of coalescence is a fascinating problem for the future, the answer to which might help to decide between these alternatives or some other possible outcome.

When the disposition of the N-mass centers in the lattice universe is almost but not quite symmetric, a situation arises much like that in a Freidmann universe when the mass distribution departs slightly from uniformity. Unless the initial conditions are very special, the magnitude of the disturbance will grow. When underwater bubbles undergo dilatational oscillations, and when the surface departs slightly from spherical symmetry at the phase of maximum expansion, then the magnitude of the disturbances ordinarily grows. In this case one can follow the phenomena far enough visually to see that prongs and spikes form. The impression is gained that the bubble changes over from contraction to expansion, not everywhere simultaneously over its surface, but more after the fashion of a glove being turned inside out one finger at a time. If the analogy is any guide, the not quite symmetrical lattice universe will be expected to show a similar behavior. One will expect first a few Schwarzschild singularities to amalgamate and then break apart, then others to fall in, amalgamate, and break apart again, and so on, with some parts of the system therefore still contracting while others have already begun reexpansion. To show the beginnings of such a behavior, a perturbation theory analysis of the regular lattice universe should suffice. To follow the later and more interesting phases of the turnabout would demand a much more elaborate scheme of analysis.

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Observables in Singular Theories by Systematic Approximation*†

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1. INTRODUCTION

In any general-relativistic theory the field variables necessarily carry some information that relates to the choice of frame of reference rather than to the physical situation. Two manifestly different fields of gravitational potentials may describe one and the same gravitational field, merely in terms of two different coordinate systems. General relativity differs from special relativity in the degree of freedom inherent in the choice of coordinate system. Whereas in a Lorentz-covariant theory the frame of reference may be chosen at one instant in time and then remains fixed (the whole freedom of choice reducing to the determination of ten parameters once and for all), the freedom in a general-relativistic theory amounts to the determination of four arbitrary functions throughout space, anew at each instant in time. As a result of this vast freedom of choice, general-relativistic dynamical laws cannot be expected[1,2] to permit the integration of the field equations in the sense that suitable initial-value conditions at one time t_0 predict the value of any component of the metric tensor $g_{\mu\nu}$, or, for that matter, of any other conventional field variable, at some space point x^s at a different time t. Nevertheless, Einstein's general theory of relativity is quite deterministic. Its field equations do determine the gravitational field from initial-value conditions for all times to come. The metric potentials are simply not the quantities that are determined completely by the physical situation.

Failure of the mathematical theory to predict the value of a field variable at a given world point corresponds to physical unobservability. It is impossible to devise an experiment that will measure some field at the world point with the coordinates x^μ, because the values of the coordinates by themselves do not identify that world point. In actual practice, a world point at which some measurement is to be made is always identified in some other manner, such as the convergence of a beam of light or the location of a material component of our instrumentation. The determination of a gravitational potential at a world point and in directions that are defined by the values of specified electromagnetic quantities represents, of course, something different from the determination of that same potential at a world point specified by nothing but

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[1] P. G. Bergmann, Phys. Rev. **75**, 680 (1949).

[2] P. G. Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1953).

coordinate values. The theory may be expected to predict (from initial-value conditions) the former, but not the latter.

A quantity that can be predicted by the theory and that is not trivial (such as the value of a numeric) will be called an *observable*.[3] In a Lorentz-covariant theory all dynamical variables may be observables in this technical sense (though in electromagnetic theory only gauge-invariant quantities are observables); in a general-relativistic theory usually none of the ordinary fields are. But if a field variable cannot even be predicted in c-number theory, then surely it cannot have an expectation value in a quantized version of the theory, either. Hence the construction of observables in our sense is a necessary preliminary for quantizing any theory in which the original variables of the theory do not have that property.

If the dynamical laws of a theory have the form of partial differential equations and if the theory is to predict observables at a time t from suitable initial-value conditions at a time t_0, then knowledge of the values of a sufficient number of observables at any intermediate time t', between t_0 and t, must permit the same prediction. We shall call such a set of observables at a time t' that permits prediction at other times a *complete set of observables*. The construction of a complete set of observables is the central theme of this paper.

