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Equations of Motion in General Relativity Theory and the Action Principle

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1. HISTORICAL INTRODUCTION

THE history of the problem of motion in general relativity theory usually, and rightly so, starts with the 1927 paper by Einstein and Grommer.[1] This showed for the first time that the equations of motion for a test particle (with mass $m \to 0$) need not be added to the field equations, but that they can be deduced from the relativistic field equations. For many years afterward, Einstein, and then Einstein with his collaborators, tackled the problem of the motion of two particles. Their problem was to find whether the equations of motion can also be deduced from the field equations. The answer was given in the 1938 paper by Einstein, Infeld, and Hoffmann[2] in which the two-body problem was solved for the first time.

Independently of us and a little later Fock[3] (1939) also deduced the equations of motion, though only the Newtonian ones, from the field equations. Later Papapetrou[4] (1951) simplified his procedure and deduced the post-Newtonian equations of motion, after Petrova[5] (1949) had done the same thing on the basis of Fock's theory. Petrova's and Papapetrou's results were the same as ours.

What are the essential similarities and differences between Einstein's theory, especially as formulated in the two later papers, (Einstein-Infeld, 1940 and 1949)[2] and the Fock-Papapetrou papers?

In general they have one essential idea in common, but two different ideas, of which only one is essential. The idea common to both sets of papers is the approximation method. We now take up the first difference.

In the school represented by Einstein the field equations in empty space are* in the usual notation:

$$G_{\alpha\beta}=R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R=0. \tag{1.1}$$

Einstein always thought that to use

$$G_{\alpha\beta}=R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}=-8\pi T_{\alpha\beta}, \tag{1.2}$$

$T_{\alpha\beta}$ being the energy-momentum tensor, instead of (1.1) is somehow in bad taste, because we do not know in (1.2) what $T_{\alpha\beta}$ is, and we mix a geometrical tensor on the left side with a physical tensor on the right side. This was the reason for Einstein's long search for a unified field theory in which such a mixture of physics and geometry would not appear.

We know that there is no solution of (1.1) representing spread-out continuous matter. Therefore, by assuming (1.1) we represent matter by means of singularities. The method first used by us consisted in forming certain two-dimensional surface integrals over surfaces enclosing these singularities. The field equations prescribed the laws by which the surfaces enclosing the singularities, and hence these singularities, moved. Therefore, these laws were deduced from the field equations in the post-Newtonian approximation.

However, Fock and Papapetrou consider (1.2) and use definite expressions for $T_{\alpha\beta}$.

This difference does not seem to me to be an essential one for the following reasons. We do not know the real distribution of matter. Neither of these methods depicts reality properly. The use of our method based on (1.1) means: if the two bodies are a great distance apart so we may assume, approximately, central symmetry of the field near one body, then the exact knowledge of density distribution inside the enclosing surface is not essential. Outside the enclosing surfaces, (1.1) is valid.

This difference between the two schools may be characterized by an example of a simple situation: that is, classical gravitational theory. There we have two kinds of equations depending on whether matter is represented by singularities or by a continuous distribution. In the first case we have Laplace's equation

$$\Delta\varphi=0; \tag{1.3}$$

in the second case Poisson's equation

$$\Delta\varphi=4\pi\rho. \tag{1.4}$$

It is more common to write Laplace's equation when thinking about its spherically symmetric solution in the form

$$\Delta\varphi=4\pi m\delta_{(3)}, \tag{1.5}$$

where m is the mass and $\delta_{(3)}$ is the three-dimensional Dirac δ function.

Let us try to use the δ functions consistently in general relativity theory. Here we have in the first approximation, using (1.1) for empty space, the

[1] A. Einstein and J. Grommer, Sitzer. deut. Akad. Wiss. Berlin 2 (1927).
[2] Einstein, Infeld, and Hoffman, Ann. Math. **39**, 66 (1938). A. Einstein and L. Infeld, Ann. Math. **41**, 797 (1940); Can. J. Math. **1**, 209 (1949).
[3] V. Fock, J. Phys. (U.S.S.R.) **1**, 81 (1939).
[4] A. Papapetrou, Proc. Phys. Soc. (London) **64**, 57 (1951).
[5] N. Petrova, J. Phys. (U.S.S.R.) **19**, 989 (1949).
* Greek indices run from 0 to 3, Latin from 1 to 3. Repetition implies summation. The quadratic form for a geodetic coordinate-system is

$$ds^2=\eta_{\alpha\beta}dx^\alpha dx^\beta;\quad \eta_{00}=1;\quad \eta_{0m}=0;\quad \eta_{mn}=-\delta_{mn}.$$

The velocity of light $c=1$.

Laplace equation which we solve by a central symmetric solution. This means that a solution of (1.1) in the first approximation valid everywhere, corresponds to that of (1.2) where the energy momentum tensor $T_{\alpha\beta}$ is proportional to Dirac's δ function. Thus the use of (1.2) with $T_{\alpha\beta}$ proportional to $\delta_{(3)}$ corresponds exactly to our previous considerations of (1.1) with singular solutions, just as Eq. (1.5) is only a different form of (1.3).

Use of (1.2) with $T_{\alpha\beta}$ proportional to $\delta_{(3)}$ functions tremendously simplifies the entire deduction of the equations of motion. This simplification was achieved in my paper[6] of 1954, but the entire procedure can still be clarified and simplified. I present here the derivation of the post-Newtonian equations of motion with almost no tedious calculations.†

Summarizing, though I believe in Einstein's idea of not using the energy momentum tensor, yet I was unfaithful to it, because I used the energy-momentum tensor as proportional to the $\delta_{(3)}$ function to express the singular solutions.

The next difference between Fock's and Einstein's school is more essential. Fock, Petrova, and Papapetrou use the harmonic coordinate system; that is the four equations

$$(-g^{\frac{1}{2}}g^{\alpha\beta})_{,\beta}=\frac{\partial}{\partial x^{\beta}}(-g^{\frac{1}{2}}g^{\alpha\beta}). \tag{1.6}$$

Fock considers the choice of this coordinate system to be extremely important, claiming that its addition to the gravitational equations (plus some conditions at infinity) restricts the coordinate system up to a Lorentz transformation. Thus, for Fock, the choice of the harmonic coordinate condition becomes a fundamental law of nature changing the character of Einstein's general relativity theory into a theory of the gravitational field, valid only in inertial coordinate systems. Others like Papapatrou[4] who based their research on Fock's work, do not go so far, but also regard the coordinate condition (1.6) as essential for deduction of the equations of motion.

In our first paper[2] we used a coordinate condition different from the harmonic one; yet we obtained the same equations of motion later obtained by Papapetrou[4] and Petrova.[5] In a few subsequent papers[6–8] we carefully analyzed the problem.

I maintain that the equations of motion have nothing whatever to do with harmonic coordinate conditions; they have much to do with the method of approximation. It is this method which uniquely determines the equations of motion up to the post Newtonian order. I will show explicitly later that violation of the harmonic coordinate condition does not change the post Newtonian equations of motion as long as we stick to the approximation procedure.

2. SOME NOTATIONS AND MATHEMATICAL PRELIMINARIES

We have a world line $\xi^k(t)$ and a field, say a scalar field φ that depends on coordinates x^k and on time $x^0=t$ and also on the $\xi^k(t)$ and their time derivatives:

$$\varphi=\varphi(x^k,t,\xi^k,\dot{\xi}^k);\quad \dot{\xi}^k=d\xi^k/dt. \tag{2.1}$$

We assume that near the line $\xi^k(t)$ the field becomes singular and has the following form:

$$\varphi=\frac{\overset{-1}{\varphi}}{\rho}+\bar{\varphi}+\overline{\varphi_{,s}}(x^s-\xi^s)+\tfrac{1}{2}\overline{\varphi_{,sr}}(x^s-\xi^s)(x^r-\xi^r)+\cdots. \tag{2.2}$$

Here

$$\rho^2=(x^s-\xi^s)(x^s-\xi^s) \tag{2.3}$$

and, therefore,

$$\bar{\varphi}=\left(\varphi-\frac{\overset{-1}{\varphi}}{\rho}\right)\Bigg|_{x^s=\xi^s}. \tag{2.4}$$

Similarly

$$\overline{\varphi_{,s}}=\frac{\partial}{\partial x^s}\left(\varphi-\frac{\overset{-1}{\varphi}}{\rho}\right)\Bigg|_{x^s=\xi^s}. \tag{2.5}$$

We must distinguish between

$$\overline{\varphi_{,s}}\quad\text{and}\quad \bar{\varphi}_{,s}=\frac{\partial\bar{\varphi}}{\partial\xi^s}=\bar{\varphi}_{,\xi^s}, \tag{2.6}$$

which generally are not equal to each other.

If φ were not singular on the curve $\xi^k(t)$ we could have defined $\bar{\varphi}$ in the following way:

$$\bar{\varphi}=\int\varphi\delta_{(3)}(x^s-\xi^s)d_{(3)}x. \tag{2.7}$$

We can narrow the definition of Dirac's δ functions so that (2.7) remains true even if φ has a singularity up to the kth order. Such δ functions can be constructed (Appendix A) as limits of ordinary functions. By use of such δ functions, we get rid of infinities without recourse to the renormalization procedure. Thus, all δ functions used here will have the property of changing φ into $\bar{\varphi}$, where $\bar{\varphi}$ is a continuous function of the ξ^k, $\dot{\xi}^k$, $\ddot{\xi}^k$. Therefore, we use (2.7) as the definition of $\bar{\varphi}$, where δ is the three-dimensional Dirac δ function, satisfying the following conditions.

1. $\delta(x)$ can be treated formally as a spherically symmetric function for which all the derivatives exist.
2. $\delta(x)=0$ for $x\neq 0$.
3. For every continuous $f(x)$ in the arbitrary region

[6] L. Infeld, Phys. Rev. **53**, 836 (1938); Can. J. Math. **5**, 17 (1953); Acta Phys. Polon. **13**, 205 (1954).

† Some of the ideas presented are due to Mr. Plebański and myself and will appear in fuller form in a book *Relativity of Motion* which we are writing.

[7] A. Einstein and L. Infeld, Ann. Math., Can. J. Math. **1**, 209 (1949).

[8] R. Teisseyre, Acta Phys. Polon. **13**, 47 (1954).

$\Omega(\mathbf{x}_0)$ forming a neighborhood of $\mathbf{x}_0$, we have

$$\int_{\Omega(\mathbf{x}_0)} d_{(3)}x\delta(\mathbf{x}-\mathbf{x}_0)f(\mathbf{x})=f(\mathbf{x}_0).$$

4. For an arbitrary neighborhood $\Omega(0)$ of the point $\mathbf{x}=0$, we have

$$\int_{\Omega(0)} d_{(3)}x\delta(\mathbf{x})|\mathbf{x}|^{-p}=0, \quad \text{for } p=1,2,\cdots k.$$

The fourth condition distinguishes *this* δ function from the usual one. Proof of the inexistence is given in Appendix A. (See also Infeld and Plebański.[9])

Thus the bars mean two things: firstly, singularities are ignored; secondly, for x^k the ξ^k's are introduced.

Returning to the problem of the difference between $\bar{\varphi}_{,s}=\partial\varphi/\partial\xi^s$ and $\overline{\varphi_{,s}}$, we have from (2.7)

$$\bar{\varphi}_{,\xi^s}=\overline{\varphi_{,\xi^s}}+\overline{\varphi_{,s}}. \tag{2.8}$$

This means that $\bar{\varphi}_{,s}$ and $\overline{\varphi_{,s}}$ are equal if and only if $\overline{\varphi_{,\xi^s}}=0$. This is certainly so if the part of φ that gives a contribution to $\bar{\varphi}$ does not depend on ξ^s.

