Three Lectures on Relativity Theory

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FIRST LECTURE: ON HOMOGENEITY, COVARIANCE, AND RELATIVITY

IN my first lecture, I try to elucidate some general notions connected with relativity theory. I speak on homogeneity, covariance, and relativity. My considerations are of a very simple nature but, nevertheless, I hope that they may be of interest, because simple notions are often the most difficult ones.

If we consider the geometrical aspect of the theory of space and time, this theory naturally divides into the theory of homogeneous (uniform) space-time and that of the nonhomogeneous (nonuniform) space-time. The former may be called Galilean space and the latter the Riemannian or Einsteinian space. (I sometimes use the word space instead of space-time.)

The property of space-time of being homogeneous means that (a) there are no privileged points in space and in time; (b) there are no privileged directions, and (c) there are no privileged inertial frames (that all frames moving uniformly and in a straight line with respect to one another are on the same footing).

The uniformity of space and time manifests itself in the existence of the Lorentz group. In particular, the equality of points in space and time corresponds to the possibility of a displacement, the equality of directions corresponds to that of spatial rotations, and the equality of inertial frames corresponds to a special Lorentz transformation. The displacements contain four parameters, the rotation three (the three angles), and the transformation to a moving frame also three (the three components of velocity). This gives together ten parameters—the maximum possible number, if we do not take into account scale transformations $x'=\lambda x$.

The statement that the Lorentz transformation leaves invariant the expression for the square of the line element is to be understood in the following sense. If one writes ds^2 as

$$ds^2=dx_0^2-dx_1^2-dx_2^2-dx_3^2$$

or

$$ds^2=\eta_{\mu\nu}dx_\mu dx_\nu$$

then, after the transformation from (x) to (x'), we have

$$ds^2=\eta_{\mu\nu}dx_\mu'dx_\nu'$$

with *the same* matrix

$$\|\eta_{\mu\nu}\|=\begin{Bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{Bmatrix}.$$

In studying the properties of homogeneous (uniform) space-time, the use of Galilean coordinates is convenient, but not essential. The property of space-time of being uniform may be as well expressed in general coordinates.

Let the substitution

$$x_\mu'=f_\mu(x_0x_1x_2x_3)\equiv f_\mu(x)$$

be performed in the expression for ds^2. Then,

$$ds^2=g_{\mu\nu}(x)dx_\mu dx_\nu$$

changes into

$$ds^2=g_{\mu\nu}'(x')dx_\mu'dx_\nu'$$

so that

$$g_{\mu\nu}'(x')=g_{\alpha\beta}\frac{\partial x_\alpha}{\partial x_\mu'}\frac{\partial x_\beta}{\partial x_\nu'}. \tag{1}$$

If the mathematical form of the functions $g_{\mu\nu}'$ is the same as that of the $g_{\mu\nu}$, that is, if

$$g_{\mu\nu}'(x')=g_{\mu\nu}(x), \tag{2}$$

then the space admits a transformation group.

For an infinitesimal transformation

$$x_\alpha'=x_\alpha+\eta^\alpha(x)$$

this leads to

$$\nabla^\mu\eta^\nu+\nabla^\nu\eta^\mu=0$$

and these equations are completely integrable if

$$R_{\mu\nu,\alpha\beta}=K(g_{\nu\alpha}g_{\mu\beta}-g_{\mu\alpha}g_{\nu\beta}),$$

that is, for a space of constant curvature. Galilean space corresponds to vanishing curvature

$$R_{\mu\nu,\alpha\beta}=0.$$

What I wish to stress is that the properties of the uniform Galilean space-time can be expressed in a generally covariant manner. On the other hand, the Einsteinian gravitation theory supposes the space-time to be nonuniform. It is just this fundamental assumption, and not the general covariance of equations, that distinguishes the gravitation theory from the theory of the Galilean space-time.

This distinction has not been sufficiently understood, or in any case not sufficiently stressed, by many physicists and, paradoxical as it may seem, by Einstein himself, although the French mathematician Cartan has drawn attention to it many years ago. Einstein called both theories relativity theories. But what is relativity? This word has been misused. It is natural to connect the notion relativity with uniformity of space and time. The **uniformity** of Galilean space with respect to positions, directions, and nonaccelerated motions may be as well termed as **relativity** of positions, of directions, and of nonaccelerated motions. That is the true content of Einstein's principle of relativity of 1905. Use of the word relativity in this sense is quite legitimate.

But, if one does this, if one connects relativity with uniformity, the relativity has nothing to do with general covariance, that is, with covariance in which (1) is true but (2) is not necessarily satisfied. This means also that, **in the theory of nonuniform space-time, there is no principle of relativity.** The generalization of the theory which consists in replacement of time by a nonuniform one means a **restriction** and not a generalization of relativity. If one uses the word relativity consistently, then the general principle of relativity is nonsensical.

In saying this, I do not want to introduce any doubt as to the validity of the wonderful Einsteinian gravitation theory, but only to stress the inconsistency of the use of the name "general relativity" when applied to gravitation theory.