Mathematically, the failure of a general-relativistic theory to permit prediction of ordinary field variables by integration expresses itself in the "singular" character of its action principle. If, as is usually the case, the Lagrangian of the action principle is a function of the field variables themselves and of their first partial derivatives, the resulting field equations will be of the second order. If we now separate the derivatives in a particular direction (say with respect to x^4) from the remainder and focus attention on the second "time" derivatives, it turns out that the field equations cannot be solved with respect to them. Even though the number of field equations equals the number of field variables, (at least) four linear combinations of the second "time" derivatives are completely unrestricted by the field equations, whereas (at least) four independent combinations of the field equations contain no reference to the second "time" derivatives at all. In a canonical (Hamiltonian) formulation, "singular" action principles lead to Hamiltonians that are only partially determined and to constraining algebraic relations between the canonical field variables.[4] Exactly the same situation is also met with in electrodynamics, where it results from the arbitrariness of the gauge frame.

Complete sets of observables are fairly easily obtained in electrodynamics and in other linear and quasi-linear theories.[5,6] Because of its much greater complexity, the general theory of relativity has so far not permitted the construction of any observables, much less a complete set. In this paper we develop a systematic method of construction by means of successive approximations, the lowest nontrivial stage corresponding to the linearized theory. As in other methods of successive approximations in physics, the question of convergence remains unanswered. Likewise, we have developed our scheme as a procedure "in the small," leaving all questions of topology aside for the time being. What we have accomplished is to formulate a procedure that at every stage leads to solvable problems. Both the method as such and its products, variables that approximate observables, lend themselves to intuitive interpretation.

The strategy of our procedure is primarily adapted to a general-relativistic theory, or more particularly to Einstein's theory of gravitation (general relativity), but is applicable to any theory whose singular character is the result of invariance properties. We shall assume that there exists a known trivial solution of the field equations (corresponding to the Minkowski metric), which consists of a constant field. The expansion is in terms of deviation from this trivial standard solution. The lowest nontrivial equations are then the linearized field equations. They correspond to an action principle that is singular (in the sense used above) and invariant with respect to a transformation group that has some similarity with the group of curvilinear coordinate transformations. At each successive stage we construct a more complex Lagrangian (nth order Lagrangian) that is singular and invariant with respect to a transformation group that increasingly resembles the group of coordinate transformations. Solving the Euler-Lagrange equations of the nth order involves correcting the $(n-1)$st-order solutions by solving a linear problem. Likewise, once we have constructed the observables corresponding to the first-order problem, constructing the observables of the next stage requires corrections that are obtained from x linear conditions. All of our work will be carried out by means of Hamiltonian (rather than Lagrangian) techniques.

Because the notation gets quite involved, we present the method in terms of a problem involving a system with a finite number of degrees of freedom, in other words a contrived "mechanical" system. Also to simplify the presentation, we assume that only first-class constraints are present.

2. INVARIANCE PROPERTIES OF THE APPROXIMATE THEORY

We assume that the exact equations of motion are derivable from an action principle, with a known Lagrangian $L(q_i,\dot{q}_i)$. The coordinate variables are then written as a power expansion in some small parameter:

$$q_i=\sum_{a=0}^{\infty} \epsilon^a q_{(i\ a)}. \tag{2.1}$$

[3] P. G. Bergmann and I. Goldberg, Phys. Rev. **98**, 531 (1955).
[4] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); **3**, 1 (1951).
[5] P. A. M. Dirac, Can. J. Phys. **33**, 650 (1955).
[6] P. G. Bergmann, Nuovo cimento **3**, 1177 (1956).

The zeroth-order variables $q_{i,0}$ are taken to be constant. They are, for example, the components of the Minkowski metric in gravitational theory. This method of approximation is very similar to the Einstein-Infeld-Hoffmann (EIH) method. The equations of motion for each variable of different order are linear nonhomogeneous equations. The nonhomogeneous part contains the variables of the previous orders. For convenience we shall call this hierarchy of equations the EIH equations. The immediate problem is to find a Lagrangian that will yield the EIH equations, to any order, as Euler equations. We shall show that when the full Lagrangian is expanded in powers of ϵ, the ath order Lagrangian will yield the first a EIH equations. To prove this write the total action

$$S=\int Ldt,$$

and vary the total q_i's. This gives

$$\delta S=\int \delta L dt=\int L^i \delta q_i dt, \quad (2.2)$$

where L^i stands for the total Euler equations and the repeated index i denotes summation (Sums over indices denoting different orders will be indicated explicitly.) Since

$$L^i=\sum_{a=1}^{\infty} \epsilon^a L^i{}_a, \quad \delta q_i=\sum_{b=1}^{\infty} \epsilon^b \delta q_{(i,\,b)}, \quad (2.3)$$

where $L^i{}_a$ are the EIH equations, (2.2) yields

$$\delta S=\int \sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \epsilon^{a+b} L^i{}_a \delta q_{(i,\,b)} =\sum_{c=2}^{\infty} \epsilon^c \sum_{a=1}^{c-1} \int L^i{}_a \delta q_{(i,\,c-a)} dt. \quad (2.4)$$

