Electricity and General Relativity

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1. INTRODUCTION

IN ITS mathematical aspects, Einstein's theory of gravity is based on two fundamental assumptions: (1) the four-dimensional space-time manifold is Riemannian; (this implies existence of a symmetric metrical tensor g_{ik}); (2) the fundamental Lagrangian of the action principle is linear in the curvature components: $L=R_{ik}g^{ik}$. The resulting field equations $R_{ik}=0$ gave full account of the gravitational phenomena but failed to include the electric and quantum phenomena. They also failed to furnish new viewpoints for the structural problems of matter since they did not allow singularity-free solutions which could have been correlated to any of the elementary particles realized in nature.[1]

Later attempts at generalizing the gravitational theory subjected the fundamental postulates (1) and (2) to close scrutiny. Einstein, and many other scientific workers following his lead, was unwilling to depart from the linear Lagrangian of the gravitational equations, since any other choice of the Lagrangian leads to field equations of fourth order, in contradiction to the fact that the basic field equations of mathematical physics do not surpass the order two. A further difficulty was that departure from Einstein's Lagrangian seemed to put the equivalence principle in jeopardy. Hence, it seemed advisable to maintain Einstein's fundamental invariant but to abandon some of the limitations of Riemannian geometry. One could base the geometry of nature on the Γ quantities of an "affine connection" (Eddington, Einstein, Schroedinger), or on a combination of the $\Gamma_{ik}{}^m$ and the g_{ik}, without demanding their symmetry with respect to i, k; (Einstein).[2]

The author's own attempts are characterized by a different departure.[3] He endeavored to leave the Riemannian geometry unchanged and replaced Einstein's linear invariant by an invariant which is *quadratic* in the curvature components. A similar attempt was made earlier by H. Weyl, but in conjunction with a modification of Riemannian geometry in favor of a more general infinitesimal geometry.[4] The author showed that integration of the field equations obtained from the (purely Riemannian) quadratic action principle gives rise to a vectorial function which has the classical properties of the electromagnetic vector potential. However, the ensuing mathematical difficulties frustrated all advance beyond the linear approximation, and the theory remained in a rudimentary stage.

Recent advances in the general Hamiltonization of field equations led to a new mathematical method which is now fully adequate to the investigation of the quadratic action principle. It can now be demonstrated that Riemannian geometry, without any encroachments, contains the entire edifice of classical electromagnetism, together with the interrelation of gravitational and electromagnetic forces. The additional non-Maxwellian terms shed new light on the nature of elementary particles, conceived as static and singularity-free solutions of the fundamental field equations.

2. HAMILTONIZATION OF FIELD EQUATIONS

In every Lorentz-invariant variational field theory the fundamental conservation laws of momentum and energy appear in the form that the Minkowskian divergence of a certain symmetric tensor of second order, the "stress-energy tensor," vanishes, in consequence of the field equations.[5] We have to assume that the basic Lagrangian does not contain the field quantities in higher than *first* derivatives. In two fundamental cases; namely, Maxwell's equations and Dirac's equation of the electron, the basic Lagrangian is even *linear* in the first partial derivatives. The resultant field equations are then of not higher than first order, in analogy to the Hamiltonian equations of dynamics. In both cases the resulting system is linear in the derivatives, with constant coefficients. Since in the dynamical case any arbitrary Lagrangian can be transformed into the Hamiltonian canonical form, should it not be likewise possible to transform an arbitrary set of *partial* differential equations, deducible from a Lagrangian, into a normal form, corresponding to Hamilton's canonical form?

To answer this question, we first interpret Hamilton's procedure in a manner which is somewhat different from the traditional approach, based on Legendre's transformation.[6] The new formulation has the advantage of much greater flexibility and is directly applicable

[1] A. Einstein, Rev. Univ. nac. Tucuman **2**, 11 (1941).

[2] For a brief account of the very extensive literature and a condensed bibliography see Sir E. Whittaker, *History of the Theories of Aether and Electricity* (Nelson and Sons, London, 1953), pp. 188–192; see also, Cornelius Lanczos, Nuovo cimento Suppl. **2**, 1193 (1955).

[3] Cornelius Lanczos, Phys. Rev. **39**, 716 (1932); **61**, 713 (1942); (subsequently quoted as I and II); Revs. Modern Phys. **21**, 497 (1949).

[4] Hermann Weyl, Math. Z. **2**, 384 (1918); Ann. Physik **59**, 101 (1919); Physik. Z. **22**, 473 (1921); see also W. Pauli, Enc. math. Wiss. **V19**, 759 (1920). In later years, under the impact of wave mechanics, Weyl lost confidence in his theory; see *Selecta Hermann Weyl* (Birkhäuser Verlag, Basel, 1956), p. 192.

[5] See, e.g., G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949), pp. 10 and 217.

[6] See C. Lanczos, *The Variational Principles of Mechanics* (Toronto University Press, Toronto, 1949), p. 161.

to the realm of partial differential equations, which is our present concern. The "momenta" p_i appear in this formulation as Lagrangian multipliers.

Considering the Lagrangian $L_0(q_i,\dot{q}_i,t)$ of a dynamical problem, we want to conceive the $\dot{q}_i$ as a second set of independent variables, let us say w_i. This is permissible, provided that in the variation we do not violate the condition

$$\dot{q}_i - w_i = 0. \tag{2.1}$$

Thus we have the Lagrangian $L_0(q_i,w_i,t)$ with the n auxiliary conditions (2.1). According to the usual procedure we multiply every auxiliary condition by an undetermined multiplier p_i and add it to the given Lagrangian. We thus get the modified Lagrangian

$$L = p_i(\dot{q}_i - w_i) + L_0(q_i,w_i,t). \tag{2.2}$$

The auxiliary conditions can be dropped since they appear as the consequence of the variational principle, varying with respect to the p_i, which are full-fledged new mechanical variables. The original Lagrangian problem is thus replaced by a new Lagrangian problem whose variables are p_i, q_i, w_i. The new Lagrangian has the "canonical form"

$$L = p_i\dot{q}_i - H, \tag{2.3}$$

where

$$H = p_i w_i - L_0(q_i,w_i,t). \tag{2.4}$$

This H is free of all derivatives. Moreover, the variables w_i appear solely in H. Hence they are purely *algebraic* variables which can be eliminated without any integration. The Euler equations applied to these variables give

$$p_i - (\partial H/\partial w_i) = 0. \tag{2.5}$$

Although these equations hold only for the actual motion, while their use for the elimination of the w_i means that we impose them also on the varied motion, this move is actually justified. The elimination involves no derivatives. Hence, the condition of "varying between definite limits" is not violated.

Eliminating the w_i from (2.5) we now obtain them as some explicit functions of p_j, q_j, t:

$$w_i = f_i(p_j,q_j,t). \tag{2.6}$$

We introduce these functions into (2.4) and arrive at a new form of H which is purely a function of the p_i, q_i, t:

$$H = H(p_i,q_i,t) \tag{2.7}$$

This construction of H is exactly the traditional one, except that the $\dot{q}_i$ have been replaced by the w_i which, however, is irrelevant since the final H does not depend on the w_i.

This method is applicable to ordinary as to partial differential equations. With the help of it any system of field equations derivable from a variational principle can be brought into a canonical form. Our present aim is restricted to the field equations of general relativity.

We thus start with a Lagrangian shaped to the demands of general relativity. We wish to see how the general canonical process operates under these circumstances. In Einstein's case the fundamental Lagrangian is $L_0 = R_{ik}g^{ik}$. However, we do not want to restrict ourselves to this special case since our aim is to study a Lagrangian which is formed of the curvature quantities in a *quadratic* rather than linear manner; (see Sec. 5). We thus leave the specific form of the Lagrangian free and assume that it is some function of the R_{ik} and the g^{ik}:

$$L_0 = f(R_{ik}, g^{ik}). \tag{2.8}$$

The tensor R_{ik} is a complicated differential operator of second order of the basic variables g_{ik}. We replace, however, R_{ik} by the algebraic variables w_{ik}, considering the equation

$$R_{ik} - w_{ik} = 0 \tag{2.9}$$

as an auxiliary condition of the variational problem. This gives the new Lagrangian

$$L = p^{ik}(R_{ik} - w_{ik}) - L_0(w_{ik}, g^{ik}). \tag{2.10}$$

Furthermore, we conceive R_{ik} as a differential operator of only *first* order by introducing the new field variables $\Gamma_{ik}{}^m$:

$$R_{ik} = \tfrac{1}{2}\left(\frac{\partial\Gamma_{i\alpha}{}^{\alpha}}{\partial x_k} + \frac{\partial\Gamma_{k\alpha}{}^{\alpha}}{\partial x_i}\right) - \frac{\partial\Gamma_{ik}{}^{\alpha}}{\partial x_\alpha} + \Gamma_{i\alpha}{}^{\beta}\Gamma_{k\beta}{}^{\alpha} - \Gamma_{ik}{}^{\beta}\Gamma_{\alpha\beta}{}^{\alpha}. \tag{2.11}$$

This is permissible, provided that we add as auxiliary conditions the equations which establish the $\Gamma_{ik}{}^m$ as functions of the g^{ik}:

$$\frac{\partial g^{ik}}{\partial x_m} + \Gamma_{\alpha m}{}^{i} g^{\alpha k} + \Gamma_{\alpha m}{}^{k} g^{\alpha i} = 0. \tag{2.12}$$

