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On the Question of a Neutrino Analog to Electric Charge

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1. INTRODUCTION

THE Maxwell equations apply[1] in a multiply connected space as well as in a simply connected space; the "handles" or "worm holes" in a multiply connected space trap lines of force, in such a way that one can associate an integrated flux of lines of force, or "charge," with each "worm hole"; and the charge so defined stays constant in time as a consequence of Maxwell's equations, regardless of the complication of the initial field values, and regardless of the way the metric changes with time, as long as the topology does not change. The theorem that the number of linearly independent constants of charge is equal to the number of worm holes (or to the second Betti number, R_2, of the 3-space) is a consequence of the well known theory of harmonic forms.[2]

This note enquires whether the two-component Dirac equation for the neutrino in curved space[3] resembles the Maxwell equations in the following sense: *does the two-component neutrino equation admit zero frequency solutions in a multiply-connected time-independent metric?* If such solutions, $\psi_1, \psi_2, \cdots$, exist, then one can construct a general solution of the following form

$$\psi = Q_1\psi_1 + Q_2\psi_2 + \cdots,$$

where the Q_k's are constants. One might then ascribe to the constants, Q_k, the name of "neutrino charge" and look for conservation properties—when the metric is allowed to vary—analogous to the conservation already proven for electric flux.

The simplest multiply-connected space has the topology of a "torus" defined by the metric in (5) and (6). This space admits a zero-frequency solution of Maxwell's equations.[4] In the same space the two-component neutrino wave equation is treated by the method of the separation of variables and it is proved that there exists no nonvanishing solution of this equation that is acceptable according to present standards. Consequently, *there is no reason to expect for the neutrino field a concept of charge having any direct analogy to electric charge.*

[1] J. A. Wheeler, Phys. Rev. **97**, 511 (1955); C. W. Misner and J. A. Wheeler (to be published).

[2] A. Lichnerowicz, *Algèbre et analyse linéaires* (Masson et Cie Paris, 1956), second edition; G. de Rham, *Variétés différentiables: formes, courants, formes harmoniques* (Hermann et Cie, Paris, 1955); W. V. D. Hodge, *The Theory and applications of Harmonic Integrals* (Cambridge University Press, Cambridge, 1952), second edition.

[3] See, for example, D. R. Brill and J. A. Wheeler, Revs .Modern Phys. **29**, 465 (1957).

[4] For zero frequency these equations reduce to two uncoupled pairs of equations, one pair for the electric field, the other for the magnetic field. It is sufficient to consider the first pair: div $\mathbf{E}=0$, curl $\mathbf{E}=0$. There is *no* solution of these equations in a spherical closed space, but in a toroidal space of the form (5) and (6), there exists one nonzero solution in which the lines of force encircle the torus in the χ direction. This is one of the simplest examples of a harmonic vector field. *Note added in proof.*—This 3-space is $S^1\times S^2$ while what is usually called a torus is $T^3=S^1\times T^2=S^1\times S^1\times S^1$.

2. TWO-COMPONENT EQUATION IN CURVED SPACE-TIME SOLUTION BY SEPARATION OF VARIABLES

Unfortunately, no theory of harmonic spinor fields exists, on which one can draw, as one can on the existing highly developed theory of harmonic vector and scalar fields. There is no harmonic *scalar* function in a closed three space which is not a trivial constant. Does the spinor neutrino field more resemble, in respect to number of independent solutions, the scalar field ($N=0$) or the vector field ($N=R_2$)? To prove that the number of nontrivial spinor fields is not R_2 as for a vector field, it is sufficient to give one contrary example. This example is the subject of the remainder of this note.

The two component specialization of the Dirac equation takes the form[5]

$$\sigma^k{}_{\dot{\mu}\rho}[\partial\psi^\rho/\partial x^k + \Gamma^\rho{}_{\alpha k}\psi^\alpha] = 0. \qquad (1)$$