Einstein himself proposed for his theory the name "general relativity," because the transformations considered in this theory are more general than the Lorentz transformations. But he omitted to state that, in the case of ordinary relativity, one has to consider transformations for which (2) must also be fulfilled, while, in the case of the so-called general relativity, this equation does not have to be taken into account. Thus, in the "general" theory, Einstein uses the word relativity simply as covariance, while in the "special" theory, the same word relativity is used as uniformity. Since covariance has nothing to do with uniformity, there arises a confusion which is very harmful to the understanding of Einstein's theory. If one uses the word relativity in both senses, then one has to admit such statements as "in general relativity there is no relativity" or "the Lagrangian form of nonrelativistic equations of motion satisfies the requirements of general relativity," etc.

This confusion is more harmful than it would seem at first glance. It leads to statements like "rotation is relative" which are obviously false, because the distinction between a geodetic and a nongeodetic is absolute and not relative.

The general covariance of equations has been considered for a long time as a specific property of the Einsteinian gravitation theory, by which it is distinguished from other physical theories. But later on, it was recognized that the covariance by itself cannot lead to any physical consequences. The true key to Einstein's discovery and the most difficult step was the limitation of the functions describing the gravitational field to geometrical ones (to the $g_{\mu\nu}$'s). Historically, the covariance requirement played a great part also, but this is because it was combined with other requirements, such as simplicity and beauty of the theory.

Nevertheless, the covariance requirement is still considered in a somewhat mystical way, as something prohibiting the use of well-defined coordinate frames, like Galilean coordinates in uniform space-time. The existence of Galilean coordinate frames is a characteristic of the inherent properties of the uniform space-time of the "special" theory. Likewise, there may be in "general" theory coordinate frames distinguished by some remarkable properties and characteristic of the kind of the nonuniform space-time considered.

In what follows, I wish to draw your attention to the fact that, for a rather general class of problems of gravitation theory, there exist such coordinate systems that may be considered as generalizations of ordinary internal systems. I mean not the local geodetic system valid in the vicinity of a point and of an instant of time, but the nonlocal generalization of the inertial frames of reference, valid throughout space.

In order to investigate whether such systems exist, it is necessary to make definite assumptions as to the physical system considered and as to the properties of space-time as a whole. This is necessary because of the nonlocal character of the problem, that requires a solution of Einstein's gravitational equations with conditions at infinity.

In the case of an isolated system of masses, it is natural to consider the system as embedded in a Galilean space-time. In a Galilean space-time, the following *theorem* holds.

Let

$$\Box\langle\psi\rangle=\frac{1}{c^2}\frac{\partial^2\langle\psi\rangle}{\partial t^2}-\left(\frac{\partial^2\langle\psi\rangle}{\partial x^2}+\frac{\partial^2\langle\psi\rangle}{\partial y^2}+\frac{\partial^2\langle\psi\rangle}{\partial z^2}\right).$$

If ψ satisfies the wave equation $\Box\psi=0$ and is finite everywhere and tends to zero at infinity like $1/r$, as well as its derivatives, and if in addition the radiation condition

$$\lim_{r\to\infty}\left\{\frac{\partial(r\psi)}{\partial r}+\frac{1}{c}\frac{\partial(r\psi)}{\partial t}\right\}=0$$

is satisfied for all values of the time t, *then ψ vanishes identically*. The radiation condition states that only outgoing waves are allowed.

A similar theorem may be proved in the case that $\Box\psi$ refers to a static Einsteinian space-time which is Galilean at infinity. It is to be supposed that the theorem holds also for a nonstatic Einsteinian space-time, though a formal proof may be difficult.

Let the space-time be such that the aforestated theorem is valid. Then, one can introduce auxiliary conditions for the coordinates in such a way that they behave like Galilean coordinates and are determined like them throughout the space-time (a Lorentz transformation remains of course arbitrary).

The auxiliary conditions are of the form $\Box x_\nu = 0$; $\nu = 0, 1, 2, 3$. But we have

$$\Box\psi = g^{\mu\nu}\frac{\partial^2\psi}{\partial x_\mu \partial x_\nu} - \Gamma^\nu \frac{\partial\psi}{\partial x_\nu},$$

where

$$\Gamma^\nu = -\frac{1}{(-g)^{\frac{1}{2}}}\frac{\partial \mathfrak{g}^{\mu\nu}}{\partial x_\mu}; \quad \mathfrak{g}^{\mu\nu} = (-g)^{\frac{1}{2}} g^{\mu\nu}.$$

Consequently, the condition is equivalent to

$$\partial \mathfrak{g}^{\mu\nu}/\partial x_\mu = 0.$$

Let the coordinates x_μ satisfy this condition. To find the most general form

$$x_\alpha' = f^\alpha(x_0 x_1 x_2 x_3)$$

we put

$$f^\alpha = a_\alpha + l_\beta a_{\alpha\beta} x_\beta + \eta^\alpha. \quad (l_0 = 1; l_1 = l_2 = l_3 = -1)$$

The linear part of this is a Lorentz transformation. Now, η^α must satisfy the wave equation $\Box\eta^\alpha = 0$, since f^α and the linear part satisfy it. Further, η^α must vanish at infinity (because the transformation must reduce there to a Lorentz transformation) and also η^α must satisfy the radiation condition (this follows from the radiation condition for the $g^{\mu\nu}$'s):

$$\eta^\alpha = \text{outgoing wave at infinity.}$$

But the conditions imposed upon η^α are so stringent that, according to the theorem, $\eta^\alpha \equiv$ zero everywhere. Thus, the whole arbitrariness of the coordinates resides in the Lorentz transformation.

