functional form of the Hamiltonian of any given system. Neither does it supply any prediction as to the functional dependence of the over-all state function of the isolated system upon the variables of the system. But neither does the classical universe of Laplace supply any prescription for the original positions and velocities of all the particles whose future behavior Laplace stood ready to predict. In other words, the relative state theory does not pretend to answer all the questions of physics. The concept of relative state does demand a totally new view of the foundational character of physics. No escape seems possible from this relative state formulation if one wants to have a complete mathematical model for the quantum mechanics that is internal to an isolated system. Apart from Everett's concept of relative states, no self-consistent system of ideas is at hand to explain what one shall mean by quantizing[4] a closed system like the universe of general relativity.

[4] C. W. Misner, Revs. Modern Phys. **29**, 497 (1957).

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Interaction of Neutrinos and Gravitational Fields

DIETER R. BRILL AND JOHN A. WHEELER

Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

1. INTRODUCTION; GRAVITATION THE ONLY FORCE IN WHICH NEUTRINOS ARE SUBJECT TO SIMPLE ANALYSIS

KNOWLEDGE of neutrinos to date is confined mainly to emission and absorption processes; that is, to the domain of elementary particle transformations. For comparison, imagine that one knew about electrons only the rate at which they are produced in beta decay, or absorbed in K-electron absorption processes, but knew nothing about the motion of electrons in electric and magnetic fields, nothing about the binding of electrons in atoms or the existence of spin-orbit coupling and very little about the stress energy tensor of the electron. What can one do to learn some fraction as much about neutrinos as one knows today about electrons?

The neutrino does not respond directly to electric or magnetic fields. Therefore, if one wishes to influence its orbit by forces subject to simple analysis one has to make use of gravitational fields. In other words, one has to consider the physics of a neutrino in a curved metric.

For this task the only available tools of analysis are theoretical. We accept the recently clarified[1] and dramatically tested[2,3] neutrino theory. We see no motive to change the theory. Instead we recall in Sec. 2 the clearly defined extension of the Dirac equation to the curved space that represents the most general gravitational field. In Sec. 3, we specialize to the neutrino with its zero mass and to the class of solutions with right-handed polarization that are demanded by the recently gained knowledge.[1–3] Section 4 separates out the radial wave equation for the motion of a neutrino in a centrally symmetric gravitational field, and identifies one term in this equation with a spin-orbit coupling. Section 5 compares and contrasts the energy level spectrum in the case of spherical symmetry for (1) an electron in an electrostatic field, (2) an electron in a gravitational field, (3) a photon in a gravitational field, and (4) a neutrino in a gravitational field. Section 6 recalls the statistical mechanics of an ensemble of neutrinos. Section 7 discusses some neutrino pair creation processes that do not depend upon beta interactions for their existence. Section 8 deals with the contribution of neutrinos to the stress energy tensor, Sec. 9 deals with the gravitational interaction of two neutrinos traveling parallel or antiparallel to each other; and Sec. 10 with the contribution to the stress energy tensor due to a neutrino in a bound orbit. Finally, Sec. 11 examines by way of illustration an object where both the creation of gravitational fields by neutrinos, and the response of neutrinos to gravitational fields come into play: a geon or entity constituted entirely of neutrinos and held together by their mutual gravitational attractions.

2. MATHEMATICS OF SPIN IN CURVED SPACE

Spinor fields have been treated in general relativity by many authors[4] and from three principal points of view (Table I). The three formalisms are in principle equivalent and must therefore in any actual problem give identical results for such well-defined quantities as

[1] T. D. Lee and C. N. Yang, Phys. Rev. **104**, 254 (1956); **105**, 1119(L) (1957).
[2] Wu, Ambler, Hayward, Hoppes, and Hudson, Phys. Rev. **105**, 1413(L) (1957).
[3] Garwin, Lederman, and Weinrich, Phys. Rev. **105**, 1415(L) (1957).

[4] See W. L. Bade and H. Jehle, Revs. Modern Phys. **25**, 714 (1953) for a general review of the literature. To their list of principal references one should add M. Riesz, Lund Univ. Math. Sem. Band 12 (1954); F. J. Belinfante, Physica **7**, 305 (1940).

TABLE I. Comparison of the three principal formalisms for description of spin in general relativity.

	General formalism		Vierbein formalism	
Components of ψ	4	2	4	2
Fundamental spin matrices ($i=1, \cdots, 4$)	γ_i	s_i	$\tilde{\gamma}_i$	$\tilde{s}_i$
Dependent on position	yes	yes	no	no
Relation between spin matrices and metric	Spin matrices conform to metric $[\gamma_i,\gamma_j]_+=2g_{ij}$	$[\tilde{s}_i,s_j]_+=2g_{ij}$	Metric transformed to locally Lorentz metric $d\tilde{x}^k=b_\alpha{}^k dx^\alpha$ $g_{ik}=b_i{}^\alpha b_k{}^\beta g_{\alpha\beta}$ Lorentz $\gamma_k=b_k{}^\alpha\tilde{\gamma}_\alpha$	$s_k=b_k{}^\alpha\tilde{s}_\alpha$ Use flat space spin matrices
Covariant under	General coordinate and similarity transformations		General coordinate transformations, and quite independent Lorentz transformations of the Vierbein differentials, $d\tilde{x}^k$	
Most general form for second of these transformations	16 parameters	4 complex parameters	6 real Lorentz parameters, with or without inversions of space or time or both.	
Formation of covariants	As in tensor analysis; see Table II for covariant derivative			
Pauli conjugate, $\psi^\dagger$	$\psi^*\cdot i\cdot$(Hermitizing matrix)		$\psi^* i\tilde{\gamma}^4$	$\psi^* i\tilde{s}^4$comp. conj.
Dirac equation for neutrino	$\gamma^\alpha\nabla_\alpha\psi=0$	$s^\alpha\nabla_\alpha\psi=0$	$\gamma^\alpha\nabla_\alpha\psi=0$	$s^\alpha\nabla_\alpha\psi=0$
Lagrangian, L	$\psi^\dagger\gamma^\alpha\nabla_\alpha\psi$	$\psi^\dagger s^\alpha\nabla_\alpha\psi$	$\psi^\dagger\gamma^\alpha\nabla_\alpha\psi$	$\psi^\dagger s^\alpha\nabla_\alpha\psi$
Current 4-vectors, s^k	$\psi^\dagger i\gamma^k\psi$	$\psi^\dagger is^k\psi$	$\psi^\dagger i\gamma^k\psi$	$\psi^\dagger is^k\psi$
Vierbein formalism for Dirac equation in polar coordinates for two simple choices of Vierbeine				
Vierbeine parallel to unit vectors in	r, θ, φ, T directions		x, y, z, T directions	
Corresponding γ matrices	$\gamma_r=(\exp\frac{1}{2}\lambda)\tilde{\gamma}_1$ $\gamma_\theta=r\tilde{\gamma}_2$ $\gamma_\varphi=r\sin\theta\tilde{\gamma}_3$ $\gamma_r=(\exp\frac{1}{2}\nu)\tilde{\gamma}_4$		$\gamma_r=(\exp\frac{1}{2}\lambda)(\sin\theta\sin\varphi\tilde{\gamma}_1+\sin\theta\cos\varphi\tilde{\gamma}_2+\cos\theta\tilde{\gamma}_3)$; similarly for γ_θ and γ_φ; $\gamma_T=(\exp\frac{1}{2}\nu)\tilde{\gamma}_4$	
Separation of Dirac wave function for simple choice of representation of the γ's	$\begin{pmatrix} F(r)f(\theta) \\ F(r)g(\theta) \\ G(r)f(\theta) \\ G(r)g(\theta) \end{pmatrix} \exp(im_j\varphi-iEt/\hbar)$		$\begin{pmatrix} F(r)Y_1(\theta)\exp[i(m_j+\frac{1}{2})\varphi-iEt/\hbar] \\ F(r)Y_2(\theta)\exp[i(m_j-\frac{1}{2})\varphi-iEt/\hbar] \\ G(r)Y_1(\theta)\exp[i(m_j+\frac{1}{2})\varphi-iEt/\hbar] \\ G(r)Y_2(\theta)\exp[i(m_j-\frac{1}{2})\varphi-iEt/\hbar] \end{pmatrix}$	
Spinor wave function is a single valued function of position	no		yes	

energy eigenvalues and density of stress and energy. We find it convenient to use here the formalism of V. Bargmann because of its generality. The equations of this formalism are covariant with respect to general coordinate transformations, and invariant under general, i.e., position-dependent, similarity transformations of the spinors. The fundamental connection between space and spin is made through a field of γ matrices which satisfy the anticommutation relationship,

$$\gamma_i\gamma_k+\gamma_k\gamma_i=2g_{ik}\mathbf{1}, \tag{1}$$

at each point in space, where g_{ij} is the metric tensor at that point, and $\mathbf{1}$ is the unit matrix. For the *signature* of the metric tensor we adopt the familiar Pauli convention $(1, 1, 1, -1)$. Let the dependence of the g_{ik} upon position be known. Then one can set up a generally covariant spinor formalism with the help of any field of 4×4 matrices γ_i that have the following properties: (1) their components are continuous functions of position in space time, (2) they satisfy (1) and (3) they transform like a vector under coordinate transformations, (4) under the spinor transformation

$$(\text{spinor})_{\text{new}}=S^{-1}(\text{spinor})_{\text{old}},$$

they undergo the similarity transformation

$$(\gamma_i)_{\text{new}}=S^{-1}(\gamma_i)_{\text{old}}S.$$

It also substantially simplifies the treatment of charged spinor fields to limit attention to real representations, γ_k, of the spin matrices, and to limit attention to spinor transformations, S, whose matrix elements are also real.

The principal feature of the mathematical formalism is a definition of covariant differentiation which is the natural generalization of the covariant differentiation of tensor analysis (Table II). In addition to the usual Christoffel symbols, $\Gamma_{ik}{}^m$, formed from the metric, g_{ik}, it is necessary to introduce four 4×4 matrices Γ_k. These quantities are uniquely determined up to an

TABLE II. Covariant differentiation of spin dependent quantities compared and contrasted with covariant differentiation of tensors.