Instead of first varying the action and substituting the expanded variables as in (2.2) and (2.3), substitute the expanded variables and then vary the action. If

$$L=\sum_{1}^{\infty} \epsilon^{a+1} L_a$$

(here we make the assumption that the Lagrangian is at least quadratic) then the action integral reads

$$S=\sum_{1}^{\infty} \epsilon^{a+1} \int L_a dt.$$

The variation of S gives

$$\delta S=\sum_{1}^{\infty} \epsilon^{a+1} \int \delta L_a dt=\sum_{a=1}^{\infty} \epsilon^{a+1} \int \sum_{b=1}^{a} L_a{}^{(i,\,b)} \delta q_{(i,\,b)} dt =\sum_{a=2}^{\infty} \epsilon^a \sum_{b=1}^{a-1} \int L_{a-1}{}^{(i,\,b)} \delta q_{(i,\,b)} dt, \quad (2.5)$$

where $L_a{}^{(i,\,b)}$ are the Euler equations of the Lagrangian L_a, corresponding to the variable $q_{(i,\,b)}$. By comparing the coefficients of ϵ in (2.4) and (2.5), we have the identity

$$\sum_{b=1}^{a-1} \int L^i{}_b \delta q_{(i,\,a-b)} dt=\sum_{b=1}^{a-1} \int L_{a-1}{}^{(i,\,b)} \delta q_{(i,\,b)} dt. \quad (2.6)$$

Since the variations in the $q_{(i,\,b)}$'s are to be taken independently, the coefficients in the integral must be identical giving

$$L^{i}{}_{a-b} \equiv L_a{}^{(i,\,b+1)} \equiv L_{a+d}{}^{(i,\,b+d+1)}. \quad (2.7)$$

This means that the $(a-b)$th EIH equation is equal to the $(b+1)$st Euler equation of the ath Lagrangian, L_a. For a fixed a we can get the first a EIH equations by letting b go from zero to $a-1$.

According to E. Noether's theorem, a theory which possesses an invariance group depending on arbitrary functions also possesses differential identities among the Euler equations, which we may call the generalized Bianchi identities. We derive the generalized Bianchi identities with an assumed transformation law, and by expanding them in a power series, show that the Euler equations of the ath Lagrangian possesses Bianchi identities as well. It will also be necessary to derive the transformation law under which the ath Lagrangian is invariant.

Assume the total Lagrangian to be invariant under the infinitesimal transformation

$$\bar{\delta} q_i=F_i{}^j q_j \xi-\dot{q}_i \xi. \quad (2.8)$$

The bar distinguishes an infinitesimal transformation from an arbitrary variation of q_i. This transformation is the analog of the gauge and coordinate transformations in electrodynamics and relativity, respectively. We get the Bianchi identities by noting that[7]

$$\bar{\delta} L \equiv \dot{Q} \equiv \frac{\partial L}{\partial q_i}\bar{\delta} q_i+\frac{\partial L}{\partial \dot{q}_i}\bar{\delta}\dot{q}_i \equiv L^i \bar{\delta} q_i+\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\bar{\delta} q_i\right). \quad (2.9)$$

Substituting (2.8) into (2.9) and differentiating once by parts we obtain

$$\left[\dot{q}_i L^i+\frac{d}{dt}(F_i{}^j L^i q_j)\right]\xi \equiv \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\bar{\delta} q_i-Q+F_i{}^j q_j L^i \xi\right]. \quad (2.10)$$

Since the ξ's are completely arbitrary, we have

$$\frac{d}{dt}(L^i F_i{}^j g_j)+L^i \dot{q}_i \equiv 0, \quad (2.11)$$

the Bianchi identities of the full theory. The power series of these identities is obtained by straight sub-

[7] P. G. Bergmann and R. Schiller, Phys. Rev. 89, 4 (1953).

stitution from (2.3)

$$F_i{}^j\frac{d}{dt}(L^i{}_1 q_{(j,0)})\equiv 0,$$
$$F_i{}^j\sum_{a=1}^{c}\frac{d}{dt}(L^i{}_a q_{(j,c-a)})+\sum_{a=1}^{c-1}L^i{}_a\dot{q}_{(i,c-a)}\equiv 0,\quad c\geqslant 2. \tag{2.12}$$

