Transfer Problems and the Reciprocity Principle

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I. INTRODUCTION

IN both the theories of radiative equilibrium and neutron diffusion we encounter transfer equations of the form

$$\boldsymbol{\Omega}\cdot\nabla\psi(\mathbf{r},\boldsymbol{\Omega})+\sigma_i(\mathbf{r})\psi_i(\mathbf{r},\boldsymbol{\Omega}) =\sum_j \int F_{ij}(\boldsymbol{\Omega},\boldsymbol{\Omega}',\mathbf{r})\sigma_j(\mathbf{r})\psi_j(\mathbf{r},\boldsymbol{\Omega}')d\boldsymbol{\Omega}'+q_i(\mathbf{r},\boldsymbol{\Omega}). \quad (1)$$

Since the most direct application of the specific results to be obtained will be to neutron diffusion we will use the terminology of that subject. Here $\psi_i(\mathbf{r},\boldsymbol{\Omega})$ is the angular flux of neutrons of energy i at $\mathbf{r}$ moving in direction $\boldsymbol{\Omega}$. $\sigma_i(\mathbf{r})$ is the total cross section for neutrons of energy i and F_{ij} is the probability that if a neutron with energy j, direction $\boldsymbol{\Omega}'$ suffers a collision at $\mathbf{r}$ a neutron of energy i, direction $\boldsymbol{\Omega}$, will appear at $\mathbf{r}$. $q_i(\mathbf{r},\boldsymbol{\Omega})$ describes any external neutron source.

While (1) is, without simplifying assumptions, exceedingly complex, a few general statements can be made. These concern the reciprocity principle and questions of uniqueness. The first of these is most important. Besides enabling us to compare different experimental situations and simplifying much of the mathematics it shows, as will be seen, how apparently difficult problems can be solved by relating them to simpler ones.

Unfortunately, even the most elegant proofs[1] have been rather complex. Indeed, there are a number of theorems all of which go by this same name. In the following it will be shown that all these theorems are consequences of a simple identity. The method of proof has a number of advantages. First, the origin of the theorems and their generalizations in more complex situations are directly apparent. Second, the equations used so powerfully by Chandrasekhar[1] to determine the angular densities are obtained at exactly the same step. In a certain sense Chandrasekhar's argument is turned around. Instead of obtaining equations for the reflection and transmission functions and then deriving a reciprocity principle we start from a slightly generalized principle. From this the equations and the specific reciprocity result follow. This permits one to see exactly what the simplifications resulting from the "Principle of Invariance" [1] are and makes possible a simple proof of this latter principle. A last advantage is that we can in this way obtain the fundamental equations for reflection and transmission functions even when reciprocity in a strict sense does not hold.

Uniqueness theorems are particularly useful in connection with equations like Eq. (1). They not only show what must be specified to achieve a unique solution but also suggest certain notations which materially simplify the obtaining of the solution. Such a theorem is proved in Sec. II for a simple situation. The result is essential for the later arguments. Fortunately the proof is quite similar to the proof of the reciprocity identity in its general form which follows in Sec. III. In that section we obtain as special cases various results which have been called "Reciprocity Principles."

In Secs. IV and V we turn to plane problems and derive Chandrasekhar's[1] equations for the reflection function for a half-space. In Sec. VI these equations are used to obtain the Green's function for one velocity neutron diffusion in the same geometry. A specific reciprocity result of Sec. III is used in Sec. VII to obtain the Green's function for two adjacent half-spaces.

In Sec. VIII Chandrasekhar's equations for a slab are obtained as another specialization of the general reciprocity identity together with certain invariance considerations.

II. A UNIQUENESS THEOREM

For simplicity in deriving the general results and in order to be able to carry through the detailed calculations for specific solutions we restrict ourselves in the body of this article to the following equation which is appropriate for one velocity neutron diffusion:

$$\boldsymbol{\Omega}\cdot\nabla\psi(\mathbf{r},\boldsymbol{\Omega})+\sigma(\mathbf{r})\psi(\mathbf{r},\boldsymbol{\Omega}) =c(\mathbf{r})\sigma(\mathbf{r})\int f(\boldsymbol{\Omega},\boldsymbol{\Omega}')\psi(\mathbf{r},\boldsymbol{\Omega}')d\boldsymbol{\Omega}'+q(\mathbf{r},\boldsymbol{\Omega}). \quad (2)$$

Here $c(\mathbf{r})$ is the number of neutrons emitted at $\mathbf{r}$ per neutron collision at $\mathbf{r}$. It is chosen so that

$$\int f(\boldsymbol{\Omega},\boldsymbol{\Omega}')d\boldsymbol{\Omega}=1. \quad (3a)$$

We also assume that

$$\int f(\boldsymbol{\Omega},\boldsymbol{\Omega}')d\boldsymbol{\Omega}'=1, \quad (3b)$$

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[1] For example, see S. Chandrasekhar, *Radiative Transfer* (Oxford University Press, New York, 1950).

and

$$f(\mathbf{\Omega},\mathbf{\Omega}') \geq 0 \quad \text{all } \mathbf{\Omega},\, \mathbf{\Omega}'. \tag{4}$$

(These conditions are satisfied for all interesting physical situations.)

Theorem.—If $c(\mathbf{r})<1$ throughout a region V, the angular density is uniquely determined by the incident angular distribution on the bounding surface S and the sources within V.

Proof.—Let ψ_1, ψ_2 be two solutions of (2) with the same incident angular distribution. The difference $\psi=\psi_1-\psi_2$ satisfies the equations,

$$\mathbf{\Omega}\cdot\nabla\psi+\sigma(\mathbf{r})\psi=c(\mathbf{r})\sigma(\mathbf{r})\int f(\mathbf{\Omega},\mathbf{\Omega}')\psi(\mathbf{r},\mathbf{\Omega}')d\mathbf{\Omega}' \tag{5}$$

and

$$\psi(\mathbf{r}_s,\Omega)=0, \quad \text{for } \mathbf{n}_0\cdot\mathbf{\Omega}<0. \tag{6}$$

Here $\mathbf{r}_s$ is an arbitrary point on S where the outward normal is $\mathbf{n}_0$.

If (5) is multiplied by $\psi(\mathbf{r},\mathbf{\Omega})$ and then integrated over V and $\mathbf{\Omega}$ we find on using Gauss' theorem that

$$I_1=I_2, \tag{7}$$

where

$$I_1=\int \mathbf{n}_0\cdot\mathbf{\Omega}\frac{[\psi(\mathbf{r},\mathbf{\Omega})]^2}{2}d\mathbf{\Omega}dS, \tag{8}$$

and

$$I_2=\int_V d\mathbf{r}\sigma(\mathbf{r})\int\int d\mathbf{\Omega}d\mathbf{\Omega}'\psi(\mathbf{r},\mathbf{\Omega})f(\mathbf{\Omega},\mathbf{\Omega}')\psi(\mathbf{r},\mathbf{\Omega}') - \int_V d\mathbf{r}\sigma(\mathbf{r})\int d\mathbf{\Omega}\psi^2(\mathbf{r},\mathbf{\Omega}). \tag{9}$$

Remembering (6) we see that

$$I_1\geq 0. \tag{10}$$

Since $f(\mathbf{\Omega},\mathbf{\Omega}')$ is positive definite [from (4)], we know that

$$\int\int d\mathbf{\Omega}d\mathbf{\Omega}'f(\mathbf{\Omega},\mathbf{\Omega}')[\psi(\mathbf{r},\mathbf{\Omega})-\psi(\mathbf{r},\mathbf{\Omega}')]^2\geq 0. \tag{11}$$

This together with the conditions of (3a) and (3b) gives

$$\int\int d\mathbf{\Omega}d\mathbf{\Omega}'\psi(\mathbf{r},\mathbf{\Omega})f(\mathbf{\Omega},\mathbf{\Omega}')\psi(\mathbf{r},\mathbf{\Omega}')\leq\int d\mathbf{\Omega}\psi^2(\mathbf{r},\mathbf{\Omega}). \tag{12}$$

Hence

$$I_2\leq\int d\mathbf{r}\sigma(\mathbf{r})[c(\mathbf{r})-1]\int d\mathbf{\Omega}\psi^2(\mathbf{r},\mathbf{\Omega})\leq 0. \tag{13}$$

The inequalities of (10) and (13) are compatible with the equality of (7) only if

$$\psi(\mathbf{r},\mathbf{\Omega})\equiv 0 \quad \text{in } V. \tag{14}$$

Two remarks should be made:

(a) The assumptions of the theorem are probably stronger than absolutely necessary. However, they are satisfactory for the applications of interest here.

(b) In case the region V extends to infinity (such as the exterior of a sphere or a half-space) the uniqueness is to be understood as applied to functions vanishing sufficiently rapidly. (Physically this means there are to be no sources at infinity.)

As a direct application of this theorem we note that to obtain the most general solution of (2) we need determine only two functions for a given region. These are $\psi_p(\mathbf{r},\mathbf{\Omega};\,\mathbf{r}_0,\mathbf{\Omega}_0)$ and $\psi_s(\mathbf{r},\mathbf{\Omega};\,\mathbf{r}_s,\mathbf{\Omega}_s)$. Here ψ_p satisfies Eq. (2) with $q(\mathbf{r},\mathbf{\Omega})=\delta(\mathbf{r}-\mathbf{r}_0)\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}_0)$ and describes zero incoming flux. ($\delta(\mathbf{r})$ is the Dirac delta function: $\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}_0)=0$, $\mathbf{\Omega}\cdot\mathbf{\Omega}_0\neq 1$ and $\int\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}_0)d\mathbf{\Omega}=1$.) ψ_s satisfies the homogeneous form of (2) with the condition

$$\psi_s(\mathbf{r},\mathbf{\Omega};\,\mathbf{r}_s,\mathbf{\Omega}_s)|_{\mathbf{r}\text{ on }S}=\delta_s(\mathbf{r}-\mathbf{r}_s)\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}_s). \tag{15}$$

Here δ_s is the delta function on the surface S and $\mathbf{\Omega}_s$ is a direction such that $\mathbf{n}_0.\ \mathbf{\Omega}_s|_{\mathbf{r}=\mathbf{r}_s}<0$.

The solution of (2) with arbitrary source $q(\mathbf{r},\mathbf{\Omega})$ and arbitrary incident flux $\psi_{\text{inc}}(\mathbf{r}_s,\mathbf{\Omega}_s)$ is clearly

$$\psi(\mathbf{r},\mathbf{\Omega})=\int_S dS\int_{\mathbf{n}_0\cdot\mathbf{\Omega}_s<0} d\mathbf{\Omega}_s\psi_s(\mathbf{r},\mathbf{\Omega};\,\mathbf{r}_s,\mathbf{\Omega}_s)\psi_{\text{inc}}(\mathbf{r}_s,\mathbf{\Omega}_s) + \int_V d\mathbf{r}_0\int d\mathbf{\Omega}_0\psi_p(\mathbf{r},\mathbf{\Omega};\,\mathbf{r}_0,\mathbf{\Omega}_0)q(\mathbf{r}_0,\mathbf{\Omega}_0). \tag{16}$$

Proof that this is the solution is obtained merely by noting that (a) the equation is satisfied; (b) the boundary conditions are satisfied; (c) the solution is unique.

In a sense to be shown below only ψ_s is necessary (ψ_p will be obtained from ψ_s).