We thus come to the conclusion that, in the case of an isolated system of masses, there is no essential difference in the coordinate question, between the so-called general and so-called special relativity theory. In both cases, arbitrary coordinates are admissible, since the equations are, or may be, written covariantly with respect to general transformations. But, in both cases, auxiliary conditions may be imposed upon the coordinates in such a way that only a Lorentz transformation remains arbitrary.

The coordinates so defined—I call them harmonic—are particularly adapted to the solution of Einstein's equations, and all the solutions that I shall discuss in the following lectures are obtained in these coordinates. But the value of the harmonic coordinates resides not only in their practical importance, but also in the fact that they help us to understand the general features of gravitation theory. Their existence shows that the usual sharp distinction between the coordinate problem in special and in general theory is somewhat artificial. In both theories, coordinates exist that are determined to a Lorentz transformation, but in both theories any other coordinate system may be used.

SECOND LECTURE: SOME APPROXIMATE SOLUTIONS OF EINSTEIN'S EQUATIONS (MOTION OF ROTATING BODIES OF FINITE SIZE)

The first approach to the problem of the motion of finite masses in connection with the solution of Einstein's nonlinear field equations was made in 1927 in a paper by Einstein and Grommer, and another paper by Einstein. More definite progress was obtained, independently and on two different lines, in the years 1938–1940 by Einstein, Infeld, and their collaborators, on the one hand, and by myself and my pupils, Petrova, Fichtenholz, and others, on the other hand. I should also like here to mention the work by Papapetrou from 1951. The ideas underlying the two research lines were widely different. Einstein's intention was, as it seems, to build a theory of elementary particles as field singularities. My aim was quite different. I wanted to find the solution of Einstein's gravitational equations for a problem of astronomical kind, where the moving bodies are of finite size, and the field has no singularities even within the bodies. I first found an approximate solution for the case of spherical nonrotating bodies, and then for the more general case of rotating bodies that are not necessarily spherical. The equations of motion for the centers of spherical nonrotating bodies turned out to be the same as the equations for point singularities obtained at the same time by Einstein, Infeld, and others. But this coincidence is due, to some extent, to good luck, because of the coordinate problem. My solution is written in a well-defined coordinate system (the harmonic one), while Einstein's and Infeld's solution corresponds to some vaguely defined coordinate system which bears a resemblance to the Newtonian one, but is modified from step to step (from approximation to approximation). Professor Infeld sees even a virtue in the fact that the coordinate system he uses is not well defined, but remains to some extent arbitrary—a standpoint with which I cannot agree.

The general problem is to find the solution to Einstein's gravitational equations, which corresponds to the motion of a given system of masses. It is clear from the start that, in order to reduce this problem to a mechanical one, it is necessary to make approximations. First of all, because of the gravitation waves, the physical system considered is not a conservative one. But, since the loss of energy by gravitational radiation is quite small, it is safe to neglect it and to consider the system as conservative. Then, the bodies themselves have in general an infinite number of degrees of freedom even in the nonrelativistic Newtonian approximation. Some important conclusions may however be drawn quite generally: thus, one may write down the integrals of the equations of motion and the asymptotic expressions for the fundamental tensor without reducing the inner degrees of freedom of the bodies to a finite number. But, in order to consider the motion of bodies as a whole, it is necessary to express the state of motion of each body in terms of a finite number of parameters.

There are thus two aspects of the problem of motion: the inner problem and the outer problem. In the Newtonian approximation, the conditions inside the bodies have no influence on the field outside them: only the total mass and the moments of inertia are important. The first relativistic approximations for the field outside requires a more detailed description of the motion inside the body, namely a description by means of Newtonian equations of motion for the continuous medium of which the body is built. This is natural, since in the first relativistic approximation one has to take into account the mass corresponding to the inner energy, and the latter may be calculated in the Newtonian approximation.

If, on the contrary, one considers the bodies as field singularities, the idealization is so far-reaching that the inner problem does not arise at all.

The formal procedure in solving Einstein's equation is to develop the solution in powers of q/c, q being a parameter of the order of the velocities involved. We have then, U being the Newtonian potential,

$$U/c^2 \sim q^2/c^2 \quad \text{and} \quad v^2/c^2 \sim q^2/c^2.$$

Since retardation is treated as a correction, the method is valid only for moderately large distances between and from the bodies, that is, for distances small compared with the wavelength emitted. If R is a distance and ω a typical frequency, we must have

$$R \ll c/\omega.$$

This condition is not independent of the condition $q \sim c$, since q and $R\omega$ are of the same order.

