Dynamics of a Lattice Universe by the Schwarzschild-Cell Method

RICHARD W. LINDQUIST* AND JOHN A. WHEELER
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

1. INTRODUCTION AND SUMMARY; THE SCHWARZSCHILD-CELL METHOD

MANY problems of electrostatics can be expressed in terms of the elementary $1/r$ potential due to a point charge. Gravitation theory provides the analog of the $1/r$ potential in Schwarzschild's expression

$$\begin{aligned} ds^2 &= \sum_{\alpha\beta} g_{\alpha\beta}dx^\alpha dx^\beta \\ &= (1-2Gm/c^2r)^{-1}dr^2+r^2[d\theta^2+\sin^2\theta d\varphi^2] \\ &\qquad -(1-2Gm/c^2r)dT^2 \quad (1) \end{aligned}$$

for the 10 gravitational or metric potentials of Einstein for the effect of an elementary concentration of mass in an asymptotically flat space-time continuum. Here G is the Newtonian constant of gravitation, 6.67×10^{-8} $\mathrm{cm^3/g\ sec^2}$, c the velocity of light (cm/sec), and m the mass in grams. The equations of gravitation theory are nonlinear. The principle of superposition, so central in electrostatic theory, does not apply. Consequently, the Schwarzschild solution has no such far-reaching realm of application as the Coulomb potential. Nevertheless there exists a class of problems that receive a simple approximate solution by way of the Schwarzschild formula.

We consider a number of mass concentrations so distributed in space and of such relative magnitudes, that the zone of influence of each can be reasonably approximated by a sphere. Inside each cell we replace the actual gravitational potentials by the expressions of Schwarzschild. This approximation demands that the distribution of gravitational influences just external to each sphere should depart relatively little from spherical symmetry. Such a treatment is inspired by the success of Wigner and Seitz[1] in analyzing electronic wave functions in crystal lattices. They approximate the elementary lattice cell by a sphere of the same volume. The wave equation for a problem of nonseparable coordinates becomes separable. Applied to a problem where the exact solution is known—where the potential is constant—the accuracy of this analysis has been tested with favorable results.[2]

From analogy with the Wigner-Seitz analysis one might expect that the gravitational potentials ought to have zero normal derivative at the boundary of each lattice cell, idealized as spherical. The Schwarzschild potentials do not satisfy this requirement. The derivative $(d/dr)(1-2Gm/c^2r)$ is proportional to the Newtonian gravitational field, Gm/r^2, and does not go to zero at a finite distance. This derivative measures the rate of acceleration of an infinitesimal test body placed at the point in question. This acceleration at the position of the cell boundary gives the acceleration of the cell boundary itself. Otherwise stated, the mass concentrations on either side of the cell boundary accelerate towards that boundary at such a rate as to nullify the discontinuity in matching of the normal derivative of the gravitational potentials that would otherwise occur. Thus the cell method in gravitation theory has an essentially new and dynamic feature that does not appear in the solid state problem. This new feature is simple and important. *It expresses the equation of motion of the mass at the center of a cell as a dynamic condition on the boundary of the cell.*

The rest of this paper elaborates this idea and applies it to the problem of the expanding universe. *The whole of the dynamics of the expansion and subsequent contraction is derived from the elementary static Schwarzschild solution (1).* For simplicity attention is limited to the case of equal masses arranged in a regular lattice in a closed space. Special interest attaches to the case of a very large number of masses, $N=600$, the largest number that can be arranged in a regular lattice. The calculated radius of maximum expansion, a_0, for this case agrees to 1.2% with the well-known result of Friedmann,

$$a_0=(4G/3\pi c^2)(\text{total mass}), \quad (2)$$

for the case of a homogeneous distribution of matter.

Briefly recapitulated, the Schwarzschild-cell method considers the dynamics of a lattice universe as a consequence of the field equations. These equations are fulfilled everywhere except at the interface between zones of influence, and fulfilled even there in an average way. Thus the field equations determine the equations of motion of the singularities that represent matter. In this sense we follow the spirit of Einstein, Infeld, and Hoffman.[3] However, our approximation method to

* National Science Foundation Predoctoral Fellow.

[1] E. P. Wigner and F. Seitz, Phys. Rev. **43**, 804 (1933); *ibid.*, **46**, 509 (1934); see also F. Seitz, *The Modern Theory of Solids* (McGraw-Hill Book Company, Inc., New York, 1940), Chap. 9.

[2] W. Shockley, Phys. Rev. **52**, 286 (1937); F. C. Von der Lage and H. A. Bethe, Phys. Rev. **71**, 612 (1947).

[3] Einstein, Infeld, and Hoffman, Ann. Math. **39**, 66 (1939). See L. Infeld and A. Schild, Revs. Modern Phys. **21**, 408 (1949) for the derivation of the equations of motion for an infinitesimal test particle moving at arbitrary speed; also L. Infeld and J. Plebanski, Bull. Acad. Polon. Sci., Class III, 4, 757 (1956).

determine the equations of motion of mass singularities from the field equations is quite different from theirs. They demand small relative velocities, and a space that is asymptotically flat, but make no symmetry requirements on the disposition of the masses. In contrast, symmetry alone is the demand of the method that decomposes all of curved space into Schwarzschild cells.

When closed space is divided up into infinitely many infinitely small cells one passes to the limit of the dust-filled universe of Friedmann. Section 2 summarizes the properties of this Friedmann solution and defines the ideas needed in comparing this uniform model with the lattice model. One introduces the idea of a "comparison uniformly curved space" or "comparison hypersphere." The boundaries of zones are marked on it that have the same reflection symmetries as do the zones of the lattice universe. Section 3 analyzes the geometry of these polyhedral zones on the comparison hypersphere. Section 4 derives the equation of motion of the boundary from two conditions: (1) the boundary of the Schwarzschild cell must be tangent to the comparison hypersphere; (2) the solid angle subtended by the Schwarzschild cell on the hypersphere must not change with time. To satisfy these two conditions, the radius of the Schwarzschild cell, and the radius of the comparison hypersphere must vary with time. The equation for the cell radius is found (Sec. 4) to be identical with that of a freely falling test particle. Section 5 shows that the radius of the comparison hypersphere has essentially the same time variation as does the radius of the ideal Friedmann universe. Section 6 analyzes possibilities for replacing the Schwarzschild cell approximation by an exact treatment of the lattice universe. Section 7 considers possible use of the variational principle for extensions of the method, and consequences of departures of a cell from perfect symmetry.

2. FRIEDMANN UNIVERSE AND THE LATTICE UNIVERSE COMPARED

A closed universe with zero cosmological constant in accord with the arguments of Einstein[4] is assumed. When this universe is homogeneous and isotropic, the metric can always be written[5]

$$ds^2=\frac{dr^2}{1-(r^2/a^2)}+r^2(d\theta^2+\sin^2\theta d\phi^2)-dT^2. \quad (3)$$

The radius of curvature, $a(T)$, depends upon time in a way that is governed by the equation of state. In the Friedmann case, where the universe is filled with a uniform dust of total mass M and zero pressure, the radius satisfies the equation of energy,

$$Mc^2+Mc^2(da/dT)^2-(4GM^2/3\pi a)=0. \quad (4)$$

[4] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1950), p. 107.

[5] H. P. Robertson, Revs. Modern Phys. **5**, 62 (1933).

