this construction can be done in conformity with the H principle. The H principle characterizes a property of the Feynman integral sufficient to guarantee that an operator form of the field equations will hold. The choice of the homogeneity of the field determines uniquely the relationship between the form of the operator field equations and the form of the classical field equations. The homogeneity of the metric field of general relativity has been defined, and the operator form of the Einstein field equations has been given. In a topologically invariant quantum field theory, such as quantized general relativity, the (Schroedinger) state functionals on equivalent hypersurfaces are equivalent, so that the Hamiltonian vanishes.

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$$\int \exp\{(i/\hbar)(\text{Einstein action})\}d\ (\text{field histories}).$$

He provided constant encouragement in pursuing the investigation; and his provoking ideas on the nature of the theory of general relativity led us to adopt the intrinsic or coordinate-free point of view which has been essential in the development of this theory. His help and advice in writing this paper are reflected in those sections which are most clearly presented.

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Reality of the Cylindrical Gravitational Waves of Einstein and Rosen

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1. INTRODUCTION

QUESTIONS have been raised whether gravitational radiation has any well-defined existence.[1] Supporting this skeptical position, Rosen has investigated further[2] the cylindrical gravitational waves first considered by him and Einstein[3] as an outgrowth of a suggestion by H. P. Robertson. A monochromatic wave, or a pulse, of cylindrical symmetry, moves inward in matter-free space, implodes on the axis, and moves out again. This is the only problem of gravitational radiation where one has an *accurate* solution of the field equations of general relativity. The problem is special enough not to illustrate all features of gravitational radiation. On the other hand, all correct general statements about gravitational radiation must obviously be compatible with this problem. This problem therefore occupies a special position in the theory of gravitational radiation.

Rosen finds an unexpected result. The pseudotensor that measures the density of gravitational energy and momentum in the cylindrical wave is everywhere zero. The significance of this finding is the subject of this paper. We conclude that many of the otherwise apparently paradoxical properties of this cylindrical wave can be understood by taking into account the analogy between gravitational waves and electromagnetic waves, and the special demands of the equivalence principle, which rules out a special role for any particular frame of reference.

Section 2 recapitulates the expressions of Einstein and Rosen and of Rosen for the metric of the cylindrical wave. Two kinds of solution are of interest: monochromatic waves and pulses. A pulse type of solution is constructed that is represented by particularly simple mathematical expressions. Section 3 reviews the proof that the pseudotensor density of gravitational

[1] A. E. Scheidigger, Revs. Modern Phys. **25**, 451 (1953).
[2] N. Rosen in *Jubilee of Relativity Theory*, edited by A. Mercier and M. Kervaire (Birkhäuser Verlag, Basel, 1956).
[3] A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937); N. Rosen, Bull. Research Council Israel **3**, 328 (1953).

momentum and energy is everywhere zero, both for monochromatic waves and for pulses. From this it does not follow that the energy of the gravitational wave is zero. Rather, this energy is *undefined* because the wave is unbounded. Were the pulse imploding, not on an infinite straight line, but on a circular line, the energy would be expected to be finite and reasonable, according to simple physical arguments. Section 4 returns to the infinitely extended case where the exact solutions are available. The curvature tensor of Riemann, R_{ijkl}, does not vanish in the region occupied by the wave; nor do the invariants formed from the components of this curvature tensor vanish. The disturbance in question is real and not removable by any change of coordinate system. In Sec. 5 influence of the gravitational wave upon the motion of an infinitesimal test particle, at rest before the arrival of a pulse, is studied. The particle receives a velocity from the ingoing pulse, but after implosion the pulse returns with reversed force and reduces the velocity to zero. The analogies are stressed with the case of a charged particle subject to the action of an electromagnetic plane wave. There the passing wave gives no net forward impulse to the particle in the approximation in which the secondary wave radiated by the particle itself is neglected. When allowance is made for radiation reaction, the equation of motion of the particle predicts a forward push. Several predictions about the response of a test particle to gravitational radiation are made which are in principle subject to check. It is concluded that the cylindrical waves of Einstein and Rosen present no real paradox; that the apparently anomalous behavior of these waves is completely consistent with the equivalence principle of general relativity.

2. CYLINDRICAL WAVES

Spherical gravitational waves, like spherical electromagnetic waves, can never be truly spherically symmetric. The polarization tensor in the one case, or the polarization vector in the other case, cannot keep a constant magnitude and change continuously in direction over the surface of a sphere, according to the fixed point theorem of topology. Cylindrical gravitation waves, like cylindrical electromagnetic waves, avoid this difficulty. The polarization at a point is described by a small ellipse normal to the direction of wave propagation. This ellipse describes the distances from a central test particle of a set of test particles which were originally at rest upon a circle before the wave fell upon them. The principal axes of this ellipse are parallel and perpendicular to the axis of the cylinder. As the phase of the oscillation advances, the long axis becomes the short one, and conversely, but the axes do not rotate. The cylindrical waves of Einstein and Rosen make no use of the other independent state of polarization, in which the principal axes are turned at 45° to the axis of the cylinder.