One more formula plays an important role later and follows from the definition (2.7):

$$\bar{\varphi}_{,0}=\frac{d\bar{\varphi}}{dt}=\overline{\varphi_{,0}}+\overline{\varphi_{,s}}\dot{\xi}^s=\overline{\varphi_{,\alpha}}\dot{\xi}^\alpha; \quad (\dot{\xi}^0=1). \tag{2.9}$$

Assume two functions, e.g., φ and ψ:

$$\begin{aligned}\varphi&=\frac{\overset{-1}{\varphi}}{\rho}+\bar{\varphi}+\overline{\varphi_{,s}}(x^s-\xi^s)+\tfrac{1}{2}\overline{\varphi_{,sr}}(x^s-\xi^s)(x^r-\xi^r)+\cdots,\\ \psi&=\frac{\overset{-3}{\psi}}{\rho^3}+\frac{\overset{-1}{\psi}}{\rho}+\bar{\psi}+\overline{\psi_{,s}}(x^s-\xi^s)+\cdots.\end{aligned} \tag{2.10}$$

We have in this case

$$\overline{\varphi\psi}=\bar{\varphi}\bar{\psi}. \tag{2.11}$$

Forming $\varphi\psi$ and ignoring singular expressions and those that vanish for $x^s=\xi^s$, we are left with the following expressions of order zero in $(x^s-\xi^s)$:

$$\bar{\varphi}\bar{\psi}+\overset{-1}{\varphi}\overline{\varphi_{,s}}\frac{x^s-\xi^s}{\rho}+\overset{-1}{\varphi}\overline{\psi_{,s}}\frac{x^s-\xi^s}{\rho}+\overset{-3}{\psi}\tfrac{1}{6}\overline{\varphi_{,srp}}\frac{(x^s-\xi^s)(x^r-\xi^r)(x^p-\xi^p)}{\rho^3}.$$

The bars over these expressions give

$$\overline{\varphi\psi}=\int\delta_{(3)}d_{(3)}x\left[\bar{\varphi}\bar{\psi}+(\overline{\varphi_{,s}}\overset{-1}{\psi}+\overline{\psi_{,s}}\overset{-1}{\varphi})\frac{x^s-\xi^s}{\rho}+\tfrac{1}{6}\overline{\varphi_{,srp}}\overset{-3}{\psi}\frac{(x^s-\xi^s)(x^r-\xi^r)(x^p-\xi^p)}{\rho^3}\right].$$

The last two expressions under the integral sign give zero, because they are products of *symmetric* δ functions and *odd* powers of $(x^s-\xi^s)$. Thus we obtain

$$\overline{\varphi\psi}=\int\bar{\varphi}\bar{\psi}\delta d_{(3)}x=\bar{\varphi}\bar{\psi}.$$

But (2.11) would not be true if φ or ψ had a singularity of order ρ^{-2}. Thus (2.11) is true if ρ appears to an *odd* power in the singular parts of φ and ψ. Therefore, we have to be cautious in applying the last equation.

Thus, we have vectors and tensors defined only *along* the curve, like $\bar{S}^\alpha$, $\bar{T}^{\alpha\beta}$, etc. We can define the metric tensor *along* the *curve* $\overline{g_{\alpha\beta}}$ and $\overline{g^{\alpha\beta}}$ and assuming (2.11) we have

$$\overline{g_{\alpha\rho}g^{\beta\rho}}=\overline{g_{\alpha\rho}}\,\overline{g^{\beta\rho}}=\delta_\alpha{}^\beta. \tag{2.12}$$

To such tensors we can apply tensor algebra and tensor analysis but *only* along the curve.

Since

$$\overline{g_{\alpha\beta}}=\int_{\Omega(3)} g_{\alpha\beta}\delta d_{(3)}x \tag{2.13}$$

(where $\Omega_{(3)}$ is a small three-dimensional neighborhood surrounding the singularity), $\overline{g_{\alpha\beta}}$ will be a tensor along the curve if $\delta_{(3)}d_{(3)}x$ is an invariant.

The four-dimensional relativistic Dirac δ function $\delta_{(4)}$, is a scalar density because of the invariant equation

$$\int\delta_{(4)}d_{(4)}x=1. \tag{2.14}$$

Now let us take

$$\delta_{(4)}=\delta_{(4)}(x^\alpha-\xi^\alpha), \tag{2.15}$$

where

$$\xi^\alpha=\xi^\alpha(\lambda), \tag{2.16}$$

λ being an invariant parameter. Then we can form an invariant density function

$$\int_{-\infty}^{+\infty}\delta_{(4)}d\lambda=\int_{-\infty}^{+\infty}\delta_{(4)}\frac{d\lambda}{d\xi^0}d\xi^0=\frac{d\lambda}{dt}\delta_{(3)}. \tag{2.17}$$

As far as the transformation properties are concerned, this is the definitions of $\delta_{(3)}$. Thus $\delta_{(3)}$ is the zero component of a density vector. From this definition it follows that

$$\begin{aligned}\int_{\Omega(3)}\delta_{(3)}d_{(3)}x&=\int_{\Omega(3)}\delta_{(3)}\frac{d\lambda}{dt}d_{(3)}x\frac{dt}{d\lambda}\\ &=\int_{\Omega(3)}\int_{t=-\infty}^{t=+\infty}\delta_{(4)}d_{(3)}x dt=\int_{\Omega(4)}\delta_{(4)}d_{(4)}x.\end{aligned} \tag{2.18}$$

Thus the space integral of $\delta_{(3)}$ is an invariant.

One more remark concerning the notation: If we have many curves we shall distinguish between them by the index written above to the left: ${}^a\xi^s$; $a=1,2\cdots p$. Then

[9] L. Infeld and J. Plebański, Bull. Acad. Polon. III 4, 689 (1956); 5, 51 (1957).

we should also write ${}^a\bar{\varphi}$, meaning:

$$ {}^a\bar{\varphi}=\int_{{}^a\Omega_{(3)}}\varphi\,{}^a\delta d_{(3)}x;\quad {}^a\delta=\delta_{(3)}(x^s-{}^a\xi^s). \qquad (2.19)$$

However, for simplicity we do not write a above the bar always understanding that it means the *first* (or only) curve: $\bar{\varphi}={}^1\bar{\varphi}$.

3. THE GRAVITATIONAL EQUATIONS

The gravitational equations expressed in contravariant tensor densities are

$$\mathbf{G}^{\alpha\beta}=\mathbf{R}^{\alpha\beta}-\tfrac{1}{2}g^{\alpha\beta}\mathbf{R}=-8\pi\mathbf{T}^{\alpha\beta}. \qquad (3.1)$$

In our case of, say, two particles moving along lines ${}^1\xi^s$ and ${}^2\xi^s$ we have

$$T^{\alpha\beta}={}^1t^{\alpha\beta}\,{}^1\delta+{}^2t^{\alpha\beta}\,{}^2\delta. \qquad (3.2)$$

We introduce an invariant and finite line element, concerning the first particle (without writing the "one" to the left)

$$\overline{ds}^2=\overline{g_{\alpha\beta}}d\xi^\alpha d\xi^\beta. \qquad (3.3)$$

From the tensor density we now form a *tensor* along a curve (the first one)

$$\frac{dt}{\overline{ds}}\int_{\Omega_{(3)}}\mathbf{T}^{\alpha\beta}d_{(3)}x=\overline{t^{\alpha\beta}}\frac{dt}{\overline{ds}}. \qquad (3.4)$$

In the next section we see that consistency of the field equations demands the following equations (Tulczyjew[10]):

$$t^{\alpha\beta}\frac{dt}{\overline{ds}}=\mu\xi^{\alpha\prime}\xi^{\beta\prime};\quad \xi^{\alpha\prime}=\frac{d}{\overline{ds}}\xi^\alpha;\quad \mu=\text{rest mass}. \qquad (3.5)$$

We show (Sec. 4) that not only has $t^{\alpha\beta}$ the form (3.5), but also that the rest mass μ is constant. For the moment, however, we use (3.5) as an *assumption*, without stipulating that μ in (3.5) is a constant.

From (3.5) follows

$$t^{\alpha\beta}=\mu\dot{\xi}^\alpha\xi^{\beta\prime}=m\dot{\xi}^\alpha\dot{\xi}^\beta;\quad \dot{\xi}^\alpha=\frac{d}{dt}(\xi^\alpha)$$
$$\frac{m}{\mu}=\frac{dt}{\overline{ds}}, \qquad (3.6)$$

and

$$\mathbf{T}^{\alpha\beta}=\sum_{a=1}^{p}{}^am\,{}^a\dot{\xi}^\alpha\,{}^a\dot{\xi}^\beta\,{}^a\delta. \qquad (3.7)$$

The right-hand side of the gravitational equations is uniquely determined by the condition that $\mathbf{T}^{\alpha\beta}$ depends linearly on the ${}^a\delta$'s.

4. THE GENERAL EQUATIONS OF MOTION

As a consequence of Bianchi's identities we always have

$$\mathbf{G}^{\alpha\beta}{}_{;\beta}=(\mathbf{R}^{\alpha\beta}-\tfrac{1}{2}g^{\alpha\beta}\mathbf{R})_{;\beta}=-8\pi\mathbf{T}^{\alpha\beta}{}_{;\beta}=0 \qquad (4.1)$$

where the semicolon indicates covariant differentiation. Taking the integral of $\mathbf{T}^{\alpha\beta}{}_{;\beta}$ over the three-dimensional region surrounding the first singularity and multiplying by $dt/\overline{ds}$, we have:

$$\frac{dt}{\overline{ds}}\int_{\Omega_{(3)}}\mathbf{T}^{\alpha\beta}{}_{;\beta}d_{(3)}x=\overline{A^\alpha}=0, \qquad (4.2)$$

where $\overline{A^\alpha}$ is a vector defined through (4.2) along the *first* curve. Generally, we have

$${}^a\overline{A^\alpha}=0;\quad \alpha=0,1,2,3;\quad a=1,2\cdots p \qquad (4.3)$$

if p is the number of singularities. Thus in (4.3) we have as a consequence of the field equations, $4p$ equations which we call the *equations of motion* of the p singularities. Indeed they contain $4p$ unknowns:

$${}^a\xi^s(t)\text{ and }{}^am(t);\quad s=1,2,3;\quad a=1,2\cdots p. \qquad (4.4)$$

Let us now write out $\overline{A^\alpha}$ explicitly (omitting the "one" on the left). We start with

$$\mathbf{T}^{\alpha\beta}{}_{;\beta}=\mathbf{T}^{\alpha\beta}{}_{,\beta}+\left\{\begin{matrix}\alpha\\ \mu\nu\end{matrix}\right\}\mathbf{T}^{\mu\nu}. \qquad (4.5)$$

Introducing here (3.8) the first right-hand expression becomes

$$\mathbf{T}^{\alpha\beta}{}_{,\beta}=\mathbf{T}^{\alpha s}{}_{,s}+\mathbf{T}^{\alpha 0}{}_{,0}=({}^1m\,{}^1\dot{\xi}^\alpha\,{}^1\dot{\xi}^s\,{}^1\delta)_{,s}+({}^1m\,{}^1\dot{\xi}^\alpha\,{}^1\delta)_{,0} \qquad (4.6)$$

+similar expressions concerning other particles. We have

$$\frac{dt}{\overline{ds}}\int_{\Omega_{(3)}}({}^1m\,{}^1\dot{\xi}^\alpha\,{}^1\dot{\xi}^s\,{}^1\delta)_{,s}d_{(3)}x=\frac{dt}{\overline{ds}}\int_{{}^1\Omega_{(3)}}\mathbf{T}^{\alpha s}{}_{,s}d_{(3)}x=0. \qquad (4.7)$$

This we can see even without explicit calculations by changing the volume integral into a surface integral which must vanish because ${}^1\delta$ vanishes on the surface of ${}^1\Omega_{(3)}$. Therefore, what remains of the integral of (4.6) is, because of (3.6),

$$\frac{dt}{\overline{ds}}\int_{{}^1\Omega_{(3)}}\mathbf{T}^{\alpha 0}{}_{,0}d_{(3)}x=\frac{dt}{\overline{ds}}\frac{d}{dt}\left[\int_{\Omega_{(3)}}{}^1m\,{}^1\dot{\xi}^\alpha\,{}^1\delta_{(3)}d_{(3)}x\right]$$
$$=\frac{dt}{\overline{ds}}\frac{d}{dt}(m\dot{\xi}^\alpha)=(\mu\xi^{\alpha\prime})'. \qquad (4.8)$$

Thus because of (4.5), (3.6), and the last equation, our equations of motion (4.2) become

$$\mu'\xi^{\alpha\prime}+\mu\xi^{\alpha\prime\prime}+\mu\overline{\left\{\begin{matrix}\alpha\\ \mu\nu\end{matrix}\right\}}\xi^{\mu\prime}\xi^{\nu\prime}=0. \qquad (4.9)$$

We show now that $\mu'=d\mu/\overline{ds}=0$, so the rest mass is constant. We assume that

$$\overline{\left\{\begin{matrix}\alpha\\ \mu\nu\end{matrix}\right\}}=\overline{g^{\alpha\rho}}\overline{[\mu\nu,\rho]}. \qquad (4.10)$$

[10] W. Tulczyjew, Bull. Acad. Polon. III, **5**, 279 (1957).