Latin indices, such as k, range from 1 to 4; and Greek indices range from 1 to 2, thus $\rho=(1,2)$. Dotted indices denote complex conjugation and are summed only against other dotted indices. The quantities $\sigma^k{}_{\dot{\mu}\rho}$ are defined by the relation

$$\sigma^{k\tau\dot{\mu}}\sigma^l{}_{\dot{\mu}\rho} + \sigma^{l\tau\dot{\mu}}\sigma^k{}_{\dot{\mu}\rho} = 2g^{kl}\delta_\rho{}^\tau \qquad (2)$$

and the metric is chosen to have the signature $(1,1,1,-1)$. Lastly, four 2×2 *connection matrices*, $\Gamma^\rho{}_{\alpha k}$, are defined up to an additive multiple of the unit matrix by the following equation

$$\partial\sigma^k{}_{\dot{\mu}\rho}/\partial x^s + \Gamma^k{}_{rs}\sigma^r{}_{\dot{\mu}\rho} - \Gamma^\tau{}_{\rho s}\sigma^k{}_{\dot{\mu}\tau} - \Gamma^{\dot{\sigma}}{}_{\dot{\mu}s}\sigma^k{}_{\dot{\sigma}\rho} = 0, \qquad (3)$$

where the $\Gamma^k{}_{rs}$ are the usual Christoffel symbols.

Restricting attention to a special class of spaces, consider an Euclidean 4-space (x,y,z,w) where the fourth-dimension w has nothing to do with time. In this flat 4-space pick out a curved three-dimensional hypersurface, or 3-space, by way of

$$[(x^2+y^2+z^2)^{\frac{1}{2}} - r]^2 + w^2/\rho^2 = 1. \qquad (4)$$

[5] We use here nearly the same spinor notation as that used by W. L. Bade and H. Jehle, Revs. Modern Phys. **25**, 714 (1953).

Here τ and ρ are parameters defining a particular space. For example, a "toroidal" space would have $\tau>1$ and $\rho>0$, while a spherical space results for $\tau=0$, $\rho=1$. Points in the three-space can be specified by the three coordinates x, y, z, or more conveniently by three orthogonal angle variables, χ, θ, φ, the latter two being analogous to the usual angles of a spherical coordinate system.

Now introduce the time variable, or "cotime," $T=ct$. The four-dimensional metric becomes

$$ds^2=A(\chi)d\chi^2+B(\chi)[d\theta^2+\sin^2\theta d\varphi^2]-dT^2, \tag{5}$$

where

$$A(\chi)=\cos^2\chi+\rho^2\sin^2\chi$$
$$B(\chi)=(\tau+\sin\chi)^2. \tag{6}$$

To satisfy (2) the 2×2 spin matrices, $\sigma^k{}_{\dot{\mu}\rho}$ are selected to have the values

$$\sigma^1{}_{\dot{\mu}\rho}=A^{-\frac{1}{2}}\sigma_x;\qquad \sigma^2{}_{\dot{\mu}\rho}=B^{-\frac{1}{2}}\sigma_y;$$
$$\sigma^3{}_{\dot{\mu}\rho}=B^{-\frac{1}{2}}\sigma_z/\sin\theta;\quad \sigma^4{}_{\dot{\mu}\rho}=1 \tag{7}$$

in terms of the standard Pauli matrices,

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix};\quad \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix};\quad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{8}$$

According to Bade and Jehle[5] the 2×2 matrices with raised spin indices have the values

$$\sigma^{1\dot{\rho}\sigma}=A^{-\frac{1}{2}}\sigma_x;\qquad \sigma^{2\dot{\rho}\sigma}=-B^{-\frac{1}{2}}\sigma_y;$$
$$\sigma^{3\dot{\rho}\sigma}=B^{-\frac{1}{2}}\sigma_z/\sin\theta;\quad \sigma^{4\dot{\rho}\sigma}=-1. \tag{9}$$

From the metric (5) the Christoffel symbols, $\Gamma^k{}_{rs}$, may be computed. Since the neutrino has no charge the usual electromagnetic fields are excluded in the following determination of the connective matrices, $\Gamma^\tau{}_{\rho k}$:

$$\Gamma^\tau{}_{\rho1}=\Gamma^\tau{}_{\rho4}=0$$
$$\Gamma^\tau{}_{\rho2}=-i(AB)^{-\frac{1}{2}}B'\sigma_z/4 \tag{10}$$
$$\Gamma^\tau{}_{\rho3}=i(AB)^{-\frac{1}{2}}\sin\theta B'\sigma_y/4-i\cos\theta\sigma_x/2.$$