	Tensors	Quantities with spinor transformation properties
Symbol for covariant differentiation with respect to x^k	Final subscript $;k$	Preceding operator ∇_k
Special form when space is flat, coordinates are Euclidean, and γ's are independent of position	$\partial/\partial x^k$	$\partial/\partial x^k$
Additional quantities needed to define covariant derivative when one or more of these conditions are not fulfilled	40 functions of position $\Gamma_{ik}{}^m$	The $\Gamma_{ik}{}^m$ and four matrices, Γ_k
Formula to determine these additional quantities from the metric	$\frac{1}{2}g^{m\alpha}\left(\frac{\partial g_{k\alpha}}{\partial x^i}+\frac{\partial g_{i\alpha}}{\partial x^k}-\frac{\partial g_{ik}}{\partial x^\alpha}\right)$	Eq. (2) or Eq. (8)
How these quantities enter into the definition of the covariant derivative	Depends upon the transformation properties of the quantity being differentiated	
Example I	Scalar, f	Spinor,
Effect of a coordinate transformation	$f_{\text{new}}=f_{\text{old}}$	$\psi_{\text{new}}=\psi_{\text{old}}$
Effect of a spinor transformation	$f_{\text{new}}=f_{\text{old}}$	$\psi_{\text{new}}=S^{-1}\psi_{\text{old}}$
Covariant derivative	$f_{;k}=\partial f/\partial x^k$	$\nabla_k\psi=\partial\psi/\partial x^k-\Gamma_k\psi$
Example II	Vector, A^s	Conjugate spinor $\psi^\dagger$
Effect of a coordinate transformation	$A^s{}_{\text{new}}=\frac{\partial\bar{x}^s}{\partial x^\alpha}A^\alpha{}_{\text{old}}$	$\psi_{\text{new}}{}^\dagger=\psi_{\text{old}}{}^\dagger$
Effect of a spinor transformation	$A^s{}_{\text{new}}=A^s{}_{\text{old}}$	$\psi_{\text{new}}{}^\dagger=\psi_{\text{old}}{}^\dagger S$
Covariant derivative	$A^s{}_{;k}=\frac{\partial A^s}{\partial x^k}+\Gamma_{k\alpha}{}^sA^\alpha$	$\nabla_k\psi^\dagger=\partial\psi^\dagger/\partial x^k+\psi^\dagger\Gamma_k$
Example III	Tensor, $M_i{}^k$	Spinor tensor, T_{ik} (such as $\gamma_i\gamma_k$)
Effect of a coordinate transformation	$M_i{}^k{}_{\text{new}}=\frac{\partial\bar{x}^k}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\bar{x}^i}M_\alpha{}^\beta{}_{\text{old}}$	$T_{ik\,\text{new}}=\frac{\partial x^\alpha}{\partial\bar{x}^i}\frac{\partial x^\beta}{\partial\bar{x}^k}T_{\alpha\beta\,\text{old}}$
Effect of a spinor transformation	$M_i{}^k{}_{\text{new}}=M_i{}^k{}_{\text{old}}$	$T_{ik\,\text{new}}=S^{-1}T_{ik\,\text{old}}S$
Covariant derivative	$M_i{}^k{}_{;l}=\frac{\partial M_i{}^k}{\partial x^l}+\Gamma_{l\alpha}{}^kM_i{}^\alpha-\Gamma_{il}{}^\alpha M_\alpha{}^k$	$\nabla_lT_{ik}=T_{ik;l}+T_{ik}\Gamma_k-\Gamma_kT_{ik}$

additive multiple of the unit matrix by

$$\partial\gamma_i/\partial x^k-\Gamma_{ik}{}^\mu\gamma_\mu-\Gamma_k\gamma_i+\gamma_i\Gamma_k=0. \tag{2}$$

The $\Gamma_{ik}{}^m$ and Γ_k together permit one to define the covariant derivative of any object of which the transformation properties for general coordinate and similarity transformations are known. This covariant derivative is denoted by ∇_k; its explicit form depends on the quantity it acts on (Table II). This covariant differentiation has the standard properties

$$\begin{aligned}\nabla_k(AB)&=(\nabla_kA)B+B(\nabla_kA),\\ \nabla_k(A^*)&=(\nabla_kA)^*,\\ \nabla_k\gamma_i&=0,\end{aligned} \tag{3}$$

where the symbol * means Hermitian adjoint (the transpose of the complex conjugate). The quantity, ∇_kA, transforms like A under similarity transformations and like a tensor of one higher rank under coordinate transformations.

The definition of the γ_i by (1) can always be fulfilled by a linear combination of the matrices of special relativity. These are constant matrices—and if one wishes, purely real matrices—that satisfy the conditions

$$\tilde{\gamma}_i\tilde{\gamma}_k+\tilde{\gamma}_k\tilde{\gamma}_i=2g_{ik\,\text{Lorentz}},$$

$$g_{ik\,\text{Lorentz}}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}, \tag{4}$$

$$\tilde{\gamma}_i{}^*=\tilde{\gamma}_i\ \ (i=1,2,3),\quad \tilde{\gamma}_4{}^*=-\tilde{\gamma}_4. \tag{5}$$

Thus, at each point, x, in 4 space it is possible to transform from the general coordinates x^i to a system $\tilde{x}^i$ ("Vierbein") whose metric is Minkowskian at that point:

$$dx^k=a^k{}_\alpha d\tilde{x}^\alpha,\quad d\tilde{x}^k=b_\beta{}^kdx^\beta. \tag{6}$$

Then (1) is satisfied by

$$\gamma_i=b_i{}^\alpha\tilde{\gamma}_\alpha. \tag{7}$$

The formalism which confines itself to solutions of (1) of the special type (7) is called Vierbein formalism. In terms of the Vierbein components $b_i^{\;j}$, $a^k_{\;l}$ the explicit solution of (2) has the simple form[5]

$$\Gamma_k = g_{\mu\alpha}[(\partial b_\nu^{\;\beta}/\partial x^k)a^\alpha_{\;\beta} - \Gamma_{\nu l}^{\;\;\alpha}]s^{\mu\nu} + a_k\mathbf{1}, \tag{8}$$

where $s^{ij}=\frac{1}{2}(\gamma^i\gamma^j-\gamma^j\gamma^i)$ and a_k is arbitrary. In the Vierbein frame a similarity transformation of the spinors is equivalent to a Lorentz transformation of the Vierbein. Thus the invariance of the formalism under similarity transformations can be understood geometrically as invariance under arbitrary Lorentz transformations of the Vierbein. These Lorentz transformations have nothing to do with any coordinate transformation, and vary arbitrarily from point to point in space.

The Dirac equation in general relativity can be written in the form

$$\gamma^\alpha\nabla_\alpha\psi+\mu\psi=0 \quad (\mu=mc/\hbar). \tag{9}$$

Here we assume the arbitrary traces of the Γ_k have been adjusted so as to account for the effects of the electromagnetic potentials.

To form real expressions, as for example for the current density, we can follow Bargmann with minor differences in notation and define the "Pauli conjugate" of ψ,

$$\psi^\dagger=\psi^*\alpha$$

by means of a "Hermitizing matrix," α, chosen so that both α itself and the four matrices, $\alpha i\gamma^k$, are Hermitian. In the Vierbein formalism we may choose the $\tilde{\gamma}^k$ to be real and choose[6] $\alpha=i\tilde{\gamma}^4$. One then can form a current density,

$$s^k=\psi^\dagger i\gamma^k\psi, \tag{10}$$

whose covariant divergence vanishes by virtue of the Dirac equation (9). The Dirac equation can be derived from the variational principle

$$\delta\int L(-g)^{\frac{1}{2}}d^4x=0, \tag{11}$$

with

$$L=\psi^\dagger\gamma^\alpha(\nabla_\alpha\psi)+\mu\psi^\dagger\psi, \tag{12}$$

by independent variation of $\psi^\dagger$ and ψ.

3. ALPHA AND BETA ROTATIONS AND TWO-COMPONENT NEUTRINO WAVE EQUATION

We set the mass term equal to zero in the Dirac equation (9) and the Lagrangian (12) to obtain the wave equation for a Dirac-type neutrino,

$$\gamma^\alpha\nabla_\alpha\psi=0. \tag{13}$$

In this special case of zero rest-mass, the wave equation and the Lagrangian are invariant under a wider class of transformations than are usually considered in the Dirac theory. The situation is analogous to the case of the charge-free electromagnetic field, F_{ik}. Consider any solution of Maxwell's equations,

$$\begin{aligned}\epsilon^{i\alpha\beta\gamma}\partial F_{\alpha\beta}/\partial x^\gamma&=0,\\ (-g)^{-\frac{1}{2}}(\partial/\partial x^\alpha)(-g)^{\frac{1}{2}}F^{\alpha k}&=0,\end{aligned} \tag{14}$$

where $\epsilon^{1234}=1$, and ϵ^{ijkl} changes sign on permutation of any two indices. From any solution of these equations in a metric of arbitrary curvature one can generate a new solution by a special kind of transformation that we may call an "α-rotation";

$$(F_{ik})_{\text{new}}=F_{ik}\cos\alpha+\tfrac{1}{2}(-g)^{-\frac{1}{2}}g_{i\sigma}g_{k\tau}\epsilon^{\sigma\tau\alpha\beta}F_{\alpha\beta}\sin\alpha, \tag{15}$$

where α is an angle that is independent of position and time. In flat space this transformation takes the form

$$\begin{aligned}\mathbf{E}_{\text{new}}&=\mathbf{E}\cos\alpha+\mathbf{H}\sin\alpha,\\ \mathbf{H}_{\text{new}}&=\mathbf{H}\cos\alpha-\mathbf{E}\sin\alpha.\end{aligned} \tag{16}$$

It is plainly not a rotation in any ordinary sense except in the special case where $\mathbf{E}$ and $\mathbf{H}$ represent the principal polarization directions of a monochromatic plane wave; then the α transformation rotates the axes of polarization by the angle α.

Similarly, let ψ be a solution of the Dirac equation for zero mass in a space-time continuum of arbitrary curvature. Then one can generate a new solution by the "β rotation,"[7]

$$\psi_{\text{new}}=\exp(\tfrac{1}{2}\beta\gamma_5)\psi=[\mathbf{1}\cos(\tfrac{1}{2}\beta)+\gamma_5\sin(\tfrac{1}{2}\beta)], \tag{17}$$

where β is a constant and

$$\begin{aligned}\gamma_5&=(-g)^{-\frac{1}{2}}(1/4!)\epsilon^{\alpha\beta\gamma\delta}\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta,\\ (\gamma_5)^2&=-\mathbf{1},\\ \gamma_5\gamma_i&=-\gamma_i\gamma_5 \quad (i=1,2,3,4).\end{aligned} \tag{18}$$

In the special case where the neutrino wave is linearly polarized in a given region of space, the β rotation turns this direction of polarization through the angle β. That $\gamma_5\psi$ is a solution of (13) follows from (3) and the anticommutation relations (18); hence the linear combination, (17), of ψ and $\gamma_5\psi$ is also a solution.