It is we hope clear how uniqueness can be proved similarly for the more general situation envisaged in (1).

III. FUNDAMENTAL IDENTITY

Let $\psi_1(\mathbf{r},\mathbf{\Omega})$ be the solution of

$$\mathbf{\Omega}\cdot\nabla\psi_1(\mathbf{r},\mathbf{\Omega})+\sigma(\mathbf{r})\psi_1 = c(\mathbf{r})\sigma(\mathbf{r})\int f(\mathbf{\Omega},\mathbf{\Omega}')\psi_1(\mathbf{r},\mathbf{\Omega}')d\mathbf{\Omega}'+q_1(\mathbf{r},\mathbf{\Omega}) \tag{17}$$

within a region V subject to $\psi_1(\mathbf{r},\mathbf{\Omega})=\psi_{1\,\text{inc}}(\mathbf{r},\mathbf{\Omega})$ (for $\mathbf{n}_0\cdot\mathbf{\Omega}<0$) on the bounding surface S. Here ψ_{inc} is a given function. Similarly we consider a solution ψ_2 corresponding to a source q_2 and a given incident flux $\psi_{2\,\text{inc}}$. In addition to ψ_2 we consider a function $\tilde{\Psi}_2(\mathbf{r},\mathbf{\Omega})$

which is the solution of

$$\boldsymbol{\Omega}\cdot\nabla\tilde{\psi}_2(\mathbf{r},\boldsymbol{\Omega})+\sigma(\mathbf{r})\tilde{\psi}_2$$
$$=c(\mathbf{r})\sigma(\mathbf{r})\int f(-\boldsymbol{\Omega}',-\boldsymbol{\Omega})\tilde{\psi}_2(\mathbf{r},\boldsymbol{\Omega}')d\boldsymbol{\Omega}'+q_2(\mathbf{r},\boldsymbol{\Omega}), \quad (18\text{a})$$

subject to

$$\tilde{\psi}_2(\mathbf{r},\boldsymbol{\Omega})=\tilde{\psi}_{2\ \mathrm{inc}}(\mathbf{r},\boldsymbol{\Omega})\equiv\psi_{2\ \mathrm{inc}}(\mathbf{r},\boldsymbol{\Omega})$$
$$(\text{for } \mathbf{n}_0\cdot\boldsymbol{\Omega}<0 \text{ on } S).$$

(The $\sim$ is associated with the indicated transposition operation on the scattering function f.) Let us substitute $-\boldsymbol{\Omega}$ for $\boldsymbol{\Omega}$ in (18a). This becomes

$$-\boldsymbol{\Omega}\cdot\nabla\tilde{\psi}_2(\mathbf{r},-\boldsymbol{\Omega})+\sigma(\mathbf{r})\tilde{\psi}_2(\mathbf{r},-\boldsymbol{\Omega})$$
$$=c(\mathbf{r})\sigma(\mathbf{r})\int f(\boldsymbol{\Omega}',\boldsymbol{\Omega})\tilde{\psi}_2(\mathbf{r},-\boldsymbol{\Omega}')d\boldsymbol{\Omega}'+q_2(\mathbf{r},-\boldsymbol{\Omega}). \quad (18\text{b})$$

Multiply (17) by $\tilde{\psi}_2(\mathbf{r},-\boldsymbol{\Omega})$ and (18b) by $\psi_1(\mathbf{r},\boldsymbol{\Omega})$. Subtract the second from the first and integrate over V and $\boldsymbol{\Omega}$. Applying Gauss' theorem we obtain

$$\int_S dS\int d\boldsymbol{\Omega}\,\mathbf{n}_0\cdot\boldsymbol{\Omega}\tilde{\psi}_2(\mathbf{r},-\boldsymbol{\Omega})\psi_1(\mathbf{r},\boldsymbol{\Omega})$$
$$=\int_V d\mathbf{r}\,c(\mathbf{r})\sigma(\mathbf{r})\int\int d\boldsymbol{\Omega}d\boldsymbol{\Omega}'\{\tilde{\psi}_2(\mathbf{r},-\boldsymbol{\Omega})f(\boldsymbol{\Omega},\boldsymbol{\Omega}')$$
$$\times\psi_1(\mathbf{r},\boldsymbol{\Omega}')-\psi_1(\mathbf{r},\boldsymbol{\Omega})f(\boldsymbol{\Omega}',\boldsymbol{\Omega})\tilde{\psi}_2(\mathbf{r},-\boldsymbol{\Omega}')\}$$
$$+\int_V d\mathbf{r}\int d\boldsymbol{\Omega}\{\tilde{\psi}_2(\mathbf{r},-\boldsymbol{\Omega})q_1(\mathbf{r},\boldsymbol{\Omega})$$
$$-\psi_1(\mathbf{r},\boldsymbol{\Omega})q_2(\mathbf{r},-\boldsymbol{\Omega})\}. \quad (19)$$

Interchanging the names of the integration variables $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}'$ shows that the first term on the right in Eq. (19) vanishes identically. Since $\psi_1(\mathbf{r},\boldsymbol{\Omega})$ and $\tilde{\psi}_2(\mathbf{r},\boldsymbol{\Omega})$ are prescribed on S for $\mathbf{n}_0\cdot\boldsymbol{\Omega}<0$, it is useful to rewrite Eq. (19) as

$$\int_S dS\int_{\mathbf{n}_0\cdot\boldsymbol{\Omega}<0} d\boldsymbol{\Omega}|\mathbf{n}_0\cdot\boldsymbol{\Omega}|\{\tilde{\psi}_{2\ \mathrm{inc}}(\mathbf{r},\boldsymbol{\Omega})\psi_1(\mathbf{r},-\boldsymbol{\Omega})$$
$$-\psi_{1\ \mathrm{inc}}(\mathbf{r},\boldsymbol{\Omega})\tilde{\psi}_2(\mathbf{r},-\boldsymbol{\Omega})\}$$
$$=\int_V d\mathbf{r}\int d\boldsymbol{\Omega}\{\tilde{\psi}_2(\mathbf{r},-\boldsymbol{\Omega})q_1(\mathbf{r},\boldsymbol{\Omega})$$
$$-\psi_1(\mathbf{r},\boldsymbol{\Omega})q_2(\mathbf{r},-\boldsymbol{\Omega})\}. \quad (20)$$

From this identity the results which have been called reciprocity principles follow as special cases. We give three examples here. In all these we suppose $f(\boldsymbol{\Omega},\boldsymbol{\Omega}')=f(-\boldsymbol{\Omega}',-\boldsymbol{\Omega})$, i.e., the scattering function has time reflection symmetry. From the uniqueness theorem we conclude that $\tilde{\psi}_2\equiv\psi_2$.

(a) Let ψ_1 correspond to a directional point source in direction $\boldsymbol{\Omega}_1$ at $\mathbf{r}_1$. ($q_1(\mathbf{r},\boldsymbol{\Omega})=\delta_2(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_1)\delta(\mathbf{r}-\mathbf{r}_1)$, $\psi_{1\ \mathrm{inc}}=0$.) We denote this solution by $\psi_p(\mathbf{r},\boldsymbol{\Omega};\mathbf{r}_1,\boldsymbol{\Omega}_1)$. Similarly let $\psi_2(\mathbf{r},\boldsymbol{\Omega})=\psi_p(\mathbf{r},\boldsymbol{\Omega};\mathbf{r}_2,\boldsymbol{\Omega}_2)$. Inserting in Eq. (20) gives

$$\psi_p(\mathbf{r}_1,-\boldsymbol{\Omega}_1;\mathbf{r}_2,\boldsymbol{\Omega}_2)=\psi_p(\mathbf{r}_2,-\boldsymbol{\Omega}_2;\mathbf{r}_1,\boldsymbol{\Omega}_1). \quad (21)$$

Thus the angular density at $\mathbf{r}_1$, direction $-\boldsymbol{\Omega}_1$, resulting from a unit point source in direction $\boldsymbol{\Omega}_2$ at $\mathbf{r}_2$, is equal to the angular density at $\mathbf{r}_2$, in direction $-\boldsymbol{\Omega}_2$, due to a unit point source in direction $\boldsymbol{\Omega}_1$ at $\mathbf{r}_1$. From this many additional results follow. For example, on integrating the two sides of (21) over $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$ we find the theorem: The density at $\mathbf{r}_1$ due to a unit isotropic source at $\mathbf{r}_2$ is equal to the density at $\mathbf{r}_2$ due to a unit isotropic source at $\mathbf{r}_1$.[2] Similarly on multiplying Eq. (21) by $\mathbf{w}\cdot\boldsymbol{\Omega}$ ($\mathbf{w}$ an arbitrary unit vector) and then integrating over $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$ we see that the component of the *current* in direction $\mathbf{w}$ at $\mathbf{r}_1$ due to an isotropic source at $\mathbf{r}_2$ is equal to the *density* at $\mathbf{r}_2$ due to the directional source $\mathbf{w}\cdot\boldsymbol{\Omega}$ at $\mathbf{r}_1$.

(b) Let $\psi_1=\psi_s(\mathbf{r},\boldsymbol{\Omega};\mathbf{r}_{1s},\boldsymbol{\Omega}_{1s})$, (i.e., $q_1=0$, $\psi_{1\ \mathrm{inc}}=\delta_s(\mathbf{r}_1-\mathbf{r}_{1s})\delta_2(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_1)$), and let $\psi_2=\psi_s(\mathbf{r},\boldsymbol{\Omega};\mathbf{r}_{2s},\boldsymbol{\Omega}_2)$. (Note that $\mathbf{n}_0\cdot\boldsymbol{\Omega}_1|\mathbf{r}_{1s}<0$ and $\mathbf{n}_0\cdot\boldsymbol{\Omega}_2|\mathbf{r}_{2s}<0$ is implied.) From Eq. (20) we find

$$|\mathbf{n}_2\cdot\boldsymbol{\Omega}_2|\psi_s(\mathbf{r}_{2s},-\boldsymbol{\Omega}_2;\mathbf{r}_{1s},\boldsymbol{\Omega}_1)$$
$$=|\mathbf{n}_1\cdot\boldsymbol{\Omega}_1|\psi(\mathbf{r}_{1s},-\boldsymbol{\Omega}_1;\mathbf{r}_{2s},\boldsymbol{\Omega}_2). \quad (22)$$

This says that the *emergent* angular distribution in direction $-\boldsymbol{\Omega}_2$ at $\mathbf{r}_{2s}$ due to an *incident* beam at $\mathbf{r}_{1s}$ in direction $\boldsymbol{\Omega}_1$ times the absolute value of the cosine of the angle between $\boldsymbol{\Omega}_2$ and the normal to S at $\mathbf{r}_{2s}$ is equal to the emergent distribution at $\mathbf{r}_{1s}$ in direction $-\boldsymbol{\Omega}_1$ due to an incident beam at $\mathbf{r}_{2s}$ in direction $\boldsymbol{\Omega}_2$ times the absolute value of the cosine of the angle between $\boldsymbol{\Omega}_1$ and the normal at $\mathbf{r}_{1s}$.