Then, multiplying (4.9) by $\bar{g}_{\alpha\sigma}\xi^{\sigma'}$, we have

$$\mu'\xi^{\alpha'}\xi^{\sigma'}\overline{g_{\alpha\sigma}}+\mu\xi^{\alpha''}\xi^{\sigma'}\overline{g_{\alpha\sigma}}+\tfrac{1}{2}\mu\overline{g_{\mu\nu,\sigma}}\xi^{\mu'}\xi^{\nu'}\xi^{\sigma'}=0. \quad (4.11)$$

Because of (2.9) this is equal to

$$\mu'+\frac{\mu}{2}\frac{d}{\overline{ds}}(\overline{g_{\alpha\beta}}\xi^{\alpha'}\xi^{\beta'})=0;\quad \mu'=0, \quad (4.12)$$

which changes (4.9) into

$$\frac{1}{\mu}\overline{A^{\alpha}}=\xi^{\alpha''}+\left\{{\overline{\alpha} \atop \mu\nu}\right\}\xi^{\mu'}\xi^{\nu'}=0. \quad (4.13)$$

Thus the rest mass must be constant.

If not for the bar above the Christoffel symbol, our Eq. (4.13) would be that of a geodetic line. For each particle their are four such equations of motion.

Let us put back t instead of $\bar{s}$ into (4.13) using (3.6). We then have:

$$\begin{aligned}&\frac{dm}{dt}+m\left\{{\overline{0} \atop \alpha\beta}\right\}\dot{\xi}^{\alpha}\dot{\xi}^{\beta}=0,\\ &\frac{d}{dt}(m\dot{\xi}^{s})+m\left\{{\overline{k} \atop \alpha\beta}\right\}\dot{\xi}^{\alpha}\dot{\xi}^{\beta}=0;\quad (\dot{\xi}^{0}=1).\end{aligned} \quad (4.14)$$

As before, we have $4p$ equations determining the ${}^{a}\xi^{k}$ and the ${}^{a}m$'s. From the first equation in (4.14) and (3.8) it follows that

$$\frac{m}{\mu}=\exp\left(-\int^{t}\left\{{\overline{0} \atop \alpha\beta}\right\}\dot{\xi}^{\alpha}\dot{\xi}^{\beta}dt\right)=\frac{dt}{\overline{ds}}, \quad (4.15)$$

since

$$\frac{\dot{m}}{m}=-\left\{{\overline{0} \atop \alpha\beta}\right\}\dot{\xi}^{\alpha}\dot{\xi}^{\beta}=\left(\log\frac{m}{\mu}\right)_{,0}. \quad (4.16)$$

From (4.15) we see that

$$\overline{ds}=dt\exp\int_{0}^{t}\left\{{\overline{0} \atop \alpha\beta}\right\}\dot{\xi}^{\alpha}\dot{\xi}^{\beta}dt.$$

This connection between $\overline{ds}$ and dt follows from the equations of motion. The normalization is such that for $t=0$, both dt and $\overline{ds}$ are equal.

Substituting the values (4.16) for $\dot{m}/m$ in the last three equations (4.14) we obtain the $3p$ equations of motion for ${}^{a}\xi^{s}$:

$$\ddot{\xi}^{s}+\left\{{\overline{s} \atop \alpha\beta}\right\}\dot{\xi}^{\alpha}\dot{\xi}^{\beta}-\left\{{\overline{0} \atop \alpha\beta}\right\}\dot{\xi}^{s}\dot{\xi}^{\alpha}\dot{\xi}^{\beta}=0;\quad \dot{\xi}^{0}=1. \quad (4.17)$$

This equation is not suitable for establishing a connection between the equations of motion and a variational principle. To do that rewrite the zeroth equation of (4.14) using (3.6)

$$\left(\log\frac{m}{\mu}\right)_{,0}=\left(\log\frac{dt}{\overline{ds}}\right)_{,0}=-\left\{{\overline{0} \atop \alpha\beta}\right\}\dot{\xi}^{\alpha}\dot{\xi}^{\beta} \quad (4.18)$$

or, since $dt/\overline{ds}=(\overline{g_{\alpha\beta}}\dot{\xi}^{\alpha}\dot{\xi}^{\beta})^{-\frac{1}{2}}$:

$$\begin{aligned}\left[\log(\overline{g_{\alpha\beta}}\dot{\xi}^{\alpha}\dot{\xi}^{\beta})^{-\frac{1}{2}}\right]_{,0}&=-\tfrac{1}{2}(\overline{g_{\alpha\beta}}\dot{\xi}^{\alpha}\dot{\xi}^{\beta})^{-1}(\overline{g_{\alpha\beta}}\dot{\xi}^{\alpha}\dot{\xi}^{\beta})_{,0}\\ &=-\left\{{\overline{0} \atop \alpha\beta}\right\}\dot{\xi}^{\alpha}\dot{\xi}^{\beta}.\end{aligned} \quad (4.19)$$

Therefore, the three equations (4.17) can also be written

$$\ddot{\xi}^{k}+\left\{{\overline{k} \atop \alpha\beta}\right\}\dot{\xi}^{\alpha}\dot{\xi}^{\beta}-\tfrac{1}{2}(\overline{g_{\alpha\beta}}\,\dot{\xi}^{\alpha}\dot{\xi}^{\beta})^{-1}(\overline{g_{\alpha\beta}}\,\dot{\xi}^{\alpha}\dot{\xi}^{\beta})_{,0}\dot{\xi}^{k}=0. \quad (4.20)$$

Let us write, for short

$$\overline{\mathfrak{L}}=(\overline{g_{\alpha\beta}}\,\dot{\xi}^{\alpha}\dot{\xi}^{\beta})^{\frac{1}{2}};\quad (\dot{\xi}^{0}=1). \quad (4.21)$$

Then we can rewrite (4.17) in the form

$$\ddot{\xi}^{\sigma}-(\log\overline{\mathfrak{L}})_{,0}\dot{\xi}^{\sigma}+\left\{{\overline{\sigma} \atop \alpha\beta}\right\}\dot{\xi}^{\alpha}\dot{\xi}^{\beta}=0. \quad (4.22)$$

The "zero" equation ($\sigma=0$) gives the known equation (4.19). We multiply (4.22) by $\overline{g_{\sigma r}}$ and assume, that for $\overline{g_{\alpha\beta}}$ and their derivatives (2.11) is always valid, that is the barred product is equal to the products of the barred expressions. Then we have, because of (2.9) the three equations of motion,

$$(\overline{g_{\sigma r}}\,\dot{\xi}^{\sigma})_{,0}-\overline{g_{\sigma r}}\,\dot{\xi}^{\sigma}(\log\overline{\mathfrak{L}})_{,0}-\tfrac{1}{2}\overline{g_{\alpha\beta,r}}\,\dot{\xi}^{\alpha}\dot{\xi}^{\beta}=0. \quad (4.23)$$

This suggests the existence of a Lagrangian: $\overline{\mathfrak{L}}$. We wish to see whether (4.23) is equivalent to

$$\frac{d}{dt}\frac{\partial\overline{\mathfrak{L}}}{\partial\dot{\xi}^{k}}-\frac{\partial\overline{\mathfrak{L}}}{\partial\xi^{k}}=0. \quad (4.24)$$

We find

$$\begin{aligned}\frac{\partial\overline{\mathfrak{L}}}{\partial\xi^{k}}&=\tfrac{1}{2}\overline{\mathfrak{L}}^{-1}\overline{g_{\alpha\beta,k}}\,\dot{\xi}^{\alpha}\dot{\xi}^{\beta},\\ \frac{\partial\overline{\mathfrak{L}}}{\partial\dot{\xi}^{k}}&=\tfrac{1}{2}\left(\frac{\partial\overline{g_{\alpha\beta}}}{\partial\dot{\xi}^{k}}\dot{\xi}^{\alpha}\dot{\xi}^{\beta}+2\overline{g_{\alpha k}}\,\dot{\xi}^{\alpha}\right)\overline{\mathfrak{L}}^{-1},\\ \frac{d}{dt}\frac{\partial\overline{\mathfrak{L}}}{\partial\dot{\xi}^{k}}&=\tfrac{1}{2}\left(\frac{\partial\overline{g_{\alpha\beta}}}{\partial\dot{\xi}^{k}}\dot{\xi}^{\alpha}\dot{\xi}^{\beta}\overline{\mathfrak{L}}^{-1}\right)_{,0}+(\overline{g_{\alpha k}}\,\dot{\xi}_{\alpha})_{,0}\overline{\mathfrak{L}}^{-1}\\ &\quad+\overline{g_{\alpha k}}\,\dot{\xi}^{\alpha}(\overline{\mathfrak{L}}^{-1})_{,0}.\end{aligned} \quad (4.25)$$

Therefore the Lagrange equation (4.24) becomes

$$(\overline{g_{\alpha k}}\,\dot{\xi}^\alpha)_{,0}-\overline{g_{\alpha k}}\,\dot{\xi}^\alpha(\log\overline{\mathfrak{L}})_{,0}-\tfrac{1}{2}\overline{g_{\alpha\beta,k}}\dot{\xi}^\alpha\dot{\xi}^\beta+\frac{1}{2}\left(\frac{\partial\overline{g_{\alpha\beta}}}{\partial\dot{\xi}^k}\dot{\xi}^\alpha\dot{\xi}^\beta\,\overline{\mathfrak{L}}^{-1}\right)_{,0}\overline{\mathfrak{L}}=0. \quad (4.26)$$

Comparing (4.26) and (4.23) we see that $\overline{\mathfrak{L}}$ is a Lagrangian if

$$\overline{g_{\alpha\beta,s}}=\overline{g_{\alpha\beta,s}}\frac{\partial\overline{g_{\alpha\beta}}}{\partial\xi^s},\quad\frac{\partial\overline{g_{\alpha\beta}}}{\partial\dot{\xi}^s}=0. \quad (4.27)$$

This means: $\overline{ds/dt}$ is a Lagrangian if the relevant part of $g_{\alpha\beta}$ does not depend on ${}^1\xi^s$ or ${}^1\dot{\xi}^s$. We shall see later that the first condition, that is the independence of the relevant part of $g_{\alpha\beta}$ from ${}^1\xi^s$ is not always fulfilled.

Thus (4.23) follows from a variational principle:

$$\delta\int_{t_1}^{t_2}\overline{\mathfrak{L}}dt=\delta\int_{t_1}^{t_2}dt\left[\int_{\Omega(3)}d_{(3)}x\delta_{(3)}(x^s-\xi^s)g_{\alpha\beta}\dot{\xi}^\alpha\dot{\xi}^\beta\right]^{\frac{1}{2}}=0, \quad (4.28)$$

if we *treat* $g_{\alpha\beta}$ *as function* of x^α *only* that is if, while varying the function under the integration sign, we ignore the possible dependence of $g_{\alpha\beta}$ on ${}^1\xi$ and ${}^1\dot{\xi}$. Then, as can also be shown by a straightforward calculation, (4.28) is equivalent to (4.23). Call the result of such variation the "geodetic line," then (4.23) is the equation of a "geodetic line."[11]

For a *test* particle—that is if $\mu\rightarrow 0$ and $g_{\alpha\beta}$ is not singular and does not depend on ${}^1\xi$ or ${}^1\dot{\xi}$—Eq. (4.23) is that of a geodetic line and the "—" means only the substitution of ${}^1\xi^s$ for x^s.

One can show (Tulczyjew[10]) that the equations of motion follow from the field equations once we assume the linear dependence of $\mathbf{T}^{\alpha\beta}$ from the δ's, that is the form (3.6) for $t_{\alpha\beta}$ and (3.7) for $\mathbf{T}_{\alpha\beta}$ is a consequence of the field equations.

From Bianchi's identities follows:

$$\frac{dt}{ds}\int_{{}^1\Omega(3)}\theta\mathbf{T}^{\alpha\beta}{}_{;\beta}d_{(3)}x=0, \quad (4.29)$$

where θ is an *arbitrary* function continuous on the worldline $\xi^s(t)$. Or, again omitting the "one" over the $t^{\alpha\beta}$'s and δ's we can write

$$\frac{dt}{ds}\int_{{}^1\Omega(3)}\theta(t^{\alpha\beta}\delta_{(3)})_{;\beta}d_{(3)}x=0. \quad (4.30)$$

[11] L. Infeld and J. Plebański, Bull. Acad. Polon. III, 4, 749 (1956).

