the electrodes and at or near the glass walls. With practically all discharges in which currents as large as a few milliamperes pass, the distribution of potential is essentially dependent on the space charges of the moving ions and electrons. However, when the currents are made sufficiently small, the effects of the space charges become negligible compared to the effects of the charges on the electrodes, and then Laplace's equation can at least theoretically be used to calculate the potential distribution. In any case, before considering the more complicated problem of the effect of space charge on the potential distribution, we should clearly understand the simpler problem of the electrostatic field produced by the charges on the electrodes and on the tube walls.

Therefore in the following pages we shall consider solutions of Laplace's and Poisson's equations for electrodes of various shapes, particularly of those types that are met with in the experimental apparatus used to study electric discharges and in the practical devices which utilize such discharges.

It will be shown also that these solutions are applicable to problems of the diffusion of ions, electrons, or atoms, or to the similar problems of heat conduction (diffusion of hot molecules).

If the potentials or the potential gradients over the surfaces of all electrodes and over the walls of the containing vessel are known, together with the distribution of space charge, the potential distribution is theoretically determined by Poisson's equation. If space charges are absent, the second member of the equation becomes zero and the solution is then given by Laplace's equation.

Forms of Poisson's equation.

For the various coordinate systems in most common use, Poisson's equation takes the following forms, V being the potential at any point and ρ the space charge density (e.s. units·cm^{-3}).

Rectangular coordinates (x, y, z)

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho. \tag{134}$$

Cylindrical Coordinates, i.e., polar coordinates in the x, y, plane, together with the unmodified z ordinate ($x = r\cos\theta$, $y = r\sin\theta$).

$$\Delta V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho. \tag{135}$$

Spherical polar coordinates. r is radius vector from the origin, ϕ the azimuth or longitude, i.e., the angle between the meridian plane and the x, z plane, θ the zenith distance or colatitude, i.e., the angle between r and the z axis. Thus $x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$.

$$\begin{aligned}\Delta V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)\\ + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = -4\pi\rho.\end{aligned} \tag{136}$$

Methods of calculating the fields.

Although theoretically any one of these equations, together with the appropriate boundary conditions, is sufficient to determine the complete potential distribution, actually no methods are known for the general solution of these equations. However, if through some symmetry requirement or other imposed condition, one of the three independent variables can be eliminated, a general solution may be obtained, although often with great difficulty.

For practical reasons, therefore, in considering electrostatic fields we must restrict ourselves to simplified cases which adapt themselves to mathematical solution. The simplest of all are those in which only one coordinate r, is involved, for example, the field distribution between parallel planes, or concentric cylinders, or spheres. In such cases, the three forms of Poisson's equation given above, Eqs. (134), (135) and (136) become

$$\Delta V = \frac{d^2 V}{dr^2} + \frac{\kappa}{r}\frac{dV}{dr} = -4\pi\rho \tag{137}$$

where κ is a parameter that is 0 for parallel planes, 1 for axial symmetry (cylinders), or 2 for spherical symmetry.

When the charges at the boundaries and the space charge distribution are known, instead of using Poisson's equation, it is also possible to calculate the potential distribution by summation of the potentials due to the separate elements of the charge. For example, any charge, e, produces at a distance, r, the potential e/r, so that at every point in space the potential V is

$$V = \int de/r. \tag{138}$$

This result is deducible from Coulomb's inverse square law for electric charges and is also derivable from Poisson's equation.

An alternative method of calculating field strengths and potential distributions, which is often convenient in simple problems, is that involving the conception of Faraday tubes or lines of force. A unit electric field is one in which the flux of force may be represented by one line of force per unit area. Thus, on a point charge, e, $4\pi e$ lines of force must terminate. In cases for which, owing to symmetry requirements, we know something of the distribution of the lines of force, it is often easy to determine the field distribution and then to obtain the potential as the integral of the field E along the line of force

$$V = \int E\,dx. \tag{139}$$

When space charges are absent, it may readily be seen by Poisson's equation (134), that there can never be an absolute maximum or minimum in space. Such a maximum can only occur in a region containing a positive space charge; that is, an excess of positive electricity, while a minimum of potential requires the presence of a negative space charge.

Since $4\pi e$ lines of force emanate from each charge, e, the electric field, E_0, close to any conducting surface is given by

$$E_0 = -4\pi\sigma \tag{140}$$

where σ is the charge per unit area.

Capacitance.

In the case of two electrodes, one of which completely surrounds the other so that all the lines of force emanating from the second pass to the first, the total charge on the inner surface of the outer electrode is the same as the total charge on the inner electrode. Thus, if C is the capacitance between the electrodes, and if V is the difference of potential between them, the charge on each electrode will be CV, and thus,

$$C = \frac{1}{V}\int \sigma dA \tag{141}$$

where dA is an element of the surface of either electrode and the integration is carried over the total surface of that electrode. In those frequent cases for which, because of symmetry, σ is known to be uniform over the surface of the electrode, we find, by combining Eqs. (140) and (141) that the electric field at the surface of either electrode is given by

$$E_0 = 4\pi CV/A_0 \tag{142}$$

where A_0 is the total surface area of that electrode.

In these equations C is in electrostatic units and is thus measured in centimeters. To convert these capacitances to micro-microfarads, they need only to be multiplied by 1.11.

Mechanical force.

Every surface on which lines of force terminate is acted on by a mechanical force which may be expressed as a negative pressure acting on the surface. The magnitude of the pressure is readily found to be

$$p = -2\pi\sigma^2 = -E_0^2/8\pi \tag{143}$$

or, if the electric field, E_0, is expressed in volts per centimeter instead of electrostatic units, the pressure is

$$p = 4.42 \times 10^{-7}E_0^2 \text{ baryes,} \tag{144}$$

the barye being the c.g.s. unit of pressure, (one dyne·cm^{-2}) which is almost exactly 10^{-6} atmosphere. A pressure corresponding to one millimeter of mercury is therefore reached when the electric field is 55,000 volts per centimeter.

Fields between planes, cylinders or spheres.

For *parallel planes* we find readily

$$\begin{aligned} E &= V_1/a \\ V &= V_1 x/a \\ \sigma &= V_1/4\pi a \end{aligned} \tag{145}$$

where V_1 is the potential difference between the two planes which are separated by the distance a, and V is the potential at an intermediate point x. The capacitance in centimeters per unit area is thus $1/4\pi a$.

For the case of *coaxial cylinders*,

$$\begin{aligned} E &= \frac{V_1}{r}\,\frac{1}{\log(r_1/r_0)} = -\frac{2e_0}{r} \\ V &= V_1\frac{\log(r/r_0)}{\log(r_1/r_0)} = -2e_0\log(r/r_0) \end{aligned} \tag{146}$$

where r_1 is the radius of the outer cylinder and r_0 is the radius of the inner cylinder, V_1 is the potential of the outer cylinder, that of the inner one being taken as zero, and e_0 is the electric charge *per unit length* on the *inner* cylinder (i.e., $2\pi r_0\sigma_0$). The capacitance per unit length is thus equal to $1/[2\log(r_1/r_0)]$.

For *concentric spheres* we have,

$$E = \frac{V_1 r_0 r_1}{r^2(r_1 - r_0)} = -\frac{e_0}{r^2}$$

$$V = V_1 \frac{r_1}{r} \frac{r - r_0}{r_1 - r_0} = e_0 \left(\frac{1}{r} - \frac{1}{r_0} \right) \tag{147}$$

$$C = \frac{r_0 r_1}{r_1 - r_0}$$

where r_0 and r_1 are the radii of the inner and outer spheres, V_1 is the potential of the outer sphere, that of the inner being taken as zero, e_0 is the *total* electric charge on the *inner* sphere (i.e., $4\pi r_0^2 \sigma_0$), and C is the capacitance between the spheres.

Relation of diffusion and heat conduction to potential distribution.

Poisson's equation is applicable also to problems of diffusion and of the conduction of heat in all cases in which steady conditions exist; that is, where the concentration of the diffusing substance, or where the temperature in a heat conduction problem, does not vary with time. Most treatises on diffusion and heat conduction deal mainly with the transient phenomena preceding the reaching of the steady state and these treatments are therefore of comparatively little interest to us in connection with gaseous discharges.

Diffusion of ions or electrons, or the ambipolar diffusion of both electrons and ions,[267] and the diffusion of metastable atoms are important phenomena in studies of gaseous discharges. Fortunately, the solutions of the cases of most interest to us are directly derivable from the corresponding solutions for the electrostatic fields. The diffusion coefficient, D, is defined by

$$q_x = D dn/dx \tag{148}$$

where n is the number of diffusing particles per unit volume at any point x, and q_x is the number of particles per unit area per second that diffuse across a plane perpendicular to the x axis. The diffusion coefficient has the dimensions $cm^2 \cdot sec^{-1}$. The methods of calculating the diffusion coefficients for electrons, ions, or atoms are given on pages 215 to 217 of Part I.

The coefficient of heat conduction, K, may be defined by

$$h_x = K dT/dx \tag{149}$$

where T is the temperature and h is the energy flow per unit area per second across a plane perpendicular to the x axis. The heat conductivity, K, may be expressed in c.g.s. units, in which case it is measured in $ergs \cdot sec^{-1} \cdot deg^{-1} \cdot cm^{-1}$, or it may be measured in watts $deg^{-1} \cdot cm^{-1}$, or in $calories \cdot sec^{-1} \cdot deg^{-1} \cdot cm^{-1}$. These are readily converted into one another if it is kept in mind that one watt $= 10^7$ ergs sec^{-1} and one calorie is 4.19 watt·sec.

If the diffusing substance or the heat passes directly through the space from one electrode to another and does not disappear or is not generated within the space, that is, if there are no sources or sinks within the space, then Laplace's equations (134), (135), (136) with $\rho = 0$ apply if we merely replace

[267] W. Schottky and J. v. Issendorff, Zeits. f. Physik **31**, 180 (1925).

V by n in the case of the diffusion problem or by T in the heat conduction problem.

In case the diffusing substance is formed or disappears in the space between the electrodes, we may let S represent the number of molecules* generated per unit volume per second at any point x, y, z. In case the substance disappears, S has a negative value. Poisson's equation then takes the form

$$\Delta n = -S/D \tag{150}$$

where Δn, in any of the systems of coordinates that we have used, may be obtained from equations (134), (135) and (136) by merely substituting n for V.

In a similar way, in problems of heat conduction, Poisson's equation becomes

$$\Delta T = -H/K \tag{151}$$

where H is the rate of generation of heat per unit volume per second, expressed in terms of the same energy units as the heat conduction, K.

Boundary conditions. The diffusion problems of most interest to us are those in which the diffusing substance originates at one electrode or throughout a volume of the gas and diffuses to, and is destroyed by, the other electrodes with which it comes in contact, or by the walls of the vessel. For example, electrons may be given off by one electrode and taken up by another, or they may be produced by ionization within the gas and diffuse to the walls. Therefore, the boundary conditions usually require that n shall be zero on all bounding surfaces except those at which the particles originate. The boundary conditions in problems of heat conduction are usually ascertained without difficulty.

In the heat conduction problems we are usually concerned with the temperature distribution which is entirely analogous to the potential distribution in the electrostatic problem and thus offers no difficulty.

In the diffusion problems that we meet in gaseous discharges, we usually wish to know the rate of flow of the diffusing substance to the walls or to some electrode. In other words, we wish to determine the quantity q, as given by Eq. (148), so that we are concerned with the concentration gradients at the surfaces of the electrodes rather than the concentration distribution within the space. We may write Eq. (141) in the following form,

$$dC/dA = \sigma/V_1 \tag{152}$$

where dC/dA represents the capacitance contribution per unit area over the surface of either one of the two electrodes, V_1 being the potential difference between the electrodes. If we have solved the problem of the potential distribution between the two electrodes with the boundary conditions $V=0$ at one electrode and V_1 at the other, then we have also solved the corresponding

* In such cases as these (as in Part I, footnote on page 125) we shall use molecule to include atoms, ions, or even electrons.

problem for diffusion from one of these electrodes to the other, the concentration being 0 at the first electrode and n_1 at the other. Thus, at every point in space,

$$\frac{1}{V_1}\frac{dV}{dx} = \frac{1}{n_1}\frac{dn}{dx}. \tag{153}$$

By combining the equations (148), (153), (140) and (152), we obtain

$$q_0 = 4\pi D n_1 dC/dA = 4\pi D n_1 \sigma / V_1 \tag{154}$$

where q_0 is the rate of arrival (molecules·sec^{-1}·cm^{-2}) of the diffusing substance at the electrode 0 when the concentration over the surface of the electrode 1 is n_1. If we let Q_0 be the total amount of the diffusing substance which reaches the whole surface of electrode 0 (molecules·sec^{-1}), then we have,

$$Q_0 = 4\pi D n_1 C \tag{155}$$

where C is the total capacitance between electrodes 0 and 1. This solution is strictly applicable only in cases in which the analogous electric problem is one where all the lines of force which emanate from one electrode pass to the other. Since, however, the walls of the vessel and usually all electrodes, except one, destroy the diffusing substance, we may consider that all of these electrodes and the walls are at zero potential in the electric problem and therefore constitute a single electrode.

The most important heat conduction problems that we may wish to consider in gaseous discharges are those in which the heat is generated in the body of the gas so that we need to apply Poisson's equation. In such cases we must also know the heat conductivity, K. If the heat is carried by the gas itself, we can determine the heat conductivity readily enough from available published data. We may wish, however, to calculate the heat that is conducted by the electrons within the gas. In such cases we may calculate K by considering that heat conduction involves essentially a diffusion of hot molecules among colder molecules. We have for the coefficient of self diffusion in a gas[268]

$$D = \tfrac{1}{3}\lambda v \tag{156}$$

Jeans shows that the viscosity of a gas is given by

$$\eta = \tfrac{1}{3}\lambda v \rho \tag{157}$$

so that

$$\eta = D\rho \tag{158}$$

where ρ, the density of the gas, is equal to nm, m being the mass of the molecule. Jeans also shows (page 318) that the heat conductivity of a monatomic gas is given by

$$K = 2.5\eta C_v \tag{159}$$

[268] Jean's Dynamical Theory of Gases, Second Edition, Cambridge 1916. p. 326. Eq. (98) on page 215, Part I, also gives this if $\lambda_1 = \lambda_2$ and $c_1 = c_2$.

where C_v, which is the specific heat of a gas at constant volume per unit mass, is equal to $3k/2m$, k being the Boltzmann constant. For a diatomic gas, the coefficient in Eq. (159) should be 1.9 instead of 2.5, and the specific heat is then $5k/2m$. We thus find for a monatomic gas,

$$K = 3.75kDn \tag{160}$$

and in the case of a diatomic gas, the coefficient should be changed to 4.75.

Although we have derived this result by a consideration of the viscosity, and no particular meaning attaches to this conception for electrons in an inert gas, the final equation that we have obtained is applicable for gas mixtures, or even for the heat conduction by a small number of electrons among a large number of gas molecules. In this case D is the diffusion coefficient of the electrons through the gas and n is the number of electrons per unit volume in the gas, and K corresponds to the heat carried by the electrons. We see that this conclusion is justified if we consider that we are dealing essentially with a problem of the diffusion of rapidly moving electrons among slower electrons.

Diffusion and heat conduction problems can only be solved in the foregoing manner if the mean free path λ of the molecules is small compared to the distances between the electrodes. When the pressure is so low that λ, although less than the dimensions of the apparatus, is comparable with them, Poisson's equation may be applied to diffusion or conduction in all the portions of the space within the device which lie at distances greater than λ from the walls or electrodes. Within a layer of thickness λ from the boundaries the conditions of free molecular flow obtain according to which the transfer of molecules across any boundary is given in terms of the average velocity, v, by

$$q = \tfrac{1}{4}n_\lambda v = n_\lambda\left(\frac{kT}{2\pi m}\right)^{1/2} \tag{161}$$

where n_λ is the number of molecules$\cdot$cm^{-3} at a distance λ from the boundary. As a result of this effect there is at any boundary a discontinuity in concentration or temperature of magnitude approximately equal to the normal gradient multiplied by λ.

Even if λ is very small compared with the dimensions of the apparatus, this discontinuity may be important when we deal with diffusion or conduction from wires or filaments so fine that their diameter is less than λ. A very full discussion of this effect and of the methods of calculating the concentration drop and the temperature drop in such cases as these was given by Langmuir[269] in his studies of the dissociation of hydrogen into atoms by hot filaments.

Two-dimensional cases.

Cylinders and wires. In experimental investigations of electron discharges or gaseous discharges, filaments are usually used as sources of electrons. Where quantitative knowledge of the electric field distribution is needed, it is

[269] I. Langmuir, Jr. Amer. Chem. Soc. **37**, p. 419 to 428 (1915).

desirable to use a straight filament at the axis of a cylindrical anode. Other electrodes are then preferably made in the form of one or more straight filaments parallel to the axis of the cylinder. The electrostatic field distribution is readily calculated in all such cases.

Field between parallel wires. Let us consider an infinitely long straight line on which there is a charge, e_1, per unit length. According to Eq. (146), the potential at any point at a distance r_1 from this line is

$$V = -2e_1 \log r_1 + \text{const.} \tag{162}$$

If this wire were alone in space, the total energy per unit length would be infinite. However, if there is another parallel infinite wire having charge $-e_1$ per unit length (or any set of such wires having the total charge $-e_1$ per unit length) the energy and the potentials become finite. We must restrict ourselves therefore to cases where the total charge per unit length for all the conductors is zero.

The potential at any point P due to two parallel lines having the charges e_1 and $-e_1$ respectively is thus $2e_1 \log (r_2/r_1)+\text{const.}$, where r_1 and r_2 are the distances from P to each of the two lines.

If we consider two fine wires of radius a_1 and a_2 respectively, these radii being small compared to the distance $2d$ between the wires, and if V_1 and V_2 are the potentials of the wires, we may derive an expression for V_1 by putting $r_2=2d$ and $r_1=a_1$, and a similar expression for V_2, and can thus eliminate the constant, getting

$$V_1 - V_2 = 2e_1 \log (4d^2/a_1a_2) \tag{163}$$

from which by Eq. (141) the capacitance per unit length between the wires C_1 is

$$C_1 = 1/[2 \log (4d^2/a_1a_2)]. \tag{164}$$

The potential at any point P is then given by

$$\frac{V - V_1}{V_2 - V_1} = \frac{\log (2r_1d/r_2a_1)}{\log (4d^2/a_1a_2)} \tag{165}$$

and the radial electric field intensities at the surfaces of the wires, in agreement with Eq. (146), are $-2e_1/a_1$ and $+2e_1/a_2$ respectively.

Electric images of wires near plane electrodes. From considerations of symmetry and also by Eq. (165), it is evident when $a_1=a_2$ and $V_2=-V_1$ that the potential is 0 everywhere over a plane that lies mid-way between the two wires and is perpendicular to the plane that includes the wires. Thus if this 0-potential plane is replaced by a conducting surface at the distance d from the wire 1, the potential distribution is still given by Eq. (165), and the charge e_1 on the wire remains unchanged. The capacitance between the wire and the plane is then

$$C_1 = 1/[2 \log (2d/a_1)]. \tag{166}$$

The problem of calculating the potential distribution between a small wire having a charge e_1 per unit length, and a parallel conducting plane at a distance d from the wire, is thus solved by replacing the plane by a second line having a charge $-e_1$ per unit length at a distance $2d$, which may be said to represent the electric image of the wire in the plane.

Radial and transverse fields in two-dimensional problems. In the problem we have just considered, the electric field close to the small wire is almost wholly radial and its magnitude is given by Eq. (146). However, strictly speaking, there is also a transverse field of magnitude E_T resulting from the action of the charge $-e_1$ of the electric image which is at a distance $2d$ and thus by Eq. (146) the magnitude of the transverse field is,

$$E_T = e_1/d. \tag{167}$$

In other two-dimensional problems where there is more than one wire, the transverse field close to one wire can be calculated by summing the potentials due to charges on all the other wires (and their images, if need be), then finding the potential gradient of this combined potential field.

