Einstein's and Other Theories of Gravitation*

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I. INTRODUCTION

EVER since it has been realized that Newton's theory of gravitation has its limitations, several theories of gravitation have been proposed from time to time. The aim of this paper is to discuss the various theories of gravitation with particular attention to Einstein's theory. We consider only the pure gravitational theories, and thus do not discuss attempts to find a unified field theory of various fields in nature.

If we leave aside all philosophical considerations, it seems reasonable that any acceptable theory of gravitation should satisfy the following requirements:

(1) It should be Lorentz covariant, because the special theory of relativity has now been well established by experiments.

(2) It should reduce to Newton's theory of gravitation as a good approximation, because Newton's theory is able to explain the observed gravitational phenomena to a fairly high degree of accuracy.

(3) It should also provide a reasonable explanation for the so-called three crucial tests.

Keeping the above requirements in mind, we first consider some attempts to construct a theory of the gravitational field in flat space similar to the theories of the electromagnetic field and the meson fields. We then show that Einstein's theory itself can be regarded as a theory of gravitation in flat space, and describe the advantages of such an approach from logical as well as practical points of view.

In this paper we deal with tensors in flat space as well as in the Riemannian space. In flat space the Greek indices will take the values 1, 2, 3, 4, and the space-time coordinates will be denoted as $x_\mu=(x_1,x_2,x_3,ict)$. In the Riemannian space the Greek indices will take the values 1, 2, 3, 0, and the space-time coordinates will be denoted as $x^\mu=(x^1,x^2,x^3,ct)$.

II. LORENTZ-COVARIANT THEORIES OF GRAVITATION

We follow the usual ideas of the field theory to see whether it is possible to find a theory of gravitation in flat space. In order that our theory of gravitation may reduce to Newton's theory as a good approximation, our gravitational field on quantization should correspond to neutral particles of vanishing rest mass and integral spin. Hence, confining attention to fields of spin 0, 1, or 2, the possible field equations for the gravitational field are

$$\Box^2 U=\kappa T, \tag{1}$$

$$\Box^2 U_\mu=\kappa T_\mu, \tag{2}$$

$$\Box^2 U_{\mu\nu}=\kappa T_{\mu\nu}, \tag{3}$$

where the scalar U, the four-vector U_μ, and the symmetrical tensor $U_{\mu\nu}$ are real field variables; T, T_μ, and $T_{\mu\nu}$ are the source functions; and κ is the coupling constant. We do not here consider the more complicated field equations corresponding to particles of spin higher than 2.

The field equation (1) for the gravitational field was first discussed by Nordström,[1] and recently a very clear presentation of this theory has been given by Bergmann.[2] In this theory the source function T is the trace of the energy-momentum tensor of matter, which includes all particles and fields except the gravitational field. This theory satisfies the first two requirements of Sec. I, but for the advance of the perihelion of planets it gives a value, which is one-sixth of Einstein's value in magnitude and opposite in sign. Since this result is definitely contrary to experiments, this simple theory of the gravitational field of spin 0 is unacceptable.

The field equation (2) is of a very familiar form. It is also well known that, in order that the energy of such a field may be positive definite, U_μ must satisfy the supplementary condition

$$\partial U_\mu/\partial x_\mu=0, \tag{4}$$

whence it follows that the source function T_μ must also satisfy the relation

$$\partial T_\mu/\partial x_\mu=0. \tag{5}$$

The only known four-vector quantity, which satisfies (5), is the current four-vector. It is, therefore, evident that the gravitational field (2) will be identical with the electromagnetic field, except that the gravitational charge of a particle might be different from its electromagnetic charge. Such a theory of the gravitational field has to be rejected, because the observed properties of the gravitational field are quite different from those of the electromagnetic field. For instance, the gravitational force between any two particles is always attractive, while the electromagnetic force between like particles is repulsive. It is rather curious that this difficulty arises in the case of the field of spin 1, while the fields of spin 0 and 2 both lead to a gravitational interaction of the observed sign between any two particles.