A typical point is designated by cylindrical coordinates, ρ, φ, and z. The product of the velocity of light and the time is represented by the cotime, T. In these coordinates the metric of Einstein and Rosen has the form

$$ds^2=e^{2\gamma-2\psi}(-dT^2+d\rho^2)+\rho^2e^{-2\psi}d\varphi^2+e^{2\psi}dz^2 \quad (1)$$

where the dilatation quantities, γ and ψ, are functions of ρ and T alone. The gravitational field equations are satisfied in empty space for a metric of the form (1) provided ψ and γ satisfy the equations,

$$\psi_{\rho\rho}+\frac{1}{\rho}\psi_\rho-\psi_{TT}=0, \quad (2)$$

$$\gamma_\rho=\rho[\psi_\rho^2+\psi_T^2], \quad (3)$$

$$\gamma_T=2\rho\psi_\rho\psi_T. \quad (4)$$

Solutions of (2) are well-known and represent cylindrical waves. It was originally thought that these waves are capable of transmitting energy to unlimited distances because one can write a solution representing progressive waves, where the first dilatation factor has the form

$$\psi=AJ_0(\omega\rho)\cos\omega t+BN_0(\omega\rho)\sin\omega t; \quad (5)$$

and the second factor, in the special case $B=A$, reduces to the expression

$$\begin{aligned}\gamma=\tfrac{1}{2}A^2\omega\rho\{&J_0(\omega\rho)J_0'(\omega\rho)+N_0(\omega\rho)N_0'(\omega\rho)\\&+\omega\rho[(J_0(\omega\rho))^2+(J_0'(\omega\rho))^2+(N_0(\omega\rho))^2\\&+(N_0'(\omega\rho))^2]+[J_0(\omega\rho)J_0'(\omega\rho)-N_0(\omega\rho)N_0'(\omega\rho)]\\&\times\cos2\omega T+[J_0(\omega\rho)N_0'(\omega\rho)+N_0(\omega\rho)J_0'(\omega\rho)]\\&\times\sin2\omega T\}-\frac{2}{\pi}A^2\omega T. \quad (6)\end{aligned}$$

The last term of (6) is aperiodic in time. It leads to a systematic change of the metric with time which was at first interpreted as due to loss of energy. Rosen gave arguments against this interpretation.[3] With Rosen, we exclude solutions that contain the irregular Bessel function, $N_0(\omega\rho)$, as not well defined at the origin.

More interesting than a monochromatic dilatation factor ψ of the form (5), with $B=0$, is a pulse formed by linear superposition of such waves.[3] We superpose such waves with an amplitude factor $A=2Ce^{-a\omega}$, thus:

$$\begin{aligned}\psi&=2C\int_0^\infty e^{-a\omega}J_0(\omega\rho)\cos\omega T d\omega\\&=C[(a-iT)^2+\rho^2]^{-\frac{1}{2}}+C[(a+iT)^2+\rho^2]^{-\frac{1}{2}}. \quad (7)\end{aligned}$$

The quantity a is an approximate measure of the pulse width.*

Consider any value of the distance ρ, which is very large compared to a. Then (7) is large only when T is

* *Note added in proof.*—We are grateful to W. B. Bonnor for a reprint of his paper, J. Math. and Mech. 6, 203 (1957) where he considers this same wave form.

near $-\rho$ and near $+\rho$. At the one moment the imploding wave is going by; at the other moment it is reexploding towards infinity.

Let T be a large negative number, and ask for the shape of the pulse at this time. Introduce the dimensionless quantity x by the formula $\rho=-T+ax$. Then the pulse has the approximate form

$$\psi\sim 2C(-2aT)^{-\frac{1}{2}}\{[x+(1+x^2)^{\frac{1}{2}}]/2(1+x^2)\}^{\frac{1}{2}}, \quad (8)$$

indicated in Fig. 1. Pulses of other shapes can easily be constructed by combining the two terms in (7) with constant phase factors, $e^{i\delta}$ and $e^{-i\delta}$, respectively. The shape of the present pulse does not change with time until it comes close to the origin. The strength evidently grows inversely as the root of the distance of the peak of the pulse from the origin, as expected for a cylindrical wave. At the origin itself the dilatation factor ψ has the value

$$\psi=2Ca/(a^2+T^2). \quad (9)$$

The second dilatation factor γ is determined up to a constant by integrating (3) and (4), as follows:

$$\gamma=\tfrac{1}{2}C^2\{a^{-2}-\rho^2[(a-iT)^2+\rho^2]^{-2}-\rho^2[(a+iT)^2+\rho^2]^{-2}$$
$$-a^{-2}(T^2+a^2-\rho^2)[T^4+2T^2(a^2-\rho^2)+(a^2+\rho^2)]^{-\frac{1}{2}}\}. \quad (10)$$

Choose a fixed time T and an arbitrarily small quantity ϵ. Try to draw a sphere around the origin of a radius R so large that the dilatations, ψ and γ, are less than ϵ everywhere over the surface of this sphere. The task is an impossible one; there is no such sphere. The metric is everywhere regular, but it does not become asymptotically flat in the sense just mentioned.