In problems involving the collection or emission of electrons from filaments, it is often important to know the potential distribution close to the surface of a wire, for this may determine whether electrons can escape from or be received by the wire. In such cases we cannot neglect the effect of the transverse field.

If we have an uncharged wire of radius a in a transverse field of magnitude E_x, the potential distribution around the wire is

$$V = E_x r(1 - a^2/r^2)\cos\theta = E_x x(1 - a^2/r^2) \tag{168}$$

where V is the potential at any point (r, θ) expressed in polar coordinates, the potential on the surface of the wire being 0. This result may readily be proved by substitution into Eq. (135).

An *uncharged* wire thus produces only a local disturbance in the potential distribution resulting from the more distant electrodes. The potential of the wire itself is, in fact, the same as the "space potential" which we may define as that which would exist, after the removal of the wire, along the line which had been occupied by the axis of the wire.

If, however, there is a charge, e_1, (per unit length) on the wire, the potential of the wire will be raised above the space potential by an amount

$$V_s = e_1/C_1 \tag{169}$$

where C_1 is the capacitance per unit length of the wire with respect to all the other electrodes connected together. According to Eqs. (146) there will then be around the wire a radial field, $-2e_1/r$, whose potential is $-2e_1 \log (r/a)$. Thus the potential distribution around the wire is

$$V = V_s[1 - 2C_1 \log (r/a)] + Er(1 - a^2/r^2)\cos\theta \tag{170}$$

the potential being referred to the space potential at $r=0$.

If V_s is made sufficiently great, either positive or negative, there will be a "saddle" in the potential distribution about the wire; that is, at a certain distance r_M from the wire there will be a line parallel to the wire along which $dV/dr=0$ and $dV/rd\theta=0$. Along this radius and its extension there is a minimum in the potential if V_s is positive, although along a line perpendicular to the radius (tangentially) the potential will be a maximum. The condition for the occurrence of the saddle is

$$|V_s| > |Ea/C_1| \cdot \tag{171}$$

The distance r_M may be found from

$$\tfrac{1}{2}(\lambda + \lambda^{-1}) = C_1 V_s/Ea \tag{172}$$

where

$$\lambda = r_M/a. \tag{173}$$

The potential V_M at the saddle is determined by

$$(V_M - V_s)/Ea = (\lambda - \lambda^{-1}) - (\lambda + \lambda^{-1}) \log \lambda. \tag{174}$$

Choosing a coordinate system in which the axis of the wire is the Z-axis and the X-axis is in the direction of the transverse field, then the values of $\partial^2 V/\partial x^2$ and $\partial^2 V/\partial y^2$ at the saddle are given by

$$-\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} = \frac{E}{a} \cdot \frac{1 - \lambda^2}{\lambda^3} \cdot \tag{175}$$

In the case of a heated electron-emitting filament when V_s is positive and greater than Ea/C_1, the number of electrons that can escape depends on the number that can get through the saddle (or pass) and thus is determined by the potential distribution in the neighborhood of the saddle, as given by Eq. (175). We shall have occasion to use these equations in a consideration of the theory of heated sounding electrodes.

TABLE XIX. *Characteristics of the potential saddle near a charged wire in a transverse electric field.*

$\lambda = r_M/a$	$C_1 V_s/Ea$	$(V_M - V_s)/Ea$	$\frac{a}{E}\frac{\partial^2 V}{\partial y^2}$
1.0	1	0	0
1.2	1.004	− 0.0006	−0.255
1.4	1.057	− 0.0041	−0.350
1.6	1.112	− 0.070	−0.381
1.8	1.178	− 0.140	−0.384
2.0	1.250	− 0.233	−0.375
2.5	1.450	− 0.557	−0.336
3.	1.667	− 0.995	−0.296
4.	2.125	− 2.142	−0.234
5.	2.600	− 3.569	−0.192
10	5.050	− 13.265	−0.099
20	10.025	− 40.099	−0.050
30	15.016	− 72.154	−0.033
50	25.010	−145.641	−0.020
100	50.005	−360.566	−0.010

Table XIX, based upon Eqs. (172), (174) and (175) may be used to calculate V_s, V_M and $\partial^2 V/\partial y^2$ in terms of the parameter λ, and may therefore be used to find r_M, V_M and $\partial^2 V/\partial y^2$ when V_s is given.

Electric images of wires in cylindrical electrodes. The electric field set up inside of a conducting cylinder of radius a by a charged wire parallel to the axis but at a distance a_1 from it, is the same as that produced in the absence of the cylindrical electrode by two wires having charges e_1 and e_2 respectively, placed at distances a_1 and a_2 from the axis, where

$$a_1 a_2 = a^2 \tag{176}$$

and

$$e_2 = -e_1. \tag{177}$$

Thus, in calculating the potential distributions, the cylinder can be replaced by the "image" of the wire in the cylinder.* The charges of the wire and of its image are equal but opposite in sign, since in this two-dimensional case all the lines of force reaching the image must have originated on the wire and have passed through the cylindrical surface.**

Cases involving several electrodes. The potential at any point can be determined in accord with Eq. (138) by summation of the potentials due to the separate electrodes so that, in general, with a system of n conductors, the potentials of the conductors are

$$\begin{aligned} V_1 &= p_{11}e_1 + p_{12}e_2 + \cdots + p_{1n}e_n \\ V_2 &= p_{21}e_1 + p_{22}e_2 + \cdots + p_{2n}e_n \end{aligned} \tag{178}$$

where e_1, e_2, etc., are the charges on the conductors and p_{11}, $p_{12} \cdots p_{hk}$ are constants called *potential coefficients*. These n equations can be solved for the e's giving

$$\begin{aligned} e_1 &= c_{11}V_1 + c_{12}V_2 + \cdots + c_{1n}V_n \\ e_2 &= c_{21}V_1 + c_{22}V_2 + \cdots + c_{2n}V_n \end{aligned} \tag{179}$$

The coefficients c_{11}, $c_{22} \cdots c_{hh}$ are the *capacitances* of the conductors, each being the charge of any given conductor when it is at unit potential, all the other conductors being at zero potential. The quantities c_{hk}, where $h \neq k$, are the *induction coefficients* or *partial capacitance coefficients*. Each represents the charge induced on any given conductor H by bringing another conductor K to unit potential, while H and all the other conductors are maintained at zero potential. These induction coefficients c_{hk} are all negative while the capacitances c_{hh} are positive.

* Except in the case of plane surfaces, electric images are not identical in position or shape with optical images.

** The image of a point charge in a sphere is located at a point which is also given by Eq. (176), a_1 and a_2 being the distances of the point and its image from the center of the sphere. In this case, however,

$$e_2 = -e_1(a_2/a) = -e_1(a_2/a_1)^{1/2}.$$

The charge on the image must be greater than $-e_1$ because a certain fraction, $(a_2-a)/a$, of the lines of force which start from the image reach out to infinity instead of passing through the spherical surface.

Gauss has proved the following "reciprocity theorem." Let $e_1, e_2, \cdots e_n$ be the charges on the conductors when their potentials are $V_1, V_2, \cdots V_n$, and let $e_1', e_2', \cdots e_n'$ be the charges when the potentials are changed to V', $V_2', \cdots V_n'$, then the relation holds

$$\sum_{h=1}^{n} e_h V_h' = \sum_{h=1}^{n} e_h' V_h. \tag{180}$$

By means of this theorem it can be shown in general that

$$c_{hk} = c_{kh} \tag{181}$$

In high vacuum electron tubes the glass walls of the tube generally take up electrons until they acquire the same potential as the cathode. In gas discharge tubes the walls are commonly at some definite potential intermediate between those of the anode and cathode. Very frequently one of the electrodes nearly completely surrounds all the others. Thus, in all these cases, we are concerned not with the capacitances between the electrodes and ground, but only with those between the electrodes. That is, in applying Eq. (179), we may usually limit ourselves to a consideration of the interior of closed systems. Thus no matter what the potentials of the individual electrodes may be, the total charge on all of them is zero:

$$e_1 + e_2 \cdots + e_n = 0, \tag{182}$$

the charge on any electrode or envelope that surrounds all the others being taken as that on its inner surface only. If all the electrodes are now brought to the same potential V', it follows from Gauss' theorem that the charge on each electrode (e') must be zero, and thus by Eq. (179)

$$\begin{aligned} c_{11} + c_{12} + \cdots + c_{1n} &= 0 \\ c_{h1} + c_{h2} + \cdots + c_{hn} &= 0 \end{aligned} \tag{183}$$

The capacitance c_{hh} of any electrode is the sum of all the induction coefficients c_{hk} applying to that electrode with their signs reversed, so that the term partial capacitance coefficient is justified.

When Eq. (182) is satisfied, we see also, from Eqs. (183) and (179), that the charge on any electrode is only dependent on the difference of potential between this electrode and the others so that then, if we use V_{21} as an abbreviation for $V_2 - V_1$, etc., Eq. (179) may then be written,

$$\begin{aligned} e_1 &= c_{12}V_{21} + c_{13}V_{31} + \cdots + c_{1n}V_{n1} \\ e_2 &= c_{21}V_{12} + c_{23}V_{32} + \cdots + c_{2n}V_{n2} \\ &\text{etc.} \end{aligned} \tag{184}$$

and the capacitances $c_{11} \cdots c_{hh}$ are eliminated.

These equations are applicable to the calculations of the amplification constants of electron tubes such as the audion. The escape of electrons from the filamentary cathode depends on the presence of an accelerating field at its surface, and thus by Eq. (140) requires that the surface be negatively charged. Consider a triode so constructed that the charge e_1 on the cathode is uniformly distributed over the cathode surface. By Eq. (184), if the potential of the cathode is taken as zero, the charge e_1 on the cathode is

$$e_1 = c_{12}V_2 + c_{13}V_3 \tag{185}$$

where the subscripts 2 and 3 apply to the grid and plate respectively. The condition that electrons shall flow from the cathode is thus that

$$V_2 > - V_3c_{13}/c_{12}. \tag{186}$$

The amplification constant μ may be defined as

$$\mu = \frac{\partial e_1}{\partial V_2}\Big/\frac{\partial e_1}{\partial V_3} = \frac{c_{12}}{c_{13}}\,. \tag{187}$$

The field at the cathode surface is affected, by changes of the grid potential, μ times as much as by a similar change in plate potential.

The induction coefficients are useful not only in calculating amplification constants, but the charges on the electrodes are found by equations like Eq. (185), and in this way, by Eq. (138), the potential distribution throughout the space may be determined.

Grid between parallel planes. Maxwell[270] has developed the theory of the screening effect of a grid 2 consisting of parallel wires, each of radius b, placed between two infinite plane electrodes 1 and 3 at distances x_1 and x_3 from the grid. His results are readily put in the form

$$\begin{aligned} 4\pi\sigma_1x_1\beta^{-1} &= -\alpha^{-1}V_{21} - x_3^{-1}V_{31} \\ 4\pi\sigma_3x_3\beta^{-1} &= -x_1^{-1}V_{13} - \alpha^{-1}V_{23} \end{aligned} \tag{188}$$

where σ_1 and σ_3 are the surface charge densities on the planes 1 and 3, and

$$\alpha = (s/2\pi)\log(s/2\pi b) \tag{189}$$

$$\beta^{-1} = x_1^{-1} + \alpha^{-1} + x_3^{-1} \tag{190}$$

s being the distance between the axes of adjacent grid wires. In this derivation it was assumed that the diameter of the grid wires, (2b), is less than 1/4 of the distance, (s), between them, and that s is small compared to x_1 and x_3.

The equations may be applied to parallel plane electrodes of finite area A, if the dimensions of these surfaces are large compared to the distance x_1+x_3 between them, so that the "edge corrections" are negligible. Comparison of Eqs. (188) with (179), remembering that $e=\sigma A$, shows that the induction coefficients are

$$\begin{aligned} c_{12} &= -A\beta/4\pi x_1\alpha \\ c_{13} &= -A\beta/4\pi x_1x_3 \\ c_{23} &= -A\beta/4\pi\alpha x_3. \end{aligned} \tag{191}$$

[270] Maxwell, Electricity and Magnetism, 1904 Edition Vol. 1, page 312.

The amplification constant μ is

$$\mu = c_{12}/c_{13} = x_3/\alpha \tag{192}$$

and is thus independent of the distance between grid and cathode.

For many purposes it is convenient to consider the grid and plate (plane 3) to be replaced by a plane electrode placed at such a position and having such a potential, V_e, that its effect on the field near the cathode (plane 1) is equivalent to that of the grid and plate.[271] Let Δ be the distance from the grid plane to the plane of this imaginary electrode which may be assumed to lie between the grid and plate. The surface charge, σ_1, thus induced on the cathode is given by

$$4\pi\sigma_1 = -V_{e1}/(x_1 + \Delta) \tag{193}$$

so that by Eq. (188)

$$\beta^{-1}V_{e1} = (1 + \Delta x_1^{-1})(\alpha^{-1}V_{21} + x_3^{-1}V_{31}) \tag{194}$$

whence by Eqs. (190) and (192)

$$V_{e1} = (V_{31} + \mu V_{21})(1 + \Delta x_1^{-1})(1 + \mu + x_3 x_1^{-1})^{-1}. \tag{195}$$

If, with Schottky, we choose $\Delta = 0$, that is, make the imaginary surface coincide with the grid, the effective potential is

$$V_{e1} = (V_{31} + \mu V_{21})(1 + \mu + x_3 x_1^{-1})^{-1} \tag{196}$$

and thus, even if the grid and plate potentials are the same, the effective potential is different from either.

It seems more useful to locate the effective plane in such a way that Δ is independent of V_{31} and V_{21} and yet gives $V_e = V_{21}$ when $V_{21} = V_{31}$. We then find

$$\Delta = x_3(1 + \mu)^{-1} \tag{197}$$

$$V_{e1} = (V_{31} + \mu V_{21})(1 + \mu)^{-1}. \tag{198}$$

Cylindrical grids. Abraham[272] has calculated the induction coefficients between the cathode, grid and plate for the case that the grid consists of n parallel wires symmetrically placed (at the corners of a regular polygon) about the cathode which coincides with the axis of the cylindrical "plate" or anode. It is assumed that the length L of the structure is large compared to the radius of the cylinder (so that end-corrections are negligible).

Let a_1, a_2, and a_3 be the radii of the cathode, of the grid cylinder, and of the anode cylinder respectively. Let b be the radius of the wires forming the grid. Then the induction coefficients are

[271] W. Schottky, Archiv. f. Electrotechnik 8, 1–31, (1919). Schottky introduced (p. 22) the concept of effective grid potential, but considered practically only cases for which the plane of the effective electrode coincides with that of the grid, i.e., $\Delta = 0$.

[272] Max Abraham, Archiv. f. Elektrotechnik 8, 42–45 (1919).

$$c_{12} = -L\lambda^{-1}\log(a_3/a_2)$$
$$c_{13} = -L\lambda^{-1}\gamma \tag{199}$$
$$c_{23} = -L\lambda^{-1}\log(a_2/a_1)$$

where

$$\gamma = n^{-1}\log(a_2/nb) \tag{200}$$

and

$$\lambda = 2\gamma[\log(a_3/a_1) + \mu\log(a_2/a_1)] \tag{201}$$

μ being the amplification constant:

$$\mu = \frac{c_{12}}{c_{13}} = \frac{n\log(a_3/a_2)}{\log(a_2/nb)}\cdot \tag{202}$$

In the derivation of these equations it was assumed that $(a_2/a_3)^n$, $(a_1/a_2)^n$, and $(nb/a_2)^2$ are small compared to unity.

Following the method used in deriving Eq. (195), we may, for many purposes, replace the grid and anode by a single cylindrical electrode of radius a_e and potential V_e which produces the same field at the cathode. By Eq. (146)

$$V_{e1} = -2(e_1/L)\log(a_e/a_1) \tag{203}$$

and then from Eqs. (199) and (184)

$$V_{e1} = 2\gamma\lambda^{-1}(V_{31} + \mu V_{21})\log(a_e/a_1). \tag{204}$$

If now, as in the case of the plane grid, we chose a_e so that V_{e1} is the same as the grid potential when the grid and anode potentials are equal, then Eq. (198) gives the effective potential V_{e1}, and a_e is given by

$$\log(a_e/a_2) = (1+\mu)^{-1}\log(a_3/a_2) \tag{205}$$

which reduces to Eq. (197) if $a_3 - a_2$ is small compared to a_2.

Single grid wire in cylinder. In many experiments with pure electron discharges as well as in gaseous discharges, it is desirable, within a tube containing a cylindrical anode of radius a_3 and a coaxial cathode of radius a_1, to use, within the anode, a third electrode in the form of a single straight filament parallel to the axis of the anode. This electrode may be used as a collector or emitter of electrons or ions. Let a_2 be the distance of this electrode from the axis and let b be its radius. This may be regarded as a special case of the problem of the cylindrical grid in which $n=1$. However, since a_2/a_3 is not negligible compared to unity, the foregoing treatment is only approximate, whereas the method of images gives a practically rigorous solution if the radii a_1 and b are small compared to a_3.

The potential distribution within the cylindrical anode is that due to the combined effect of charge e_1/L per unit length at the axis, of a line charge e_2/L at a distance a_2 from the axis, and of a line charge $-e_2/L$ at the distance a_3^2/a_2 from the axis in the same plane as the other two line charges.* These

* See Eq. (176).

charges e_1 and e_2 are readily found by Eqs. (184) as soon as we know the induction coefficients c_{12}, c_{13}, and c_{23}.

The solution of this problem shows that instead of the value of γ given by Eq. (200), we must now use

$$\gamma = \log \left[a_2 b^{-1}(1 - a_2^2 a_3^{-2})\right]. \tag{206}$$

In terms of this new γ the amplification constant μ becomes

$$\mu = \gamma^{-1} \log (a_3/a_2) \tag{207}$$

and c_{12}, c_{13}, c_{23} and λ are given as before by Eqs. (199) and (201).

Slat grid and anode. Kingdon and Langmuir[273] have found that the deactivation by positive ion bombardment of sensitized cathodes in traces of alkali vapors can be avoided by the special electrode arrangement shown in cross-sections in Fig. 35. The filament at the axis is represented by the point at the center of the figure. The grid consists of the four radial slats g, connected together. Interleaved with these, and spaced further from the filament, are the four radial anode slats a. In practice the anode slats for convenience are mounted on the inside of a cylinder as in Fig. 35(b).

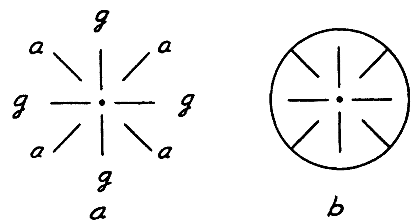

Fig. 35. Cross-sections of electrode arrangements showing anode and grid in slats.

H. M. Mott-Smith and L. Tonks,[274] assuming the slats are semi-infinite planes, have calculated the induction coefficient c_{12} and c_{13} (1, 2 and 3 refer to cathode, grid and anode respectively)

$$\begin{aligned} c_{12} &= L\mu/2(1 + \mu) \log (r_e/r_1) \\ c_{13} &= L/2(1 + \mu) \log (r_e/r_1) \end{aligned} \tag{208}$$

where L is the length; μ is the amplification constant (c_{12}/c_{13}); r_1 is the radius of the cathode filament and r_e is the "effective radius" of the grid-anode structure, i.e., that of a cylinder which would give the same field at the cathode as the actual electrodes, if these were brought to the same potential. The values of μ and r_e are

$$\mu = \frac{\pi}{\sin^{-1}\left[2\rho^{n/2}(\rho^n + 1)^{-1}\right]} - 1 \tag{209}$$

$$r_e = \left[4/(1 + \rho^n)\right]^{1/n} r_3 \tag{210}$$

[273] K. H. Kingdon and I. Langmuir, U. S. Patent No. 1,648,312 Nov. 8, 1927.