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[1] G. Nordström, Ann. Physik 43, 1101 (1914).
[2] O. Bergmann, Am. J. Phys. 24, 38 (1956).

Birkhoff[3] has suggested an interesting theory of gravitation, which is based on a field equation of the form (3), where the source function $T_{\mu\nu}$ is the symmetrical energy-momentum tensor of matter. Birkhoff's theory satisfies the three requirements mentioned in Sec. I. Nevertheless, as pointed out by Weyl,[4] this theory suffers from serious difficulties. The essential difficulty here is that Birkhoff's gravitational field does not have a positive definite energy, and therefore on quantization it corresponds to particles of positive as well as negative energies. But particles of negative energy are not permissible in the quantum theory of fields, because otherwise the state of vacuum will become unstable due to the spontaneous production of real particles of positive and negative energies.

The above-mentioned difficulty in Birkhoff's theory arises from the fact that we cannot impose the supplementary condition

$$\partial U_{\mu\nu}/\partial x_\nu = 0 \tag{6}$$

on Birkhoff's gravitational field. For it would follow from (3) and (6) that

$$\partial T_{\mu\nu}/\partial x_\nu = 0, \tag{7}$$

which is not possible, because the energy-momentum tensor of matter alone cannot satisfy the conservation equation. In fact, instead of (7) we must have

$$\frac{\partial (T_{\mu\nu}+t_{\mu\nu})}{\partial x_\nu} = 0, \tag{8}$$

where $t_{\mu\nu}$ is the energy-momentum tensor of the gravitational field. This shows that the supplementary condition (6) will be compatible with the gravitational field equation, provided that we replace (3) by

$$\square^2 U_{\mu\nu} = \kappa (T_{\mu\nu}+t_{\mu\nu}). \tag{9}$$

In the absence of matter, the above field equation reduces to

$$\square^2 U_{\mu\nu} = \kappa t_{\mu\nu}, \tag{10}$$

which is a nonlinear equation for the gravitational field. Moreover, when we try to derive this field equation by the usual variational principle, we find[5] that the required Lagrangian density L has to be an infinite series of the form

$$L = L_0 + \kappa L_1 + \kappa^2 L_2 + \cdots + \kappa^n L_n + \cdots, \tag{11}$$

where the terms in L_n consist of a product of $(n+2)$ factors, each factor being $U_{\mu\nu}$ or its derivative. This peculiar situation in the case of the gravitational field of spin 2 is of particular interest in the next section.

III. EINSTEIN'S THEORY OF GRAVITATION

The most widely accepted theory of gravitation at the present time is due to Einstein. Einstein's theory not only satisfies the requirements of Sec. I, but it also possesses a beautiful mathematical structure. Usually Einstein's theory is regarded as a theory of gravitation in the Riemannian space, which makes this theory strikingly different from other field theories. We shall, however, show that Einstein's theory can also be treated as a theory of gravitation in flat space.

The Lagrangian density for Einstein's gravitational field in the Riemannian space is given by

$$\mathfrak{L} = -\kappa^{-2}\mathfrak{g}^{\mu\nu}\left(\begin{Bmatrix}\alpha\\ \mu\beta\end{Bmatrix}\begin{Bmatrix}\beta\\ \nu\alpha\end{Bmatrix} - \begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\begin{Bmatrix}\beta\\ \alpha\beta\end{Bmatrix}\right), \tag{12}$$

where

$$\mathfrak{g}^{\mu\nu} = (-g)^{\frac{1}{2}} g^{\mu\nu}, \tag{13}$$

and

$$\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix} = \tfrac{1}{2} g^{\alpha\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right). \tag{14}$$