On the other hand, in order to get convenient expressions for the potential from the bodies, we must suppose the distances R to be large compared with the dimensions L of the bodies. We thus have

$$L \ll R \ll c/\omega.$$

Inside the body, the quantity U/c^2 may attain the value α/L, where α is the gravitational radius of the body. Since this is to be small, we have $\alpha \ll L$, and thus

$$\alpha \ll L \ll R \ll c/\omega.$$

To solve Einstein's equations, step by step, we proceed as follows. In the zeroth approximation, we consider the metrics as Euclidean, and the coordinates as Galilean, and we take for the energy tensor the expressions,

$$T^{00} = \frac{1}{c^2}\rho, \quad T^{0i} = \frac{1}{c^2}\rho v_i \quad (i=1,2,3),$$

(ρ density), while the values of the other components T^{ik} are irrelevant [they are of the order $T^{ik} \square \rho(q^2/c^2)$]. The foregoing expressions satisfy the relation

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x_i} = 0$$

since

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0$$

(it is summed from 1 to 3 over Latin indexes appearing twice). To make the next step, we calculate the Newtonian potential

$$U = \int \frac{\gamma \rho'}{|\mathbf{r}-\mathbf{r}'|} dV'$$

and the corresponding vector potential

$$U_i = \int \frac{\gamma(\rho v_i)'}{|\mathbf{r}-\mathbf{r}'|} dV'.$$

This gives for the metric tensor the approximation

$$g_{00} = c^2 - 2U; \quad g_{0i} = \frac{4}{c^2} U_i$$

$$g_{ik} = -\left(1 + \frac{2U}{c^2}\right)\delta_{ik}$$

from which the quantities $\mathfrak{g}^{\mu\nu} = (-g)^{\frac{1}{2}} g^{\mu\nu}$ are easily calculated, namely

$$\mathfrak{g}^{00} = \frac{1}{c} + \frac{4U}{c^3}; \quad \mathfrak{g}^{0i} = \frac{4U_i}{c^3}; \quad \mathfrak{g}^{ik} = -c\delta_{ik}.$$

These quantities satisfy the equation,

$$\frac{\partial \mathfrak{g}^{00}}{\partial t} + \frac{\partial \mathfrak{g}^{0i}}{\partial x_i} = 0.$$

In the approximation considered, the coordinates are thus harmonic.

Proceeding to the next step, we must take into account the corrections in the expression for the divergence of a tensor. We have now

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0k}}{\partial x_k} + \frac{1}{c^2}\frac{\partial U}{\partial t} T^{00} = 0$$

and

$$\frac{\partial T^{0i}}{\partial t} + \frac{\partial T^{ik}}{\partial x_k} - \frac{\partial U}{\partial x_i} T^{00} = 0.$$

Supposing the bodies to be elastic, we may take for the energy tensor the expressions

$$c^2T^{00}=\rho\left\{1+\frac{1}{c^2}(\tfrac{1}{2}v^2+\Pi-U)\right\}$$

$$c^2T^{0i}=\rho v_i\left\{1+\frac{1}{c^2}(\tfrac{1}{2}v^2+\Pi-U)\right\}-\frac{1}{c^2}p_{ik}v_k$$

$$c^2T^{ik}=\rho v_iv_k-p_{ik}.$$

These expressions are obtained by adding to that part of mass density, which is conserved, the variable part of the mass density and also by adding terms corresponding to the energy flux. The notion of energy flux was first introduced by the Russian physicist Umow as early as in 1874 so that the expressions above are essentially the same as in the Umow paper.

The quantities ρ etc. entering into these expressions satisfy the nonrelativistic equations of motion, namely

$$\rho\frac{dv_i}{dt}-\rho\frac{\partial U}{\partial x_i}=\frac{\partial p_{ik}}{\partial x_k}$$

as well as the continuity equation

$$\frac{\partial \rho}{\partial t}=\frac{\partial(\rho v_i)}{\partial x_i}=0.$$

The letter Π denotes the elastic energy density that satisfies the equation

$$\rho\frac{d\Pi}{dt}=\tfrac{1}{2}p_{ik}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right).$$

In virtue of these equations, it is easily proved that the divergence of the energy tensor vanishes in the required approximation.

Now, the calculations can proceed in different directions. Firstly, one may obtain, from the Einsteinian equations

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R=-\frac{8\pi\gamma}{c^2}T^{\mu\nu}$$

the next approximation for the $\mathfrak{g}^{\mu\nu}$'s. Secondly, one may obtain the equations of motion for the bodies without solving Einstein's equations explicitly. One can also write down the integrals of the equations of motion. Since the calculations are rather tedious, I shall only briefly indicate their principal idea and the results.

In a harmonic coordinate system, where the condition

$$\partial\mathfrak{g}^{\mu\nu}/\partial x_\mu=0$$

is satisfied, we have

$$g(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R)=\tfrac{1}{2}\mathfrak{g}^{\alpha\beta}\frac{\partial^2\mathfrak{g}^{\mu\nu}}{\partial x_\alpha\partial x_\beta}+\cdots,$$

where the members denoted by dots do not contain second derivatives. If one uses this expression, the vanishing of divergence of the tensor $T^{\mu\nu}$ is no longer an identity, but is only satisfied in virtue of the harmonicity condition. Approximately, we have

$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{\partial\mathfrak{g}^{\mu\nu}}{\partial x_\mu}=\frac{16\pi\gamma}{c^3}g\nabla_\mu T^{\mu\nu},$$

where Δ is the Euclidean operator. In order that $\partial\mathfrak{g}^{\mu\nu}/\partial x_\mu$ should decrease rapidly with the distance from the mass m_a, it is necessary that

$$\int_{(a)} g\nabla_\mu T^{\mu\nu}(dx)^3=0;\quad (dx)^3=dx_1dx_2dx_3$$

where the integral is taken over a volume enclosing the mass m_a. This gives the equations of motion for the centers of inertia of the separate bodies. To obtain equations for rotational motion, one has to write

$$\int_{(a)} g(x_i\nabla_\mu T^{\mu k}-x_k\nabla_\mu T^{\mu i})(dx)^3=0.$$

I shall not write down the equations of motion in explicit form because they are complicated. But it is worth noting that the equations of motion for the centers of mass can be written in Lagrangian form.