[274] Unpublished work in General Electric Research Laboratory.

where n is the number of grid (or anode) slats, i.e., 4 in the case illustrated in Fig. 35; r_2 and r_3 are the distances of the inner edges of grid and anode slats from the axis, and

$$\rho = r_3/r_2. \tag{211}$$

Experimental determinations of μ and the anode impedance (derivable from c_{12} and c_{13}) for actual tubes constructed as shown in Fig. 35 have given values in good agreement with these equations.

Electrode at one end of a long conducting tube.[275]

A two-electrode problem, sometimes of importance in the study of electric discharges in gases, is that of the potential distribution between the inner surface of a long conducting cylinder and a disk-shaped electrode within the cylinder near one end. Assuming axial symmetry, Laplace's equation (135) may be applied. By substitution of

$$V = U(r) \cdot G(z) \tag{212}$$

in this equation, two equations are obtained from which U and G are found. Thus a particular solution of the problem is

$$V = V_0 J_0 \,(2.405r/a) \exp{(-\, 2.405z/a)} \tag{213}$$

where a is the radius of the tube and V is the potential at any point at a distance r from the axis of the tube and at a distance z measured along the axis, the potential of the cylinder having been taken as zero. V_0 is the potential at the point $r=0$, $z=0$, and $J_0(u)$ is the zeroth order Bessel function of u. The coefficient 2.405 is the first root of the equation $J_0(u)=0$. Since $J_0(0)=1$, we see that the potential falls to one ϵ^{th} value for each increment of distance Δz along the tube where $\Delta z = 0.4156\, a$, or the potential falls to $1/10^{th}$ value for each increment of $0.96\, a$.

Eq. (213) is a particular solution which is applicable only when the distance from the disk electrode is large compared to a, the tube radius, and is then independent of the size or shape of this electrode.

This equation is applicable also to the diffusion of any substance along a tube whose walls absorb the diffusing substance. The number of molecules which diffuse per second across any given cross-section is

$$3.2619\, Dn_0 a \exp{(-\, 2.405z/a)} \tag{214}$$

where n_0 is the number of molecules per cm^3 at the point $r=0$, $z=0$. The rate of diffusion to the cylindrical surface (molecules$\cdot$cm$^{-2}\cdot$sec^{-1}) is

$$1.2485\, (Dn_0/a) \exp{(-\, 2.405\, z/a)} \tag{215}$$

Two others solutions of electrostatic problems have already been given in Part I. On page 151 an equation was given for the potential distribution near a checker-board surface having alternate squares at potentials 0 and V_0. On

[275] W. Schottky and J. v. Issendorff, Zeits. f. Physik **31**, 163–201 (1925) have treated this case (on p. 180) as a diffusion problem.

page 168 the potential distributions near conducting solids bounded by dihedral angles and near ridge-like elevations on planes, were considered.

Kunz and Bayley[275] have given the solutions for the field produced by a wire placed between parallel planes or located in a tube of rectangular section.

Effect of uniform space charge on potential distribution.

Although in gaseous discharges the space charge when present is ordinarily not uniform, it will help in forming a conception of the important rôle of space charge to consider the cases of parallel planes, or coaxial cylinders, or spheres between which there is a uniform space charge, ρ.

Two successive integrations of Eq. (137), considering ρ to be constant, give

$$V = Ar^{1-\kappa}(1-\kappa)^{-1} - 2\pi\rho r^2 (1+\kappa)^{-1} + C \tag{216}$$

where A and C are the integration constants. For the cylindrical case ($\kappa=1$) this is indeterminate, but direct integration of Eq. (137) gives,

$$V = A \log r - \pi\rho r^2 + C. \tag{217}$$

In each of these equations the first term may be regarded as the contribution of the charges on the electrodes and the 2nd term that of the space charge. If there is no internal electrode within the cylinder or sphere, or if, in the case of planes, they are at the same potential, the constants A and C become zero and the general solution for all three cases is

$$V = -2\pi\rho r^2 (1+\kappa)^{-1} \tag{218}$$

the distances being measured from the center of symmetry and the potential being taken as zero at this center. Placing $\rho = ne$ and inserting the value $e = 4.77\times10^{-10}$ e.s.u. $= 1.43\times10^{-7}$ volt cm, we find, if r is in cm,

$$V = \mp 8.99 \times 10^{-7}\, nr^2(1+\kappa)^{-1} \text{ volts} \tag{219}$$

the $-$ sign being taken for positive ion, and the $+$ sign for electron, space charges, n being the equivalent number of electron charges per cm^3.

With $n = 10^9$ electrons per cm^3 corresponding to an average distance of 0.01 mm between electrons, we thus see that $V = 900\ r^2$ volts, while with electrons 0.1 mm apart ($n = 10^6$), $V = 0.9\ r^2$. Large potential gradients can thus result from relatively low concentrations of electrons or ions.

When there are two or more electrodes at different potentials, a potential maximum or minimum in space occurs only when the magnitude of the space charge exceeds a definite value. For example, consider two parallel plane electrodes at a distance a apart and let V_a be the difference of potential. If $\rho = 0$ there is a linear potential distribution between the planes, but as ρ increases (positive space charge), the potential distribution becomes parabolic in accordance with Eq. (216), and at a certain value of ρ, the potential gradient at the more positive electrode (anode) becomes zero. A further increase

[275] J. Kunz and P. L. Bayley, Phys. Rev. **17**, 147 (1921).

in ρ produces a potential maximum near the anode. Measuring distances and potentials from the potential maximum, the distribution is given by Eq. (218) with $\kappa=0$. Thus the condition for the occurrence of a maximum is $\rho > V_a/2\pi a^2$.

We shall see in Chapter VI that the presence of a potential maximum or minimum in the space between the electrodes often produces profound modifications in the character of a gaseous discharge, since low velocity electrons (or positive ions) become trapped within the region of potential maximum (or minimum) and their accumulation nearly destroys the space charge within this region.

For the cases of cylinders and spheres, the conditions for the occurrence of a potential maximum or minimum near the internal electrode are also obtainable directly from Eqs. (218) or (219). For example, consider a cylinder of radius a, and at zero potential, containing a uniform distribution of n singly charged positive ions per cm^3. At the cylinder axis the potential V_a is a maximum and is obtained from Eq. (219) by taking the + sign and placing $r=a$ and $\kappa=1$. If now we introduce at the cylinder axis a small wire and make its potential V_a which is the "space potential" at this point, it will produce practically no change in the potential distribution: the potential gradient at the surface of the wire is zero and by Eq. (140) the wire is uncharged. If the potential of the wire is made *less* than the space potential, a potential maximum develops, but with a potential above V_a, the maximum in space disappears. Thus Eq. (219) shows that when a cylindrical anode having a radius of 1 cm contains 10^8 electrons per cm^3 and an axial cathode filament, there will be a potential minimum close to the cathode surface, if the potential difference between the cathode and anode is less than 45 volts.

This method cannot be used to determine the conditions for the occurrence of a potential maximum or minimum near the surface of the outer cylinder. The constants A and C in Eq. (217), however, may be so determined as to make V and dV/dr both zero at a particular value of r, say r_0. The equation then becomes

$$V = 2\pi\rho r_0^2[\log_\epsilon (r/r_0) + \tfrac{1}{2}(1 - r^2/r_0^2)]. \qquad (220)$$

If the voltage V is known for some particular value of r, say at the anode, then this equation with these values of V and r can be solved for ρ and the value ρ_0 thus obtained gives the space charge which, if exceeded, gives a potential minimum at the cathode of radius r_0.

This method may be used either when the potential minimum is near the outer or near the inner cylinder.

Chapter III. The Flow of Current in High Vacuum When Space Charge is Negligible

The simplest types of electric discharges are those occurring in such high vacuum that the motions of the charged particles, electrons or ions, which carry the current are not modified by collisions between these carriers and gas molecules. When the current is carried wholly by electrons, the discharge is usually referred to as a pure electron discharge. No electrons or ions are generated in the vacuous space and the carriers move from one electrode, which we may call the emitter, to another electrode, called the collector, solely under the influence of the electric field in the space.

In a unipolar discharge of this type, let n be the number of charged particles per unit volume and let e be the charge on each particle. If v_x is the average drift velocity of these particles at any point in space parallel to the X axis, then the drift current density I_x is given by

$$I_x = nev_x. \tag{221}$$

A similar equation may be written for the components of velocity and current density parallel to each of the other axes Y and Z. Thus the drift velocity v_d and the drift current density I_d are vectors whose components are v_x, v_y, v_z, I_x, I_y and I_z. In general the direction of the drift velocity at any point does not necessarily coincide with the direction of the electric field at that point. If the lines of force that originate at the emitter and pass to the collector are curved, the electrons can obviously not follow such curved paths for the curvature would result in a centrifugal force in a direction perpendicular to the path. Thus, to hold the electron in a path coinciding with the lines of force would require a field component perpendicular to the lines of force, but this is incompatible with the definition of lines of force.

If the electrons escape from the surface of the emitter without initial velocities, then the kinetic energy of an electron when it reaches any given point in space is equal to Ve, where V is the potential of that point with respect to the emitter. Thus we have,

$$\tfrac{1}{2}mv^2 = Ve. \tag{222}$$

Although the velocities with which the electrons are emitted are so low that we are often justified in neglecting them, there are many cases where we must take these velocities into account. In general, the particles emitted have an initial velocity distribution corresponding to Maxwell's distribution law.* At any point in space, therefore, the kinetic energy of an electron is equal to Ve plus the kinetic energy with which that electron was emitted.

It is important to recognize that the velocity, v, which enters Eq. (222) is a scalar quantity as distinguished from the vector velocity v_d in Eq. (221). However, when the lines of force connecting the emitter and collector are straight, the distinction between v_d and v loses its significance and for these cases, characterized by special kinds of symmetry, the mathematical treatment is very greatly simplified.

The equations that we shall derive for unipolar discharges in high vacuum will naturally be applicable both to the pure electron discharge, where electrons are emitted from the emitter, and to the pure positive ion discharge, where the current is carried wholly by the positive ions originating at the

* Part I, page 204.

emitter. Such positive ion discharges may be obtained** by using a Kunsman emitter, or by using, for example, a tungsten filament in presence of a very low pressure of an alkali metal, whose pressure may be so low that the number of ions generated in space is negligible.

We shall also have occasion to discuss bipolar discharges in which electrons are emitted from one electrode and positive ions are emitted from the other electrode. The theory of such bipolar discharges will prove to be useful in subsequent studies of gas discharges.

The simplest type of high vacuum discharge is that in which the vacuous space is bounded wholly by the two electrodes, emitter and collector, so that all the lines of force that originate at the emitter terminate on the collector. In a great number of electric discharges, however, the space is bounded also by glass walls or by metallic surfaces of a third electrode. The potential distribution over an insulating surface, such as that of glass, may thus play an important part in determining the potential distribution in space. In most unipolar discharges, at least as a first approximation, we may consider that an insulating surface receives electrons or ions from the emitter until it becomes charged to the same potential as the emitter. Thus, with the convention we have used of taking the potential of the emitter to be zero, we may assume insulated surfaces also to have zero potential. We shall find, however, some cases, both in high vacuum and in the presence of gas, where the potential distribution on glass surfaces becomes an important factor in determining the character of the discharge.

According to Eq. (221), current cannot flow without necessitating the presence of a definite space charge. If, however, the current density is so low that the total charge in the space is very small compared to the charges on the electrodes and on the glass walls, the potential distribution will be determined wholly by Laplace's equation, in accord with the principles discussed in Chapter II. If the potential distribution is such that the electric field at the surface of the emitter accelerates the emitted particles, i.e., if there is an accelerating field, it will often happen that every particle which is emitted passes to the collector so that the current will then be independent of the potential distribution and will be equal to the *saturation current* characteristic of the emitter as determined by its temperature. However, in other cases, some of the electrons which escape from the emitter describe orbits in space and return to the emitter. We shall return later to a consideration of the conditions under which this occurs.

If the potential distribution within the device is such that there is a retarding field at the surface of the emitter, electrons can only pass to the collector if they possess sufficient initial kinetic energy to enable them to move against the retarding field. Thus, if a particle is emitted with a velocity, v, it cannot move against the retarding field into a region which has a retarding voltage (with respect to the emitter) greater than V where V is given by Eq. (222). However, it must be kept in mind that all particles having this

** See Part I, page 140.

velocity, v, are not *necessarily* able to travel into a region corresponding to a potential difference less than V. This is readily seen if we remember that the electron ceases to move against a retarding field as soon as its *velocity component* in the direction of the field has fallen to zero, even if it has a large transverse velocity.

With increasing current density I, the total charge in the space must finally become comparable with the charge on the electrodes and on the glass walls. Many of the lines of force originating at the emitter then terminate on the charges in space. The potential distribution can thus no longer be determined by Laplace's equation, but requires Poisson's equation. Because of this space charge, it now becomes possible (see Chapter II, page 214) that an absolute minimum (in the case of positive ions, a maximum) in space may occur. The accelerating field at the surface of an emitter may thus fall to zero and it is even possible that close to the surface of an emitter there is a retarding field, although at a greater distance there is an accelerating field with a surface of minimum potential lying between these two regions.

Such a region of retarding field close to the emitter thus constitutes a potential barrier across which only electrons having sufficiently large normal velocity components are able to pass. The total current that flows from emitter to collector is thus less than the saturation current and it is then said that the current is limited by space charge. In Chapter IV we shall consider in detail this limitation of current for electrodes of several different shapes.

Before treating this space charge problem, however, let us analyze more carefully the flow of current with accelerating and retarding field when space charge is insufficient to cause the appearance of a potential barrier.

Electron swarms as ideal gas.

The repulsion between electrons, varying inversely as the square of the distance, falls off so slowly with increasing distance that the force which acts on any given electron in a swarm is due mainly to the more distant electrons, for the number of such electrons, in successive spherical shells (of uniform thickness), *increases* with the square of the distances. The effect of the distant electrons can, however, be taken into account by calculating the potential distribution by Poisson's equation, considering the space charge ρ as a continuous quantity, ignoring the fact that it is built up of discrete charges. The forces due to electrons which lie within distances comparable to the average separation of the electrons (approximately $n^{-1/3}$, where n is the number of electrons per cm^3) are, however, quite different from those that would result from a continuous distribution of charge. These fluctuating forces may be regarded as analogous to those that result from collisions between gas molecules according to the kinetic theory. They tend to bring about a random distribution of electron velocities and establish a state of thermal equilibrium among the electrons.

In 1918 and 1919 there was an active discussion[276] as to whether the ideal

[276] M. v. Laue, Jahrb. d. Radioakt. u Elektronik, **15**, 205, 257 (1918); Ann. d. Physik **58**, 695 (1919); Phys. Zeits. **20**, 202 (1919). W. Schottky, Phys. Zeits. **20**, 49, 220 (1919).

gas laws could be properly applied to an electron gas and whether the distributions of velocities and concentrations were in accord with the Maxwell-Boltzmann laws. It was found that the electron gas behaves like an ideal one in a field of force if the mean kinetic energy $(3/2)kT$ is large compared to the potential energy of two electrons at the average distance $n^{-1/3}$. This condition requires that

$$n \ll 7 \times 10^8 T^3. \tag{223}$$

With electrons in thermal equilibrium at 1000°, the gas laws are thus applicable for all concentrations less than about 10^{17} electrons·cm^{-3}. We see from Eq. (219) that such concentrations could exist only in regions of exceedingly small dimensions and with enormous potential gradients, for example, a layer 10^{-5} cm thick would produce a field of 10^6 volts·cm^{-1}.

Modern quantum theory* gives another reason why the classical kinetic theory is not applicable to an electron gas of high concentration. From equations given in K. K. Darrow's article on Statistical Theories of Matter, Radiation ond Electricity,[277] it can readily be shown that the velocity distribution of electrons in equilibrium at a temperature T is accurately given by the classical kinetic theory if

$$10mkT \gg h^2 n^{2/3}. \tag{224}$$

Inserting the numerical values of m, k and h, this condition becomes

$$T \gg 3.5 \times 10^{-11} n^{2/3} \tag{225}$$

With electron temperatures greater than 300°K, the classical theory is thus applicable for all electron concentrations less than about 10^{19} per cm^3.

In determining whether electrons emitted by one electrode may reach another, it is frequently necessary to calculate the orbits that the electrons describe in space. With central force fields such as those between coaxial cylinders or concentric spheres, these calculations are based on the principles of the *conservation of energy* and of *angular momentum*. Collisions between electrons, or any appreciable interactions between neighboring electrons, will make it impossible to apply these principles to individual electrons.

An estimate of the magnitude of these interactions can be made by considering the "free path" of an electron moving with velocity v among a swarm of electrons of uniform concentration n, the velocities of these electrons being considered negligible compared to v.

The deflections of the given electron from its normal orbit are of two kinds: single large deflections due to close encounters, and the accumulated effects of numerous small deflections due to fluctuations in the fields of the more distant electrons.

Single encounters.

Consider an electron A moving with a velocity v (corresponding to a fall through potential V) in a path which, without deflection, would carry it to a

* See references 119, 120 on p. 163 of Part I.

[277] K. K. Darrow, Rev. of Modern Physics **1**, 90–155 (1929). See especially pages 117–8.

minimum distance p from another electron B, which is initially at rest. Then a rigorous analysis of the movements of both electrons under the inverse square law of force gives for the angular deflection θ of electron A from its original straight path resulting from the encounter,

$$\sin\theta = (1+\alpha^2)^{-1/2} \tag{226}$$

where α is defined by

$$\alpha = Vp/e = 7.00 \times 10^6 Vp \tag{227}$$

if V is in volts and p in cm.

The free path λ_θ may now be defined as the average distance that the electron A can move through a swarm of electrons B before it suffers an angular deflection exceeding θ. We find

$$\lambda_\theta = 1/\pi n p^2 = V^2(\sin^2\theta)/\pi n e^2(1-\sin^2\theta). \tag{228}$$

For small deflections, where $\sin^2\theta \ll 1$, this becomes

$$\lambda_\theta = V^2\theta^2/\pi n e^2 = 4.74 \times 10^9 V^2\theta^2/n \text{ cm} \tag{229}$$

if V is in volts, θ in degrees, and n in electrons $\cdot$ cm^{-3}. As an example, we find that 10 volt electrons would travel through an electron swarm containing 10^{11} electrons$\cdot$cm^{-3} for an average distance of 19 cm before suffering a deflection greater than 2° as a result of a single encounter.

Multiple small deflections.

The mean square angular deflection θ of an electron, after moving a distance t through an electron swarm, is given[278] by

$$\theta^2 = (2\pi e^2 nt/V^2)\log(\alpha/\alpha_0) \tag{230}$$

or with θ in degrees and n, V, t in the usual units:

$$\theta = 3.116 \times 10^{-5}[(nt)^{1/2}/V][\log_{10}(\alpha/\alpha_0)]^{1/2}. \tag{231}$$

This is derived on the assumption that the electrons in the swarm are distributed with *uniform probability* throughout a cylinder of radius p, of which the path of the moving electron A is the axis. The deflections produced by electrons which lie within a certain small distance p_0 of the path of A are excluded since these produce large deflections which can be better considered by the theory of single encounters. The values of α and α_0 are then found from p and p_0 by Eq. (227).