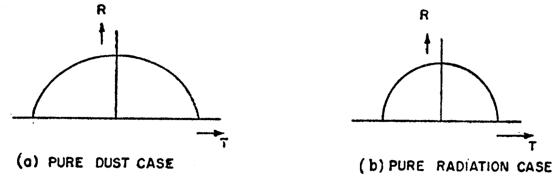

FIG. 1. Radius of the universe, a, as a function of cotime, T=(velocity of light)·(time), for a homogeneous isotropic universe. The system expands outward and falls back together again under gravitational attraction, but the detailed dynamics depends upon the equation of state of the medium. In Friedmann's universe (case a) the pressure is zero. In the case of pure radiation (case b) the pressure is equal to one third of the energy density, and the circle $a^2+T^2=a_0^2$ takes the place of Friedmann's cycloid. The intermediate case has been treated numerically by G. Gamow, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 10 (1953).

The maximum radius is

$$a_0=4GM/3\pi c^2. \quad (2a)$$

The relation between radius, a, and cotime, T =(velocity of light)·(time), is a cycloid (Fig. 1) given parametrically in the form

$$a=(a_0/2)(1+\cos\eta)$$
$$T=(a_0/2)(\eta+\sin\eta). \quad (5)$$

The three dimensional curved space of Friedmann at any one cotime, T, is equivalent to the three-dimensional hypersurface of a sphere in a four-dimensional *Euclidean* space. The extra dimension has no physical meaning; it is an imbedding dimension. Introducing Cartesian coordinates,

$$u_1=a\sin\chi\sin\theta\sin\varphi,$$
$$u_2=a\sin\chi\sin\theta\cos\varphi,$$
$$u_3=a\sin\chi\cos\theta,$$
$$u_4=a\cos\chi,$$

and letting

$$r=a\sin\chi, \quad (6)$$

gives the metric (3) the simple form

$$ds^2=du_1^2+du_2^2+du_3^2+du_4^2-dT^2.$$

On this sphere we mark out the vertices of a regular figure to give a geometry that can be compared with the geometry of the lattice universe. Particular dust particles specify these N vertices. Every vertex can be equidistant from its nearest neighbors only when N=5, 8, 16, 24, 120 or 600.[6]

The case $N=8$ gives one of the simplest of these arrangements. For any typical one, P, of the eight points there is another one, A, of the eight which may be called its antipode. The remaining six particles are most conveniently named as N(north), S(south), E, W,

[6] See H. M. S. Coxeter, *Regular Polytopes* (Methuen and Company, Ltd., London, 1948), in particular Table I (ii) on pp. 292–293, for an enumeration of the regular polytopes in four dimensions.

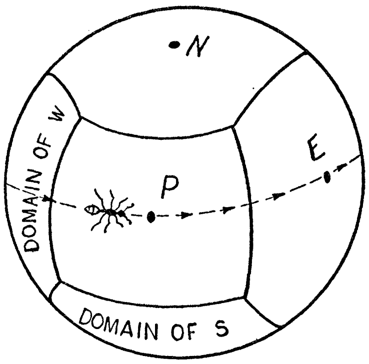

FIG. 2. Two-dimensional analog of a closed lattice universe. The third dimension in the drawing merely provides a dimensionality high enough to allow imbedding a two-dimensional space with the topology of a sphere. The superfluous radial coordinate has no relation to the time coordinate. There are six lattice cells: P, E, A (antipode), W and N, S. Each is like a square in its count of edges and corners but is a deformed square. The distortion is such that the angle at a corner is 120° instead of the normal 90°. In terms of distances:

All distances in terms of radius, a, of sphere	Center of cell to center of edge	Center of cell to corner
Lattice cell of Fig. 2	$\frac{2\pi a}{8}=0.7854a$	$a\sin^{-1}\frac{6^{\frac{1}{2}}}{3}=0.9553a$
Circular domain on surface of sphere with same center and same area	$a\cos^{-1}\left(\frac{2}{3}\right)=0.8411a$	
Ratios	0.9338	1.1359
Square of same area in flat space $\left(=\frac{4\pi}{6}a^2\right)$	$\left(\frac{\pi}{6}\right)^{\frac{1}{2}}a=0.7236R$	$\left(\frac{\pi}{3}\right)^{\frac{1}{2}}a=1.0233a$
Circle of same area in flat space	$\left(\frac{2}{3}\right)^{\frac{1}{2}}a=0.8165a$	
Ratios	0.8862	1.2533

U(up) and D(down). Proceeding from P to any one of these six nearest neighbors, say E and continuing on in the same direction, then one comes next to the antipodal mass, A; then to W and finally back to P. Likewise, proceeding upward from P, one comes in turn to U, A, D and back to P; and so on. Figure 2 sketches the analog of this lattice arrangement for a two-dimensional closed surface imbedded in a three-dimensional Euclidean space.

An infinitesimal test particle ordinarily lies closer to one of the lattice centers, P, than to another. It may be said to belong to the zone of influence of P, or to the *lattice cell* centered on P. Between P and E lies a two-dimensional array of points which are equidistant from P and E: the *boundary* or *interface* between the domains of P and E.

From Friedmann's simple metric with its uniform curvature we turn to the metric of a lattice universe, where all the mass is concentrated into N centers. The curvature of the metric, R_{ijkl}, now varies from place to place as indicated qualitatively in Fig. 3. However, the contracted curvature tensor, $R_{ik}=g^{\alpha\beta}R_{i\alpha k\beta}$ vanishes throughout the mass-free space between the vertices. Moreover the symmetry group of the vertices is completely unchanged, as are the reflection symmetries of the metric at the cell boundaries.

How is one to deal with this metric with its complicated variation in space? Can one simplify the problem by limiting attention to those parts of space that lie infinitesimally close to zonal boundaries, with their reflection symmetries? No, even the metric in these limited regions of space does not conform to the surface of a comparison hypersphere. One naturally chooses the radius of this comparison hypersphere so that points halfway between nearest neighbors will lie outside the comparison hypersphere, and points equidistant from three nearest neighbors will lie inside.

To simplify the metric in an individual cell we have to simplify the geometry of its boundary. We *replace the typical cell of the lattice universe by a Schwarzschild cell* that (1) has a spherically symmetric metric:

$$ds^2=[1-(2m^*/r)]^{-1}dr^2+r^2[d\theta^2+\sin^2\theta(d\varphi)^2] -[1-(2m^*/r)]dT^2, \quad (7)$$

m^* = mass of one singularity expressed in units of length,

$$=(G/c^2)(\text{mass of singularity})=Gm/c^2, \text{ and } m=(\text{total mass})/(\text{number of vertices})=M/N; \quad (8)$$

(2) has a spherically symmetric boundary whose radius r depends on time, and (3) conformally joins at this radius r onto a *comparison hypersphere* of a radius, a, that also depends on time. This radius defines what we mean by the radius of the lattice universe. To state in quantitative terms this joining or *tangency* condition now requires examination of geometry of the comparison hypersphere.

There is a second motivation to study geometry on the hypersphere. There one can define polyhedral zones and compare them with spherical zones to test quantitatively how nearly the dimensions of the two objects agree. Such a comparison is not possible for the lattice space itself because its metric is not known. Therefore the analogous comparison on the surface of the hypersphere—and in flat space—supply our only simple means to estimate the accuracy of the Wigner-Seitz approximation.

