Gravitational Field of an Axially Symmetric System in First Approximation

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THE question as to whether a material system can radiate gravitational waves, on the basis of the field equations of the general theory of relativity, does not appear to have been answered conclusively. The present work is intended as a first step in looking for the answer.

To investigate the possibility of a physical system radiating gravitational waves it is desirable to choose a simple system, one with axial symmetry. If the field of such a system is described by means of a spherical polar coordinate system $(x^1,x^2,x^3,x^4)\equiv(r,\vartheta,\varphi,t)$,then by a suitable choice of coordinates one can satisfy two conditions: (a) the metric tensor $g_{\mu\nu}$ is independent of the angle φ; (b) it is diagonal.

The field equations in the empty space surrounding the system,[1]

$$G_{\mu\nu}\equiv R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=0, \quad (1)$$

give a set of 7 equations, since $G_{\mu\nu}$ vanishes identically if $(\mu,\nu)=(1,3)$, $(2,3)$, $(3,4)$. These equations serve to determine the 4 diagonal components of $g_{\mu\nu}$. Among the equations there exist 3 identities, the contracted Bianchi identities, except for the one with index 3, which is trivial.

The field equations are nonlinear and difficult to solve exactly. It is proposed to solve them by the method of successive approximations. As a beginning, the first approximation is obtained here. Let us write the line element in the form

$$ds^2=-(1+\rho)dr^2-r^2(1+\sigma)d\vartheta^2 -r^2\sin^2\vartheta(1+\tau)d\varphi^2+(1+\mu)dt^2, \quad (2)$$

where ρ, σ, τ, and μ are functions of r, ϑ, and t and are considered (along with their derivatives) to be small of the first order. The linear approximation of the field equations has the following form (indexes denoting partial differentiation):

$$\sigma_{44}+\tau_{44}-\frac{1}{r^2}(\tau_{22}+\mu_{22})+\frac{\mathrm{ctg}\vartheta}{r^2}(\sigma_2-2\tau_2-\mu_2) -\frac{1}{r}(\sigma_1+\tau_1+2\mu_1)+\frac{2}{r^2}(\rho-\sigma)=0, \quad (3a)$$

$$-\tau_{11}-\mu_{11}+\rho_{44}+\tau_{44}-\frac{\mathrm{ctg}\vartheta}{r^2}(\rho_2+\mu_2) +\frac{1}{r}(\rho_1-2\tau_1-\mu_1)=0, \quad (3b)$$

$$-\sigma_{11}-\mu_{11}+\rho_{44}+\sigma_{44}-\frac{1}{r^2}(\rho_{22}+\mu_{22}) +\frac{1}{r}(\rho_1-2\sigma_1-\mu_1)=0, \quad (3c)$$

$$\sigma_{11}+\tau_{11}+\frac{1}{r^2}(\rho_{22}+\tau_{22})+\frac{\mathrm{ctg}\vartheta}{r^2}(\rho_2-\sigma_2+2\tau_2) -\frac{1}{r}(2\rho_1-2\sigma_1-3\tau_1)-\frac{2}{r^2}(\rho-\sigma)=0, \quad (3d)$$

$$\tau_{12}+\mu_{12}-\frac{1}{r}(\rho_2+\mu_2)-\mathrm{ctg}\vartheta(\sigma_1-\tau_1)=0, \quad (3e)$$

$$\sigma_{14}+\tau_{14}-\frac{1}{r}(2\rho_4-\sigma_4-\tau_4)=0, \quad (3f)$$

$$\rho_{24}+\tau_{24}-\mathrm{ctg}\vartheta(\sigma_4-\tau_4)=0. \quad (3g)$$

Integrating (3f) and (3g) with respect to t and taking the functions of integration to vanish, we get

$$\sigma_1+\tau_1-\frac{1}{r}(2\rho-\sigma-\tau)=0, \quad (4a)$$

$$\rho_2+\tau_2-\mathrm{ctg}\vartheta(\sigma-\tau)=0. \quad (4b)$$

Differentiating (4a), one obtains an equation which can be put into the form

$$\sigma_{11}+\tau_{11}-\frac{2}{r}(\rho_1-\sigma_1-\tau_1)=0, \quad (5a)$$

while differentiating (4b) gives

$$\rho_{12}+\tau_{12}-\mathrm{ctg}\vartheta(\sigma_1-\tau_1)=0. \quad (5b)$$

Subtracting (5b) from (3e) and integrating with respect to ϑ (with the function of integration taken to vanish), one gets

$$\mu_1-\rho_1-\frac{1}{r}(\rho+\mu)=0, \quad (4c)$$

and from this by differentiation one gets a relation which can be written

$$\mu_{11}-\rho_{11}-\frac{2}{r}\rho_1=0. \quad (5c)$$

[1] H. Dingle, Proc. Natl. Acad. Sci. U. S. 19, 559 (1933).