To discuss the Bianchi identities of the approximate theory, we derive the transformation law for the $q_{(i,a)}$ of different order. By writing

$$\xi=\sum_{a=0}^{\infty}\epsilon^a\xi_a, \tag{2.13}$$

and by replacing q_i and $\dot{q}_i$ by their power expansion in (2.8), we get

$$\bar{\delta}q_i=\sum_{a=1}^{\infty}\epsilon^a\bar{\delta}q_{(i,a)}$$
$$=F_i{}^j\sum_{a=0}^{\infty}\epsilon^a q_{(j,a)}\sum_{b=0}^{\infty}\epsilon^b\dot{\xi}_b-\sum_{a=1}^{\infty}\sum_{b=0}^{\infty}\epsilon^{a+b}\dot{q}_{(i,a)}\xi_b. \tag{2.14}$$

Reordering terms in the sums and equating the different coefficients of ϵ^a we obtain[8]

$$\bar{\delta}q_{(i,a)}=F_i{}^j\sum_{b=0}^{a}q_{(j,a-b)}\dot{\xi}_b-\sum_{b=0}^{a-1}\dot{q}_{(i,a-b)}\xi_b. \tag{2.15}$$

The requirement that $q_{(i,0)}$ equal a fixed constant results in ξ_0 being a constant. In general relativity, likewise, the $g_{(\mu\nu,0)}$ will remain Minkowskian only if the zeroth-order coordinate transformations are Lorentz transformations.

We can establish whether the transformation law (2.15) will leave a particular Lagrangian L_b invariant. To do so we assume that the Lagrangian *is* in fact invariant under the law (2.15), and then show that the resulting Bianchi "identities" are actually identically satisfied by virtue of (2.12).[9]

We perform the identical operations that we used to obtain (2.11), except that we replace L by L_a, and the transformation law (2.8) by (2.15):

$$\sum_{b=1}^{a}\sum_{c=1}^{b}F_i{}^j\frac{d}{dt}(L_a{}^{(i,b)}q_{(j,b-c)})\xi_c+\sum_{b=1}^{a}\sum_{c=1}^{b-1}L_a{}^{(i,b)}\dot{q}_{(i,b-c)}\xi_c\equiv 0. \tag{2.16}$$

If we now interchange the sums, being careful to use the correct new ranges of summation, then because the ξ_c's are arbitrary, we obtain

$$F_i{}^j\frac{d}{dt}(L_a{}^{(i,a)})q_{(j,0)}\equiv 0,$$
$$\sum_{b=c}^{a}F_i{}^j\frac{d}{dt}(L_a{}^{(i,b)}q_{(j,b-c)})+\sum_{b=c+1}^{a}L_a{}^{(i,b)}\dot{q}_{(i,b-c)}\equiv 0. \tag{2.17}$$

After use of (2.7) and some relabeling of the indices, (2.17) reduces exactly to (2.12). This completes the proof that the approximate theory possesses the same invariance properties as the full theory to any order in the approximation. Although in the full theory there is only one Bianchi identity for each ξ, in the "truncated" theory to order a, there are a identities. The reason for this is that the transformation group now depends on the a arbitrary functions, ξ_a, instead of just one.

3. ABSENCE OF FALSE CONSTRAINTS

The matrix $L^{ij}\equiv\partial^2L/\partial\dot{q}_i\partial\dot{q}_j$ is singular in all theories that possess an invariance group depending on arbitrary functions; the converse, however, is not true. L^{ij} may be singular and possess no invariance group; in that case the singularity corresponds to second-class constraints (constraints having nonvanishing Poisson brackets with each other) in the canonical formalism. With our assumption of only one arbitrary function in the invariant transformation law and the existence of only first-class constraints, the equation $L^{ij}u_i=0$ has only one independent solution and there are exactly two first-class constraints.[10] The first constraint (called *primary* constraint) expresses the fact that the velocities are not uniquely determined by the q's and p's. The second constraint (*secondary* constraint) arises from the consistency requirement that the time derivative of the primary constraint must vanish. The matrix corresponding to L^{ij} in the approximate theory is

$$L_e{}^{(i,a)(j,b)}\equiv\frac{\partial^2 L_e}{\partial\dot{q}_{(i,a)}\partial\dot{q}_{(j,b)}}.$$

Since the Lagrangian L_e is invariant under e transformations, the equation

$$L_e{}^{(i,a)(j,b)}u_{(i,a)}=0 \tag{3.1}$$

must have at least e solutions. The question arises as to whether the approximation procedure introduces additional solutions to (3.1), that is, whether it introduces second-class constraints. We will prove that this is not the case.

