REVIEWS OF MODERN PHYSICS VOLUME 29, NUMBER 2 APRIL, 1957

Relativity

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1. SPECIAL RELATIVITY

1(a) Basic Principles

THE theory of relativity cannot be separated from the rest of physics, for every physical theory is supposed to conform to the basic relativistic principles, and any concrete physical problem involves a synthesis of relativity and some specific physical theory.

To mention one historically important example: When Sommerfeld first derived the fine structure of the hydrogen spectrum, his success was widely considered to be primarily a confirmation of relativistic dynamics. Today we know that the relativistic basis of his analysis was much more secure than the quantum theoretical one. After all, ten years were to pass before the electronic spin was discovered, not to mention wave mechanics or Dirac's theory of the electron.

I shall concentrate here on some of the basic principles relevant to all applications, and on some instances where relativity was crucial for the development of new physical theories.

Special relativity originated from electrodynamics. But only the rudiments of electrodynamics, namely, the fundamental laws of light propagation, were actually used for its foundation. This was the strength of Einstein's position compared to that of Lorentz and Poincaré. Both Lorentz and Poincaré put electrodynamics very much into the foreground, on the assumption that in the last analysis all interactions are electrodynamic interactions (with the possible exception of gravitation). The development of relativity, however, has shown that the theory is in no way restricted to electrodynamics and that it is quite independent of our views on the ultimate nature of the interaction between elementary particles.

For the purpose of this talk, I define the postulate of special relativity as the assertion that in the absence of gravitation the laws of physics have the same form in all inertial frames of reference and that any two inertial frames are connected by a linear coordinate transformation (inhomogeneous Lorentz transformation)

$$x^{i\prime}=a^i+\lambda_j{}^i x^j \quad (i,j=1,\cdots,4) \tag{1.1}$$

which leaves the Minkowskian metric

$$ds^2=-(dx^1)^2-(dx^2)^2-(dx^3)^2+(dx^4)^2 \quad (x^4=ct) \tag{1.2}$$

invariant.

To make this statement unambiguous, several remarks must be added.

(1) Einstein's definition of the space-time coordinates—in particular, his definition of simultaneity—is explicitly assumed. Thus the postulate contains the law of light propagation in free space as well as relativistic kinematics.

(2) In a given inertial frame, K, the state of a physical system must be given by a well-defined set of quantities, say, $f_1, \cdots, f_n$ (which may or may not be functions of space-time), and the transformation law of the f_α must be explicitly stated. This means that in a second inertial frame, K', the same state is described by n similar quantities f'_α which are related to the original f_α by n equations

$$f'_\alpha=F_\alpha(f_1,\cdots,f_n), \tag{1.3}$$

where the form of the functions F_α is determined by the Lorentz transformation connecting K and K'. (Dirac's equation shows how intricate the transformation law may prove to be.) I list here indiscriminately all quantities describing the system, such as hydrodynamical and electromagnetic ones, and the equations (1.3) imply in no way a close relationship between the various f_α.

(3) Passive and active Lorentz transformations. The transformations considered so far—relating the descriptions of the same physical situation in two different frames K and K'—may be termed *passive* Lorentz transformations. Since the laws of physics have the same form in K and K', it follows that the f'_α describe also a possible situation in K, different, in general, from that described by the f_α. This leads to an interpretation of (1.3) as an *active* transformation. If, in a fixed frame K, $\{f_1,\cdots,f_n\}$ is an admissible state of the physical system in question, so is $\{f'_1,\cdots,f'_n\}$.

(4) Finally, not all Lorentz transformations need be admitted. The minimal requirement is that a theory be invariant with respect to the "restricted" Lorentz group, i.e., the group of all transformations which involve neither time reversal nor spatial reflexions. [In terms of the coefficients in (1.1) these are defined by: (i) $\lambda_4{}^4>0$, (ii) the determinant of the $\lambda_j{}^i$ is equal to 1.]

As is well known, three other Lorentz transformations may be adjoined to the restricted Lorentz group:

$$(A) \quad x^{i\prime}=-x^i \ (i\neq 4); \quad x^{4\prime}=x^4 \quad \text{(reflexion)}$$

$$(B) \quad x^{i\prime}=x^i \ (i\neq 4); \quad x^{4\prime}=-x^4 \quad \text{(time reversal)}$$

$$(C) \quad x^{i\prime}=-x^i \quad (i=1,\cdots,4). \quad \text{(total inversion)}$$

Note that C is the product of A and B.

The distinction of the different possibilities is particularly important for the "active" interpretation. Thus invariance with respect to time reversal expresses the reversibility of the phenomena described by a given theory. In the past one has considered at least the group of all transformations that do not invert the time (i.e., the "restricted" group together with A), but the recent observations on "strange" particles have led to the suggestion of dropping[1]—tentatively—the spatial reflexion A.*

1(b) Experimental Verification

Regarding the experimental verification of relativistic kinematics, I refer to Robertson's analysis of the observational basis of the Lorentz transformations.[2] It concerns three basic optical experiments, and it is, in my opinion, a particular virtue of this analysis that it depends as little as possible on any specific physical theory. As in the definition of the Lorentz group only the basic laws of light propagation are involved. The three experiments are (1) the Michelson-Morley experiment, (2) the experiment by Kennedy and Thorndike[3] (whose arrangement is similar to that of the Michelson Morley experiment, with the distinction, however, that the two arms along which the light is propagated are as different in length as feasible), (3) the experiment by Ives and Stilwell on the transverse Doppler effect, which exhibits the time dilation required by the Lorentz transformation. Robertson has shown that under very modest assumptions one can deduce from the results of these experiments—to the extent that such a thing can be deduced at all—the validity of the Lorentz transformation (and, hence, of relativistic kinematics) including terms of order v^2/c^2.

Apart from these optical experiments, application and verification of special relativity have, of course, been mostly connected with elementary particles. This holds in particular for relativistic dynamics as distinguished from relativistic kinematics.

A most impressive confirmation of the kinematic effect of time dilation is provided by a comparison of the decay time of π mesons in flight with that of π mesons at rest.[4]

Today there is no longer any need to stress the experimental verification of relativistic dynamics. Particles of extreme relativistic energies are an everyday occurrence in physical laboratories, and no failure of relativistic dynamics is known. This applies specifically to the most important consequence of relativity, namely, the equivalence of energy and mass.

1(c) Significance of Relativity for Quantum Theory

Let me mention a few instances where relativity has been crucial for the development of quantum theory. I might start with the very notion of a photon, for the relation $E=cp$ between the energy and the momentum of a particle can certainly not be understood on the basis of Newtonian mechanics. (Planck's book on heat radiation contains a proof that light cannot be assumed to consist of particles, because such an assumption would be incompatible with the relation $P=\frac{1}{3}u$ between the radiation pressure P and the density of radiation energy, u. Planck's analysis rests, of course, on the Newtonian formula $E=p^2/2m$.)