Thus this leads to the following:

$$\overline{A^\alpha}\,\overline{\theta}+\overline{A^{\alpha\beta}}\,\overline{\theta_{,}{}^{\beta}}=0. \quad (4.31)$$

Let us start by calculating $\overline{A^{\alpha\beta}}$. Because

$$\delta_{,0}=-\delta_{,s}\dot{\xi}^s, \quad (4.32)$$

we have

$$\overline{\theta_{,\beta}}\,\overline{A^{\alpha\beta}}=\frac{dt}{\overline{ds}}(-t^{\alpha s}+t^{\alpha 0}\dot{\xi}^s)\overline{\theta_{,s}}. \quad (4.33)$$

Since $\dot{\xi}^0=1$, this can be written

$$\overline{\theta_{,\beta}}\,\overline{A^{\alpha\beta}}=\frac{dt}{\overline{ds}}(-t^{\alpha\beta}+t^{\alpha 0}\dot{\xi}^\beta)\overline{\theta_{,\beta}}. \quad (4.34)$$

Because $\overline{\theta_{,\beta}}$ is arbitrary, we have

$$\overline{A^{\alpha\beta}}=\frac{dt}{\overline{ds}}(-t^{\alpha\beta}+t^{\alpha 0}\dot{\xi}^\beta)=0. \quad (4.35)$$

Putting here $\alpha=0$ we find

$$t^{0\beta}=t^{00}\dot{\xi}^\beta, \quad (4.36)$$

therefore, generally

$$t^{\alpha\beta}=t^{00}\dot{\xi}^\alpha\dot{\xi}^\beta=m\dot{\xi}^\alpha\dot{\xi}^\beta, \quad (4.37)$$

which is the proof of the theorem. Obviously $\overline{A^\alpha}=0$ gives the equations of motion.

5. THE APPROXIMATION METHOD

We solve the field equations and formulate the equations of motion explicitly by means of the approximation method to be described.

Let us assume a function developed in a power series in the parameter $\lambda=(1/c)$ (assume c arbitrary and not equal to "one").

$$\varphi={}_0\varphi+{}_1\varphi+{}_2\varphi+\cdots. \quad (5.1)$$

The indices written as left subscripts indicate the order of λ absorbed by the φ's.

If the function $\varphi(x^\mu)$ varies rapidly in space but slowly with x^0, then we are justified in not treating all its derivatives in the same manner. The derivatives with respect to x^0 will be of a higher order than the space derivatives. We can formalize this procedure by assuming

$$\frac{\partial}{\partial x^0}({}_l\varphi)={}_{l+1}\varphi_{,0}; \quad (5.2)$$

that is differentiation with respect to x^0 raises the power of λ, absorbed by the φ's, by one.

The problem now is: with what order should we start the power development of the quantities appearing in the field equations?

The quantity ξ^s will start (by an obvious convention) with the order "zero." It will be an unknown quantity determined by the equations of motion; we shall for the moment not develop ξ^s into a power series. Thus $\dot{\xi}^s$ will be of the order "one" and $\ddot{\xi}^s$ of the order "two."

From this follows

$$ {}^a m = {}_2{}^a m + {}_4{}^a m + {}_6{}^a m + \cdots. \quad (5.3)$$

To begin with, ${}_2 m$ is not pure convention. Indeed in the Newtonian approximation, which we hope to obtain, we have in the chosen units

$$\text{mass} \times \text{acceleration} = \frac{\text{mass} \times \text{mass}}{(\text{distance})^2}.$$

Since the acceleration is of order two, the orders of both sides will be equal, only if the order of mass is also "two."

In all the power developments we take into account only even (as in ${}^a m$), or only odd powers of λ (Infeld,[6] 1938).

Thus, because of the order with which we start ${}^a m$ and ξ^s, we have

$$\begin{aligned} \mathbf{T}^{00} &= {}_2\mathbf{T}^{00} + {}_4\mathbf{T}^{00} + {}_6\mathbf{T}^{00} + \cdots, \\ \mathbf{T}^{0m} &= {}_3\mathbf{T}^{0m} + {}_5\mathbf{T}^{0m} + \cdots, \\ \mathbf{T}^{mn} &= {}_4\mathbf{T}^{mn} + {}_6T^{mn} + \cdots. \end{aligned} \quad (5.4)$$

Now as to the g's we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}; \quad g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}. \quad (5.5)$$

From the gravitational equation follows:

$$\mathbf{R}_{\alpha\beta} = -8\pi(\mathbf{T}_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta}\mathbf{T}), \quad (5.6)$$

where

$$\mathbf{R}_{\alpha\beta} = R_{\alpha\beta}\sqrt{-g},$$

$$R_{\mu\nu} = -\left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\}_{,\rho} + \left\{ \begin{matrix} \rho \\ \mu\rho \end{matrix} \right\}_{,\nu} + \left\{ \begin{matrix} \rho \\ \mu\sigma \end{matrix} \right\}\left\{ \begin{matrix} \sigma \\ \rho\nu \end{matrix} \right\} - \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\}\left\{ \begin{matrix} \sigma \\ \rho\sigma \end{matrix} \right\}. \quad (5.7)$$

From the right-hand side of Eq. (5.6) follows: R_{00} and R_{mn} start with the order "2" and R_{0m} with the order "3." The lowest order expressions on the left-hand side are

$$\begin{aligned} &\text{for } R_{00}: \quad -\tfrac{1}{2}h_{00,\,ss}, \\ &\text{for } R_{mn}: \quad -\tfrac{1}{2}h_{mn,\,ss} + \tfrac{1}{2}h_{ms,\,ns} + \tfrac{1}{2}h_{ns,\,ms} + \tfrac{1}{2}h_{00,\,mn} - \tfrac{1}{2}h_{ss,\,mn}, \\ &\text{for } R_{0m}: \quad -\tfrac{1}{2}h_{0m,\,ss} + \tfrac{1}{2}h_{0s,\,ms} + \tfrac{1}{2}h_{ms,\,0s} - \tfrac{1}{2}h_{ss,\,m0}. \end{aligned} \quad (5.8)$$

Therefore,

$$\begin{aligned} h_{00} &= {}_2h_{00} + {}_4h_{00} + \cdots, \\ h_{0m} &= {}_3h_{0m} + {}_5h_{0m} + \cdots, \\ h_{mn} &= {}_2h_{mn} + {}_4h_{mn} + \cdots. \end{aligned} \quad (5.9)$$

All the functions that appear later are obtained from the h's by summation, multiplication, differentiation. To every component, the following rule applies: any component having an odd (even) number of zero suffixes has only odd (even) powers of λ in its expansion.

6. THE NEWTONIAN EQUATIONS OF MOTION

We try to find the equations of motion in the lowest (Newtonian) approximation. We do it in such a way as to make the generalization to the post-Newtonian approximation as simple as possible.

Because of (5.6) and (5.8), the field equations of the lowest order are in h_{00}

$$\begin{aligned} -\tfrac{1}{2}\,{}_2h_{00,\,ss} &= -8\pi({}_2\mathbf{T}^{00} - \tfrac{1}{2}\,{}_2\mathbf{T}^{00}) \\ &= -4\pi\,{}_2\mathbf{T}^{00} = -4\pi \sum_{a=1}^{p} {}_2{}^a m\,{}^a\delta, \end{aligned} \quad (6.1)$$

or

$${}_2h_{00,\,ss} = 8\pi \sum_{a=1}^{p} {}_2{}^a m\,{}^a\delta. \quad (6.2)$$

As the solution of this equation we take the Newtonian field, that is

$${}_2h_{00} = -2\,{}_2{}^1 m\,{}^1r^{-1} - 2\,{}_2{}^2 m\,{}^2r^{-1} - \cdots, \quad (6.3)$$

where

$${}^a r^2 = (x^s - {}^a\xi^s)(x^s - {}^a\xi^s). \quad (6.4)$$

We also write briefly:

$${}_2h_{00} = \varphi \quad (6.5)$$

and in the two-body case, which we assume for the sake of simplicity‡

$$\begin{aligned} \varphi &= f + g, \\ f &= -2\,{}^1m\,{}^1r^{-1}; \quad g = -2\,{}^2m\,{}^2r^{-1}, \\ {}^1\xi &= \xi; \quad {}^2\xi = \eta. \end{aligned} \quad (6.6)$$

Because of (5.5) and

$$g^{\alpha\rho}g_{\beta\rho} = \delta^\alpha{}_\beta, \quad (6.7)$$

we have

$${}_2h^{00} = -\varphi. \quad (6.8)$$

The relevant part of φ, that is g, does not depend on ξ; therefore, we have

$$\overline{{}_2h_{00,\,s}} = {}_2\bar{h}_{00,\,s} = \overline{g_{,\,s}} = \frac{\partial\bar{g}}{\partial\xi^s} = \bar{g}_{,\,\xi^s}. \quad (6.9)$$

There is no reason to assume that ${}_2m$ is a constant. This follows, however, from (4.15), or, the first equation in (4.14).

Because of (6.9) and because in the lowest order we do not have any products of the h's, the conditions (4.10) and (4.27) are satisfied, this means that the path is a "geodetic line" and can be deduced from a Lagrangian:

$$\bar{\mathfrak{L}} = (\overline{g_{\alpha\beta}}\,\dot{\xi}^\alpha\dot{\xi}^\beta)^{\frac{1}{2}}; \quad {}_2\bar{\mathfrak{L}} = (1 - \dot{\xi}^s\dot{\xi}^s + \bar{g})^{\frac{1}{2}}, \quad (6.10)$$

$$\begin{aligned} {}_2\bar{\mathfrak{L}} &= -\tfrac{1}{2}\dot{\xi}^s\dot{\xi}^s + \tfrac{1}{2}\bar{g} = -\tfrac{1}{2}\dot{\xi}^s\dot{\xi}^s - \frac{{}^2m}{r}, \\ r^2 &= (\xi^s - \eta^s)(\xi^s - \eta^s). \end{aligned} \quad (6.11)$$

‡ There is, I hope, no danger in confusing this g in (6.6) with $g = |g_{\alpha\beta}|$.

Thus the equations of motion, up to the second order, are

$$\frac{d}{dt}\frac{\partial\overline{\mathfrak{L}}}{\partial\dot{\xi}^s}-\frac{\partial\overline{\mathfrak{L}}}{\partial\xi^s}=0, \tag{6.12}$$

that is

$$\ddot{\xi}^s=\frac{\partial}{\partial\xi^s}({}^2m/r). \tag{6.13}$$

The Lagrangian $\overline{\mathfrak{L}^{**}}$ for *both* particles is

$${}_4\overline{\mathfrak{L}^{**}}=-\tfrac{1}{2}\,{}^1m\dot{\xi}^s\dot{\xi}^s-\tfrac{1}{2}\,{}^2m\dot{\eta}^s\dot{\eta}^s-\frac{{}^1m\,{}^2m}{r}. \tag{6.14}$$

The Newtonian Lagrangian for both particles is of the fourth order. Since $(m/\mu)=(dt/\overline{ds})$, we have here

$${}_2m={}_2\mu,\quad {}_4{}^1m=\tfrac{1}{2}\,{}_2{}^1m\dot{\xi}^s\dot{\xi}^s+{}^1m\,{}^2m/r. \tag{6.15}$$

We also find the equations of motion directly by going back to (4.17). We then have simply

$$\ddot{\xi}^s+\overline{\left\{\begin{matrix}s\\00\end{matrix}\right\}}=0 \tag{6.16}$$

or

$$\ddot{\xi}^s-\overline{[00,s]}=\ddot{\xi}^s+\tfrac{1}{2}\overline{g_{,s}}=0, \tag{6.17}$$

which is identical with (6.13).

7. TRANSITION TO THE NEXT APPROXIMATION

To find the equations of motion up to the fourth order, besides ${}_2h_{00}$, we must also know

$${}_2h_{mn},\ {}_3h_{0m},\ {}_4h_{00}. \tag{7.1}$$

The first two are easy. The left-hand side of the corresponding equations is written out in (5.8) and the right-hand side is given by (5.6) and it is

$$\begin{aligned}&\text{for } mn: && -4\pi\delta_{mn}({}_2{}^1m\,{}^1\delta+{}_2{}^2m\,{}^2\delta),\\ &\text{for } 0m: && 8\pi({}_2{}^1m\dot{\xi}^m\,{}^1\delta+{}_2{}^2m\dot{\eta}^m\,{}^2\delta).\end{aligned} \tag{7.2}$$

Therefore, for ${}_2h_{mn}$ we have the equations

$$-\tfrac{1}{2}\,{}_2h_{mn,\,ss}+\tfrac{1}{2}\,{}_2h_{ms,\,ns}+\tfrac{1}{2}\,{}_2h_{ns,\,ms}-\tfrac{1}{2}\,{}_2h_{ss,\,mn}+\tfrac{1}{2}\varphi_{,\,mn}=-\tfrac{1}{2}\delta_{mn}\varphi_{,\,ss}. \tag{7.3}$$

We are looking for a solution of a Newtonian character. Such a solution is

$${}_2h_{mn}=\delta_{mn}\varphi. \tag{7.4}$$

The choice of these Newtonian solutions for ${}_2h_{00}$ and ${}_2h_{mn}$ must be regarded as part of our approximation procedure (Sec. 9).