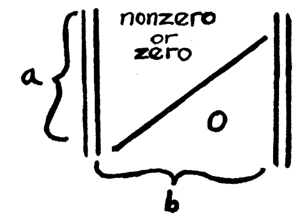
Because the original Lagrangian is assumed to be quadratic in the velocities, and because the expansion

[8] By showing that the Jacobi identity is satisfied, the group property of the transformation (2.15) can be verified.

[9] The fact that the Bianchi identities imply the invariance of the theory can be seen by replacing $\dot{Q}$ by F in Eq. (2.9). Continuing in the same manner as in the text, we get an equation similar to (2.11). If the quantities called the Bianchi identities *are* identically zero, the F must be equal to $\dot{Q}$, which is the condition for an invariant transformation.

[10] Bergmann, Janis, Goldberg, and Newman, Phys. Rev. **103**, 807 (1956).

was written $L=\sum \epsilon^{a+1}L_a$, the matrix $L_e{}^{(i,a)(j,b)}$ takes the form



(3.2)

where every element stands for an $n\times n$ matrix, $i, j=1\cdots n$. Accordingly, (3.1) can be written

$$\sum_{d=1}^{c} L_e{}^{(j,e-c+1)(i,d)}u_{(i,d)}=0, \quad c=1\cdots e. \tag{3.3}$$

We now express $L^{ij}u_i$ in a power series in ϵ, and equate the coefficients to zero. These coefficients will equal (3.3); because of the uniqueness of the power expansion of u_i. Before we can perform this derivation, we need a lemma, which is also important in the next section.

If we consider x as a general variable, standing for q, $\dot{q}$, or p and write $x=\sum \epsilon^a x_a$, then any function of x can be expressed as a power series in ϵ, $f(x)=\sum \epsilon^a f_a(x_b)$. Our lemma can be stated in the form

$$\frac{\partial f_e}{\partial x_{e-d}}\equiv\frac{\partial f_{b+d}}{\partial x_b}\equiv\frac{\partial f_{c+d}}{\partial x_c}. \tag{3.4}$$

The proof is simple. We write $f(x)=f(\sum \epsilon^a x_a)$ and note

$$\frac{\partial f}{\partial x}=\epsilon^{-b}\frac{\partial f}{\partial x_b}=\epsilon^{-c}\frac{\partial f}{\partial x_c}. \tag{3.5}$$

The indicated differentiation operations are then performed on the expanded $f(x)$, and the coefficients of different powers of ϵ are compared; hence (3.4).

We are now ready to expand the equation $L^{ij}u_j=0$. If

$$u_i=\sum_{a=0}^{\infty} \epsilon^a u_{(i,a+1)} \tag{3.6}$$

(the form of the summation is just for convenience) and

$$L^{ij}=\sum_{a=0}^{\infty} \epsilon^a \frac{\partial^2 L_{a+b+c-1}}{\partial \dot{q}_{(i,b)}\partial \dot{q}_{(j,c)}} =\sum_{a=0}^{\infty} \epsilon^a L_{a+b+c-1}{}^{(i,b)(j,c)}, \quad b, c \text{ arbitrary}, \tag{3.7}$$

where we have used the power expansion of L and (3.5) twice, then

$$L^{ij}u_i=\sum_{d=0}^{\infty}\sum_{a=0}^{\infty} \epsilon^{a+d}L_{d+b+c-1}{}^{(i,b)(j,c)}u_{(i,a+1)} =\sum_{f=0}^{\infty}\epsilon^f\sum_{a=0}^{f} L_{f-a+b+c-1}{}^{(i,b)(j,c)}u_{(i,a+1)}. \tag{3.8}$$

After equating the coefficients of ϵ to zero and using (3.4) in the following form

$$\frac{\partial^2 L_{f-a+b+c-1}}{\partial \dot{q}_{(i,b)}\partial \dot{q}_{(j,c)}}\equiv\frac{\partial^2 L_e}{\partial \dot{q}_{(i,b)}\partial \dot{q}_{(j,e-f+a-b+1)}}\equiv L_e{}^{(i,b)(j,e-f+a-b+1)},$$

we obtain

$$\sum_{a=0}^{f} L_e{}^{(i,b)(j,e-f+a-b+1)}u_{(i,a+1)}=0. \tag{3.9}$$

By relabeling the indices and setting b equal to $a+1$, we obtain exactly (3.3). The arbitrariness in the choice of b yields, for example, the following e linearly independent null vectors of L

$$\begin{Vmatrix} u_{(i,1)} \\ \cdot \\ \cdot \\ \cdot \\ u_{(i,e)} \end{Vmatrix}, \begin{Vmatrix} 0 \\ u_{(i,1)} \\ \cdot \\ \cdot \\ u_{(i,e-1)} \end{Vmatrix}, \cdots \begin{Vmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ u_{(i,1)} \end{Vmatrix}. \tag{3.10}$$