The second instance I want to mention is the discovery of de Broglie's relation. It was the requirement of Lorentz invariance that led from Planck's equation $E=h\nu$ to the relation $p=h/\lambda$.

The third is the inauguration of relativistic quantum mechanics by Dirac. Not only did Dirac's equation provide the correct description of the kinematics and the dynamics of the electron, but it also led to the concept of antiparticles and to the prediction of the creation and annihilation of pairs, thus turning the apparent paradox of negative energy states into the most remarkable success of the theory.

Finally, all of quantum field theory belongs here, because relativity has played an increasingly important role in its development, particularly during the last ten years.

In addition to these instances, where relativity has been of vital importance for the discovery or the formulation of new laws of physics, there are some general results which give a deeper insight into the structure of quantum theory. Firstly, the connection between Lorentz invariance and conservation laws. Conservation laws follow whenever the field equations are derived from a Lorentz invariant Lagrangian. (Admittedly, this connection is not specifically quantum theoretical, but is equally valid for a classical field theory.) Secondly, Pauli's analysis of the relation between spin and statistics,[5] which is based on Lorentz

[1] See C. N. Yang's report on new particles.

* *Note added in proof.*—In the meantime the nonconservation of parity has been brilliantly and successfully demonstrated. By themselves, however, the experiments do not prove that spatial reflexion must be omitted from the invariance group of special relativity, because—as was pointed out in the previous remarks—the invariance of a theory with respect to a given Lorentz transformation can only be adequately discussed if the transformation laws (1.3) are explicitly stated for *all* physical quantities. According to the highly satisfactory interpretation of the recent experiments by Lee, Oehme, and Yang (to appear in Phys. Rev.) the theory remains invariant if spatial reflexion is coupled with charge conjugation. Thus the change in our theoretical views does not lead to the omission of spatial reflexion from the list of admissible Lorentz transformations, but it rather leads to a radical alteration of those transformation laws (1.3) which correspond to spatial reflexion, an alteration involving the transition from particles to antiparticles. Since heretofore charge conjugation by itself had been considered a symmetry operation of the theory the total number of symmetry operations has been cut in half.

[2] See reference 12 (bibliography).

[3] See H. P. Robertson's remark in the discussion following this talk, p. 173.

[4] See reference 13 (bibliography). Here the factor $(1-\beta^2)^{-\frac{1}{2}}$ is of the order 1.5.

[5] See reference 14 (bibliography).

invariance, and thirdly Wigner's investigation of the representations of the inhomogeneous Lorentz group, which resulted in the classification of all free relativistic systems that obey the basic laws of quantum theory.[6]

Relativity did not start out as a specifically atomic theory, but it has survived the transition from classical to quantum theoretical concepts. This, I believe, is not a coincidence, for those laws of physics which express a basic "invariance" or "symmetry" of physical phenomena seem to be our most fundamental ones. (Even in Newtonian physics Euclidean geometry—and, consequently, also the group of orthogonal transformations—plays a much more important role than the traditional expositions seem to indicate.)

The present difficulties of relativistic quantum theory are familiar, and it is not known how they will be resolved. If we shall have to abandon the Minkowskian description of the space-time continuum the notion of Lorentz invariance will no longer be directly applicable. But there is no doubt that even then the basic relativistic postulate of the equivalence of all inertial frames will remain valid in some—suitably modified—form.

2. GENERAL RELATIVITY

This second part is concerned with the classical theory of general relativity as originally formulated by Einstein. Various attempts to go beyond this theory are surveyed in Parts III and IV.†

I shall analyze the logical and mathematical structure of general relativity in much greater detail than I did in the case of special relativity in Part I. For one thing, special relativity is an every-day working tool of many theoretical physicists, whereas general relativity is not. For another, many of the details are relevant to the discussion in Parts III and IV.

Familiarity with the elements of Riemannian geometry is assumed—to the extent that they are treated in Einstein's "Meaning of Relativity." I also follow Einstein's notation.

2(a) Basic Principles

General relativity starts with Einstein's equivalence principle. It follows from the equality of inertial and gravitational mass that for purely mechanical phenomena a uniformly accelerated frame of reference is equivalent to a frame at rest which carries a uniform gravitational field. ("Acceleration" and "rest" are taken here with respect to an inertial frame of Newtonian mechanics.) Einstein's principle postulates the equivalence of such frames for *all* physical phenomena and thereby makes gravitation an integral part of general relativity. The analysis of even the simplest cases shows that coordinate differences are no longer directly measured by clocks and measuring rods. Thus, one is led to the introduction of general coordinate systems for the local description of the space-time continuum.

General relativity rests, then, on the following assumptions:

I. General covariance. The laws of physics have the same form in all coordinate systems.

II. Local validity of special relativity. The laws of special relativity hold locally in a coordinate system with vanishing gravitational field.

To these must be added

III. The precise form of the field equations.

Let me begin the analysis with assumption II.

First of all it is meant to imply the existence, in the neighborhood of a given point in the space-time continuum, of a frame of reference in which no gravitational field is present. In this Lorentz frame we have a Minkowskian interval

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2 \tag{2.1}$$

which is measured by "ideal" clocks or measuring rods according as $ds^2 > 0$ or $ds^2 < 0$. In an arbitrary coordinate system this interval takes the form

$$ds^2 = g_{ik} dx^i dx^k \quad (g_{ik} = g_{ki}) \tag{2.2}$$

the components g_{ik} of the metric tensor being functions of the coordinates.[7] Here is the origin of the Riemannian structure of the space-time continuum, which is fundamental for general relativity.

Secondly, in a Lorentz frame the world line of a test particle is a time-like straight line in the absence of external forces, so that $d^2x^i/ds^2 = 0$. For a general coordinate system this leads to the *principle of the geodesic line*: In a purely gravitational field, the world line of a test particle is given by the equations

$$\frac{d^2x^i}{ds^2} + \Gamma_{jk}{}^i \frac{dx^j}{ds}\frac{dx_k}{ds} = 0 \tag{2.3}$$

$$g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 1. \tag{2.3a}$$

Here, $\Gamma_{jk}{}^i$ are the coefficients of the *affine connection* defined by the metric tensor,

$$\Gamma_{jk}{}^i = \frac{1}{2} g^{im}\left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right) \tag{2.4}$$

and s/c is the proper time of the particle. In a given coordinate system, the gravitational field (which determines the motion of a particle) is determined by

[6] See reference 15, also 16(a), (b).

† *Note added in proof.*—The July issue of the *Reviews of Modern Physics* will contain a number of papers on various aspects of general relativity, in particular on the quantization of covariant theories, a question which is only briefly discussed in part IV of the present article.

[7] In a very interesting paper C. Møller discusses the question to what extent real physical systems may approximate "ideal" clocks (reference 17, bibliography).

the coefficients Γ and, hence, by the derivatives $\partial g_{ij}/\partial x^k$. The g_{ik} themselves act as the potentials of the gravitational field and thus appear in their dual role describing both the metrical properties of the space-time continuum and the dynamical action of gravity.