(c) Let $\psi_1=\psi_p(\mathbf{r},\boldsymbol{\Omega};\mathbf{r}_1,\boldsymbol{\Omega}_1)$, $\psi_2=\psi_s(\mathbf{r},\boldsymbol{\Omega};\mathbf{r}_{2s},\boldsymbol{\Omega}_s)$. Substituting in Eq. (20) we find

$$\psi_p(\mathbf{r}_{2s},-\boldsymbol{\Omega}_2;\mathbf{r}_1,\boldsymbol{\Omega}_1)=\frac{1}{|\mathbf{n}_2\cdot\boldsymbol{\Omega}_2|}\psi_s(\mathbf{r}_1,-\boldsymbol{\Omega}_1;\mathbf{r}_{2s},\boldsymbol{\Omega}_2). \quad (23)$$

Thus, the emergent angular distribution in direction $-\boldsymbol{\Omega}_2$ at $\mathbf{r}_{2s}$ due to a point source of direction $\boldsymbol{\Omega}_1$ at $\mathbf{r}_1$ is the product of the reciprocal of the absolute value of the cosine of the angle between the normal at $\mathbf{r}_{2s}$ and $\boldsymbol{\Omega}_2$ and the angular density at $\mathbf{r}_1$, direction $-\boldsymbol{\Omega}_1$ due to an incident beam in direction $\boldsymbol{\Omega}_2$ at $\mathbf{r}_{2s}$. Hence, if we are primarily interested in emerging angular distributions *only* the function ψ_s is necessary.

A particular consequence of Eq. (23) that will be used below is obtained by integrating both sides over all $\boldsymbol{\Omega}_1$. This yields: the emergent angular distribution at $\mathbf{r}_{2s}$, direction $-\boldsymbol{\Omega}_2$ is $1/(4\pi|\mathbf{n}_2\cdot\boldsymbol{\Omega}_2|)$ times the

[2] By "density" and "current" we mean $\int\psi(\mathbf{r},\boldsymbol{\Omega})d\boldsymbol{\Omega}$ and $\int\boldsymbol{\Omega}\psi(\mathbf{r},\boldsymbol{\Omega})d\boldsymbol{\Omega}$, respectively.

density at $\mathbf{r}_1$ due to an incident beam in direction $\mathbf{\Omega}_2$ at $\mathbf{r}_{2s}$.

IV. PROBLEMS WITH PLANE SYMMETRY

In these problems Eq. (2) simplifies (on introducing the optical thickness[3]) to

$$\mu\frac{\partial\psi(z,\mathbf{\Omega})}{\partial z}+\psi(z,\mathbf{\Omega})=c(z)\int f(\mathbf{\Omega},\mathbf{\Omega}')\psi(z,\mathbf{\Omega}')d\mathbf{\Omega}'+q(z,\mathbf{\Omega}), \quad (24)$$

where μ is the cosine of the angle between $\mathbf{\Omega}$ and the positive z direction. The equation for $\tilde{\psi}$ is the same with $f(\mathbf{\Omega},\mathbf{\Omega}')$ replaced by $f(-\mathbf{\Omega}', -\mathbf{\Omega})$.

We can, of course, appropriately specialize Eq. (20) for this symmetry. It is, however, convenient to rederive the result. Since incident and emergent directions will be characterized by $\mu>0$ or $\mu<0$ it is suggested by the uniqueness theorem to make the decomposition (used by Chandrasekhar[1]) of ψ into $\psi_\pm$ defined by

$$\psi_\pm(z,\mathbf{\Omega})=\psi(z,\pm\mathbf{\Omega}), \quad (\mu>0). \quad (25)$$

Similarly we decompose our sources into $q_\pm(z,\mathbf{\Omega})$. For a given source q_1 Eq. (24) becomes the two equations

$$\pm\mu\frac{\partial\psi_\pm^{(1)}(z,\mathbf{\Omega})}{\partial z}+\psi_\pm^{(1)}=c(z)\int\{f(\pm\mathbf{\Omega},\mathbf{\Omega}')\psi_+^{(1)}(z,\mathbf{\Omega}') \quad (26\text{a, b})$$
$$+f(\pm\mathbf{\Omega},-\mathbf{\Omega}')\psi_-^{(1)}(z,\mathbf{\Omega}')\}d\mathbf{\Omega}'+q_\pm^{(1)}(z,\mathbf{\Omega}).$$

(Here all $\mathbf{\Omega}$ are restricted to $\mu>0$.)

For the functions $\tilde{\psi}_\pm^{(2)}$ corresponding to a different source $q^{(2)}$ we have

$$\pm\mu\frac{\partial\tilde{\psi}_\pm^{(2)}}{\partial z}(z,\mathbf{\Omega})+\tilde{\psi}_\pm^{(2)}=c(z)\int\{f(-\mathbf{\Omega}',\mp\mathbf{\Omega})\tilde{\psi}_+^{(2)}(z,\mathbf{\Omega}') \quad (27\text{a, b})$$
$$+f(\mathbf{\Omega}',\mp\mathbf{\Omega})\tilde{\psi}_-^{(2)}(z,\mathbf{\Omega}')\}d\mathbf{\Omega}'+q_\pm^{(2)}(z,\mathbf{\Omega}).$$

Multiply Eq. (26a) by $\tilde{\psi}_-^{(2)}(z,\mathbf{\Omega})$, Eq. (26b) by $\tilde{\psi}_+^{(2)}(z,\mathbf{\Omega})$, Eq. (27a) by $\psi_-^{(1)}(z,\mathbf{\Omega})$, and Eq. (27b) by $\psi_+^{(1)}(z,\mathbf{\Omega})$. Subtract the sum of the resulting second two equations from the sum of the resulting first two. Integrating $\mathbf{\Omega}$ over all angles with $\mu>0$ and z between a and b $(a>b)$ we find

$$\int d\mathbf{\Omega}\mu\{\tilde{\psi}_-^{(2)}(z,\mathbf{\Omega})\psi_+^{(1)}(z,\mathbf{\Omega})-\tilde{\psi}_+^{(2)}(z,\mathbf{\Omega})\psi_-^{(1)}(z,\mathbf{\Omega})\}_a{}^b$$
$$=\int_a^b dz\int d\mathbf{\Omega}\{\tilde{\psi}_-^{(2)}(z,\mathbf{\Omega})q_+^{(1)}(z,\mathbf{\Omega})+\tilde{\psi}_+^{(2)}(z,\mathbf{\Omega})q_-^{(1)}(z,\mathbf{\Omega})-\psi_-^{(1)}(z,\mathbf{\Omega})q_+^{(2)}(z,\mathbf{\Omega})$$
$$-\psi_+^{(1)}(z,\mathbf{\Omega})q_-^{(2)}(z,\mathbf{\Omega})\}. \quad (28)$$

Specializing to various sources and incident angular distributions the analogs of the theorems in Sec. III are obtained. It will be seen, though, that even more information is contained in these equations.

V. HALF-SPACE PROBLEMS—BASIC EQUATIONS

The fundamental problem of this type is: find functions $(\psi_\pm(z,\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger))$ which satisfy Eqs. (26a, b) (with $q_\pm=0$) for $z\geq\bar{z}$ such that $\psi_+(\bar{z},\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger)=\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}^\dagger)$.

A convenient reformulation of this problem is the following: Find functions satisfying Eqs. (26a, b) everywhere which agree with the above $\psi_\pm(z,\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger)$ for $z>\bar{z}$ and which vanish for $z<\bar{z}$. Clearly this can only be achieved by having plane sources at $z=\bar{z}$. To see what these sources must be we can write our equations as

$$\pm\mu\frac{\partial\psi_\pm}{\partial z}(z,\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger)+\psi_\pm=c(z)\int\{\ \}+\lambda_\pm(\mathbf{\Omega})\delta(z-\bar{z}).$$

Integrating these equations from $\bar{z}-\epsilon$ to $\bar{z}+\epsilon$ (ϵ is infinitesimal) gives

$$\pm\mu\{\psi_\pm(\bar{z}+\epsilon,\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger)-\psi_\pm(\bar{z}-\epsilon,\mathbf{\Omega};\bar{z},\mathbf{e}^\dagger)\}=\lambda_\pm(\mathbf{\Omega}). \quad (29\text{a, b})$$

Since

$$\psi_\pm(\bar{z}-\epsilon,\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger)=0$$

and

$$\psi_+(\bar{z}+\epsilon,\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger)=\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}^\dagger),$$

we have

$$\lambda_+(\mathbf{\Omega})=\mu\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}^\dagger), \quad (30\text{a})$$

$$\lambda_-(\mathbf{\Omega})=-\mu\psi_-(\bar{z},\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger). \quad (30\text{b})$$

Let us now apply Eq. (28) to $\psi_\pm(z,\mathbf{\Omega};z_1,\mathbf{\Omega}_1)$ and $\tilde{\psi}_\pm(z,\mathbf{\Omega};z_2,\mathbf{\Omega}_2)$ where $z_2>z_1$. The sources are

$$q_+^{(1,2)}(z,\mathbf{\Omega})=\mu\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}_{1,2})\delta(z-z_{1,2}),$$
$$q_-^{(1)}(z,\mathbf{\Omega})=-\mu\psi_-(z_1,\mathbf{\Omega};z_1,\mathbf{e}_1)\delta(z-z_1), \quad (31)$$
$$q_-^{(2)}(z,\mathbf{\Omega})=-\mu\tilde{\psi}_-(z_2,\mathbf{\Omega};z_2,\mathbf{\Omega}_2)\delta(z-z_2).$$

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[3] For example, see Case, deHoffmann, and Placzek, *Introduction to the Theory of Neutron Diffusion* (U. S. Government Printing Office, Washington, D. C., 1953), Vol. I.

If we remember that $\tilde{\psi}_\pm(z,\mathbf{\Omega};z_2,\mathbf{\Omega}_2)=0$ for $z<z_2$ and require the functions to vanish as $z\to\infty$ we obtain

$$\mu_2\psi_-(z_2,\mathbf{\Omega}_2;z_1,\mathbf{\Omega}_1)=\int d\mathbf{\Omega}\mu\tilde{\psi}_-(z_2,\mathbf{\Omega};z_2,\mathbf{\Omega}_2)\psi_+(z_2,\mathbf{\Omega};z_1,\mathbf{\Omega}_1). \quad (32)$$

Suppose $z_2\to z_1$. Then $\psi_+(z_2,\mathbf{\Omega};z_1,\mathbf{\Omega}_1)\to\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}_1)$ and we find the relation

$$\mu_2\psi_-(z_1,\mathbf{\Omega}_2;z_1,\mathbf{\Omega}_1)=\mu_1\tilde{\psi}_-(z_1,\mathbf{\Omega}_1;z_1,\mathbf{\Omega}_2). \quad (33a)$$

If in particular the reversal symmetry holds $(f(\mathbf{\Omega},\mathbf{\Omega}')=f(-\mathbf{\Omega}',-\mathbf{\Omega}))$, we have $\psi=\tilde{\psi}$ and Eq. (33a) becomes

$$\mu_2\psi_-(z_1,\mathbf{\Omega}_2;z_1,\mathbf{\Omega}_1)=\mu_1\psi_-(z_1,\mathbf{\Omega}_1;z_1,\mathbf{\Omega}_2). \quad (33b)$$

Up to this point there has been absolutely no use of "The Principle of Invariance." Equation (32) is generally valid. However, as Chandrasekhar[1] has so fully shown, considerable further progress is possible if we can call on this invariance. In the present context this means *only* the following theorem.