No change whatever in the metric can lift the degeneracy between the spin polarization states ψ and $\exp(\frac{1}{2}\beta\gamma_5)\psi$. An analogous situation occurs in the physics of a two electron system. No allowable system of forces can ever produce a difference in energy between the states $u(x_1,x_2)$ and $\exp(i\gamma P_{12})u(x_1,x_2)$, where P_{12} is the permutation operator. Nature apparently does not ever permit an irrevocable degeneracy of this kind. Only the combination $u(x_1,x_2)-u(x_2,x_1)$ is allowed. Assume similarly that nature rules out a duplicity of spin states for the neutrino that could never be split by any gravita-

[5] For the solution for general γ fields, see O. Klein, Arch. Math., Astr. och Phys. **34**, 1 (1947).

[6] For the general Hermitizing matrix, see W. Kofink, Math. Z. **51**, 702 (1949).

[7] The possibility of this transformation in *flat* space is well known.

tional field, however strong. More specifically, assume that the only allowed state, ψ, for the neutrino is a state that is transformed into a multiple of itself by every β rotation:

$$\exp(\tfrac{1}{2}\beta\gamma_5)\psi_{\text{allowed}}=e^{i\,(\text{constant})}\psi_{\text{allowed}}. \tag{19}$$

Then one concludes that the allowed state functions necessarily have *circular polarization*, in the sense that the expression

$$\psi_{\text{allowed}}=\psi_c=(1+i\gamma_5)\psi, \tag{20}$$

constitutes a mixture with 90° phase difference of the states with rotations of 0° and 180°. To change the sign of i in (20) is only to interchange the definitions of positive and negative energy states. Lee and Yang[1] have recently given different arguments for considering all neutrinos to have right-handed circular polarization. Their considerations have received dramatic verification.[2,3] The conclusion appears inescapably that neutrinos possess only one state of polarization, which is circular.

The allowable spinor state functions satisfy the condition

$$(1-i\gamma_5)\psi_c=0, \tag{21}$$

and in a suitable representation have only two nonzero components. They can be described by Pauli's spinors of two components, as Lee and Yang show. They introduce the two-component wave equation

$$H\psi=c\boldsymbol{\sigma}\cdot\mathbf{p}\psi=i\hbar\partial\psi/\partial t, \tag{22}$$

where $\boldsymbol{\sigma}$ are the three 2×2 Pauli spin matrices. To write this equation in generally covariant form, it is convenient to introduce four 2×2 matrices s_i which satisfy the conditions

$$[\bar{s}_i,s_j]_+\equiv\bar{s}_is_j+\bar{s}_js_i=2g_{ij}, \tag{23}$$

where the bar denotes complex conjugation. Then the covariant form of the Pauli-Lee-Yang equation is

$$s^\alpha\nabla_\alpha\psi=0. \tag{24}$$

The correct interpretation of the s^i and the covariant derivative ∇_i is well known from spinor analysis.[8]

A beta rotation is not the only means to generate a new solution of the neutrino wave equation (13) from a general solution, ψ. Let a representation be employed in which the basic matrices are all real, $\gamma^i{}_{\text{complex conjugate}}=\gamma^i$, and let the complex conjugate of (13) be taken; then it follows at once that $\psi_{\text{complex conjugate}}$ satisfies the wave equation as well as does ψ itself. When ψ represents a positive energy state, $\psi_{\text{complex conjugate}}$ of course represents a negative energy state; but there is ordinarily no well-defined distinction between the two kinds of states in a metric which varies both with space and with time.

How does one know that the basic matrices can still be taken real when the three-dimensional space under consideration has an arbitrary curvature and topology? We assume for simplicity that time has the topology of a straight line. We are indebted to Professor V. Bargmann for informing us of a theorem cited by Hopf at the International Congress of Mathematicians, Cambridge, Massachusetts (Vol. I, p. 193 of the 1950 Proceedings) to the effect that one can always in a *three*-dimensional space define three mutually orthogonal nonsingular vector fields—a construction that is well known to be impossible on the *two*-dimensional closed surface of a sphere. Then one has only to take over the standard four *real* $\tilde{\gamma}$ matrices of flat space and employ the Vierbein formalism to have four real matrices,

$$\gamma_i=b_i{}^\alpha\tilde{\gamma}_\alpha, \tag{25}$$

in the curved space under consideration.

[8] See, for example, W. L. Bade and H. Jehle, Revs. Modern Phys. **25**, 714 (1953).

4. MOTION OF A NEUTRINO IN A SPHERICALLY SYMMETRICAL GRAVITATION FIELD

We explore the reaction of the neutrino to a gravitational field in the simplest known case: a spherically symmetric metric of the Schwarzschild type,

$$g_{ij}=\begin{pmatrix} e^\lambda & 0 & 0 & 0\\ 0 & r^2 & 0 & 0\\ 0 & 0 & r^2\sin^2\theta & 0\\ 0 & 0 & 0 & -e^\nu\end{pmatrix}. \tag{26}$$

Here the coordinates are $x^1=r$, $x^2=\theta$, $x^3=\varphi$, $x^4=ct=T$. The dilation functions $\lambda(r)$ and $\nu(r)$ are assumed to depend upon distance in an arbitrary way; they are not limited to the special Schwarzschild solution for a localized concentration of mass,

$$e^{-\lambda}=e^\nu=1-2GM/c^2r. \tag{27}$$

In order to write the Dirac equation in this metric we choose a field of γ matrices of the type (7). Two choices for these matrices are simple: (1) Vierbein axes parallel to the r, θ, and φ axes at each point, so that the desired Dirac matrices, γ^i are expressed in terms of the standard Dirac matrices, $\tilde{\gamma}^i$, for a Cartesian coordinate system, by the formulas

$$\gamma_1=e^{\frac{1}{2}\lambda}\tilde{\gamma}_1,\quad \gamma_2=r\tilde{\gamma}_2,\quad \gamma_3=r\sin\theta\tilde{\gamma}_3,\quad \gamma_4=e^{\frac{1}{2}\nu}\tilde{\gamma}_4, \tag{28}$$

and (2), Vierbein axes parallel to some rectangular coordinate system:

$$\begin{aligned}
\gamma_1&=e^{\frac{1}{2}\lambda}(\sin\theta\cos\varphi\tilde{\gamma}_1+\sin\theta\sin\varphi\tilde{\gamma}_2+\cos\theta\tilde{\gamma}_3),\\
\gamma_2&=r(\cos\theta\cos\varphi\tilde{\gamma}_1+\cos\theta\sin\varphi\tilde{\gamma}_2-\sin\theta\tilde{\gamma}_3),\\
\gamma_3&=r\sin\theta(-\sin\varphi\tilde{\gamma}_1+\cos\varphi\tilde{\gamma}_2),\\
\gamma_4&=e^{\frac{1}{2}\nu}\tilde{\gamma}_4.
\end{aligned} \tag{29}$$

The two choices lead to the same radial equation, but to a different dependence of the components of the spinor wave function upon the angles (Table I). In case (2) the angular dependence agrees with that predicted in the familiar case of special relativity by standard formulas.[9] Such ambiguities in the wave function are to be expected whenever one uses a formalism which is invariant under similarity transformations that are quite independent of coordinate transformations. The physically meaningful quantities, such as the current density, will, of course, agree for both choices, for the similarity transformation leaves these quantities invariant. We show the explicit calculations for our choice 1. The Γ_k are found from (2) or (8),

$$\begin{aligned}&\Gamma_1=0;\quad \Gamma_2=\tfrac{1}{2}e^{-\frac{1}{2}\lambda}\tilde{\gamma}_1\tilde{\gamma}_2;\\ &\Gamma_3=\tfrac{1}{2}(\sin\theta e^{-\frac{1}{2}\lambda}\tilde{\gamma}_1\tilde{\gamma}_3+\cos\theta\tilde{\gamma}_2\tilde{\gamma}_3);\\ &\Gamma_4=(\nu'/4)e^{\frac{1}{2}(\nu-\lambda)}\tilde{\gamma}_1\tilde{\gamma}_4,\end{aligned}\tag{30}$$

where $\nu'=d\nu/dr$.

The Dirac equation for a particle of mass $\mu\hbar/c$ takes the form

$$\begin{aligned}0&=\gamma^\alpha\left(\frac{\partial}{\partial x^\alpha}-\Gamma_\alpha\right)\psi+\mu\psi\\ &=[\tilde{\gamma}_1e^{-\frac{1}{2}\lambda}(\partial/\partial r+r^{-1}+\nu'/4)+(\tilde{\gamma}_2/r)(\partial/\partial\theta+\tfrac{1}{2}\cot\theta)\\ &\quad+(\tilde{\gamma}_3/r\sin\theta)(\partial/\partial\varphi)-e^{-\frac{1}{2}\nu}\tilde{\gamma}_4(\partial/\partial T)]\psi+\mu\psi.\end{aligned}\tag{31}$$

We can rewrite this equation, following Schrödinger, as

$$i\partial\omega/\partial T=h\omega,\tag{32}$$

with

$$\omega=\exp(\nu/4)r(\sin\theta)^{\frac{1}{2}}\psi,\tag{33}$$

and

$$e^{-\frac{1}{2}\nu}h=\tilde{\gamma}_4\tilde{\gamma}_1e^{-\frac{1}{2}\lambda}\partial/i\partial r+(\tilde{\gamma}_1/r)K-i\tilde{\gamma}_4\mu,\tag{34}$$

$$K=\tilde{\gamma}_1\tilde{\gamma}_4\tilde{\gamma}_2\partial/i\partial\theta+\tilde{\gamma}_1\tilde{\gamma}_4\tilde{\gamma}_3\partial/i\sin\theta\partial\varphi.\tag{35}$$

The operator K is Hermitian and commutes with h. We can therefore choose simultaneous eigenfunctions of h and K, and separate the μth component of the wave function, ω_μ, into radial, angular and time factors,

$$\omega_\mu=R_\mu(r)\Theta_\mu(\theta,\varphi)\exp(-ihT).\tag{36}$$

Here the quantity h represents (energy/$\hbar c$). The angular factor, Θ, is determined by the requirement

$$K\omega=k\omega,\tag{37}$$

where k is constant. This equation for eigenstates of the angular motion has been investigated by Schrödinger.[10] He finds, in agreement with the conventional treatment of the Dirac equation in a central field, a spectrum of positive and negative integral eigenvalues k.