In the neighborhood of a given point P_0 of the space-time continuum a coordinate system ("geodesic" coordinate system) can be introduced such that at P_0 (1) the line element has the Minkowskian form (2.1) and (2) all coefficients $\Gamma_{jk}{}^i$ vanish (i.e., the g_{ik} are constant in first approximation). This well-known mathematical fact clearly demonstrates the dependence of the gravitational field on the frame of reference, and it also permits us to express the "local validity" of special relativity in a precise mathematical form. The coordinate system referred to in assumption II is such a geodesic coordinate system, for which the gravitational field vanishes at P_0.

With the help of the principle of general covariance the equations of special relativity can then be "translated" into general relativity in an unambiguous way if we assume the general relativistic equations to contain no higher than first-order derivatives of the g_{ik}, and if no new quantities, in addition to the metric tensor, are introduced to describe the system in question.[8]

Examples are Maxwell's equations, the equations of hydrodynamics; furthermore, the equations of motion for a charged particle in an external electromagnetic field, which are obtained by inserting the relativistic Lorentz force on the right-hand side of (2.3). (One might also mention the world line of a light ray, which is a null geodesic as follows from Maxwell's equation.)

2(b) Field Equations of Gravitation

To establish the field equations we must go beyond special relativity. In the case of a weak static gravitational field, (2.3) go over into the Newtonian equations of motion if one sets in first approximation

$$g_{44}\sim 1+2\phi/c^2, \tag{2.5}$$

where ϕ is the classical potential of Newtonian gravitation. The general relativistic equations ought to be obtained by generalizing Poisson's equation

$$\nabla^2\phi=4\pi\gamma\rho$$

(γ = Newtonian gravitational constant, ρ = mass density of the sources of the gravitational field). In view of the special relativistic principle of the equivalence of mass and energy ρ must be replaced by the total energy density of the sources, or rather by the total (symmetric) energy-momentum tensor T_{ij} of the sources. Hence, we are led to a tensor equation of the form

$$H_{ij}=T_{ij}, \tag{2.6}$$

where the components of the symmetric tensor H_{ij} are universal functions (i.e., the same in every coordinate system) of the g_{ik} and their derivatives. To maintain the analogy with Poisson's equation the following—very modest—assumption is added: H_{ij} contains no higher than second-order derivatives of the g_{ik}, and it is linear in the second derivatives. The mathematical analysis shows that H_{ij} has then necessarily the form

$$H_{ij}=a_1R_{ij}+a_2g_{ij}R+a_3g_{ij},$$

where R_{ij} is the contracted curvature tensor, R the curvature scalar, and the a_i are constants.[9]

In a Lorentz frame the energy-momentum tensor satisfies the conservation law,

$$\partial T^{ij}/\partial x^j=0, \tag{2.7}$$

which implies for a general coordinate system

$$T^{ij}{}_{;j}=0. \tag{2.8}$$

Here, and in the sequel, the semicolon indicates covariant differentiation. In view of the assumed field equations (2.6) we must then have $H^{ij}{}_{;j}=0$, which is generally true if and only if $a_2=-\frac{1}{2}a_1$, for in every Riemannian space the curvature tensor satisfies the "Bianchi" identities

$$G^{ij}{}_{;j}=0 \quad (G_{ij}=R_{ij}-\tfrac{1}{2}g_{ij}R). \tag{2.9}$$

Dividing by a_1, we obtain finally

$$G_{ij}+\Lambda g_{ij}=-\kappa T_{ij} \tag{2.10}$$

(with $\Lambda=a_3/a_1$, $\kappa=-1/a_1$). Comparison with Poisson's equation for a weak gravitational field (produced by a mass distribution of low density) shows that

$$\kappa=8\pi\gamma/c^4$$

if T_{ij} has the dimension of an energy density.

This completes the derivation of the field equations to the extent that they are determined by the assumptions made. The cosmological constant Λ—which has the dimension $1/(\text{length})^2$—remains undetermined. Einstein introduced it—reluctantly—in 1917 in order to account for a static universe on the basis of general relativity, but later, particularly when the idea of a static universe had to be abandoned, he strongly advocated discarding it. I believe that many physicists share this view today. Although the astronomers will probably have the last word on this question, I may indicate some of the objections: (1) A universal length $\Lambda^{-\frac{1}{2}}$ is introduced whose value is in no way determined by the theory. (2) Even in the absence of any sources ($T_{ij}=0$) the Minkowskian metric form is no longer a solution of the field equations.—In any event, the length $\Lambda^{-\frac{1}{2}}$ has cosmic dimensions so that Λ may be disregarded in all applications outside of cosmology. Accordingly, I shall omit it throughout most of the following discussion.

[8] Only if such a qualification is added does the principle of general covariance have an unambiguous meaning. [See also the discussion in 1(a).]

[9] See Pauli, reference 2 (bibliography) art. 17, p. 598.

In a complete description of a physical system, T_{ij} will depend on other field variables (electromagnetic, hydrodynamic, etc.) which in turn satisfy an additional set of field equations (Maxwell's, Euler's etc.)

2(c) Consequences of the Covariance of the Field Equations

The covariance of the relativistic field equations has striking consequences. I shall confine myself to the gravitational equations, but what I have to say holds—with minor modifications—for any covariant set of equations, in particular for any general relativistic unified field theory.

Continuation properties[10]

Consider a pure gravitational field ($T_{ij}=0$) with vanishing cosmological constant. Then

$$G_{ij}=0.$$

Given the "initial values" g_{ik} and $\partial g_{ik}/\partial x^4$ on the hypersurface S which, in a suitable coordinate system, has the equation $x^4=0$. To what extent do the field equations determine the continuation of the g_{ik} in the x^4 direction? The covariance of the equations implies immediately that the continuation cannot be *uniquely* determined, for if $g_{ik}(x)$ is a solution with the given initial values, so is any field $g'_{ik}(x)$ obtained from $g_{ik}(x)$ by a (continuous) change of the coordinate system outside S.

If, everywhere on S, $g_{44}\neq 0$ (geometrically: if S nowhere touches the local light cone $g_{ik}dx^idx^k=0$) the covariance of the theory may also be utilized as follows: Assuming the x^4 direction time-like, we may introduce a coordinate system in which

$$g_{44}=g^{44}=1; \quad g_{4\alpha}=g^{4\alpha}=0 \quad (\alpha=1,2,3).$$

Then we are left with ten equations ($G_{ik}=0$) for only six unknown functions $g_{\alpha\beta}(\alpha,\beta=1,2,3)$. One finds that six equations (*viz.*, $G_{\alpha\beta}=0$) determine the continuation in the x^4 direction, i.e., they may be solved for the derivatives $\partial^2 g_{\alpha\beta}/(\partial x^4)^2$, while the remaining four ($G_{4\alpha}=0$, $G_{44}=0$) are conditions to be satisfied on a section $x^4=\text{const}$.