Theorem.—If $c(z)$ is independent of z,

$$\psi_\pm(z,\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger)=\psi_\pm(z-\bar{z},\,\mathbf{\Omega};0,\,\mathbf{\Omega}^\dagger).$$

Proof.—$\psi_\pm(z-\bar{z},\,\mathbf{\Omega};0,\,\mathbf{\Omega}^\dagger)$ satisfy exactly the same equations and boundary conditions as $\psi_\pm(z,\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger)$. The uniqueness theorem then tells us they are identical.

Define S and $\tilde{S}$ by

$$S(\mathbf{\Omega},\mathbf{\Omega}')=\mu\psi_-(0,\mathbf{\Omega};0,\mathbf{\Omega}'),$$
$$\tilde{S}(\mathbf{\Omega},\mathbf{\Omega}')=\mu\psi_-(0,\mathbf{\Omega};0,\mathbf{\Omega}'),$$

and put $z_1=0$, $z_2=z$ in Eq. (32). We find

$$\mu\psi_-(z,\mathbf{\Omega};0,\mathbf{\Omega}_1)=\int d\mathbf{\Omega}'\tilde{S}(\mathbf{\Omega}',\mathbf{\Omega})\psi_+(z,\mathbf{\Omega}';0,\mathbf{\Omega}_1). \quad (34a)$$

From Eq. (34a) complete knowledge of S and $\tilde{S}$ can be obtained. For example, if we put $z=0$ this becomes

$$S(\mathbf{\Omega},\mathbf{\Omega}_1)=\tilde{S}(\mathbf{\Omega}_1,\mathbf{\Omega}), \quad (35a)$$

which can be used to rewrite Eq. (34a) as

$$\mu\psi_-(z,\mathbf{\Omega};0,\mathbf{\Omega}_1)=\int d\mathbf{\Omega}'S(\mathbf{\Omega},\mathbf{\Omega}')\psi_+(z,\mathbf{\Omega}';0,\mathbf{\Omega}_1). \quad (34b)$$

[It is convenient that all "reversed" functions have disappeared from the fundamental Eq. (34b) even though no symmetry assumptions on the scattering law have been made.] We note that Eq. (34b) is just Chandrasekhar's starting point for treating half-space problems.

Further information can be obtained from Eq. (34b) by considering the limit $z\to\infty$. Since in this limit we are far from the boundary $z=0$ it is to be expected that ψ will tend towards an infinite medium solution of the homogeneous Eq. (26a, b). Translational invariance implies such a solution in as exponential function of z times a function of $\mathbf{\Omega}$. The requirement $\psi\to0$ as $z\to\infty$ shows the argument of the exponential to be negative (i.e., $\psi_\pm\to\phi_\pm(\mathbf{\Omega})e^{-\kappa z}$ where $\kappa\geq0$ and $\phi_\pm(\mathbf{\Omega})e^{-\kappa z}$ is a solution of the homogeneous transfer equation). Substituting in Eq. (34b) we find that

$$\mu\phi_-(\mathbf{\Omega})=\int d\mathbf{\Omega}'S(\mathbf{\Omega},\mathbf{\Omega}')\phi_+(\mathbf{\Omega}'). \quad (35b)$$

The remaining information needed can be obtained by following Chandrasekhar's procedure and differentiating Eq. (34b) with respect to z. The transfer equations [Eq. (26a, b)] can then be used to eliminate $\partial\psi_\pm/\partial z$. Taking the limit $z\to0$ and remembering that

$$\lim_{z\to0}\psi_+(z,\mathbf{\Omega};0,\mathbf{\Omega}_1)=\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}_1),$$

and

$$\lim_{z\to0}\psi_-(z,\mathbf{\Omega};0,\mathbf{\Omega}_1)=S(\mathbf{\Omega},\mathbf{\Omega}_1)/\mu,$$

we obtain

$$\left(\frac{1}{\mu_0}+\frac{1}{\mu_1}\right)S(\mathbf{\Omega}_0,\mathbf{\Omega}_1)=c\int\int d\mathbf{\Omega}'d\mathbf{\Omega}''f(\mathbf{\Omega}',-\mathbf{\Omega}'')\times\left[\delta_2(-\mathbf{\Omega}_0\cdot\mathbf{\Omega}')+\frac{S(\mathbf{\Omega}_0,\mathbf{\Omega}')}{\mu'}\right]\times\left[\delta_2(-\mathbf{\Omega}''\cdot\mathbf{\Omega}_1)+\frac{S(\mathbf{\Omega}'',\mathbf{\Omega}_1)}{\mu''}\right]. \quad (36)$$

Here, for compactness in writing, we use the conventions that integrals are over *all* directions and $S(\mathbf{\Omega},\mathbf{\Omega}')$ is taken zero for μ or $\mu'<0$.

Thus, using only the reciprocity identity and the translational invariance the nonlinear integral equation [Eq. (36) and the condition on the solution Eq. (35b)] are obtained. Chandrasekhar[1] has shown in many cases that these uniquely determine S.[4] Given S the determination of $\psi_\pm(z,\mathbf{\Omega};0,\mathbf{\Omega}_1)$ is well known and straightforward. Examples are given below.

For further applications of the reciprocity relations it is useful to specialize f. Let us assume, as is true in many cases of interest, that

$$f(\mathbf{\Omega}',\mathbf{\Omega}'')=\Phi_1(\mu')\Phi_2(\mu'')/4\pi,$$

where Φ_1, Φ_2 are even functions. (The physical origin of this even requirement will be seen later.) Then Eq. (36) becomes

$$\left(\frac{1}{\mu_0}+\frac{1}{\mu_1}\right)S(\mathbf{\Omega}_0,\mathbf{\Omega}_1)=\frac{c}{4\pi}\mathcal{H}(\mu_0)\mathcal{G}(\mu_1), \quad (37)$$

[4] A general proof is possible. Since it introduces many concepts foreign to the present discussion and is applicable to a wider class of equations it will be presented elsewhere.

where

$$\mathcal{H}(\mu_0)=\Phi_1(\mu_0)+\int \frac{S(\mathbf{\Omega}_0,\mathbf{\Omega}')}{\mu'}\Phi_1(\mu')d\Omega', \quad (38a)$$

and

$$\mathcal{G}(\mu_1)=\Phi_2(\mu_1)+\int \frac{S(\mathbf{\Omega}'',\mathbf{\Omega}_1)}{\mu''}\Phi_2(\mu'')d\Omega''. \quad (38b)$$

Substituting for S from Eq. (37) gives the coupled integral equations

$$\mathcal{H}(\mu_0)=\Phi_1(\mu_0)+\frac{c}{2}\mu_0\mathcal{H}(\mu_0)\int_0^1 \frac{\Phi_1(\mu')\mathcal{G}(\mu')d\mu'}{\mu_0+\mu'}, \quad (39a)$$

$$\mathcal{G}(\mu_1)=\Phi_2(\mu_1)+\frac{c}{2}\mu_1\mathcal{G}(\mu_1)\int_0^1 \frac{\Phi_2(\mu'')\mathcal{H}(\mu'')d\mu''}{\mu_1+\mu''}. \quad (39b)$$

These are readily reducible to a single integral equation. Let $\mathcal{H}(\mu)=\Phi_1(\mu)H(\mu)$, $\mathcal{G}(\mu)=\Phi_2(\mu)H(\mu)$. The coupled equations are then equivalent to

$$H(\mu)=1+\frac{c}{2}\mu H(\mu)\int_0^1 \frac{\Psi(\mu')H(\mu')d\mu'}{\mu+\mu'}, \quad (40)$$

where

$$\Psi(\mu')=\Phi_1(\mu')\Phi_2(\mu'). \quad (41)$$

The equation for H is thus just in the standard form discussed in detail by Chandrasekhar.[1]

For S we have

$$S(\mathbf{\Omega}_0,\mathbf{\Omega}_1)=\frac{c}{4\pi}\frac{\mu_0\mu_1}{\mu_0+\mu_1}\Phi_1(\mu_0)\Phi_2(\mu_1)H(\mu_0)H(\mu_1); \quad (42a)$$

while

$$\tilde{S}(\mathbf{\Omega}_0,\mathbf{\Omega}_1)=\frac{c}{4\pi}\frac{\mu_0\mu_1}{\mu_0+\mu_1}\Phi_2(\mu_0)\Phi_1(\mu_1)H(\mu_0)H(\mu_1). \quad (42b)$$

VI. HALF-SPACE GREEN'S FUNCTION

For simplicity we restrict ourselves in the following example to isotropic scattering (i.e., $\Phi_1=\Phi_2=1$). Clearly[5] the discussion can be restricted to isotropic sources. The problem is then to find the solution $(\psi_{p\pm}(z,\mathbf{\Omega};z_0))$ of

$$\pm\mu\frac{\partial\psi_{p\pm}}{\partial z}(z,\mathbf{\Omega};z_0)+\psi_{p\pm} =\frac{c}{4\pi}\int d\mathbf{\Omega}'\{\psi_{p+}(z,\mathbf{\Omega}')+\psi_{p-}(z,\mathbf{\Omega}')\}d\mathbf{\Omega}'+\delta(z-z_0)/4\pi, \quad (43)$$

subject to

$$\lim_{z\to 0}\psi_{p+}(z,\mathbf{\Omega};z_0)=0, \quad (44)$$

and

$$\lim_{z\to\infty}\psi_{p\pm}(z,\mathbf{\Omega};z_0)=0.$$

[5] For example, see reference 3, p. 122.

The remark at the end of Sec. III states that

$$\psi_{p-}(0,\mathbf{\Omega};z_0)=\frac{1}{4\pi\mu}\rho(z_0;0,\mu), \quad (45)$$

where $\rho(z_0;0,\mu)=\int d\mathbf{\Omega}'\psi(z_0,\mathbf{\Omega}';0,\mathbf{\Omega})$ is the density at z_0 due to an incident beam in direction $\mathbf{\Omega}$ on the plane $z=0$. ρ is obtained in the following well-known way. Formal integration of Eqs. (26a, b) yields

$$\psi_+(z,\mathbf{\Omega})=e^{-z/\mu}\left\{\psi_+(0,\mathbf{\Omega})+\frac{1}{\mu}\int_0^z e^{+z'/\mu}Q_+(z',\mathbf{\Omega})dz'\right\}, \quad (46a)$$

and

$$\psi_-(z,\mathbf{\Omega})=e^{z/\mu}\left\{\frac{1}{\mu}\int_z^\infty e^{-z'/\mu}Q_-(z',\mathbf{\Omega})dz'\right\}, \quad (46b)$$

where

$$Q_\pm(z,\mathbf{\Omega})=c(z)\int\{f(\pm\mathbf{\Omega},\mathbf{\Omega}')\psi_+(z,\mathbf{\Omega}') +f(\pm\mathbf{\Omega},-\mathbf{\Omega}')\psi_-(z,\mathbf{\Omega}')\}d\mathbf{\Omega}'+q_\pm(z,\mathbf{\Omega}). \quad (47)$$

In the present instance of isotropic scattering, constant c, and isotropic sources $q_0(z)/4\pi$ this says

$$\mu\psi_-(0,\mu)=\int_0^\infty e^{-z'/\mu}\left\{\frac{c}{4\pi}\rho(z')+\frac{q_0(z')}{4\pi}\right\}dz', \quad (48)$$

or

$$\frac{1}{p}\psi_-(0,1/p)=\text{Laplace transform of }\left\{\frac{c}{4\pi}\rho+\frac{q_0}{4\pi}\right\}. \quad (49)$$