Only the two matrices $\tilde{\gamma}_4$ and $\tilde{\gamma}_1$ remain explicitly in the radial equation (34) after the operator K is replaced by the number k. They can therefore be represented by 2×2 matrices, and the radial factor by a two-component spinor

$$\tilde{\gamma}_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \tilde{\gamma}_4=\begin{pmatrix}i&0\\0&-i\end{pmatrix},\quad R=\begin{pmatrix}F\\G\end{pmatrix}.\tag{38}$$

In this representation the radial equation for an electron in a Schwarzschild metric, with an electric potential energy $V=vhc$, becomes a modification,

$$\begin{aligned}&[e^{-\frac{1}{2}\nu}(h-v)+\mu]F-e^{-\frac{1}{2}\lambda}dG/dr-(k/r)G=0,\\ &[e^{-\frac{1}{2}\nu}(h-v)-\mu]G+e^{-\frac{1}{2}\lambda}dF/dr-(k/r)F=0,\end{aligned}\tag{39}$$

of the familiar radial equations[11] for an electron in a centrally symmetric potential. For the neutrino, of course, we annul both v and μ.

5. COMPARISON OF ENERGY LEVELS OF ELECTRON IN ELECTROSTATIC AND GRAVITATIONAL FIELDS AND NEUTRINO AND PHOTON IN GRAVITATIONAL FIELD

In order to gain some qualitative understanding of the behavior of electrons and neutrinos in gravitational fields we first consider the case of an electron in a gravitational field. We take over the Schwarzschild solution (27) for the metric outside a mass M. We substitute this metric into the radial wave equation (39) and neglect terms of order $1/c^2$ and higher. We find $(E-V)=\hbar c(h-v)$ is replaced by $(E-V-\phi)$, where $\phi=-(GM/r)(E/c^2)$ is the "gravitational potential energy" of a particle of energy E. Therefore in this approximation the energy levels of an electron of positive energy remain unchanged when the electric potential is replaced by an equally strong gravitational potential. The Bohr formulas for energy levels and radii of circular orbits,

$$\begin{aligned}E&=mc^2-\tfrac{1}{2}m(Ze^2/n\hbar)^2,\\ r&=n^2\hbar^2/mZe^2,\end{aligned}\tag{40}$$

are replaced by the corresponding formulas

$$\begin{aligned}E&=mc^2-\tfrac{1}{2}m(GMm/n\hbar)^2,\\ r&=n^2\hbar^2/GMm^2.\end{aligned}\tag{41}$$

Deviations from this behavior, such as arise from gravitational spin-orbit coupling, are to be expected only when the calculated velocity of the electron in the lowest Bohr orbit is comparable to the speed of light:

$$GMm/\hbar\sim c,$$

or

$$Mm\sim\hbar c/G=(2.18\times10^{-5}\text{ g})^2,$$

or

$$M\sim5\times10^{19}\text{ g}.\tag{42}$$

For the analysis to apply in such an extreme case it would be necessary that this attracting mass should be confined within a distance of the order of the Schwarz-

[9] See, for example, H. A. Bethe, *Handbuch der Physik* (Julius Springer Verlag, Berlin, 1953, second edition), Vol. 24, Part 1.

[10] E. Schrödinger, Commentationes Pontif. Acad. Sci. 2, 321 (1938).

[11] See for example, L. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), second edition.

schild radius of this mass, or the Compton radius of the electron,

$$r_{\text{Schw}}=2GM/c^2=2\hbar/mc=7.7\times10^{-11}\text{ cm}$$

—a condition impossible of attainment even with matter of nuclear density.

The foregoing weak field analysis demands a binding energy for the particle small compared to its rest energy. It obviously will not apply to the neutrino with its zero rest mass. Moreover, bound orbits lie in the energy region between $+mc^2$ and $-mc^2$ and will cease to exist for an object with zero rest mass. The wave function for such an object never falls off exponentially in the region where the metric has become flat. However, it is possible to construct a metric with an inner region, a barrier region, and an outer region, in such a way that the neutrino wave function falls off exponentially in the barrier region. Then leakage from the inner region to the outer region is greatly inhibited. Effectively bound proper states of long life then exist for the neutrino in the inner region.

Such trapping of neutrinos is illustrated especially simply in the metric of the thin shell spherical geon,[12]

$$\begin{aligned}&e^{\nu}=e^{-\lambda}=1-(2GM/c^2r)\quad\text{for}\quad r>(9/4)(GM/c^2),\\&e^{\nu}=1/9;\quad e^{-\lambda}=1\quad\text{for}\quad r<(9/4)(GM/c^2).\end{aligned}\tag{43}$$

With the abbreviations

$$\begin{aligned}dr^*&=e^{\frac{1}{2}\lambda-\frac{1}{2}\nu}dr,\\ \rho&=(c^2/GM)r,\\ h\ (\text{cm}^{-1})&=\text{energy}/\hbar c,\\ \epsilon\ (\text{dimensionless})&=(GM/c^2)h\\&=(GM/\hbar c^3)\ (\text{energy}),\end{aligned}\tag{44}$$

we rewrite the two first-order wave equations (39) for zero rest mass and zero electrostatic potential. We then eliminate one of the two dependent variables to obtain a single equation of the second order for the other variable; either

$$d^2F/d\rho^{*2}+[\epsilon^2-\xi(\rho)]F=0,$$

with

$$\xi(\rho)=e^{\nu}k^2/\rho^2-e^{\nu-\frac{1}{2}\lambda}k/\rho^2+(k/\rho)(d/d\rho^*)e^{\frac{1}{2}\nu};\tag{45}$$

or

$$d^2G/d\rho^{*2}+[\epsilon^2-\eta(\rho)]G=0,$$

with

$$\eta(\rho)=e^{\nu}k^2/\rho^2+e^{\nu-\frac{1}{2}\lambda}k/\rho^2-(k/\rho)(d/d\rho^*)e^{\frac{1}{2}\nu}.\tag{46}$$

The last term on the right-hand sides of the dimensionless effective potentials $\xi(\rho)$ and $\eta(\rho)$ of (45) and (46) has the character of a spin-orbit coupling. As in the case of an electron moving under electrostatic forces, where the spin-orbit coupling is proportional to the angular momentum and to the radial derivative of the potential, so here one term in the effective potential experienced by the neutrino is proportional to the angular momentum parameter, k, and also proportional to the radial derivative of the metric quantity, $e^{\frac{1}{2}\nu}=(-g_{44})^{\frac{1}{2}}$. However, this term appears with opposite sign in the second order wave equations for the two components, F and G, of the *same* state function. From the fact that the two different equations have the same eigenvalues, it follows that the last *two* coupling terms together in (45) and in (46) produce no net effect on the energy levels. Otherwise stated, the energy eigenvalues are completely invariant against the change from k to $-k$. This degeneracy is the same as the fundamental polarization degeneracy that was discussed in Sec. 4. The demand that the neutrino have right-handed polarization means that the allowed state function is given neither by the solution of (45) and (46) for positive k, nor by the solution for negative k, but by the proper linear combination of these two solutions. However, (45) and (46) give the correct pieces out of which to construct the allowed total wave function, and also gives the correct energy eigenvalues.

It is now appropriate to check that the same radial equations are obtained in the two component formalism. For spin matrices, s_i, that satisfy the equations of definition (23), we make a choice analogous to the choice (1) of (28):

$$\begin{aligned}s_1&=e^{\frac{1}{2}\lambda}\begin{pmatrix}0&1\\1&0\end{pmatrix}=e^{\frac{1}{2}\lambda}\tilde{s}_1;\\ s_2&=r\begin{pmatrix}i&0\\0&i\end{pmatrix}=r\tilde{s}_2;\\ s_3&=r\sin\theta\begin{pmatrix}-1&0\\0&1\end{pmatrix}=r\sin\theta\tilde{s}_3;\\ s_4&=e^{\frac{1}{2}\nu}\begin{pmatrix}0&1\\-1&0\end{pmatrix}=e^{\frac{1}{2}\nu}\tilde{s}_4.\end{aligned}\tag{47}$$

To define covariant differentiation we need 2×2 matrices, Γ_k, that satisfy the analog of (2)

$$\partial s^i/\partial x^k-\Gamma_{ik}{}^{\mu}s_{\mu}-\Gamma_{k\ \text{comp. conj.}}s_i+s_i\Gamma_k=0,\tag{48}$$

namely,

$$\begin{aligned}&\Gamma_1=0;\quad \Gamma_2=\tfrac{1}{2}e^{-\frac{1}{2}\lambda}\tilde{s}_{1\ \text{comp. conj.}}\tilde{s}_2;\\&\Gamma_3=\tfrac{1}{2}(\sin\theta e^{-\frac{1}{2}\lambda}\tilde{s}_{1\ \text{comp. conj.}}\tilde{s}_3+\cos\theta\tilde{s}_{2\ \text{comp. conj.}}\tilde{s}_3);\\&\Gamma_4=(\nu'/4)e^{\frac{1}{2}\nu-\frac{1}{2}\lambda}\tilde{s}_{1\ \text{comp. conj.}}\tilde{s}_4.\end{aligned}\tag{49}$$

These representation-independent expressions evidently can also be generated from the Γ_k's, of the Dirac formalism (30) by a simple change: every product of the form $\tilde{\gamma}_i\tilde{\gamma}_j$ is replaced by a product of the form $\tilde{s}_{i\ \text{comp. conj.}}\tilde{s}_j$. The Dirac equation for a particle of mass m in the present formalism operates on a pair of spinors, ψ and φ, each of two components,

$$\begin{aligned}s^{\alpha}\nabla_{\alpha}\psi+\mu\varphi&=0,\\ s^{\alpha}{}_{\text{comp. conj.}}\nabla_{\alpha}\varphi+\mu\psi&=0,\end{aligned}\tag{50}$$

[12] J. A. Wheeler, Phys. Rev. 97, 511 (1955).

but the Pauli-Lee-Yang neutrino equation has the simpler form

$$0=s^\alpha\nabla_\alpha\psi=s^\alpha(\partial/\partial x^\alpha-\Gamma_\alpha)\psi. \qquad (51)$$