Finally, once the conditions $G_{4\alpha}=0$, $G_{44}=0$ are satisfied for $x^4=0$ and the $g_{\alpha\beta}$ are determined from the six equations $G_{\alpha\beta}=0$, the field equations must insure the validity of $G_{4\alpha}=0$, $G_{44}=0$ also outside the hypersurface S. That this is indeed the case follows from the Bianchi identities $G^{ij}{}_{;j}=0$ [see Eq. (2.9)] which, in the coordinate system we have chosen, imply $\partial\mathfrak{G}_{4\alpha}/\partial x^4=0$, $\partial\mathfrak{G}_{44}/\partial x^4=0$, where $\mathfrak{G}^{ij}=|g|^{\frac{1}{2}}G^{ij}$ (g=determinant of the g_{ik}).

It should be emphasized that these are purely local considerations. Strictly speaking, they hold for a sufficiently small part of S and for sufficiently small values of x^4. Little is known on the very difficult question of how far the g_{ik} may be continued free of singularities.

Identities and Conservation Laws[11]

The covariance of the field equations, by itself, does not insure the existence of the necessary identities. But if the field equations are derived from a covariant variational principle, the identities as well as the conservation laws may be deduced. This fact demonstrates the importance of a variational principle for covariant field equations.

Einstein's gravitational equations (2.10) are obtained from

$$\delta\int\mathfrak{L}d^4x=0; \quad \mathfrak{L}=\mathfrak{R}-\Lambda|g|^{\frac{1}{2}}-\kappa\mathfrak{L}_s. \tag{2.11}$$

Here, g is the determinant of the g_{ik}, $|g|$ its absolute value, $\mathfrak{R}=|g|^{\frac{1}{2}}R$, and $\mathfrak{L}_s$ is the nongravitational part of the Lagrangian, the subscript s referring to the sources of the gravitational field. For pure electromagnetism,

$$\mathfrak{L}_s=\tfrac{1}{2}|g|^{\frac{1}{2}}f_{ik}f^{ik}; \quad f_{ik}=\frac{\partial\phi_i}{\partial x^k}-\frac{\partial\phi_k}{\partial x^i}. \tag{2.11a}$$

The quantities to be varied are (1) the components g_{ik} of the metric tensor and (2) whatever additional field variables are contained in $\mathfrak{L}_s$ (e.g., the four components of the vector potential in the electromagnetic case). Variation of the g_{ik} yields the gravitational equations, variation of the other variables the remaining field equations (for example, Maxwell's equation). As usual, the variations of the field variables (and, if necessary, of some of their derivatives) are supposed to vanish at the boundary of the fixed four-dimensional region over which the integration is extended.

I shall illustrate the procedure for the derivation of identities for the pure gravitational field, i.e., for $\mathfrak{L}=\mathfrak{R}$ (setting $\Lambda=0$), the essential fact being that $\mathfrak{R}$ is a scalar density.[12]

We first define the following transformation of fields. (1) Assume that with every point x of some region in space-time a point y is correlated by the equations

$$y^i=y^i(x) \quad \text{or} \quad y=y(x).$$

(2) This point transformation induces a corresponding transformation of scalars, vectors, etc.: A scalar field

[10] See reference 3 (bibliography), Chap. II, in particular pp. 31–32.

[11] See reference 2 (bibliography), art. 23.

[12] While the essential features of the method are correctly described here the particular form (B), (C) of the conservation laws is obtained if we follow the customary procedure of replacing $\mathfrak{R}$ by a Lagrangian $\mathfrak{L}'$ which contains only first-order derivatives of the g_{ik}. $\mathfrak{L}'$ is formed by subtracting from $\mathfrak{R}$ a suitable divergence: $\mathfrak{L}'=\mathfrak{R}-\partial\mathfrak{R}^i/\partial x^i$. [See reference 2, art. 23 or reference 4, p. 196 (bibliography).] $\mathfrak{L}'$ is no longer a scalar density, but the variation $\delta\mathfrak{L}'$ induced by (2.14) is still a divergence. The comparison of (2.14a) and (2.14b) remains the crucial step, but the variation of $\mathfrak{R}$ must now be expressed by the variation of $\mathfrak{L}'$ and of $\mathfrak{R}^i$.

$\phi(x)$ is transformed into

$$\phi'(x)=\phi(y(x)),$$

a covariant vector field $v_i(x)$ into

$$v_i'(x)=\frac{\partial y^k}{\partial x_i}v_k(y(x)),\tag{2.12}$$

contravariant vectors and tensors of all ranks and types being transformed accordingly.

$\mathfrak{R}$ is a universal function of the g_{ik} and their derivatives. If the "transformed" values $g'_{ik}(x)$, $\partial g'_{ik}/\partial x^l$ etc., are inserted, one obtains

$$\mathfrak{R}'(x)=\mathfrak{R}(g_{ik}',\partial g'_{ik}/\partial x^l,\cdots)=J\cdot\mathfrak{R}(y(x))\tag{2.13}$$

(J is the Jacobian $\partial(y^1,\cdots,y^4)/\partial(x^1,\cdots,x^4)$) because $\mathfrak{R}$ is a *scalar density*. Incidentally, it is precisely at this point that the covariance of the theory is used.

Introducing a family of transformations

$$y^i(x,\epsilon)=x^i+\epsilon\xi^i(x),$$

we find that, for a given choice of the functions ξ^i, the transformed field variables are turned into functions of x and ϵ. Their variations are the first-order terms of their expansions in powers of ϵ, thus [from (2.12)]

$$\begin{aligned}\delta v_i&=\epsilon\frac{\partial}{\partial\epsilon}v_i'(x,\epsilon)\,|_{\epsilon=0}=\epsilon\left(\frac{\partial\xi^k}{\partial x^i}v_k+\frac{\partial v_i}{\partial x^k}\xi^k\right)\\ \delta g_{ij}&=\epsilon\left(\frac{\partial\xi^k}{\partial x_i}g_{kj}+\frac{\partial\xi^k}{\partial x^j}g_{ik}+\frac{\partial g_{ij}}{\partial x^k}\xi^k\right).\end{aligned}\tag{2.14}$$

The variation of $\mathfrak{R}$ may now be expressed in two ways. On one hand,

$$\delta\mathfrak{R}=\frac{\partial\mathfrak{R}}{\partial g_{ij}}\delta g_{ij}+\frac{\partial\mathfrak{R}}{\partial g_{ij,\,k}}\frac{\partial}{\partial x^k}(\delta g_{ij})+\cdots\qquad\left(g_{ij,\,k}=\frac{\partial g_{ij}}{\partial x^k}\right)$$

or, using the standard device of variational calculus,

$$\delta\mathfrak{R}=-\mathfrak{G}^{ij}\delta g_{ij}+\frac{\partial\mathfrak{B}^i}{\partial x^i}\tag{2.14a}$$

($\mathfrak{G}^{ij}$ are the left-hand sides of the field equations, $\mathfrak{B}^i$ are expressions linear in the variations δg_{ij} and their derivatives.) On the other hand, since $\mathfrak{R}$ is a scalar density [see (2.13)],

$$\delta\mathfrak{R}=\epsilon\left(\frac{\partial\xi^i}{\partial x^i}\mathfrak{R}+\frac{\partial\mathfrak{R}}{\partial x^i}\xi^i\right)=\epsilon\frac{\partial}{\partial x^i}(\xi^i\mathfrak{R}).\tag{2.14b}$$

The last two expressions for $\delta\mathfrak{R}$ coincide for every choice of ξ^i if the variations (2.14) are inserted in (2.14a). Equating the coefficients of the ξ^i and of their derivatives of various orders in (2.14a) and (2.14b) one obtains, therefore, a series of relations which hold *identically*, i.e., whether or not the field equations are satisfied.