Applying the Fourier-Mellin inversion theorem gives

$$\frac{c}{4\pi}\rho(z_0;0,\mu)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{1}{p}\psi_-(0,1/p;0,\mu)e^{pz_0}dp. \quad (50)$$

In Sec. V it was found that

$$\frac{1}{p}\psi_-(0,1/p;0,\mu)=\frac{c}{4\pi}\frac{H(\mu)H(1/p)}{p+1/\mu}, \quad (51)$$

where

$$H(u)=1+\frac{c}{2}\mu H(\mu)\int_0^1\frac{H(\mu')d\mu'}{\mu+\mu'}. \quad (52)$$

Therefore,

$$\rho(z_0;0,\mu)=\frac{H(\mu)}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{H(1/p)e^{pz_0}}{p+1/\mu}dp. \quad (53)$$

The work of Chandrasekhar[1] and others[3] enables us to regard $H(\mu)$ $(0\le\mu\le1)$ as a known function. ("Known" here means in the sense of numerical tables. While it is true that analytical formulas are available these are so complicated as to be virtually useless.) The main problem is to express the contour integral in terms of "known" functions. This can be done with the aid of the condition given by Eq. (35) which has so far been ignored. The infinite medium form of the homogeneous equations (26a, b) for isotropic scattering and constant c have the unique solution which vanishes for $z \to \infty$

$$\psi_{\pm}(z,\mathbf{\Omega}) = (\text{constant})[1/(1\mp\kappa\mu)]e^{-\kappa z},$$

where κ is the positive root of the equation

$$1 - c(\tanh^{-1}\kappa)/\kappa = 0.$$

Hence in Eq. (35b) we have

$$\phi_{\pm}(\mathbf{\Omega}) = 1/(1\mp\kappa\mu). \tag{54}$$

Substituting the expression for S in terms of the H function we find from Eq. (35b) that

$$\int_0^1 \frac{\mu' H(\mu')d\mu'}{(\mu+\mu')(1-\kappa\mu')} = \frac{2}{c}\frac{1}{H(\mu)}\frac{1}{(1+\kappa\mu)}. \tag{55}$$

However, on decomposing into partial fractions we have the identity

$$\frac{\mu'}{(\mu+\mu')(1-\kappa\mu')} = \frac{1}{1+\kappa\mu}\left\{\frac{-\mu}{\mu+\mu'} + \frac{1}{\kappa}\frac{1}{(1/\kappa)-\mu'}\right\}. \tag{56}$$

On substituting this into Eq. (55) we obtain integrals readily evaluated on recalling the integral equation [Eq. (52)] satisfied by $H(\mu)$. The result is

$$\int_0^1 \frac{\mu' H(\mu')d\mu'}{(\mu+\mu')(1-\kappa\mu')} = \frac{2}{c}\frac{1}{1+\kappa\mu}\left\{\frac{1}{H(\mu)} - \frac{1}{H(-1/\kappa)}\right\}. \tag{57}$$

Comparing Eqs. (57) and (55) we see that $H(p)$ has a pole at $p = -1/\kappa$.

The integral in Eq. (53) can now be evaluated in the limit of large z_0. Deforming the contour to the left we see that the dominant term in an asymptotic expansion of $\rho(z_0; 0,\mu)$ is given by the singularity of $H(1/p)$ furthest to the right in the complex plane. This is just the pole at $p = -\kappa$. Hence

$$\rho(z_0; 0,\mu) \underset{z_0\to\infty}{\longrightarrow} (\text{constant})e^{-\kappa z_0}\frac{\mu H(\mu)}{1-\kappa\mu}. \tag{58}$$

The reciprocity relation (Eq. 45) says then that the emergent angular distribution from a half-space with a plane source far from the boundary is proportional to $H(\mu)/(1-\kappa\mu)$. Alternatively, $H(\mu)/(1-\kappa\mu)$ is the emergent angular distribution for the Milne problem (i.e., the problem of finding a solution of the homogeneous transfer equation for the half-space which is $\sim e^{\kappa z}$ for large z). Thus

$$\rho_m(z) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{H(1/p)e^{pz}}{p-\kappa}dp. \tag{59}$$

This is a "known" function with thoroughly investigated properties.[3] (The normalization[6] chosen here is such that $\rho_m(0) = 1$.)

Comparing Eqs. (53) and (59) it is apparent that

$$\left(\frac{d}{dz_0} + \frac{1}{\mu}\right)\frac{\rho(z_0; 0,\mu)}{H(\mu)} = \eta_m(z_0), \tag{60}$$

where we define $\eta_m(z_0)$ by

$$\eta_m(z_0) = \left(\frac{d}{dz_0} - \kappa\right)\rho_m(z_0).$$

An elementary integration then gives

$$\rho(z_0; 0,\mu) = H(\mu)\left\{e^{-z_0/\mu} + \int_0^{z_0} e^{-(z_0-z')/\mu}\eta_m(z')dz'\right\}. \tag{61}$$

The emergent angular distribution for a unit isotropic plane source at z_0 is given by Eq. (45) in conjunction with Eq. 61).

The density $\rho(z;z_0)$ resulting from an isotropic plane source is obtained by inverting the Laplace transform of Eq. (49). Thus

$$c\rho(z; z_0) + \delta(z-z_0) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{pz}H(1/p) \times\left\{e^{-pz_0} + \int_0^{z_0} e^{-p(z_0-z')}\eta_m(z')dz'\right\}dp. \tag{62}$$

The contour integral is readily evaluated since it is seen from Eq. (59) that

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} H(1/p)e^{pz}dp = \delta(z-z_0) + S(z)\eta_m(z), \tag{63}$$

where

$$S(z) = 0, \quad z<0$$
$$\quad = 1, \quad z>0.$$

Using this result in Eq. (62) it follows that

$$c\rho(z; z_0) = \eta_m(|z-z_0|) + \int_{|z-z_0|/2}^{(z+z_0)/2} \eta_m\left(z' + \frac{|z-z_0|}{2}\right) \times\eta_m\left(z' - \frac{|z-z_0|}{2}\right)dz'. \tag{64}$$ [7]

A more convenient form of this equation is one in which the corrections to the infinite medium plane

[6] The solution of the Milne problem given in reference 3 is $[\kappa/(1-c)^{\frac{1}{2}}]\rho_m(z)$.

[7] A result equivalent to this has been obtained by B. Davison and G. Placzek, Canadian Report MT-118.

source solution due to the boundary $z=0$ are given explicitly. This can be obtained as follows: Let z, $z_0\to\infty$ while $|z-z_0|$ is kept finite. The result must certainly be the infinite medium plane source solution[8] $\rho_\infty(z-z_0)$ i.e.,

$$c\rho_\infty(z-z_0)=\eta_m(|z-z_0|)+\int_{|z-z_0|/2}^{\infty}\eta_m\left(z'+\frac{|z-z_0|}{2}\right)\times\eta_m\left(z'-\frac{|z-z_0|}{2}\right)dz'. \quad (65)$$

Therefore, on making a simplifying change of variable in the additional term in Eq. (64) we have

$$\rho(z;z_0)=\rho_\infty(z-z_0)-\frac{1}{c}\int_0^\infty \eta_m(\bar{z}+z)\eta_m(\bar{z}+z_0)d\bar{z}. \quad (66)$$

If desired the angular distributions can be obtained using Eqs. (66) and (46a, b).

VII. GREEN'S FUNCTION FOR ADJACENT HALF-SPACES

Generalizing the problem considered in the previous section we will here construct the Green's function for adjacent half-spaces. For simplicity we again restrict ourselves to isotropic scattering. The media for $z>0$ (region 1) and $z<0$ (region 2) are to be characterized by constants c_1 and c_2, respectively. The unit isotropic source will be at z_0 where $z_0>0$.

It is useful to start with the simpler problem of a unit directional plane source at $z=0$ emitting in direction Ω_1 into region 1 ($\mu_1>0$). The first example of a reciprocity relation discussed in Sec. III relates the density at z_0 due to this source to the emergent angular distribution at $z=0$ due to the unit isotropic source at z_0. Specifically

$$\psi_{p-}(0,\mathbf{\Omega}_1;z_0)=\rho(z_0;0,\mathbf{\Omega}_1)/4\pi. \quad (67)$$

Our program will be the following

(1) We will find the emergent distribution from region 1 due to the directional source at $z=0$.

(2) From this emergent distribution the density at z_0 due to the directional source will be obtained.

(3) Using (67) the emergent distribution from "1" due to the isotropic plane source is then known. From this we can determine:

(a) The density for $z>0$.
(b) The emergent distribution from "2."

(4) From the emergent distribution from "2" the density for $z<0$ due to the isotropic source will be found.

[8] This is discussed in detail in reference 3.

The first step is then to solve the equations

$$\pm\mu\frac{\partial}{\partial z}\psi_\pm(z,\mathbf{\Omega};0,\mathbf{\Omega}_1)+\psi_\pm=\frac{c(z)}{4\pi}\int\{\psi_+(z,\mathbf{\Omega}')+\psi_-(z,\mathbf{\Omega}')\}d\mathbf{\Omega}'+\begin{cases}\delta(z)\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}_1)\\0,\end{cases} \quad (68\text{a, b})$$

where

$$c(z)=c_1,\quad z>0$$
$$\quad=c_2,\quad z<0.$$

Denote the solutions for $z>0$ and $z<0$ by superscripts [1] and [2], respectively. Integrating Eqs. (68a, b) from slightly less than to slightly greater than zero gives

$$\pm\mu[\psi_\pm^{(1)}(0,\mathbf{\Omega})-\psi_\pm^{(2)}(0,\mathbf{\Omega})]=\begin{cases}\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}_1) & (69\text{a})\\0. & (69\text{b})\end{cases}$$

From the discussion in Sec. V we also know that

$$\psi_-^{(1)}(0,\mathbf{\Omega})=\int d\mathbf{\Omega}'\frac{S^{(1)}(\mathbf{\Omega},\mathbf{\Omega}')}{\mu}\psi_+^{(1)}(0,\mathbf{\Omega}'), \quad (70\text{a})$$

and

$$\psi_+^{(2)}(0,\mathbf{\Omega})=\int d\mathbf{\Omega}'\frac{S^{(2)}(\mathbf{\Omega},\mathbf{\Omega}')}{\mu}\psi_-^{(2)}(0,\mathbf{\Omega}'), \quad (70\text{b})$$

where $S^{(1)}$, $S^{(2)}$ are the S functions corresponding to c_1 and c_2.