The next step is to calculate ${}_3h_{0m}$. Again (5.8) and (7.2) gives

$$-\tfrac{1}{2}\,{}_3h_{0n,\,ss}+\tfrac{1}{2}\,{}_3h_{0s,\,ns}+\tfrac{1}{2}\,{}_3h_{ns,\,0s}-\tfrac{1}{2}\,{}_3h_{ss,\,n0}=8\pi({}_2{}^1m\dot{\xi}^n\delta^1+{}_2{}^2m\dot{\eta}^n\delta^2). \tag{7.5}$$

If (7.4) is introduced into the last equation we have

$$-\tfrac{1}{2}\,{}_3h_{0n,\,ss}+\tfrac{1}{2}\,{}_3h_{0s,\,ns}-\varphi_{,\,n0}=8\pi({}_2{}^1m\dot{\xi}^n\,{}^1\delta+{}_2m^2\dot{\eta}^{n2}\delta). \tag{7.6}$$

The solution which we adopt here is

$${}_3h_{0n}=-2f\dot{\xi}^n-2g\dot{\eta}^n. \tag{7.7}$$

This is not the only possible solution. Section 9 considers a more general solution and its influence upon the equations of motion.

Calculation of ${}_4h_{00}$ is more troublesome (Appendix B). Here we quote the relevant expressions of ${}_4h_{00}$, that is those that give a contribution to $\overline{{}_4h_{00}}$ and $\overline{{}_4h_{00,\,\alpha}}$. They are

$${}_4h_{00}\sim 2({}^2m)^2({}^2r)^{-2}-3\dot{\eta}^s\dot{\eta}^s\,{}^2m({}^2r)^{-1}-{}^2m\,{}^2r_{,\,00}+2\,{}^1m\,{}^2m(r\,{}^2r)^{-1}. \tag{7.8}$$

Now we should like to see whether these expressions for ${}_2h_{mn}$, ${}_3h_{0n}$, ${}_4h_{00}$ are such that they make the equations of motion equivalent to those of a "geodetic line," that is, if the conditions

$$\overline{\left\{\begin{matrix}\alpha\\\beta\rho\end{matrix}\right\}}=\overline{g^{\alpha\sigma}}\overline{[\beta\rho,\sigma]}, \tag{7.9}$$

$$\overline{g_{\alpha\beta,\,s}}=\overline{g_{\alpha\beta}}_{,\,s};\quad \partial\overline{g_{\alpha\beta}}/\partial\dot{\xi}^r=0 \tag{7.10}$$

are satisfied. Both conditions are satisfied for ${}_2h_{mn}$ and ${}_3h_{0m}$. These expressions have a singularity of order $1/r$, an odd singularity; therefore, for them, (7.9) is satisfied. In their relevant part neither ξ nor $\dot{\xi}$ appears explicitly; therefore, for them, (7.10) is satisfied. But this is not true of ${}_4h_{00}$. It has an even singularity because of the appearance of $({}^1r)^{-2}$. Yet this does not matter, since ${}_4h_{00}$ appears only *linearly* in the equations of motion. Therefore, we can disregard the condition (7.9) for ${}_4h_{00}$. But is the condition (7.10) satisfied for ${}_4h_{00}$? It is certainly satisfied for the first two expressions, that is, for $2({}^2m)^2({}^2r)^{-2}-3\dot{\eta}^s\dot{\eta}^s\,{}^2m({}^2r)^{-1}$, since neither of these two expressions depends on $\dot{\xi}$, ξ. We must be more careful with the third expression:

$$-{}^2m\,{}^2r_{,\,00}=-{}^2m\,{}^2r_{,\,\eta^s\eta^r}\dot{\eta}^s\dot{\eta}^r-{}^2m\,{}^2r_{,\,\eta^s}\ddot{\eta}^s. \tag{7.11}$$

The first expression on the right-hand side does not depend on ξ, $\dot{\xi}$, but the second expression contains $\ddot{\eta}^s$. Here we may introduce for ${}^2m\ddot{\eta}^s=-{}^1m\ddot{\xi}^s$, the Newtonian value, since the mistake will be of order "6". Thus we can put

$$\begin{aligned}\alpha&=-{}^2m\,{}^2r_{,\,\eta^s}\ddot{\eta}^s={}^1m\,{}^2m\,{}^2r_{,\,\eta^s}\left(\frac{1}{r}\right)_{,\,\xi^s}\\&=[{}^1m\,{}^2m(x^s-\eta^s)({}^2r)^{-1}](\xi^s-\eta^s)r^{-3}.\end{aligned} \tag{7.12}$$

This expression, differentiated with respect to x^m

gives zero at the point $x^s = \xi^s$. Thus

$$\overline{\alpha_{,m}} = 0. \tag{7.13}$$

But

$$\bar{\alpha}_{,m} = \frac{\partial \bar{\alpha}}{\partial \xi^m}\, {}^1m\, {}^2m (r^{-2})_{,\xi^m}.$$

A similar situation occurs in the last expression in ${}_4h_{00}$:

$$\beta = 2\, {}^1m\, {}^2m (r\, {}^2r)^{-1}. \tag{7.14}$$

We have

$$\overline{\beta_{,s}} = 2\, {}^1m\, {}^2mr^{-1} \overline{\left(\frac{1}{r^2}\right)_{,s}} = 2\, {}^1m\, {}^2mr^{-1}(r^{-1})_{,\xi^s},$$
$$\bar{\beta}_{,s} = 2\, {}^1m\, {}^2m (r^{-2})_{,\xi^s}. \tag{7.15}$$

Therefore

$$\bar{\beta}_{,s} = 2\overline{\beta_{,s}}. \tag{7.16}$$

However, we can easily find an auxiliary field which we shall denote by $h_{00}{}^*$ such that

$${}_4\bar{h}^*{}_{00,m} = \overline{{}_4h_{00,m}}. \tag{7.17}$$

From this it follows that such $\overline{{}_4h_{00}{}^*}$ is

$$\overline{{}_4h^*{}_{00}} = 2({}^2m)^2 r^{-2} - 3\dot{\eta}^s \dot{\eta}^s\, {}^2mr^{-1} - {}^2mr_{,\xi^s\xi^r} \dot{\eta}^s \dot{\eta}^r + {}^1m\, {}^2mr^{-2}. \tag{7.18}$$

8. THE POST NEWTONIAN EQUATIONS OF MOTION

In the general equations of motion (4.17) ${}_4h_{00}$ will appear only *once* up to the fourth order, that is in the expression $\frac{1}{2}\, \overline{{}_4h_{00,m}}$. This means, because of (7.17), that a Lagrangian up to the fourth order exists§:

$$\overline{\mathfrak{L}^*} = \overline{ds^*}/dt, \tag{8.1}$$

with

$$\left(\frac{\overline{ds^*}}{dt}\right)^2 = \overline{g_{\alpha\beta}{}^*}\, \dot{\xi}^\alpha \dot{\xi}^\beta \tag{8.2}$$

in which only $\overline{g_{00}{}^*} \neq \overline{g_{00}}$ and

$$g_{00}{}^* = \eta_{00} + {}_2h_{00} + {}_4h_{00}{}^* = 1 + \varphi + {}_4h_{00}{}^*. \tag{8.3}$$

Therefore,

$$\overline{\mathfrak{L}^*} = (1 + \bar{\varphi} + \overline{{}_4h^*{}_{00}} - \dot{\xi}^s \dot{\xi}^s + \bar{\varphi}\, \dot{\xi}^s \dot{\xi}^s + \overline{{}_3h_{s0}}\, \dot{\xi}^s)^{\frac{1}{2}}. \tag{8.4}$$

§ Compare I. G. Fichtenholz [J. Phys. (U.S.S.R.) **27**, 563 (1954)] where the Lagrangian is found mechanically from the explicit equations of motion and not, as here, the equation of motion from the Lagrangian.

Here we introduce the values

$$\bar{\varphi} = -2\, {}^2mr^{-1}; \quad \overline{{}_3h_{0s}} = 4\, {}^2mr^{-1}\dot{\eta}^s;$$
$$\overline{{}_4h_{00}{}^*} = 2({}^2m)^2 r^{-2} - 3\dot{\eta}^s \dot{\eta}^s\, {}^2mr^{-1} - {}^2m\, r_{,\xi^s\xi^r} \dot{\eta}^s \dot{\eta}^r + {}^1m\, {}^2mr^{-1}. \tag{8.5}$$

Remembering that up to the fourth order

$$(1 + {}_2a + {}_4a)^{\frac{1}{2}} = 1 + \tfrac{1}{2}({}_2a + {}_4a) - \tfrac{1}{8}\, {}_2a^2 \tag{8.6}$$

we obtain from (8.4) and (8.5) the Lagrangian:

$$\overline{\mathfrak{L}^*} = \overline{{}_2\mathfrak{L}^*} + \overline{{}_4\mathfrak{L}^*} = -\tfrac{1}{2}\dot{\xi}^s \dot{\xi}^s - {}^2mr^{-1} - \tfrac{3}{2}\, {}^2mr^{-1}(\dot{\xi}^s \dot{\xi}^s + \dot{\eta}^s \dot{\eta}^s) + 4\, {}^2mr^{-1} \dot{\xi}^s \dot{\eta}^s - \tfrac{1}{8}(\dot{\xi}^s \dot{\xi}^s)^2 + \frac{1}{2}\frac{{}^2m({}^1m + {}^2m)}{r^2} - \tfrac{1}{2}\, {}^2mr_{,\xi^s\xi^r} \dot{\eta}^s \dot{\eta}^r. \tag{8.7}$$

This is supposed to be the Lagrangian for the *first* particle. But we wish to find the Lagrangian for *both* particles. This means *first* a Lagrangian which gives the same equations of motion as the Lagrangian (8.7); *secondly*, a Lagrangian which is invariant with respect to a transformation, changing

$${}^1m\xi \rightleftarrows {}^2m\eta. \tag{8.8}$$

Let us multiply $\overline{\mathfrak{L}^*}$ in (8.7) by ${}^1m$. The equations of motion will be the same. Let us add

$$-\tfrac{1}{2}\, {}^2m\dot{\eta}^s \dot{\eta}^s - \tfrac{1}{8}\, {}^2m(\dot{\eta}^s \dot{\eta}^s)^2. \tag{8.9}$$

With this addition, the equations of motion for the first particle will still be the same, since there is no contribution from (8.9). With these changes, the only expression not invariant with respect to the change (8.8) will be the last one in (8.7), that is

$$-\tfrac{1}{2}\, {}^1m\, {}^2mr_{,\xi^s\xi^r} \dot{\eta}^s \dot{\eta}^r, \tag{8.10}$$

But instead of it we can write

$$\tfrac{1}{2}\, {}^1m\, {}^2mr_{,\xi^s\eta^r} \dot{\xi}^s \dot{\eta}^r, \tag{8.11}$$

which is invariant with respect to the change (8.8), and gives *the same* contributions to the equations of motion.