Because of the electric fields of the electrons in the swarm, these electrons will be more uniformly distributed than if the *probability* of their occurrence were uniform throughout space. This tendency towards a uniform distribution may be taken into account by assuming that there are no fluctuations in density beyond a certain distance λ_D, the Debye distance, given by Eq. (131)

[278] This is based upon an equation discussed by I. Langmuir and H. A. Jones, Phys. Rev. **31**, p. 390–1 (1928), derived by the method of H. A. Wilson, Proc. Roy. Soc. **A102**, 9 (1923).

of Part I. Thus the p used to calculate α in Eq. (231) by Eq. (227) may be identified with λ_D. If the electrons in the swarm, instead of having velocities of thermal agitation corresponding to a temperature T, have average energies corresponding to V_B volts, we may put

$$p = \lambda_D = 530\,(V_B/n)^{1/2}. \tag{232}$$

In the example which we used to illustrate Eq. (229), we considered the distance that electrons would travel before being deflected 2° by a single encounter. For such small angles Eq. (226) may be written

$$\alpha = 57.3/\theta$$

θ now being expressed in degrees, and thus for an encounter which gives a 2° deflection, $\alpha = 28.6$.

For electrons of 10-volt velocity, Eq. (227) then gives $p = 4.1\times10^{-7}$ cm as the "target radius" corresponding to an encounter that produces deflections greater than 2°. These are the values of α and p that we have referred to as α_0 and p_0.

If the electrons in the swarm ($n = 10^{11}$) have velocities corresponding to 1000°, the Debye distance λ_D, by Eq. (131), Part I, is 4.9×10^{-4} cm. Substituting this for p in Eq. (227) with $V = 10$ volts gives $\alpha = 34000$.

Putting these in Eq. (231) we find that θ, the mean deflection produced in a path $t = 19$ cm long, by the fluctuations in density of the more distant electrons ($p > 4.1\times10^{-7}$) is 13.2°—about 6 times as much as the average deflection (2°) produced by the nearer electrons ($p < 4.1\times10^{-7}$).

If the electrons in the swarm have higher velocities, corresponding to say 10 volts, the value of p may be roughly estimated by Eq. (232) to be $p = 5.3\times10^{-3}$, which gives by Eq. (227) and Eq. (231) an average deflection of 17.6° instead of 13.2°.

Although with concentrations as high as 10^{11}, the orbits of electrons may thus be considerably modified in a manner that cannot be taken into account by assuming a continuous distribution of charge (Poisson's equation), it is clear that with the lower concentrations ordinarily present in pure electron discharges,* the orbit theory may be quite accurately applied.

Limitation of current by orbital motions of electrons.

In determining whether the current from one electrode to another will correspond to the saturation current or to some lower value, we need to consider the various types of orbits that are possible among the electrons in the space between the electrodes.

Let us consider a portion of space in which there is a given potential distribution and which is bounded by two equipotential surfaces A and B having respectively the potentials V_A and V_B. Let this space or field contain electrons in thermal equilibrium at the temperature T, so that the distribution of velocities and concentrations is in accord with the Maxwell-Boltzmann

* A current of 10 ma per cm² of 10-volt electrons corresponds to only $n = 6\times10^8$.

laws. In order that there may be thermal equilibrium, it is necessary that the bounding surfaces A and B shall have certain properties. They may, for example, be perfect reflectors of electrons (either specular or diffuse reflection) in which case they must not emit electrons. Or they may absorb all incident electrons, but must then act like electron-emitting metals at the temperature T, the current density of the emission current being such that the emission equals the absorption on each surface element. We will denote the *complete* Maxwell-Boltzmann distribution which then exists by the symbol *MBD*.

In an *MBD*, the distribution of velocities of the electrons in any small element of volume is isotropic, i.e., the velocities are distributed with equal probability in all directions. The number of electrons per unit volume having speeds (regardless of direction) lying between v and $v+dv$ is given by*

$$nF(v)dv = 4\pi n(m/2\pi kT)^{3/2} v^2 \exp(-mv^2/2kT)dv \quad (233)$$

and thus the relative numbers lying within given ranges of velocity are the same at all points within the field. The average energy of the electrons within any given volume is $(3/2)kT$. The average velocity is

$$\bar{v} = (8kT/\pi m)^{1/2}. \quad (234)$$

Consider any imaginary plane cutting the field. A certain number of electrons pass per second per unit area through any surface element of this plane from one side to the other while an equal number pass in the opposite direction. Because of the isotropic velocity distribution, the current density at any point corresponding to this electron flow is independent of the orientation of the imaginary plane which passes through this point. We may thus speak of a *random current density* I which is given by

$$I = \tfrac{1}{4}ne\bar{v}. \quad (235)$$

The factor 1/4 results from the fact that 1/2 of the electrons in any element of volume are approaching the plane with an average velocity component normal to this plane of $(1/2)\bar{v}$. Combining Eq. (235) and (234) gives

$$I = ne(kT/2\pi m)^{1/2}. \quad (236)$$

If at any point in the field there are n electrons per cm^3, the random electron current density at this point is thus

$$I = 2.478 \times 10^{-14} nT^{1/2} \text{ amperes}\cdot\text{cm}^{-2}. \quad (237)$$

Because of the presence of the electric field, the concentration of electrons cannot be uniform but varies in accord with the Boltzmann equation**

$$n = n_0 \exp(Ve/kT) \quad (238)$$

where n_0 is the concentration at a surface at which the potential is zero.

* This is equivalent to Eq. (71) on p. 205, Part I.

** See Eq. (77), page 206, Part I.

By Eq. (238), n must be uniform over any equipotential surface such as A or B.

If there is a potential difference between A and B, however, the concentrations of electrons at the two surfaces must differ and by Eq. (236) the random current densities must differ in the same ratio, thus

$$I_B = I_A \exp (V_{BA}e/kT) \tag{239}$$

where V_{BA} is the potential of B with respect to A, and e, the charge of an electron, is regarded as a positive quantity. Thus for positive ions $-V_{BA}$ should replace V_{BA}. When the surfaces A and B absorb all incident electrons, they must have electron emissions which are not only uniform over each surface, but the emission current densities I_A and I_B must be such as to satisfy Eq. (239). With any current densities which do not fulfill this condition, a current must flow from one surface to the other so that the thermal equilibrium and the *MBD* are destroyed.

All these equations which characterize the *MBD* are valid no matter what may be the mechanism by which the distribution is brought about. Elastic collisions of electrons with each other or with gas molecules will not disturb the *MBD*, but serve only as a "catalyst" to establish equilibrium conditions more rapidly after any disturbance in equilibrium has been produced by other means.

Let us now consider that the density of the electrons and of gas molecules is so low that collisions, or interactions between individual electrons, may be neglected.* The *MBD* may be brought about by interactions with the surfaces A or B or by temporarily introducing a catalyst into the field. The electrons then describe free orbits in the given field. If the velocity and direction of motion of an electron at one point of its orbit is given, the whole orbit is thus fixed.

It is characteristic of an *MBD* that it includes particles moving along every possible orbit within the field. The totality of orbits may be divided into 5 classes, viz: *AA*, *AB*, *BA*, *BB* and *O*, the first letter designating the surface where the orbit begins and the second that of the surface where it ends. The group *O* consists of orbits which do not intersect either A or B. Electrons in such orbits may be called trapped electrons because they cannot reach either surface. The *BA* orbits are identical with the *AB* orbits except that the sign of the velocity with which the electron passes any given point is reversed.

The electrons that pass through any given surface may be classified according to the orbits they describe. Let i_{AF} and i_{BF} be the electron currents that flow respectively from the A and B surfaces into the field F and let i_{FA} and i_{FB} be the currents of electrons incident on these surfaces, all these currents being regarded as having positive values regardless of the actual directions of the electrons. Then we may place

* See Chapter IV for a further discussion of this restriction.

$$\left.\begin{aligned} i_{AF} &= i_{AB} + i_{AA}; \quad i_{FA} = i_{BA} + i_{AA}; \\ i_{BF} &= i_{BA} + i_{BB}; \quad i_{FB} = i_{AB} + i_{BB}; \end{aligned}\right\} \tag{240}$$

where the currents i_{AA}, i_{AB}, i_{BA} and i_{BB} are those of electrons moving in orbits of the AA, AB, BA and BB types. According to the principle of reversibility,[279] when there is thermal equilibrium

$$i_{AB} = i_{BA}. \tag{241}$$

Under these conditions it also follows that

$$i_{AF} = i_{FA}; \quad i_{BF} = i_{FB}, \tag{242}$$

and since these currents must be uniformly distributed over any equipotential surface, we have

$$i_{AF} = I_A S_A \text{ and } i_{BF} = I_B S_B \tag{243}$$

where S_A and S_B are the areas of the surfaces A and B. Although i_{AF} and i_{BF} are uniformly distributed over the surfaces, this will not in general be true of i_{AB}, i_{AA} and i_{BB}. Combining Eq. (243) with Eq. (239), we have

$$\frac{i_{BF}}{i_{AF}} = \frac{S_B}{S_A} \exp\,(V_{BA}e/kT). \tag{244}$$

Let λ_{AB} be the fraction of the total current that leaves A which reaches B, and λ_{BA} the corresponding fraction for the current from B to A. Then, by Eqs. (240)

$$\lambda_{AB} = i_{AB}/i_{AF}; \quad \lambda_{BA} = i_{BA}/i_{BF} \tag{245}$$

and

$$\left.\begin{aligned} (\lambda_{AB})^{-1} &= 1 + (i_{AA}/i_{AB}) \\ (\lambda_{BA})^{-1} &= 1 + (i_{BB}/i_{BA}) \end{aligned}\right\} \tag{246}$$

By combining Eqs. (245) and (244), we find that

$$\lambda_{AB} = (S_B/S_A)\lambda_{BA} \exp\,(V_{BA}e/kT). \tag{247}$$

It thus appears that if we can find means for calculating the relative numbers of electrons that describe AA and AB orbits, i.e., (i_{AA}/i_{AB}), for any given potential difference V_{BA}, then we can, from these equations, determine the relative numbers of electrons that describe BB orbits.

So far we have considered only equilibrium conditions. By imposing two restrictions on our problem, however, we may use the foregoing equations to calculate the flow of current in certain cases where thermal equilibrium and the complete MBD do not exist.

The first restriction that we impose is that the current densities shall be so low that space charge is unimportant; that is, we assume that the potential distribution which determines the shapes of the electron orbits can be ob-

[279] I. Langmuir, Jr. Amer. Chem. Soc. **38**, 2221–2295 (1916). See particularly p. 2253 and footnote on p. 2262.

tained by a solution of Laplace's equation and thus is not changed when electrons having certain types of orbits are removed from the field. In Chapter IV we shall consider the modifications produced by space charge.

As a second restriction we assume that both A and B absorb all incident electrons, in other words, they are non-reflecting surfaces. Then i_{AF} and i_{BF} in Eq. (240) represent the saturation currents from the surfaces A and B which we may now regard as two metallic electrodes.

Under these conditions, the currents i_{AA}, i_{BA} and i_{BB} are independent of one another in the sense that the removal of one group does not directly alter the others. For example, if, without changing its temperature, we change the character of one surface, say B, so that it no longer emits the current density required by Eq. (239), we alter i_{BA} and i_{BB}, but do not thereby change the currents i_{AB} and i_{AA}.

Reciprocal relations between emitter and collector.

Eq. (247) is obviously not directly applicable to a case where the thermal equilibrium is destroyed by cooling one of the electrodes, but we may use it to compare two cases, in one of which B is cooled, while in the other A is cooled.

To begin with, let us assume that A and B are two electrodes at the temperature T between which there is a difference of potential V_{BA} and that their electron emissions are such that Eq. (239) is fulfilled, giving thermal equilibrium. Then Eq. (247) is applicable.

Now let electrode B be cooled until it ceases to emit electrons so that i_{BA} and i_{BB} become zero. This, however, does not change i_{AA} and i_{AB} and therefore λ_{AB}, as given by Eq. (246), has the same value as when A and B were both at the temperature T.

Finally let us cool electrode A until it ceases to emit, but bring B again to its original temperature T. By similar reasoning, we find that λ_{BA} now has the same value as when both A and B were at T.

Thus Eq. (247) is applicable to cases where equilibrium is destroyed by cooling first one and then the other of two electrodes, if we interpret λ_{AB} to mean the fraction of electrons emitted from A which reach B when A is at temperature T and λ_{BA} the fraction emitted by B which reach A when B is at temperature T. Since space charge is negligible, the values of λ_{AB} and λ_{BA} will be independent of I_A and I_B, the emission current densities, and therefore it is no longer necessary for the validity of Eq. (247) that these emissions shall satisfy Eq. (239).

Equation (247) thus gives us a *reciprocal relation* by which, for non-reflecting electrodes of any shape having a given potential difference, we can calculate the relative change in current produced by transferring the property of electron emission from one electrode to the other. If, before this transfer, the electrons flow in an accelerating field, after the transfer they flow in the opposite direction against a retarding field.

Since the temperature T and the potential V_{BA} occur in Eq. (247) only as a ratio, it is evident that the reciprocal relation can be applied when the temperature T_A to which A is heated is different from the temperature T_B to which B is heated, provided the potential differences V_{BA} in the two cases are changed in the ratio T_A/T_B.

Saturation currents in accelerating fields.

If the shapes of the electrodes and the potential distribution in the field are such that orbits of the AA type are impossible, then it follows from Eq. (246) that $\lambda_{AB}=1$, so that every electron emitted by A passes to B. Evidently, when B is the cold electrode, a *necessary and sufficient condition* for the existence of a true *saturation current* from A is that AA orbits shall be impossible in the field. Similarly, to obtain the saturation current from B, with A cold, orbits of BB type must be impossible.

Since electrons of very low velocity are able to move only small distances against retarding fields, AA orbits will always exist if there is a retarding field for electrons near the surface of A. Thus, saturation currents from A can be obtained only when there is an accelerating field, i.e., when $V_{BA}>>0$. This condition, although necessary, is not sufficient to determine the existence of the saturation current.

The possible orbits in the field must include all straight lines that can be drawn in the field, since electrons of infinitely high velocity must describe such paths. Therefore AA orbits will always exist if any part of the A surface is concave. Strictly speaking, therefore, saturation currents cannot be obtained from such electrodes, even with strong accelerating fields.

Boltzmann relation for currents in retarding fields.

Under the field conditions which gave saturation current from A, (impossibility of AA orbits so that $\lambda_{AB}=1$), we may now calculate by Eqs. (247) and (245) the current i_{BA} that can flow from B to A when B is at temperature T while A is cold. Placing $\lambda_{AB}=1$ gives

$$i_{BA} = i_{BF}(S_A/S_B) \exp (V_{AB}e/kT) \tag{248}$$

i_{BF} being the saturation current $(i_{BA}+i_{BB})$ corresponding to the full emission from the hot electrode, and V_{AB} having been put in place of $-V_{BA}$. The current density received at electrode A is $I_A=i_{BA}/S_A$, while the emission current density at B is $I_B=i_{BF}/S_B$. In terms of I_A and I_B, Eq. (248) becomes identical with Eq. (239) which we derived directly from the Boltzmann equation. We see now, however, that this equation is only applicable for the flow of electrons in retarding fields when the conditions are such that no AA orbits are possible, (i.e., those which begin and terminate on the collector).

The reciprocal relation between emitter and collector thus shows that the conditions in regard to potential distribution and shapes of electrodes that are necessary and sufficient to give saturation currents, when one electrode acts as emitter, are identical with those that are necessary and sufficient for flow of current in accord with the Boltzmann equation when the other electrode is made emitter.

Let us now apply these principles to cases where the two electrodes are:

(1) Parallel planes; (2) Coaxial cylinders; (3) Concentric spheres. In each of the three cases, all the electron orbits curve towards the more positive electrode and thus cannot start and end on the more negative electrode unless this electrode has a concave surface.

We may thus conclude that with planes, coaxial cylinders and concentric spheres with one emitting and one non-emitting electrode, the current is saturated when electrons (or ions) are accelerated from a non-reflecting emitter which is not concave. When, however, electrons (or ions) are retarded as they approach a non-reflecting collector which is not concave, the current i is given by the following relation based on the Boltzmann equation:

$$i = S_C I_E \epsilon^{\eta}; \ \eta < 0, \tag{249}$$

where S_C is the surface area of the collector, I_E is the emission current density corresponding to saturation, and η is a parameter proportional to the voltage difference between collector and emitter defined by

$$\eta = V_{CE}e/kT = 11600 V_{CE}/T \tag{250}$$

if V_{CE} is in volts. When Eq. (249) applies, η is always a negative quantity.

We have now to consider the important class of cases where both AA and BB orbits exist, for example, cases where electrons are accelerated from a concave emitter or are acted on by a retarding field as they approach a concave collector.

Coaxial cylinders and concentric spheres.

Case I. Electrons accelerated inwards from an external emitter. The general methods of calculating the currents of various types, i_{AB}, i_{AA}, i_{BB}, etc., that flow under various conditions have been discussed[280] by Mott-Smith and Langmuir. For the flow of electrons with an accelerating field from an external emitter to an inner positively charged collector (anode) the current was found[281] to be

$$i = S_C I_E f \tag{251}$$

where S_C is the surface area of the collector, I_E the saturation emission density from the emitter and f is a function of the radii of the two electrodes and the potential.

For *coaxial cylinders*,

$$f = (r_0/r)P(\phi^{1/2}) + \epsilon^{\eta}[1 - P(\{\phi + \eta\}^{1/2})] \tag{252}$$

where $\eta > 0$, and is defined by Eq. (250), r_0 is the radius of the emitter, r is that of the collector and

$$\phi = r^2\eta(r_0^2 - r^2)^{-1} \tag{253}$$

$$\phi + \eta = r_0^2\eta(r_0^2 - r^2)^{-1}. \tag{254}$$

[280] H. M. Mott-Smith and I. Langmuir, Phys. Rev. **28,** 727–63, (1926). In Eq. (28a) on p. 738, which is equivalent to Eq. (252) given below, the factor e^{η} was inadvertently omitted from the last term.

[281] I. Langmuir and H. M. Mott-Smith, Gen. Elec. Rev. **27,** 454 (1924).

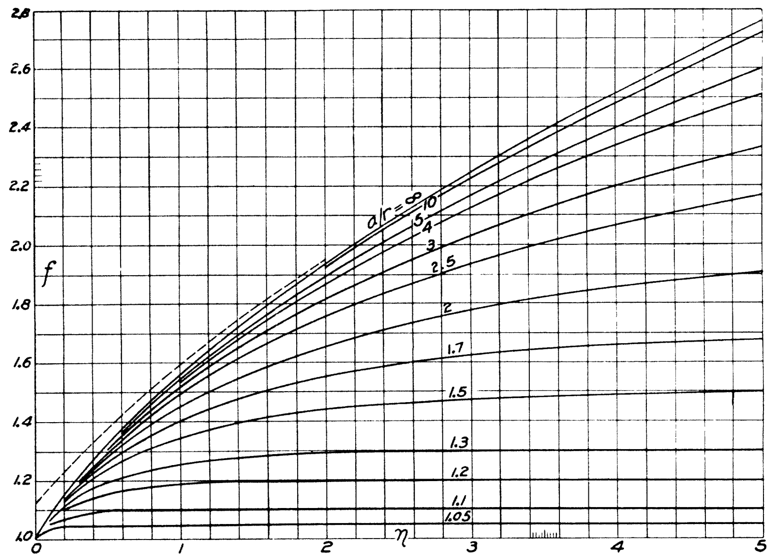

Fig. 36. Plot of f as function of η calculated from Eq. 252.