This equation is obtained in explicit form from (31) by annulling the mass term, $\mu\psi$ and replacing $\tilde{\gamma}_i$ everywhere by $\tilde{s}_i$. Again introduce a new form for the wave function,

$$\omega=\exp(\nu/4)r(\sin\theta)^{\frac{1}{2}}\psi, \qquad (52)$$

as in (33). Also multiply the equation for ω through by $\tilde{s}_{4\text{ comp. conj.}}$ and note that

$$\tilde{s}_{4\text{ comp. conj.}}\tilde{s}_4=-\mathbf{1},$$

and recall that the quantities

$$\tilde{s}_{4\text{ comp. conj.}}\tilde{s}_i=\sigma_i, \quad (i=1,2,3) \qquad (53)$$

are the Pauli spin matrices. Then the general relativity form of the Pauli-Lee-Yang neutrino equation in a centrally symmetric metric becomes

$$(\sigma_1 e^{-\frac{1}{2}\lambda}\partial/\partial r+\sigma_2\partial/r\partial\theta +\sigma_3\partial/r\sin\theta\partial\varphi+e^{-\frac{1}{2}\nu}\partial/\partial T)\omega=0. \qquad (54)$$

We consider a solution, ω, which depends upon azimuthal angle and time as

$$\exp(im_j\varphi-ihT),$$

so that (54) reduces to

$$(\sigma_1 e^{-\frac{1}{2}\lambda}\partial/\partial r+\sigma_2\partial/r\partial\theta+im_j\sigma_3/r\sin\theta-ihe^{-\frac{1}{2}\nu})\omega=0. \qquad (55)$$

We can no longer find an angular operator, K, that commutes with (55). Instead, we proceed as follows. (1) We *temporarily* introduce an explicit representation for the Pauli matrices, σ_i, and express (55) as two coupled first-order equations for two unknown functions. (2) We deduce the character of the solution from what we already know of the solution in the Dirac formalism. (3) Out of this solution we construct the mixed density matrix, $M(1,2)=\psi(2)\psi^\dagger(1)$. From that matrix we can calculate the expectation value of any physical quantity. The neutrino flux, for example, is given by the trace

$$\begin{aligned}(\text{neutrino flux})^k&=\psi^\dagger is^k\psi\\ &=Tr[is^kM(1,1)]. \end{aligned}\qquad (56)$$

(4) This density matrix can be expressed as a linear combination of the four matrices, σ_i and $\mathbf{1}$, and thus translated back to a form independent of any special representation of the Pauli matrices. It then takes the form

$$\begin{aligned}M(1,2)&=\psi(2)\psi^\dagger(1)\\ &=\exp[-\tfrac{1}{4}\nu(r_1)-\tfrac{1}{4}\nu(r_2)](r_1r_2)^{-1}\\ &\quad\cdot\{[\bar{F}(r_1)F(r_2)+\bar{G}(r_1)G(r_2)]\\ &\quad\times[\bar{\Theta}_1(1)\Theta_1(2)+\bar{\Theta}_2(1)\Theta_2(2)\\ &\quad+i[\bar{G}(r_1)F(r_2)-\bar{F}(r_1)G(r_2)]\\ &\quad\times[\bar{\Theta}_1(1)\Theta_1(2)-\bar{\Theta}_2(1)\Theta_2(2)]\}\mathbf{1}(is^4)+\text{etc.}\end{aligned} \qquad (57)$$

where F and G are the radial functions already introduced. The angular factors in (57) have the form

$$\begin{aligned}\Theta_1(1)&=f(\theta_1)\exp(im_j\varphi_1)/(\sin\theta_1)^{\frac{1}{2}},\\ \Theta_2(1)&=g(\theta_1)\exp(im_j\varphi_1)/(\sin\theta_1)^{\frac{1}{2}},\ \text{etc.,}\end{aligned} \qquad (58)$$

where f and g are the angular functions of Schrödinger.[10] Such quantities as the number, flux and stress and energy density of neutrinos follow in a straightforward way from the density matrix (57) according to the pattern of (56), based on two component spinors. Alternatively the same answers can be obtained from the Dirac formalism, with which we shall generally work.

In either 2 or 4 component formalisms the energy levels of a trapped neutrino are found by solution of an eigenvalue equation which can be taken to be (45). We assume the geon metric (43). Then the dimensionless effective potential, $\xi(\rho)$, has the form

$$\xi(\rho)=[1-(2/\rho)](k^2/\rho^2)-[1-(2/\rho)]^{1/2}(k/\rho)[1-(3/\rho)]$$

outside the radius $\rho=2.25$; and inside it becomes

$$\xi(\rho)=k(k-1)/9\rho^2. \qquad (59)$$

This potential is sketched in Fig. 1 for several values of the positive integer, k. Semistable bound states occur only for values of k of the order of 10 and larger. For such values of k terms in k can be neglected relative to terms in k^2. Then in the outer region $\xi(\rho)$ attains its maximum value

$$\xi_{\max}(\rho_{\max})=k^2/27$$

at

$$\rho_{\max}=3. \qquad (60)$$

Its minimum

$$\xi_{\min}(\rho_{\min})=4^2k^2/9^3$$

occurs at

$$\rho_{\min}=9/4. \qquad (61)$$

The dimensionless vibration frequency or energy, ϵ, of the nth semistable bound state can be estimated from the JWKB approximation formula,

$$(n+\tfrac{1}{2})\pi=\int_a^b[\epsilon^2-\xi(\rho)]^{\frac{1}{2}}d\rho^*, \qquad (62)$$

where a and b are the classical turning points of the "bound" state of highest energy. The integral exists only when ϵ lies between $\xi_{\min}{}^{\frac{1}{2}}=2k/27$ and $\xi_{\max}{}^{\frac{1}{2}}=k/3^{\frac{3}{2}}$ (the two dashed limits in the right-hand part of Fig. 1). The total number, N_k, of bound states of type k can be estimated by setting ϵ^2 equal to the maximum of $\xi(\rho)$ and calculating (62), with the result

$$\begin{aligned}N=\pi^{-1}k\bigg\{&\int_{9/4}^{3}[(1/27)-(1/\rho^2)+(2/\rho^3)]^{\frac{1}{2}}[1-(2/\rho)]^{\frac{1}{2}}d\rho\\ &+\int_{\sqrt{3}}^{9/4}[(1/27)-(1/9\rho^2)]^{\frac{1}{2}}3d\rho-\tfrac{1}{2}\\ &=0.15k-0.5.\end{aligned} \qquad (63)$$

For each value of k the spectrum of allowed values for the dimensionless energy parameter, ϵ, stretches from a minimum value a little above $\xi_{\min}{}^{\frac{1}{2}}$ (proportional to k) to a maximum value a little less than $\xi_{\max}{}^{\frac{1}{2}}$ (also proportional to k). In this interval the number of levels, N_k, is also approximately proportional to k. These qualitative features of the level scheme are shown in Fig. 1.

In the absence of a gravitational field the neutrino spectrum reduces to the natural analog of the spectrum of the free electron. For each wave number, (k_x,k_y,k_z), there is one state of right-handed circular polarization and of positive energy; and a second state of negative energy and left handed circular polarization. All states of negative energy are to be considered as filled. Absence of a neutrino from a negative energy is to be interpreted as the presence of an antineutrino. According to hole theory the momentum (or angular momentum) of the real physical antineutrino is the negative of the momentum (or angular momentum) of the missing negative energy neutrino. For that negative energy neutrino the momentum and spin angular momentum are *opposite* in direction, according to the equation, $H=c(\boldsymbol{\sigma}\cdot\mathbf{p})$, of Lee and Yang. Therefore, as they show, the momentum and spin angular momentum have *opposite* directions for the real physical antineutrino.

6. STATISTICAL MECHANICS AND THERMAL EQUILIBRIUM OF NEUTRINOS

Gamow and Schoenberg[13] have given reason to believe that neutrino emission determines the rate of gravitational contraction of a heavy star in late phases of evolution, after the normal sources of thermonuclear energy have been exhausted. In oversimplified terms, hot neutrons change to cooler protons plus electrons plus neutrinos; hot protons and electrons change to cooler neutrons and antineutrinos. The medium continually loses energy by emission of neutrinos and antineutrinos—Gamow's "Urca process." The ν's and $\bar{\nu}$'s escape so much more readily than photons that they alone determine the rate of energy liberation and gravitational contraction. In late phases of such gravitational contraction the density might rise to a point where the opacity of matter even to neutrinos begins to make itself felt. For example, consider the point where nuclear densities have been reached, of the order of 10^{38} nucleons/cm^3; and assume a neutrino interception cross section, 10^{-43} cm^2, of the order of that found by Reines and Cowan[14]; then the mean free path of a neutrino will be of the order of only 1 km. Under such conditions one has to speak of an opacity with respect to neutrinos and a local neutrino temperature along lines familiar from the theory of transfer of heat by electromagnetic radiation. One does not have to consider this particular problem of stellar interiors to raise the question: what is the equilibrium distribution of neutrinos? The specification of this equilibrium demands one more quantity for neutrinos than for photons: one has to give both the temperature and a suitably defined Fermi energy. The probability $W(E)$ that a neutrino state of energy, E,

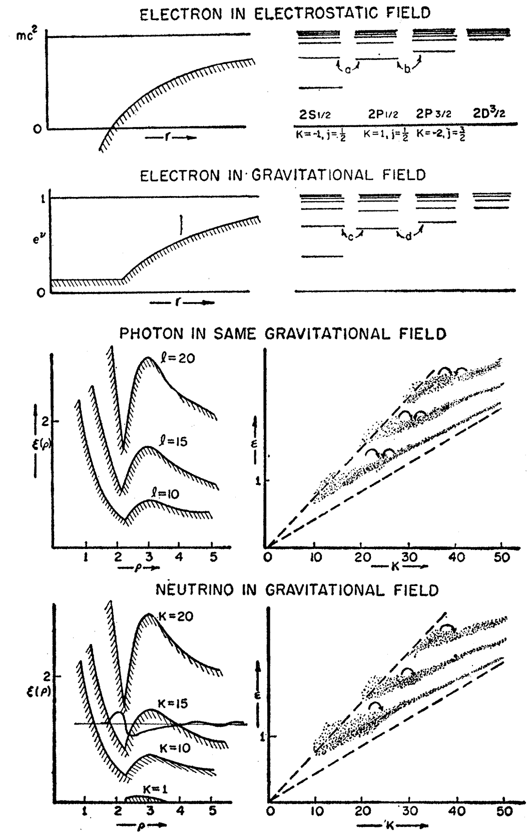

FIG. 1. Comparison of energy levels of an electron in an electrostatic field and in a gravitational field, and of a photon and a neutrino in a gravitational field. The left-hand diagram in these four cases gives, respectively, the electrostatic potential; the Einstein gravitational potential, $-g_{44}=e^{\nu}$ for the "geon metric"; the dimensionless effective potential, $\xi(\rho)=l(l+1)e^{\nu}/\rho^2$ for photons; and the corresponding quantity for neutrinos [the $\xi(\rho)$ of (59)]. The bound states of the electron lie lower than mc^2 and are therefore stable against electron escape. The energy splitting at (a) arises from Lamb shift and any other perturbative effects that raise the effective potential near the origin compared to the ideal Coulomb potential. Similarly, a splitting arises at (c) because s-wave electrons respond more than p-wave electrons to the central flat region of the metric. The splittings at (b) and (d) arise from spin-orbit coupling. Photons and neutrinos can be trapped only in states of greater or lesser life time, never in completely stable levels. The wave function of a trapped state falls off exponentially in the region of the effective potential barrier, lower left. Zero or small angular momentum corresponds to motion along, or nearly along, the radius vector—a kind of motion that always leads to escape of the photon or neutrino. The diagrams lower right show schematically both the position and the width of the semistable bound states. Each photon state can be occupied by any number of photons of either independent state of polarization. Each neutrino state can be occupied by only one neutrino. The corresponding antineutrino state has the same frequency and energy and can also be occupied. There is nothing to compare with the spin-orbit splitting of the electron states: the angular momentum of the neutrino always precesses in such a way as to stay parallel to the neutrino momentum.