From (8.10) we have the following addition to the equations of motion of the first particle:

$$\tfrac{1}{2}\, {}^1m\, {}^2mr_{,\xi^s\xi^r\xi^n} \dot{\eta}^s \dot{\eta}^r = -\tfrac{1}{2}\, {}^1m\, {}^2mr_{,\xi^s\eta^r\xi^n} \dot{\eta}^s \dot{\eta}^r. \tag{8.12}$$

From (8.11) we have the addition:

$$\tfrac{1}{2}\, {}^1m\, {}^2m(r_{,\xi^n\eta^r}\dot{\eta}^r)_{,0} - \tfrac{1}{2}\, {}^1m\, {}^2mr_{,\xi^s\eta^r\xi^n} \dot{\xi}^s \dot{\eta}^r$$
$$= \tfrac{1}{2}\, {}^1m\, {}^2m(r_{,\xi^n\eta^r\xi^s} \dot{\eta}^r \dot{\xi}^s + r_{,\xi^n\eta^r\eta^s} \dot{\eta}^r \dot{\eta}^s + r_{,\xi^n\eta^r} \ddot{\eta}^r - r_{,\xi^s\eta^r\xi^n} \dot{\xi}^s \dot{\eta}^r)$$
$$= \tfrac{1}{2}\, {}^1m\, {}^2m(r_{,\xi^n\eta^r\eta^s} \dot{\eta}^r \dot{\eta}^s + r_{,\xi^n\eta^r} \ddot{\eta}^r). \tag{8.13}$$

But the last expression in (8.13) equals

$$\tfrac{1}{2}\,{}^1m\,{}^2mr_{,\xi^n\xi^r}\ddot{\eta}^r = \tfrac{1}{2}\,{}^1m\,{}^2m\left(-\frac{\delta_{nr}}{r}+\frac{(\xi^n-\eta^n)(\xi^r-\eta^r)}{r^3}\right)\ddot{\eta}^r. \quad (8.14)$$

Since

$$\ddot{\eta}^r=\frac{{}^1m(\xi^r-\eta^r)}{r^3}, \quad (8.15)$$

we see that (8.14) vanishes and, therefore, (8.10) and (8.11) give the same contributions to the equations of motion.

Therefore, if we call the final Lagrangian for two particles $\overline{\mathfrak{L}^{**}}$, we have

$$\overline{\mathfrak{L}^{**}}={}_4\overline{\mathfrak{L}^{**}}+{}_6\overline{\mathfrak{L}^{**}}=-\tfrac{1}{2}\,{}^1m\dot{\xi}^s\dot{\xi}^s-\tfrac{1}{2}\,{}^2m\dot{\eta}^s\dot{\eta}^s-\frac{{}^1m\,{}^2m}{r}$$
$$-\frac{3}{2}\frac{{}^1m\,{}^2m}{r}(\dot{\xi}^s\dot{\xi}^s+\dot{\eta}^s\dot{\eta}^s)+4\frac{{}^1m\,{}^2m}{r}\dot{\xi}^s\dot{\eta}^s$$
$$-\tfrac{1}{8}\,{}^1m(\dot{\xi}^s\dot{\xi}^s)^2-\tfrac{1}{8}\,{}^2m(\dot{\eta}^s\dot{\eta}^s)^2$$
$$+\frac{1}{2}\frac{{}^1m\,{}^2m({}^1m+{}^2m)}{r^2}+\tfrac{1}{2}\,{}^1m\,{}^2mr_{,\xi^s\eta^r}\dot{\xi}^s\dot{\eta}^r. \quad (8.16)$$

From this Lagrangian the equations of motion for the first particle are:

$$\ddot{\xi}^n-{}^2m\left(\frac{1}{r}\right)_{,\xi^n}$$
$$={}^2m\left\{\left[\dot{\xi}^s\dot{\xi}^s+\tfrac{3}{2}\dot{\eta}^s\dot{\eta}^s-4\dot{\xi}^s\dot{\eta}^s-4\frac{{}^2m}{r}-5\frac{{}^1m}{r}\right]\left(\frac{1}{r}\right)_{,\xi^n}\right.$$
$$+[4\dot{\xi}^s(\dot{\eta}^n-\dot{\xi}^n)+3\dot{\xi}^n\dot{\eta}^s-4\dot{\eta}^n\dot{\eta}^s]\left(\frac{1}{r}\right)_{,\xi^s}$$
$$\left.+\tfrac{1}{2}r_{,\xi^s\xi^r\xi^n}\dot{\eta}^s\dot{\eta}^r\right\}. \quad (8.17)$$

The equations of motion for the other particle are obtained by replacing

$${}^1m\;{}^2m,\;\xi,\;\eta \quad\text{by}\quad {}^2m,\;{}^1m,\;\eta,\;\xi, \quad (8.18)$$

respectively.

The generalization of this result to p particles is almost trivial, if we take into account the changes caused by the addition of these particles in $\overline{h_{00}}^*$ (Appendix B). These additional expressions are due, say, in the case of three particles, to interaction between the second and third particles; that is, in the equations of motion for the first particle they will give a contribution proportional to ${}^1m\,{}^2m\,{}^3m$. These expressions appear in $\overline{\mathfrak{L}^{**}}$ from two sources: from ${}_4\overline{h_{00}}^*$ and from $\overline{\varphi^2}$ in (8.6). If we now denote the "distance" from the ath to the bth particle by:

$$({}^{(ab)}r)^2=({}^a\xi^s-{}^b\xi^s)({}^a\xi^s-{}^b\xi^s), \quad (8.19)$$

then we have the Lagrangian for the p particles:

$$\overline{\mathfrak{L}^{**}}={}_4\overline{\mathfrak{L}^{**}}+{}_6\overline{\mathfrak{L}^{**}}$$
$$=-\tfrac{1}{2}\sum_{a=1}^{p}{}^am\,{}^a\dot{\xi}^s\,{}^a\dot{\xi}^s-\tfrac{1}{2}\sum_{\substack{a,b=1\\a\neq b}}^{p}{}'\,{}^am\,{}^bm({}^{(ab)}r)^{-1}$$
$$-\tfrac{3}{4}\sum_{\substack{a,b=1\\a\neq b}}^{p}{}'\,{}^am\,{}^bm({}^{(ab)}r)^{-1}({}^a\dot{\xi}^s\,{}^a\dot{\xi}^s+{}^b\dot{\xi}^s\,{}^b\dot{\xi}^s)$$
$$+2\sum_{\substack{a,b=1\\a\neq b}}^{p}{}'\,{}^am\,{}^bm({}^{(ab)}r)^{-1}\,{}^a\dot{\xi}^s\,{}^b\dot{\xi}^s$$
$$-\tfrac{1}{8}\sum_{a=1}^{p}{}^am({}^a\dot{\xi}^s\,{}^a\dot{\xi}^s)^2$$
$$+\tfrac{1}{4}\sum_{\substack{a,b=1\\a\neq b}}^{p}{}'\,{}^am\,{}^bm({}^am+{}^bm)({}^{(ab)}r)^{-2}$$
$$+\sum_{\substack{a,b=1\\a\neq b}}^{p}{}'\,{}^am\,{}^bm({}^{(ab)}r_{,\xi^s\xi^r})\,{}^a\dot{\xi}^s\,{}^b\dot{\xi}^r$$
$$+\tfrac{1}{6}\sum_{\substack{a,b,c=1\\a\neq b\neq c}}^{p}{}''\,{}^am\,{}^bm\,{}^cm[({}^{(ab)}r\;{}^{(ac)}r)^{-1}$$
$$+({}^{(bc)}r\;{}^{(ba)}r)^{-1}+({}^{(ca)}r\;{}^{(cb)}r)^{-1}]. \quad (8.20)$$

In the case of two particles the Lagrangian (8.20) reduces to (8.16). The only new expression appearing in (8.20) is the last one; in the case of three particles it is equal to

$${}^1m(\tfrac{1}{2}\,\overline{{}_4s_{00}}-\tfrac{1}{4}\bar{g}\bar{k})$$

using the notation of Appendix B where $\overline{{}_4s_{00}}$ is the change in ${}_4\overline{h_{00}}^*$ caused by the interaction of the second and third particles and k is for the third particle what f and g are for the first and second. Thus $-\frac{1}{4}\bar{g}\bar{k}$ is the contribution to the Lagrangian of the interaction between the second and third particles coming from $-\frac{1}{8}\overline{\varphi^2}$ in (8.6).

9. ON THE CHOICE OF THE COORDINATE SYSTEM

The harmonic coordinate condition is

$$[(-g)^{\frac{1}{2}}g^{\mu\nu}]_{,\nu}=0, \quad (9.1)$$

which in our case means

$${}_2h^{mn}{}_{,n}=0;\quad {}_3h^{0m}{}_{,m}+{}_2h^{00}{}_{,0}=0. \quad (9.2)$$

None of these conditions are fulfilled in our coordinate system.

The values for ${}_2h_{00}$ and ${}_2h_{mn}$ accepted by us here were the Newtonian values. Our convention is that their choice characterized our approximation procedure. Yet, with some justification, this approach may be regarded as too formal. Instead of our values for ${}_2h_{mn}$ we could have chosen

$${}_2h_{mn}' = {}_2h_{mn} + {}_2a_{m,n} + {}_2a_{n,m}$$

the a's being arbitrary functions. This change from ${}_2h_{mn}$ to ${}_2h_{mn}'$ could also be induced by a change in a coordinate system that does not disturb the approximation procedure. The physical meaning of the choice ${}_2a_m = 0$ is that we assume the existence of a coordinate system in which each of the two bodies reveals its spherical symmetry; a coordinate system in which for ${}_2m \to 0$, $\dot{\xi} \to 0$, the field goes over into that defined by the Schwarzschild solution in an isotropic coordinate system. The choice of such a coordinate system is implicitly assumed by our approximation procedure. However this choice of ${}_2a_m = 0$ refers only to the beginning of our approximation procedure; therefore it does not refer to ${}_3h_{0m}$. If we replace ${}_3h_{0m}$ by

$${}_3h_{0m}' = {}_3h_{0m} + {}_3a_{0,m} \tag{9.3}$$

${}_3a_0$ being an arbitrary function of x^α, then Eq. (7.5) is fulfilled just as well. This change in ${}_3h_{0m}$ can be induced by a change in a coordinate system which does not disturb our approximation procedure. Such a change also induces a simple change in ${}_4h_{00}$ (Appendix B):

$${}_4h_{00}' = {}_4h_{00} + 2\,{}_4a_{0,0}. \tag{9.4}$$

Therefore, it would seem that the Lagrangian and with it the equations of motion would change. The expressions that change in the Lagrangian (8.4) are

$$\bar{h}_{0s}\dot{\xi}^s + \tfrac{1}{2}\,\overline{{}_4h_{00}^*} \tag{9.5}$$

and the change induced by them in the Lagrangian is, because of (9.3) and (9.4),

$$\Delta\overline{\mathfrak{L}^*} = \overline{a_{0,s}}\,\dot{\xi}^s + \overline{a_{0,0}} = \frac{d\overline{a_0}}{dt}. \tag{9.6}$$

Therefore,

$$\delta\int_{t_1}^{t_2} \Delta\overline{\mathfrak{L}^*}\,dt = \delta(\overline{a_0})\Big|_{t_1}^{t_2} = 0. \tag{9.7}$$

This means: the equations of motion are uniquely determined up to the fourth approximation by the field equations and by our approximation procedure. Neither the harmonic coordinate condition nor any other coordinate condition played any role in our derivation of the equations of motion.

10. THE GENERAL THEORY

Now we formulate the general theory,‖ according to which we proceeded in our special case and in which we found the equations of motion of the second and fourth order. Such a general theory is of little practical value, since there would be hardly any physical meaning in developing the calculations one step further. Moreover, it seems—and we discuss this later—that by proper choice of the coordinate system we can annihilate all contributions to the equations of motion beyond the fourth order. From the formal point of view it is important to know that the procedure can be pushed as far as we wish. Of course we do not know anything about its convergence.