At this stage our Lagrangian appears in the following form:

$$L = \tfrac{1}{2}p^{ik}\left(\frac{\partial\Gamma_{i\alpha}{}^{\alpha}}{\partial x_k} + \frac{\partial\Gamma_{k\alpha}{}^{\alpha}}{\partial x_i} - 2\frac{\partial\Gamma_{ik}{}^{\alpha}}{\partial x_\alpha}\right) + \gamma_{ik}{}^{m}\frac{\partial g^{ik}}{\partial x_m} - H_1 - H_0 \tag{2.13}$$

where

$$H_1 = p^{ik}(\Gamma_{ik}{}^{\beta}\Gamma_{\beta\alpha}{}^{\alpha} - \Gamma_{i\alpha}{}^{\beta}\Gamma_{k\beta}{}^{\alpha}) - \gamma_{ik}{}^{m}(\Gamma_{\alpha m}{}^{i} g^{\alpha k} + \Gamma_{\alpha m}{}^{k} g^{\alpha i}) \tag{2.14}$$

$$H_0 = p^{ik}w_{ik} - f(w_{ik}, g^{ik}). \tag{2.15}$$

The field quantities of the variational problem are the g^{ik}, p^{ik}, $\Gamma_{ik}{}^m$, $\gamma_{ik}{}^m$ and the w_{ik}, all quantities which are symmetric in i, k. However, the w_{ik} can be eliminated with the help of the equations

$$p^{ik} = \partial f/\partial w_{ik}. \tag{2.16}$$

Substituting the result of this elimination in (2.15) we obtain H_0 as a function of g^{ik}, p^{ik}:

$$H_0 = H_0(p^{ik}, g^{ik}). \tag{2.17}$$

Hence our field variables are: the 10 g^{ik} and 10 p^{ik}; the 40 $\Gamma_{ik}{}^m$ and 40 $\gamma_{ik}{}^m$, altogether 100 variables. In terms of an abstract "configuration space" we could conceive these variables as the components $\psi_1, \psi_2, \cdots \psi_N$ of a "vector" in a space of $N=100$ dimensions.

The canonical integrand (2.13) contains the partial derivatives of the field variables in a particularly simple, namely purely *linear* form. The highly nonlinear character of the relativistic equations is thrown completely into the Hamiltonian function $H=H_0+H_1$, but even here H_1 is of not higher than *third* degree in the field variables.

In terms of the abstract vector ψ, the canonical integrand may be written in the following homogeneous fashion:

$$L = \psi_i \alpha_{ik}{}^m \frac{\partial \psi_k}{\partial x_m} - H(\psi_1, \psi_2, \cdots \psi_N) \tag{2.18}$$

where the $\alpha_{ik}{}^m$ indicate a set of numerical matrices. The lower indices i, k belong to the configuration space of the ψ_i, while the upper index m belongs to the space of the x_m. The term in which the $\alpha_{ik}{}^m$ appears is variationally equivalent to

$$\tfrac{1}{2}\alpha_{ik}{}^m\left(\psi_i \frac{\partial \psi_k}{\partial x_m} - \psi_k \frac{\partial \psi_i}{\partial x_m}\right). \tag{2.19}$$

This shows that the matrices $\alpha_{ik}{}^m$ can be conceived as anti-symmetric in i, k. If the operator $\partial/\partial x_m$ is replaced by the self-adjoint operator $\partial/i\partial x_m$, the matrices $i\alpha_{ik}{}^m$ become purely imaginary. Addition of a real symmetric part would not change anything since we have added not more than a pure divergence which is variationally deletable. Thus generally we may conceive the matrices $i\alpha_{ik}{}^1, \cdots i\alpha_{ik}{}^4$ as 4 given numerical Hermitian matrices. In Dirac's case of the electron they become 4 by 4 matrices while in the case of general relativity we get a system of four 100 by 100 matrices; (although in a given particular problem great simplifications can take place, of course).

Even such a highly complicated system of nonlinear field equations as the field equations of general relativity may still be written in the form of a Dirac equation, if the four special Dirac matrices are replaced by a much more elaborate but still purely numerical set of matrices.[7]

[7] The analogy is still more pronounced on formulating the canonical system in terms of 50 *complex* field variables $p^{ik}+ig^{ik}$, and

$$\Gamma_{ik}{}^m + i(\gamma_{ik}{}^m - \tfrac{1}{3}\gamma_{i\alpha}{}^\alpha \delta_k{}^m - \tfrac{1}{3}\gamma_{k\alpha}{}^\alpha \delta_i{}^m).$$

The 100 real ψ_i components are thus reducible to 50 complex ψ_i, in harmony with the complex nature of wave-mechanical field variables.

3. FIELD EQUATIONS OF GENERAL RELATIVITY IN CANONICAL FORM

In view of the fact that the $\Gamma_{ik}{}^m$ quantities do not form a genuine tensor, one might think that the Lagrangian multipliers p^{ik} and $\gamma_{ik}{}^m$ will likewise lack tensor character. However, the determining equations for these field variables are deduced by *varying* the g^{ik} and the $\Gamma_{ik}{}^m$ and we know that *variationally* the $\Gamma_{ik}{}^m$ *do* behave like a genuine tensor, covariant in i, k, contravariant in m. For this reason the additional field variables p^{ik} and $\gamma_{ik}{}^m$ become genuine tensors of second, respectively third rank. Variation with respect to the $\Gamma_{ik}{}^m$ is particularly interesting since it does not involve the unknown function $H_0(p^{ik}, g^{ik})$ which is free of the $\Gamma_{ik}{}^m$.

From now on only *covariant* operations appear in our deductions. For the purpose of denoting a covariant derivative we follow Einstein's procedure, although replacing his "semicolon" by a simple "comma," since a distinction between ordinary and covariant differentiation is not necessary. (In the few cases when ordinary derivatives are encountered, the usual notation $\partial/\partial x_i$ will be used.)

Variation of the $\Gamma_{ik}{}^m$ yields the following 40 equations:

$$p_{,m}{}^{ik} - \tfrac{1}{2}(p^{i\alpha}{}_{,\alpha}\delta_m{}^k + p^{k\alpha}{}_{,\alpha}\delta_m{}^i) + \gamma_{m\alpha}{}^i g^{\alpha k} + \gamma_{m\alpha}{}^k g^{\alpha i} = 0. \tag{3.1}$$

To this we add the 40 "conjugate" equations, obtained by varying the $\gamma_{ik}{}^m$:

$$\frac{\partial g^{ik}}{\partial x_m} + \Gamma_{m\alpha}{}^i g^{\alpha k} + \Gamma_{m\alpha}{}^k g^{\alpha i} = 0. \tag{3.2}$$

So far we have obtained 80 equations of the canonical system. The remaining 20 equations are obtained by varying with respect to the g^{ik} and the p^{ik}. The former variation yields

$$-\gamma_{ik}{}^\alpha{}_{,\alpha} = \frac{\partial H_0}{\partial g^{ik}} - \tfrac{1}{2}H_0 g_{ik}. \tag{3.3}$$

The latter variation yields

$$\tfrac{1}{2}\left(\frac{\partial \Gamma_{i\alpha}{}^\alpha}{\partial x_k} + \frac{\partial \Gamma_{k\alpha}{}^\alpha}{\partial x_i} - 2\frac{\partial \Gamma_{ik}{}^\alpha}{\partial x_\alpha}\right) + \Gamma_{i\alpha}{}^\beta \Gamma_{k\beta}{}^\alpha - \Gamma_{ik}{}^\beta \Gamma_{\beta\alpha}{}^\alpha = \frac{\partial H_0}{\partial p^{ik}}. \tag{3.4}$$

All these equations are *linear* in the derivatives, with constant coefficients. The nonlinearity enters only *algebraically*, in the terms which do not contain derivatives, in conformity with the canonical equations of ordinary dynamics.

4. EINSTEIN'S LINEAR ACTION PRINCIPLE

We first encounter the canonical system in a 1925 paper of Einstein[8], when he became interested in the

[8] A. Einstein, Sitzber. preuss. Akad. Wiss. 414 (1925).

theory of "affine connection," as advocated by A. S. Eddington.[9] Einstein proposes[10] not to discard the metrical tensor in favor of the $\Gamma_{ik}{}^m$ quantities, but to determine both the g_{ik} and the $\Gamma_{ik}{}^m$ from a common variational principle in which the g_{ik} and $\Gamma_{ik}{}^m$ are treated as independent field variables, but not assuming their symmetry with respect to i, k. He remarks that for the case of pure gravity (i.e., symmetric g_{ik} and $\Gamma_{ik}{}^m$), one obtains the most satisfactory derivation of the ordinary gravitational equations $R_{ik}=0$. Einstein does not use any Lagrangian multipliers and his system (for the case of symmetry) contains only 50 instead of 100 field variables. Why is it that in his case the additional variables do not come into evidence?

Einstein chose the action principle

$$L_0=R_{ik}g^{ik}. \tag{4.1}$$

In this case

$$f(w_{ik},g^{ik})=w_{ik}g^{ik} \tag{4.2}$$

and

$$H_0=(p^{ik}-g^{ik})w_{ik}. \tag{4.3}$$

In view of the linearity of H_0 in w_{ik} we do not succeed now with the elimination scheme. Nor is that necessary in the present case. Addition of H_0 to the Lagrangian can be conceived as a tool to maintain the auxiliary condition

$$p^{ik}=g^{ik}. \tag{4.4}$$

By introducing this condition in (2.13) we dispense with the field variables p^{ik} and also with H_0, although the Lagrangian factors $\gamma_{ik}{}^m$ still remain. But now (3.1) yields for the present case a complete *vanishing* of the $\gamma_{ik}{}^m$, since the covariant derivatives of the g_{ik} are zero. By maintaining the condition $\gamma_{ik}{}^m=0$ during the variation we lose the Lagrangian multipliers and (3.2), but we are entitled to do so because (3.1) take over the role of the equations (3.2).