Here B' is an abbreviation for $dB/d\chi$. Substituting into (1) we obtain the first-order two-component neutrino wave equation

$$A^{-\frac{1}{2}}\sigma_x(\partial/\partial\chi)B^{\frac{1}{2}}\psi+(\sin\theta)^{-\frac{1}{2}}\sigma_y(\partial/\partial\theta)(\sin\theta)^{\frac{1}{2}}\psi$$
$$+(\sin\theta)^{-1}\sigma_z\partial\psi/2\varphi+\partial\psi/\partial T=0. \tag{11}$$

Our interest centers around harmonic or zero-frequency solutions; hence the last term in (11) will be put equal to zero. The solution of (11) proceeds by assuming a space dependence of the form

$$\psi=B^{-\frac{1}{2}}\begin{pmatrix}f(\theta)\\g(\theta)\end{pmatrix}\exp\left(k\int_{\chi_0}^{\chi}(A/B)^{\frac{1}{2}}d\chi+im\varphi\right). \tag{12}$$

Substituting into (11) and multiplying by σ_y the "radial" or χ dependent part of ψ comes out as a common factor, leaving only an equation in the angle θ:

$$[(\sin\theta)^{-\frac{1}{2}}(d/d\theta)(\sin\theta)^{\frac{1}{2}}-ik\sigma_z-(m\sigma_x/\sin\theta)]\begin{pmatrix}f\\g\end{pmatrix}=0 \tag{13}$$

equivalent to the two equations

$$(\sin\theta)^{-\frac{1}{2}}(d/d\theta)(\sin\theta)^{\frac{1}{2}}f-ikf-(mg/\sin\theta)=0$$
$$(\sin\theta)^{-\frac{1}{2}}(d/d\theta)(\sin\theta)^{\frac{1}{2}}g+ikg-(mf/\sin\theta)=0. \tag{14}$$

The radial factor in (12) is only acceptable if the number k has one or another purely imaginary value or is zero, whereas the angular factor is only acceptable if k is a real nonzero integer. No acceptable solution meets both requirements.

The radial factor has the value

$$\psi_\chi=B^{-\frac{1}{2}}\exp\left(k\int_{\chi_0}^{\chi}(A/B)^{\frac{1}{2}}d\chi\right). \tag{15}$$

This radial factor should be nonsingular, and should be periodic with period 2π in χ. Therefore k must have the purely imaginary or zero value

$$k=(2\pi i)\cdot(\text{integer})\Big/\int_0^{2\pi}(A/B)^{\frac{1}{2}}d\chi, \tag{16}$$

where the integrand never changes sign and never even goes to zero.

Equations for the angular part of ψ similar to (14) have been investigated by Schrödinger.[6] They differ from ours because of our choice of representation of the matrices, $\sigma^k{}_{\dot{\mu}\rho}$. In no other respect but this formal one do the angular equations differ from the familiar angular equations for the Dirac electron. General relativity effects put in no appearance. Only those states meet present standards of acceptability for which k is a nonzero integer,[7] and for which m has one of the values

$$m=-|k|-\tfrac{1}{2},\ -|k|-\tfrac{3}{2},\ \cdots,\ |k|-\tfrac{3}{2},\ |k|-\tfrac{1}{2}.$$

The value $k=0$ is not among the allowed values. Consequently *there exists no harmonic spinor field in the given metric.*

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[6] E. Schrödinger, Commentationes Pontif. Acad. Sci. **2**, 321 (1938).

[7] For a table of values of k and corresponding values of j and l, see for example L. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1940), first edition, p. 322. We investigated the two independent solutions for $k=0$, $m=0$,

$$\begin{pmatrix}f\\g\end{pmatrix}=(\sin\theta)^{-\frac{1}{2}}\begin{pmatrix}1\\0\end{pmatrix}\quad\text{and}\quad(\sin\theta)^{-\frac{1}{2}}\begin{pmatrix}0\\1\end{pmatrix}.$$

Both are square integrable. However, operation on either by the angular momentum operators that transform $\psi(k,m)$ to $\psi(k,m+1)$ or $\psi(k,m-1)$ *ought* to give a zero result but instead give divergent functions. In other words, the angular momentum operators become non-Hermitian operators if the state $k=0$, $m=0$ is accepted.