From the equations of motion, the ten classical integrals may be obtained (namely, the momentum and energy integral, the moment of momentum, and the center of gravity integral). These integrals may be deduced by integrating the previous expressions over the whole space instead of over the region where a single mass lies.

The integrals have the following form. Let

$$W=\tfrac{1}{2}\gamma\int\rho'|\mathbf{r}-\mathbf{r}'|(dx')^3$$

so that

$$\Delta W=U=\gamma\int\frac{\rho'}{|\mathbf{r}-\mathbf{r}'|}(dx')^3,$$

where U is the Newtonian potential, and let U_i be, as before, the vector potential.

We put

$$G_i=\left[\rho+\frac{\rho}{c^2}(\tfrac{1}{2}v^2+\Pi+3U)\right]v_i-\frac{1}{c^2}p_{ij}v_j-\frac{4}{c^2}\rho U_i-\frac{1}{c^2}\rho\frac{\partial^2W}{\partial x_i\partial t}.$$

Then, the quantity,

$$P_i=\int G_i(dx)^3,$$

which is to be interpreted as the momentum of the system, is constant in virtue of the equations of motion.

If we put

$$M_{ik}=\int(x_iG_k-x_kG_i)(dx)^3,$$

then M_{ik} is the angular momentum and is also constant.

The energy integral is

$$M=\int\rho\left\{1+\frac{1}{c^2}(\tfrac{1}{2}v^2+\Pi-\tfrac{1}{2}U)\right\}(dx)^3.$$

If one uses the Lagrangian form of the equations of motion one can obtain the next term (with respect to q^2/c^2) in the energy integral also.

Finally, for the integral of the center of mass, we have

$$MX_i-P_it=K_i,$$

where the center of mass is defined by

$$MX_i=\int x_i\rho\left\{1+\frac{1}{c^2}(\tfrac{1}{2}v^2+\Pi-\tfrac{1}{2}U)\right\}(dx)^3.$$

All the quantities M, P_i, M_{ik}, K_i are constant in virtue of the relativistic equations of motion.

The values of these constants are essential for the asymptotic behavior of the metric tensor at large distances from the system of masses. We write these expressions for moderately large distances (not in the wave region). For the space components $\mathfrak{g}^{ik}$, we have

$$\mathfrak{g}^{ik}=-c\delta_{ik}+\frac{2\gamma}{c^3r}\frac{d^2}{dt^2}\int\rho x_ix_k(dx)^3$$
$$+\frac{2\gamma}{c^3}\frac{x_j}{r^3}\frac{d}{dt}\int\rho(x_jx_iv_k+x_jx_kv_i-x_ix_kv_j)(dx)^3+\frac{\gamma^2M^2}{c^3r^4}x_ix_k.$$

Introducing the generalized moments of inertia

$$D_{ji}=\int\rho x_ix_j\left\{1+\frac{1}{c^2}(\tfrac{1}{2}v^2+\Pi-\tfrac{1}{2}U)\right\}(dx)^3+\frac{7}{2c^3}\delta_{ij}\int\rho W(dx)^3$$

and also the function

$$W_i=\tfrac{1}{2}\gamma\int(\rho v_i)'|\mathbf{r}-\mathbf{r}'|(dx')^3$$

which allows us to take into account the retardation in the vector potential, then we obtain

$$\mathfrak{g}^{0i}=\frac{4\gamma}{c^2r}P_i+\frac{2\gamma x_j}{c^3r^3}M_{ji}-\frac{\partial^2}{\partial x_j\partial t}\frac{2\gamma}{c^3r}D_{ji}+\frac{\partial^2}{\partial x_j\partial x_k}\left(\frac{2\gamma}{c^3r}\right)$$
$$\times\int\rho v_ix_jx_k(dx)^3+\frac{7\gamma^2MP_i}{c^5r^2}+\frac{\gamma^2MP_kx_ix_k}{c^5r^4}+\frac{4}{c^5}\frac{\partial^2W_i}{\partial t^2}.$$

The last three terms are correction terms of the order q^2/c^2 with respect to the main terms.

For the component $\mathfrak{g}^{00}$ of the fundamental tensor, we obtain finally

$$\mathfrak{g}^{00}=\frac{1}{c}+\frac{4\gamma M}{c^3r}+\frac{4\gamma x_j}{c^3r^3}MX_j+\frac{\partial^2}{\partial x_j\partial x_k}\left(\frac{2\gamma}{c^3r}\right)D_{jk}$$
$$-\frac{\partial^3}{\partial x_i\partial x_j\partial x_k}\left(\frac{2\gamma}{c^3r}\right)\int\rho x_ix_jx_k(dx)^3+\frac{7\gamma^2M^2}{c^5r^2}$$
$$+\frac{14\gamma^2M^2X_jx_j}{c^5r^4}+\frac{4}{c^5}\frac{\partial^2W}{\partial t^2}.$$

All the expressions are of course approximate. It is easy to see that $\mathfrak{g}^{00}$ and $\mathfrak{g}^{0i}$ verify the relation

$$\frac{\partial\mathfrak{g}^{00}}{\partial t}+\frac{\partial\mathfrak{g}^{0i}}{\partial x_i}=0.$$

The corresponding relation for $\mathfrak{g}^{0i}$ and $\mathfrak{g}^{ik}$ is also verified up to terms containing c^3 in the denominator (terms of higher order have been neglected in $\mathfrak{g}^{ik}$). Thus, the coordinates used are in fact harmonic.