3. VANISHING OF THE PSEUDOTENSOR DENSITY OF ENERGY AND MOMENTUM

In evaluating the stress energy pseudotensor, we consider not only the monochromatic wave (5) and the pulse (7), not only the most general solution of the gravitational field equations (2)–(4), for empty space, but also the case where nongravitational energy is present. Then the field equations (2), (3), (4) no longer pertain. However, we limit ourselves to the case where the stress-energy tensor, T_{ik}, of this extra energy has cylindrical symmetry and where a metric of the form (1) still applies.

The pseudotensor, t_{ik}, of gravitational stress and energy is known not to be uniquely defined by the local conservation law, or requirement of zero divergence:

$$(-g)^{-\frac{1}{2}}(\partial/\partial x^\alpha)(-g)^{\frac{1}{2}}(T_k{}^\alpha+t_k{}^\alpha)=0 \quad (k=1,2,3,4). \quad (11)$$

We adopt the familiar Einstein choice for the stress energy pseudotensor, formulated by Tolman[4] in the following convenient language: (1) Define a Lagrange

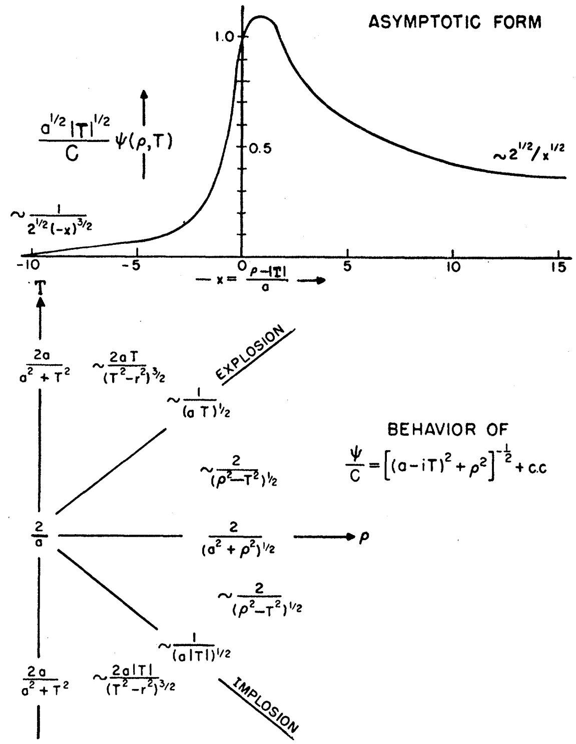

FIG. 1. The pulse-like cylindrical gravitational wave of Eq. (7) is concentrated near the radius $\rho=-T$ on the implosion run, for negative values of T; and concentrated near $\rho=+T$ when it reexpands out from the axis of symmetry. The upper diagram shows the pulse shape valid for both inward and outward runs when ρ and T are large compared with the pulse width, a. The lower diagram gives approximate representations of the pulse valid in the selected regions of the (ρ,T) diagram. The symbols r and ρ are used indiscriminately. The asymptotic value of the dilatation constant γ within the forward and backward light cone is taken to be zero, from which it follows that the asymptotic value of γ everywhere in the neutral zone is the constant

$$\gamma_{\text{asym}}=\int_0^\infty \rho(\psi_\rho{}^2+\psi_T{}^2)d\rho.$$

density function,

$$\mathfrak{L}=(-g)^{\frac{1}{2}}g^{\alpha\beta}(\Gamma_{\alpha\beta}{}^\mu\Gamma_{\mu\nu}{}^\nu-\Gamma_{\alpha\nu}{}^\mu\Gamma_{\beta\mu}{}^\nu). \quad (12)$$

(2) Define the quantity

$$s_i{}^{km}=(c^4/8\pi G)g^{k\lambda}\partial\mathfrak{L}/\partial g_{,m}{}^{i\lambda} \quad (13)$$

where $g_{,m}{}^{il}$ is an abbreviation for $\partial g^{il}/\partial x^m$. Then the divergence of this quantity gives an expression for the total stress energy pseudotensor,

$$T_i{}^k+t_i{}^k=(-g)^{-\frac{1}{2}}\partial s_i{}^{k\mu}/\partial x^\mu, \quad (14)$$

that satisfies the conservation law (11).