The matrix (3.2), being of order e can have at most e independent null vectors; if there were other null vectors, they would have to come from the n by n submatrices. Since the power expansion of L is unique and there is only one u_i which is a null vector, there can be only one of each type of the e vectors (3.10). This proves that there are only e null vectors of $L_e{}^{(i,a)(j,b)}$ and since the most general transformation of the "truncated" theory to order e contains e arbitrary functions, there can be no second class constraints.

4. CANONICAL FORMALISM

Construction of a Hamiltonian formalism from the Lagrangian L_e proceeds along lines indicated by Dirac.[4] It differs from the usual procedure in that the Hamiltonian is not a unique function of the canonical variables; a linear combination of primary constraints can be added. Though construction of the Hamiltonian is easy, it must be shown what relationship it bears to the expansion of the complete Hamiltonian formalism.

First write the expansion of the exact p_i in the form

$$p_i=\sum_{1}^{\infty} \epsilon^a \bar{p}_{(i,a)}, \tag{4.1}$$

and try to connect the $\bar{p}_{(i,a)}$'s with the

$$p_{e(i,a)}=\frac{\partial L_e}{\partial \dot{q}_{(i,a)}}.$$

This can be done as follows

$$\begin{aligned} p_i&=\frac{\partial L}{\partial \dot{q}_i}=\sum_{a=b}^{\infty}\epsilon^{a-b+1}\frac{\partial L_a}{\partial \dot{q}_{(i,b)}}=\sum_{c=1}^{\infty}\epsilon^c\frac{\partial L_{b+c-1}}{\partial \dot{q}_{(i,b)}} \\ &=\sum_{c=1}^{\infty}\epsilon^c\frac{\partial L_e}{\partial \dot{q}_{(i,e-c+1)}}=\sum_{c=1}^{\infty}\epsilon^c p_{e(i,e-c+1)}, \end{aligned} \tag{4.2}$$

using (3.4) and (3.5). Hence from (4.1) and (4.2)

$$\bar{p}_{(i,a)}=p_{e(i,e-a+1)}. \tag{4.3}$$

In the power expansion of any function $A(q,p)$, the $\bar{p}_{(i,a)}$'s must be replaced with the aid of (4.3) in order to obtain the variable or variables in the approximate theory which are equivalent to the A in the full theory. It is easy to show that the eth term in the Hamiltonian constructed in this manner is equal to the Hamiltonian formed from the eth Lagrangian,

$$H_e=\sum_{a=1}^{e} p_{e(i,a)}\dot{q}_{(i,a)}-L_e. \tag{4.4}$$

We must obtain a connection between the Poisson bracket of two functions in the full theory and the Poisson bracket in the approximate theory. This relationship may be stated in its simplest form by the equation

$$(A,B)=\sum_{1}^{\infty}\epsilon^{e-1}(A_e,B_e)^e, \tag{4.5}$$

where the bracket with the superscript e means the Poisson bracket of A_e with B_e using the variables $q_{(i,1)}\cdots q_{(i,e)}$ and $p_{e(i,1)}\cdots p_{e(i,e)}$. In the proof of (4.5) we suppress the coordinate index i.

Using (3.5), we can write

$$\frac{\partial A}{\partial q}=\sum_{a=b}^{\infty}\epsilon^{a-b}\frac{\partial A_a}{\partial q_b}=\sum_{e=0}^{\infty}\epsilon^{e}\frac{\partial A_{b+e}}{\partial q_b},$$
$$\frac{\partial B}{\partial p}=\sum_{c=d}^{\infty}\epsilon^{c-d}\frac{\partial B_c}{\partial \bar{p}_d}=\sum_{f=0}^{\infty}\epsilon^{f}\frac{\partial B_{d+f}}{\partial \bar{p}_d}, \tag{4.6}$$

from which follows

$$\frac{\partial A}{\partial q}\frac{\partial B}{\partial p}=\sum_{g=0}^{\infty}\epsilon^{g}\sum_{e=0}^{g}\frac{\partial A_{b+e}}{\partial q_b}\frac{\partial B_{d+g-e}}{\partial \bar{p}_d}. \tag{4.7}$$