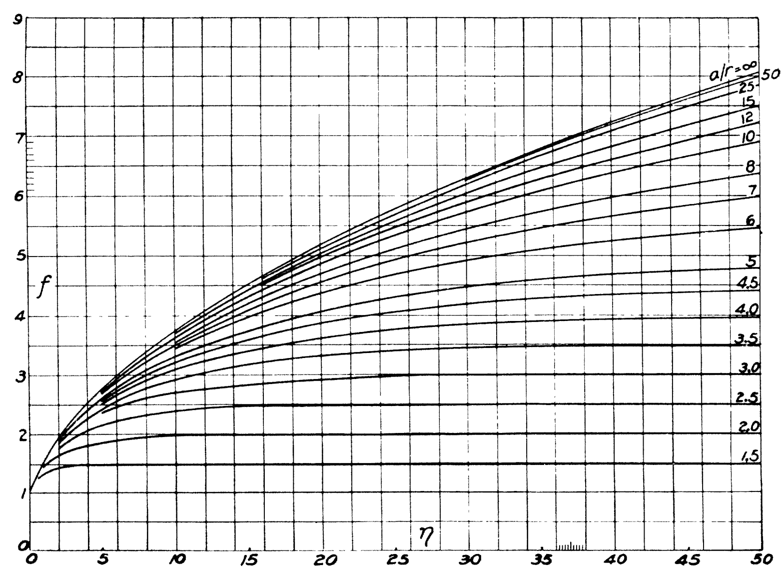

Fig. 37. Plot of f as function of η calculated from Eq. 252.

Here P ( ) stands for the probability integral defined by

$$P(x) = 2\pi^{-1/2}\int_0^x \epsilon^{-y^2}dy. \tag{255}$$

We shall see in a later chapter that these equations prove useful in determining the concentrations of electrons in gaseous discharges and therefore to avoid the rather laborious calculations of the function f, we have prepared the family of curves of Figs. 36, 37 and 38 which give f as a function of η for various values of r_0/r, which on these curves is given as a/r.

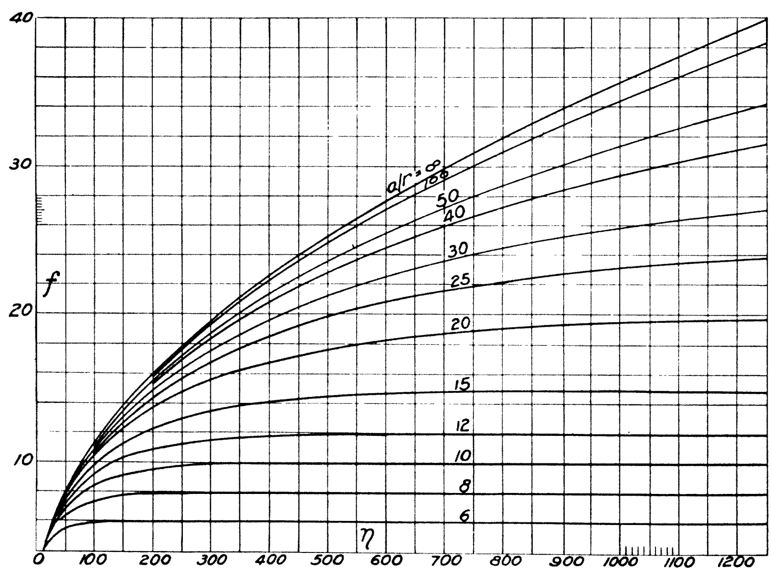

Fig. 38. Plot of f as function of η calculated from Eq. 252.

Under certain conditions, Eq. (252) approaches simple limiting forms by which the values of f are sometimes more conveniently obtained than by the curves.

As $\phi+\eta$ increases to values greater than about 3, f is given with rapidly increasing accuracy by

$$f = (r_0/r)P(\phi^{1/2}) + \epsilon^{-\phi}\pi^{-1/2}(\eta + \phi)^{-1/2}. \tag{256}$$

If $\phi > 3$, this expression becomes practically

$$f = r_0/r. \tag{257}$$

Substituting this in Eq. (251), remembering that $S_E/S_C = r_0/r$, we have

$$i = S_E I_E = i_{EF} \equiv i_0. \tag{258}$$

Thus, although strictly speaking the current from a concave emitter cannot be completely saturated, the current for all practical purposes has reached saturation value i_0 when $\phi > 3$ or

$$\eta > 3(r_0/r)^2 - 3. \tag{259}$$

To get approximate saturation from an external cylindrical emitter of 1 cm radius having a temperature of 1000°K, would take 1 volt on an internal collector of 0.5 cm radius, 26 volts if the radius were 1 mm and 2600 volts if a wire 0.1 mm radius were used as collector. These voltages increase in proportion to the temperature of the emitter, and are needed wholly to prevent the electrons from describing orbits which will carry them back to the emitter (not because of space charge).

The other limiting case of interest to us is that obtained when the radius of the emitter is large compared to that of the collector and the voltages applied are far less than those just considered. As r_0/r increases to infinity, f approaches a limiting value given by

$$f = 2\eta^{1/2}\pi^{-1/2} + \epsilon^{\eta}[1 - P(\eta^{1/2})]. \tag{260}$$

The topmost curves, marked $a/r = \infty$, in Figs. 36, 37 and 38 have been calculated by this equation.

An expansion of Eq. (252) in terms of r/r_0 gives

$$f = 2\pi^{-1/2}[\eta + 1 - 2\eta^2 r^2/3r_0^2]^{1/2} \tag{261}$$

which gives an excellent approximation for f whenever $3r_0^2/2\eta r^2$ and η are both large compared to unity. For large values of r_0/r it reduces to

$$f = 2\pi^{-1/2}(\eta + 1)^{1/2} \tag{262}$$

which agrees excellently (error < 1 percent) with Eq. (260) when $\eta > 3$. The dotted line in Fig. 36 gives the values of f according to Eq. (262) for $\eta < 2$.

For the corresponding flow of electrons between *concentric spheres*, the value of f to be used in Eq. (251) is

$$f = (r_0/r)^2(1 - \epsilon^{-\phi}) + \epsilon^{-\phi};\ \eta > 0. \tag{263}$$

When $\phi > 5$, this reduces to (error < 1 percent)

$$f = (r_0/r)^2 \tag{264}$$

and by Eq. (251) the current becomes practically equal to the saturation current from the external emitter.

With collectors of small radius, i.e., as r_0/r increases to infinity, f approaches another limiting value

$$f = \eta + 1. \tag{265}$$

Case II. Electrons flowing from an internal emitter to an external collector against a retarding field. The currents that flow under these conditions can be calculated from the values of f we have just found for Case I, by means of the reciprocal relation expressed by Eq. (247). Let us identify the outer (concave) electrode as A and the inner (convex) electrode as B. In Case I, A is the emitter while in Case II, B is emitter. The fraction λ_{AB} defined by Eq. (245) is the ratio of the current i given by Eq. (251), to the saturation current $i_0 = S_E I_E$ and thus

$$\lambda_{AB} = (S_B/S_A)f \tag{266}$$

where f applies to Case I.

In Case II, the fraction of the saturation current i_0 which flows from the internal emitter to the external collector is

$$i = i_0\lambda_{BA};\ V_{BA} > 0. \tag{267}$$

We may now substitute these values of λ_{AB} and λ_{BA} in Eq. (247) and then, still using η as a measure of the *collector* potential in accord with Eq. (250), we obtain

$$i = i_0 f\epsilon^{\eta};\ \eta < 0. \tag{268}$$

This equation allows us to calculate the current for Case II for cylinders or spheres in terms of the values of f given by Eqs. (252) to (265) or obtained from the curves of Figs. 36, 37 and 38. It should be noted that the values of r_0, r and η used to calculate f in this way are respectively: r_A, r_B and $V_{BA}e/kT$, where A refers to the external electrode. In Eq. (268), however,

$$\eta = V_{AB}e/kT \tag{269}$$

so that $\eta < 0$, whereas in calculating f, we must take $\eta > 0$.

The general expression for i which may be obtained for cylinders for Case II from Eq. (268) by introducing the value of f from Eq. (252), has already been derived by W. Schottky[282] and applied by him and by Germer[283] in experimental proof that the electrons emitted by a hot filament have a Maxwellian distribution of velocities.

In all these experiments the emitter was a small wire at the axis of a relatively large negatively charged cylindrical collector. Under such conditions a simple, yet accurate, equation for the current can be calculated from the value of f from Eqs. (262) and (268), giving

$$i = 2\pi^{-1/2} i_0 (1-\eta)^{1/2}\epsilon^{\eta};\ \eta < 0. \tag{270}$$

The corresponding equation for spheres, from Eqs. (265) and (268) is

$$i = i_0(1-\eta)\epsilon^{\eta};\ \eta < 0. \tag{271}$$

[282] W. Schottky, Ann. d. Physik **44**, 1011 (1914).
[283] L. H. Germer, Phys. Rev. **25**, 795–807 (1925).

Where r/r_0 is not much larger than unity, we found in Case I, currents close to saturation. Under similar conditions we now find from Eqs. (257), (264) and (268) that the current in Case II is given by the equation of the Boltzmann type, Eq. (249), although the collector in this case has a concave surface.

The general relationships between the effects of accelerating and retarding fields, and the curvatures of the electrode surfaces, for electron currents in high vacuum where space charge is negligible, are summarized in Figs. 39 and 40. The collector voltage (proportional to η) is plotted as abscissa and the

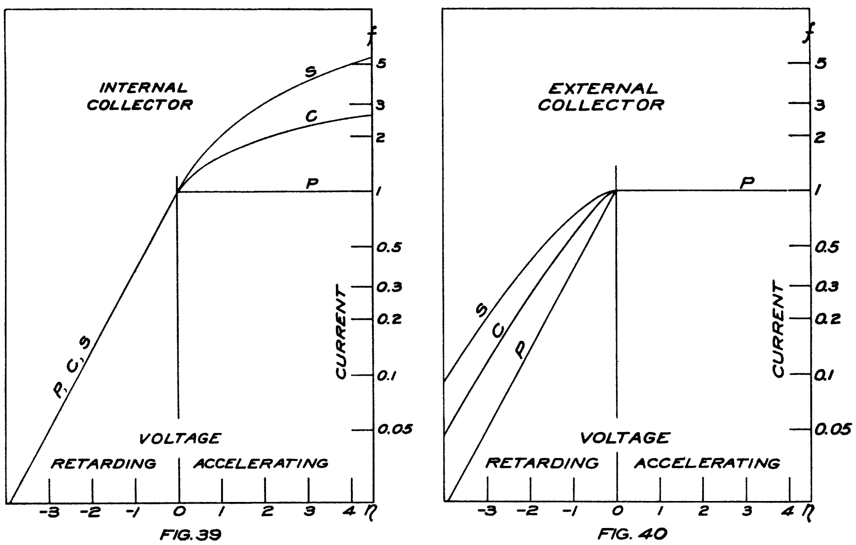

Fig. 39 and 40. Current-voltage relations for planes (P) cylinders (C) and spheres (S). Internal electrodes are assumed to be of small radius.

logarithm of the electron current as ordinate. Plotting in this way, as we see from Eq. (249), the current-voltage curve corresponding to the Boltzmann equation is a straight line whose slope is

$$\frac{d \log i}{dV} = \frac{e}{kT} = \frac{11606}{T} \text{ volts}^{-1} \tag{272}$$

or if ordinary logarithms are used

$$\frac{d \log_{10} i}{dV} = \frac{5040}{T} \text{ volt}^{-1}. \tag{273}$$

The curves of Figs. 39 and 40 were calculated by Eqs. (260), (265) and (268), assuming the radius of the internal electrode to be small compared to that of the external electrode.

Range of validity of foregoing equations. In deriving Eqs. (252) and (263), which determine the limitations of current brought about by orbital motions of the electrons between cylinders or spheres, it was not necessary to use Laplace's equation or to assume any particular potential distribution between the electrodes. The equations were, in fact, derived by calculating the radial and tangential components, u and v, of the electron velocities at the emitter and at the collector without needing to know these quantities in the space between the electrodes. At the emitter, the distribution of u and v was obtained from Maxwell's distribution law and at the collector they were obtainable from the former values by applying the laws of the conservation of energy and of angular momentum.

It is clear, however, that some restrictions upon the potential distribution are necessary. Schottky[284] pointed out that the method fails if, owing to space charge, a region of potential minimum exists between emitter and collector, the current then being "limited by space charge" instead of "orbital motions." (See Chapter IV.) Davisson was the first to state the conditions which the potential distribution must fulfill in order that the law of conservation of angular momentum may be correctly applied by the method used in deriving Eqs. (252) and (263). In this derivation the current was calculated by determining the number of electrons in orbits which intersect both the A and B surfaces: moving outward from the A surface (A being emitter) and into the B surface. It may happen, however, that an orbit which intersects both A and B consists of two branches (as, for example, the two branches of a hyperbola) separated in a radial direction by a region in which the radial velocity u has imaginary values. Obviously, such orbits must not be included among AB orbits, but one branch should be classed as an AA orbit, while the other is a BB orbit. Thus, as Davisson pointed out, no orbit can properly be classed as an AB orbit unless $u^2 > 0$ for every value of the radius vector lying between r_A and r_B. This condition for electron currents is fulfilled if the voltage V at every point between the electrodes satisfies the relation

$$V \geqq V_B \frac{(r_A/r)^2 - 1}{(r_A/r_B)^2 - 1}; \; r_B < r < r_A \tag{274}$$

where A is the outer electrode, whose potential is taken to be zero and B is the inner electrode at a positive potential V_B. For positive ion currents V and V_B will be negative and the direction of the inequality sign should be reversed.

Mott-Smith and Langmuir[285] derived this relationship independently and showed that the same equation is applicable for spherical electrodes and for internal as well as external collectors.

Investigation shows that Davisson's condition expressed by Eq. (274) is fulfilled not only when the potential distribution is that given by Laplace's equation (no space charge) but also with currents so large that the space charge is *nearly* sufficient to bring the potential gradient at the surface of the emitter to zero. In Chapter IV this matter will be discussed in more detail.

[284] W. Schottky, Ann. d. Physik **44**, 1011 (1914).

[285] H. M. Mott-Smith and I. Langmuir, Phys. Rev. **28**, 727–63 (1926), particularly pages 749–54.

In deriving the *reciprocal relation* between emitter and collector, expressed by Eq. (247), we assumed that space charge was negligible and thus the removal of electrons in AA or in AB orbits would not change the number of those describing BA orbits. It now apears, at least in the cases of coaxial cylinders and concentric spheres, that this restriction was unnecessarily severe. We have found in these cases that λ_{AB} and λ_{BA} are unchanged by the presence of space charge (the electrode potentials remaining fixed) provided the Davisson condition is still fulfilled. Thus the reciprocal relation is valid until the current density becomes so great that the Davisson condition is violated.

Effect of electron reflection at the electrodes.

The equations that we have derived for electron currents have been based on the assumption that emitter and collector absorb all incident electrons. There are two effects to be considered. Electrons reflected from the collector may return to the emitter and thus decrease the net current. When the emitter reflects electrons, it is no longer necessary that the emitted electrons shall have a Maxwellian velocity distribution and thus equations such as Eq. (252) may no longer be accurate. Consider first equilibrium conditions under which the complete MBD must exist. Then at the surface of electrode A, which we shall later take to be the emitter,

$$i_{AF} = i_{AA} + i_{AB} = i_0 + Ri_{BA} \tag{275}$$

where i_0 is the saturation emission and R is the effective reflection coefficient for electrons which have the velocity distribution present among the electrons in BA orbits. Now the electrons constituting the current Ri_{BA} will in general not have a Maxwellian velocity distribution, for the reflection coefficient is a function of velocity and probably the angle of incidence of the impinging electrons (Part I, p. 171–4). The total current i_{AF}, however, under equilibrium conditions must consist of electrons having an MBD. Therefore by Eq. (275), electrons constituting the saturation current i_0 will only have a true Maxwellian distribution when the reflection coefficient from the emitter is zero.

If the collector is at a positive potential (*accelerating field*) electrons reflected at its surface will not be able to return to the emitter since reflection generally involves energy loss. The reflected electrons increase the space charge, but unless this becomes sufficient to bring the potential gradient at the emitter to zero, the current from emitter to collector will not be modified.

When the collector is at a negative potential (*retarding field*), the field will tend to draw any reflected electrons back to the emitter. Thus if the collector *does not have a concave surface*, so that all reflected electrons return, the current will be

$$i = (1 - R)S_C I_B \epsilon^{\eta}. \tag{276}$$

This equation will thus apply in the case of retarding fields between plane electrodes, and between coaxial cylinders or concentric spheres if the collector is the internal electrode. Although the reflection coefficient R is in general a function of the average velocity of the incident electrons (and their velocity distribution), it should be noted in this case that the velocity distribution of the incident electrons corresponds to half a Maxwellian distribution and the average energy ($2kT$) is independent of the retarding voltage used. Thus, in spite of reflection from the collector, the current varies in proportion to ϵ^{η}. Reflection from the emitter, however, might alter the Maxwellian distribution of the emitted electrons and thus cause the $i \sim \epsilon^{\eta}$ relation to fail.

With *retarding fields* and a *concave collector*, for example, a negatively charged external cylindrical collector, electrons reflected from the collector may not be able to return to the emitter even with the accelerating field which acts on them. The velocities of the reflected electrons will usually be far higher than if they were emitted thermally so that the limitation by orbital motion, Eq. (252), will make the fraction of the reflected electrons which return to the emitter small, if the emitter radius is small compared to that of the collector. Under such conditions the effect of reflection tends to be negligible unless it brings about a limitation of current by space charge (see Chapter IV).

Effects of magnetic fields on the flow of electron currents.

A magnetic field of strength H, expressed in electromagnetic units (gauss) exerts a force eHv on a moving electron. Here v is the *component* of the electron's velocity in a direction perpendicular to the magnetic filed. This force acts in a direction perpendicular to the direction of motion of the electron and also perpendicular to the magnetic field. Thus any component of motion of the electron parallel to the magnetic field remains unaltered by this field.

Consider a magnetic field parallel to the Z-axis acting on an electron moving with velocity v in the XY plane. The electron then describes a circular path in this plane of such radius, r, that the force due to the magnetic field balances the centrifugal force. Thus

$$r = mv/eH = 5.65 \times 10^{-8} v/H \text{ cm}. \tag{277}$$

If we express the electron velocity in terms of equivalent volts, V,

$$r = 3.354 V^{1/2}/H \text{ cm}. \tag{278}$$

The direction of rotation is clockwise as seen in the direction of the magnetic lines of force. In presence of the magnetic field and of an electric field of strength X, in a direction parallel to the X-axis, the electron describes[286] a trochoidal path in the XY-plane, this motion being that of a point moving in a circle whose center drifts in a direction parallel to the Y-axis with a velocity X/H. If X is expressed in volts$\cdot$cm^{-1}, this drift velocity corresponds to

$$(m/2e)X^2/H^2 = 2.826 X^2/H^2 \text{ volts}. \tag{279}$$

[286] See "Conduction of Electricity Through Gases," J. J. Thomson, Cambridge Univ. Press, 2nd Edition (1906), p. 111 or 3rd Edition, p. 223.

A component of electric field parallel to the magnetic field (Z-axis) causes the same acceleration of the electrons as if no magnetic field were present.

Let us now consider the flow of electrons from a plane emitter to a parallel plane collector, with a magnetic field parallel to the plane surfaces. If the collector has a positive potential large compared to $T/11600$ volts (i.e., kT/e), the initial velocities of the electrons may be neglected. The electrons thus describe cycloidal paths. Each electron starts to move toward the collector, but its path becomes curved by the action of the magnetic field so that after reaching a certain distance x_0 from the emitter it again returns to the collector. This maximum distance is

$$x_0 = (2m/e)X/H^2 = 11.31X/H^2 \text{ cm}. \tag{280}$$

If V is the potential difference between two plane electrodes separated by the distance x, we may put $X = V/x$, and then from Eq. (280) we can conclude that current can flow between the electrodes only if

$$x < (2m/e)^{1/2}V^{1/2}/H = 3.363V^{1/2}/H. \tag{281}$$

With a given H and x, the current should be zero until V has reached a critical value and then rise abruptly to saturation. If V and x are held constant, the current will be saturated until H reaches a critical value and then falls abruptly to zero.

Theoretical and experimental studies of the effects of magnetic fields on the flow of current between coaxial cylinders have been made by Hull[287] in his investigations of the magnetron. The magnetic field was assumed to be parallel to the axis of the cylinders.