In the case of $\mathfrak{R}$ these relations may be reduced to the following three sets:

$$\text{(A)}\quad \mathfrak{G}_i{}^j{}_{;j}\equiv\frac{\partial\mathfrak{G}_i{}^j}{\partial x^j}-\frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\mathfrak{G}^{jk}\equiv 0.\qquad(\mathfrak{G}_i{}^j=|g|^{\frac{1}{2}}G_i{}^j)$$

$$\text{(B)}\quad \frac{\partial}{\partial x^j}(\mathfrak{G}_i{}^j-\mathfrak{U}_i{}^j)\equiv 0.$$

$$\text{(C)}\quad\begin{cases}\mathfrak{G}_i{}^j-\mathfrak{U}_i{}^j\equiv-\dfrac{\partial\mathfrak{A}_i{}^{jk}}{\partial x^k},\\[2ex] \mathfrak{A}_i{}^{jk}=-\mathfrak{A}_i{}^{kj}=|g|^{-\frac{1}{2}}g_{il}\dfrac{\partial}{\partial x^m}(\mathfrak{g}^{jl}\mathfrak{g}^{mk}-\mathfrak{g}^{jm}\mathfrak{g}^{lk}).\end{cases}$$

I turn now to the interpretation of these relations. (A) are the Bianchi identities whose significance for the field equations I have discussed already. The equations (B) constitute *conservation laws* for energy and momentum. (The $\mathfrak{U}_i{}^j$ are quadratic in the gravitational field).[13] Using the gravitational field equations (with $\Lambda=0$)

$$\mathfrak{G}_i{}^j=-\kappa\mathfrak{T}_i{}^j=-\kappa|g|^{\frac{1}{2}}T_i{}^j$$

we may rewrite (B) in the form

$$\frac{\partial\mathfrak{S}_i{}^j}{\partial x^j}=0;\quad \mathfrak{S}_i{}^j=\mathfrak{T}_i{}^j+\mathfrak{t}_i{}^j,\tag{2.15}$$

where $\mathfrak{T}_i{}^j$ is the energy-momentum density of the gravitation producing field, while $\mathfrak{t}_i{}^j=-\kappa^{-1}\mathfrak{U}_i{}^j$ may be considered the gravitational part of the total energy-momentum density. The $\mathfrak{t}_i{}^j$ form a "pseudo-tensor" density, i.e., they transform like the components of a tensor density only under *linear* coordinate transformations. (At any given point in space-time, the $\mathfrak{t}_i{}^j$ vanish in a geodesic coordinate system, in which the gravitational field vanishes.)

Regarding the equations (C) we see, first of all, that the conservation equations (B) follow from them, in view of the antisymmetry of $\mathfrak{A}_i{}^{jk}$ in j and k. Furthermore, they greatly facilitate and clarify the transition to the conservation laws for total energy and momentum. These may be defined for an isolated system whose metric is "asymptotically Minkowskian," so that, in a suitable frame of reference, the g_{ik} approach the special relativistic values sufficiently fast as $r=((x^1)^2+(x^2)^2+(x^3)^2)^{\frac{1}{2}}\rightarrow\infty$. Then the integrals

$$J_i=\int\mathfrak{S}_i{}^4d^3x$$

(extended over a three-dimensional section $x^4=\text{const}$) exist, and in virtue of the local conservation laws $\partial\mathfrak{S}_i{}^j/\partial x^j=0$ they are independent of the "time" x^4.

[13] Explicit expressions for $\mathfrak{U}_i{}^j$ in reference 2 (bibliography), p. 622, Eq. (185).

It follows from (C) that then

$$J_i=\lim_{a\to\infty}\left(\kappa^{-1}\int_{S_a}\mathfrak{A}_i^{\beta 4}n_\beta d\sigma\right)\quad(\beta=1,2,3).$$

(S_a is the "sphere" $r=a$, and n_β are the components of its outward normal.) These last integrals are, in many cases, easy to evaluate, and they show explicitly that total energy and momentum are uniquely determined by the values of the g_{ik} for large r, irrespective of the specific nature of the system considered.[14] Likewise, if we apply a coordinate transformation which does not change the asymptotically Minkowskian character of the metric and is thus asymptotically (for large r) a Lorentz transformation, the J_i themselves undergo this same Lorentz transformation.[15]

2(d) Problem of Motion

Of the later developments in the theory of general relativity proper (i.e., excluding for the present the attempts at constructing a unified field theory) I shall mention only the analysis of the problem of motion, which is of considerable interest.

This problem concerns two related, but separate, questions. The first is the motion of a test particle (a particle of vanishing mass) in a given gravitational field. The second is the n-body problem where the task is to find both the gravitational field produced by n particles and the motion of these particles. I shall briefly review some of the more recent work in this direction.

Motion of a Test Particle

We have seen that the principle of the geodesic line is one of the assumptions of the theory, adopted from special relativity. Rather early the question was raised whether it could be derived from the field equations. Infeld and Schild[16] treat this problem as follows: Given a gravitational field g_{ik} free from singularities. The world line L_0 of a test particle is regarded as a limiting case (for $m\to 0$) of the world line $L_{(m)}$ of a particle of mass m, which is treated as a singularity of the field. Thus, no energy-momentum tensor (T_{ij}) need be introduced, and outside $L_{(m)}$ the free field equations $G_{ij}=0$ (or $R_{ij}=0$) are satisfied. Both the field and the world line $L_{(m)}$ depend on m. As $m\to 0$, it is assumed that the field converges to the given field g_{ik}, and that the world line approaches a limiting position L_0, the test particle's world line. Infeld and Schild show that L_0 is necessarily a *geodesic line* of the metric g_{ik}.

n-Body Problem

This problem, which is considerably more comprehensive and also more difficult, has been investigated by two groups: By Einstein and his co-workers[17] (in particular Infeld), and by Fock[18] and some of his students. Both groups use approximation methods adapted to the assumption that, in a suitable frame of reference, the velocities are small compared to the velocity of light and that the gravitational field is relatively weak (so that the Minkowskian metric may serve as a starting point for the approximation procedure). In addition, both assume an asymptotically Minkowskian metric, hence treat the n bodies as an isolated system.