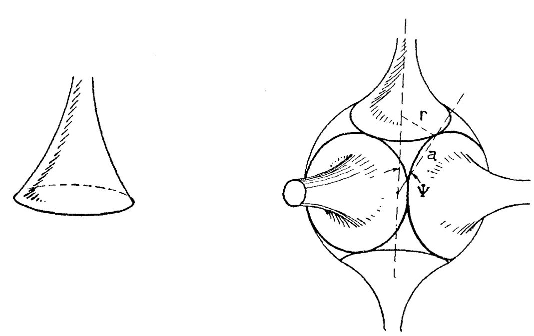

FIG. 3. Qualitative character of the space in a lattice universe. The space is closed up, but not by an everywhere uniform curvature as in the Friedmann universe.

3. GEOMETRY OF SPHERICAL AND POLYHEDRAL CELLS IN UNIFORMLY CURVED SPACE

The shape of the typical lattice cell in uniformly curved space is easily visualized by considering the two-dimensional analog (Fig. 2) of the three-dimensional lattice universe containing eight particles. From that analogy, or from direct analysis, one sees that the faces of the cube-like lattice cell meet at a dihedral angle of 120°, rather than the familiar 90° (Figs. 2 and 4). An edge is common to three cells, not to the four expected from the familiar picture of stacked blocks. At a corner only four lattice cells meet, instead of the usual eight; and only four edges, instead of the usual six. These four edges radiate from the corner at the regular tetrahedral angles. The corner of a lattice cell appears blunt when compared with the corner of a cube. To approximate the lattice cell as a sphere is expected to be slightly better than the replacement of a cube by a sphere as done in solid state physics. This conclusion is borne out by the comparison of distances given in Table I.

Similar distance comparisons for the other regular lattices in a space of uniform curvature appear in Table II. Thus the lattice cell of a spherical polytope

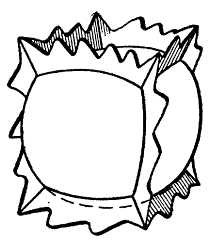

FIG. 4. Shape of typical cell in case of eight particle lattice universe. The cell is like a deformed cube. Three cells meet at an edge rather than the four of Euciidean geometry, and at a corner four cells meet rather than six.

deviates from a sphere considerably less than does its counterpart in flat space. This makes it reasonable to develop the spherical cell approximation.

To obtain a measure of the size of a spherical cell, both absolute and relative to the total hypersurface, we imbed the curved three-dimensional space at a fixed moment of time into a Euclidean four-dimensional space. Denote by a the radius of the hypersphere and by r_0 the radius of the spherical cell, both as measured in the flat imbedding space. Denote by ψ the angular separation of the center of the spherical cell and its boundary as seen from the center of the hypersphere:

$$\sin\psi = r_0/a. \tag{9}$$

The cell subtends at the center of the hypersphere a hypersolid angle

$$\Psi = \frac{\int d(\text{three-dimensional volume})}{a^3} = \int_{\text{inside}} \sin^2\chi \sin\theta d\chi d\theta d\varphi = \pi(2\psi - \sin 2\psi) \tag{10}$$

TABLE I. Comparison of (1) lattice cell dimensions for the case of eight mass centers with (2) dimensions of a spherical cell of the same volume in curved space, under assumption of *uniform* curvature of the space, i.e., uniform mass density. In the case when the mass of each cell is concentrated at the center of that cell, the geometry of space is changed—and changed in a time dependent way. Then these dimensions no longer apply exactly, and have primarily only illustrative value.

All distances in terms of radius of curvature, a, of space	Center of cell to center of face	Center of cell to center of edge	Center of cell to corner
Lattice cell of volume $(1/8)2\pi^2a^3$	$0.785a$	$0.955a$	$1.047a$
Spherical domain in curved space with same center and volume		$0.883a$	
Ratios	0.889	1.082	1.186
Cube of same volume in flat space	$0.676a$	$0.955a$	$1.170a$
Sphere of same volume in flat space		$0.838a$	
Ratios	0.806	1.140	1.396

in analogy to the familiar formula in three-dimensional geometry

$$\Omega = 2\pi(1-\cos\psi)$$

for solid angle subtended by a cone of half-angle ψ. Out of the whole hypersurface or three volume of the hypersphere,

$$2\pi^2a^3, \tag{11}$$

the spherical cell occupies the fraction

$$\Psi/\Psi_{\text{total}} = (2\psi - \sin 2\psi)/2\pi. \tag{12}$$

We wish to cut away the uniformly curved space occupied by a spherical cell and replace that cell by a Schwarzschild cell. In the Schwarzschild cell the metric is nonuniform but still spherically symmetrical. We want the Schwarzschild cell to have the same spherical boundary as the uniformly curved spherical cell that it replaces, and to be "tangent" to the hypersphere at the point of join.

How big shall the spherical boundary be chosen? Two alternative conditions suggest themselves for defining the angle ψ:

TABLE II. Distance ratios as measures of the "roundness" of a lattice cell in curved space. For comparison the same distance ratios are also given for flat space for a cell with the same number of faces, edges and corners. The cells are most nearly spherical when there are 120 mass centers.

Number of identical cells	Name that would be given to one such cell in flat space	Ratio distance to: face or corner / "radius" of sphere of same volume: Curved space	Flat space	Curved space	Flat space
5	tetrahedron	0.862	0.671	1.246	2.013
8	cube	0.889	0.806	1.186	1.396
16	tetrahedron	0.763	0.671	1.525	2.013
24	octahedron	0.880	0.846	1.320	1.465
120	dodecahedron	0.917	0.910	1.133	1.146
600	tetrahedron	0.679	0.671	1.947	2.013

Condition I: The boundary sphere shall cut out of the hypersphere $1/N$ of the total solid angle.

Condition II: The boundary sphere associated with one mass concentration shall just touch the boundary spheres of all its nearest neighbors.

Condition I is the direct generalization to curved space of the standard Wigner-Seitz approximation. The boundary angle ψ is fixed by

$$\frac{\Psi}{\Psi_{\text{total}}}=\frac{1}{N}=\frac{2\psi-\sin 2\psi}{2\pi}. \quad (13)$$

N spherical cells, each of which will fit into $1/N$th the hypersurface—or three volume—of the hypersphere will in some places overlap and in other places leave regions of "no man's land." When the cells are merely cutouts from the hypersurface, they have uniform curvature and in the regions of overlap meet every reasonable requirement of "tangency." Consider the other case when the regions inside the boundary spheres are Schwarzschild cells, *adjusted here and always to tangency to the comparison hypersphere at their spherical boundaries*. Where the boundaries of two spherical cells interact, the two Schwarzschild metrics are tangent because at that point both are tangent to the hypersphere. Where the cells overlap, the intersecting Schwarzschild metrics depart from tangency in one sense; and they depart from tangency in the opposite sense wherever the intersection lies outside the spherical boundaries of the cells. Condition I makes the metrics of two Schwarzschild cells be tangent to each other in an average way, it being understood that at the spherical boundary of each cell the Schwarzschild metric has been made tangent to the comparison hypersphere in a way defined later.

In contrast, Condition II gives a matching only at the one point of contact midway between neighboring mass centers. Away from this point of contact, in the now enlarged "no man's land" between spherical cells, two extrapolated Schwarzschild metrics at their intersection depart more and more from tangency. The departure always has the same sign. Consequently, Condition II will be a less reasonable criterion for cell size than Condition I. Thus far, we have examined the two criteria for cell size from the point of view of matching of boundary conditions thinking of the metric inside as of Schwarzschild form. Now imagine instead a uniformly curved space inside of these boundaries. Then the tangency requirements are automatically satisfied. Another point of view offers itself to compare the two criteria for cell size. Table III shows that the polyhedral cell dimensions are not as well matched by Condition II as those spheres of Condition I. One is not in a position so easily to calculate distances when the metric has the nonuniform curvature of the true lattice space, but it is reasonable to believe that Condition I continues to give the more reasonable cell dimensions, as it also better satisfies on the average the tangency requirement.