With the aid of (3.4), this can be rewritten

$$\frac{\partial A}{\partial q}\frac{\partial B}{\partial p}=\sum_{g=0}^{\infty}\epsilon^{g}\sum_{e=0}^{g}\frac{\partial A_b}{\partial q_{b-e}}\frac{\partial B_d}{\partial q_{d-g+e}}. \tag{4.8}$$

The $\bar{p}$'s can be eliminated by the equation $\bar{p}_{d-g+e}=p_{a(a-d+g-e+1)}$, obtained from (4.3), yielding

$$\frac{\partial A}{\partial q}\frac{\partial B}{\partial p}=\sum_{g=0}^{\infty}\epsilon^{g}\sum_{e=0}^{g}\frac{\partial A_b}{\partial q_{b-e}}\frac{\partial B_d}{\partial p_{a(a-d+g-e+1)}}. \tag{4.9}$$

Since the b, d, and a are arbitrary, we choose them to be equal to $g+1$, with the result

$$\frac{\partial A}{\partial q}\frac{\partial B}{\partial p}=\sum_{g=0}^{\infty}\sum_{e=0}^{g}\frac{\partial A_{g+1}}{\partial q_{g+1-e}}\frac{\partial B_{g+1}}{\partial p_{g+1(g+1-e)}}. \tag{4.10}$$

By relabeling the indices we obtain

$$\frac{\partial A}{\partial q}\frac{\partial B}{\partial p}=\sum_{a=1}^{\infty}\epsilon^{a-1}\sum_{d=1}^{a}\frac{\partial A_a}{\partial q_d}\frac{\partial B_a}{\partial p_{a(d)}}. \tag{4.11}$$

After antisymmetrizing with respect to A and B, (4.5) results.

Another relation which is important in the next section is

$$(A_a,B_b)^c\equiv(A_a,B_{b+1})^{c+1}\equiv(A_{a+1},B_b)^{c+1},\quad c\geqslant a,\,b. \tag{4.12}$$

This can be proven simply by a direct calculation. It must be emphasized that the coefficients of the expanded functions of the q's and p's, though unique functions of the q_a's and $\bar{p}_a$'s, are not unique functions of the unbarred $p_{e(a)}$'s; the subscript e being arbitrary; (4.12) is to be understood as an identity in the $\bar{p}_a$'s.

The consistency of our approximation method with the full Hamiltonian theory can be tested. From the Poisson bracket $(q,p)=1$, it should be possible to derive the relations $(q_a,p_{e(b)})^e=\delta_{ab}$. That this is true can be seen from (4.5) and (4.12). When A and B are replaced by q and p in (4.5), we get the result

$$(q_1,\bar{p}_1)^1=(q_1,p_{1(1)})^1=1,$$
$$(q_e,\bar{p}_e)^e=(q_e,p_{e(1)})^e=0, \tag{4.13}$$

where we have used (4.3) to eliminate the barred p's. With the aid of (4.12) applied to (4.13) we can obtain the entire system of Poisson bracket relations.

5. INVARIANTS IN THE APPROXIMATE THEORY

When a classical theory contains a general invariance group (depending on arbitrary functions rather than arbitrary parameters), the velocities cannot be solved as unique functions of the canonical variables from the equation

$$p_i=\frac{\partial L(q,\dot{q})}{\partial \dot{q}_i}. \tag{5.1}$$

The velocities can be eliminated from some of them yielding relationships between the canonical variables that must be satisfied if the Hamiltonian formalism is to be equivalent to the Lagrangian formalism. These relations, called primary first-class constraints, are denoted by $C_s(q,p)=0$. The number of these constraints is equal to the number of arbitrary functions in the invariant transformation law. For consistency, the time derivative of these constraints must vanish when the constraints themselves are satisfied. If this does not happen automatically then additional constraints must be imposed, $(C_s,H)=0$. These are called *first-class* secondary constraints. It is assumed here that no tertiary constraints appear. When we deal with the eth-order Lagrangian L_e, e primary constraints and e secondary constraints (if the full theory possesses one primary constraint) will appear.

According to Bergmann and Goldberg[3] the canonical variables are not the most appropriate variables for constructing the Hamiltonian formalism and transformation theory; the "observables" are the (maximum number of) independent functions of the canonical variables that have vanishing Poisson brackets with the constraints. They do not change their value or form under the invariant transformations of the theory. All quantities of physical interest can be expressed as functions of the observables. Their number in a given theory can be calculated readily to be $(2n-2n_0)$, $2n$ being the number of canonical variables and n_0 the number of constraints (this is not true if some of the constraints are second-class). In our case, due to our assumptions, the full theory possesses $(2n-2)$ observables and the eth approximation $(2n-2)e$.