In the wave region, the foregoing expressions are to be modified. I consider this modification in the next lecture.

THIRD LECTURE: ON GRAVITATIONAL WAVES FROM A SYSTEM OF MOVING BODIES

In my previous lecture, I have given asymptotic expressions for the metric tensor, valid at moderately large distances from the system of moving bodies. (The distances must be large compared with dimensions of the system, but still small compared with the wavelength of the gravitational waves emitted.) There are questions, however, for which the knowledge of the asymptotic values of the metric tensor at very large distances is required, that is, at distances of the order of the wavelength, or even large, compared with the wavelength. These questions are of a theoretical nature, but nevertheless they are of interest, because, the gravitational equations being nonlinear, it is not so simple to establish even the existence of gravitational waves.

In the following, I intend to show that there exist solutions corresponding to spherical waves emitted by the system of moving bodies. This is due to the particular structure of gravitational equations in which the components $\mathfrak{g}^{\mu\nu}$ of the metric tensor are at the same time coefficients in the wave operator and unknown functions to which this operator applies.

Einstein's equation in harmonic coordinates may be written in the form

$$g(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R)=\tfrac{1}{2}\mathfrak{g}^{\alpha\beta}\frac{\partial^2\mathfrak{g}^{\mu\nu}}{\partial x_\alpha\partial x_\beta}-c^2N^{\mu\nu}=-\frac{8\pi\gamma g}{c^2}T^{\mu\nu},$$

where $N^{\mu\nu}$ is a quadratic form in the first derivatives of the $\mathfrak{g}^{\alpha\beta}$'s. As the zeroth approximation to the wave operator we choose

$$\Box^0\psi=\frac{1}{c^2}\left(1+\frac{4\alpha}{r}\right)\frac{\partial^2\psi}{\partial t^2}-\left(\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\psi}{\partial r}+\frac{\Delta^*\psi}{r^2}\right)$$

where

$$\Delta^*\psi=\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\psi}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2\psi}{\partial\varphi^2}$$

is the angular part of the ordinary Laplace operator in spherical coordinates. In the coefficients of the operator $\Box^0$ only statical terms of the order $1/r$ are retained.

We want to study spherical waves going out from the system. For such waves, the term $\Delta^*\psi/r^2$ in the wave operator is small as compared with other terms and we can replace the wave equation by

$$\frac{1}{c^2}\left(1+\frac{4\alpha}{r}\right)\frac{\partial^2\psi}{\partial t^2}-\left(\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\psi}{\partial r}\right)=0.$$

If we write

$$r\psi=f$$

and introduce the independent variable

$$r^*=r+2\alpha(\lg r-\lg r_0)$$

we have approximately

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}-\frac{\partial^2 f}{\partial r^{*2}}=0.$$

The solution that corresponds to the outgoing wave is of the form

$$f=f(\tau,\mathbf{n}),$$

where $\mathbf{n}$ is the unit vector in the direction *from* the system, with components $n_i=(x_i/r)$ and τ is the quantity

$$\tau=t-\frac{1}{c}r^*=t-\frac{1}{c}\left(r+2\alpha\lg\frac{r}{z_0}\right).$$

Thus, the outgoing wave has the asymptotic form

$$\psi=\frac{1}{r}f(\tau,\mathbf{n}).$$

For a function of this form, derivatives with respect to the coordinates and the time can be calculated by neglecting the dependence of ψ on $\mathbf{n}$ and r and by taking into account only the dependence on τ.

Putting

$$k_\alpha=\partial\tau/\partial x_\alpha,$$

and thus

$$k_0=1;\quad k_i\cong -n_i/c,$$

we have approximately

$$\partial\psi/\partial x_\alpha=k_\alpha\dot{\psi},\qquad (*)$$

where the point indicates the derivatives with respect to τ (or to t).