We have now decided that the spherical boundary of one Schwarzschild cell is to cut out $1/N$th of the volume of the comparison hyperspace (13). Next we formulate the requirement that the cell be *tangent* to the hypersphere at the point of join. Imagine a surveyor told to find the fraction of the earth's area enclosed within a circular kingdom, but only allowed to make measurements on or near its periphery. His solution is simple: Measure the circumference of the kingdom, and compare it with that of an infinitesimally smaller circle also on the earth's surface. Also measure the difference in their radii. If the earth were flat, one would find

$$\frac{d(\text{circumference})}{d(\text{radial distance})}=2\pi.$$

However, due to its finite radius a, one finds

$$\frac{d(\text{circumference})}{d(\text{radial distance})}=\frac{d(2\pi a \sin\psi)}{a d\psi}=2\pi \cos\psi. \quad (14)$$

The extension of this method to the problem of a sphere on the surface of a given hypersphere requires no essentially different ideas. Choose any great circle on this sphere, and compare its circumference with that of the corresponding great circle on an infinitesimally smaller one. Let d (radial distance) denote the difference in their radii *as measured in an invariant manner along the surface of the hypersphere*. One is led again to (14). In view of the complete symmetry of the problem, one clearly obtains the same value of ψ regardless of which great circle is chosen for the measurement.

Thus there is an invariant way to say that one Schwarzschild cell shall cut out $1/N$th of the tangent hypersphere, and to say this entirely in terms of *measurements in and near the cell boundary*:

$$\frac{1}{2\pi}\frac{d(\text{circumference})}{d(\text{radial distance})}=\cos\psi, \quad (15)$$

TABLE III. Comparison of polyhedral and spherical cells in the closed three space that forms the hypersurface of a sphere in 4 dimensional *Euclidean* space.

	Distance from center of cell in units of hypersphere radius				
				Radius of sphere that replaces cell according to	
N	Center of face	Center of edge	Center of cell	Condition I	Condition II
5	0.912	1.150	1.315	1.057	0.912
8	0.785	0.955	1.047	0.883	0.785
16	0.524	0.785	1.047	0.686	0.524
24	0.524	0.617	0.785	0.595	0.524
120	0.314	0.365	0.388	0.343	0.314
600	0.135	0.232	0.388	0.199	0.135

where ψ is *defined* by

$$\frac{2\psi-\sin 2\psi}{2\pi}=\frac{1}{N}. \qquad (16)$$

In addition, there is an invariant way to define the radius, a, of the hypersphere—or what we call the radius of the universe—in terms of measurements at the cell boundary:

$$a=\frac{\text{(cell radius)}}{\sin\psi}=\frac{\text{(circumference of great circle)}}{2\pi\sin\psi}. \qquad (17)$$

4. DYNAMICS OF THE LATTICE DEFINED BY BOUNDARY CONDITIONS

The problem of matching metrics at the boundary between two cells has been replaced by the problem of matching the metric in one Schwarzschild cell to the metric of the comparison hypersphere. From this matching condition we now derive the change with time in the radius of the cell and the radius of the universe—in other words, the dynamics of expansion and recontraction—using (15), (16), and (17), which depend only on behavior of the metric near the cell boundary.

In the Schwarzschild metric (7), a great circle of radius r has the circumference $2\pi r$. A great circle of radius $r-\epsilon$ has the circumference $2\pi(r-\epsilon)$. The invariant infinitesimal distance between one circle and the other is

$$\epsilon[1-(2m^*/r)]^{-\frac{1}{2}}.$$

These two observations made near the boundary fix the fraction of the whole hypersurface occupied by one Schwarzschild zone:

$$\frac{1}{2\pi}\frac{d\text{(circumference)}}{d\text{(radial distance)}}=[1-(2m^*/r)]^{\frac{1}{2}}=\cos\psi_N, \qquad (18)$$

where ψ_N is fixed by N, according to (16). Thus, the radius r of the Schwarzschild cell cannot depend upon time, and we end up with a universe that has no dynamics at all! Obviously we have made an oversight somewhere.

The difficulty arises because the two circles were compared at equal values of the Schwarzschild time coordinate. We tacitly assumed that the Schwarzschild coordinates, r and T, are well adapted to describing the space-time continuum inside one cell, but they are not. Their inappropriateness appears even more clearly for r than for T. As the size of the lattice cell grows or shrinks, the range of the r coordinate increases or decreases. It appears as if one has to add on or chop off space at the cell boundary! It is much more natural to think of a new pair of space and cotime coordinates, ρ and τ, which are functions of r and T with the following properties. The boundary of the lattice cell is described by the statement that $\rho=\rho(r,T)=\text{constant}$ at that point. The point $\rho-\epsilon$ in the "coordinate patch" of one lattice cell is to be identified with the point $\rho+\epsilon$ in the extrapolated coordinate patch of another lattice cell. This identification is not to be made for equal values of the Schwarzschild time T because then the two coordinate patches do not *overlap* but only *intersect* (Fig. 5). This failure to overlap can be stated this way: The coordinate surface, $T=\text{constant}$, does not stand perpendicularly to the boundary surface, $\rho=\text{constant}$, but *obliquely* to it. The appropriate new time variable, $\tau=\tau(r,T)$, has the property that the surface, $\tau=\text{constant}$, *does* stand perpendicularly to the boundary surface, $\rho=\text{constant}$. This orthogonality of the two surfaces is demanded by the condition of mirror symmetry at the zone boundary.

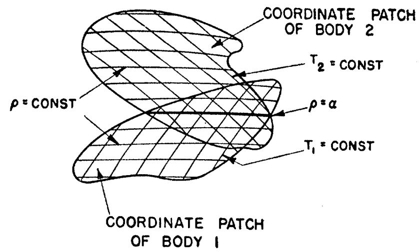

FIG. 5. The coordinate patches of two lattice cells merely intersect at the boundary, $\rho=\alpha$, and do not overlap as they should when the surfaces in question belong—as in the figure—to equal values of the Schwarzschild time, $T_1=T_2=\text{const}$. Real overlap or tangency demands instead equal values of a new time coordinate τ, not shown. Surfaces of constant τ intersect the zone boundary perpendicularly; surfaces of constant T, like those shown here, make oblique intersections.

We later construct a coordinate system, ρ, τ, that satisfies the orthogonality requirement. However, for formulation of the boundary condition at the surface such detail is not needed. It is enough to demand that the two circles of slightly different sizes be compared at the same value of the τ coordinate; in other words, that the ring-like surface connecting the two circles shall be orthogonal—in the four-dimensional sense—to the boundary of the lattice cell. This condition, being invariant, can also be formulated in the original Schwarzschild coordinate system.

The boundary will move in the interval of *Schwarzschild* cotime dT from r to $r+dr$. Describe this change by the four vector $(dr,0,0,dT)$ (in contravariant components). At this stage the ratio, dr/dT, of the components of this vector is not known. Construct a new vector that is (1) perpendicular to this vector, and (2) perpendicular to the two independent vectors that lie in the boundary $(0,d\theta,0,0)$ and $(0,0,d\varphi,0)$. The new vector is uniquely determined up to a multiplicative factor, and in one convenient normalization has the contravariant components

$$(-g^{rr}dT,0,0,g^{TT}dr).$$

This vector, being normal to the boundary, lies in the surface of constant τ.