[13] G. Gamow and M. Schoenberg, Phys. Rev. **59**, 539 (1941).
[14] F. Reines and C. Cowan, Jr., Phys. Rev. **92**, 830 (1953).

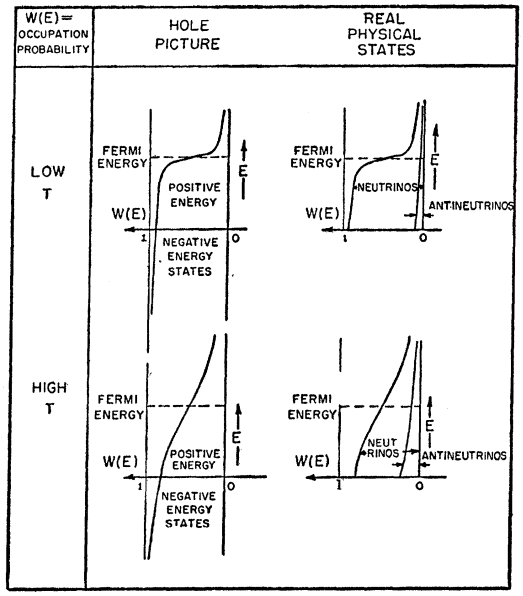

FIG. 2. Influence of temperature and Fermi energy on the equilibrium distribution of neutrinos and antineutrinos.

shall be occupied is given by[15]

$$W(E)=1/[1+e^{-\eta+E/T}], \tag{64}$$

where T represents the temperature (in energy units) and ηT represents the Fermi energy.

Translating from hole theory to the physically observable neutrino and antineutrino states (Fig. 2), one has the expression,

$$dn_+=[4\pi k^2dk/(2\pi)^3]/[1+e^{-\eta+k\hbar c/T}], \tag{65}$$

for the average number of neutrinos per unit volume in the interval, dk, of circular wave number; and for the corresponding number of antineutrinos,

$$\begin{aligned} dn_-&=[4\pi k^2dk/(2\pi)^3][1-W(-k\hbar c)] \\ &=[4\pi k^2dk/(2\pi)^3]/[1+e^{\eta+k\hbar c/T}]. \end{aligned} \tag{66}$$

The quantities of greatest interest are the total density of neutrinos,

$$n_+=(T^3/2\pi^2\hbar^3c^3)\int_0^\infty \frac{x^2dx}{1+e^{x-\eta}}, \tag{67}$$

the total density of antineutrinos (given by the same expression with the sign of η reversed), and the total energy density,

$$\epsilon=(T^4/2\pi^2\hbar^3c^3)\left[\int_0^\infty \frac{x^3dx}{1+e^{x-\eta}}+\int_0^\infty \frac{x^3dx}{1+e^{x+\eta}}\right]. \tag{68}$$

If the Fermi energy is zero, the integrals have the simple values

$$n_+=n_-=(T^3/2\pi^2\hbar^3c^3)\times 1.803,$$

and

$$\begin{aligned} \epsilon_\nu&=(T^4/2\pi^2\hbar^3c^3)2\times 5.757 \\ &=(7\pi^2/120)(T^4/\hbar^3c^3). \end{aligned} \tag{69}$$

In contrast, the energy content of electromagnetic blackbody radiation is

$$\epsilon_{em}=(8\pi^2/120)(T^4/\hbar^3c^3). \tag{70}$$

The neutrino energy is smaller than the electromagnetic energy by the factor $\frac{7}{8}$. The number of accessible states of a given wave number is the same for the two kinds of radiation: two spin polarizations for photons; and for the more penetrating radiation, one neutrino and one antineutrino state. Moreover, the occupation probability for either kind of state follows the same limiting Boltzmann formula for states of high energy $W(E)\sim e^{-E/T}$. But any given state of low energy can be occupied by many photons, and at most one neutrino: hence the advantage for electromagnetic energy compared to neutrino energy. These relationships are entirely changed when the Fermi energy of the neutrinos has a nonzero value. In that case the energy density of the neutrinos can be made to have any arbitrarily large value.

7. CREATION OF NEUTRINO PAIRS

In this paper we disregard all processes for creation or disappearance of neutrinos which depend upon transformations of the elementary particles, considering only the response of neutrinos to the curvature of the metric. More specifically, we limit attention to processes in which—according to hole theory—a neutrino is raised from a negative energy state to a positive energy state, or where, in physical terms, a neutrino-antineutrino pair is created.

Is there anything special about the gravitational field which makes it incapable of raising a neutrino from a negative energy state to a positive energy state? Otherwise stated, does there exist any quantity, Q, which will commute or anticommute with the operator in the wave equation (51) and which will serve to distinguish positive and negative energy states? To construct such an operator one has only the four independent 2×2 spin matrices. Among these only the unit matrix commutes with (51). From it nothing of interest can be constructed. We therefore expect that there is no point of principle which prevents transitions from negative energy to positive energy states. In other words, it is as much out of the question in neutrino physics as it is in electron physics to make a well-defined distinction between negative and positive energy states when general gravitational or electromagnetic fields are at work.

The mechanism of the transitions in the neutrino case is simple. Just as a static disturbance in the metric deflects a neutrino, bringing about a transition between states of different momenta, so a time-varying disturbance causes a transition between states of different

[15] E. Fermi, Z. Physik 36, 902 (1926).

TABLE III. Order-of-magnitude estimate of cross section for creation of a pair by the collision of two quanta of equal but opposite momenta.

Process	$\gamma+\gamma\rightarrow e^{+}+e^{-}$	$G+G\rightarrow\nu+\bar{\nu}$
Energy of one quantum	$E=\hbar c/\lambdabar$	$E=\hbar c/\lambdabar$
Localization volume	$\sim\lambdabar^3$	$\sim\lambdabar^3$
Energy density	$\sim\hbar c/\lambdabar^4$	$\sim\hbar c/\lambdabar^4$
Relevant field	electric	gravitational
Square of field	$\sim(\hbar c)^{\frac{1}{2}}/\lambdabar^2$	$\sim(\hbar cG)^{\frac{1}{2}}/\lambdabar^2$
Available potential in region of energy concentration	$\delta A\sim(\hbar c)^{\frac{1}{2}}/\lambdabar$	$\delta g\sim(\hbar G/c^3)^{\frac{1}{2}}/\lambdabar$
Potential required to produce transition from $E_1=-\hbar c/\lambdabar^2$ to $E_2=+\hbar c/\lambdabar^2$ with nearly 100% probability	$\delta A\sim\hbar c/\lambdabar e$	$\delta g\sim1$
Available disturbance/required disturbance	$\sim e/(\hbar c)^{\frac{1}{2}}$	$\sim(\hbar G/c^3)^{\frac{1}{2}}/\lambdabar$
Number of times this factor enters into matrix element	2	2
Number of times matrix elements occur in transition probability	2	2
Pair creation cross section for 100% creation probability	$\sim\lambdabar^2$	$\sim\lambdabar^2$
Resulting estimate for cross section for pair creation	$\sigma\sim(e^2/\hbar c)^2\lambdabar^2$ $\sim(e^2/mc^2)^2(mc^2/E)^2$	$\sigma\sim(\hbar G/c^3)^2/\lambdabar^2$ $\sim(1.6\times10^{-33}\text{ cm})^4/\lambdabar^2$ $\sim(GE/c^4)^2$
Asymptotic behavior of accurate formula for cross section, high energy, unpolarized radiation	$\sigma\sim2\pi(e^2/mc^2)^2(mc^2/E)^2$ $\times\ln[2E/e^{\frac{1}{2}}mc^2]$	Not yet calculated

energies. Let such a time-varying disturbance in the free gravitational field be analyzed into Fourier components in the weak field approximation. According to the laws of conservation of momentum and energy, no single monochromatic disturbance will be able to produce transitions of neutrinos from negative energy states to positive energy states (apart from the singular case where all three momenta lie along the same line). Real pair production demands collaboration of at least two of these monochromatic disturbances, traveling in different directions. As in the case where two photons collide to produce an electron pair,[16] so here it is simplest to analyze the two quantum process in a frame of reference in which the momenta are equal and opposite. For an order-of-magnitude estimate of the cross section for pair creation it is not necessary to evaluate matrix elements in detail. Instead, one can follow the reasoning sketched by Bohr and Rosenfeld in their discussion of vacuum polarization.[17] Let the two concentrations of energy be pictured as localized in two regions of space, of dimensions $\sim\lambdabar$, and as moving towards each other with velocity c, so that they overlap and collaborate only during a time interval $\sim\lambdabar/c$. Then the calculation outlined in Table III leads to a cross section of the order

$$\begin{aligned}\sigma&\sim(GE/c^4)^2\\&\sim(1.6\times10^{-33}\text{ cm})^4/\lambdabar^2,\end{aligned}\tag{71}$$

for creation of a $(\nu,\bar{\nu})$ pair by two gravitons, each of energy E. The cross section is fantastically small for quanta of any familiar energies. As the energy is increased, a domain of wavelengths is ultimately reached, $\lambdabar\sim(\hbar G/c^3)\sim1.6\times10^{-33}$ cm, at which any normal analysis of gravitational disturbances into waves would appear to be ruled out: the disturbances in the metric have reached the order of magnitude, $\delta g\sim1$, where nonlinear effects completely dominate the analysis. Even at such incredibly high energies the estimated cross section (71) only attains a value of the order

$$\sigma\sim10^{-66}\text{ cm}^2.\tag{72}$$

The principle of microscopic reversibility gives a cross section also of the order (71) for the process

$$\nu+\bar{\nu}\rightarrow G+G.\tag{73}$$

The existence of the inelastic process (73) implies, according to dispersion theory,[18] the existence of an elastic scattering process

$$\nu+\bar{\nu}(\text{via}\rightarrow G+G)\rightarrow\nu'+\bar{\nu}'.\tag{74}$$

[16] G. Breit and J. A. Wheeler, Phys. Rev. **46**, 1087 (1934). See E. J. Williams, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **13**, 44 (1935) for the observable physical consequences of this pair creation process.