We now put

$$\mathfrak{g}^{\mu\nu} = \epsilon^{\mu\nu} - \kappa\gamma^{\mu\nu}, \tag{15}$$

where $\epsilon^{\mu\nu}$ is the Minkowskian metrical tensor

$$\epsilon^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We can then express the tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ as infinite series in powers of κ as[6]

$$\begin{aligned} g^{\mu\nu} = \epsilon^{\mu\nu} &+ \kappa(-\gamma^{\mu\nu} + \tfrac{1}{2}\epsilon^{\mu\nu}\epsilon_{\alpha\beta}\gamma^{\alpha\beta}) \\ &+ \kappa^2(-\tfrac{1}{2}\epsilon_{\alpha\beta}\gamma^{\alpha\beta}\gamma^{\mu\nu} + \tfrac{1}{4}\epsilon^{\mu\nu}\epsilon_{\alpha\rho}\epsilon_{\beta\lambda}\gamma^{\alpha\beta}\gamma^{\lambda\rho} \\ &+ \tfrac{1}{8}\epsilon^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\lambda\rho}\gamma^{\alpha\beta}\gamma^{\lambda\rho}) + O(\kappa^3), \end{aligned} \tag{16}$$

$$\begin{aligned} g_{\mu\nu} = \epsilon_{\mu\nu} &+ \kappa(-\tfrac{1}{2}\epsilon_{\mu\nu}\epsilon_{\alpha\beta}\gamma^{\alpha\beta} + \epsilon_{\mu\alpha}\epsilon_{\nu\beta}\gamma^{\alpha\beta}) \\ &+ \kappa^2(\epsilon_{\alpha\mu}\epsilon_{\lambda\nu}\epsilon_{\beta\rho}\gamma^{\alpha\beta}\gamma^{\lambda\rho} - \tfrac{1}{2}\epsilon_{\alpha\mu}\epsilon_{\beta\nu}\epsilon_{\lambda\rho}\gamma^{\alpha\beta}\gamma^{\lambda\rho} \\ &+ \tfrac{1}{8}\epsilon_{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\lambda\rho}\gamma^{\alpha\beta}\gamma^{\lambda\rho} - \tfrac{1}{4}\epsilon_{\mu\nu}\epsilon_{\alpha\lambda}\epsilon_{\beta\rho}\gamma^{\alpha\beta}\gamma^{\lambda\rho}) + O(\kappa^3), \end{aligned} \tag{17}$$

where $\epsilon_{\mu\nu}$ is reciprocal to $\epsilon^{\mu\nu}$, and $O(\kappa^3)$ contains third and higher powers of κ.

If we substitute (15), (16), and (17) in (12), the Lagrangian density $\mathfrak{L}$ becomes an infinite series in powers of κ, each term of which is simply a function of $\gamma^{\mu\nu}$ and the Minkowskian metrical tensors $\epsilon^{\mu\nu}$ and $\epsilon_{\mu\nu}$. Thus, we obtain a flat-space expansion for $\mathfrak{L}$, which can be more conveniently expressed by using the usual flat space notation. In this way we find

$$\begin{aligned} \mathfrak{L} = -\frac{1}{4}&\left[\frac{\partial\gamma_{\mu\nu}}{\partial x_\lambda}\frac{\partial\gamma_{\mu\nu}}{\partial x_\lambda} - \frac{1}{2}\frac{\partial\gamma_{\alpha\alpha}}{\partial x_\lambda}\frac{\partial\gamma_{\beta\beta}}{\partial x_\lambda} - 2\frac{\partial\gamma_{\alpha\mu}}{\partial x_\nu}\frac{\partial\gamma_{\alpha\nu}}{\partial x_\mu}\right] \\ &+ \kappa\gamma_{\mu\nu}\left[\frac{\partial\gamma_{\alpha\beta}}{\partial x_\mu}\frac{\partial\gamma_{\alpha\beta}}{\partial x_\nu} - \frac{1}{2}\frac{\partial\gamma_{\alpha\alpha}}{\partial x_\mu}\frac{\partial\gamma_{\beta\beta}}{\partial x_\nu} + 2\frac{\partial\gamma_{\mu\alpha}}{\partial x_\beta}\frac{\partial\gamma_{\nu\beta}}{\partial x_\alpha}\right. \\ &\left. + \frac{\partial\gamma_{\beta\beta}}{\partial x_\alpha}\frac{\partial\gamma_{\mu\nu}}{\partial x_\alpha} - 2\frac{\partial\gamma_{\mu\beta}}{\partial x_\alpha}\frac{\partial\gamma_{\nu\beta}}{\partial x_\alpha}\right] + O(\kappa^2). \end{aligned} \tag{18}$$