From the standpoint of the general theory the case of Einstein's linear action principle represents a *degenerate* case in which the 50 conjugate field variables become tautological. The objection that any other choice of the action principle leads to differential equations of fourth order for the g_{ik}, obtains a different meaning on viewing the problem from the vantage point of the canonical equations. The canonical equations remain of first order under all circumstances, and it is only the vanishing of the conjugate variables which characterizes the Einsteinian system. Since, however, the 100 field variables of the general scheme are replaceable by 50 complex field variables,[7] we can regard the action principle of Einstein as that extreme case in which the generally complex field variables are reduced to real variables.

[9] A. S. Eddington, Proc. Roy. Soc. (London) **A99**, 104 (1921).

[10] The general program of this paper is remarkably close to the last efforts of Einstein toward a unified field theory; see Albert Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 1955), fifth edition, p. 154.

5. QUADRATIC ACTION PRINCIPLE

We now depart from Einstein's linear action principle and introduce the following Lagrangian[11]

$$\begin{aligned}L_0&=\tfrac{1}{2}(R_{ik}R^{ik}+\beta R^2)\\&=\tfrac{1}{2}(w_{ik}w^{ik}+\beta w^2),\end{aligned} \tag{5.1}$$

where β is an unspecified numerical constant. We have to eliminate w_{ik} from the Hamiltonian

$$H_0=p^{ik}w_{ik}-\tfrac{1}{2}(w^{ik}w_{ik}+\beta w^2). \tag{5.2}$$

The elimination yields

$$H_0=\tfrac{1}{2}(p^{ik}p_{ik}+\sigma p^2) \tag{5.3}$$

with

$$\sigma=-\frac{\beta}{1+4\beta}. \tag{5.4}$$

We do not use the full canonical scheme of Sec. 2, in order to facilitate comparison with the classical results of general relativity. We allow differential operators of second order by conceiving R_{ik} as a direct function of the g_{ik}, without interjecting the $\Gamma_{ik}{}^m$ as an additional set of independent variables. Then also the conjugate $\gamma_{ik}{}^m$ will not appear. Our action variables are solely the 10 g_{ik} and the 10 p^{ik}. The Lagrangian (2.13) will therefore simplify to

$$\begin{aligned}L&=p^{ik}R_{ik}-H_0\\&=p^{ik}R_{ik}-\tfrac{1}{2}(p^{ik}p_{ik}+\sigma p^2),\end{aligned} \tag{5.5}$$

where R_{ik} is defined by (2.11) on substituting for the $\Gamma_{ik}{}^m$ the usual expressions in terms of the g_{ik} and their first derivatives. Hence from now on R_{ik} will be regarded as a second-order differential operator of the g_{ik}.

In order to perform the variation with respect to the g_{ik} and the p_{ik}, we need a certain invariant differential operator of second order, generated by variation of R_{ik}. This expression has been derived previously[12] and may be written as follows:

$$E_{ik}(\gamma)=\delta R_{ik}=\tfrac{1}{2}(\Delta\gamma_{ik}-\gamma_i{}^\alpha{}_{,k\alpha}-\gamma_k{}^\alpha{}_{,i\alpha}+\gamma_\alpha{}^\alpha{}_{,ik}) \tag{5.6}$$

where $\gamma_{ik}=\delta g_{ik}$ and Δ denotes the invariant Laplace operator

$$\Delta u_{i\cdots p}=g^{\alpha\beta}u_{i\cdots p,\,\alpha\beta}. \tag{5.7}$$

We also need the adjoint of this operator, likewise given before [see I (2.10)]:

$$D_{ik}(p)=\tfrac{1}{2}(\Delta p_{ik}-p_i{}^\alpha{}_{,k\alpha}-p_k{}^\alpha{}_{,i\alpha}+p^{\alpha\beta}{}_{,\alpha\beta}g_{ik}). \tag{5.8}$$

In terms of this operator the variation with respect to g_{ik} yields:

$$D^{ik}(p)-(p^{i\alpha}p_\alpha{}^k+\sigma pp^{ik})+\tfrac{1}{2}Lg^{ik}=0 \tag{5.9}$$

while the variation with respect to p^{ik} gives

$$R_{ik}=p_{ik}+\sigma pg_{ik}. \tag{5.10}$$

[11] Concerning the uniqueness of this choice see II, p. 714.

[12] See I (2.6) and (2.12). The present E corresponds to $\frac{1}{2}D$, the present D to $\frac{1}{2}F$ of the previous paper.

In consequence of (5.10) we may put

$$L=H_0. \tag{5.11}$$

On the basis of this relation (5.9) may be written as follows:

$$D^{ik}(p)-(p^{i\alpha}p_\alpha{}^k-\tfrac{1}{4}p^{\alpha\beta}p_{\alpha\beta}g_{ik}) -\sigma p(p^{ik}-\tfrac{1}{4}pg^{ik})=0. \tag{5.12}$$

The 20 equations (5.10) and (5.12) are the fundamental field equations of our problem.

With the help of these equations we demonstrate two fundamental consequences of the quadratic action principle, obtained earlier by different tools; [see II (2.13) and (2.15)]. Replace the notation p_{ik} by $\bar{p}_{ik}$ and put

$$\bar{p}_{ik}=p_{ik}+\lambda g_{ik}. \tag{5.13}$$

Then Eq. (5.10) changes to

$$R_{ik}=(1+4\sigma)\lambda g_{ik}+p_{ik}+\sigma p g_{ik} \tag{5.14}$$

while the modification of the left side of (5.12) is only the addition of the following term:

$$-2\lambda(1+2\sigma)(p^{ik}-\tfrac{1}{4}pg^{ik}). \tag{5.15}$$

The inhomogeneous part of the substitution drops out completely of the (5.12) and $p_{ik}=0$ is still a possible solution of the field equations. This means that the so-called "cosmological equations"

$$R_{ik}=(1+4\sigma)\lambda g_{ik} \tag{5.16}$$

with an arbitrary constant λ represent an *exact solution* of the field equations. Any solution of (5.12), [corrected by the term (5.15)], which is not zero but small, can be conceived as a small deformation of the fundamental cosmological solution (5.16). The constant λ, usually considered as completely negligible and modifying only the structure of the world in cosmic dimensions, will in our later discussions play the role of a fundamental *atomic* constant; (see Sec. 9).

As a second consequence, multiply (5.12) by g_{ik}. We get

$$\Delta p+2p^{\alpha\beta}{}_{,\alpha\beta}=0. \tag{5.17}$$

On the other hand, taking the divergence of the divergence condition for the "metrical matter tensor"

$$T_{ik}=R_{ik}-\tfrac{1}{2}Rg_{ik} \tag{5.18}$$

we obtain the relation

$$R^{\alpha\beta}{}_{,\alpha\beta}=\tfrac{1}{2}\Delta R \tag{5.19}$$

which means, in view of (5.10)

$$p^{\alpha\beta}{}_{,\alpha\beta}=\tfrac{1}{2}(1+2\sigma)\Delta p. \tag{5.20}$$

Substitution in (5.17) gives

$$(1+\sigma)\Delta p=0. \tag{5.21}$$

which may also be written in the form

$$\frac{1+\sigma}{1+4\sigma}\Delta R=0. \tag{5.22}$$

We see that $R=\text{const}$ is an *exact first integral* of the field equations. The special choice $\sigma=-1$ of the free constant σ leads to a *degeneracy*. We lose one of the determining equations of our system and thus the solution becomes under-determined; (cf. Sec. 7).

6. FREE VECTORIAL FUNCTION φ_i

In problems of Hamiltonian dynamics the theory of canonical transformations is frequently of paramount importance.[13] Instead of trying to integrate the dynamical equations directly, we solve them indirectly by performing transformations of the dynamical variables which retain the normal form of the dynamical equations but simplify the Hamiltonian function H. The canonical nature of the transformation demands that the differential form $p_i\dot{q}_i$ shall change by a complete derivative only. In this case the canonical equations are preserved but the transformation modifies the form of the Hamiltonian function H.

In our present problem the derivative part of the Lagrangian (5.5) is contained in the first term. A canonical transformation of the variables p_{ik} and g_{ik} should have the property that it should change the term $p^{ik}R_{ik}$ by a pure divergence. The new Lagrangian will then be of the same form as the original one, but with a modified Hamiltonian H. We will now take advantage of the fact that the divergence of the metrical matter tensor (5.18) *vanishes identically*. We denote once more the original p_{ik} by $\bar{p}_{ik}$ and apply the following transformation:

$$\bar{p}_{ik}=p_{ik}+\tfrac{1}{2}(\varphi_{i,k}+\varphi_{k,i}-\varphi^{\alpha}{}_{,\alpha}g_{ik}). \tag{6.1}$$

A simple calculation shows that the added terms contribute to L a pure divergence which can be omitted. Hence our new Lagrangian becomes

$$L=p^{ik}R_{ik}-H, \tag{6.2}$$

where H, compared with the previous H_0, is augmented by further terms, partly linear and partly quadratic in φ_i. First consider the *quadratic* terms only. They amount to the following addition H_1 to the previous H_0[14]:

$$H_1=\tfrac{1}{8}(\varphi_{i,k}+\varphi_{k,i})(\varphi^{i,k}+\varphi^{k,i})+\tfrac{1}{2}\sigma(\varphi^{\alpha}{}_{,\alpha})^2. \tag{6.3}$$

The first term can be transformed as follows:

$$\begin{aligned}\tfrac{1}{8}(\varphi_{i,k}+\varphi_{k,i})(\varphi^{i,k}+\varphi^{k,i})&=\tfrac{1}{4}(\varphi_{i,k}+\varphi_{k,i})\varphi^{i,k}\\ &=\tfrac{1}{4}(\varphi_{i,k}-\varphi_{k,i})\varphi^{i,k}+\tfrac{1}{2}\varphi_{k,i}\varphi^{i,k}\\ &=\tfrac{1}{8}F_{ik}F^{ik}+\tfrac{1}{2}\varphi_{k,i}\varphi^{i,k},\end{aligned} \tag{6.4}$$