In the above expression for $\Box^0\psi$ only the static part of $\mathfrak{g}^{\mu\nu}$ has been taken into account, but the non-static part (which we denote by $C^{\mu\nu}$) may be of the same order of magnitude. We may put

$$\mathfrak{g}^{00}=\frac{1}{c}+\frac{4\alpha}{cr}+C^{00}$$

$$\mathfrak{g}^{0i}=C^{0i}$$

$$\mathfrak{g}^{ik}=-c\delta_{ik}+C^{ik}$$

where the $C^{\mu\nu}$'s are of the same asymptotic form as ψ, so that

$$\partial C^{\mu\nu}/\partial x_\alpha=k_\alpha\dot{C}^{\mu\nu}.$$

But the derivatives of the static part of the $\mathfrak{g}^{\mu\nu}$'s decrease as $1/r^2$ and are to be neglected as compared with the derivatives of the $C^{\mu\nu}$'s. Thus, we may write

$$\partial\mathfrak{g}^{\mu\nu}/\partial x_\alpha=k_\alpha\dot{C}^{\mu\nu}.\qquad (**)$$

Now, the harmonicity condition gives

$$\partial\mathfrak{g}^{\mu\nu}/\partial x_\mu=k_\mu\dot{C}^{\mu\nu}=0$$

and this may be integrated (with respect to τ or to t). We thus obtain

$$k_\mu C^{\mu\nu}=0.$$

The integration constant has been put equal to zero, because the the static part of the $\mathfrak{g}^{\mu\nu}$'s is considered separately. From the last equation we have

$$C^{0i}=\frac{n_k}{c}C^{ik};\quad C^{00}=\frac{n_i n_k}{c^2}C^{ik}.$$

With the corrected values of $\mathfrak{g}^{\mu\nu}$, the expression for the wave operator becomes

$$\frac{1}{c}\mathfrak{g}^{\alpha\beta}\frac{\partial^2\psi}{\partial x_\alpha\partial x_\beta}=\Box^0\psi+\frac{1}{c}C^{\alpha\beta}\frac{\partial^2\psi}{\partial x_\alpha\partial x_\beta}.$$

But, if ψ is an outgoing wave and satisfies (*), then

$$C^{\alpha\beta}\frac{\partial^2\psi}{\partial x_\alpha\partial x_\beta}=k_\alpha k_\beta C^{\alpha\beta}\ddot{\psi}=0$$

separately. Thus, in the wave operator applied to an outgoing wave, the coefficients may be replaced by their static values. This justifies the asymptotic form of wave function used in the preceding calculations. (This form was obtained with the static values of the $\mathfrak{g}^{\mu\nu}$'s.)

We now proceed to the calculations of that part of the Einstein tensor that is quadratic in the first derivatives of the $\mathfrak{g}^{\mu\nu}$. Using the expression (**) for $\partial\mathfrak{g}^{\mu\nu}/\partial x_\alpha$ and raising and lowering the indexes with Galilean values of the metric tensor we obtain

$$N^{\mu\nu}=\frac{1}{4c^2}k^\mu k^\nu(\dot{C}^{\alpha\beta}\dot{C}_{\alpha\beta}-\tfrac{1}{2}\dot{C}_\alpha{}^\alpha\dot{C}_\beta{}^\beta).$$

The quantity

$$\sigma_g=\frac{1}{32\pi\gamma}(\dot{C}^{\alpha\beta}\dot{C}_{\alpha\beta}-\tfrac{1}{2}\dot{C}_\alpha{}^\alpha\dot{C}_\beta{}^\beta)$$

is to be interpreted as the energy density of the gravitational waves. We have

$$N^{\mu\nu}=\frac{8\pi\gamma}{c^2}\sigma_g k^\mu k^\nu.$$

We may also take into account the electromagnetic radiation. Putting

$$\sigma_{\text{em}}=(E^2+H^2)/8\pi,$$

we have for the energy tensor in the wave zone

$$T^{\mu\nu}=\sigma_{\text{em}}k^\mu k^\nu.$$

Introducing the total density

$$\sigma=\sigma_g+\sigma_{\text{em}}$$

of radiation energy, we obtain the Einstein equations in the form

$$\frac{1}{2c}\Box\mathfrak{g}^{\mu\nu}=\frac{8\pi\gamma}{c^2}\sigma k^\mu k^\nu.$$

From this, it is clear that the correction to the $\mathfrak{g}^{\mu\nu}$'s due to the radiation terms in Einstein's equations (to the terms $N^{\mu\nu}$ and $T^{\mu\nu}$) is of the form $hk^\mu k^\nu$, where h satisfies the equation

$$\Box h=\frac{16\pi\gamma}{c}\sigma.$$

This correction modifies by the same amount the values of $C^{\mu\nu}$ and it might seem that this modification alters the value of the gravitational energy density σ_g. If it were so, then the integration problem would not be solved and should be taken up anew. As a matter of fact, it happens, however, that, owing to the relation $k_\mu\dot{C}^{\mu\nu}=0$, the expression for σ_g is *invariant* with respect to the transformation

$$\dot{C}^{\alpha\beta}\rightarrow\dot{C}^{\alpha\beta}+\lambda k^\alpha k^\beta.$$

Thus, the right-hand side of the equation for $\mathfrak{g}^{\mu\nu}$ is not to be modified.