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[3] G. D. Birkhoff, Proc. Natl. Acad. Sci. U. S. **29**, 231 (1943) and **30**, 324 (1944). See also M. Moshinsky, Phys. Rev. **80**, 514 (1950).
[4] H. Weyl, Am. J. Math. **66**, 591 (1944).
[5] S. N. Gupta, Phys. Rev. **96**, 1683 (1954).

[6] For mathematical details of the contents of this section, see S. N. Gupta, Proc. Phys. Soc. (London) **A65**, 608 (1952).

The above expansion can, of course, be obtained up to any desired power of κ. It is also possible to show that in flat space the gravitational field equation can be expressed as

$$\Box^2\gamma_{\mu\nu}=\kappa(T_{\mu\nu}+t_{\mu\nu}) \tag{19}$$

with the supplementary condition

$$\partial\gamma_{\mu\nu}/\partial x_\nu=0, \tag{20}$$

where $T_{\mu\nu}$ and $t_{\mu\nu}$ are the symmetrical energy-momentum tensors for the matter field and the gravitational field respectively.[7] The extremely simple appearance of the field equation (19) is rather deceptive, because $t_{\mu\nu}$ in fact consists of an infinite series in powers of κ.

Comparing (19), (20), and (18) with (9), (6), and (11), we find that Einstein's field in flat space has just those properties, which a Lorentz-covariant gravitational field of spin 2 has to satisfy. Thus, we have shown that Einstein's theory can be reduced to a theory of gravitation in flat space by an expansion of the gravitational Lagrangian density as an infinite series in powers of the gravitational coupling constant κ. From a mathematical point of view, the theory remains unchanged when we pass over from the Riemannian space to the flat space. But, from a philosophic point of view, the above procedure implies a departure from Einstein's ideas in some respects.

[7] It is well known that in the Riemannian space the quantity $t_{\mu\nu}$ is not a tensor. However, when we pass over to the flat space and confine ourselves to the Lorentz transformations, we can regard $t_{\mu\nu}$ as a tensor.

IV. QUANTIZATION OF THE GRAVITATIONAL FIELD

We have seen in the preceding section that we can treat Einstein's theory as a theory of gravitation in flat space. Such a treatment has two great advantages. Firstly, it provides us with a more uniform description of the gravitational and the electromagnetic fields. Secondly, it enables us to carry out the quantization of Einstein's gravitational field by following the same procedure as we use for the electromagnetic field, as has been shown by the author.[8] On quantization, Einstein's gravitational field corresponds to gravitational quanta or gravitons of vanishing rest-mass and spin 2, and it is possible to calculate the interaction of these gravitons and other particles in the usual way. In such a quantized theory the nonlinearity of the gravitational field appears as a direct interaction between the gravitons.

An interesting application of the above theory of the quantization of the gravitational field has recently been carried out by Corinaldesi.[9] He has calculated the gravitational potential between two particles of spin 0 due to the exchange of a graviton, and he has then used this potential to find the two-body equation of motion under the influence of the mutual gravitational fields of these bodies. In this way he has derived in a remarkably straightforward way exactly the same equations of motion as obtained by Einstein, Infeld, and Hoffmann.[10]

Thus, the treatment of Einstein's gravitational field as a field in flat space and its quantization are also very useful from a practical point of view.

It is a pleasure to thank Professor John A. Wheeler for valuable discussions.

[8] S. N. Gupta, Proc. Phys. Soc. (London) **A65**, 161, 608 (1952).
[9] E. Corinaldesi, Nuovo cimento **1**, 1289 (1955) and **2**, 168 (1955); Proc. Phys. Soc. (London) **A69**, 189 (1956).
[10] Einstein, Infeld, and Hoffmann, Ann. Math. **39**, 66 (1938).