Before we formulate the general theory let us recall what has been done here. We had the Newtonian equations of motion:

$$\begin{aligned} \frac{dt}{ds}\int {}_3\mathbf{T}^{0\nu}{}_{;\nu}\,d_{(3)}x &= {}_3\overline{A^0} = 0; \\ \frac{dt}{ds}\int {}_4\mathbf{T}^{n\nu}{}_{;\nu}\,d_{(3)}x &= {}_4\bar{A}^n = 0. \end{aligned} \tag{10.1}$$

Since ${}_2{}^1m$ appears as a factor, we called these equations (after dividing them by ${}_2{}^1m$) the equations of the *second* order. But here, since it is multiplied by ${}_2{}^1m$, it appears as an equation of the fourth order. For this section, therefore, let us rename the order of the equations of motion calling the *Newtonian* equations of motion those of the *fourth* order and the post-Newtonian equations those of the *sixth* order. Let us also put generally

$$\begin{aligned} \frac{dt}{ds}\int {}_{2n-1}T^{0\nu}{}_{;\nu}\,d_{(3)}x &= {}_{2n-1}\overline{A^0}; \\ \frac{dt}{ds}\int {}_{2n}\mathbf{T}^{n\nu}{}_{;\nu}\,d_{(3)}x &= {}_{2n}\overline{A^n}. \end{aligned} \tag{10.2}$$

Thus for our post-Newtonian equations of motion we have

$${}_3\overline{A^0} + {}_5\overline{A^0} = 0; \quad {}_4\overline{A^n} + {}_6\overline{A^n} = 0. \tag{10.3}$$

These equations gave us

$${}^am = {}_2{}^am + {}_4{}^am; \quad {}^a\xi^s = {}_0{}^a\xi^s + {}_2{}^a\xi^s, \tag{10.4}$$

where ${}_0{}^a\xi^s$ is the motion in the Newtonian approximation. To find these equations explicitly we used the Newtonian equation of motion in ${}_6\overline{A^m}$, since the use of ${}^a\xi^s$ instead of ${}_0{}^a\xi^s$ would give a contribution of the 8th order to the equations of motion. We express this idea in symbols and write instead of (10.3):

$$\begin{aligned} {}_3A^0({}_0\xi^k + {}_2\xi^k) + {}_5A^0({}_0\xi^k) &= 0; \\ {}_4A^m({}_0\xi^k + {}_2\xi^k) + {}_6A^m({}_0\xi^k) &= 0. \end{aligned} \tag{10.5}$$

Thus ${}_4A^m({}_0\xi + {}_2\xi)$ also gives a contribution of the order

‖ The ideas presented here are a few years old. A. E. Scheidegger [Revs. Modern Phys. **25**, 451 (1953)] refers to them in Sec. 5 stating that they were suggested by me. I found a more explicit formulation of similar ideas in a thesis by B. Rameswararao (thesis, Banares Hindu University, 1955). An alternative general theory was given in a paper by Plebański and myself [Bull. Acad. Polon. III, 4, 755 (1956)].

six. The field was solved so as to obtain (10.4) in

$$m,\ n \text{ up to the order } {}_2h_{mn}({}_0\xi),$$
$$0,\ m \text{ up to the order } {}_3h_{0m}({}_0\xi),$$
$$0,\ 0 \text{ up to the order } {}_4h_{00}({}_0\xi).$$

Suppose that we wish to go one step further. We then have the equations of motion,

$$\begin{aligned}&{}_3A^0({}_0\xi+{}_2\xi+{}_4\xi)+{}_5A^0({}_0\xi+{}_2\xi)+{}_7A^0({}_0\xi)=0,\\&{}_4A^m({}_0\xi+{}_2\xi+{}_4\xi)+{}_6A^m({}_0\xi+{}_2\xi)+{}_8A^m(\xi)=0.\end{aligned}\tag{10.6}$$

In ${}_3A$, ${}_5A$, ${}_4A$, ${}_6A$, the argument in (10.6) is different from that in (10.5); therefore, they give contributions *up* to the eighth order. But to find ${}_8A^m({}_0\xi)$ we have to know

$${}_4h_{mn},\ {}_5h_{0m},\ {}_6h_{00},\tag{10.7}$$

all functions of ${}_0\xi$. Thus simply denoting

$$\mathbf{Q}^{\alpha\beta}=-8\pi(\mathbf{T}^{\alpha\beta}-\tfrac{1}{2}g^{\alpha\beta}\mathbf{T})\tag{10.8}$$

we have to solve the equations (omitting the a's above the ξ's):

$$\begin{aligned}&{}_2\mathbf{R}^{mn}({}_0\xi+{}_2\xi)+{}_4\mathbf{R}^{mn}({}_0\xi)={}_2\mathbf{Q}^{mn}({}_0\xi+{}_2\xi)+{}_4\mathbf{Q}^{mn}({}_0\xi),\\&{}_3\mathbf{R}^{0m}({}_0\xi+{}_2\xi)+{}_5\mathbf{R}^{0m}({}_r\xi)={}_3\mathbf{Q}^{0m}({}_0\xi+{}_2\xi)+{}_5\mathbf{Q}^{0m}({}_0\xi),\\&{}_2\mathbf{R}^{00}({}_0\xi+{}_2\xi+{}_4\xi)+{}_4\mathbf{R}^{00}(\xi_0+{}_2\xi)+{}_6\mathbf{R}^{00}({}_0\xi)\\&\quad={}_2\mathbf{Q}^{00}({}_0\xi+{}_2\xi+{}_4\xi)+{}_4\mathbf{Q}^{00}({}_0\xi+{}_2\xi)+{}_6\mathbf{Q}^{00}({}_0\xi).\end{aligned}\tag{10.9}$$

This would seem to be an impossible task, since to solve the last Eq. (10.9) we would have to know ${}_4\xi$, which we wish to find by Eq. (10.6). However, this is not so, because we have

$${}_2\mathbf{R}^{00}=-\tfrac{1}{2}\ {}_2h_{00,\,ss}=-4\pi\sum_{a=1}^{p}{}^am\ {}^a\delta={}_2Q^{00}\tag{10.10}$$

for *arbitrary* motion. Thus we may rewrite the last equation (10.9):

$${}_4\mathbf{R}^{00}({}_0\xi+{}_2\xi)+{}_6\mathbf{R}^{00}({}_0\xi)={}_4\mathbf{Q}^{00}({}_0\xi+{}_2\xi)+{}_6\mathbf{Q}^{00}({}_0\xi).\tag{10.11}$$

Collecting here the expressions of the sixth order we find ${}_6h_{00}({}_0\xi)$. Similarly we can find ${}_5h_{0m}$ and ${}_4h_{mn}$. Thus we can push the approximation one step further.

We can now formulate the general theory. In the development of $g_{\alpha\beta}$, R, T, we took into account *arbitrary* motion. Under this assumption we developed, say

$$T^{mn}={}_4T^{mn}+{}_6T^{mn}+{}_8T^{mn}+\cdots.\tag{10.12}$$

But, instead of *arbitrary* motion, let us put into the arguments *certain* motion developed into power series:

$$\xi={}_0\xi+{}_2\xi+{}_4\xi+\cdots\tag{10.13}$$

and write, say,

$${}_{2s}T({}_0\xi+{}_2\xi+{}_4\xi).\tag{10.14}$$

If developed properly, the above expression gives contributions of the order $2s+2$ and $2s+4$. Assume that we have solved the equations of motion of the order $2r$

$$\begin{aligned}&{}_3A^0({}_0\xi+{}_2\xi+\cdots{}_{2r-4}\xi)+{}_5A^0({}_0\xi+\cdots{}_{2r-6}\xi)\\&\qquad+\cdots{}_{2r-1}A^0({}_0\xi)=0,\\&{}_4A^m({}_0\xi+\cdots{}_{2r-4}\xi)+{}_6A^m({}_0\xi+\cdots{}_{2r-6}\xi)\\&\qquad+\cdots{}_{2r}A^m({}_0\xi)=0.\end{aligned}\tag{10.15}$$

This means, that we have solved the field equations in

$$m,\ n \text{ up to the order } {}_{2r-4}h_{mn},$$
$$0,\ m \text{ up to the order } {}_{2r-3}h_{0m},$$
$$0,0 \text{ up to the order } {}_{2r-2}h_{00}.$$

Now we wish to solve the equation of motion of order $2r+2$:

$$\begin{aligned}&{}_3A^0({}_0\xi+\cdots{}_{2r-2}\xi)+\cdots{}_{2r+1}A^0({}_0\xi)=0,\\&{}_4A^m({}_0\xi+\cdots{}_{2r-2}\xi)+\cdots{}_{2r+2}A^m({}_0\xi)=0.\end{aligned}\tag{10.16}$$

The arguments in ${}_3A^0$, $\cdots{}_{2r-1}A^0$, ${}_4A^m$, $\cdots{}_{2r}A^m$ are different in (10.16) from those in (10.15). Therefore, they give contributions of order $2r+1$ and $2r+2$. But to find ${}_{2r+2}A^m({}_0\xi)$ we have to know

$${}_{2r-2}h_{mn},\ {}_{2r-1}h_{0m},\ {}_{2r}h_{00}.$$

We have, therefore, to solve the equations

$$\begin{aligned}&{}_2\mathbf{R}^{mn}({}_0\xi+\cdots{}_{2r-4}\xi)+\cdots{}_{2r-2}\mathbf{R}^{mn}({}_0\xi)\\&\quad={}_2\mathbf{Q}^{mn}({}_0\xi+\cdots{}_{2r-4}\xi)+\cdots{}_{2r-2}\mathbf{Q}^{mn}({}_0\xi),\\&{}_3\mathbf{R}^{0m}({}_0\xi+\cdots{}_{2r-4}\xi)+\cdots{}_{2r-1}\mathbf{R}^{0m}({}_0\xi)\\&\quad={}_3\mathbf{Q}^{0m}({}_0\xi+\cdots{}_{2r-4}\xi)+\cdots{}_{2r-1}\mathbf{Q}^{0m}({}_0\xi),\\&{}_2\mathbf{R}^{00}({}_0\xi+\cdots{}_{2r-2}\xi)+\cdots{}_{2r}\mathbf{R}^{00}({}_0\xi)\\&\quad={}_2\mathbf{Q}^{00}({}_0\xi+\cdots{}_{2r-2}\xi)+\cdots{}_{2r}\mathbf{Q}^{00}({}_0\xi).\end{aligned}\tag{10.17}$$

Everywhere in these equations, with the exception of ${}_2\mathbf{R}^{00}$ and ${}_2\mathbf{Q}^{00}$, we substitute the motion already known. However, ${}_2\mathbf{R}^{00}={}_2\mathbf{Q}^{00}$ for an *arbitrary* motion. Then we can replace the last equations in (10.17) by

$$\begin{aligned}&{}_4\mathbf{R}^{00}({}_0\xi+\cdots{}_{2r-4}\xi)+\cdots{}_{2r}\mathbf{R}^{00}({}_0\xi)\\&\quad={}_4\mathbf{Q}^{00}({}_0\xi+\cdots{}_{2r-4}\xi)+\cdots{}_{2r}\mathbf{Q}^{00}({}_0\xi).\end{aligned}$$

In ${}_{2r-2}\mathbf{R}^{mn}({}_0\xi)$ the expression ${}_{2r-2}h_{mn}$ appears, for the first time. In ${}_{2r-1}\mathbf{R}^{0m}$ the expression ${}_{2r-1}h_{0m}$ appears and finally in ${}_{2r}\mathbf{R}^{00}$ the expression ${}_{2r}h_{00}$ appears. Collecting all the contributions of highest order in these equations and putting them equal to zero we can find ${}_{2r-2}h_{mn}$, ${}_{2r-1}h_{0m}$, ${}_{2r}h_{00}$; for ${}_{2r}h_{00}$ the equation is purely a Poisson equation!

Looking back at Eqs. (5.8), we see that if ${}_{2r-1}h_{0m}$, ${}_{2r}h_{mn}$ is a solution of (10.17), then

$$\begin{aligned}&{}_{2r-1}h_{0m}'={}_{2r-1}h_{0m}+{}_{2r-1}a_{0,\,m},\\&{}_{2r}h_{mn}'={}_{2r}h_{mn}+{}_{2r}a_{m,\,n}+{}_{2r}a_{n,\,m}\end{aligned}\tag{10.18}$$

is also a solution.

For example, let us put $r=2$; that is,

$${}_3h_{0m}'={}_3h_{0m}+{}_3a_{0,\,m},\quad {}_4h_{mn}'={}_4h_{mn}+{}_4a_{m,\,n}+{}_4a_{n,\,m}.$$

The choice of these functions can always be achieved by a coordinate transformation from a coordinate system in which the a's equal zero. Then in the equations of motion of the 8th order, the derivatives of these four functions will appear. Generally, they can be so chosen as to annihilate the expressions of the 8th order in the equations of motion. But it is difficult to judge whether such a coordinate system would have any physical meaning. In any case, up to the post-Newtonian approximation, the choice of the coordinate system does not play any role as long as we stick to our approximation procedure by which its beginning is determined, that is, the choice of ${}_2h_{00}$ and ${}_2h_{mn}$.