Return to the comparison of the two circles. One circle passes through the point with coordinates $(r,0,0,T)$. The other circle was previously erroneously taken to pass through a point $(r-\epsilon, 0, 0, T)$ at the *same* Schwarzschild time. Instead, we must take a circle at the same value of τ, passing through the point $(r-g^{rr}dT, 0, 0, T+g^{TT}dr)$. The measurements on the two circles near the zone boundary now provide an invariant measure of the fraction of the hypersphere spanned by the Schwarzschild cell:

$$\cos\psi_N=\frac{1}{2\pi}\frac{d(\text{circumference})}{d(\text{radial distance})}$$
$$=\frac{g^{rr}dT}{[g_{rr}(g^{rr}dT)^2+g_{TT}(g^{TT}dr)^2]^{\frac{1}{2}}}$$
$$=\frac{g^{rr}dT}{[g^{rr}g^{TT}]^{\frac{1}{2}}[g_{TT}(dT)^2+g_{rr}(dr)^2]^{\frac{1}{2}}}$$
$$=\left(\frac{g^{rr}}{g^{TT}}\right)^{\frac{1}{2}}\frac{dT}{d(\text{proper cotime})}. \quad (19)$$

The angle ψ_N and the hypersolid angle Ψ spanned by the cell must remain constant in time; *the motion of the boundary must adjust itself to this match-up requirement.* Therefore (19) determines the equation of motion of the boundary:

$$\frac{dT}{d(\text{proper cotime})}=\frac{\cos\psi_N}{[1-(2m^*/r)]} \quad (20)$$

or

$$(dr/dT)=\pm(\cos\psi)^{-1}[1-(2m^*/r)][(2m^*/r)-\sin^2\psi]^{\frac{1}{2}}. \quad (21)$$

This completes the derivation of the dynamics of expansion from the metric match-up conditions.

The equation of motion of the boundary, (20) or (21), is identical with the law of conservation of energy for a unit test particle thrown out radially from the mass, m. The expression on the left side of (20)—after multiplication by c^2—gives the rest plus kinetic energy of the test particle,

$$\frac{c^2}{[1-(v^2/c^2)]^{\frac{1}{2}}}=c^2\frac{dT}{d(\text{proper cotime})}, \quad (22)$$

and the expression on the right similarly gives the relativistic generalization of its total energy diminished by its potential energy in the gravitational field. This integrated form of the equation of motion is less familiar than the differential form, analogous to Newton's equation of motion,

$$d^2x^i/ds^2+\Gamma_{\alpha\beta}{}^i(dx^\alpha/ds)(dx^\beta/ds)=0 \quad (23)$$

or

$$d^2T/ds^2+g^{TT}(dg_{TT}/dr)(dr/ds)(dT/ds)=0 \quad (24)$$

[compare (20)], and

$$d^2r/ds^2+\tfrac{1}{2}g^{rr}(dg_{rr}/dr)(dr/ds)^2-\tfrac{1}{2}g^{rr}(dg_{TT}/dr)(dT/ds)^2=0, \quad (25)$$

[compare (21)]. However the content is the same: the entire time dependence of the boundary between two lattice cells can be described by the behavior of a particle that falls towards both mass points simultaneously under the action of their gravitational attractions.

The integrated form of the equation of motion, (21), is less complete than the differential form because it possesses the singular solution

$$r=2m^*/\sin^2\psi_N=\text{a constant}. \quad (26)$$

This singular solution already appeared (18) in the first, improperly formulated, attempt to derive the dynamics of the lattice cell from conditions at the boundary. Similar examples of singular solutions are common in classical physics, if one deals only with first integrals of the equations of motion. For example, a particle tossed straight up in a uniform gravitational field, that came to rest at the top of its flight—*and stayed at rest*—would satisfy the law of conservation of energy. However, it would violate Newton's second law of motion. Likewise an infinitesimal test particle permanently at the location (26) cannot satisfy the geodesic equation of motion (23). In mathematical terms, (26) is a singular solution of equations in the sense that it represents the *envelope* of the trajectories of test particles which reach the same distance of maximum excursion r at different values of T.

5. RADIUS OF THE LATTICE CELL AND OF THE UNIVERSE AS FUNCTIONS OF THE TIME

Equation (21) describes a motion of the cell boundary that changes from expansion to contraction when the last term vanishes; that is, when the cell radius, r, reaches a maximum value, R, defined by

$$r_{\max}\equiv R=2m^*/\sin^2\psi. \quad (27)$$

TABLE IV. Maximum radius, a_0, attained during expansion of the lattice universe (as defined and calculated by the Schwarzschild-cell approximation method) compared with Friedmann's value for the maximum radius of a universe uniformly filled with dust. The radius is expressed relative to the Schwarzschild "radius" $2M^*=2GM/c^2$ *defined* by the mass of the whole universe and also relative to the Schwarzschild radius $2m^*=2Gm/c^2=2M^*/N$ associated with the mass of one lattice cell.

	Condition I			Condition II		
N	ψ	$a_0/2m^*$	$a_0/2M^*$	ψ	$a_0/2m^*$	$a_0/2M^*$
5	60.59°	1.5126	0.3025	52.24°	2.0239	0.4048
8	50.60°	2.1672	0.2709	45°	2.9293	0.3535
16	39.34°	3.9256	0.2454	30°	8.0000	0.5000
24	34.10°	5.6750	0.2365	30°	8.0000	0.3333
120	19.63°	26.376	0.2198	18°	33.889	0.2824
600	11.42°	128.82	0.2147	7.66°	406.06	0.6768
∞ [a]	0°		0.2122	0°		0.2122

[a] Friedmann universe.

In terms of r_{max} the equation of motion of the boundary takes the form

$$dT=\frac{\pm[1-(2m^*/R)]^{\frac{1}{2}}dr}{[1-(2m^*/r)][(2m^*/r)-(2m^*/R)]^{\frac{1}{2}}}. \quad (28)$$

Direct integration leads to a closed formula[7] connecting the cell radius with the Schwarzschild cotime T:

$$T=\left[\frac{(R-2m^*)(R-r)r}{2m^*}\right]^{\frac{1}{2}}+\left[\frac{R-2m^*}{2m^*}\right]^{\frac{1}{2}}(R+4m^*)\text{ arc cos}\left(\frac{r}{R}\right)^{\frac{1}{2}}+2m^*\ln\frac{[r(R-2m^*)]^{\frac{1}{2}}+[2m^*(R-r)]^{\frac{1}{2}}}{[r(R-2m^*)]^{\frac{1}{2}}-[2m^*(R-r)]^{\frac{1}{2}}}. \quad (29)$$

More interesting is the connection between the cell radius and the proper cotime τ measured on an infinitesimal test particle that sits at the zone interface, falling simultaneously towards both mass centers:

$$d\tau=\frac{dT}{(dT/d\tau)}=\frac{dr}{[(2m^*/r)-(2m^*/R)]^{\frac{1}{2}}}, \quad (30)$$

a connection which integrates to

$$\tau=(R/2m^*)^{\frac{1}{2}}[r^{\frac{1}{2}}(R-r)^{\frac{1}{2}}+R\text{ arc cos}(r/R)^{\frac{1}{2}}]. \quad (31)$$