Einstein treats the n bodies as point particles, i.e., as singularities of the gravitational field and has, therefore, merely to satisfy the free field equations ($R_{ik}=0$) outside the singularities, so that he can avoid the introduction of a specific energy-momentum tensor. In addition, it is explicitly assumed that each singularity has the character of a pole—not that of a dipole or higher pole—and that no spontaneous radiation is emitted. The masses of the particles appear as integration constants, and it follows from the analysis that they are time independent. The first approximation yields the Newtonian potential and the Newtonian law of motion. The second approximation yields additional terms, from which, for example, the advance of the perihelion of a planetary orbit may be derived.

Fock works from the start with extended bodies described by a suitable energy-momentum tensor T_{ij} and deals thus with a combined gravitational and hydrodynamical problem. If the linear dimensions of the n bodies are small compared to their mutual distances, he obtains the same results as Einstein. He has, however, gone further by allowing not only translatory but also rotatory motion, and has also investigated the (gravitational) radiation problem in detail.

2(e) Mathematical Investigations of the Free Field Equations

Since the gravitational field equations are obtained by generalizing Poisson's equation

$$\nabla^2\phi=4\pi\gamma\rho$$

the energy-momentum tensor replacing the material

[14] For a static field whose metric form coincides with the Schwarzschild line element for large r one finds $J_4=mc^2$, $J_i=0 (i\neq 4)$. Here m is the mass constant that appears in the Schwarzschild line element.

[15] The connection between the covariance of the field equations and the identities and conservation laws was recognized very early. The method described here was developed and studied by Einstein, Hilbert, Klein, Lorentz, and others. [For full references see Pauli, art. 23.) But it was not until 1939 that an explicit form for (C) was first derived (by Freud, reference 18 (bibliography).] The particularly simple expressions used here are due to Landau and Lifschitz [reference 5 (bibliography), Eq. (11–78)].—For recent investigations on this subject see references 35a, 19, 20, 33 (bibliography).

[16] See reference 21 (bibliography).

[17] See reference 22 (bibliography).

[18] The whole Chapter VI of Fock's book (reference 6, bibliography) is devoted to the problem of motion.

density ρ, one may expect similar results for them, in particular, that for vanishing T_{ij} no stationary gravitational field exists. I refer, of course, to a field free of singularities. This has been proved under very general assumptions. The strongest results are due to Lichnérowicz.[19] They relate to a field which, in a suitable frame of reference, is independent of the time like variable x^4. The assumptions on the three-dimensional space S of the coordinates x^1, x^2, x^3 are of two kinds: Either S is *closed*, in which case no further conditions are needed; or S is *infinite* (topologically Euclidean) and, in addition, *asymptotically Minkowskian*. In both cases the Riemann curvature tensor vanishes, and the space-time continuum is (at least locally) Minkowskian, although for a closed S it is, of course, topologically different from the Minkowskian world of special relativity.

Very little is known on the nonstationary case. (The classical analog would be the wave equation—instead of Laplace's equation—which does admit solutions with the proper asymptotic behavior.) This question certainly deserves further study. It would be highly interesting to know what kind of solutions, if any, are possible.

In this context I should like to mention Wheeler's work,[20] which deals no longer with the pure gravitational field, but with the Einstein-Maxwell theory, the combined gravitational and electromagnetic equations for a field free of charges and currents. Wheeler constructs solutions, his "geons" (short for gravitational-electromagnetic entities), which have the properties of "bodies": The electromagnetic field remains confined to a finite region under the gravitational attraction produced by its own energy density. The mathematical analysis is partly based on approximations, but it has been made highly probable that rigorous solutions of this kind exist.

Due to the "weakness" of gravitational interaction the ratio of the linear dimension R and the mass M of a geon has the uncomfortably small order of magnitude

$$\gamma/c^2 = 0.74\times10^{-28}\ \text{cm/g}$$

so that either the extension or the density of a geon is tremendous.

2(f) Verification of General Relativity by Astronomical Observations[21]

There are the three classical astronomical effects. (1) The observations on the advance of the Mercury perihelion are now in very good agreement with the theory. (2) As to the deflection of light in the gravitational field of the sun, it is generally admitted that an effect of the predicted order of magnitude exists, but there is no unanimity among astronomers as to whether the theory is quantitatively confirmed.[22] (3) Finally, there is the gravitational shift of spectral lines. The theory predicts

$$(\nu_0-\nu_e)/\nu_e = (\phi_e-\phi_0)/c^2.$$

Here, ν_e and ν_0 are emitted and observed frequencies, respectively. The emitting light source and the observer are assumed at rest in a static gravitational field. ϕ_e and ϕ_0 are the corresponding values of the Newtonian potential, which provides an excellent approximation in all cases of interest. For a star, $\phi_e = -\gamma M/R$ (M = mass, R = radius of the star), while the terrestrial ϕ_0 is negligible. Thus we have the red shift

$$(\nu_0-\nu_e)/\nu_e = -\gamma M/Rc^2$$

which for the sun has the order of magnitude 2×10^{-6}. For rays coming from the limb of the solar disk, the observed red shift has the theoretical value, but it is smaller for all other rays from the sun. Although an explanation for this behavior may be given, this is still an inconclusive result. The theory seems best confirmed for some stars of very high density (e.g., the companion of Sirius), for which the red shift should indeed be considerably bigger than it is for the sun. Unfortunately, in these cases M and R are not very accurately known.

It may be that in the not too distant future terrestrial experiments will, at long last, become possible. For an observer on the earth, and the emitting light source at rest at height h, we have a "violet" shift

$$(\nu_0-\nu_e)/\nu_e = gh/c^2$$

(g = gravitational acceleration on the earth's surface). If $h = 1$ km, this shift is of the order 10^{-13}, still outside the range of the most accurate detecting devices. Here the artificial satellites[23] may provide the solution in magnifying the effect by a factor 10^3. (The Doppler effect due to the satellite's motion changes the shift into a red shift, as long as h is smaller than half the earth's radius, but does not appreciably alter the order of magnitude of the effect.)

Unfortunately, I do not have the time to talk here about the field of cosmology, the field on which general relativity has had by far the most fruitful and stimulating effect and which is now advancing so rapidly, due to the recent extraordinary achievements of astronomy. Let me stress just one fact. For a long time, as you know, the observed value of the Hubble constant was very disquieting because the age of the universe appeared smaller than the age of the earth's crust as determined from radioactive measurements. During the last few years, however—mainly through Baade's work[24]—the distance scale of distant nebulae has more

[19] See reference 3, Chap. VIII (bibliography).
[20] See reference 23 (bibliography).
[21] For this whole section compare Ginsburg's excellent survey, reference 24 (bibliography).
[22] See R. Trumpler's report to the Berne conference, reference 11 (bibliography). Trumpler considers the observational data in good agreement with the theory.
[23] See references 24 and 25 (bibliography).
[24] See W. Baade's report to the Berne conference, reference 11 (bibliography).

than doubled. According to recent determinations of the Hubble constant, $1/H$ is of the order of 5×10^9 years. If this is confirmed, then the main difficulty will have disappeared.