Eliminating $\psi_+^{(1)}$ and $\psi_-^{(2)}$ from Eqs. (70a, b) by means of the relations of Eqs. (69a, b) we obtain the integral equations

$$\psi_-^{(1)}(0,\mathbf{\Omega})=\frac{c_1}{4\pi}\frac{H_1(\mu)H_1(\mu_1)}{\mu+\mu_1}+\frac{c_1}{2}H_1(\mu)\int_0^1\frac{\mu' H_1(\mu')}{\mu+\mu'}\psi_-^{(2)}(0,\mathbf{\Omega}')d\mu',$$
$$\psi_+^{(2)}(0,\mathbf{\Omega})=\frac{c_2}{2}H_2(\mu)\int_0^1\frac{\mu' H_2(\mu')}{\mu+\mu'}\psi_-^{(1)}(0,\mathbf{\Omega}')d\mu'. \quad (71)$$

Fortunately, these equations are readily solved. The identity

$$\frac{\mu'}{(\mu+\mu')(\mu'+a)}=\frac{1}{\mu-a}\left\{\frac{\mu}{\mu+\mu'}-\frac{a}{\mu'+a}\right\}$$

together with the integral equation satisfied by an H function gives

$$\int_0^1\frac{\mu' H(\mu')du'}{(\mu+\mu')(\mu'+a)}=\frac{2}{c}\frac{1}{\mu-a}\left\{\frac{1}{H(a)}-\frac{1}{H(\mu)}\right\}. \quad (72)$$

This suggests that

$$\psi_-^{(1)}(0,\mathbf{\Omega})=\{\alpha_1+\beta_1H_1(\mu)/H_2(\mu)\}/(\mu+\mu_1),$$
$$\psi_+^{(2)}(0,\mathbf{\Omega})=\{\alpha_2+\beta_2H_2(\mu)/H_1(\mu)\}/(\mu-\mu_1), \quad (73)$$

where α_1, α_2, β_1, β_2 are constants to be determined.[9] Substituting Eq. (73) into Eq. (71) gives

$$\{\alpha_1+\beta_1H_1(\mu)/H_2(\mu)\}/(\mu+\mu_1)=\frac{H_1(\mu)}{(\mu+\mu_1)}\left[\frac{c_1}{4\pi}H_1(\mu_1)+\frac{\alpha_2}{H_1(-\mu_1)}+\frac{\beta_2c_1/c_2}{H_2(-\mu_1)}\right]-\frac{\alpha_2}{(\mu+\mu_1)}-\frac{\beta_2H_1(\mu)c_1/c_2}{(\mu+\mu_1)H_2(\mu)},$$
$$\{\alpha_2+\beta_2H_2(\mu)/H_1(\mu)\}/(\mu-\mu_1)=\frac{H_2(\mu)}{(\mu-\mu_1)}\left[\frac{\alpha_1}{H_2(\mu_1)}+\frac{\beta_1c_2/c_1}{H_1(\mu_1)}\right]-\frac{\alpha_1}{(\mu-\mu_1)}-\frac{\beta_1H_2(\mu)c_2/c_1}{(\mu-\mu_1)H_1(\mu)}. \quad (74)$$

These equations are compatible if

$$\alpha_1=-\alpha_2,\quad \beta_1=-\beta_2c_1/c_2,$$
$$\frac{c_1}{4\pi}H_1(\mu_1)+\frac{\alpha_2}{H_1(-\mu_1)}+\frac{\beta_2c_1/c_2}{H_2(-\mu_1)}=0,$$

and

$$\frac{\alpha_1}{H_2(\mu_1)}+\frac{\beta_1c_2/c_1}{H_1(\mu_1)}=0.$$

Solving for the constants we obtain

$$\beta_1=\frac{c_1}{4\pi}\frac{H_1(\mu_1)}{H_2(\mu_1)}\{\ \},\quad \beta_2=-\frac{c_2}{4\pi}\frac{H_1(\mu_1)}{H_2(\mu_1)}\{\ \},$$
$$\alpha_1=-\frac{c_2}{4\pi}\{\ \},\qquad \alpha_2=\frac{c_2}{4\pi}\{\ \},$$

where

$$\{\ \}=\left[\frac{1}{H_2(\mu_1)H_2(-\mu_1)}-\frac{c_2/c_1}{H_1(\mu_1)H_1(-\mu_1)}\right]^{-1}. \quad (75)$$

The expression of Eq. (75) simplifies on using the identity[10]

$$\frac{1}{H(z)H(-z)}=1-c(\tanh^{-1}z)/z, \quad (76)$$

which follows directly from the integral equation for the H functions. Thus we obtain

$$\psi_-^{(1)}=\frac{-c_1c_2}{4\pi(c_1-c_2)}\frac{1}{(\mu+\mu_1)}\left\{1-\frac{c_1H_1(\mu_1)H_1(\mu)}{c_2H_2(\mu_1)H_2(\mu)}\right\}, \quad (77a)$$

$$\psi_+^{(2)}=\frac{-c_1c_2}{4\pi(c_1-c_2)}\frac{1}{(\mu-\mu_1)}\left\{-1+\frac{H_1(\mu_1)H_2(\mu)}{H_2(\mu_1)H_1(\mu)}\right\}. \quad (77b)$$

Inverting Eq. (49) shows the neutron density at $z_0>0$ due to the directional source at $z=0$ is given by

$$\frac{c_1}{4\pi}\rho(z_0;0,\mathbf{\Omega}_1)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{\psi_-^{(1)}(0,1/p;0,\mathbf{\Omega}_1)}{p}e^{pz_0}dp, \quad (78)$$

or

$$\rho(z_0;0,\mathbf{\Omega}_1)=J_1+J_2,$$

where

$$J_1=\frac{-c_2}{c_1-c_2}\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{e^{pz_0}dp}{1+p\mu_1}=\frac{-c_2}{c_1-c_2}\frac{e^{-z_0/\mu_1}}{\mu_1}, \quad (79)$$

and

$$J_2=\frac{c_1}{c_1-c_2}\frac{H_1(\mu_1)}{H_2(\mu_1)}\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{H_1(1/p)e^{pz_0}dp}{H_2(1/p)(1+p\mu_1)}. \quad (80)$$

As in Sec. VI, J_2 can be related to "known" functions. When $z_0\to\infty$ the principal contribution in Eq. (78) comes from the pole of $H_1(1/p)$ at $p=-\kappa_1$.

$$[1-c_1(\tanh^{-1}\kappa_1)/\kappa_1=0.]$$

Hence

$$\rho(z_0;0,\mathbf{\Omega}_1)\underset{z_0\to\infty}{\longrightarrow}(\text{constant})\frac{e^{-\kappa_1z_0}}{1-\kappa_1\mu_1}\frac{H_1(\mu_1)}{H_2(\mu_1)}. \quad (81)$$

Thus, by Eq. (67), $H_1(\mu_1)/H_2(\mu_1)(1-\kappa_1\mu_1)$ is the emergent distribution from region "1" resulting from a source located far from the boundary. In other words, this is the emergent distribution for the generalized Milne problem—the problem of finding the solution of the homogeneous form of Eq. (68) subject to the condition of behaving as e^{κ_1z} for $z\to\infty$. Let us call the density for $z>0$ corresponding to this problem $\bar{\rho}_m^{(1)}(z)$. The inverse of the Laplace transformation gives

$$\bar{\rho}_m^{(1)}(z)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{H_1(1/p)e^{pz}dp}{(p-\kappa_1)H_2(1/p)}. \quad (82)$$

This can be expressed in terms of the known functions $H_2(\mu)$ and $\rho_m^{(1)}(z)$. [Here $\rho_m^{(1)}(z)$ is defined as in Eq. (59) by

$$\rho_m^{(1)}(z)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{H_1(1/p)}{p-\kappa_1}e^{pz}dp.]$$

Thus, on using the integral equation for H_2 in the form

$$\frac{1}{H_2(1/p)}=1-\frac{c_2}{2}\int_0^1\frac{H_2(\mu')d\mu'}{1+p\mu'} \quad (83)$$

[9] This form is also suggested by the solution obtained for a similar set of equations by S. Chandrasekhar, Can. J. Research **A29**, 14 (1951).

[10] See reference 1, p. 116.

it is found that

$$\bar{\rho}_m^{(1)}(z)=\rho_m^{(1)}(z)-\frac{c_2}{2}\int_0^z \rho_m^{(1)}(z')dz' \times\int_0^1 e^{-(z-z')/\mu'}\frac{H_2(\mu')d\mu'}{\mu'}. \quad (84)$$

In terms of $\bar{\eta}_m^{(1)}(z)\equiv(d/dz-\kappa_1)\bar{\rho}_m^{(1)}(z)$ we can write Eq. (80) as

$$J_2=\frac{c_1}{c_1-c_2}\frac{H_1(\mu_1)}{H_2(\mu_1)}\frac{1}{\mu_1} \times\left\{e^{-z_0/\mu_1}+\int_0^{z_0} e^{-(z_0-z')/\mu_1}\bar{\eta}_m^{(1)}(z')dz'\right\}. \quad (85)$$

Using Eq. (67) the emergent angular distribution from region "1" due to a unit isotropic plane source at $z_0>0$ is

$$\psi_{p-}(0,\mathbf{\Omega}_1;z_0)=\frac{-c_2}{4\pi(c_1-c_2)}\frac{e^{-z_0/\mu_1}}{\mu_1} +\frac{c_1}{4\pi(c_1-c_2)}\frac{H_1(\mu_1)}{H_2(\mu_1)}\frac{1}{\mu_1}\{\ \}, \quad (86)$$

where { } is the same as in Eq. (85).

The inversion theorem gives for the density at $z>0$

$$\bar{\rho}(z;z_0)=\frac{1}{c_1-c_2}\bar{\eta}_m^{(1)}(|z-z_0|) +\frac{1}{c_1-c_2}\int_{|z-z_0|/2}^{(z+z_0)/2}\bar{\eta}_m^{(1)}\left(z'+\frac{|z-z_0|}{2}\right) \times\bar{\eta}_m^{(1)}\left(z'-\frac{|z-z_0|}{2}\right)dz'. \quad (87)$$

Again considerations of z, $z_0\to\infty$, $|z-z_0|$ finite, identifies the terms which describe a plane source in an infinite medium of properties "1" ($\rho_\infty^{(1)}(z-z_0)$) and the correction terms due to the boundary. The result is

$$\bar{\rho}(z;z_0)=\rho_\infty^{(1)}(z-z_0)-\frac{1}{c_1-c_2}\int_0^\infty \bar{\eta}_m^{(1)}(z'+z_0) \times\bar{\eta}_m^{(1)}(z'+z)dz'. \quad (z,z_0>0). \quad (88)$$

[It is immediately apparent that Eq. (88) reduces to Eq. (66) when $c_2\to0$ and to $\rho_\infty^{(1)}(z-z_0)$ when $c_2\to c_1$.]