Magnetron. The critical magnetic field required to stop the flow of electron current from a central cathode to a coaxial anode of relatively large radius r is

$$H = (8m/e)^{1/2}V^{1/2}/r = 6.726V^{1/2}/r \text{ gauss}. \tag{282}$$

The effect of initial velocities of the electrons was shown to be negligible in all practical cases of interest. The experiments gave results in agreement with the theory within the limits of accuracy prescribed by the degree of symmetry of the apparatus.

Inverted magnetron. In the case of electrons flowing from an outer cylindrical cathode toward an internal coaxial anode, the effect of initial velocities corresponding to thermal agitation is by no means negligible. Consider an electron emitted with a tangential initial velocity v_i and let V_i be the potential needed to give to an electron this velocity. Then the longitudinal magnetic field needed to prevent this electron from reaching the inner cylinder is

$$H = (8m/e)^{1/2}(1/r_0)^2(rV^{1/2} \pm r_0V_i^{1/2}) \tag{283}$$

where r_0 is the radius of the emitter, r is that of the collector, and V is the accelerating potential of the collector. The + or the − sign is used according

[287] A. W. Hull, Phys. Rev. **18,** 31–57 (1921) and Jr. Amer. Inst. E. E. Sept. 1921

as the direction of the initial tangential velocity is opposite to or is the same as the direction of the magnetic deflection.

The effects of the radial components of the initial velocities are of far less importance than the tangential components. The complete theory is, however, given by Hull, not only for cylindrical but also for plane electrodes.

Experiments again gave results in good agreement with the theory. With an anode at 110 volts having a radius 1/25th of that of the surrounding cylindrical cathode, a field of only 8 gauss brought the current to half value, but this cut-off, because of the effect of initial velocities, was gradual instead of being sharp as in the case of the external anode. The shape of the curves thus obtained (i as function of H) should, in fact, with a carefully constructed cylindrical cathode, furnish an exceptionally accurate means of determining the velocity distribution of the emitted particles.

The equations that have been derived by Hull for the critical magnetic field are not dependent on any specific assumptions regarding the potential distribution and therefore require no modification as the potential distribution is altered by space charge. However, just as in the absence of magnetic field, no electron can pass from emitter to collector unless the expression for its radial velocity u shows that $u^2>0$ for all values of the radius lying between r_0 and r. This should furnish a kind of generalized Davisson relation analogous to Eq. (274).

The *Maxwell-Boltzmann distribution* of a system of electrons in equilibrium at temperature T, which is characterized by Eqs. (233), (234), (236) and (238), is not altered by a steady magnetic field, although the orbits of the electrons are changed. The methods which we used in studying the flow of electrons between the electrodes A and B, dividing the orbits into 5 types, may still be used in presence of magnetic fields.

Thus the reciprocal relation expressed by Eq. (247) or (268) is applicable in a magnetic field (even if the field is not uniform or symmetrically arranged with respect to the electrodes) provided the generalized Davisson condition is fulfilled.

For example, in an "inverted magnetron" (inner anode) it was found experimentally that with +110 volts on the anode, the current was brought down to ½ the saturation value by a longitudinal magnetic field of 8 gauss. We may conclude from Eq. (247) that if the internal electrode had then been heated to the temperature previously used for outer electrode, and the latter had then been cooled without altering the potentials of the electrodes or the magnetic field, the current under the new conditions would be ½ that calculated by the Boltzmann Eq. (249). Since the current decreases gradually with increasing magnetic field in the inverted magnetron with accelerating field, the same must be true of the current from an internal emitter to an external collector against a retarding field.

Similarly, since the cut-off for the ordinary magnetron (external anode) is very sharp at a definite field strength H, we conclude that with an internal collector at a negative potential, the current would be given accurately by the Boltzmann equation, Eq. (249), until the magnetic field increases to this same value of H, and then the current must drop abruptly to zero.

By dimensional reasoning, applied to geometrically similar devices of size r, or directly from Eqs. (247) (282) and (283), we see that λ_{AB} or λ_{BA} (the fraction of the emitted current that reaches the collector) is a function of (V_{BA}/T) and $(Hr/T^{1/2})$ only. In this way, the reciprocal relation may be used when $T_A \neq T_B$ or the effect of changing the emitter temperature can be calculated in terms of the effects produced by changing V_{BA} or H.

Chapter IV. The Effect of Space Charge on the Flow of Current in High Vacuum

The current of electrons that flows in high vacuum from a hot cathode to a positively charged electrode (anode) increases at first in proportion to the emission of the cathode and will equal this emission (saturation current) if the cathode surface is not concave.

By raising the cathode temperature, the space charge can thus be increased in proportion to the emission. This, however, cannot go on indefinitely, for finally, in accord with Poisson's Equation, a potential minimum must appear in space, and the cathode will be surrounded by a potential barrier which limits the escape of electrons.

This limitation of current by space charge begins when the emission is raised to a critical value which first brings the potential gradient to zero at some point of the cathode surface.

Space charge equation for parallel planes.

The effect of space charge in limiting the current that can flow between electrodes in high vacuum is best illustrated by considering the passage of electrons from a heated plane cathode at zero potential, to a parallel plane anode at potential V_a.

Let us assume, to begin with, that the initial velocities of the electrons as they leave the cathode can be neglected and therefore at any point at potential V

$$\tfrac{1}{2}mv^2 = Ve. \tag{284}$$

If I is the current density, the space charge ρ is

$$\rho = I/v. \tag{285}$$

Eliminating v between these equations to get ρ and substituting this value in Poisson's equation, gives

$$\Delta V = 2\cdot 2^{1/2}\pi(m/e)^{1/2}IV^{-1/2}. \tag{286}$$

For the case of parallel planes where ΔV is equal to d^2V/dx^2, integration gives

$$(dV/dx)^2 = E_0^2 + 8\pi(2m/e)^{1/2}IV^{1/2} \tag{287}$$

where E_0, the integration constant, is equal to the electric field intensity at the surface of the cathode.

With the cathode at such low temperature that emission is negligible, $dV/dx = E_0$, so that there is a linear potential distribution between the electrodes as indicated by Curve I, Fig. 41. By raising the cathode temperature so that I increases, the field near the anode increases, and must thus decrease near the cathode. The curve giving the potential V as a function of x, the distance from the cathode, becomes concave upwards (d^2V/dx^2 being positive), and finally when I becomes sufficiently large, E_0 must decrease to zero.

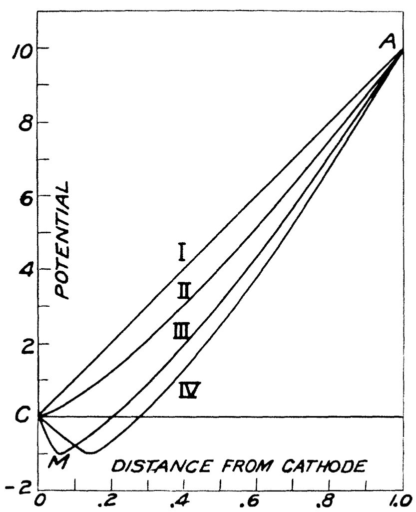

Fig. 41. Potential distribution between plane cathode C and parallel plane anode A.

Until this happens, however, there is everywhere an accelerating field so that every electron emitted must pass to the anode and the current is limited only by the cathode emission, the current being the saturation current I_s. If E_0 becomes negative, there is a retarding field at the cathode and thus, since the initial velocity is assumed to be zero, the current would have to be zero. It is clear, therefore, that no matter how large I_s may become, the current that flows between the electrodes cannot increase beyond the definite value which would make $E_0 = 0$. Imposing this condition in Eq. (287) we have,

$$dV/dx = (8\pi)^{1/2}(2m/e)^{1/4}I^{1/2}V^{1/4}. \tag{288}$$

Integration gives[288]

$$I = \frac{2^{1/2}}{9\pi}\left(\frac{e}{m}\right)^{1/2}\frac{V^{3/2}}{x^2}. \tag{289}$$

[288] This equation was first derived by C. D. Child, (Phys. Rev. **32,** 492 (1911)) in a theoretical and experimental investigation of the magnitude of currents that could be carried by positive ions in arcs at low pressures. It was independently derived, and applied to electron currents in high vacuum by I. Langmuir, Phys. Rev. **2,** 450–486 (1913) and Phys. Zeits. **15,** 348, (1914).

This equation is applicable to unipolar currents carried either by electrons or by positive or negative ions which leave the emitting surface with negligible initial velocities. For electrons we may take $e/m = 5.279 \times 10^{17}$ e.s. units[289] and expressing V in volts and x in cm, Eq. (289) becomes

$$I = 2.334 \times 10^{-6} V^{3/2} x^{-2} \text{ amps}\cdot\text{cm}^{-2}. \qquad (290)$$

If, however, the carriers are singly charged ions having a "molecular weight" M, (oxygen atom $= 16$), then we must put

$$e/m = 2.893 \times 10^{14} M^{-1} \text{ e.s. units} \qquad (291)$$

and our space-charge equation is

$$I = 5.462 \times 10^{-8} M^{-1/2} V^{3/2} x^{-2} \text{ amps}\cdot\text{cm}^{-2}. \qquad (292)$$

With doubly charged ions the currents would be $2^{1/2}$ times greater.

The current densities given by these equations are those which will bring E_0, the field intensity at the cathode, to zero.

In this derivation V has been treated as a function of x while I is constant. The potential V thus varies in proportion to $x^{4/3}$ as indicated by curve II in Fig. 41, while the field intensity E is proportional to $x^{1/3}$.

If, however, in the equation we let x be the distance between the plane electrodes and V the difference in potential between them, the equation expresses the relation between the current and the voltage often referred to as the 3/2-power law.

There are thus two limitations to the current: it cannot exceed either the saturation current I_s, or the space-charge limited value given by Eq. (290). If the potential difference between the electrodes is raised gradually from 0, the current-voltage curve shows two distinct regions. In the first or space-charge region, the current increases in proportion to $V^{3/2}$, is inversely proportional to x^2 and is independent of the cathode material or temperature. In the second region the current is equal to the saturation current I_s and thus varies with cathode temperature or material, but is independent of V or x.

Objection is sometimes raised to the derivation of this equation on the ground that if $v = 0$ and $E_0 = 0$ at the cathode surface, then no current could flow. This difficulty seems rather academic if we consider that $I = \rho v$ and under our assumed conditions this takes the indeterminate form $\infty \times 0$. A complete justification of Eq. (290) is found in the fact that a more rigorous derivation based on a complete Maxwellian distribution of the velocities of the emitted electrons leads, as we shall see, to an equation which becomes identical with Eq. (290) when the initial velocities of the electrons approach zero.

Effects of initial velocities. We know, of course, that the electrons are actually emitted from hot cathodes with appreciable velocities and therefore Eq. (290) is to be regarded only as an approximation.

[289] Birge, Rev. Modern Phys. **1**, 1 (1929); C. T. Perry and E. L. Chaffee, Phys. Rev. **36**, 904–18 (1930).

As a second approximation let us consider what changes would be brought about in Eq. (290) if the electrons were all emitted with a definite velocity component v_0 in a direction normal to the cathode surface.[290] Velocity components parallel to the cathode surface may be present but do not need to be taken into account in the following derivations.

If the current is to be limited by space charge, a *fraction* of the electrons must return to the cathode. This can occur only if there exists between the cathode and anode a retarding field just sufficient to bring the electrons to rest; i.e., there must be a potential minimum (M in Fig. 41) in the space at a certain distance, say x_0, from the cathode, where the potential is $-V_0$, V_0 being the volt-equivalent of the initial velocity v_0.

If we measure V and x from the potential minimum instead of from the cathode, it is evident that Eq. (284) is still applicable. In Eq. (287) we may put $E_0=0$ (at $x=0$) and thus in the region between M and the anode A the potential distribution is given as before by Eqs. (288), (289) and (290).

Between M and the cathode C, electrons are moving in two opposite directions, the current density of electrons away from the cathode being I_s and I_s-I in the opposite direction, where I_s corresponds to the saturation current and I to the actual current flowing to the anode. Since the space charge produced by a given current in either direction is the same, the space charge ρ at any point is the same as though there were a current $2I_s-I$ in one direction only. Thus the potential distribution between C and M is given by Eq. (289) or (290) if we replace I by $2I_s-I$.

Thus we have

$$2I_s - I = 2.33 \times 10^{-6} V_0^{3/2} x_0^{-2} \tag{293}$$

and

$$I = 2.33 \times 10^{-6} V_a^{3/2} x_a^{-2} \tag{294}$$

where x_a is the distance from M to A and V_a is the potential of A with respect to M. Dividing one equation by the other and placing $x=x_0+x_a$ and $V=V_a-V_0$ (the potential difference between A and C), we get

$$x_0 = x/[1 + \{1 + (V/V_0)\}^{3/4}\{2(I_s/I) - 1\}^{1/2}] \tag{295}$$

or if $V_0 \ll V$,

$$\frac{x_0}{x} = \left(\frac{V_0}{V}\right)^{3/4}\left(\frac{I}{2I_s - I}\right)^{1/2} \tag{296}$$

approximately.

Curves III and IV in Fig. 41 have been calculated for the case $V=10$ volts, $V_0=1$ volt, $x=1$ cm, I_s/I being taken equal to 5 for curve III and 1 for curve IV.

[290] J. J. Thomson, in the 2nd Edition of his book, Conduction of Electricity Through Gases (1906) p. 223 (also p. 372 in 3rd Edition, 1928) made this assumption in treating the space charge problem between parallel planes. He obtained an equation like Eq. (287) except that it was in far more complicated form.

Under the conditions we have assumed, where the emitted electrons all have the same velocity component v_0 normal to the cathode surface, the space charge at M is theoretically infinite just as it previously was at the cathode surface in the case $v_0 = 0$. The surface of minimum potential thus acts exactly as a cathode giving a space-charge limited current I to the anode A. On the other side, facing C, it receives a current I_s and gives off a current $I_s - I$. A *"virtual cathode"* of this sort has its position and effective potential fixed by the number and the velocities of the electrons that are projected towards it by the neighboring actual electrodes. We shall find other cases of virtual cathodes in considering space charge currents in three-electrode tubes.

The assumption that the electrons are emitted from the cathode with a uniform normal velocity component v_0 is still objectionable in that it leads to an infinite space charge density at M. However, the foregoing analysis is useful in introducing the conception of the "virtual cathode" and showing how the effect of initial velocities can be taken into account merely by measuring V and x from M instead of from C.

Space charge equations for electrons emitted with Maxwellian velocity distribution. The electrons given off by a hot cathode actually have a Maxwellian velocity distribution.* On this basis Schottky[291] obtained an approximate solution for x_0, the distance from the cathode to the potential minimum when the current is limited by space charge. Complete solutions of the problem of the potential distribution have been obtained by Epstein, Fry and Langmuir.[292] The following summary of results is based on Langmuir's treatment of the problem, but an attempt is made to put the equations in more convenient form than in prior publications.

With x as abscissa to measure distances from the cathode surface, we let x_1, x_2 and x_M be the abscissas of the cathode, anode and surface of minimum potential, respectively. Similarly, we let V be the potential at any surface at x and V_1, V_2 and V_M at cathode, anode and x_M. The saturation current density is I_s and the actual current density to the anode is I, as limited by space charge. The potential V_M at the minimum can be calculated directly by means of the Boltzmann equation* which can be put in the form

$$V_1 - V_M = (T/5040) \log_{10} (I_s/I) \text{ volts.} \tag{297}$$

The problem of determining the potential distribution is greatly simplified by choosing, as origin for a $V(x)$ plot, the point x_M, V_M, and also by replacing x and V by dimensionless parameters ξ and η which serve as measures of x and V in terms of units of distance and potential appropriate to the particular case considered. Thus we put

* See Part I, p. 204–207.

[291] W. Schottky, Phys. Zeits. **15,** 526 and 624 (1914).

[292] P. S. Epstein, Verh. d. D. Phys. Ges. **21,** 85 (1919). Tables are given by which the potential distribution may be obtained. However, all the equations involving the Boltzmann constant k involve an error which may be corrected by replacing k by $2k$. T. C. Fry, Phys. Rev. **17,** 441 (1921) and ibid., **22,** 445 (1923). I. Langmuir, Phys. Rev. **21,** 419 (1923).

* See Eqs. (9) and (10) Part I, page 139.

$$\eta = e(V - V_M)/kT = (11606/T)(V - V_M) \tag{298}$$

$$\xi = 9.174 \times 10^5 T^{-3/4} I^{1/2}(x - x_M) \tag{299}$$

where I is in amps. · cm^{-2}, T in degrees K, x in cm.**

An equivalent expression for ξ is

$$\xi = (8\pi n_M e/V_e)^{1/2}(x - x_M) \tag{300}$$

where n_M is the electron concentration (electrons cm^{-3}) at the potential minimum and V_e is the potential equivalent to the electron temperature T in accord with Eq. (8), Part I. The insertion of numerical values of e and V_e (in terms of T) gives

$$\xi = 0.2042(n_M/T)^{1/2}(x - x_M). \tag{301}$$

Comparison of Eqs. (299) and (301) gives for the electron concentration at x_M

$$n_M = (\pi m/2kT)^{1/2}(I/e) = 2.018 \cdot 10^{13} I T^{-1/2} \tag{302}$$

if I is in amps. · cm^{-2}. This is also directly derivable from Eq. (74) Part I, remembering that in the present case at x_M we have only half of a complete Maxwellian distribution.

The electron concentration n is given by

$$n = n_M[1 \mp 1 \pm \text{erf}\, \eta^{1/2}]\epsilon^\eta \tag{303}$$

where the error function is defined by

$$\text{erf}\, x = 2\pi^{-1/2} \int_x^\infty \exp(-y^2)dy. \tag{304}$$

The upper or lower signs are to be taken according as $x - x_M$ is + or − respectively.

If $\eta > 3$, a good approximation is

$$n = n_M[(1 \mp 1)\epsilon^\eta \pm (\pi\eta)^{-1/2}] \tag{305}$$

Langmuir[293] has published a table of ξ as a function of η, from which data the Curve I of Fig. 42 has been prepared.

The procedure for calculating the potential distribution when I, I_s, T are given is as follows. First calculate η_1 (the value of η at the cathode surface) by the relation

$$\eta_1 = 2.303 \log_{10}(I_s/I) \tag{306}$$

which is easily derivable from Eqs. (297) and (298), and then from Curve I of Fig. 42 get the corresponding value of ξ_1, which will be negative. Substitu-

** The constant 9.174×10^5 is equal to $4(\pi/2k)^{3/4} m^{1/4} e^{1/2}$ where m is the mass of the electron. In order to obtain I in amps · cm^{-2}, e should be taken to be $1.4300 = 4.770 \times 10^{-10} \times 0.1\ c$, c being the velocity of light 2.998×10^{10} cm · sec^{-1}.

[293] I. Langmuir, Phys. Rev. **21**, 419 (1923).

tion of ξ_1, T and I in Eq. (299) gives $x_M - x_1$. For any other value of x the corresponding ξ can be found; the curve gives η for this point and Eq. (298) then gives $V - V_M$. Comparison with Eq. (297) gives V at the desired point.