Calculation of (12), (13), and (14) is simplified by the diagonal character of the metric. We find

$$\partial\mathfrak{L}/\partial g_4{}^{44}=0;$$
$$\partial\mathfrak{L}/\partial g_1{}^{44}=\tfrac{1}{2}(-g)^{\frac{1}{2}}g^{11}g_{44}(g^{22}\partial g_{22}/\partial x^1+g^{33}\partial g_{33}/\partial x^1) \quad (15)$$

and thence the values of $s_4{}^{41}$ and $s_4{}^{44}$. We have no interest in the values of $s_4{}^{42}$ and $s_4{}^{43}$ because their derivatives with respect to $x^2=\theta$ and $x^3=\varphi$, respectively, are zero.

[4] R. C. Tolman, Phys. Rev. **35**, 875 (1930); see C. Møller, *The Theory of Relativity* (Clarendon Press, Oxford, 1952) for a summary. Also, see especially L. Landau and E. Lifshitz, *The Classical Theory of Fields*, translated by M. Hamermesh (Addison-Wesley Press, Reading, Massachusetts, 1951), Chap. 11.

We end up with an identically zero value for the energy density,

$$T_4{}^4+t_4{}^4=(-g)^{-\frac{1}{2}}(c^4/16\pi G)(\partial/\partial x^1)(-g)^{\frac{1}{2}}g^{11}\times(g^{22}\partial g_{22}/\partial x^1+g^{33}\partial g_{33}/\partial x^1)$$
$$=(c^4/8\pi G)(\partial/\partial\rho)(-1+2\rho\partial\psi/\partial\rho-2\rho\partial\psi/\partial\rho)=0. \quad (16)$$

In other words, when no matter is present, the gravitational pseudoenergy density itself is zero, as has already been shown by Rosen.[2] When matter is present, the right-hand sides of the field equations (2), (3), (4) have to be corrected, but the calculation (16) of the energy density remains valid. The gravitational field automatically adjusts itself so that the pseudoenergy density of the gravitational field is equal and opposite to the density of all other forms of energy. In a similar fashion, the gravitational contribution to the density of momentum in the radial direction compensates exactly all other contributions to this momentum,

$$T_4{}^1+t_4{}^1=0. \quad (17)$$

The results (16) and (17), while impressive, are meaningless. The quantity T_{ik} is a tensor, but t_{ik} is not. A coordinate transformation produces no change in the values of the four independent invariants I_p of the tensor T_{ik}, defined by

$$\lambda^4+\lambda^3I_1+\lambda^2I_2+\lambda I_3+I_4=\begin{vmatrix}(\lambda+T_1{}^1) & T_1{}^2 & T_1{}^3 & T_1{}^4\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ T_4{}^1 & T_4{}^2 & T_4{}^3 & (\lambda+T_4{}^4)\end{vmatrix}. \quad (18)$$

The corresponding quantities, i_1, i_2, i_3, i_4 for the pseudotensor t_{ik} are not invariant; they can be given arbitrary values at a point by suitable coordinate transformation or by suitably altering the definition (14) of the pseudoenergy tensor, consistently with the conservation law (11) or (3) by any combination of (1) and (2).

In physical terms, it has no well defined sense to say that the density of gravitational energy is zero, or that the flow of gravitational energy vanishes. No experiment could ever test whether these quantities are zero. The density of *other* forms of energy can be measured by the gravitational fields they produce. But gravitational field energy itself does not appear as a source term in Einstein's field equations, so there is no unambiguous way to speak of *localized* gravitational energy density as a source of a measurable gravitational field.

Only total energy has a well-defined significance, and even this only when very special conditions are met. There is no such quantity as total energy, for example, in a closed universe[4]; there the integrated conservation laws reduce to the trivial identity, zero equals zero. To define total energy it is necessary that the metric become asymptotically flat. More precisely, a coordinate system must exist in which the metric quantities go to the Euclidean flat space values at least as fast as 1/(distance from origin). Then the gravitational field falls off as fast as $1/r^2$ or faster, and a surface integral allows one to compute the mass or energy of the system unambiguously. This condition is not met in the case of an infinite cylindrical gravitational wave.†

The wave must be limited to a finite region of space to have a well-defined energy. Consider therefore a gravitational wave imploding, not on a straight line, but on a circular line. Let the thickness of the pulse a

† *Note added in proof.*—We are indebted to Professor M. Fierz for permission to quote from his letter to us of May 14, 1957.

"So we have $\gamma(\infty,T)=$constant, i.e., independent of T [from Eq. (4)]. Let $\psi(\rho,T)$ be a wave-packet fulfilling Eq. (2); we may assume that $\psi(\rho,T)=0$ for $\rho>R$—naturally this is not strictly true, but practically we may imagine ψ to have a finite spread. Now for $\rho\gg R$ we have

$$ds^2=e^{2\gamma}(d\rho^2-dT^2)+\rho^2d\varphi^2+dz^2,$$

where γ is a *positive constant*, namely the 'energy' in the ordinary sense of the wave field ψ:

$$\gamma(\infty,T)=\int_0^\infty\rho(\psi_\rho{}^2+\psi_T{}^2)d\rho=\text{const}$$