The emergent angular distribution from region "2" due to the plane source at z_0 in region "1" is

$$\psi_{p+}(0,\mathbf{\Omega};z_0)=\int\frac{S^{(2)}(\mathbf{\Omega},\mathbf{\Omega}')}{\mu}\psi_{p-}(0,\mathbf{\Omega}';z_0)d\Omega'. \quad (89)$$

Expressing $S^{(2)}$ in terms of H_2, using the integral equation for H_2 and remembering that $z_0>0$ we find

$$\psi_{p+}(0,\mathbf{\Omega};z_0)=\frac{-c_2}{4\pi(c_1-c_2)}\frac{H_2(\mu)}{H_1(\mu)}\frac{1}{2\pi i} \times\int_{\beta-i\infty}^{\beta+i\infty}\frac{e^{pz_0}}{1-\mu p}\frac{H_1(1/p)}{H_2(1/p)}dp. \quad (90a)$$

With the aid of Eq. (82) it is seen that

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{e^{pz_0}}{1-\mu p}\frac{H_1(1/p)}{H_2(1/p)}dp =\frac{e^{z_0/\mu}}{\mu}\int_{z_0}^\infty e^{-z'/\mu}\bar{\eta}_m^{(1)}(z')dz' =\frac{e^{z_0/\mu}}{\mu}\left\{\frac{H_1(\mu)}{H_2(\mu)}-\int_0^{z_0} e^{-z'/\mu}\bar{\eta}_m^{(1)}(z')dz'\right\}.$$

Hence

$$\psi_{p+}(0,\mathbf{\Omega};z_0)=\frac{-c_2}{4\pi(c_1-c_2)}\frac{1}{\mu}\left\{e^{z_0/\mu}-\frac{H_2(\mu)}{H_1(\mu)} \times\int_0^{z_0} e^{(z_0-z')/\mu}\bar{\eta}_m^{(1)}(z')dz'\right\}. \quad (90b)$$

The Fourier-Mellin inversion theorem then gives

$$\bar{\rho}(z;z_0)=\frac{-1}{c_1-c_2}\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{-pz}\left\{e^{pz_0}-\frac{H_2(1/p)}{H_1(1/p)} \times\int_0^{z_0} e^{p(z_0-z')}\bar{\eta}_m^{(1)}(z')dz'\right\}, \quad (z<0,\ z_0>0). \quad (91)$$

Changing the order of integration yields

$$\bar{\rho}(z;z_0)=\frac{1}{c_1-c_2}\int_0^{z_0}\bar{\eta}_m^{(1)}(z')dz'\frac{1}{2\pi i} \times\int_{\beta-i\infty}^{\beta+i\infty} e^{p(z_0-z'-z)}\frac{H_2(1/p)}{H_1(1/p)}dp. \quad (92)$$

However, it is readily shown that

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{H_2(1/p)}{H_1(1/p)}e^{-pz}dp=\bar{\eta}_m^{(2)}(z)\equiv\left(\frac{d}{dz}-\kappa_1\right)\bar{\rho}_m^{(2)}(z), \quad (93)$$

where $\bar{\rho}_m^{(2)}(z)$ is the density in region "2" in the generalized Milne problem. Expressed in terms of $\bar{\eta}_m^{(2)}$

Eq. (92) is

$$\bar{\rho}(z;z_0)=\frac{1}{c_1-c_2}\int_{(z-z_0)/2}^{(z+z_0)/2}\bar{\eta}_m^{(1)}\left(z'-\frac{z-z_0}{2}\right)\times\bar{\eta}_m^{(2)}\left(z'+\frac{z-z_0}{2}\right)dz',$$

where $z<0$, $z_0>0$. [Explicit formulas for $\bar{\rho}_m^{(2)}$, analogous to Eq. (84) for $\bar{\rho}_m^{(1)}$, in terms of $H_1(\mu)$ and the solution of the standard Milne problem for a medium of properties "2" ($\rho_m^{(2)}$) are readily obtained.]

VIII. EQUATIONS FOR A PLANE SLAB

Finally it will be shown that Chandrasekhar's equations[11] which determine the reflection and transmission of a finite slab follow directly from the identity of Eq. (28).

The problem is to determine the solution of the homogeneous form of Eq. (26) in a region $\bar{z}\leq z\leq\bar{z}+\tau$ subject to the boundary conditions

$$\psi_{\text{inc}}(\bar{z},\mathbf{\Omega})\equiv\psi_+(\bar{z},\mathbf{\Omega})=\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}^\dagger),$$
$$\psi_{\text{inc}}(\bar{z}+\tau,\Omega)\equiv\psi_-(\bar{z}+\tau,\Omega)=0. \quad (94)$$

Leaving the τ dependence implicit, we denote the solution by $\Psi_\pm(z,\mathbf{\Omega},\bar{z},\mathbf{\Omega}^\dagger)$. As before we consider a solution which coincides with Ψ for $\bar{z}<z<\bar{z}+\tau$ and vanishes outside this region. Clearly it is necessary to introduce the sources

$$q_+(z,\mathbf{\Omega})=\mu\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}^\dagger)\delta(z-\bar{z})-\mu\Psi_+(\bar{z}+\tau,\,\mathbf{\Omega};\,\bar{z},\,\mathbf{\Omega}^\dagger)\delta(z-(\bar{z}+\tau)),\quad (95)$$
$$q_-(z,\mathbf{\Omega})=-\mu\Psi_-(\bar{z},\mathbf{\Omega};\,\bar{z},\mathbf{\Omega}^\dagger)\delta(z-\bar{z}).$$

Applying the identity of Eq. (28) to $\Psi_\pm(z,\mathbf{\Omega};z_1,\mathbf{\Omega}_1)$ and $\tilde{\Psi}_\pm(z,\mathbf{\Omega};z_2,\mathbf{\Omega}_2)$ where $z_1+\tau\geq z_2\geq z_1$ results in

$$\mu_2\Psi_-(z_2,\mathbf{\Omega}_2;z_1,\mathbf{\Omega}_1)=\int d\mathbf{\Omega}\mu\tilde{\Psi}_-(z_2,\mathbf{\Omega};z_2,\mathbf{\Omega}_2)\Psi_+(z_2,\mathbf{\Omega};z_1,\mathbf{\Omega}_1)-\int d\mathbf{\Omega}\mu\Psi_+(z_1+\tau,\mathbf{\Omega};z_1,\mathbf{\Omega}_1)\times\tilde{\Psi}_-(z_1+\tau,\,\mathbf{\Omega};\,z_2,\,\mathbf{\Omega}_2).\quad (96a)$$

Assuming c independent of z we know from the "Principle of Invariance" that

$$\Psi(z,\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger)=\Psi(z-\bar{z},\,\mathbf{\Omega};0,\,\mathbf{\Omega}^\dagger).$$

If

$$S(\mathbf{\Omega},\mathbf{\Omega}')=\mu\Psi_-(0,\mathbf{\Omega};0,\mathbf{\Omega}'),\quad\tilde{S}(\mathbf{\Omega},\mathbf{\Omega}')=\mu\tilde{\Psi}_-(0,\mathbf{\Omega};0,\mathbf{\Omega}'),$$
$$T(\mathbf{\Omega},\mathbf{\Omega}')=\mu\Psi_+(\tau,\mathbf{\Omega};0,\mathbf{\Omega}'),\quad\tilde{T}(\mathbf{\Omega},\mathbf{\Omega}')=\mu\tilde{\Psi}_+(\tau,\mathbf{\Omega};0,\mathbf{\Omega}'),$$

it is possible to rewrite Eq. (96a) in the form

$$\mu\Psi_-(z,\mathbf{\Omega};0,\mathbf{\Omega}_1)=\int d\mathbf{\Omega}'\tilde{S}(\mathbf{\Omega}',\mathbf{\Omega})\Psi_+(z,\mathbf{\Omega}';0,\mathbf{\Omega}_1)-\int d\mathbf{\Omega}'T(\mathbf{\Omega}',\mathbf{\Omega})\tilde{\Psi}_-(\tau-z,\,\mathbf{\Omega}';0,\,\mathbf{\Omega}_1).\quad (96b)$$

Putting $z=0$ and remembering the boundary condition of Eq. (94) one finds

$$S(\mathbf{\Omega},\mathbf{\Omega}_1)=\tilde{S}(\mathbf{\Omega}_1,\mathbf{\Omega}).\quad (96d)$$

Therefore, if $f(\mathbf{\Omega},\mathbf{\Omega}')$ has the time reflection symmetry S is symmetrical. In any event it is permissible to rewrite Eq. (96b) as

$$\mu\Psi_-(z,\mathbf{\Omega};0,\mathbf{\Omega}_1)=\int d\mathbf{\Omega}'S(\mathbf{\Omega},\mathbf{\Omega}')\Psi_+(z,\mathbf{\Omega}';0,\mathbf{\Omega}_1)-\int d\mathbf{\Omega}'T(\mathbf{\Omega}',\mathbf{\Omega})\tilde{\Psi}_-(\tau-z,\,\mathbf{\Omega}';0,\,\mathbf{\Omega}).\quad (96c)$$

Differentiate this equation with respect to z. Use the transfer equations to eliminate derivatives and put $z=0$. Remembering the boundary conditions and the definitions of S and T one obtains

$$\left(\frac{1}{\mu_0}+\frac{1}{\mu_1}\right)S(\mathbf{\Omega}_0,\mathbf{\Omega}_1)=c\int\int d\mathbf{\Omega}'d\mathbf{\Omega}''\{\ \},$$

where

$$\{\ \}=f(\mathbf{\Omega}',-\mathbf{\Omega}'')\left[\delta_2(-\mathbf{\Omega}_0\cdot\mathbf{\Omega}')+\frac{S(\mathbf{\Omega}_0,\mathbf{\Omega}')}{\mu'}\right]\times\left[\delta_2(-\mathbf{\Omega}''\cdot\mathbf{\Omega}_1)+\frac{S(\mathbf{\Omega}'',\mathbf{\Omega}_1)}{\mu''}\right]-f(-\mathbf{\Omega}',\,\mathbf{\Omega}'')\frac{\tilde{T}(\mathbf{\Omega}',\mathbf{\Omega}_0)}{\mu'}\frac{T(\mathbf{\Omega}'',\mathbf{\Omega}_1)}{\mu''}.\quad (97)$$

[The same convention is used as in Eq. (36).]

The remaining conditions to determine S and T are obtained by comparing a solution with a beam incident on the slab from the left to one in which the incidence is from the right. Let $\tilde{\Phi}(z,\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger)$ be the solution of the homogeneous Eq. (27) for $\bar{z}<z<\bar{z}+\tau$ subject to the boundary conditions

$$\tilde{\Phi}_-(\tau+\bar{z},\,\mathbf{\Omega};\,\bar{z},\,\mathbf{\Omega}^\dagger)=\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}^\dagger),$$
$$\tilde{\Phi}_+(\bar{z},\mathbf{\Omega};\bar{z},\mathbf{\Omega}^\dagger)=0.$$

Again consider a solution of Eq. (27) everywhere which vanishes except for $\bar{z}<z<\bar{z}+\tau$ where it coincides with

[11] See reference 1, Chap. VII.