Finally, by combining (3a), (3b), (3c), (3d), (5a), and (5c), one gets

$$\rho_{11}+\frac{2}{r}\rho_1+\frac{1}{r^2}\rho_{22}+\frac{\mathrm{ctg}\vartheta}{r^2}\rho_2-\rho_{44}=0 \tag{6}$$

that is, a wave equation for ρ.

The general solution of (6) is, of course, well known. It can be expressed, for example, as an expansion in spherical (zonal) harmonics. We limit ourselves here to the term of lowest order that can be expected to give a nontrivial result, that corresponding to an axial quadrupole source at the origin. The method can be extended to multipoles of higher order. We also take the solution to be sinusoidal in time and to represent an outgoing wave. Using the complex form (in the end the real part will be taken), the solution can be written

$$\rho=Ch(x)P_2(\cos\vartheta)e^{-i\omega t} \tag{7}$$

where the frequency ω and C are constants, $x=\omega r$, and

$$h(x)=\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}[J_{\frac{5}{2}}(x)-iJ_{-\frac{5}{2}}(x)] \tag{8}$$

with $J_n(x)$ denoting the Bessel function of order n. Once ρ has been found, the remaining unknowns can be determined by means of the field equations.

The constant C is to be determined in terms of the strength of the source. For this purpose we take a very simple model of the source, in the form of an infinitesimal system consisting of two massive particles situated on the Z axis and connected by a nearly mass-less spring. We assume that the energy-momentum-stress tensor has for its nonvanishing components in the Cartesian coordinate system T'_{33}, T'_{34}, and T'_{44}, with $x^3=z$. Representing the mass density of a particle by a Dirac delta function and neglecting the static part of the tensor, we take

$$T'_{44}=\tfrac{1}{2}p\frac{\partial^2\delta(\mathbf{r})}{\partial z^2}, \tag{9}$$

where

$$p=p_0e^{-i\omega t}, \tag{10}$$

and $\delta(\mathbf{r})$ is the three-dimensional Dirac delta function. From (9) it follows that

$$p=\int z^2T'_{44}d\tau. \tag{11}$$

The remaining components of the tensor are determined by integrating the equations of motion

$$T'^{\mu\nu}{}_{;\nu}=0, \tag{12}$$

which in first approximation have the form

$$T'_{33,3}-T'_{34,4}=0, \tag{13a}$$

$$T'_{43,3}-T'_{44,4}=0. \tag{13b}$$

One obtains

$$T'_{33}=-\tfrac{1}{2}\omega^2p\delta(\mathbf{r}), \tag{14a}$$

$$T'_{34}=-\frac{i}{2}\omega p\frac{\partial(\mathbf{r})}{\partial z}. \tag{14b}$$

Transforming the components to the polar coordinate system and also expressing the three-dimensional delta function $\delta(\mathbf{r})$ in terms of the one-dimensional delta function $\delta(r)$, one gets the following nonvanishing components:

$$T_{11}=-\frac{1}{4\pi}\omega^2p\frac{\cos^2\vartheta}{r^2}\delta(r), \tag{15a}$$

$$T_{12}=\frac{1}{4\pi}\omega^2p\frac{\cos\vartheta\sin\vartheta}{r}\delta(r), \tag{15b}$$

$$T_{14}=\frac{3}{4\pi}i\omega p\frac{\cos^2\vartheta}{r^3}\delta(r), \tag{15c}$$

$$T_{22}=\frac{1}{4\pi}\omega^2p\sin^2\vartheta\delta(r), \tag{15d}$$

$$T_{24}=\frac{3}{4\pi}i\omega p\frac{\sin\vartheta\cos\vartheta}{r^2}\delta(r), \tag{15e}$$