[where γ is normalized to zero at the origin]. Putting $\rho'=e^\gamma\rho$, $T'=e^\gamma T$, $\varphi'=e^{-\gamma}\varphi$, we have

$$ds^2=-dT'^2+d\rho'^2+\rho'^2d\varphi'^2+dz^2.$$

This looks pseudo-Euclidean. *But* the range of φ' is not $0\rightarrow 2\pi$, but $0\rightarrow e^{-\gamma}2\pi$. This means that the geometry on a 'plane' $T=$const, $z=$const is the one on a *conical surface*. This holds as well as we want for any T if ρ (or ρ') is big enough. So, if we assume ψ to be regular everywhere the geometry is not 'regular' at infinity ($\rho\rightarrow\infty$), that is, it is not pseudo-Euclidean though the curvature vanishes for big ρ. Looking at the problem in this way, it is evident that the deviation from [Euclidean character] is not due to the coordinate system. This peculiar behavior is a property in the large."

This beautiful analysis of Fierz indicates that there is a subtle sense in which one can define an energy-like quantity per unit length of a cylindrically symmetrical gravitational wave, and indicates further that this energy-like quantity is positive definite. There is a most interesting analogy between the conical space of Fierz, and the curved space of the Schwarzschild metric, in the following sense: In both cases there is a departure from a Euclidean character, the integrated magnitude of which over a surface of radius ρ or r is independent of r and measures the total energy included within that surface. Of course, the area of the bounding surface increases faster with r in the second case than the first; correspondingly, the departure from the Euclidean character remains constant (as measured by $\Delta\varphi$) in the cylindrical case, and falls off with r in the spherical case. See also the discussion by Fierz at the end of reference 2.—We owe to Dr. Charles Misner an important additional remark about the definability of energy in a system that is not asymptotically Euclidean, and especially in a closed universe. Landau and Lifshitz show [reference 4, Eq. (11–88)] that the energy-momentum four vector can be expressed in the form of an integral over a closed 2-dimensional surface bounding a 3-dimensional space-like volume $P^\beta=\int h^{\beta0\alpha}dS^*{}_{0\alpha}$. When the three-dimensional space is closed, then the bounding surface S can be shrunk to zero. This circumstance might appear to argue, as indicated in the text, that the law of energy conservation reduces to the trivial identity, $0=0$. However, no closed surface can be covered without singularity by a single coordinate system. One has to use at least two coordinate patches. But $h^{\beta0\alpha}$ depends upon the gravitational stress-energy pseudotensor. Therefore its values in the two or more coordinate patches in the regions where they overlap are not related to each other as are the values of a true tensor. Consequently, it is *not* clear—Dr. Misner points out—that the energy integral will vanish. The remarks of Fierz and Misner raise the following question. Does there exist in relativity theory for a closed space a quite generally defined energy-like quantity that has not yet been clearly formulated?

be chosen very small compared to the radius, b, of the circle. For simplicity, let the initial conditions for the problem be formulated at the moment, $T=0$, of maximum implosion. The dilatation factors at this time we require to follow formulas like (7) and (10),

$$\psi \doteq 2C(a^2+\rho^2)^{-\frac{1}{2}}; \qquad \partial\psi/\partial T=0$$
$$\gamma \doteq (C^2/a^2)[(1+a^2/\rho^2)^{-2}-1]; \quad \partial\gamma/\partial T=0 \quad (19)$$

provided that the distance ρ—now measured from the circular line—is small compared to the radius, b, of that line. We now have a toroidal gravitational wave instead of a cylindrical gravitational wave. For times before $T=0$ this wave implodes on the circle, and for later moments the wave explodes out from the circle. If the wave is weak enough ($C\ll a$), the familiar linearized theory of gravitational waves, or even Huygens' analysis, can be applied to discuss its behavior. It will approach asymptotically to a spherical wave pattern for very large values of the time. The wave will behave qualitatively in the same way when the dimensionless measure of strength C/a is larger, but then the analysis will be more complicated.

The point at issue is this: will the toroidal gravitational wave have any energy? By conservation laws the answer to this question at one time will answer it for all times. Therefore take the time $T=0$ of maximum concentration of the wave. The disturbance being confined to a limited region of space, one may assume the metric to be asymptotically flat. This permits definition of the total energy or mass of the gravitational wave. But is this energy unique? May it not be assigned at will by choosing arbitrary values for the coefficients of $1/r$ in the asymptotic formulas for the space part of the metric

$$ds^2 \sim (1+GM/2c^2r)^4(dx^2+dy^2+dz^2). \quad (20)$$

Are not the initial conditions for the gravitational wave at the time $T=0$ a matter of arbitrary choice? Does the existence of this freedom mean that it has no sense to talk of gravitational energy and gravitational waves?