This is the equation for a cycloid, as appears from the parametric representation

$$\left.\begin{aligned}r&=\tfrac{1}{2}R(1+\cos\eta)=(M^*/N\sin^3\psi_N)(1+\cos\eta)\\ \tau&=\tfrac{1}{2}R(R/2m^*)^{\frac{1}{2}}(\eta+\sin\eta)\\ &=(M^*/N\sin^3\psi_N)(\eta+\sin\eta)\end{aligned}\right\}. \quad (32)$$

The radius of the universe,

$$\begin{aligned}a&=(\text{radius of cell boundary in the imbedding Euclidean space})/\sin\psi_N\\ &=(1/2\pi)(\text{circumference of great circle about this sphere})/\sin\psi_N\\ &=r/\sin\psi_N\end{aligned} \quad (33)$$

is also obtained as a function of proper cotime in parametric form directly from (32),

$$a=(M^*/N\sin^3\psi_N)(1+\cos\eta). \quad (34)$$

The Schwarzschild-cell method predicts a cycloidal relation between radius of the universe and proper cotime which is identical to that found by Friedmann (5), except for the connection between maximum radius and mass. When the mass is distributed uniformly, as in Friedmann's model, the maximum radius has the value

$$a_0=4GM/3\pi c^2=4M^*/3\pi. \quad (35)$$

When the mass is concentrated into N centers, and when the concept of "radius of the universe" is identified with the concept of "radius of the hypersphere tangent to a Schwarzschild cell at a zone boundary" then the maximum radius has the value (34)

$$a_0=2M^*/N\sin^3\psi_N. \quad (36)$$

Table IV lists the maximum radius calculated from (36) for the six regular lattice universes, using the two prescriptions discussed in Sec. 3 to define the size of the sphere that bounds a typical lattice zone. Condition I predicts a maximum radius that steadily approaches the Friedmann value with increasing N (Fig. 6) and that is only 1.2% above the Friedmann value for $N=600$. Thus, the Friedmann universe is the natural idealization of a universe containing a very large number of very small masses.

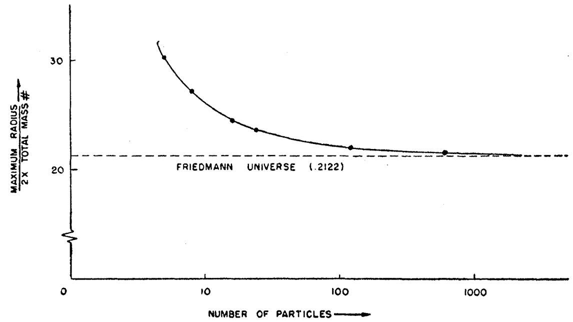

FIG. 6. Maximum radius of N-particle lattice universe, defined and derived in the Schwarzschild-cell approximation, compared with maximum radius of a universe filled with the same uniformly distributed mass.

These results do not depend upon any particular choice of coordinate system. However, there is a coordinate system (ρ,θ,ϕ,τ) especially suited to describe dynamics within a Schwarzschild cell. Consider not merely one test particle that reaches its maximum r value, $r=R$, at $T=\tau=0$, but a whole collection of radially moving test particles that reach different maximum r values, R_1, R_2, $\cdots$, at $T=\tau=0$. Require that the geodesic of a test particle which is momentarily at rest at any point $r=R_i$ in a Schwarzschild field at time $t=\tau=0$ shall be given by $d\rho_i=0$. Thus ρ_i is to be interpreted as a "labeling coordinate" which distinguishes between the several test particles, and which depends on the value, R_i, of their maximum radii but not on their subsequent dynamics. For simplicity we choose the undetermined function $\rho_1(R_i)$ to be R_i itself. Also define the new cotime coordinate, τ, to be proper cotime measured along the appropriate test particle geodesic from the moment of maximum expansion. Then (29) and (31) define the transformation of coordinates $(r,T)\rightarrow(\rho,\tau)$ on

[7] Mr. Nicola N. Khuri kindly independently calculated this integral for us, which presumably also appears somewhere in the extensive literature of general relativity.

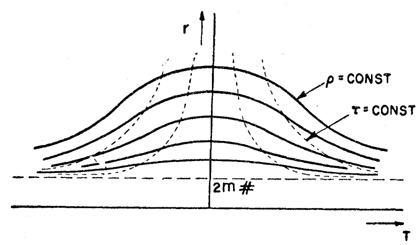

FIG. 7. Qualitative plot of the test particle geodesics $\rho=$const and the space-like geodesics $\tau=$const as functions of r and T.

replacing R by ρ. In the dimensionless units

$$r'=r/2m^* \quad \rho'=\rho/2m^*$$
$$T'=T/2m^* \quad \tau'=\tau/2m^*,$$

the transformation equations take the implicit form

$$T'=[(\rho'-1)(\rho'-r')r']^{\frac{1}{2}}+(\rho'-1)^{\frac{1}{2}}(\rho'+2)\text{ arc }\cos(r'/\rho')^{\frac{1}{2}}+\ln\frac{(r')^{\frac{1}{2}}(\rho'-1)^{\frac{1}{2}}+(\rho'-r')^{\frac{1}{2}}}{(r')^{\frac{1}{2}}(\rho'-1)^{\frac{1}{2}}-(\rho'-r')^{\frac{1}{2}}} \quad (37)$$

and

$$\tau'=(\rho')^{\frac{1}{2}}[(r')^{\frac{1}{2}}(\rho'-r')^{\frac{1}{2}}+\rho'\text{ arc }\cos(r'/\rho')^{\frac{1}{2}}]. \quad (38)$$

In the new dimensionless coordinates, ρ' and τ', the dimensionless metric, $ds'=ds/2m^*$, takes the form

$$(ds')^2=\left[\frac{(\rho'-r')}{4r'\rho'(\rho'-1)}\right]\left[(3\rho'-r')\left(\frac{r'}{\rho'-r'}\right)^{\frac{1}{2}}+3\rho'\text{ arc }\cos\left(\frac{r'}{\rho'}\right)^{\frac{1}{2}}\right]^2(d\rho')^2+(r')^2[d\theta^2+\sin^2\theta d\varphi^2]-(d\tau')^2, \quad (39)$$

where r' is to be expressed in terms of ρ' and τ' from (38). The literature contains other coordinate systems, adapted to the study of other features of a Schwarzschild singularity.[8] The present coordinate system is well adapted to description of the expansion and recontraction process (Fig. 7).

6. RIGOROUS FORMULATION OF THE INITIAL VALUE PROBLEM

In principle, the problem of the dynamics of a closed N-body universe could be solved rigorously by first solving the Lichnerowicz[9] initial value equations, and then using the field equations to determine the metric at later times. This would provide a solution to the dynamical problem at least for a finite length of time.[10]

A class of solutions of the initial value problem has recently been found by Faurès and Misner,[11] under the assumption that $\partial g_{\mu\nu}/\partial T=0$, that is, that all mass centers are momentarily at rest on the initial surface $T=T_0$. These solutions provide an accurate description of the N-particle universe at the moment of maximum expansion. In isotropic coordinates, Misner's solution in the absence of charges is given by the expression

$$ds^2=\left(1+\sum_{i=1}^{N-1}\frac{\alpha_i}{|\mathbf{r}_i|}\right)^4(dx^2+dy^2+dz^2), \quad (40)$$

where $|\mathbf{r}_i|$ is the distance from the ith particle and $\alpha_i>0$. The Nth particle is located at infinity; in effect one is looking in through its Schwarzschild singularity at the rest of the universe. By comparing this expression with the standard Schwarzschild form in isotropic coordinates,

$$ds^2=\left(1+\frac{m^*}{2r}\right)^4(dx^2+dy^2+dz^2)-\left(1-\frac{m^*}{2r}\right)^2\left(1+\frac{m^*}{2r}\right)^{-2}dT^2, \quad (41)$$

as the distance $|\mathbf{r}_i|$ shrinks to zero, one finds the mass of the ith particle is given by

$$m_i^*=2\alpha_i\sum_{j\neq i}\left(1+\frac{\alpha_j}{|\mathbf{r}_{ij}|}\right) \quad \text{for} \quad i,j=1,2,\cdots,N-1$$
$$m_N^*=2\sum_i\alpha_i. \quad (42)$$

The location of the Schwarzschild singularity surrounding each particle can be defined as the surface of minimum area.