[13] See reference 6, Chap. VII.
[14] The notation $\varphi^{i,k}$ refers to $\varphi^{i}{}_{,\alpha}g^{\alpha k}$.

where

$$F_{ik}=\varphi_{i,k}-\varphi_{k,i}=\frac{\partial\varphi_i}{\partial x_k}-\frac{\partial\varphi_k}{\partial x_i}. \quad (6.5)$$

Moreover,

$$\begin{aligned}\varphi^{i,k}\varphi_{k,i}&=(\varphi^{i,k}\varphi_k)_{,i}-\varphi^{i,k}{}_{,i}\varphi_k\\&=(\varphi^{i,k}\varphi_k)_{,i}-\varphi^i{}_{,ik}\varphi^k+R_{ik}\varphi^i\varphi^k\\&=(\varphi^i{}_{,k}\varphi^k-\varphi^k{}_{,k}\varphi^i)_{,i}+(\varphi^\alpha{}_{,\alpha})^2+R_{ik}\varphi^i\varphi^k. \quad (6.6)\end{aligned}$$

Omitting a divergence, which is variationally zero, we can write

$$H_1=\tfrac{1}{8}F_{ik}F^{ik}+\tfrac{1}{2}(1+\sigma)(\varphi^\alpha{}_{,\alpha})^2+\tfrac{1}{2}R_{ik}\varphi^i\varphi^k. \quad (6.7)$$

The last term contains once more the differential operator R_{ik} which should be united with the first term. Hence, our final transformation will not be (6.1) but

$$\bar{p}_{ik}=p_{ik}+\tfrac{1}{2}(\varphi_{i,k}+\varphi_{k,i}-\varphi^\alpha{}_{,\alpha}g_{ik}+\varphi_i\varphi_k). \quad (6.8)$$

We know in advance that the term $p^{ik}R_{ik}$ will remain unchanged. The Hamiltonian H_0, however, has to be augmented by further terms which will be partly linear in the p^{ik} and partly independent of them. Those terms which are linear in the p^{ik}, can be combined with the first term of the Lagrangian by introducing a modified differential operator R_{ik}^*. Then the Hamiltonian H_1 has to absorb only those terms which depend on φ_i alone. Carrying through the calculations, we arrive at the following result:

$$L=p^{ik}R_{ik}^*-(H_0+H_1), \quad (6.9)$$

where

$$\begin{aligned}R_{ik}^*=R_{ik}&-\tfrac{1}{2}(\varphi_{i,k}+\varphi_{k,i}+\varphi_i\varphi_k)\\&+\tfrac{1}{2}[(1+2\sigma)\varphi^\alpha{}_{,\alpha}-\sigma\varphi^\alpha\varphi_\alpha]g_{ik} \quad (6.10)\end{aligned}$$

and

$$H_1=\tfrac{1}{8}F_{ik}F^{ik}+\tfrac{1}{2}(1+\sigma)(\varphi^\alpha{}_{,\alpha}-\tfrac{1}{2}\varphi^\alpha\varphi_\alpha)^2 \quad (6.11)$$

while H_0 is still defined by (5.3).

The transformation (6.8) correlates to any $\bar{p}_{ik}$ a new p_{ik}, considering φ_i as an arbitrarily prescribed (although continuous and differentiable) vector function. Varying with respect to the p_{ik} and the g_{ik}, considering the φ_i as given, we obtain the previous field equations, although now expressed in the new p_{ik}. What happens if we include the φ_i among the field variables by adding their free variation to the free variations of p_{ik} and g_{ik}? The transformation equation (6.8) shows that if we keep p_{ik} constant but vary φ_i arbitrarily, we obtain a certain special type of variation for the $\bar{p}_{ik}$. The field equations guarantee the vanishing of the first variation of our action integral for *any* variation of the $\bar{p}_{ik}$. Hence addition of the φ_i to our field variables merely adds four more equations which are tautological since they are satisfied in consequence of the field equations. Thus we do not lose in generality if we add the φ_i to the field variables of our problem. The 24 field equations thus derived do not yield in fact more than 20 independent equations, obtained by varying the p_{ik} and the g_{ik}.

7. THEORY OF WEYL

We now discuss a remarkable relation between our purely Riemannian action principle and the theory of Hermann Weyl,[4] developed on the basis of an infinitesimal geometry which is more general than Riemann's geometry. Weyl constructed a geometry which enlarges the Riemannian line-element g_{ik} by a further vectorial quantity φ_i, which in physical interpretation becomes the electromagnetic vector potential. The fundamental field equations of this geometry are developed from an action principle which is quadratic in the curvature quantities, in full analogy to the assumptions of the present theory. The general quadratic action principle was considered by W. Pauli,[15] who developed all the relevant mathematical and physical consequences of Weyl's theory. It is composed of a linear superposition of four terms, with three arbitrary constants, and may be written, according to Pauli, in the following form [see Pauli's Eq. (32), p. 462]:

$$L_p=-\tfrac{1}{2}\left[k_1\tilde{R}_{iklm}\tilde{R}^{iklm}+k_2\tilde{R}_{ik}\tilde{R}^{ik}+\frac{k_3}{2}\tilde{R}^2+\tfrac{1}{4}F_{ik}F^{ik}\right], \quad (7.1)$$

where

$$\begin{aligned}\tilde{R}_{ik}=R_{ik}&-\tfrac{1}{2}(\varphi_{i,k}+\varphi_{k,i}+\varphi_i\varphi_k)\\&-\tfrac{1}{2}(\varphi^\alpha{}_{,\alpha}-\varphi^\alpha\varphi_\alpha)g_{ik}. \quad (7.2)\end{aligned}$$

In fact it is unnecessary to include the term with $\tilde{R}_{iklm}$ since one can show[16] that variation of the first invariant is equivalent to that of a certain linear combination of the other three invariants. Hence, it is permissible to put in advance $k_1=0$, retaining only the two essential constants k_2 and k_3. In the Riemannian case, where a similar reduction takes place,[17] only *one* essential constant remains, *viz.* the β of (3.1).

To faciliate a comparison between our purely Riemannian Lagrangian (6.9) and Weyl's Lagrangian (7.1), we now eliminate the p_{ik} from our action principle, thus returning once more to the Lagrangian instead of Hamiltonian formulation of action. The elimination gives the new Lagrangian

$$L_0'=\tfrac{1}{2}(R_{ik}^*R^{*ik}+\beta R^{*2})-H_1 \quad (7.3)$$

in which only the g_{ik} and the φ_i remain as field variables. Comparison of the R_{ik}^* [see (6.10)] with the $\tilde{R}_{ik}$ of Weyl's theory shows a remarkable analogy. The general structure of the terms is exactly the same in both cases, except that the numerical coefficients within the last term do not agree. For the special choice $\sigma=-1$ the two expressions become *exactly* identical. But even for other values of σ the difference is of *second* order only, if we assume the customary "Lorentz

[15] W. Pauli, Physik. Z. **20**, 457 (1919).
[16] See R. Bach, Math. Z. **9**, 110 (1921).
[17] See C. Lanczos, Ann. Math. **39**, 842 (1938).

condition" for the vector potential:

$$\varphi^{\alpha}{}_{,\alpha}=0. \tag{7.4}$$

And yet, the origins of R_{ik}^* and $\tilde{R}_{ik}$ are widely different. The first expression came about as the result of a canonical transformation (with φ_i=arbitrary), the second on the basis of a certain type of non-Riemannian geometry. Discarding for the moment the small difference between the two kinds of curvature quantities, Weyl's modification of Riemannian geometry appears tautological, except for the fact that Weyl's action principle permits *two* free constants k_2 and k_3, while the Riemannian case leads to the *single* free constant β. This greater freedom of Weyl's geometry is not necessarily an advantage since the two free constants k_2 and k_3 cannot be correlated to definite physical constants. Pauli arrives at a decision on the basis that the results of Einstein's gravitational theory should be maintained for the case of vanishing φ_i. He thus makes the choice $k_1=k_2=0$; [see his equation (50), p. 465]; Weyl subsequently adopted this choice.[18]

In the Riemannian case we do not have the freedom of k_2. The comparison of (7.3) with (7.1) shows that here

$$k_2=-1 \tag{7.5}$$

while the constants k_3 and β are in the relation

$$k_3=-2\beta=2\sigma/(1+4\sigma). \tag{7.6}$$

Although (7.5) prohibits the choice $k_2=0$, we can interpret Pauli's choice as the limiting case

$$k_2/k_3=0 \tag{7.7}$$

which means

$$\beta=\infty,\quad \sigma=-\tfrac{1}{4}. \tag{7.8}$$

There exists a deep-seated difference between the theory of Weyl and the theory here advocated. In Weyl's geometry the "principle of gauge-invariance" invokes a certain pre-established harmony between the g_{ik} and the φ_i, chosen in such manner that at every point of the manifold a proportionality factor of the g_{ik} must remain undetermined; correspondingly, the φ_i can also be determined up to a scalar function only. Einstein[19] found Weyl's theory unacceptable on account of the arbitrariness of a common factor of the g_{ik}. He pointed out that the tremendous consistency of the spectral lines throughout the universe demonstrates the inevitability of an absolute measuring rod. The present theory is not affected by this criticism of Einstein since the g_{ik} are in our case completely determined. On the other hand, the vector potential φ_i remains completely undetermined. This shows that the limiting case $\sigma=-1$, i.e.,

$$\beta=-\tfrac{1}{3};\quad k_1=0,\quad k_2=-1,\quad k_3=\tfrac{2}{3} \tag{7.9}$$

which is a point of tangency between the two theories, must be singular from both the Riemannian and the Weylian approach. From the Riemannian standpoint we must lose one of our equations in determining the metrical tensor, from the Weylian standpoint we must lose the determining equation for the vector potential.