No such complete freedom exists.[5] Let $g_{\alpha\beta}$ ($\alpha,\beta=1,2,3$) denote the space part of the metric at the time $T=0$, and let $R^{(3)}$ denote the curvature invariant of this three-dimensional space. Furthermore, introduce six quantities, $\varphi_{\alpha\beta}=\varphi_{\beta\alpha}$ ($\alpha,\beta=1,2,3$), with the trace $\varphi=\varphi_\gamma{}^\gamma$, to measure the time rate of change of the metric,

$$\partial g_{\alpha\beta}/\partial T=\varphi_{\alpha\beta}. \quad (21)$$

Then the three-dimensional "stretching rate tensor," $\varphi_{\alpha\beta}$ has to satisfy in the three space, $T=0$, the equations of Lichnerowicz and Fourès

$$(\varphi_\alpha{}^\beta-\delta_\alpha{}^\beta\varphi)_{;\beta}=0 \quad (\alpha=1,2,3)$$
$$\varphi^2-\varphi_{\alpha\beta}\varphi^{\alpha\beta}+R^{(3)}=0. \quad (22)$$

At $T=0$, the first time derivative of all the metric quantities vanishes. It follows from (22) that the *three-dimensional curvature invariant of the metric* $g_{\alpha\beta}$ *must vanish everywhere* at the time $T=0$:

$$R^{(3)}=0 \quad \text{at} \quad T=0. \quad (23)$$

This demand is automatically and accurately fulfilled by the line element (20). But this line element is the asymptotic form of the metric for the toroidal wave. Thus (23) is a differential equation for the metric that connects its asymptotic behavior (specified by the parameter M) with its behavior in the region of energy concentration, so *(23) provides a means to determine the "mass" or energy of the gravitational wave.*

The most general pure gravitational wave that is symmetric with respect to the time $T=0$ satisfies (23). A particular class of such solutions can be written in the form

$$ds^2=f^4(x,y,z)(dx^2+dy^2+dz^2), \quad (24)$$

where the Laplacian of f must vanish:[6]

$$\nabla^2 f=0.$$

Apart from the trivial constant solution, all solutions of this equation diverge somewhere, as illustrated by (20). The everywhere regular solutions of (23) can therefore not be written in the conformal form (24). We have not so far found a scheme to construct and catalog all such regular solutions. The solution of this problem is central to further study of time-symmetric gravitational waves.

4. THE CYLINDRICAL GRAVITATIONAL WAVE PRODUCES A NONZERO CURVATURE

From now on we regard the cylindrical gravitational wave as the idealized limit of a toroidal gravitational wave, of finite and well-defined energy, when the radius b of the torus is extremely large in comparison with all other relevant physical dimensions.

To see that this wave is not a fictitious "coordinate" wave in a really flat space, it is enough to consider the Riemann curvature *tensor*, R_{ijkl}. Were the space really flat, this tensor would vanish in one coordinate system and therefore in all coordinate systems. However, by direct calculation from the metric (1) [without use of the field equations (2), (3), (4)] we find the components

TABLE I. Alteration in separation of nearby test particles as an appropriate invariant way to describe the polarization properties of the cylindrical gravitational wave.

Does separation at right change as determined by alteration in	Nature of separation of the two infinitesimal test particles $\delta\rho$	δz	$\rho e^{-\psi}\delta\varphi$
Coordinates?	yes	no	$\delta\varphi$ unchanged
Invariant intervals?	yes	yes	yes

[5] See Y. Fourès-Bruhat, J. Rational Mech. Anal. 5, 951 (1956).

[6] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1926), p. 90, Eq. (28.7).

$$R_1{}^4{}_{14}=\gamma_{TT}-\gamma_{\rho\rho}-\psi_{TT}+\psi_{\rho\rho}$$
$$R_1{}^2{}_{12}=\frac{\gamma_\rho+\psi_\rho}{\rho}-\psi_\rho\gamma_\rho+\psi_{\rho\rho}-\psi_T\gamma_T+\psi_T{}^2$$
$$R_1{}^2{}_{42}=-\psi_\rho\gamma_T+\psi_{T\rho}+\gamma_T/\rho-\psi_T\gamma_\rho+\psi_T\psi_\rho$$
$$R_1{}^3{}_{13}=\psi_T\gamma_T-\psi_T{}^2+\psi_\rho\gamma_\rho-2\psi_\rho{}^2-\psi_{\rho\rho}$$
$$R_1{}^3{}_{43}=\psi_T\gamma_\rho-3\psi_T\psi_\rho+\psi_\rho\gamma_T-\psi_{\rho T} \tag{25}$$
$$R_2{}^3{}_{23}=\rho^{-2\gamma}\rho^2[\psi_\rho{}^2-\psi_T{}^2-\psi_\rho/\rho]$$
$$R_2{}^4{}_{42}=\rho^{-2\gamma}\rho^2\left[\psi_{TT}-\psi_T\gamma_T+\frac{\gamma_\rho-\psi_\rho}{\rho}-\psi_\rho\gamma_\rho+\psi_\rho{}^2\right]$$
$$R_2{}^4{}_{12}=\rho^{-2\gamma}\rho^2\left[\psi_{T\rho}-\psi_T\gamma_\rho+\psi_T/\rho+\frac{\gamma_T-\psi_T}{\rho}-\psi_\rho\gamma_T+\psi_\rho\psi_T\right]$$
$$R_3{}^4{}_{43}=\rho^{4\psi-2\gamma}[\psi_T\gamma_T-2\psi_T{}^2-\psi_{TT}+\psi_\rho\gamma_\rho-\psi_\rho{}^2].$$

