Remark on Cosmological Models

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IT is unusual to hypothesize that the four-dimensional space-time universe of general relativity is compact (i.e., "finite"). But in such a case several interesting conclusions can be drawn. In the first place, if the mass distribution is assumed to be continuous, so that the metric tensor has no singularities, then the Euler-Poincaré characteristic of the universe must be zero.[1] This implies, for example, that *the universe cannot be a four-dimensional sphere*. It also implies that *a finite universe cannot be simply-connected*, in the sense that its first Betti number cannot vanish. (This is reminiscent of Professor J. A. Wheeler's nonsimply-connected models.)

In the second place, it seems to be generally known that *in a finite cosmology there must exist a closed curve in space-time whose tangent vector at every point is time-like*. Professor L. Markus has indicated a proof to us. Let V_4 denote the 4-manifold of the universe. Now, on V_4 construct a continuous, nowhere vanishing field of time-like vectors.[2,3] By Birkhoff's fundamental theorem on the existence of recurrent orbits in compact dynamical systems,[4] there must exist either an orbit of the type sought or else an "almost-closed" time-like orbit which can serve for the construction of such a closed orbit by an obvious procedure.

A more standard hypothesis, however, is that the universe V_4 is not compact, but is the topological product of the infinite real line (a time axis) with a 3-manifold V_3. The manifold V_3 is often assumed to be compact, and any local (hence experimentally verifiable) condition which implies compactness is of much interest. For example, if V_3 has constant curvature K, then V_3 is compact if K is positive,[5] and in this case is a 3-sphere if its first Betti number vanishes, and in general admits the 3-sphere as a covering space.

We wish to point out a new method for studying the topology of manifolds such as V_3 and V_4. This method consists of the construction of a continuous, nowhere vanishing, irrotational vector field on the manifold under consideration. Once such a vector field has been constructed, we can assert that either the manifold is noncompact (i.e., open or "infinite"), or that it cannot be simply connected.

We shall prove a slight generalization of this theorem; but first, let us note that a similar, but much more restrictive and less easily applicable condition is a trivial consequence of Hodge's well-known theorem that the number of linearly independent harmonic vector fields on a compact Riemannian manifold is equal to its first Betti number. For if after constructing on our manifold an irrotational vector field (which is nontrivial, but may vanish at more than one point), we then ascertain that it is also solenoidal (i.e., of vanishing divergence), then the vector field must be harmonic.[6]

Theorem 1 (Hodge). *Let V_n be an n-dimensional Riemannian manifold* (with positive definite metric tensor), *and let F denote a nontrivial class C_2 vector field defined on V_n. Suppose that the curl and the divergence of F both vanish identically; or equivalently, suppose that the field F satisfies the generalized Laplace equation for harmonic vector fields. Then, if V_n is compact, its first Betti number is not zero.*

Corollary (Bochner-Myers). *If V_n is orientable and has positive definite Ricci curvature throughout, then its first Betti number vanishes.*[7]

Recall that the curl tensor of a vector field is independent of the metric tensor, and so is a nonmetric notion. Accordingly, *the following theorem applies equally well to V_4 with its indefinite hyperbolic metric as to V_3 with its positive definite Riemannian metric.*

Theorem 2. *Let V_n be an n-dimensional differentiable manifold, and let F be a continuous, class C^1 vector field defined on V_n. Suppose that F vanishes at most once and that its curl vanishes identically on V_n. Then, either V_n is noncompact, or V_n is compact and its first Betti number does not vanish. In either case, of course, if F actually vanishes nowhere, the Euler-Poincaré characteristic of V_n is zero.*

For nonvanishing F this theorem is a consequence of a more general theorem[8] which applies, for example, to manifolds with boundary. In fact, by a generalization to arbitrary flows of a theorem proved by Lichnerowicz for a very special class of flows,[9] we can prove[10] that V_n is homeomorphic to the product of the real line with an $(n-1)$-dimensional space which is a connected subset of a V_{n-1}. But in the present case, because we are deal-

[1] L. Markus, Ann. Math. **62**, 411–417 (1955).

[2] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Électromagnétism* (Masson et Cie, Paris, 1955), pp. 6, 7.

[3] N. Steenrod, *The Topology of Fibre Bundles* (Princeton University Press, Princeton, 1951), p. 207.

[4] G. D. Birkhoff, *Dynamical Systems* (American Mathematical Society, New York, 1927).

[5] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1949), pp. 84, 203.

[6] K. Yano and S. Bochner, "Curvature and Betti Numbers," Annals of Mathematics Studies No. 32, Princeton, 1953, p. 56.

[7] See reference 6, p. 37.

[8] R. W. Bass, "A criterion for the existence of unstable sets," a RIAS preprint, December 26, 1956. Copies available on request.

[9] See reference 2, p. 79.

[10] R. W. Bass, "Topological dynamics and nonlinear differential equations," a RIAS preprint (to be published).

ing with a manifold, there is a much simpler proof. We wish to thank Professor Kervaire for pointing out to us this simpler proof during the 1957 North Carolina Conference on Gravitation and General Relativity. The proof runs as follows. If V_n is simply connected, then the generalized Stokes theorem assures us that there exists on V_n a single-valued scalar potential function of which F is the gradient field. (See the survey of vector analysis in the paper by H. Weyl.[11]) But if V_n is compact, this potential function must assume both its maximum and minimum values on M, and at these extreme points the gradient must vanish. This contradicts the hypothesis that F has at most one zero on V_n, and so proves the theorem.

It is possible that Theorems 1 and 2 have applications to the study of specific cosmological models. In fact, there are many ways of constructing on V_3 or on V_4 continuous vector fields which are unique once the indefinite metric (or set of gravitational potentials) for V_4 has been specified.

Professor J. A. Wheeler has pointed out to us an application of Theorem 2 to V_4.

Theorem 3. *Consider the combined Einstein-Maxwell field theory on V_4. If the vector field $u_i = \epsilon_{ijkl} R_m{}^{j;l} R^{mk} / R_{pq} R^{pq}$ is defined every where and of class C^1 on V_4, and if u_i does not vanish more than once, then the universe V_4 cannot be compact.*

The vector field u_i, which was defined by Dr. C. Misner, is essentially the gradient of the ratio (in a certain coordinate system) of the electric to the magnetic field strength. Dr. Misner has shown that Maxwell's equations imply that $u_{i;j} - u_{j;i} \equiv 0$. Hence, Theorem 3 follows from Theorem 2.

[11] H. Weyl, Duke Math. J. 7, 411–444 (1940).