Closer examination corroborates these predictions. We have seen in Sec. 5 that $\sigma=-1$ is a singular case for which the determining equation (5.22) of the scalar curvature R is lost. This is in harmony with Weyl's geometry in which R remains undetermined on account of the freedom of an arbitrary factor of the g_{ik}. Let us see, what we get for the vector potential in Weyl's theory. Variation of the vector potential in Pauli's action principle (7.1) yields the following relation[20]:

$$\tfrac{1}{2}(1+2k_1+k_2)\frac{\partial g^{\frac{1}{2}}F^{i\alpha}}{g^{\frac{1}{2}}\partial x_\alpha} =(k_1+k_2+\tfrac{3}{2}k_3)(\tilde{R}_{,\alpha}g^{\alpha i}+\tilde{R}\varphi^i). \tag{7.10}$$

For the case $k_1=0$, $k_2=-1$, the left side of the equation disappears. The right side would then demand the complete vanishing of φ_i, except if the choice $k_3=\frac{2}{3}$ is made, which makes the right side vanish too. This choice corresponds exactly to our singular case $\sigma=-1$, as (7.6) demonstrates. For other values of σ, the left side of (7.10) is still zero but the right side is now corrected by further terms because of the modifications which have to be applied to the quantities of Weyl. The exact relation between R_{ik}^* and $\tilde{R}_{ik}$ is given as follows:

$$R_{ik}^*=\tilde{R}_{ik}+(1+\sigma)Qg_{ik}, \tag{7.11}$$

where the scalar Q is a certain non-Maxwellian quantity which plays a characteristic and fundamental role in the present theory

$$Q=\varphi^{\alpha}{}_{,\alpha}-\tfrac{1}{2}\varphi^{\alpha}\varphi_{\alpha}. \tag{7.12}$$

Moreover, our Lagrangian (7.3) and Pauli's Lagrangian (7.1) (for the case $k_1=0$, $k_2=-1$, $k_3=-2\beta$) are in the following relation to each other:

$$L_0'=L_p+(1+3\beta)Q(\tilde{R}+\tfrac{3}{2}\phi). \tag{7.13}$$

The correction term on the right side is such that its variation with respect to φ_i exactly counteracts the right side of (7.10) and once more we obtain the empty equation 0=0 which leaves φ_i undetermined.

8. VECTOR POTENTIAL AND METRICAL TENSOR

The vectorial function φ_i—interpreted in Weyl's theory as the vector potential of the electromagnetic field—appears in our investigation as a quantity which

[18] See H. Weyl, *Space, Time, Matter* (Methuen and Company, London, 1922), p. 295.

[19] Albert Einstein, Physik. Z. **21**, 651 (1920).

[20] See Eqs. (26), (29), (30), and (32) of Pauli's paper; Pauli's equation (26) should on the right side contain the factor $(1+2c_1+c_2)$. The error is caused by the identity (B) on p. 460, which in fact holds for $\bar{R}^{ki}$ and not for $\bar{R}^{ik}$. At the author's request Professor Pauli kindly rechecked his calculations and corroborated the author's findings.

originates in a certain canonical transformation. The inner meaning of φ_i does not reveal itself by this method. We now approach the same problem differently to obtain further clues concerning the true significance of the vector potential.

One of the earliest investigations of Einstein[21] brought already to light a peculiar feature of the contracted curvature tensor. Integration of the gravitational equations succeeds in a natural way only if one normalizes the reference system in a very special manner, in spite of the general covariance of the fundamental equations. Einstein's normalization for infinitesimal fields was afterwards extended by the author[22] to fields of arbitrary strength. The normalization condition takes the form

$$\frac{1}{g^{\frac{1}{2}}}\frac{\partial g^{\frac{1}{2}}g^{i\alpha}}{\partial x_\alpha}=0. \tag{8.1}$$

The effect of this normalization is that the differential operator R_{ik} becomes reducible to a greatly simplified operator B_{ik} which has the merit that its essential part (which contains the second derivatives) is *separated* in the components g_{ik}. Moreover, this operator is free of any inner identities. It can be prescribed freely, without having to satisfy any conservation laws,—in marked contrast to the original operator R_{ik}.

The general expression of B_{ik} can be written down as follows:

$$B_{ik}=\tfrac{1}{2}\frac{\partial^2 g_{ik}}{\partial x_\alpha \partial x_\beta}g^{\alpha\beta}+\tfrac{1}{2}\Gamma_{\alpha\beta}{}^{\rho}\left(\frac{\partial g^{\alpha\beta}}{\partial x_k}g_{\rho i}+\frac{\partial g^{\alpha\beta}}{\partial x_i}g_{\rho k}\right)-\Gamma_{i\alpha}{}^{\rho}\Gamma_{k\beta}{}^{\sigma}g_{\rho\sigma}g^{\alpha\beta}. \tag{8.2}$$

Without restricting our reference system by any special condition we can say quite generally that the operator R_{ik} may be split in a definite noncovariant manner into the simplified operator B_{ik} plus the symmetrized gradient of a certain vectorial quantity. We can put

$$R_{ik}=B_{ik}+\tfrac{1}{2}(V_{i,k}+V_{k,i}) \tag{8.3}$$

where

$$V^i=\frac{1}{g^{\frac{1}{2}}}\frac{\partial g^{\frac{1}{2}}g^{i\alpha}}{\partial x_\alpha}. \tag{8.4}$$

(The operation "comma" is taken in the usual invariant sense, although $V_i=V^\alpha g_{\alpha i}$ is not a true vector.)

We introduce this B_{ik} in our discussions, replacing R_{ik} by the expression (8.3). We substitute (8.3) in our quadratic Lagrangian (5.1) and take into account the divergence-free character of the matter tensor (5.18). By a similar procedure as that employed in Sec. 6 we transform L_0 as follows:

$$L_0=\tfrac{1}{2}[B^{ik}B_{ik}+\beta B^2+(1+2\beta)BV-V^iV^kB_{ik}-\tfrac{1}{4}W^{ik}W_{ik}+\beta V^2+\tfrac{1}{2}VV^\alpha V_\alpha], \tag{8.5}$$

where we have put

$$V=V^\alpha{}_{,\alpha} \tag{8.6}$$

$$W_{ik}=\frac{\partial V_i}{\partial x_k}-\frac{\partial V_k}{\partial x_i}. \tag{8.7}$$

We now proceed to the canonical method by replacing B_{ik} and V^i by the algebraic variables w_{ik} and φ^i. This gives rise to the Lagrangian multipliers p^{ik} and ρ_i and we obtain the new Lagrangian

$$L=p^{ik}(B_{ik}-w_{ik})+\rho_i(V^i-\varphi^i)+L_0, \tag{8.8}$$

where L_0 is the above expression (8.5), but replacing B_{ik} by $w_i{}^\alpha$ and V_i by φ_i. We recognize that L is purely algebraic in the variables w_{ik} which can thus be eliminated. (The same is not true for φ_i which enters in L_0 with its first derivatives.) By putting the partial derivative of L with respect to w_{ik} equal to zero we obtain the equation

$$p^{ik}=w^{ik}+\beta w g^{ik}+\tfrac{1}{2}(1+2\beta)\varphi g^{ik}-\tfrac{1}{2}\varphi^i\varphi^k \tag{8.9}$$

from which w_{ik} can be eliminated. After the elimination the final Lagrangian becomes

$$L=p^{ik}B_{ik}{}^*-(H_0+H_1)+\rho_i\left(\frac{\partial g^{\frac{1}{2}}g^{i\alpha}}{g^{\frac{1}{2}}\partial x_\alpha}-\psi^i\right) \tag{8.10}$$

where

$$B_{ik}{}^*=B_{ik}-\tfrac{1}{2}\varphi_i\varphi_k+\tfrac{1}{2}[(1+2\sigma)\varphi^\alpha{}_{,\alpha}-\sigma\varphi^\alpha\varphi_\alpha]g_{ik} \tag{8.11}$$

while H_0 and H_1 are again defined by (5.3) and (6.11).

On comparing the new Lagrangian (8.10) with the expression (6.9) found in Sec. 6, we see the close resemblance. The last term of (8.10) was not present in the earlier treatment. Moreover, the definition of $R_{ik}{}^*$ according to (6.10) contains, if compared with (8.11), the additional term $-\tfrac{1}{2}(\varphi_{i,k}+\varphi_{k,i})$, but then this term is absorbed by B_{ik} as (8.3) shows.

Variation with respect to φ_i gives an equation which can be interpreted as the Maxwellian equation for the vector potential, augmented by certain correction terms:

$$\tfrac{1}{2}\frac{\partial g^{\frac{1}{2}}F^{i\alpha}}{g^{\frac{1}{2}}\partial x_\alpha}+[(1+\sigma)Q-\tfrac{1}{2}(1+2\sigma)p]_{,\alpha}g^{\alpha i}+(1+\sigma)Q\varphi^i-(p^{i\alpha}+\sigma p g^{i\alpha})\varphi_\alpha=\rho^i. \tag{8.12}$$

How does the variational principle determine the right side of this equation, the "electric current"? Variation with respect to ρ_i gives the condition

$$\frac{\partial g^{\frac{1}{2}}g^{i\alpha}}{g^{\frac{1}{2}}\partial x_\alpha}=\varphi^i \tag{8 13}$$

which says that the normalization vector, usually put equal to zero, [see 8.1)], should in actual fact be equated to the vector potential. But what can we say about ρ_i which enters the Lagrangian in a linear way

[21] Albert Einstein, Sitzber. preuss. Akad. Wiss. 688 (1916).
[22] Cornelius Lanczos, Physik. Z. **23**, 537 (1922); Z. Physik **13**, 7 (1923).

only? It so happens that ρ_i is determined in an *indirect* way.