Outside of cosmology, the impact of general relativity on the rest of physics has not been nearly so great as that of special relativity. One of the reasons is undoubtedly the smallness of the observable effects predicted by the theory and, generally speaking, the weakness of gravitational interactions, particularly in comparison with other interactions between elementary particles. The smallness of direct gravitational effects, however, does not prove that the principles of general relativity are unimportant even for microscopic phenomena. If all frames of reference are equivalent with respect to the laws of macrophysics, it is hard to believe that this equivalence has no counterpart in microphysics, even though the mathematical form in which the equivalence is expressed may greatly differ from the customary one.

To quote Klein[25]: "In atomic physics there has been a tendency to neglect the importance of general relativistic invariance on account of the small influence of gravitational forces on ordinary phenomena. Quite apart from the fact that gravitation will probably be important at very small distances[26] such a point of view would seem no more justified than would be the neglect of atomic spin in atomic structure because in most problems the direct action of spin magnetic moments is very small."

3. UNIFIED FIELD THEORIES

I shall now take up the attempts to go beyond the theory of general relativity in its original form. There are two main trends: on one hand, the search for a unified field theory, on the other, the attempts to quantize the gravitational equations.

Throughout the years many unified theories have been suggested and investigated. It is not my intention to consider them in any detail; I shall rather try to characterize some of the main ideas and procedures which have been developed.

General relativity, as we have seen, leads in an essentially unambiguous way only to a theory of the gravitational field. Accordingly, in the field equations,

$$G_{ij}=-\kappa T_{ij},$$

only the left-hand side is determined. General relativity can accommodate virtually any additional field and its energy-momentum tensor, but it provides no method for choosing among the various possibilities.

When Weyl, in 1918, developed the first unified field theory[27] the unification of various fields of physics had progressed very far. The study of atomic phenomena, in particular, had revealed the predominance of electromagnetic interactions, and it seemed probable that in the last analysis all physical forces could be reduced to electromagnetic and gravitational ones. It was therefore very tempting to complete the synthesis by combining gravitation and electromagnetism in a more comprehensive entity. The unification was to achieve two things: (1) to deduce, at least in principle, all physical interactions from one law, (2) to modify the field equations in such a way that they would admit solutions corresponding to stable charged particles.

In the course of time, specifically after the advent of quantum mechanics, the program which I have sketched has undergone many changes and has been interpreted in many different ways. There was the possibility of interpreting the "particles" either as part of the field or as genuine point singularities; there was the possibility of interpreting a unified field theory either as a c-number theory later to be quantized or as a field theory which implicitly contained the quantum laws already. Also the view that the gravitational and electromagnetic fields were the only fundamental ones was no longer generally accepted.

So far, none of these theories has actually attained the objectives of the original program. To a large extent the work has been concerned with the search for a mathematical structure—more comprehensive than the four dimensional Riemannian space—on which to base a unified or generalized field theory, even a theory that would effect only a partial unification.

While in general relativity the equivalence principle and the principle of general covariance lead directly to the metric structure of space-time and also to the explicit form of the field equations, no such general principles are available to guide the search for a more comprehensive structure. As a result, the search has proceeded in quite different directions.

I shall discuss here only two types of theories, both of which have been actively investigated in recent years.

3(a) Five-Dimensional Theory of Kaluza-Klein[28]

This theory starts from the following observation, which is far from trivial. The fourteen field variables—namely, the ten g_{ik} and the four components ϕ_i of the electromagnetic potential—and the fourteen field equations that describe the gravitational and electromagnetic fields in a four-dimensional space-time continuum S_4 may be interpreted in terms of a suitable *five-dimensional* Riemannian space S_5.

S_5 is characterized by the following property: In an appropriate coordinate system all components $\bar{g}_{\alpha\beta}$ of

[25] Reference 32, p. 17 (bibliography).

[26] Landau points out that for this reason quantum electrodynamics may not be considered a closed system, because for very high cut-off momenta (i.e., for very small distances) gravitational effects can no longer be neglected. See reference 26, p. 60 (bibliography).

[27] See reference 7 (bibliography). This theory was later abandoned by Weyl and will not be considered here.

[28] The original papers are references 27 and 28 (bibliography).

its metric tensor[29] are *independent* of the fifth coordinate x^5.

Through every point P of S_5 passes a curve $C: x^i = \text{const}$ $(i=1,\cdots,4)$, and any vector a^α at P may be decomposed into two vectors $a_\perp$ and $a_\parallel$ which are, respectively, parallel and perpendicular to C. For their lengths $l_\parallel$ and $l_\perp$ one obtains

$$l_\parallel^2 = (\bar{g}_{\alpha 5} a^\alpha)^2 / \bar{g}_{55};$$
$$l_\perp^2 = \bar{h}_{ij} a^i a^j; \quad \bar{h}_{ij} = \bar{g}_{ij} - \bar{g}_{i5}\bar{g}_{j5}/\bar{g}_{55}.$$

The correspondence of the field variables in S_4 with the metric in S_5 is then defined by

$$\begin{aligned} g_{ij} &= \bar{h}_{ij} \quad (i,j=1,\cdots,4), \\ 2^{\frac{1}{2}}\phi_i &= \bar{g}_{i5}/\bar{g}_{55} \quad (i=1,\cdots,4). \end{aligned} \tag{3.1}$$

This identification[30] is justified by the following facts:

Transformation Properties

There are two types of coordinate transformations in S_5 which leave the curves C unchanged:

$$x^{i\prime} = F^i(x^1,\cdots,x^4); \quad x^{5\prime} = x^5; \tag{3.2a}$$

$$x^{i\prime} = x^i; \quad x^{5\prime} = x^5 + \zeta(x^1,\cdots,x^4). \tag{3.2b}$$

Under (3.2a), $\bar{h}_{ij}$, ϕ_i, $\bar{g}_{55}$ transform respectively like a tensor, a vector, and a scalar in four-dimensional space. Under (3.2b), $\bar{h}_{ij}$ and $\bar{g}_{55}$ are unchanged, but ϕ_i is subjected to a *gauge transformation*

$$\phi_i' = \phi_i - 2^{-\frac{1}{2}}\partial\zeta/\partial x^i.$$

Field Equations

In the original form of the theory Kaluza added the condition

$$\bar{g}_{55} = -1,$$

thus reducing the number of independent $\bar{g}_{\alpha\beta}$ to fourteen, the number of field variables. Under this assumption one finds for $|\bar{g}|^{\frac{1}{2}}\bar{R}$ ($\bar{R}$ the curvature scalar of S_5, and $\bar{g}$ the determinant of the metric tensor) the expression

$$|\bar{g}|^{\frac{1}{2}}\bar{R} = |g|^{\frac{1}{2}}(R - \tfrac{1}{2}f_{ik}f^{ik}) \tag{3.3}$$