Making use of the field equations (2), (3), (4) further simplifies these expressions, but does not reduce them all to zero. Existence of these nonzero components proves that the cylindrical gravitational waves propagate real curvature in space.

This departure from flatness produces real physical effects; the curvature modifies[7] the invariant distance between nearby infinitesimal test particles. Consequently objective existence must be attributed to the gravitational wave that produces these effects.

5. RESPONSE OF TEST PARTICLES TO THE CYLINDRICAL GRAVITATIONAL WAVE

An infinitesimal test particle follows the geodesic equation of motion

$$d^2x^i/ds^2+\Gamma_{\alpha\beta}{}^i(dx^\alpha/ds)(dx^\beta/ds)=0 \tag{26}$$

where the quantities $\Gamma_{jk}{}^i$ may be regarded as components of the gravitational field—dependent, of course, according to the equivalence principle, on the choice of the coordinate system. In the coordinate system (1) a test particle which is instantaneously at rest ($dx^1/ds=0$; $dx^4/ds=1$) experiences at that moment the acceleration

$$\begin{aligned}d^2\rho/dT^2&=-\Gamma_{44}{}^1=\tfrac{1}{2}g^{11}\partial g_{44}/\partial x^1=-\partial\gamma/\partial\rho+\partial\psi/\partial\rho\\&=C^2\rho\left\{\frac{\rho^2-(a-iT)^2}{[(a-iT)^2+\rho^2]^3}+\frac{\rho^2-(a+iT)^2}{[(a+iT)^2+\rho^2]^3}\right.\\&\quad\left.+2\frac{\rho^2+a^2+T^2}{[\rho^2+(a-iT)^2]^{\frac{3}{2}}[\rho^2+(a+iT)^2]^{\frac{3}{2}}}\right\}\\&\quad+C\rho\{[(a-iT)^2+\rho^2]^{-\frac{3}{2}}+[(a+iT)^2+\rho^2]^{-\frac{3}{2}}\}\end{aligned} \tag{27}$$

from the pulse type of gravitational wave. In particular, a particle once at rest at the axis of the cylinder remains forever at rest.

[7] F. Pirani, report at Chapel Hill conference (to be published).

To treat the motion of a particle that is some distance from the axis, we analyze the equation of motion to the first order of approximation: (1) we neglect terms in C^2, such as γ and its derivatives, in comparison to terms in C; (2) we evaluate the forces not at the new position, but at the old position; (3) we treat dx^4/ds as equal to 1; and (4) we write the pulse in the form

$$\psi \doteq f(\rho+T) \tag{28}$$

for negative values of T; and in the form

$$\psi=f(\rho-T) \tag{29}$$

for positive values of T. In this approximation the velocity has the value

$$\begin{aligned}d\rho/dT&\doteq\int_{-\infty}^{T}(d^2\rho/dT^2)dT\\&\doteq\int_{-\infty}^{T}(\partial\psi/\partial T)dT=+\psi(\rho_0,T)\quad\text{for}\quad T<0\\d\rho/dT&\doteq-\psi(\rho_0,T)\quad\text{for}\quad T>0.\end{aligned} \tag{30}$$

The particle experiences a sharp outward push (see Fig. 1) at about $T=-\rho$; then a weaker inward acceleration that brings the particle to rest at the time $T=0$. Then the inward slow acceleration resumes, the particle gains more and more velocity; and finally, at about $T=+\rho$, the particle experiences a final sharp outward acceleration that reduces it to rest. The outward migration of the particle up to the time $T=0$ is just balanced by the subsequent inward migration,

$$\Delta\rho=\int_{-\infty}^{0}(d\rho/dT)dT\doteq\int_{-\infty}^{0}\psi dT=C\pi. \tag{31}$$

The particle ends up where it started. A calculation to the first order that is really accurate does not replace $\partial\psi/\partial\rho$ by $\partial\psi/\partial T$ and gives for the migration, not (31), but

$$\Delta\rho=\int_{-\infty}^{0}\int_{-\infty}^{T'}(\partial\psi/\partial\rho)dT'dT=2C$$

for large ρ for the change from $T=-\infty$ to $T=0$.