The radiation energy density σ is of the form

$$\sigma=\frac{\sigma_0(\tau,\mathbf{n})}{r^2}.$$

Inserting this in the equation for h, we find the following asymptotic expression for this quantity

$$h=\frac{8\pi\gamma\lg r}{r}\int_{\tau_0}^{\tau}\sigma_0(\tau,\mathbf{n})d\tau+\frac{h_0(\tau,\mathbf{n})}{r}.$$

This may be written as

$$h=\frac{2\gamma}{cr}(\lg r\,\Delta E+\epsilon),$$

where $\Delta E(d\Omega/n\pi)$ is the energy loss during the time $\tau-\tau_0$ in the solid angle $d\Omega$ lying in the direction $\mathbf{n}$. Superposing a solution of the homogeneous wave equation, we may write

$$C^{\mu\nu}=\frac{2\gamma}{c^3r}f^{\mu\nu}(\tau,\mathbf{n})+\frac{2\gamma}{cr}(\lg r\Delta E+\epsilon)k^\mu k^\nu.$$

We thus have in the wave region

$$\mathfrak{g}^{00}=\frac{1}{c}+\frac{4\gamma M}{c^3r}+\frac{2\gamma}{c^3r}n_in_kf^{ik}+\frac{2\gamma}{c^3r}(\lg r\,\Delta E+\epsilon)$$

$$\mathfrak{g}^{0i}=\frac{2\gamma}{c^4r}n_kf^{ik}+\frac{2\gamma}{c^4r}n_i(\lg r\,\Delta E+\epsilon)$$

$$\mathfrak{g}^{ik}=-c\delta_{ik}+\frac{2\gamma}{c^3r}f^{ik}+\frac{2\gamma}{c^3r}n_in_k(\lg r\,\Delta E+\epsilon)$$

since

$$f^{0i}=\frac{n_k}{c}f^{ik};\quad f^{00}=\frac{n_in_k}{c^2}f^{ik}.$$

We now may compare the expressions just obtained with the former expressions valid at moderately large distances from the system.

If we introduce the moments of inertia

$$D_{ik}(t)=\int\rho x_ix_k(dx)^3$$

and replace t by τ, then we can put

$$f^{ik}=(d^2/d\tau^2)D_{ik}(\tau).$$

Neglecting terms in ΔE and ϵ that are very small (in spite of the logarithm), we find that the formulas

$$\mathfrak{g}^{00}=\frac{1}{c}+\frac{4\gamma M}{c^3r}+\frac{2\gamma}{c^3}\frac{\partial^2}{\partial x_i\partial x_k}\frac{D_{ik}(\tau)}{r}$$

$$\mathfrak{g}^{0i}=-\frac{2\gamma}{c^3}\frac{\partial^2}{\partial x_k\partial t}\frac{D_{ik}(\tau)}{r}$$

$$\mathfrak{g}^{ik}=-c\delta_{ik}+\frac{2\gamma}{c^3}\frac{\partial^2}{\partial t^2}\frac{D_{ik}(\tau)}{r}$$

give approximately correct values for the $g^{\mu\nu}$'s in the wave zone as well as at moderate distances from the system.

The harmonicity condition is satisfied rigorously.

I should like to add some remarks as to the energy pseudo-tensor and the general formulation of conservation laws.

In virtue of Einstein's equations, one can introduce the following symmetrical pseudo-tensor

$$U^{\mu\nu}=\frac{1}{16\pi\gamma}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}(g^{\alpha\beta}g^{\mu\nu}-g^{\alpha\mu}g^{\beta\nu}).$$

This pseudo-tensor is symmetrical and satisfies the equation,

$$\partial U^{\mu\nu}/\partial x_\mu = 0,$$

identically.

Now consider the integrals

$$*M=c^2\int U^{00}(dx)^3$$

$$*P^i=c^2\int U^{0i}(dx)^3$$

$$*M^{ik}=c^2\int (x_i U^{0k}-x_\mu U^{0i})(dx)^3$$

$$*M^{i0}=c^2\int (x_i U^{00}-tU^{0i})(dx)^3$$

$$=*M*X^i-t*P^i$$

taken over a finite volume V. The time derivatives of these integrals may be transformed into surface integrals. The surface enclosing the volume V (which contains all the masses) is to be taken at moderately large distances from the system of masses. Then the surface integrals will be very small and thus the volume integrals nearly constant.

But the constancy of the volume integrals may be verified by direct calculation. The integrals themselves, and not only their time derivatives, may be transformed into surface integrals. If they are taken over a distant surface, only asymptotic values of the $g^{\mu\nu}$'s are to be used. Inserting these in the integrals we obtain for the quantities $*M$, $*P^i$, $*M^{ik}$, $*M^{i0}$ just the constants M, P_i, M_{ik}', K_i entering in the asymptotic expressions, while all other terms in $g^{\mu\nu}$, like D_{ik}, etc., cancel out. We have thus a direct verification of the asymptotic expressions for the fundamental tensor.

As to the energy loss due to gravitational radiation, it can be calculated as usual with the help of the energy pseudo tensor which is approximately equal to

$$U^{\mu\nu}=\sigma_g k^\mu k^\nu.$$

Introducing the quantities,

$$B_{ik}=\frac{d^3}{d\tau^3}\{D_{ik}(\tau)-\tfrac{1}{3}\delta_{ik}D_{jj}(\tau)\},$$

we obtain, for the rate of energy loss,

$$\frac{dW}{dt}=-\frac{\gamma}{5c^5}B_{ik}B_{ik},$$

which is an extremely small quantity owing to the very large value of the constant

$$5c^3/\gamma=2.10^{39}\text{ gram/sec.}$$

In these three lectures, I have given only a very brief account of the problems in relativity theory that have interested me during recent years. A more detailed treatment of these problems, with all the calculations involved, is contained in my book *Theory of Space, Time, and Gravitation* published in Russian in 1955, and which will appear in English translation in 1957 or 1958.