To calculate the current as a function of anode potential when I_s, T and the distance $x_2 - x_1$ between anode and cathode are given, the foregoing

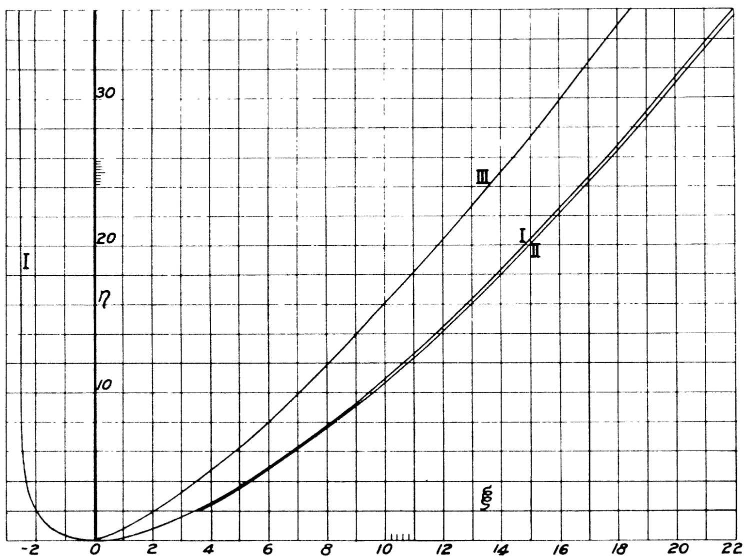

Fig. 42. Curve I gives ξ as function of η. ξ and η measure distance and potential respectively in terms of specially defined units. Curve II corresponds to Eq. (310), while Curve III corresponds to Eq. (289).

$\xi = 0$ represents position of potential minimum M.
$+\xi$ applies to space between M and anode.
$-\xi$ applies to space between M and cathode.

method is used to calculate the potential V at $x = x_2$ for a series of chosen values of I, and thus a curve is produced giving the desired relation between I and V_2.

The field intensity $E = -dV/dx$ at any point can be found without use of the function ξ. It is given by

$$E^2 = 8\pi e n_M V_e[(1 \mp 1 \pm \operatorname{erf} \eta^{1/2})\epsilon^\eta - 1 \pm 2\pi^{-1/2}\eta^{1/2}] \tag{307}$$

using the same convention for the signs as in Eq. (303). To obtain E in volts·cm^{-1}, the factor in front of the bracket can be placed

$$8\pi e n_M V_e = 3.097 \times 10^{-10} n_M T = 6250 \cdot I T^{1/2}. \tag{308}$$

Langmuir has shown, for large values of η and $x > x_M$ that ξ can be expanded in terms of inverse powers of η:—

$$\xi = (2/3)2^{1/2}\pi^{1/4}\eta^{3/4} + 1.6685\eta^{1/4} \cdots \tag{309}$$

Substituting this in Eq. (299), squaring and combining with Eq. (298) gives

$$I = \frac{2^{1/2}}{9\pi}\left(\frac{e}{m}\right)^{1/2} \frac{(V - V_M)^{3/2}}{(x - x_M)^2}(1 + 2.658\eta^{-1/2}) \tag{310}$$

which is the usual three-halves power law, as given by Eq. (289), except that the distances and voltages are measured from the potential minimum and the correction factor given in parenthesis is added. The 2nd term in this correction factor has its origin in the 2nd term of the expansion for ξ, (Eq. 309)), whereas the usual three-halves power equation results from the first term.

To show the relative magnitude of the errors made by using the uncorrected equation, Curve III in Fig. 42 has been drawn to give the values of the 1st term in Eq. (309), while Curve II has been obtained by multiplying this first term by $(1+2.658\eta^{-1/2})^{1/2}$ which is equivalent to using the approximation made in Eq. (310). Examination of these curves shows that the ordinary space charge equation (289) or (290) involves rather large errors; even when η is as large as 100 these errors amount to 10 percent. The first order correction term employed in Eq. (310), however, makes the agreement (Curve II) good even for values of η as small as 1 where the error in ξ is about 2 percent.

For nearly all practical cases, therefore, in which the effect of the initial velocities needs to be considered, Eq. (310) is sufficiently accurate. Inserting the numerical values as in Eq. (290) to express I in amps.$\cdot$cm^{-2}, V in volts and x in cm we have

$$I = 2.334 \times 10^{-6}\frac{(V - V_M)^{3/2}}{(x - x_M)^2}[1 + 0.0247T^{1/2}(V - V_M)^{-1/2}]. \tag{311}$$

In using this equation V_M is found directly from Eq. (297); putting $V_1 = 0$ if we choose to measure voltages from the cathode. Measuring distances (x) from the cathode we can find x_M from Eq. (299) which may for convenience be written

$$x_M = 1.090 \times 10^{-6}(-\xi_1)T^{3/4}I^{-1/2}\ \text{cm}. \tag{312}$$

The quantity $(-\xi_1)$, according to Fig. 42, cannot exceed 2.55 and yet is greater than 2.0 if $I_s/I > 7$. Taking $-\xi_1 = 2.55$, we get as a convenient expression

$$x_M < 0.0016(1000I)^{-1/2}(T/1000)^{3/4}\ \text{cm} \tag{313}$$

and thus with a current density of 1 ma per cm^2 and a cathode temperature of 1000° K, x_M is only 0.0016 cm.

The equations that we have derived for the potential distribution between parallel planes are only applicable when there is actually a potential minimum between the electrodes. When the retarding field extends to the anode, the current is no longer limited by space charge and the Boltzmann equation alone is sufficient to determine the current that flows. The potential distributions in some cases of retarding fields extending from cathode to anode have been treated by Langmuir[294] and by v. Laue.[295]

Space charge equations for cylinders and spheres.

In considering the space charge limitation of current flowing between concentric cylinders or spheres, we have to distinguish between the functions of the two electrodes. For example, we may have electron currents from a hot wire cathode at the axis of a cylindrical anode, or we may wish to treat the inverted case of a small cylindrical anode inside of a large electron-emitting cylindrical cathode. Since, however, we may want to apply our equations to the flow of positive ions as well as electrons, it is better not to refer to the two electrons as cathode and anode, but as emitter and collector.

As in the derivation of Eqs. (289) and (290), let us assume that the carriers of current escape from the emitter without initial velocity so that the velocity is given by Eq. (284) if we take the potential of the emitter to be zero. Using Poisson's equation in the form of Eq. (137), we now obtain in place of Eq. (286)

$$\frac{d^2V}{dr^2} + \frac{\kappa}{r}\frac{dV}{dr} = 2(2)^{1/2}\pi(m/e)^{1/2}IV^{-1/2} \tag{314}$$

where κ is 1 for cylinders and 2 for spheres.

Let us now replace the variables V and r by two new dimensionless variables γ and β defined by

$$\gamma = \log_\epsilon (r/r_0) \tag{315}$$

$$I = \frac{1}{9\pi}\left(\frac{2e}{m}\right)^{1/2}\frac{V^{3/2}}{r^2\beta^2} \tag{316}$$

where r_0 is the radius of the emitter.

Elimination of I and r from Eq. (314) by means of these two equations gives

$$\frac{d^2V}{d\gamma^2} + (\kappa - 1)\frac{dV}{d\gamma} = \frac{4}{9}\frac{V}{\beta^2} \tag{317}$$

Since the current i between emitter and collector is independent of r, the current density I varies inversely as r^κ or in proportion to $\epsilon^{-\kappa\gamma}$ so that

$$dI/d\gamma = -\kappa I. \tag{318}$$

Introduction in this equation of the value of I from Eq. (316) gives

$$3\frac{dV}{d\gamma} = 2(2 - \kappa)V + \frac{4V}{\beta}\frac{d\beta}{d\gamma}\cdot \tag{319}$$

[294] I. Langmuir, Phys. Rev. **21,** 419 (1923).
[295] M. v. Laue, Jahrb. d. Radioakt. u. Elektronik **15,** 205 (1918).

Differentiation gives an expression for $d^2V/d\gamma^2$. By substituting this and the value of $dV/d\gamma$ from Eq. (319) in Eq. (317), V is eliminated and we obtain,

$$3\beta\frac{d^2\beta}{d\gamma^2} + \left(\frac{d\beta}{d\gamma}\right)^2 + (5 - \kappa)\beta\frac{d\beta}{d\gamma} + \beta^2(1 + 0.5\kappa - 0.5\kappa^2) = 1. \qquad (320)$$

For cylinders ($\kappa = 1$), this gives,

$$3\beta\frac{d^2\beta}{d\gamma^2} + \left(\frac{d\beta}{d\gamma}\right)^2 + 4\beta\frac{d\beta}{d\gamma} + \beta^2 = 1. \qquad (321)$$

For spheres ($\kappa = 2$), to prevent confusion, we will replace β by α and thus find

$$3\alpha\frac{d^2\alpha}{d\gamma^2} + \left(\frac{d\alpha}{d\gamma}\right)^2 + 3\alpha\frac{d\alpha}{d\gamma} = 1. \qquad (322)$$

To solve our space-charge problem we need to integrate these equations, subject to the boundary conditions: V and $dV/dr = 0$ at $r = r_0$ or $\gamma = 0$.

Solutions of these equations may be obtained in the form of power series in γ as follows:[296]

$$\beta = \gamma - (2/5)\gamma^2 + (11/120)\gamma^3 - (47/3300)\gamma^4 + 0.00168\gamma^5 \cdots \qquad (323)$$

and

$$\alpha = \gamma - 0.3\gamma^2 + 0.075\gamma^3 - 0.01432\gamma^4 + 0.00216\gamma^5 \cdots \qquad (324)$$

That these solutions satisfy the boundary conditions can be proved by substituting the values of α or β into Eq. (319), noting that $dV/d\gamma = r\,dV/dr$ and that V/β approaches zero as γ and V approach zero, since $V^{3/4}/\beta$ remains finite according to Eq. (316).

For larger values of γ where the series for α and β converge too slowly, the following approximate equations may be used which become more accurate as $|\gamma|$ increases.

Cylindrical case. External collector ($\gamma > 0$). Examination of the values of β calculated from the series of Eq. (323) using positive values of γ shows that as r/r_0 increases, β rises to a maximum and then decreases to a minimum.

For large values of γ the second term in Eq. (321) can be neglected and the equation can then be integrated giving,

$$\beta = 1 + A(r_0/r)^{2/3} \sin\left[(2^{1/2}/3)\log_\epsilon(Br/r_0)\right]. \qquad (324)$$

The integration constants A and B have been determined empirically from

[296] The series for β was published by I. Langmuir Phys. Rev. **2**, 450 (1913) and Phys. Zeits. **15**, 348 (1914). In the Phys. Rev. paper, through a printer's error, the coefficient of the second term was incorrectly given. The method of calculating the coefficients of this series was described by I. Langmuir and K. B. Blodgett, Phys. Rev. **22**, 347 (1923), the coefficients of 14 terms being given. The series for α was derived later by I. Langmuir and K. B. Blodgett, Phys. Rev. **24**, 49 (1924). Values of β^2 and α^2 were tabulated in these papers.

values of β found from the series of Eq. (323): $A=0.9769$, $B=0.08383$. For convenient calculation the equation may then be written,

$$\beta = 1 + 0.9769(r_0/r)^{2/3} \sin\left[1.0854 \log_{10}(r/11.93r_0)\right]. \tag{325}$$

This equation gives β accurate to 1 percent at $r/r_0=12$ and within 0.1 percent at $r/r_0=44$. Thus β is a damped harmonic function of γ, rapidly approaching the limit $\beta=1$. The following table gives some of the maximum and minimum values of β, and the values of γ and r/r_0 at which $\beta=1$.

TABLE XX. *Maxima and minima of* β. (External collector).

γ	r/r_0	$\beta-1$	β^2
2.4234	11.284	0	1.00000
3.7827	44.05	0.04621	1.09455
9.143	9.352×10^3	0	1.00000
10.449	3.451×10^4	-5.321×10^{-4}	0.999894
15.808	7.332×10^6	0	1.0
17.113	2.706×10^7	$+6.258\times10^{-6}$	1.000013

Between the successive maxima and minima or *vice versa*, the radius r increases 784 fold and the values $\beta-1$ change in the ratio 85.02:(−1). The curve marked β^2 in Fig. 43 gives values of β^2 for this case up to $r/r_0=1000$.

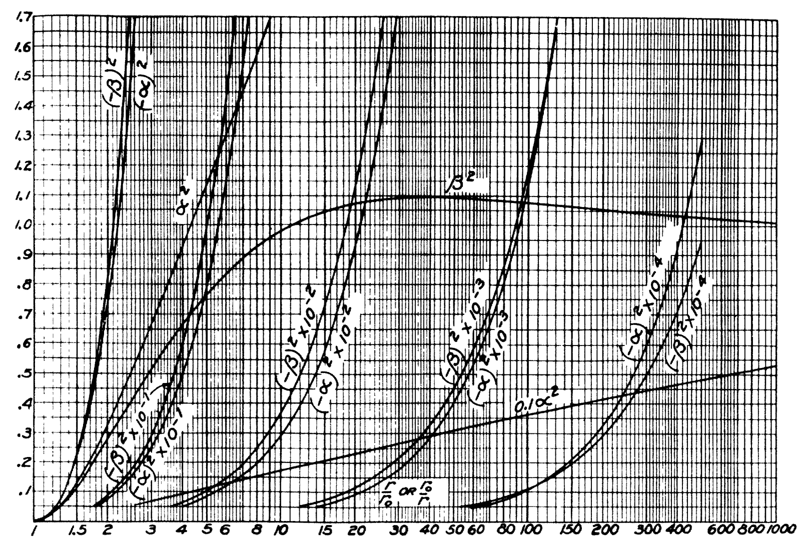

Fig. 43. Plot of β^2, $(-\beta)^2$, α^2, $(-\alpha)^2$ as functions of radius. r_0 = radius of emitter; r = radius of collector. β^2 and α^2 apply to cases for which $r>r_0$. $(-\beta)^2$ and $(-\alpha)^2$ apply to cases for which $r_0>r$.

Cylindrical case. Internal collector. ($\gamma<0$). When r_0/r is large, the values of β^2 are given by

$$\beta^2 = 4.6712(r_0/r)\left[\log_{10}(r_0/1.4142r)\right]^{3/2}. \tag{327}$$

By inserting this value of β^2 in Eq. (316) and comparing the result with Eq. (146), it appears that the electric field near the collector is the same as that which would be produced in the absence of space charge by an emitting cylinder having a radius 0.7071 r_0. In this inverted case, the effect of space

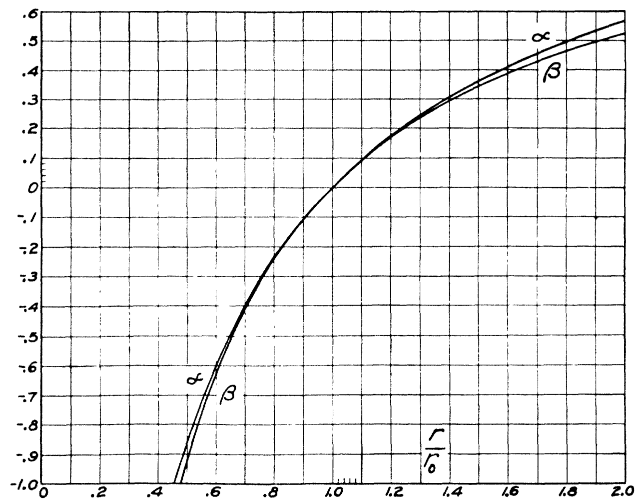

Fig. 44. Plot of β and α as functions of r/r_0 in the neighborhood of $r/r_0=1$.

charge near the collector is therefore negligible, but the space charge near the emitter has an effect equivalent to a reduction in radius of the emitter in the ratio 1.4142:1.

Spherical case. External collector. ($\gamma>0$). For very large values of r/r_0, the following equation gives a good approximation,

$$\alpha^2 = (2/3)\gamma + 0.5158 \log_{10}(3.885\gamma). \tag{328}$$

The error in α^2 calculated by this equation is about 0.5 percent for $\gamma=6$ and 0.02 percent at $\gamma=12$.

Spherical case. Internal collector. ($\gamma<0$.) For large values of r_0/r, α^2 is given quite accurately by*

$$\alpha^2 = [1.11(r_0/r) - 1.64]^{3/2}. \tag{329}$$

The error in α^2 involved in this approximation is 0.5 percent at $r_0/r=9$ and 0.02 percent at $r_0/r=20$. Comparison of this equation with Eqs. (316) and (147) shows that the electric field near the collector is the same as that which would be produced in the absence of space charge by an emitting sphere having a radius 0.677 r_0.

Space charge equation for cylinders (*Negligible initial velocities*). The current density I at any point between the cylinders is given by Eq. (316). We

* In the paper by I. Langmuir and K. B. Blodgett, Phys. Rev. **24**, 49 (1924), in the equation on p. 54, Eq. (15), $y=\alpha^2/2$ was inadvertently used in place of α^2 but the tabulated values were correct.

are usually interested in knowing the current i flowing between the cylinders which is given by

$$i = \frac{2}{9}\left(\frac{2e}{m}\right)^{1/2} \frac{LV^{3/2}}{r\beta^2} \tag{330}$$

where r is the collector radius and L, the length of the cylinders, is so great compared to the radius that end-corrections are negligible.

For unipolar currents carried by electrons we may put $e/m = 5.279 \times 10^{17}$ e.s. units and with V in volts the equation becomes

$$i = 14.66 \times 10^{-6} LV^{3/2}/(r\beta^2) \text{ amperes.} \tag{331}$$

If, however, the currents are carried wholly by singly charged ions of mass m, the equation becomes

$$i = 14.66 \times 10^{-6}(m_e/m)^{1/2} LV^{3/2}/(r\beta^2) \text{ amperes} \tag{332}$$

which may be written

$$i = 3.432 \times 10^{-7} M^{-1/2} LV^{3/2}/(r\beta^2) \tag{332–a}$$

where M is the "molecular weight" of the ion (oxygen atom = 16).

The values of β^2 to be used in these equations may be calculated by the methods already given or may be read off the curves of Fig. 43. When r/r_0 is not far from unity α^2 or β^2 is more accurately obtained from the values of α and β given in Fig. 44.

These equations may be used to calculate the space charge limited current between cylinders of given radii and at known potentials, or, considering i to be constant, they may be solved for V, for any chosen values of r/r_0 (and therefore β^2), and thus the potential distribution can be calculated.

In the theory of electric discharges in uniformly ionized gases, we shall see that a cylindrical collector of radius r in such a gas is surrounded by a sheath of radius r_0 through which the current is carried either by positive ions or by electrons, but not by both. Thus the space charge equations (331) and (332) are applicable, the outer surface of the sheath constituting a virtual emitter. As the applied voltage V changes, the current density I_0 at the outer surface of the sheath remains constant so that the collector current i varies according to

$$i = 2\pi r_0 L I_0. \tag{333}$$

If the current i is measured at given voltage V, the sheath radius r_0 can be calculated by getting β^2 from the known values of i, L, V and r by Eq. (331), and then getting the corresponding value of r_0/r from the curves marked $(-\beta)^2$ in Fig. 43. The current density I_0 is then obtained by Eq. (333). It is usually found to be independent of the applied voltage.

It often happens that I_0 is known and it is desired[297] to calculate the current i that will flow to a collector of known radius r with a given applied

[297] I. Langmuir and H. A. Jones, Phys. Rev. **31**, 366 (1928).

potential V. Eq. (333) cannot be applied directly since we do not know r_0. Eliminating i between Eqs. (333) and (332), we have

$$(r_0/r)\beta^2 = 2.334 \times 10^{-6}(m_e/m)^{1/2}V^{3/2}/(I_0 r^2). \tag{334}$$

Thus from the known values of V, I_0 and r, the value of $(r_0/r)\,\beta^2$ is found, and from this by means of the curves in Fig. 45, r_0/r and hence r_0 is obtained. Eq. (333) then gives i.