We have analyzed this metric in detail only for the five- and eight-body universes. To illustrate we deal only with the five-particle case. Take four masses at the points

$$(1,1,1),\ (1,-1,-1),\ (-1,1,-1)$$

and

$$(-1,-1,1).$$

The requirement of symmetry, and the condition that all five masses have the same value, gives the unique result

$$\alpha_i=2\sqrt{2};\quad m_i^*=16\sqrt{2} \quad \text{for all } i.$$

Hence a possible choice of metric for the moment of maximum expansion is

$$ds^2=\left(1+2\sqrt{2}\sum_{i=1}^{4}\frac{1}{|\mathbf{r}_i|}\right)^4(dx^2+dy^2+dz^2), \quad (43)$$

which rigorously satisfies the initial value requirements.

One can use this metric to estimate the validity of

[8] G. Lemaître, Ann. soc. sci. Bruxelles **53A**, 51 (1933); J. L Synge, Proc. Roy. Irish Acad. **A53**, 83 (1950); A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).

[9] A. Lichnerowicz, *Théories relativistes de la gravitation et de l'electromagnétisme* (Masson et Cie, Paris, 1955).

[10] A. Raychaudhuri, Phys. Rev. **98**, 1123 (1955); A. Komar, Phys. Rev. **104**, 544 (1956).

[11] Y. Faurès-Bruhat, J. Rational Mech. Anal. **5**, 951 (1956); C. W. Misner (to be published).

the two approximate match-up conditions (Conditions I and II) of Sec. 3. The most convenient quantity to compute for comparison with the rigorous solution is the invariant distance $2d$ between two Schwarzschild singularities, as measured along a geodesic between them. For the case of a pure Schwarzschild field this distance is given by

$$\frac{d}{2m^*}=\frac{1}{2m^*}\int_{2m^*}^{2m^*\xi}\left(1-\frac{2m^*}{r}\right)^{-\frac{1}{2}}dr$$
$$=(\xi^2-\xi)^{\frac{1}{2}}+\frac{1}{2}\ln\left[\frac{\xi^{\frac{1}{2}}+(\xi-1)^{\frac{1}{2}}}{\xi^{\frac{1}{2}}-(\xi-1)^{\frac{1}{2}}}\right]. \quad (44)$$

The radial coordinate, $r=2m^*\xi$, of the interface between two zone boundaries may be defined in two ways: (a) as the radius of the boundary sphere, or (b) as the radial coordinate of the midpoint between two mass particles, i.e., the center of the interface between two nonspherical zones. For Condition II both definitions are the same; the maximum radius is determined by

$$\xi_2=1/\sin^2\psi_2. \quad (45)$$

For Condition I one finds

$$\xi_1=1/\sin^2\psi_1, \quad (46a)$$

using the alternative (a), but a more complicated expression:

$$\xi_1\cot\psi_2=3\cot\psi_1+\cot^3\psi_1-2(\xi_1-1)^{\frac{1}{2}} \quad (46b)$$

with alternative (b).

With the rigorous metric (43), symmetry shows that a radial line drawn from one singularity at $r=\sqrt{3}+0.844$ to the Schwarzschild radius of the fifth particle at 11.314 will be a suitable geodesic. Therefore

$$2d=\int_{\sqrt{3}+0.844}^{11.314}\left(1+\frac{2\sqrt{2}}{r-\sqrt{3}}+\frac{6\sqrt{2}}{l}\right)^2 dr \quad (47)$$

with

$$l^2=8/3+(r+\sqrt{3}/3)^2.$$

Evaluation of this integral gives

$$2d=83.70$$

or, in dimensionless units,

$$d/2m^*=0.925. \quad (48)$$

Values of this dimensionless ratio, as determined from the Schwarzschild cell approximation and from the rigorous metric, are given in Table V for the five and eight-body universes. Once again the Schwarzschild-cell treatment gives more reasonable results using Condition I, which allows the spheres of influence to overlap slightly.

Given the rigorous metric (43) at a reasonably fine lattice of points, an electronic computer can in principle calculate the metric at ensuing time intervals. Such a calculation may become practicable in the future. Recently progress has been made at Livermore, Los Alamos, and Princeton in treating problems of hydrodynamics where the velocity, pressure, and density depend on two space coordinates and one time coordinate. In the present problem the metric quantities depend upon one more space coordinate. It would be easier first to treat the problem of two mass singularities in an asymptotically flat universe. This problem idealizes the head on collision of two masses in a closed universe by assuming that the two masses in question are very far away from all the other masses that curve up the metric into a closed space.

TABLE V. Distances along a geodesic, at the moment of maximum expansion, from the Schwarzschild singularity to the boundary of a zone, in units of the Schwarzschild radius, $2m^*=2Gm/c^2$, of one of the masses.

Number of particles in lattice universe		5	8
Distance in Schwarzschild metric from singularity to spherical boundary, where radius of sphere is determined by Condition	I	1.187	1.812
	II	1.694	2.295
Distance in Schwarzschild metric from one singularity to point halfway to a nearest neighbor (center of interface between two zones), using Condition	I	0.961	1.599
	II	1.694	2.295
Distance in rigorous metric (43) from one singularity to point halfway to a nearest neighbor		0.925	1.609

7. POSSIBLE GENERALIZATIONS

Variational Formulation

Can the problem of a lattice universe be formulated in a way that does not make such heavy demands on the symmetry of the elementary lattice cell? An analogous familiar problem is to solve

$$\nabla^2\psi+k^2\psi=0, \quad (49)$$

in a two-dimensional region subject to the condition that ψ vanish on a boundary of unsymmetric shape, such as a triangle. One approximate method of solution uses the variation principle,

$$k^2=\text{minimum of }\frac{\int(\nabla\psi)^2dxdy}{\int\psi^2dxdy}. \quad (50)$$

For example, from the center of gravity of the triangle lines are drawn to the three corners to divide the triangle into three zones, I, II, III. In zone I a trial solution is defined by

$$\psi_{\text{trial I}}=\frac{\sin(k\cdot\text{distance from outer edge to }x,y)}{\sin(k\cdot\text{distance from outer edge to center})}; \quad (51)$$

similarly in II and III. The trial solution is everywhere continuous and satisfies the boundary conditions; it also satisfies the differential equation (49) everywhere except at the boundaries between zones. This permits one to express the right-hand side of (50) in terms of integrals taken exclusively along zone boundaries. In each such integral there enters the jump in the normal derivative of ψ across the boundary. These jump integrals are easily evaluated, giving a simple approximate eigenvalue for the wave equation in the region in question.