$$T_{44}=\frac{3}{4\pi}p\frac{5\cos^2\vartheta-1}{r^4}\delta(r). \tag{15f}$$

To take account of the source, (1) should be replaced by

$$G_{\mu\nu}=-T_{\mu\nu}, \tag{16}$$

in units such that $8\pi k/c^2=1$. If one follows through the steps previously used in obtaining the wave equation for ρ, (6), one finds that now the corresponding equation has the form

$$\nabla^2\rho+\omega^2\rho=\frac{3}{2\pi}p\left[\frac{5\cos^2\vartheta-4}{r^4}+f(r)\right]\delta(r), \tag{17}$$

where $f(r)$ is an arbitrary function. On the other hand, the solution for ρ previously obtained, as given in (7), for small values of r can be written in the form

$$\rho\simeq-\frac{3iC}{2\omega^3}\frac{\partial^2\left(\frac{1}{r}\right)}{\partial z^2}e^{-i\omega t}, \tag{18}$$

from which one gets

$$\nabla^2\rho\simeq\frac{9iC}{\omega^3}\frac{5\cos^2\vartheta-1}{r^4}\delta(r)e^{-i\omega t}. \tag{19}$$

Comparing (17) with (19) and using the expression for p given by (10), one finds that

$$C=-\frac{i\omega^3}{6\pi}p_0. \tag{20}$$

The solution for all four unknowns, expressed in terms of real functions, is found to be given by

$$\rho=-\frac{\omega^3}{6\pi}p_0[j_2(x)\sin\omega t+j_{-2}(x)\cos\omega t]P_2(\cos\vartheta), \tag{21a}$$

$$\sigma=-\frac{\omega^3}{12\pi}p_0\{[(3k_2(x)+j_2(x))\sin\omega t+(3k_{-2}(x)+j_{-2}(x))\cos\omega t]P_2(\cos\vartheta)-(k_2(x)+j_2(x))\sin\omega t-(k_{-2}(x)+j_{-2}(x))\cos\omega t\}, \tag{21b}$$

$$\tau=-\frac{\omega^3}{12\pi}p_0\{[(k_2(x)-j_2(x))\sin\omega t+(k_{-2}(x)-j_{-2}(x))\cos\omega t]P_2(\cos\vartheta)+(k_2(x)+j_2(x))\sin\omega t+(k_{-2}(x)+j_{-2}(x))\cos\omega t\}, \tag{21c}$$

$$\mu=-\frac{\omega^3}{6\pi}p_0[(2q_2(x)+j_2(x))\sin\omega t+(2q_{-2}(x)+j_{-2}(x))\cos\omega t]P_2(\cos\vartheta). \tag{21d}$$

Here

$$j_n(x)=\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x), \tag{22a}$$

$$k_n(x)=\frac{1}{x}\int_\infty^x j_n(u)du, \tag{22b}$$

$$q_n(x)=x\int_\infty^x u^{-2}j_n(u)du. \tag{22c}$$

In the limit of a static system ($\omega=0$) one finds that μ is proportional to the Newtonian potential, as is to be expected. It will be noted that in the above solution ρ, μ, and $\sigma+\tau$ are of a form corresponding to a quadrupole wave, i.e., they depend on the angle through $P_2(\cos\theta)$. On the other hand, the dependence of $\sigma-\tau$ on the angle is given by $\sin^2\theta$.

We see then that, in the linear approximation, the field equations admit a solution describing gravitational waves emitted by an oscillating physical system. A calculation shows that these waves are real, that is, the Riemann-Christoffel tensor is different from zero, so that they cannot be removed by a coordinate transformation.

To calculate the rate of emission of energy by the material system, one can make use of the gravitational energy-momentum-stress pseudotensor $t_\nu{}^\mu$. For this it is desirable to go over to a Cartesian coordinate system, since in that case the first-order terms in the metric tensor will give second-order terms in $t_\nu{}^\mu$, which will be the terms of lowest order, and these will not depend on terms of higher order in the metric tensor. The calculation is somewhat tedious, and the final result for the rate of energy emission is

$$N=\frac{1}{120\pi}\omega^6 p_0{}^2,$$

in the present units, or

$$N=\frac{k}{15c^5}\omega^6 p_0{}^2,$$

in the usual units. This result agrees with that obtained by others.[2]

It is planned to use the above solution as the starting point for a more accurate calculation. The interesting question is whether the exact equations have a solution going over into the above for sufficiently weak fields.

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[2] L. Landau and E. Lifshitz, *Classical Theory of Fields* (Addison-Wesley Press, Cambridge, 1951), p. 331.