The purpose of this section is to show how to construct the $(e+1)$th observables of the $(e+1)$-st approximation if we know the e observables of the eth approximation. If we write the observables as $O_a{}^u$, a going from 1 to e and u going from 1 to $(2n-2)$ and the constraints $C_a{}^s$, s going from 1 to 2, we shall show that the O_a's are the power expansion of the full O's. By replacing A and B in (4.5) by O^u and C^s and remembering that the full observables and the constraints must commute, we obtain

$$(O_a{}^u, C_a{}^s)^a = 0, \quad a = 1 \cdots e. \tag{5.2}$$

Repeated application of the identity (4.12) to (5.2) yields

$$(O_a{}^u, C_b{}^s)^e = 0. \tag{5.3}$$

This is just the condition for the $O_a{}^u$'s to be observables.

The first e observables of the $(e+1)$ approximation are just the e observables that we already know, the only difference being that all $p_{e(a)}$ are to be replaced by $p_{e+1(a+1)}$. That they commute with the constraint $C_{e+1}{}^s$ is proven by applying (4.12) once to (5.3) and taking $b=e$.

The problem that remains to be solved is how to construct the $(e+1)$st observables. In what follows we consider all functions as functions of the $\bar{p}$'s to simplify the notation. From (3.4)

$$\frac{\partial O_{e+1}{}^u}{\partial q_{(i,\,a+1)}} = \frac{\partial O_e{}^u}{\partial q_{(i,\,a)}}, \quad \frac{\partial O_{e+1}{}^u}{\partial \bar{p}_{(i,\,a+1)}} = \frac{\partial O_e{}^u}{\partial \bar{p}_{(i,\,a)}}. \tag{5.4}$$

These expressions indicate that the dependence of $O_{e+1}{}^u$ on all the q_a's and $\bar{p}_a$'s except for $a=1$, is uniquely determined. In other words, we now determine a function of $q_{(i,1)}$, and $\bar{p}_{(i,1)}$, which must be added to the already determined part of $O_{e+1}{}^u$. We can narrow the possibilities further by noting that $O_{e+1}{}^u$ are the coefficients of ϵ^{e+1} in a power series, and hence the function we are seeking must be homogeneous of the $(e+1)$st degree in the $q_{(i,1)}$ and $\bar{p}_{(i,1)}$. Thus we see that only the numerical coefficients are not determined. They, however, may be determined by the requirement

$$(O_{e+1}{}^u, C_a{}^s)^{e+1} = 0. \tag{5.5}$$

6. CONCLUSION

In the present paper we have developed an approximation procedure that purports to lead to the solution of the following purely abstract problem. Given a field y_A of several parameters x, with an (infinitesimal) transformation law under a group of transformations whose elements depend on one or several arbitrary functions of the x; find a complete set of functions (or functionals) of the $y_A(x)$ that are invariant with respect to the transformation group. An invariant, in this sense, differs from a scalar field in that its value is to remain unchanged if the transformed functions of x, $y_A(x)$ are substituted into its law of formation instead of the original ones, whereas a scalar retains its numerical value at a fixed point P (whose coordinates x will change their values as a result of the transformation). To the extent that the law of formation of an invariant makes any reference to the values of the parameters x at all, the same numerical values are to be inserted in both cases.

If the x are invariant under the group of transformations (as they are, e.g., in the gauge group), invariant and scalar are equivalent. But if the transformation group involves coordinate transformations in x space, they are not.

In this paper we have developed our approximation procedure within the Hamiltonian formalism. Originally it had been hoped that the approximation procedure would lead directly to a prescription for quantization, so that one could also speak of a quantum theory correct up to the ath order. Our discussion of the Poisson bracket in the c-number theory shows that such a procedure is not automatic. We shall endeavor to develop such a theory, either with the help of our approximation procedure in the canonical formalism or by Lagrangian quantization.

The motivation of this paper is closely related to work by Komar,[11] Géhéniau,[12] and Janis.[13] Komar and Géhéniau have proposed to construct invariants with the help of ideas specific to Riemannian geometry. Their ideas appear to apply primarily to pure gravitation theory, where they may be particularly useful; they do not represent a general physically oriented search for observables in singular theories. Janis, on the other hand, proposes to exploit a generalized Fermi-type approach involving subsidiary conditions, whose innocuous character is explicitly demonstrated.

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