Let us express in the Lagrangian (8.10) the operator B_{ik} in terms of R_{ik} with the help of (8.3), then perform an integration by parts. We arrive exactly at the invariant (6.9), augmented by a term which has the form of the last term of (8.10), except that ρ_i is replaced by a ρ_i' which is related to ρ_i as follows:

$$\rho_i' = \rho_i + p_i{}^{\alpha}{}_{,\alpha}. \tag{8.14}$$

This Lagrangian satisfies the condition of general covariance, except for the single term

$$\rho_i' \frac{\partial g^{\frac{1}{2}} g^{i\alpha}}{g^{\frac{1}{2}} \partial x_\alpha}. \tag{8.15}$$

Let us vary with respect to g^{ik}. The variation of the completely covariant part of L yields a certain symmetric tensor A_{ik}. Adding the variation of the term (8.15), the resulting equation becomes

$$A_{ik} - \tfrac{1}{2}\left(\frac{\partial \rho_i'}{\partial x_k} + \frac{\partial \rho_k'}{\partial x_i} - \frac{\partial \rho_\alpha'}{\partial x_\beta} g^{\alpha\beta} g_{ik}\right) = 0. \tag{8.16}$$

We take the covariant divergence of this equation. The divergence of A_{ik} vanishes identically from the general principles of tensor calculus. We thus obtain for ρ_i' a linear and homogeneous partial differential equation of second order. We can assume that this equation has no solution which is regular throughout the space-time manifold and vanishes at infinity. Hence we can put $\rho_i = 0$ and we obtain

$$\rho^i = -p^{i\alpha}{}_{,\alpha}. \tag{8.17}$$

The current vector ρ^i is thus uniquely determined.

We return to our Lagrangian (8.10) and vary with respect to p^{ik}. This gives the equation

$$B_{ik}{}^* = p_{ik} + \sigma p g_{ik}. \tag{8.18}$$

Then we vary with respect to the g_{ik}. We obtain a linear differential operator of second order in the p^{ik} which becomes equal to quantities *quadratic* in the φ_i. Integration of this equation establishes the p^{ik} in terms of the φ_i. The quadratic dependence has the consequence that for weak φ_i fields the p^{ik} become infinitesimal of second order. This leads to the conclusion that the non-Maxwellian terms of (8.12) drop to quantities of second and third order. Hence, for weak fields the determining equation of the vector potential becomes

$$\frac{\partial g^{\frac{1}{2}} F^{i\alpha}}{g^{\frac{1}{2}} \partial x_\alpha} = 0, \tag{8.19}$$

in harmony with the Maxwellian equations of empty space. If in (8.18) we go to the same degree of approximation, we obtain for weak fields the simplified equation

$$B_{ik} = 0. \tag{8.20}$$

Let us consider an electrostatic field. Here the vector potential is reduced to the scalar potential which decreases to zero with r^{-1}. Integration of the field equations offers series difficulties. Integration of the fourth equation of (8.13) would demand that the g_{i4} ($i=1,2,3$) go to infinity with the order x_i/r, in contradiction to the boundary condition that they have to vanish for $r=\infty$. Moreover, the corresponding B_{i4} quantities do not become zero and thus (8.20) are not satisfied. On the other hand, if we consider the centrally symmetric and static solutions of (8.20), we find that φ_4 vanishes identically. Existence of a free electric charge is thus in contradiction to the demands of infinitesimal fields. This difficulty was recognized by the author at a very early stage of his speculations (compare I, page 735). It was baffling that on the one hand the quadratic action principle seemed to offer the vector potential as a free gift, on the other it took it away by denying the possibility of a free charge. The proper explanation of the puzzle could not be found as long as the adequate mathematical method for the treatment of strong fields was missing. Discovery of the canonical method puts us in the position to see the wider implications of the problem. The proper role of the vector potential φ_i in relation to the structural problems of matter can now be established.

9. ATOMISTIC STRUCTURE OF MATTER

Up to now the quadratic action principle shows little relation to the classical results of general relativity. In general relativity the problem of matter appears in the following context. Any generally covariant Lagrangian has the following property. It contains the components g_{ik} of the metrical tensor. Varying with respect to the g^{ik}, we obtain a symmetric tensor whose divergence is zero. This tensor we call the "physical matter tensor." If for example L is identified with Maxwell's fundamental Lagrangian $-F_{ik}F^{ik}$, the variation with respect to g^{ik} yields Maxwell's "stress-tensor" S_{ik}, also called the "energy-momentum tensor" of the electromagnetic field. If matter is conceived in purely electric terms, this tensor in itself can be equated to the physical matter tensor. Generally L may be composed of further terms, in which case S_{ik} will be complemented by additional tensors. The sum of these tensors represents the physical matter tensor which we wish to denote by P_{ik}.

Einstein discovered the purely metrical origin of gravitational forces and established in the "metrical matter tensor" T_{ik} [see (5.18)] a symmetric tensor of second order whose divergence vanished identically. It could be derived variationally by varying the scalar Riemannian curvature R with respect to the g^{ik}. The equivalence principle demanded that *all forms of energy* *act also gravitationally*. The mathematical formulation of this principle takes the following form:

$$T_{ik} = -\kappa P_{ik}. \qquad (9.1)$$

It expresses the equilibrium between gravitational energy and all other forms of energy. The constant κ is a dimensioned scale factor by which energy density can be reduced to curvature. Its numerical value in the cm gram system is (the time being measured in natural units, i.e., c^{-1} sec):

$$\kappa = \frac{8\pi k}{c^2} = 1.863 \cdot 10^{-27} \text{ (cm/g)}. \qquad (9.2)$$

From the standpoint of an action principle Einstein's fundamental equation should be written in the form

$$T_{ik} + \kappa P_{ik} = 0. \qquad (9.3)$$

The significance of this equation is that the original physical Lagrangian L_0 is changed to

$$L = L_0 + (1/\kappa^{-1} R). \qquad (9.4)$$

Moreover, the metrical tensor g_{ik} is added to the previous physical variables as additional variables of the field. However, such a purely external coupling of "metrical" and "physical" quantities never satisfied Einstein. He tried to find some geometrical interpretation of the "physical" part L_0 of the Lagrangian in order to conceive the *entire* L as one unified quantity.

Looking at our Lagrangian (8.10), deduced from an action principle which is quadratic in the curvature quantities but free of any external additions—except for quantities which are dictated by the mathematical nature of the problem—we are at first sight at a loss to see any relation to a Lagrangian of the form (9.4). Our Lagrangian contains the surplus variables p_{ik} which make a direct comparison difficult. We now make use of the fact, deduced earlier in Sec. 5 [see (5.16)], that in the absence of the potentials φ_i

$$p_{ik} = \lambda g_{ik} \qquad (9.5)$$

is an *exact solution* of the field equations. At variance with the usual concepts we equate λ to an *excessively large* constant by considering it as the reciprocal square of a *length of subatomic dimensions*. Under these circumstances p_{ik} changes but little by the presence of the φ_i and we obtain a good approximation of our Lagrangian by considering (9.5) as an *exact law*, although in reality it holds only in approximation. We now introduce this solution into our Lagrangian (8.10) and thus reduce our Lagrangian problem from the original 10+10+4=24 variables to a problem of only 10+4=14 variables, *viz.* the 10 g_{ik} and the 4 φ_i. Now a direct comparison with Einstein's action principle becomes possible.

Our new Lagrangian is composed of a number of parts which we write down and analyze separately:

$$\begin{aligned} L_1 &= \lambda R \\ L_2 &= -2\lambda^2(1+4\sigma) \\ L_3 &= -\tfrac{1}{8} F_{ik} F^{ik} \\ L_4 &= -\tfrac{1}{2}(1+\sigma)(\varphi^\alpha{}_{,\alpha} - \tfrac{1}{2}\varphi^\alpha \varphi_\alpha)^2 \\ L_5 &= -\frac{\lambda}{2}(1+4\sigma)\varphi^\alpha \varphi_\alpha \\ L_6 &= \rho_i \left(\frac{\partial g^{\frac{1}{2}} g^{i\alpha}}{g^{\frac{1}{2}} \partial x_\alpha} - \varphi^i \right). \end{aligned} \qquad (9.6)$$

The first invariant appears originally in the form $\lambda B_{ik} g^{ik}$ but this differs from $\lambda R_{ik} g^{ik}$ by a divergence only and is thus replaceable by λR. Hence *Einstein's linear invariant comes into strong focus*. The largness of λ causes a practical *linearization* of our originally quadratic action principle, thus bringing it in harmony with demands of the equivalence principle.

The second invariant represents the traditional "cosmological term." If we wanted to adhere to the traditional view that the cosmological constant is of practically negligible dimensions, demanded only for cosmological purposes, we would choose σ as exceedingly near to $-\frac{1}{4}$, i.e. β as excessively large. This means an overwhelming emphasis of the invariant R^2 compared with the invariant $R_{ik}R^{ik}$. We are then back at the choice of Pauli and Weyl; [see (7.8)]. In this case we lose not only L_2 but also L_5. We deprive ourselves of an entirely essential building block for the construction of material particles of atomic size. Furthermore, we once more experience the above discussed difficulty concerning the free charge of an electrostatic field.

We depart from the traditional theory by assuming that the "effective cosmological constant"

$$\mu = -\lambda(1+4\sigma) \qquad (9.7)$$

is not small but excessively large (although possibly much smaller than λ itself) by considering it as the reciprocal square of a length of atomic dimensions. (We assume μ to be positive which puts σ to the negative side of $-\frac{1}{4}$.) This means that the metrical substructure of the world, characterized by the equation [see (5.10)]

$$R_{ik} = -\mu g_{ik} \qquad (9.8)$$

is very far from being flat, owing to the presence of a tremendous negative pressure which is uniformly present throughout the universe. The implications of this assumption will be discussed in the following section.