In analogy one can try to use the variational principle of general relativity,

$$\delta \int R(-g)^{\frac{1}{2}} dx^1 dx^2 dx^3 dx^4 = 0, \tag{52}$$

to formulate the Schwarzschild-cell method more precisely. The trial metric is constructed by joining together N-Schwarzschild metrics into an over-all metric of the appropriate symmetry. The boundaries are defined by the intersections of the Schwarzschild cells. They are not spherical. They are perhaps most easily visualized in the appropriate Euclidean imbedding space. The trial metric is uniquely determined by a single unknown function. It can be considered to represent the radius of the universe as a function of time, or the separation of nearest neighbors as a function of time. The variational integral (52) can be expressed in terms of the properties of this unknown function. The variational principle leads to a differential equation for its determination. This procedure gives the best possible Schwarzschild cell approximation to the accurate solution, "best" in the sense of the variational principle. We see no reason to expect from this method results substantially better than or different from those that come from a more intuitive formulation of the Schwarzschild cell method.

The dependent variables do not match smoothly at zone boundaries in the variational formulation of the problem, either of the lattice universe or of the wave equation (49). The variational principle minimizes the discrepancy only on the average, not locally. Therefore one cannot expect the spherical lattice cell approximation to satisfy the boundary conditions of O'Brien and Synge exactly.[12]

These authors deduced the boundary conditions that must be satisfied by the metric and its derivative at any three-dimensional surface of discontinuity in space-time. If this surface is defined by

$$x^4 = 0,$$

where the coordinate x^4 may be either space-like or time-like, the quantities

$$g_{\mu\nu}, \frac{\partial g_{\mu\nu}}{\partial x^i}, \frac{\partial^2 g_{\mu\nu}}{\partial x^i \partial x^j} \quad \text{and} \quad \frac{\partial g_{ij}}{\partial x^4} \begin{pmatrix} i, j = 1, 2, 3 \\ \mu, \nu = 1, 2, 3, 4 \end{pmatrix} \tag{53}$$

must all be continuous. Thus one can only permit jumps in the four derivatives

$$\partial g_{4\mu}/\partial x^4. \tag{54}$$

The variational treatment gives up the continuity requirements on the quantities (53) locally, but satisfies them on the average.

Coalescence of Schwarzschild Singularities

What happens to the lattice universe at late stages of the contraction? The interface between two lattice cells is specified in the Schwarzschild-cell approximation by the motion of an infinitesimal test particle falling towards one Schwarzschild singularity; but it is well known that the test particle will arrive at the singularity in a finite proper time[13] [compare (30) or (32)] even though the necessary interval of Schwarzschild time is infinite. It will no longer be reasonable to approximate a lattice cell as a spherically symmetric region when the separation of two lattice cells becomes comparable to the radius of the Schwarzschild singularity at the center of each. From Table IV it appears that this degree of approach will be attained relatively soon in the case of the 5-mass system, and relatively late for the system of 600 mass centers. The surface of singularity will be deformed from its normal spherical form, $r = 2m^*$. This deformation might trigger off an exponentially growing disturbance in the singularity, causing it to become violently disturbed and possibly to explode. Such an outcome seems unlikely in view of considerations in another paper on the stability of a Schwarzschild singularity.[14]

If stability obtains, three alternatives suggest themselves for the manner of approach of the singularities: (1) Approaching faces flatten more and more. The remaining space continuum has the character of a thin film separating polyhedral dice from each other. Ultimately this film is pressed to nothingness. (2) The film becomes ever thinner but never disappears. (3) Approaching faces bulge out to meet each other. On contact they join. The regions interior to the N-Schwarzschild singularities suddenly become connected to each other. Equidistant from neighboring singularities there are still regions of space. Moreover, there still remain channels between these regions. However, a little later these channels are pinched off and the regions of space become islands. The islands continue to shrink in size and eventually disappear.

To analyze by way of the field equations the limiting analytical forms for the metric near a point or surface

[12] S. O'Brien and J. L. Synge, *Jump Conditions at Discontinuities in General Relativity* (Dublin, Institute for Advanced Studies, 1952).

[13] See reference 8. Synge cites there H. P. Robertson's announcement of the same conclusion in 1939.

[14] T. Regge and J. A. Wheeler (to be published).

of coalescence is a fascinating problem for the future, the answer to which might help to decide between these alternatives or some other possible outcome.

When the disposition of the N-mass centers in the lattice universe is almost but not quite symmetric, a situation arises much like that in a Freidmann universe when the mass distribution departs slightly from uniformity. Unless the initial conditions are very special, the magnitude of the disturbance will grow. When underwater bubbles undergo dilatational oscillations, and when the surface departs slightly from spherical symmetry at the phase of maximum expansion, then the magnitude of the disturbances ordinarily grows. In this case one can follow the phenomena far enough visually to see that prongs and spikes form. The impression is gained that the bubble changes over from contraction to expansion, not everywhere simultaneously over its surface, but more after the fashion of a glove being turned inside out one finger at a time. If the analogy is any guide, the not quite symmetrical lattice universe will be expected to show a similar behavior. One will expect first a few Schwarzschild singularities to amalgamate and then break apart, then others to fall in, amalgamate, and break apart again, and so on, with some parts of the system therefore still contracting while others have already begun reexpansion. To show the beginnings of such a behavior, a perturbation theory analysis of the regular lattice universe should suffice. To follow the later and more interesting phases of the turnabout would demand a much more elaborate scheme of analysis.

ACKNOWLEDGMENTS

We are indebted to Charles Misner for many discussions of the problem of the lattice universe and for his help in constructing the special metric of Sec. 6.

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Observables in Singular Theories by Systematic Approximation*†

EZRA NEWMAN‡ AND PETER G. BERGMANN

Syracuse University, Syracuse, New York

1. INTRODUCTION

IN any general-relativistic theory the field variables necessarily carry some information that relates to the choice of frame of reference rather than to the physical situation. Two manifestly different fields of gravitational potentials may describe one and the same gravitational field, merely in terms of two different coordinate systems. General relativity differs from special relativity in the degree of freedom inherent in the choice of coordinate system. Whereas in a Lorentz-covariant theory the frame of reference may be chosen at one instant in time and then remains fixed (the whole freedom of choice reducing to the determination of ten parameters once and for all), the freedom in a general-relativistic theory amounts to the determination of four arbitrary functions throughout space, anew at each instant in time. As a result of this vast freedom of choice, general-relativistic dynamical laws cannot be expected[1,2] to permit the integration of the field equations in the sense that suitable initial-value conditions at one time t_0 predict the value of any component of the metric tensor $g_{\mu\nu}$, or, for that matter, of any other conventional field variable, at some space point x^s at a different time t. Nevertheless, Einstein's general theory of relativity is quite deterministic. Its field equations do determine the gravitational field from initial-value conditions for all times to come. The metric potentials are simply not the quantities that are determined completely by the physical situation.

Failure of the mathematical theory to predict the value of a field variable at a given world point corresponds to physical unobservability. It is impossible to devise an experiment that will measure some field at the world point with the coordinates x^μ, because the values of the coordinates by themselves do not identify that world point. In actual practice, a world point at which some measurement is to be made is always identified in some other manner, such as the convergence of a beam of light or the location of a material component of our instrumentation. The determination of a gravitational potential at a world point and in directions that are defined by the values of specified electromagnetic quantities represents, of course, something different from the determination of that same potential at a world point specified by nothing but

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‡ Now at University of Pittsburgh, Pittsburgh, Pennsylvania.

[1] P. G. Bergmann, Phys. Rev. **75**, 680 (1949).

[2] P. G. Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1953).