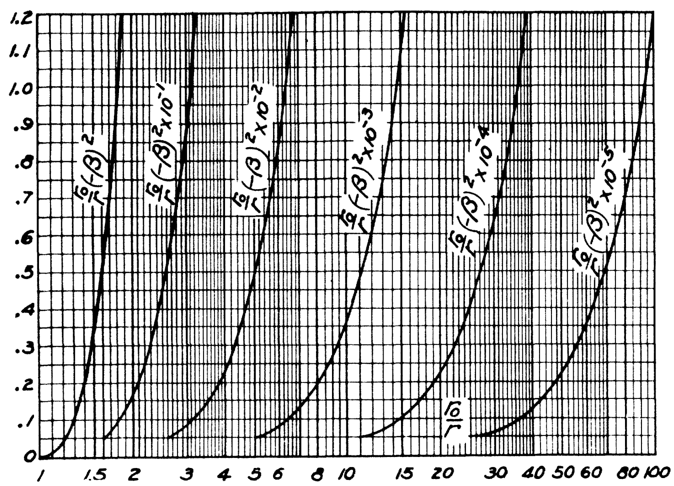

Fig. 45. Plot of $(-\beta)^2(r_0/r)$ as function of r_0/r.

Space charge equations for spheres (Negligible initial velocities). Instead of using Eq. (316) which gives the current density I, it is more convenient to use the following expression for the total current i.

$$i = \frac{4}{9}\left(\frac{2e}{m}\right)^{1/2}\frac{V^{3/2}}{\alpha^2}\,. \tag{335}$$

For electrons, when V is expressed in volts, this becomes

$$i = 29.34 \times 10^{-6}V^{3/2}\alpha^{-2} \text{ amperes.} \tag{336}$$

For ions of mass m, the coefficient is decreased in the ratio $(m/m_e)^{1/2}$ giving,

$$i = 6.864 \times 10^{-7}M^{-1/2}V^{3/2}\alpha^{-2} \tag{337}$$

where M is the "molecular weight" of the ions.

Space charge currents between equidistant curved surfaces.

Comparison of Eq. (316) which gives the space charge current density for either cylinders or spheres, with Eq. (289), which applies to parallel planes, shows that they are identical except that in the former $r^2\beta^2$ replaces x^2, x being the distance between the planes. The reason for this is evident if we consider that when r and r_0 are nearly equal, γ, as defined by Eq. (315), is approximately equal to $(r-r_0)/r$ or to x/r. Eqs. (323) and (324), when the terms involving the higher powers of γ are neglected, show that $\beta=\gamma$ for both cylinders and spheres and therefore $r^2\beta^2=x^2$.

Thus equidistant curved surfaces having a radius of curvature large compared to their separation may be treated approximately as though they were plane surfaces. Careful consideration[298] of the effects of the terms involving γ^2 and γ^3 shows that very accurate values of the space charge current between equidistant curved surfaces may be obtained by using the equation for *parallel planes* if the area used in calculating the current density I from the total current i is taken to be the area of a surface 4/5ths of the distance from the emitter to the collector surfaces. The fractional error in the current thus calculated is of the order of $-0.03x^2\sigma^2$ where σ is the "total curvature" of the surface ($1/r$ for cylinders and $2/r$ for spheres).

The 3/2-power law for electrodes of any shape.

In all three cases, parallel planes, coaxial cylinders and concentric spheres, neglecting initial velocities we have seen that the space-charge-limited current varies in proportion to $V^{3/2}$. The following reasoning proves that this relation holds for electrodes of any shape provided the effect of the initial velocities of the emitted particles may be neglected.

Let us assume that the emitter is at zero potential and that space in which the discharge takes place is bounded by surfaces which are either at zero potential or at the potential V_1, (the collector potential). Then if there are no initial velocities, the problem of the flow of current as limited by space charge, involves the simultaneous solution of the three equations

$$\tfrac{1}{2}mv^2 = Ve \qquad (338)$$

$$I = \bar{v}\rho \qquad (339)$$

$$\Delta V = -4\pi\rho \qquad (340)$$

with the boundary conditions $V=0$ over the emitter surface and either 0 or V_1 over all other bounding surfaces, and also the normal component of the field intensity is zero at the emitter surface.

It should be noted that v in Eq. (338), which is the velocity in the direction of motion, enters here as a scalar quantity, whereas in Eq. (339), $\bar{v}$ and I are essentially vectors, the current density being different in different directions. In the cases of planes, cylinders, and spheres, this distinction has not been necessary since the paths of the electrons or ions have always been straight and have coincided with the lines of electric force. For electrodes of other shapes, however, the electrons describe orbits which will cut across the lines of electric force. In fact, at each point the centrifugal force mv^2/R due to the curvature of any electron path must equal the electric force acting in a direction normal to the path (R being the radius of curvature).

Let us now consider what must be the effect of increasing the potential at every point in space in a definite ratio, $n:1$, thus replacing V by nV. Equa-

[298] I. Langmuir and K. B. Blodgett, Phys. Rev. **24**, 57 (1924).

tion (340) shows that ρ, the space charge, must increase n-fold. Equation (338) then indicates that the velocity v must everywhere be increased $n^{1/2}$-fold. The boundary conditions $V=0$ and gradient $E=0$ at the emitter surface are obviously not altered when V is increased n-fold.

If we increase the voltage everywhere n-fold, the electric force, and therefore also the centrifugal force, mv^2/R, must increase n-fold. But we have just seen that v^2 itself increases n-fold and therefore the radius of curvature and the shapes of the paths are unchanged.

For this reason, whatever the distinction between v and $\bar{v}$ may be, their ratio remains unchanged, and thus both v and $\bar{v}$ must be proportional to $n^{1/2}$. Equation (339) then indicates that I varies with $n^{3/2}$. In other words, the current, limited by space charge, varies with $V^{3/2}$ for electrodes of any shape.

The conditions under which this conclusion is valid should be examined more closely. In order that Eq. (338) may apply, it is necessary not only that the initial velocities shall be zero, but that the whole of the emitting surface shall be at uniform potential. The potential drop along a filamentary cathode will thus cause deviations from the 3/2-power law which we shall consider later.

If there are only two electrodes and one surrounds the other, our assumed boundary conditions are fulfilled. If we have two electrodes in a glass envelope, the glass surface receives electrons from the cathode until it becomes charged to cathode potential. For all such insulated surfaces, if we neglect initial velocities, we may place $V=0$ and our boundary conditions are again fulfilled so that the 3/2-power law should apply. However, we shall see that with electrodes whose mutual capacitance is low compared to that between the glass surface and the emitter, the tendency of the glass to acquire negative potentials because of the initial velocities of the electrons is a factor that causes much larger deviations than in the case where one electrode encloses the other.

Effect of size. If in any high vacuum device having electrodes of any shape, and in which the unipolar current is limited by space charge, all dimensions are increased n-fold, but the potentials of the electrodes are kept unchanged, the total current remains unchanged. This follows from Eq. (340) which shows that if V remains unchanged, ρ must be inversely proportional to n^2 and thus by Eq. (339), the current density I is everywhere inversely proportional to n^2, and the current i is independent of the size of the device.

Effects of initial velocities on space charge currents between cylinders.

Case I. *External collector.* Schottky[299] pointed out that the correction of the three-halves power law for cylinders, due to the initial velocities of the electrons, must be much smaller than for parallel planes, as given, for example, by the correction factor in Eq. (311). The effect of initial velocities may be approximately taken into account by considering that the space

[299] W. Schottky, Phys. Zeits. **15**, 624 (1914).

charge at every point is reduced in the ratio $[V/(V+V_0)]^{1/2}$ where V_0 is the volt-equivalent of the average initial velocity in a radial direction.

It is thus possible to show[300] that the current between an internal emitter of small diameter and an external collector of large diameter can be approximately calculated by Eq. (331) or (332) by replacing V by an effective value V_e given by

$$V_e = V - V_M + \frac{V_0}{4}\left(\log_e \frac{V}{\lambda V_0}\right)^2 \tag{341}$$

where V is the anode potential (cathode taken as zero), V_M is the potential at the potential minimum as obtained from Eq. (297) by putting $V_1=0$, λ is a numerical factor that may be estimated to have the value 1.5 and V_0 is given by

$$V_0 = (3/2)kT/e = T/7733 \text{ volts}. \tag{342}$$

The radius r_M of the cylinder which is the locus of the minimum in the potential near the emitter, can be estimated by considering this surface to be a "virtual emitter," using a method similar to that employed in deriving Eqs. (293) and (295). Between the emitter ($r=r_0$) and the virtual emitter ($r=r_M$), the electrons (or ions) are in a retarding field, the effective current being $2i_s-i$ where i_s is the saturation emission current, so that in applying the space charge equation (331) we use values of β corresponding to the case of the internal collector. Beyond $r=r_M$, however, the electrons are in an accelerating field and the case for the external collector is applicable. Thus we find

$$2(i_s/i) - 1 = [(V_0 - V_M)/(V - V_M)]^{3/2}(r/r_0)(\beta/\beta_0)^2 \tag{343}$$

where the subscript 0 applies to the emitter, and β_0 and β are functions of r_M/r_0 and of r/r_M respectively.

Since we may take r/r_M to be large, β will be close to unity and its value will be nearly independent of r/r_M. If V is so large that we may neglect V_M compared to V and if we take the potential of the emitter to be zero, we have

$$\beta_0^2 = \beta^2(r/r_0)(-V_M/V)^{3/2}[2(i_s/i) - 1]^{-1} \tag{344}$$

from which r_M/r_0 can be obtained from the curve marked β in Fig. 44. β_0 in this case refers only to that half of the curve where β has negative values, and r_M/r_0 is equivalent to $1/(r/r_0)$ on this curve.

As an example consider a filament of 0.2 mm diameter at the axis of a cylindrical anode 20 mm diameter. According to Eq. (331), neglecting the effect of initial velocities, the maximum current that can flow with $V=50$ volts is 4.83 milliamps. per cm of length. Taking the cathode temperature to be 2000°K, the average initial velocity in the radial direction according to (342) corresponds to $V_0=0.26$ volts. If the emission from the filament is 24.1 ma per cm (about 5 times the "space-charge current"), we find from Eq. (297) that the potential at the potential minimum is $V_M=-0.28$ volts and

[300] I. Langmuir, Phys. Rev., **21**, 419–35 (1923), see especially p. 430.

thus by Eq. (341) the effective potential, allowing for the initial velocities, is 51.8 volts. From this, by Eq. (331), the space-charge current is 5.09 $ma \cdot cm^{-1}$ as compared with 4.83 obtained by neglecting initial velocities. According to Eq. (344), $(-\beta_0)^2 = 0.00502$ or $\beta_0 = -0.071$, and by Fig. 44 this gives $r_M/r_0 = 1.0715$. The minimum potential thus lies so close to the cathode surface that we are justified in having failed to distinguish between r_M and r_0 in calculating β^2 for the purpose of getting i.

Case II. Internal collector. The conservation of angular momentum corresponding to the initial velocities requires that the tangential velocities of the electrons shall increase as the electrons approach the collector, (in proportion to $1/r$). With a collector of small size, a considerable fraction of the electrons thus pass close to the collector without striking it.

We have seen in Chapter III from Eqs. (251) and (252) that the fraction of the emitted electrons that reach an internal cylindrical collector remains constant as the emission increases until the Davisson condition, Eq. (274), is no longer fulfilled. When this occurs,* the current has risen so nearly to the critical value needed to give $dV/dr = 0$ at the emitter, that we may without appreciable error consider these two conditions to be equivalent. For currents less than this we have, by Eq. (251),

$$i_C = S_C I_E f = S_E I_E (r/r_0) f \tag{345}$$

where i_C is the current flowing to the collector and $S_E I_E$ is the saturation current from the emitter.

When the emission has risen to such value that $dV/dr = 0$ at the emitter surface (or at the minimum potential surface close to it), the current is limited by space charge and will remain nearly independent of the cathode emission. The critical value of the emission current density I_E which gives $dV/dr = 0$ may be found by means of the space charge equation (330), in accord with the following reasoning.

The discussion of Eq. (327) has shown that the effect of space charge near an internal collector is unimportant as compared to that near the emitter. Therefore the space charge produced by electrons which move from the emitter to the collector in strictly radial directions (no initial velocities) is not materially different from that due to electrons which miss the collector because of orbital motions, except that the latter in returning again to the emitter contribute as much on the outward as on the inward journey. Thus, as in the derivation of Eq. (293), we must replace i in the space charge equation, (330), by $2I_E S_E - i_C$. In this way, with the value of i_C from (345), we find

$$I_E S_E (2 - fr/r_0) = (2 \cdot 2^{1/2}/9)(e/m)^{1/2} V^{3/2}/(r\beta^2) \tag{346}$$

By eliminating $I_E S_E$ between this equation and Eq. (345), we find that the critical current which gives $dV/dr = 0$ at the emitter has the value

* See forthcoming paper by I Langmuir and L. Tonks in Phys. Rev.

$$i_C = \frac{2\cdot 2^{1/2}}{9}\left(\frac{e}{m}\right)^{1/2}\cdot\frac{V^{3/2}}{r\beta^2}\cdot\frac{1}{(2r/(r_0f)) - 1}\cdot \tag{347}$$

This* gives the required limitation of current by space charge, taking into account the orbital motions resulting from the initial velocities. Here the value of f is to be calculated by Eqs. (252), (257) or (262). In case I_ES_E, the saturation current i_S from the emitter, is greater than the value given by Eq. (347), there will be a minimum potential surface or potential barrier close to the emitter, whose potential V_M in accord with Eq. (297) can be put equal to

$$V_M = -(T/5040)\log_{10}(i_Srf/(i_Cr_0)) \tag{348}$$

where the value of i_C is given as a first approximation by Eq. (347). To allow for the effect of this potential barrier in calculating i_C, we may now replace V in Eq. (347) by an effective value V_e given by

$$V_e = V - V_M. \tag{349}$$

Space charge limitation of bipolar currents.

If positive ions are generated at the anode of a high vacuum discharge with hot cathode, the partial neutralization of the electron space charge allows the electron current to rise above that calculated from the space charge equations already given.

It will be instructive to calculate the magnitude and other characteristics of this effect for the case of parallel plane electrodes. We shall see later that this theory leads to an understanding of phenomena at hot cathodes in ionized gases.

Consider an infinite plane cathode C at zero potential and a similar parallel plane anode A at the potential V_A and at a distance a from C. Let the cathode emit a surplus of electrons without initial velocities. The current is then limited by space charge and the current per unit area, I_0, is given by Eq. (289).

Now let the anode A emit positive ions (without initial velocities) with a uniform distribution over its surface, the current density being I_p. Because of the partial neutralization of the electron space charge, the electron current will increase to a new value, say, I_e per unit area. Assuming no recombinations nor collisions between electrons and ions, we wish to determine the relation between I_e and I_p.

The solution of this space charge problem[301] gives

$$(I_e/I_0)^{1/2} = (3/4)\int_0^1 [y^{1/2} + \alpha\{(1-y)^{1/2} - 1\}]^{-1/2}dy \tag{350}$$

where

$$\alpha = \frac{I_p}{I_e}\left(\frac{m_p}{m_e}\right)^{1/2} \tag{351}$$

m_p/m_e being the ratio of the masses of the ions and electrons.

[301] I. Langmuir, Phys. Rev. **33**, 954–89 (1929). See especially p. 956–60.

The potential distribution between the electrodes is given by

$$x/a = (3/4)(I_0/I_e)^{1/2}\int_0^{V/V_A}[y^{1/2}+\alpha\{(1-y)^{1/2}-1\}]^{-1/2}dy \quad (352)$$

where x is the distance from the cathode to a point whose potential is V.

Table XXI gives x/a for various values of V/V_A and α. The lowest line gives I_e/I_0 as a function of α.

TABLE XXI. *Potential distribution between plane cathode emitting surplus of electrons and parallel plane anode which emits given numbers of ions.* Table of values of x/a, the fraction of the distance to the anode, at which the potential is a given fraction of the anode potential.

V/V_A	$\alpha=0$	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1.0$
0	0	0	0	0	0	0	0
0.02	0.0532	0.0513	0.0491	0.0467	0.0438	0.0419	0.0396
0.05	.1057	.1022	.0981	.0934	.0879	.0842	.0798
0.1	.1778	.1723	.1661	.1588	.1498	.1437	.1367
0.2	.2991	.2911	.2823	.2714	.2573	.2477	.2363
0.3	.4054	.3962	.3855	.3721	.3546	.3423	.3274
0.4	.5030	.4932	.4815	.4667	.4467	.4324	.4146
0.5	.5946	.5847	.5731	.5580	.5371	.5218	.5000
0.6	.6817	.6723	.6612	.6461	.6245	.6080	.5854
0.7	.7653	.7570	.7471	.7332	.7123	.6958	.6726
0.8	.8459	.8395	.8314	.8198	.8016	.7861	.7637
0.9	.9240	.9201	.9149	.9074	.8940	.8813	.8633
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
I_e/I_0	1.0000	1.0839	1.1872	1.3237	1.5186	1.6644	1.8605

These results show that when $\alpha=1$, the potential gradient at the anode becomes zero just as it is at the cathode, and thus the positive ion current as well as the electron current is limited by space charge. Values of α greater than unity give imaginary values for the potentials and are thus impossible. For $\alpha=1$, the potential distribution curve is symmetrical about its mid-point ($x/a=0.5$, $V/V_A=0.5$).

From the values of I_e/I_0, in the last row of Table XXI, we see that the electron current increases as more positive ions are emitted from the anode until the positive ion current also becomes limited by space charge. When this occurs, the electron current and the positive ion current are each 1.860 times as great as the currents of electrons or ions that could flow (with the same applied potentials) if carriers of the opposite sign were absent.

It is interesting to inquire how large is the effect of single positive ions emitted from the anode, in causing an increased electron flow from the cathode.

Differentiation of Eq. (350) gives

$$dI_e/dI_p = 0.378(m_p/m_e)^{1/2} \text{ for } \alpha = 0 \quad (353)$$

and

$$dI_e/dI_p = 3.455(m_p/m_e)^{1/2} \text{ for } \alpha = 1 \quad (354)$$

A plot of I_e/I_0 as function of α from the data of Table XXI shows that the slope of the curve increases gradually from 0.378 at $\alpha = 0$ up to 3.455 at $\alpha = 1$. Thus the effectiveness of the ions in raising the electron current increases as the field strength decreases in the region where they originate, but only up to a certain limiting value. Of course when $\alpha = 1$ the further increase in the electron current is stopped by the space charge limitation of the ion current.

The square root of the ratio of the masses of the ions and the electrons is 607 for mercury vapor, 271 for argon, and 60.8 for hydrogen, and therefore each positive ion of these gases liberated at the anode will increase the number of electrons that cross the space by 229, 102 or 23 respectively in the case of a pure electron discharge ($\alpha = 0$).

The effect has also been determined of generating the ions not at the anode but in the space between cathode and anode. Single ions introduced into a pure electron discharge at a point 4/9ths of the distance from cathode to anode produce a maximum effect, 0.582 $(m_p/m_e)^{1/2}$, in increasing the electron current.

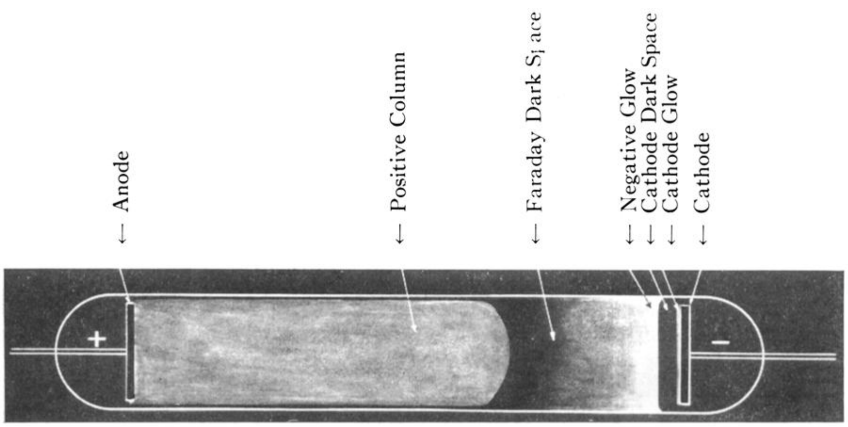

Fig. 34. Typical glow discharge in a gas at about 1 mm pressure.