[17] N. Bohr and L. Rosenfeld, Phys. Rev. **78**, 794 (1950).

[18] J. S. Toll, "The dispersion relation for light and its application to problems involving electron pairs," Ph.D. thesis, Princeton, 1952, unpublished; see also J. Toll and J. A. Wheeler, Phys. Rev. **81**, 654(A) (1951).

To estimate the cross section for this process even for neutrino energies, $\hbar\omega$, small compared to the critical energy $(\hbar c^5/G)^{\frac{1}{2}}$, it is not legitimate to apply

$$(d\sigma/d\Omega)_{\text{elastic forward at }\omega}=\left[\frac{\omega^2}{2\pi^2 c}\int_0^\infty\frac{\sigma_{\text{abs}}(\omega')d\omega'}{\omega'^2-\omega^2}\right]^2,\quad(75)$$

because the principal contribution to the integral comes from virtual processes at energies, $\hbar\omega'$, comparable to and larger than the critical energy.

We have not considered processes that belong in the realm of true elementary particle physics; processes such as

$$\nu+\bar{\nu}\rightarrow\mu^{+}+e^{-},\quad(76)$$

and

$$\nu+\bar{\nu}(\text{via virtual}\rightarrow\mu^{+}+e^{-})\rightarrow\nu'+\bar{\nu}'.\quad(77)$$

In the frame where the total momentum is zero the first process has a threshold at a neutrino energy,

$$E=104mc^2.\quad(78)$$

The calculated cross section rises well above threshold according to the asymptotic formula[12]

$$\sigma_{\text{abs}}\sim(g^2/\hbar^2c^4)(E/mc^2)^2,\quad(79)$$

where the beta-coupling constant has the familiar value

$$g\sim10^{-49}\text{ erg cm}^3.\quad(80)$$

From the absorption process one infers via causality arguments the existence of the scattering process (77). Applied to this process, the dispersion integral (76) is only logarithmically divergent. Attribute to the divergent logarithm the conventional value "10." Then one estimates[12] a scattering cross section of the order

$$(d\sigma/d\Omega)_{\text{forward elastic}}\sim\text{“100”}(E^3g^2/\hbar^5c^5)^2.\quad(81)$$

It appears reasonable to conclude that both $(\nu,\bar{\nu})$ scattering processes are negligible at any reasonable energies in comparison with the two absorption processes. The two absorption cross sections have the same E^2 energy dependence, interestingly enough. Their ratio is the dimensionless number

$$\frac{\sigma(\nu+\bar{\nu}\rightarrow\mu^{+}+e^{-})}{\sigma(\nu+\bar{\nu}\rightarrow G+G)}\sim\frac{(g^2/\hbar^2c^4)(E/\hbar)^2}{(GE/c^4)^2}=(gc^2/G\hbar^2)^2\sim10^{34},\quad(82)$$

a testimonial to the well-known great ratio between beta couplings and gravitational couplings.

8. DENSITY OF STRESS, MOMENTUM, AND ENERGY

So far we have considered the response of a neutrino (or electron) to a given metric, either static or varying in time. Now we ask for the response of the gravitational field to the neutrino (or electron). The Einstein field equations connect the gravitational field and the change in the metric with the stress-momentum-energy tensor, T_{ik}. This tensor has been given for example by Pauli[19] in the case of flat space. Rosenfeld[20] has discussed this tensor in curved space from the viewpoint of the Vierbein formalism and derived some of its components. We follow his very general approach. The Lagrangian density in general relativity is proportional to the quantity

$$[(c^4/G)R+L](-g)^{-\frac{1}{2}},\quad(83)$$

where R is the curvature scalar and L arises from other field variables in addition to the g_{ik}. The requirement that the Lagrangian be an extremum with respect to variations of the g_{ik} leads to the Einstein field equations of the form

$$G_{ik}(\equiv R_{ik}-\tfrac{1}{2}g_{ik}R)=\frac{8\pi G}{c^4}T_{ik},\quad(84)$$

where the stress energy tensor, T_{ik}, is defined for ordinary fields as the variational derivative

$$T_{ik}=(1/8\pi)(-g)^{\frac{1}{2}}\frac{\delta(-g)^{-\frac{1}{2}}L}{\delta g^{ik}}=(1/8\pi)\left(\frac{\delta L}{\delta g^{ik}}-\tfrac{1}{2}g_{ik}L\right),\quad(85)$$

and more generally, for spinor fields, by the equation

$$\int T_{\mu\nu}\delta g^{\mu\nu}(-g)^{\frac{1}{2}}d^4x=(1/8\pi)\delta\int L(-g)^{\frac{1}{2}}d^4x.\quad(86)$$

The procedure for finding this variation for spinor fields is somewhat more subtle than for c-number fields. The variation of the g_{ik} reflects itself not only where they are used to form invariants, but also in variations of the spinors which are necessary to preserve the fundamental relationship (1): we must have

$$[\delta\gamma_i,\gamma_j]_{+}+[\gamma_i,\delta\gamma_j]_{+}=2\delta g_{ij}\mathbf{1}.\quad(87)$$

One solution is[21]

$$\delta\gamma^i=\tfrac{1}{2}\gamma_\alpha\delta g^{\alpha i}.\quad(88)$$

Since any other solution can be obtained from this one by a similarity transformation, the above is general enough for our purposes. This variation of the γ^i leads, through (2), to the variation of the Γ_k,

$$\delta\Gamma_k=\tfrac{1}{8}(g_{\nu\sigma}\delta\Gamma_{\mu k}{}^{\sigma}-g_{\mu\sigma}\delta\Gamma_{\nu\rho}{}^{\sigma})s^{\mu\nu},\quad(89)$$

where s^{ij} is an abbreviation for $\frac{1}{2}(\gamma^i\gamma^j-\gamma^j\gamma^i)$. For the

[19] W. Pauli, *Handbuch der Physik* (Julius Springer Verlag, Berlin, 1933), second edition; Vol. 24, Part 1, p. 235.

[20] L. Rosenfeld, Acad. roy. Belg. **18**, No. 6 (1940). We are indebted to Bryce DeWitt for the opportunity to see an unpublished manuscript of his written several years ago in which the result (92) has already been given.

[21] W. Pauli, Ann. Phys. **18**, 337 (1933).

Lagrangian (7), which vanishes when the field equations hold, we find the variation

$$\delta\int L(-g)^{\frac{1}{2}}d^4x=\int \delta L(-g)^{\frac{1}{2}}d^4x$$

$$=\int[\psi^\dagger\delta\gamma^k\nabla_k\psi+\psi^\dagger\gamma^k\delta\Gamma_k\psi](-g)^{\frac{1}{2}}d^4x$$

$$=\int[\tfrac{1}{2}\psi^\dagger\gamma_\nu\nabla_\mu\psi\delta g^{\mu\nu}+\tfrac{1}{8}(g^{\rho\nu}s^\mu-g^{\rho\mu}s^\nu)$$

$$\times(g_{\mu\sigma}\delta\Gamma_{\nu\rho}{}^\sigma-g_{\nu\sigma}\delta\Gamma_{\mu\rho}{}^\sigma)](-g)^{\frac{1}{2}}d^4x, \quad (90)$$

where s^k is the previously defined four vector of flow and density. The $\delta\Gamma_{\nu\rho}{}^\sigma$ are expressible in terms of the $\delta g^{\mu\nu}$ and their derivatives. The derivatives are removed by partial integration, and the various terms in the second part of the integrand collected to form the covariant derivative of the typical component of the flow vector, s^k:

$$\delta\int L(-g)^{\frac{1}{2}}d^4x$$

$$=\int[\tfrac{1}{2}\psi^\dagger\gamma_\nu\nabla_\mu\psi-\tfrac{1}{4}\nabla_\mu s_\nu]\delta g^{\mu\nu}(-g)^{\frac{1}{2}}d^4x. \quad (91)$$

Comparing (91) with (86), we find the momentum-energy tensor

$$T_{ik}=-(\hbar c/4)[\psi^\dagger\gamma_i\nabla_k\psi-(\nabla_k\psi^\dagger)\gamma_i\psi$$
$$+\psi^\dagger\gamma_k\nabla_i\psi-(\nabla_i\psi^\dagger)\gamma_k\psi]. \quad (92)$$

In the case of special relativity, where $\nabla_k=\partial/\partial x^k$, this result—which holds both for the electron and the neutrino—reduces to the energy-momentum tensor given by Pauli.[19] The trace of the energy-momentum tensor (92) reduces to

$$T_\alpha{}^\alpha=\mu\hbar c\psi^\dagger\psi \quad (93)$$

as a consequence of the field equations. It is remarkable that this trace vanishes for the neutrino field ($\mu=0$) just as it does for the electromagnetic field.

One can use this result to derive the expressions already given for the density of neutrino energy under conditions of thermodynamic equilibrium.

9. INTERACTION BETWEEN TWO PENCILS ON NEUTRINOS

We consider a second application of the energy momentum tensor. We ask for the effect of a directed beam of neutrinos in an otherwise nearly flat space. For simplicity, let the beam move in the x direction. Take $k_1=-k_4=k^4=k=E/\hbar c$. Then the neutrino wave has the form

$$\psi=u\exp[ik(x^1-x^4)]. \quad (94)$$

Here u is a slowly varying spinor function of position which (1) is nearly constant in the domain of the pencil of radiation—very large compared to $\lambda\!\!\!^-=1/k$—and (2) falls off smoothly and strongly outside of this region. The gradient of ψ contains terms that come from the gradient of u, and others from the gradient of the exponential; but the second terms are overwhelmingly more important than the first ones. We conclude that all components of T_{ik} with i or k equal to 2 or 3 are negligible. The significant components have the value

$$T_{14}=T_{44}=T_{11}=\hbar ck|\psi|^2, \quad (95)$$

since the flow-density four vector is light-like. Tolman and Ehrenfest have investigated[22] the metric due to a pencil like concentration of electromagnetic energy of identical character. Therefore the results which they derived for light also hold for neutrinos. Two photons or neutrinos, or one electron and one neutrino, attract when their propagation vectors are antiparallel with twice the Newtonian value, and not at all when their propagation vectors are parallel. Therefore the neutrinos in a toroidal neutrino geon will be in their most stable configuration when half go around one way and half go around the other way.