We now come to L_3 which is Maxwell's invariant. The standard theory which recognizes electricity and gravity as the two fundamental phenomena of nature, operates with L_1 and L_3 only. As we know, no stable particle can be constructed on this basis. But in our present considerations we have also L_4 and L_5 at our disposal. What kind of forces are generated by these invariants? Varying with respect to g^{ik}, we obtain a tensor of the following form:

$$\Theta_{ik}=\alpha(\varphi_i\varphi_k-\tfrac{1}{2}\varphi^\alpha\varphi_\alpha g_{ik}). \quad (9.9)$$

Interpreting φ_i as a velocity field[23] we recognize in this matter tensor the action of inertial masses, coupled with a pressure which is $\frac{1}{2}$ of the matter density. The forces thus generated are of a *mechanical* nature. We can thus answer the age-old puzzle: what kind of forces balance the tremendous electrostatic repulsion inside the electron which should lead to its explosion? The answer is that these forces are of a *mechanical* kind. Let us write down the equation which determines the vector potential [see (8.12)]:

$$\frac{1}{2}\frac{\partial g^{\frac{1}{2}}F^{i\alpha}}{g^{\frac{1}{2}}\partial x_\alpha}+[\mu+(1+\sigma)Q]\varphi^i+(1+\sigma)Q,_{\alpha}g^{\alpha i}=0. \quad (9.10)$$

Let us examine this equation from the standpoint of static and spherically symmetric solutions, paying particular attention to the equation $i=4$ which determines the "scalar potential" φ_4. The scalar Q becomes negligibly small in weak fields (i.e., small φ_i); near the center $r=0$, however, it becomes very large. Moreover, we can conceive the equation (9.10) for $i=4$ as a homogeneous linear differential equation for φ_4 which contains, however, an *adjustable constant* because of the amplitude factor of Q which is freely disposable. We obtain a regular *eigenvalue problem* of a nonlinear type.[24] Apart from the realm of very small r we obtain the solution

$$\varphi_4=e^{-(2\mu)^{\frac{1}{2}}r}/r, \quad (9.11)$$

and see that instead of the Coulomb potential we obtain a potential of the Yukawa kind.[25] But the singularity at the origin is avoided, due to the cooperation of the scalar Q which puts a very strong *negative* mass near the center of the particle, thus balancing the strong forces at the center.

We come to the conclusion that we are able to construct a *spherically symmetric and static electric particle which has no singularity anywhere.* The physical matter tensor generated by this particle has an entirely *insular* character. Matter is concentrated in little lumps of very small extension and a direct interaction between matter and matter is not possible. The Lorentz force is traditionally derived from the Maxwellian matter tensor, on the basis of the conservation laws. Hence it represents a matter-matter interaction. In the present theory an influence from particle to particle can only occur through *metrical* signals. Hence the electromagnetic vector potential must be of *metrical* origin and the Lorentz force must represent a matter-metric and not a matter-matter interaction.

This is foreshadowed by (8.13) which shows that the vector φ_i cannot be of the nature of a generating function since it is itself obtained from other field variables by differentiation. In fact this equation closely resembles the equation of Lorentz which connects the electric field quantities with the electric current vector:

$$\frac{\partial g^{\frac{1}{2}}F^{i\alpha}}{g^{\frac{1}{2}}\partial x_\alpha}=\rho^i. \quad (9.12)$$

The difference is that g^{ik} is a symmetric, F^{ik} an antisymmetric tensor. The analogy goes still deeper. The significance of (8.13) is that we normalize our reference system in a suitable way. This normalization will involve a proper coordinate transformation. Assuming that the φ_i are small, the transformation assumes an *infinitesimal* character. Such a transformation is characterized by the law

$$\bar{g}_{ik}=g_{ik}-\epsilon(A_{i,k}+A_{k,i}) \quad (9.13)$$

where ϵ is a small parameter. Thus the essential change demanded by our reorientation is that the role of the customary antisymmetric combination $F_{ik}=A_{i,k}-A_{k,i}$ is taken over by the symmetric combination

$$G_{ik}=A_{i,k}+A_{k,i}. \quad (9.14)$$

The customary "nabla equation" between vector potential and current vector remains essentially unchanged, except that the continuity equation for the current vector is not a consequence of the connecting equation since the left side of the equation is free of any identities.

This reclassification of electric quantities within the framework of general relativity leads to surprising results. The vector φ_i, previously treated as the electromagnetic vector potential, becomes in actual fact the vector of the *electric current*. It is surprising to see that it is not the vector potential A_i but the electric current φ_i which has a direct influence on the matter tensor by replacing the vector potential in the Maxwellian stress tensor. This tensor is complemented, however, by additional mechanical terms. In the theory of Lorentz the electric current is a purely *extraneous* quantity, foisted on the field without any organic relation to the basic field quantities. It is not more than an "asylum ignorantiae," in the words of Einstein. In the present theory the electric current becomes a basic field quantity, organically related to the field and determined by an eigenvalue problem. The electron, before a "stranger of electrodynamics" (Einstein[26]), appears

[23] This is strictly speaking incorrect, because even in a static field the components φ_i ($i=1,2,3$) are not zero. Since, however, the particle is small and we can average over the φ_i, the *average values* of the space components vanish and what remains is φ_4 which is tangential to the world line of the particle.

[24] The author's computations are not yet completed and will be reported at another occasion. The symmetry of the basic Lagrangian with respect to $\pm\varphi_4$ holds only if the metrical action of the φ_i is neglected. The mass-inequality of positive and negative electricity is not outside the scope of the present theory.

[25] See reference 5, p. 45.

[26] See A. Sommerfeld, *Electrodynamics* (Academic Press, Inc., New York, 1952), p. 236.

with a well-defined structure in which all infinities are avoided.

The reorientation of the electromagnetic quantities that is here advocated leads to a modified formulation of the dynamical law of the electron. The equation "time rate of change of momentum equals moving force" is retained but the "moving force" cannot be the Lorentz force since the antisymmetric combination $A_{i,k}-A_{k,i}$ has no significance. Its place is taken by the symmetric combination (9.14) according to the law

$$\frac{dp_i}{ds}=F_i=e(A_{\alpha,i}+A_{i,\alpha})\frac{dx^\alpha}{ds}, \tag{9.15}$$

where A_i is the vector potential of the external field. The "momentum" of the electrically charged particle is defined as follows:

$$p_i=m_0\frac{dx^\alpha}{ds}g_{\alpha i}+2eA_i. \tag{9.16}$$

If the correction term is transferred from the left to the right, the sum $A_{\alpha,i}+A_{i,\alpha}$ changes to the difference $A_{\alpha,i}-A_{i,\alpha}$ and the Lorentz force is once more restored.

The field momentum (9.16) differs from the customary expression[27] by the factor 2. The celebrated Einstein–de Haas e/m experiment which directly measures the correction to which the mechanical momentum is subjected in an external magnetic field, gave twice the expected value, in agreement with the law (9.16).[28]

10. METRICAL SUBSTRUCTURE

The most radical, though inevitable, departure of the present theory from the traditional views lies in the assumption that the material particles represent not more than a weak superstructure on a metrical substructure which is exceedingly strong. It is characterized by the equation

$$R_{ik}=-\mu g_{ik}, \tag{10.1}$$

where μ is an exceedingly large constant. We have here a development somewhat analogous to Einstein's famous theoretical prediction[29] in 1907 that the traditional "kinetic energy" of Newtonian physics is not more than a very small modification of an exceedingly large energy source of the amount m_0c^2, connected with the very existence of the mass m_0. This enormous energy-source remains latent under ordinary circumstances because our measurements involve energy *differences* only and under such conditions the rest-mass usually drops out, due to the great stability of the atomic building blocks of the universe.

Equation (10.1) similarly assumes that all physical events are only small modifications of a fundamental structure which could be compared to a huge "rest-mass" of the universe. But is such an assumption not preposterous when we know that our ordinary geometry is so nearly Euclidian? Would not a law of the form (10.1) lead to a tremendous *explosion* of the universe? In actual fact this would only happen if the matter maintaining this tremendous curvature would act in a *coherent* way. But we can conceive the picture of a gas of very high temperature whose molecules fly around in all directions with practically light velocity and which create a huge uniform pressure, apparently due to the action of some very large force, but in actual fact caused by the statistical kinetic action of a large number of molecules. An explosion of tremendous power would occur only if by some miracle all the molecules could be directed into parallel paths.

We carry over this statistical picture in order to demonstrate that a metrical substructure of the form (10.1) is actually feasible, in spite of the apparently almost Euclidian character of our metric. We utilize a statistical device used by Lorentz to show that the macroscopic equations of Maxwell can be explained on the basis of electron theory, if we allow the statistical interaction of a huge number of electric charges in a region which is macroscopically small but yet large compared with the dimensions of atoms and molecules. Using a similar picture we want to calculate the *average values* of the curvature components, assuming that the metric is far from stationary. In fact, we want to assume that there exists a metrical radiation which is composed of exceedingly high frequencies and which propagates irregularly in every direction. We will now cut out a four-dimensional cube at some point of the world and evaluate the average values of the curvature components within the cube. The edge of this cube is chosen large compared with the wavelengths of the radiation. These wavelengths are so small that the cube can be of subatomic size.