APPENDIX A¶

To distinguish between *our* δ function and *Dirac's* δ function we shall here denote the former by δ_1. Our aim is to give a "realistic" model, showing how to construct a sequence of $\delta_1(\epsilon)$ so that $\delta_1=\lim_{\epsilon\to 0}\delta_1(\epsilon)$ and such that for every ϵ:

$$\int_\infty \delta_1(\epsilon)d_{(3)}x=1;\quad \int \delta_1(\epsilon)r^{-p}d_{(3)}x=0;\quad p=1,2\cdots k. \tag{A.1}$$

Such a model can be gained from a model $\delta(\epsilon)$ of an ordinary Dirac δ function satisfying the following conditions:

$$\delta(\epsilon)=\delta(\epsilon,r)=\epsilon^{-3}\Delta(r/\epsilon), \tag{A.2}$$

where $\Delta(r/\epsilon)$ is such that

$$\frac{1}{4\pi}D^{(p)}=\int_0^\infty \Delta(z)z^{-p+2}dz;\quad p=1,2\cdots k \tag{A.3}$$

always exists, and

$$D^{(0)}=\int \delta(\epsilon)d_{(3)}x=4\pi\int_0^\infty z^2\Delta(z)dz=1. \tag{A.4}$$

If $\delta(\epsilon)$ does not have this property, it can be made to have it by multiplying it by $(r/\epsilon)^k$ and renormalizing.** Thus with such δ's we can form the model of our δ_1 function in the following way:

$$\delta_1(\epsilon,r)=\frac{1}{k!}\left(\frac{\partial}{\partial\epsilon}\right)^k\left[\epsilon^{k-3}\Delta\left(\frac{r}{\epsilon}\right)\right]. \tag{A.5}$$

We have to show that such a choice of δ_1, satisfies (A.1). To do so, let us start with the first equation (A.1):

$$\int \delta_1 d_{(3)}x=\frac{4\pi}{k!}\left(\frac{\partial}{\partial\epsilon}\right)^k\epsilon^k\int_0^\infty \Delta(z)z^2dz=D^{(0)}=1. \tag{A.6}$$

Now as to the second equation (A.1):

$$\int \delta_1 r^{-p}d_{(3)}x=\frac{4\pi}{k!}\left(\frac{\partial}{\partial\epsilon}\right)^k\epsilon^{k-p}\int_0^\infty \Delta(z)z^{2-p}dz =\frac{1}{k!}\left(\frac{\partial}{\partial\epsilon}\right)^k\epsilon^{k-p}D^{(p)}=0, \tag{A.7}$$

for every integer p, if

$$1\leq p\leq k.$$

Thus the δ_1's defined by (A.5) satisfy (A.1).

This procedure can easily be generalized. We introduce the modified Dirac functions δ_2 from the conditions

$$\int \delta_2 r^{-p}d_{(3)}x=\omega_{(p)};\quad p=1,2\cdots k, \tag{A.8}$$

where the ω are arbitrarily *prescribed* numbers. The realistic $\delta_2(\epsilon)$ satisfying (A.8) in the limit $\epsilon\to 0$ is the following:

$$\delta_2(\epsilon)=\sum_{s=0}^k (D^{(k-s)})^{-1}\frac{\omega_{(k-s)}}{s!}\left(\frac{\partial}{\partial\epsilon}\right)^s\left[\epsilon^{k-3}\Delta\left(\frac{r}{\epsilon}\right)\right]. \tag{A.9}$$

We find

$$\int \delta_2(\epsilon)r^{-p}d_{(3)}x =4\pi\sum_{s=0}^k (D^{(k-s)})^{-1}\frac{\omega_{(k-s)}}{s!}\left(\frac{\partial}{\partial\epsilon}\right)^s\epsilon^{k-p}\int_0^\infty z^{-p+2}\Delta(z)dz =\sum_{s=0}^k (D^{(k-s)})^{-1}\frac{\omega_{(k-s)}}{s!}\left(\frac{\partial}{\partial\epsilon}\right)^s\epsilon^{k-p}D^{(p)}. \tag{A.10}$$

This is different from zero and finite for $\epsilon\to 0$, only for $s=k-p$. We have:

$$\int \delta_2 r^{-p}d_{(3)}x=\omega_{(p)}+O(\epsilon)$$

and for $\epsilon\to 0$ we have (A.8).

The use of Dirac's functions requires the prescription for the values of $\omega_{(p)}$'s. The one used in this paper is the most convenient for our purpose; it requires $\omega_{(p)}=0$; $p=1,2\cdots k$.

APPENDIX B

R_{00} *up* to the fourth approximation equals

$$R_{00}=-\tfrac{1}{2}\varphi_{,ss}-\tfrac{1}{2}\,{}_4h_{00,ss}+{}_4h_{0s,0s}-\tfrac{3}{2}\varphi_{,00}+\tfrac{1}{2}\varphi_{,s}\varphi_{,s}-\tfrac{1}{2}\varphi\varphi_{,ss}. \tag{B.1}$$

Therefore

$$R^{00}=R_{00}(1-2h_{00})=R_{00}(1-2\varphi),\qquad \mathbf{R}^{00}=R^{00}\sqrt{-g}=(1-\varphi)R^{00}=R_{00}(1-3\varphi). \tag{B.2}$$

In (B.2) exceptionally $g=|g_{\alpha\beta}|$. We have

$$\mathbf{R}^{00}=-\tfrac{1}{2}\varphi_{,ss}+\varphi\varphi_{,ss}+\tfrac{1}{2}\varphi_{,s}\varphi_{,s}+\tfrac{1}{2}\varphi_{,00}-\tfrac{1}{2}\,{}_4h_{00,ss}. \tag{B.3}$$

¶ This is an abbreviated and changed version of two papers written in collaboration with J. Plebański.[9]

** E.g., Let us take $\delta(\epsilon)=(2\pi)^{-\frac{3}{2}}\epsilon^{-3}\exp(-\frac{1}{2}r^2\epsilon^{-2})$, that is $\Delta(z)=(2\pi)^{-\frac{3}{2}}\exp(-\frac{1}{2}z^2)$. Such $\Delta(z)$ shall be changed into

$$\Delta(z)=z^k(2\pi)^{-1}2^{-(k+3/2)}\left[\Gamma\left(\frac{k+3}{2}\right)\right]^{-1}\exp(-\tfrac{1}{2}z^2).$$

The right-hand side of our gravitational equations

$$\mathbf{R}^{\alpha\beta}=-8\pi(\mathbf{T}^{\alpha\beta}-\tfrac{1}{2}g^{\alpha\beta}\mathbf{T}) \tag{B.4}$$

is up to the fourth order for the "zero-zero" component:

$$-8\pi({}_2\mathbf{T}^{00}-\tfrac{1}{2}(\eta^{00}-\varphi)\,{}_2\mathbf{T}^{00}(\eta^{00}+\varphi)+{}_4\mathbf{T}^{00}-\tfrac{1}{2}\,{}_4\mathbf{T}^{00}+\tfrac{1}{2}\,{}_4\mathbf{T}^{ss})=-4\pi({}_2\mathbf{T}^{00}+{}_4\mathbf{T}^{00}+{}_4\mathbf{T}^{ss}). \tag{B.5}$$

Because

$${}_4{}^1m=\tfrac{1}{2}\,{}_2{}^1m\dot{\xi}^s\dot{\xi}^s+\frac{{}^1m\,{}^2m}{r},\quad {}_4{}^2m=\tfrac{1}{2}\,{}_2{}^2m\dot{\eta}^s\dot{\eta}^s+\frac{{}^1m\,{}^2m}{r},$$

we have for the right-hand side:

$$-4\pi\left({}_2{}^1m\,{}^1\delta+{}_2{}^2m\,{}^2\delta+\tfrac{3}{2}\,{}^1m\dot{\xi}^s\dot{\xi}^s\,{}^1\delta+\tfrac{3}{2}\,{}^2m\dot{\eta}^s\dot{\eta}^s\,{}^2\delta+\frac{{}^1m\,{}^2m}{r}{}^1\delta+\frac{{}^2m\,{}^1m}{r}{}^2\delta\right). \tag{B.6}$$

On both sides we take only expressions (a) of the order four, (b) those that give a finite contribution to $\overline{{}_4h_{00,m}}$. Thus this relevant part of ${}_4h_{00}$ satisfies the equation

$${}_4h_{00,ss}=2fg_{,ss}+2gg_{,ss}+g_{,00}+g_{,s}g_{,s}+8\pi\left(\tfrac{3}{2}\,{}_2{}^2m\dot{\eta}^s\dot{\eta}^s\,{}^2\delta+\frac{{}^2m\,{}^1m}{r}{}^2\delta\right). \tag{B.7}$$

Because

$$g_{,ss}=8\pi\,{}^2m\,{}^2\delta;\quad f=-2\,{}^1m({}^1r)^{-1}, \tag{B.8}$$

we have for the contribution of ${}_4C$ to ${}_4h$ coming from the first two expressions:

$${}_4C_{,ss}=-4\pi\,{}^2\delta a. \tag{B.9}$$

Generally, the solution of

$${}_4C_{,ss}=-4\pi\,{}^2\delta a \tag{B.10}$$

is

$${}_4C={}^2\bar{a}({}^2r)^{-1}, \tag{B.11}$$

where

$${}^2\bar{a}=\int_{{}^2\Omega(3)}a\,{}^2\delta d_{(3)}x. \tag{B.12}$$

Thus in our case:

$${}_4C=\frac{4\,{}^1m\,{}^2m}{r\,{}^2r}. \tag{B.13}$$

Therefore,

$${}_4h_{00,ss}=g_{,00}+g_{,s}g_{,s}+8\pi\left(\tfrac{3}{2}\,{}^2m\dot{\eta}^s\dot{\eta}^s\,{}^2\delta-\frac{{}^1m\,{}^2m}{r}{}^2\delta\right). \tag{B.14}$$

Finally, therefore, we have

$${}_4h_{00}\to -{}^2m\,{}^2r_{,00}+2({}^2m)^2({}^2r)^{-2}-3\,{}^2m\dot{\eta}^s\dot{\eta}^s({}^2r)^{-1}+2\,{}^1m\,{}^2m({}^2rr)^{-1}. \tag{B.15}$$

Let us now generalize ${}_4h_{00}$ for *three* particles, again looking only for expressions which give a contribution to ${}_4h_{00,m}$. The only nontrivial expressions of this kind are those proportional to ${}^2m\,{}^3m$.

We denote by ${}^{(ab)}r$, the "distance" between the a and b particle††:

$$({}^{(ab)}r)^2=({}^a\xi^s-{}^b\xi^s)({}^a\xi^s-{}^b\xi^s), \tag{B.16}$$

and ask: what contributions to (B.4) come from the third particle and are proportional to ${}^2m\,{}^3m$? We now have

$$\varphi={}_2h_{00}=f+g+k,$$
$$f=-2\,{}^1m({}^1r)^{-1};\quad g=-2\,{}^2m({}^2r)^{-1};\quad k=-2\,{}^3m({}^3r)^{-1}. \tag{B.17}$$

Then the additional expressions for which we look in (B.3) are:

$$\tfrac{1}{2}(gk)_{,ss}+\tfrac{1}{2}gk_{,ss}+\tfrac{1}{2}kg_{,ss}-\tfrac{1}{2}s_{00,ss} \tag{B.18}$$

where s_{00} denotes the additional expression in ${}_4h_{00}$. The additional expressions in (B.5) because of (B.6) are

$$-4\,{}^2m\,{}^3m({}^{(23)}r)^{-1}({}^2\delta+{}^3\delta). \tag{B.19}$$

Therefore, the additional expression to ${}_4h_{00,ss}$ is:

$$s_{00,ss}\sim 8\pi\,{}^2m\,{}^3m({}^{(23)}r)^{-1}({}^2\delta+{}^3\delta)+(gk)_{,ss}+gk_{,ss}+kg_{,ss}. \tag{B.20}$$

From this we find

$$s_{00}=2\,{}^2m\,{}^3m[({}^{(23)}r)^{-1}({}^3r)^{-1}+({}^{(32)}r)^{-1}({}^2r)^{-1}+2({}^2r\,{}^3r)^{-1}]. \tag{B.21}$$

Therefore, as we see, s_{00} does not depend explicitly on ${}^1\xi$. Therefore,

$$\overline{s_{00,m}}=\overline{s_{00,m}} \tag{B.22}$$

and

$$\tfrac{1}{2}\,\overline{s_{00}}-\tfrac{1}{4}\,\overline{gk}={}^2m\,{}^3m[({}^{(23)}r\,{}^{(31)}r)^{-1}+({}^{(23)}r\,{}^{(21)}r)^{-1}+({}^{(12)}r\,{}^{(13)}r)^{-1}]. \tag{B.23}$$

The last very simple question with which we shall deal here is the change from ${}_3h_{0m}$ to ${}_3h_{0m}'$:

$${}_3h_{0m}'={}_3h_{0m}+{}_3a_{0,m}. \tag{B.24}$$

Putting this in (B1), we have

$${}_4h_{00}'={}_4h_{00}+2\,{}_4a_{0,0} \tag{B.25}$$

which is identical with (9.4).

†† Previously ${}^{(12)}r=r$.