10. STRESS, MOMENTUM, AND ENERGY OF A NEUTRINO IN A TRAPPED STATE

As a third application of the stress energy tensor we consider the motion of a neutrino in a nearly bound orbit in a central field, as in Sec. 5. Let the time dependence of the wave function be described by the factor $\exp(-iET/\hbar c)$, and neglect any small imaginary part of E that describes the slow leakage of the state out of the zone of trapping. Then the stress energy tensor takes the form

$$\begin{aligned} T_r{}^r&=(i\hbar c/2)e^{-\frac{1}{2}\lambda}[(\partial\psi^*/\partial r)\tilde{\gamma}_4\tilde{\gamma}_1\psi \\ &\qquad -\psi^*\tilde{\gamma}_4\tilde{\gamma}_1(\partial\psi/\partial r)], \\ T_T{}^T&=-e^{-\frac{1}{2}\nu}E\psi^*\psi, \\ T_\theta{}^\theta+T_\varphi{}^\varphi&=-[T_r{}^r+T_T{}^T]. \end{aligned} \quad (96)$$

Require that ψ be an eigenstate of the z component of the total angular momentum. Then these expressions are independent of ϕ. It is therefore a simple matter to sum or average them over various orientations of the orbit provided the contributions add incoherently. It is easy to give conditions which make this incoherent addition legitimate.[12] The effect of the addition or averaging is to make $\langle T_r{}^r\rangle$, $\langle T_T{}^T\rangle$, $\langle T_\theta{}^\theta\rangle$, and $\langle T_\varphi{}^\varphi\rangle$ the only nonzero components, and to eliminate the dependence upon angle. Then these surviving components of the averaged stress-momentum-energy tensor depend only upon the radial components of the neutrino wave function. When this wave function has the form (36), using the wave equation (32), these averages take

[22] See, for example, R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, New York, 1934), p. 272.

the form

$$\langle T_r{}^r\rangle = (e^{-\frac{1}{2}\nu}/r^2)[e^{-\frac{1}{2}\nu}ER^*R - (\hbar k/r)R^*\tilde{\gamma}_1 R]$$
$$\langle T_T{}^T\rangle = -(e^{-\nu}/r^2)ER^*R, \tag{97}$$
$$\langle T_\theta{}^\theta\rangle = \langle T_\varphi{}^\varphi\rangle = (\hbar k/r^2)e^{-\frac{1}{2}\nu}R^*\tilde{\gamma}_1 R;$$

however, this form applies only to a Dirac neutrino. The normalization of (36) is such that the fourth component of the flow-density vector integrates to unity:

$$1 = \int s^4 d(\text{volume})$$
$$= -\int [\langle T_T{}^T\rangle/E] d(\text{volume})$$
$$= \int_0^\infty R^*Re^{\frac{1}{2}\lambda-\frac{1}{2}\nu}dr \int \frac{\Theta^*\Theta}{\sin\theta}\sin\theta d\theta d\varphi. \tag{98}$$

The radial and angular integrals are separately normalized to unity here and hereafter.

The wave function for the right-handed neutrino, as already noted, does not have the form, ω, of (36), but rather, the form, $2^{-\frac{1}{2}}(1-i\gamma_5)\omega$ of a linear combination of two parts. Each part is individually separable in the product form but their sum is not. Let R continue to denote the radial factor in the first of these two parts. Then the normalization (98) will continue to apply. The form (97) will not be valid for the stress-energy tensor for any individual neutrino state; new interference terms have to be added. However, when the average is made, as before, over orbits of all orientations, and use is made of the angular properties of the functions, as deduced by Schrödinger, then these interference terms drop out, and (97) continues to apply to the Lee-Yang neutrino.

11. NEUTRINO GEON

So far as one knows, geons have nothing to do either with elementary particle physics or with astrophysics. They have interest for quite another reason. Gravitational fields have a nonlinear character that has many unexplored consequences. A few of these consequences come to light in the analysis of a geon.[23] A collection of immaterial energy holds itself together by its own gravitational attraction. The analysis of such an object or "geon" combines both major topics of the present paper: the response of neutrinos to gravitational fields, and the production of gravitational fields by neutrinos.

Geons of toroidal form appear in principle to be more stable than spherical geons, at least when comparable numbers of quanta go around in the two opposite directions. The case in which the angular momentum is zero has been analyzed by Ernst.[24]

In the opposite extreme case where all the neutrinos go around the same way, then the right-handed character of the neutrinos implies that the resulting toroidal system is not mirror symmetric with respect to reflections in its own plane. It would be interesting to investigate the consequences of this asymmetry to see whether it shows up in the gravitational field near the toroidal geon.

Among spherically symmetrical geons perhaps the simplest to consider is a thermal geon.[23] A thermal neutrino geon differs from a thermal electromagnetic geon only in a trivial respect. The photon energy density in a thermal geon is

$$T_T{}^T = (\pi^2/15)(T^4/\hbar^3c^3)e^{-2\nu}(\omega/4\pi), \tag{99}$$

where T is the temperature (in energy units), $-e^{-\nu(r)} = g_{44}$ is the familiar factor in the metric, and $\omega/4\pi$ is the solid angle within which those rays travel which are trapped. The neutrino energy density in a thermal neutrino geon is similarly

$$T_T{}^T = (\pi^2/15)(T^4/\hbar^3c^3)e^{-2\nu}(\omega/4\pi)f(\eta). \tag{100}$$

Here the dimensionless factor,

$$f(\eta) = (15/2\pi^4)\left[\int_0^\infty \frac{x^3dx}{1+e^{x-\eta}} + \int_0^\infty \frac{x^3dx}{1+e^{x+\eta}}\right], \tag{101}$$

is uniquely fixed by the Fermi energy, $E_F = \eta T$ of the trapped neutrinos. All the analysis of thermal electromagnetic geons[23] can be taken over immediately to thermal neutrino geons. For this purpose it is only necessary to redefine the characteristic scale of length, R_T, as

$$R_T = (15\hbar^3c^7/8\pi^3 f(\eta)GT^4)^{\frac{1}{2}}, \tag{102}$$

and the characteristic unit of mass as

$$M_T = (15\hbar^3c^{11}/8\pi^3 f(\eta)G^3T^4)^{\frac{1}{2}}. \tag{103}$$

Then all the calculated graphs of reference 23 apply at once, and a complete description of the metric and energy distribution is available for a thermal neutrino geon.

In the other simple limiting case of a spherical neutrino geon, all of the energy is concentrated in a thin spherical shell. In ray language, all of the neutrinos are executing circular orbits of the same radius. Orbits of all orientations occur with equal probability. It is a matter of choice—that is, a matter of initial conditions—whether neutrino orbits of many different energies are occupied, or only states of a single energy are filled. We take the second option to simplify the analysis, although it demands a higher energy per neutrino to provide a given total mass. The wavelength of the neutrinos is then very small compared to the geon radius so the quantum number, k, is very large.

To formulate equations for the self-consistent metric, we proceed as in reference 12, introducing the ab-

[23] For further discussion, especially of unexplored analogies between gravitation physics and hydrodynamics, see E. Power and J. A. Wheeler, Revs. Modern Phys. **29**, 480 (1957).
[24] F. Ernst, Revs. Modern Phys. **29**, 496 (1957).

breviations

$$\begin{aligned} \Omega &= E/\hbar c, \\ \rho &= \Omega r, \\ d\rho^* &= e^{\frac{1}{2}\lambda-\frac{1}{2}\nu}d\rho, \\ e^{-\lambda} &= 1-2\rho^{-1}L(\rho), \end{aligned} \tag{104}$$

where $L(\rho)$ is a parameter that measures the mass included within the distance r;

$$e^{\lambda+\nu}=Q^2(\rho);$$

and

$$\begin{pmatrix} f(\rho) \\ g(\rho) \end{pmatrix} = [8\pi G(2k-1)E/c^4]^{\frac{1}{2}} \begin{pmatrix} F(r) \\ G(r) \end{pmatrix}, \tag{105}$$

where the radial functions F and G have the normalization

$$\int_0^\infty e^{\frac{1}{2}\lambda-\frac{1}{2}\nu}(F^2+G^2)dr=1. \tag{106}$$

The wave equation takes the form

$$d^2f/d\rho^{*2}+[1-(e^\nu k^2/\rho^2)+(d/d\rho^*)(e^{\frac{1}{2}\nu}k/\rho)]f=0, \tag{107}$$

and the Einstein field equations lead to the formulas

$$\begin{aligned} dL/d\rho^* &= (1/2Q)(f^2+g^2) \\ &= (1/2Q)[f^2+(df/d\rho^*)^2+(e^\nu k^2/\rho^2)f^2 \\ &\qquad -(e^{\frac{1}{2}\nu}k/\rho)df/d\rho^*], \\ dQ/d\rho^* &= (\rho-2L)^{-1}[f^2+(df/d\rho^*)^2]. \end{aligned} \tag{108}$$

These equations for the self-consistent field of a thermal geon are very similar to (38), (40), and (41) of reference 12 for electromagnetic geons. When we neglect terms in k compared to terms in k^2, the two systems of equations become identical. The solution for the case of large angular momentum[12] leads to the geon metric already used (43) in discussing the response of neutrinos to a given gravitational field. In summary, geons can be constructed in principle out of neutrinos in much the same way that they can be constructed in principle out of photons.

12. CONCLUSION

From our analysis of some of the interactions between neutrinos and gravitational fields we conclude that neutrino physics has an interesting character in and by itself, even when attention is withdrawn from all beta-ray transformations. The behavior of neutrinos has become a little clearer, but the mystery why spinors occur in nature is left as pressing as ever. What is there about the description of the geometry of space which is not already adequately covered by ordinary scalars, vectors, and tensors of standard tensor analysis? To this question the mathematics of spinor fields gives a well known answer: spinors allow one to describe rotations at one point in space completely independently of rotations at all other points in space—rotations that have nothing to do with the coordinate transformations that are treated in the usual tensor analysis. Fully to see at work this machinery of independent rotations at each point in space, we do best to consider the spinor field in a general curved space, as in this paper. But the deeper part of such rotations in the description of nature is still mysterious.

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