The migration in coordinate gives no true measure of the change in distance between test particle and origin. To the first order this distance at the time $T=0$ is

$$\begin{aligned}\int(g_{11})^{\frac{1}{2}}d\rho&=\int_0^{\rho_0+\Delta\rho}e^{\gamma-\psi}d\rho\doteq\int_0^{\rho_0+\Delta\rho}(1-\psi)d\rho\\&=\rho_0+\Delta\rho-\int_0^{\rho}\psi d\rho\\&\doteq\rho_0+2C-2C\ln(2\rho/a).\end{aligned} \tag{32}$$

The particle moves, not farther from the origin, but closer to it.‡ These results warn how dangerous it is to draw any conclusions about motions of test particles from changes in their *coordinates*. One must instead use invariant intervals between particles to measure the effect of the wave. We note (Table I) that the principal axes of the polarization ellipse lie in the z and φ directions, despite the fact that these coordinates do *not* change, while ρ does.

Can energy be extracted out of the gravitational wave? Yes, because the distance δz between nearby test particles changes with time. This change can be used to drive an engine. Where does the energy come from, it having been agreed that the pseudotensor of gravitational energy density and energy flow is zero in the chosen coordinate system? The maximum possible rate of absorption of energy by the engine will be governed by the rate at which the gravitational forces do work upon the test particles. These forces will be proportional to the first power of the mass of the typical test particle. The inward energy flux onto the test system ought therefore to be proportional to the first power of the mass of the test particle. Consider these quantities in the weak field approximation. Let Γ_{back} symbolize the gravitational field of the cylindrical wave. Likewise let Γ_{test} symbolize the gravitational field due to the moving test particles. Then the energy density is qualitatively of the form

$$\Gamma_{\text{back}}^2+2\Gamma_{\text{back}}\Gamma_{\text{test}}+\Gamma_{\text{test}}^2.$$

The first term, due to the background field, can be given arbitrary values by suitable choice of the coordinate system, and happens to vanish in the coordinate system (1). The middle term describes the flow of energy into the absorbing system.

Details of the energy flow require elaborate analysis. By analogy with the problem of an electromagnetic wave working on one or two test particles: (1) response of the particles, measured by changes in their invariant separation, is proportional to the first power of the field strength in the weak field approximation; (2) loss of energy from the driving wave to the test particle system is describable in terms of an interference between the primary wave and the secondary or scattered wave[8]; (3) when there is no true absorption in the test system, but only scattering, then the scattered energy, of second order in the field of the test particle, is the same in integrated value but is opposite in sign to the loss of energy from the primary wave, represented by the term linear in the field of the test particle; (4) the primary wave moves the test particle only transversely to the first order in the primary field strength, and to the second order in the primary field strength there is also a forward and backward response of the test particle.[9] A charged particle responding to an electromagnetic wave executes a figure eight motion—without however undergoing any net forward motion in this second approximation. Only when the radiative reaction of the secondary wave on the test particle is taken into account does one find a phase lag in the figure eight motion and a net forward impulse imparted to the charge. Radiation pressure comes in only when the scattering is taken into account. All these results are consistent with the conservation laws.

Similarities between gravitational and electromagnetic waves thus make it simple to draw a number of reasonable inferences. The significance of these inferences has a much more subtle character in the gravitational case than in the electromagnetic case. Neither field energy densities nor test particle motions have a meaning independent of the choice of coordinate systems. The simple observable consequences of wave action are instead *changes* in the separation of various nearby test particles—changes that are related to the covariant components of the curvature tensor R_{ijkl}, rather than to the values of the noncovariant field strengths, $\Gamma^i{}_{jk}$. This more complex character of the observation problem in gravitation physics is in full accord with Einstein's principle of equivalence.

This principle does not deny a physical reality to gravitational radiation but on the contrary leads to the only well defined way there is to express the influence of this radiation: in terms of its effect upon invariant space time intervals, such as the interval between two test bodies.

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‡ *Note added in proof.*—To the second order the velocity of the test particle originally at ρ_0 follows from integration of

$$d^2\rho/dT^2 \doteq (\partial\psi/\partial\rho)_\rho+(d\rho/dT)(\partial\psi/\partial T)_{\rho_0}-(\partial\gamma/\partial\rho)_{\rho_0}.$$

Here it is legitimate in the same order to write

$$(\partial\psi/\partial\rho)_\rho \doteq \psi_0'+(\rho-\rho_0)\psi_0''.$$

The integration gives for the velocity at the time $T=+\infty$ the values $-2C^2/a^2$ and $-2\pi C^2\rho_0/a^3$ for values of ρ_0 large and small compared to a, respectively. This finite velocity produces in the course of an infinite time an infinite displacement. This circumstance does not affect the validity of the expansion in powers of the wave strength, C.

[8] See Bohr, Peierls, and Placzek, Nature **144**, 200 (1939); N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), second edition, Chap. VIII; and N. F. Mott, Proc. Roy. Soc. (London) **A133**, 228 (1931).

[9] We are indebted to Professor Robert Thompson for illuminating discussions on this point a number of years ago.