$\tilde{\Phi}$. The sources that must be introduced are

$$q_+(z,\mathbf{\Omega})=-\mu\tilde{\Phi}_+(\bar{z}+\tau,\,\mathbf{\Omega}\,;\,\bar{z},\,\mathbf{\Omega}^\dagger)\delta(z-(\bar{z}+\tau)),$$
$$q_-(z,\mathbf{\Omega})=\mu\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}^\dagger)\delta(z-(\bar{z}+\tau))-\mu\tilde{\Phi}_-(\bar{z},\mathbf{\Omega}\,;\,\bar{z},\mathbf{\Omega}^\dagger)\delta(z-\bar{z}). \quad (98)$$

Let $\psi_\pm^{(1)}=\Psi_\pm(z,\mathbf{\Omega}\,;\,z_1,\mathbf{\Omega}_1)$ and $\tilde{\psi}_\pm^{(2)}=\tilde{\Phi}_\pm(z,\mathbf{\Omega}\,;\,z_2,\mathbf{\Omega}_2)$ in the identity of Eq. (28) (where $z_1\leq z_2\leq z_1+\tau$). The result is

$$\int d\mathbf{\Omega}\mu\Psi_+(z_2,\mathbf{\Omega}\,;\,z_1,\mathbf{\Omega}_1)\tilde{\Phi}_-(z_2,\mathbf{\Omega}\,;\,z_2,\mathbf{\Omega}_2)$$
$$=\int d\mathbf{\Omega}\mu\tilde{\Phi}_-(z_1+\tau,\,\mathbf{\Omega}\,;\,z_2,\,\mathbf{\Omega}_2)\Psi_+(z_1+\tau,\,\mathbf{\Omega}\,;\,z_1,\,\mathbf{\Omega}_1). \quad (99)$$

Suppose $\tilde{\Psi}_\pm(z,\mathbf{\Omega})$ give a solution of Eqs. (27) with constant c. Clearly $\tilde{\Psi}'_\pm(z,\mathbf{\Omega})=\tilde{\Psi}_\mp(-z,\,\Omega)$ is a solution of the equations obtained from Eq. (27) by replacing $f(\mathbf{\Omega},\mathbf{\Omega}')$ by $f(-\mathbf{\Omega},\,-\mathbf{\Omega}')$. Let us *assume* $f(\mathbf{\Omega},\mathbf{\Omega}')=f(-\mathbf{\Omega},\,-\mathbf{\Omega}')$. (This corresponds to invariance under spatial reflections. It is satisfied for all important applications.[12]) From the uniqueness theorem it follows that for constant c

$$\tilde{\Phi}_\mp(z,\mathbf{\Omega}\,;\,\bar{z},\mathbf{\Omega}^\dagger)=\tilde{\Psi}_\pm(\bar{z}+\tau-z,\,\mathbf{\Omega}\,;\,0,\,\mathbf{\Omega}^\dagger). \quad (100)$$

Inserting this in Eq. (99) and using the translational invariance one concludes that

$$\int d\Omega\tilde{T}(\mathbf{\Omega},\mathbf{\Omega}_2)\Psi_+(z,\mathbf{\Omega}\,;\,0,\mathbf{\Omega}_1)=\int d\mathbf{\Omega}T(\mathbf{\Omega},\mathbf{\Omega}_1)\tilde{\Psi}_+(z,\mathbf{\Omega}\,;\,0,\mathbf{\Omega}_2). \quad (101)$$

Putting $z=0$ gives

$$\tilde{T}(\mathbf{\Omega}_1,\mathbf{\Omega}_2)=T(\mathbf{\Omega}_2,\mathbf{\Omega}_1). \quad (102)$$

Differentiate Eq. (101) with respect to z. Eliminate derivatives by means of the transfer equation and then put $z=0$. One obtains

$$\frac{T(\mathbf{\Omega}_0,\mathbf{\Omega}_1)}{\mu_0}-\frac{\tilde{T}(\mathbf{\Omega}_1,\mathbf{\Omega}_0)}{\mu_1}=c\int\int d\mathbf{\Omega}d\mathbf{\Omega}'\{\ \}, \quad (103)$$

where

$$\{\ \}=f(\mathbf{\Omega}',\,-\mathbf{\Omega})\frac{T(\mathbf{\Omega},\mathbf{\Omega}_1)}{\mu}\left[\delta_2(-\mathbf{\Omega}'\cdot\mathbf{\Omega}_0)+\frac{\tilde{S}(\mathbf{\Omega}',\mathbf{\Omega}_0)}{\mu'}\right]-f(\mathbf{\Omega},\,-\mathbf{\Omega}')\frac{\tilde{T}(\mathbf{\Omega},\mathbf{\Omega}_0)}{\mu}\left[\delta_2(-\mathbf{\Omega}'\cdot\mathbf{\Omega}_1)+\frac{S(\mathbf{\Omega}',\mathbf{\Omega}_1)}{\mu'}\right].$$

Finally, using the symmetry properties [Eqs. (96d) and (102)] we obtain as the fundamental equations for S and T

$$\left(\frac{1}{\mu_0}+\frac{1}{\mu_1}\right)S(\mathbf{\Omega}_0,\mathbf{\Omega}_1)=c\int\int d\mathbf{\Omega}d\mathbf{\Omega}'f(\mathbf{\Omega},\,-\mathbf{\Omega}')\{\ \}_1, \quad (104a)$$

$$\left(\frac{1}{\mu_0}-\frac{1}{\mu_1}\right)T(\mathbf{\Omega}_0,\mathbf{\Omega}_1)=c\int\int d\mathbf{\Omega}d\mathbf{\Omega}'\{\ \}_2, \quad (105)$$

$$\{\ \}_1=\left[\delta_2(-\mathbf{\Omega}_0\cdot\mathbf{\Omega})+\frac{S(\mathbf{\Omega}_0,\mathbf{\Omega})}{\mu}\right]\times\left[\delta_2(-\mathbf{\Omega}'\cdot\mathbf{\Omega}_1)+\frac{S(\mathbf{\Omega}',\mathbf{\Omega}_1)}{\mu'}\right]-\frac{T(\mathbf{\Omega}_0,\mathbf{\Omega})}{\mu}\frac{T(\mathbf{\Omega}',\mathbf{\Omega}_1)}{\mu'},$$

and

$$\{\ \}_2=f(\mathbf{\Omega}',\,-\mathbf{\Omega})\frac{T(\mathbf{\Omega},\mathbf{\Omega}_1)}{\mu}\left[\delta_2(-\mathbf{\Omega}'\cdot\mathbf{\Omega}_0)+\frac{S(\mathbf{\Omega}_0,\mathbf{\Omega}')}{\mu'}\right]-f(\mathbf{\Omega},\,-\mathbf{\Omega}')\frac{T(\mathbf{\Omega}_0,\mathbf{\Omega})}{\mu}\left[\delta_2(-\mathbf{\Omega}'\cdot\mathbf{\Omega}_1)+\frac{S(\mathbf{\Omega}',\mathbf{\Omega}_1)}{\mu'}\right].$$

It is remarkable that Eqs. (104a) and (105) have no explicit dependence on τ. (Exactly the same equations hold for slabs of all thicknesses.) Clearly some additional condition must be added to make the solution of the problem for a given slab unique. However, Eq. (105) tells, in essence, what the condition is. Solving for T in terms of the integral on the right it is seen that $T(\mathbf{\Omega}_0,\mathbf{\Omega}_1)$ is determined only up to a term proportional to $\delta(\mu_0-\mu_1)$. This must describe the direct contribution of the incident beam. Thus the correct form of Eq. (105) including the condition necessary to make Eqs. (104a) and (105) determinate is

$$T(\mathbf{\Omega}_0,\mathbf{\Omega}_1)=\mu_1\delta_2(\mathbf{\Omega}_0\cdot\mathbf{\Omega}_1)e^{-z/\mu_1}+\frac{c\mu_0\mu_1}{\mu_1-\mu_0}\int\int d\mathbf{\Omega}d\mathbf{\Omega}'\{\ \}_2. \quad (104b)$$

Equations (104a, b) are then the fundamental equations which determine the reflection and transmission for a slab. To recapitulate: we have assumed $c=$ constant (translational invariance), $f(\mathbf{\Omega},\mathbf{\Omega}')=f(-\mathbf{\Omega},\,-\mathbf{\Omega}')$ (spatial inversion invariance), but we have not assumed $f(\mathbf{\Omega},\mathbf{\Omega}')=f(-\mathbf{\Omega}',\,-\mathbf{\Omega})$ (time reflection invariance). If we do make this additional assumption we have also proved that S and T are symmetric. The essential difference between this and Chandrasekhar's[1] derivation is that we start from Eqs. (96c) and (101)—which follow immediately from the reciprocity identity and translational invariance. There is never any need to consider solutions of slab problems with different thicknesses.

APPENDIX

Since reciprocity relations are important for much more complicated situations than those treated above (where analytic, or semianalytic, solutions can be

[12] This is why the functions Φ_1, Φ_2 in Sec. V are usually even.

found), it seems worthwhile to sketch the proof of such theorems for the general case described by Eq. (1).

Let $\psi_i^{(1)}(\mathbf{r},\mathbf{\Omega})$ and $\tilde{\psi}_i^{(2)}(\mathbf{r},\mathbf{\Omega})$ satisfy

$$\mathbf{\Omega}\cdot\nabla\psi_i^{(1)}(\mathbf{r},\mathbf{\Omega})+\sigma_i(\mathbf{r})\psi_i^{(1)}=\sum_j\int F_{ij}(\mathbf{\Omega},\mathbf{\Omega}',r)\times\sigma_j(\mathbf{r})\psi_j^{(1)}(\mathbf{r},\mathbf{\Omega}')d\mathbf{\Omega}'+q_i^{(1)}(\mathbf{r},\mathbf{\Omega}), \quad \text{(A1)}$$

and

$$-\mathbf{\Omega}\cdot\nabla\tilde{\psi}_i^{(2)}(\mathbf{r},-\mathbf{\Omega})+\sigma_i(\mathbf{r})\tilde{\psi}_i^{(2)}=\sum_j\int F_{ji}(\mathbf{\Omega}',\mathbf{\Omega},\mathbf{r})\times\sigma_i(\mathbf{r})\tilde{\psi}_j^{(2)}(\mathbf{r},-\mathbf{\Omega}')d\mathbf{\Omega}'+q_i^{(2)}(\mathbf{r},-\mathbf{\Omega}). \quad \text{(A2)}$$

Multiply Eq. (A1) by $\tilde{\psi}_i^{(2)}(\mathbf{r},-\mathbf{\Omega})$, Eq. (A2) by $\psi_i^{(1)}(\mathbf{r},\mathbf{\Omega})$. Subtract, sum over i and integrate over the volume V under consideration and all $\mathbf{\Omega}$. The result is

$$\int dS\int d\mathbf{\Omega}\mathbf{n}_0\cdot\mathbf{\Omega}\sum_i\psi_i^{(1)}(\mathbf{r},\mathbf{\Omega})\tilde{\psi}_i^{(2)}(\mathbf{r},-\mathbf{\Omega})=\int d\mathbf{r}\int d\mathbf{\Omega}\sum_i[\tilde{\psi}_i^{(2)}(\mathbf{r},-\mathbf{\Omega})q_i^{(1)}(\mathbf{r},\mathbf{\Omega})-\psi_i^{(1)}(\mathbf{r},\mathbf{\Omega})q_i^{(2)}(\mathbf{r},-\mathbf{\Omega})]. \quad \text{(A3)}$$

Specializing to various point sources and incident distribution it is possible to obtain a large number of relations between solutions of different problems.

If we define a matrix $\mathfrak{F}(\mathbf{\Omega},\mathbf{\Omega}',\mathbf{r})$ by

$$\mathfrak{F}_{ij}(\mathbf{\Omega},\mathbf{\Omega}',\mathbf{r})=F_{ij}(\mathbf{\Omega},\mathbf{\Omega}',\mathbf{r})\phi_j(\mathbf{r})$$

we might say that reciprocity in a strict sense exists provided there is a nonsingular matrix $Q(\mathbf{\Omega})$ such that

$$Q(\mathbf{\Omega})\mathfrak{F}^\dagger(-\mathbf{\Omega}',-\mathbf{\Omega},\mathbf{r})Q^{-1}(\mathbf{\Omega}')=\mathfrak{F}(\mathbf{\Omega},\mathbf{\Omega}',\mathbf{r}), \quad \text{(A4)}$$

where $\mathfrak{F}^\dagger$ denotes the transpose of $\mathfrak{F}$.

Under these conditions,

$$Q\tilde{\psi}=\psi, \quad \text{(A5)}$$

and the identities implied in Eq. (A3) will relate various solutions of the same equation. (A particular example of this is the reciprocity relation proved by Chandrasekhar[13] for the case of Rayleigh scattering of partially polarized light.)

[13] Reference 1, Chap. VII, Sec. 52.