For present purposes we depart from the previous real line-element of the signature $(-1, -1, -1, +1)$ and introduce Minkowski's Euclidian line-element $g_{ik}=\delta_{ik}$, considering $x_4=it$ as an imaginary variable. We consider a metrical tensor which differs from the Euclidian values by small amounts only

$$g_{ik}=\delta_{ik}+\gamma_{ik}. \tag{10.2}$$

However, although the deviations themselves are small, the Γ quantities can become *very large* because the γ_{ik} are composed of a spectrum of very high frequencies. We start with Riemann's curvature tensor R_{imkn} which can be written in terms of an auxiliary tensor A_{ikmn} defined as follows:

$$R_{imkn}=A_{ikmn}-A_{inmk} \tag{10.3}$$

$$A_{ikmn}=\tfrac{1}{2}\left(\frac{\partial^2 g_{ik}}{\partial x_m\partial x_n}+\frac{\partial^2 g_{mn}}{\partial x_i\partial x_k}\right)+\begin{bmatrix} ik \\ \alpha \end{bmatrix}\begin{bmatrix} mn \\ \beta \end{bmatrix}g^{\alpha\beta}. \tag{10.4}$$

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[27] See, e.g., H. Goldstein, *Classical Mechanics* (Addison-Wesley Press, Cambridge, 1951), p. 49.

[28] The customary explanation refers the discrepancy to the "spin" of the electron. The spin action of the electron is deeply interwoven with the here discussed re-evaluation of electric quantities. The deeper analysis of this problem transcends, however, the limitations of the present paper.

[29] Albert Einstein, Jahr. Radioakt. 4, 411 (1907).

The terms which are derivatives, cannot contribute to the average values since the derivative of a periodic function is once more periodic and its average value zero. It is sufficient to consider the last term of (10.4), and this again can be written with practically sufficient accuracy in the form

$$a_{ikmn}=\begin{bmatrix} ik \\ \alpha \end{bmatrix}\begin{bmatrix} mn \\ \alpha \end{bmatrix}. \quad (10.5)$$

We now make the statistical assumption that the various components of the tensor quantity

$$\Gamma_{ik,\,m}=\begin{bmatrix} ik \\ m \end{bmatrix} \quad (10.6)$$

are statistically uncorrelated, so taking the average values of products involving two different components gives zero. Average values of the squares of components cannot give zero since the average of both $\sin^2\omega x$ and $\cos^2\omega x$ is $\frac{1}{2}$. (In a purely Euclidian world these averages would be composed of a sum of squares and be necessarily positive. In a Minkowskian world some of the terms become negative, because of the differentiation with respect to the imaginary x_4.) In consequence of this statistical behavior average values of a_{ikmn} will become proportional to $\delta_{im}\delta_{kn}+\delta_{in}\delta_{km}$. We assume that the factor of proportionality is a universal constant C:

$$\bar{a}_{ikmn}=C(\delta_{im}\delta_{kn}+\delta_{in}\delta_{km}). \quad (10.7)$$

This gives

$$\bar{R}_{imkn}=C(\delta_{in}\delta_{km}-\delta_{ik}\delta_{mn}). \quad (10.8)$$

The constant C is of the order of magnitude $(\epsilon\omega)^2$ and can become very large, in spite of the smallness of ϵ. The inhabitants of this world, who cannot measure local values but only averages—since the elementary particles which they can use as measuring rods are themselves large compared with the basic wavelength—will find that their metric is nearly Euclidian and yet the curvature of their world uniform but exceedingly high. This would not be possible in terms of local values, of course. However, the *contracted* curvature tensor R_{ik} may come out as proportional to g_{ik} (with a very high proportionality factor) even in the local sense. Hence, it is not absurd to assume that the substructure of the world is characterized by the law (10.1) and yet the g_{ik} are of the nature of constants, modified only by oscillatory variations of exceedingly high frequencies but small amplitudes.[30]

11. SUMMARY

The present paper investigates the possibilities of a Riemannian geometry characterized by an action principle which is not linear in the curvature components, as the choice made by Einstein, but quadratic. Such a Lagrangian is advocated by the property that the basic action integral remains invariant not only with respect to arbitrary coordinate transformations but also with respect to an arbitrary choice of the scale in which lengths are measured. Hermann Weyl advocated such an action principle at an earlier date, but in connection with a proposed generalization of Riemann's geometry. The author's aim—spreading over a span of more than 25 years—was to draw all the possible logical conclusions from the quadratic action principle, without abandoning Riemann's geometry.

Mathematical analysis of this problem was carried out by a new mathematical method, based on the method of the Lagrangian multiplier, by which the problem is reduced to a set of field equations of first order in an increased number of variables. This demands 100 variables. It seemed more convenient, however, to admit differential operators of second order and formulate the problem in 20 variables. The results can be summarized as follows.

(1) Einstein's field equations $R_{ik}=0$ are included among the solutions but so are the more general "cosmological equations" $R_{ik}=\lambda g_{ik}$. The merit of the latter solution is that it introduces a dimensioned quantity λ which normalizes the lengths of the universe. To assume that the normal length of the universe is of cosmic magnitude means that we miss the opportunity for the explanation of atomism. The present theory assumes that the fundamental unit of length is of subatomic dimensions. This makes the cosmological constant exceedingly large which means that the substructure of the world is not flat but highly curved.

(2) Weyl's generalization of Riemann's geometry for the purpose of introducing the vector potential turns out to be unnecessary since the vector potential appears in the action principle quite naturally as the result of a canonical transformation. Later, a purely metrical interpretation of the vector potential is found, on the basis of splitting the contracted curvature tensor R_{ik} into a simplified operator B_{ik}, which is free of all identities, and a second part which involves the symmetrized gradient of a coordinate-dependent vectorial quantity V^i. It is found that this vector, in the ordinary gravitational theory normalized to zero, should actually be equated to the vector potential φ^i.

(3) In view of the largeness of λ we succeed with an approximate linearization of the action principle and a direct comparison with classical relativity becomes possible. A deeper analysis of the determining equation of φ_i reveals that φ_i was wrongly interpreted as the vector potential of the electromagnetic field. It is in actual fact the electric *current vector*. At variance to the theory of Lorentz in which the current vector is foisted on the field as an extraneous quantity without definite structure, the vector φ_i is determined by a differential equation of second order which possesses

[30] For another approach to the problem of the substructure, based on a Fourier analysis of the metrical manifold as a whole, see the author's earlier paper "Matter waves and electricity," reference 3, II, where the idea of a cosmological constant of subatomic dimensions was discussed for the first time.

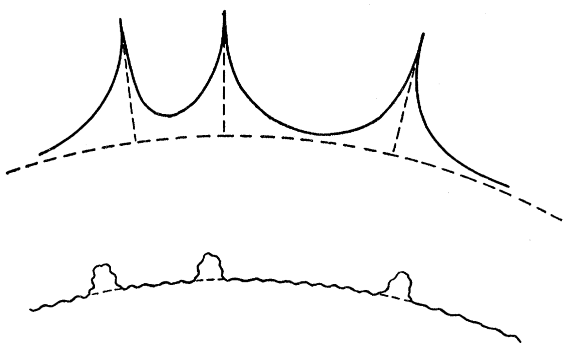

FIG. 1.

static and spherically symmetric eigensolutions. A static and stable electron can thus be constructed, free of all infinities. The forces which counteract the strong repulsion of the electrostatic forces, are of a mechanical kind.

(4) The matter tensor is reduced to very small portions of space and does not have that diffused character that the Maxwellian stress tensor attributed to it. The role of the vector potential in the Maxwellian tensor is taken over by the current vector. Matter becomes strictly insular and the dynamical influence from particle to particle occurs purely through the medium of the metric. The Lorentz force loses its primary significance and comes about by the interaction of two terms on opposite sides of the equation of motion, the one belonging to the momentum of the particle, the other containing the symmetric combination $A_{i,k}+A_{k,i}$, instead of the traditional antisymmetric combination $A_{i,k}-A_{k,i}$, which in the present theory has no primary significance.

(5) The superstructure of the metric in the form of material particles is a weak modification of the substructure which represents a tremendous reservoir of momentum and energy in potential form. That under ordinary circumstances the weak modification becomes of paramount importance and the much stronger substructure remains latent, finds its explanation in the fact that the superstructure is of a *static* character while the substructure represents a statistically distributed radiation field of exceedingly high frequency which is comparable to a capricious sequence of constant seismographical "tremors," too small to be observable under ordinary macroscopic circumstances but of paramount importance for the question of radiation, i.e. the interaction of particle and field. It seems thus justified to call the metrical substructure, composed of a seething cauldron of ultra-high-frequency oscillations. the "tremorfield." Figure 1 gives a schematic illustration of the general modification of ideas that the present theory suggests, compared with the traditional concepts. In the traditional view we have a smooth background, almost completely flat, except for a very slight curvature, demanded by the cosmologically closed nature of the universe. Erected on this background we find the material particles, growing to infinity near the center (or at the center) and decreasing in strength toward the periphery, but the decrease is mild enough to make a direct matter-matter interaction possible; (the Lorentz force obtained from the conservation law of the matter tensor). The second picture illustrates a highly agitated "rippled" background and the material particles grow out of this agitated background as little humps, completely isolated from each other. The signals emanating from a particle and reaching another one are of a purely metrical nature and belong thus to the realm of gravitational action. The motion of a particle takes place due to the response of the particle to the external metric which does not occur according to the geodesic principle.

The tremendous complication which comes into the picture due to the presence of the substructure demonstrates that our conceptual understanding of physical action cannot be brought down to a simple and elementary level. The problem of radiation is much more than a secondary interaction between field and particle, induced by the energy loss accompanying the accelerated motion of a particle. The momentum-energy exchange between substructure and superstructure opens a new perspective for the deeper understanding of the mysterious quantum phenomena and it seems possible that Heisenberg's uncertainty principle is deeply interwoven with the statistical nature of the tremorfield. To find the solution of these problems on the basis of the fundamental field equations may be far in the future. The author's aim in this investigation was only to outline a new solution of the problem of general relativity which may finally bring gravity, electricity and quantum physics into one unified structure, based on the immortal thought constructions of Albert Einstein.