$(f_{ik} = \partial\phi_i/\partial x^k - \partial\phi_k/\partial x^i)$, i.e., precisely the Lagrangian $\mathfrak{R} - \mathfrak{L}_s$ of the Einstein-Maxwell theory [see Eq. (2.11a)] in units where $\kappa = 1$. The field equations themselves may be expressed by the contracted curvature tensor of S_5. The gravitational equations read

$$\bar{G}^{ij} \equiv G^{ij} + T^{ij} = 0$$

(where T^{ij} is the Maxwellian energy-momentum tensor), and Maxwell's equations are

$$\bar{G}_5{}^i \equiv 2^{-\frac{1}{2}} f^{ij}{}_{;j} = 0.$$

The world line of a charged test particle in four-dimensional space-time corresponds to a geodesic in S_5. Such a geodesic intersects each curve C it meets at a fixed angle which is determined by the specific charge e/m of the test particle.

So far the "five-dimensional" theory merely provides a reinterpretation of the Einstein-Maxwell theory in quasi geometrical terms without changing its content in any way.[31] Nevertheless, the relations discovered by Kaluza and Klein have repeatedly attracted the attention of mathematicians and physicists, and they have suggested genuine generalizations.

The theory of Jordan-Theory.[32] If Kaluza's assumption $\bar{g}_{55} = -1$ is dropped a fifteenth variable, say, $\chi = -\bar{g}_{55}$, which is a scalar in four-dimensional space-time, is introduced into the theory. What is its significance? The expression for the curvature scalar $\bar{R}$ changes from (3.3) to

$$\bar{R} = R - \tfrac{1}{2}\chi f_{ik}f^{ik} + g^{ik}(\chi_i/\chi)_{;k}. \tag{3.4}$$

$(\chi_i = \partial\chi/\partial x^i)$, which suggests the interpretation of χ as a *variable* gravitational constant. On this basis Jordan has developed an extensive theory. It is based on a variational principle whose Lagrangian is slightly more general than $|\bar{g}|^{\frac{1}{2}}\bar{R}$ and contains two adjustable constant parameters. In addition to the fourteen Einstein-Maxwell equations—which are, of course, modified—there appears a fifteenth field equation for χ.

Dirac[33] first advanced the idea of a variable gravitational constant. He argued, roughly, as follows. For two electrons the ratio of the Coulomb force and the Newtonian gravitational force is the dimensionless number $e^2/\gamma m^2$ of the order 10^{42}. It is highly unlikely that a constant of this size may be explained by any theory.

On the other hand, numbers of this order are obtained if astronomical magnitudes are expressed in atomic units (for example, m_0 = proton mass, l_0 = electron radius, $t_0 = l_0/c$). Thus, we find $\sim 10^{40}$ for the present age of the universe (of some billion years), $\sim 10^{-40}$ for the average material density in the universe, etc. The difficulty of explaining the value of $e^2/\gamma m^2$ disappears, therefore, if γ is not a constant but inversely proportional to the age of the universe, while the atomic units remain unchanged. This argues for a time dependence of the gravitational constant, and in a relativistic theory γ must then be treated as a function of space-time.

In a very interesting critical analysis Fierz[34] has pointed out that Jordan's theory in its present form (which includes only the pure electromagnetic field) does not yet lead to an unambiguous interpretation of

[29] In Sec. 3(a), barred quantities refer to the metric in S_5. Greek indices run from 1 to 5, Latin indices from 1 to 4.

[30] Kaluza identified the metric of S_4 with $\bar{g}_{ij}$ $(i,j=1,\ldots,4)$ and therefore obtained the Einstein-Maxwell equations only as a first approximation for weak fields. The equations (3.1) are due to Klein.—The sign of $\bar{g}_{55}$ must be so chosen that the curves C are space-like.

[31] This holds also for the mathematically more elegant versions of the theory. The "projective" theory is described in reference 8 (bibliography), Chap. III.

[32] See Jordan's book [reference 8 (bibliography)] and reference 29 (bibliography).

[33] See reference 30 (bibliography).

[34] See reference 31 (bibliography).

either χ as gravitational constant or g_{ij} as the true metric tensor. For an unambiguous interpretation it would be necessary to make specific assumptions about the action on material particles. Depending on these assumptions also the ratios of different atomic units of length may or may not be constant.

Apart from any particular theoretical views it would be very worthwhile to attack the question of the time dependence of γ by experiment comparing "astronomical" and "atomic" times.[35]

Klein's theory.[36]—O. Klein has generalized the original Kaluza-Klein theory in an entirely different way. He drops the condition that the field quantities be independent of x^5, and develops a truly five-dimensional theory. The fifth coordinate has a quantum theoretical significance.

As I have mentioned, the world lines of charged particles correspond to geodesics in Kaluza's S_5. These may be obtained from the Lagrangian

$$L=\bar{g}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \quad (\dot{x}^\alpha=dx^\alpha/d\tau)$$

(τ is an auxiliary parameter), and in suitable units the specific charge e/m of the particle equals the momentum $p_5=\partial L/\partial\dot{x}^5$. The fifth coordinate is thus conjugate to the charge, and in view of the quantization of charge Klein assumes x^5 to be an angular variable, or, in other words, he assumes the field variables to be periodic in x^5. Expanding each field variable in a Fourier series in x^5, one obtains a sequence of Fourier coefficients which are functions of the space-time coordinate x^i. They presumably refer to different particles and are to be quantized. Particles with half-integral spin may also be incorporated in the theory.

3(b) Nonsymmetric Theories of Einstein and Schrödinger[37]

In contrast to the theory of Kaluza and Klein, these theories operate in the ordinary four-dimensional space-time continuum.

While the basic element of the mathematical structure of general relativity is the Riemannian metric ds^2 of the space-time continuum, the mathematical reinvestigation of infinitesimal geometry—which was greatly stimulated by Einstein's theory—has revealed the more fundamental structural element of the affine connection (or "affinity") or the parallel displacement of vectors defined by the equations

$$\delta a^i=-\Gamma_{jk}{}^i a^j dx^k.$$

In the hands of Levi-Cività and of his successors this new concept has, indeed, led to a very remarkable mathematical progress. It is more fundamental because it remains applicable if no metric is defined. The curvature tensor, in particular, is directly defined in terms of the Γ's.

Quite generally one finds that the symmetric part

$${}^s\Gamma_{jk}{}^i=\tfrac{1}{2}(\Gamma_{jk}{}^i+\Gamma_{kj}{}^i) \tag{3.5a}$$

of an affinity is itself an affinity, while the antisymmetric part

$${}^a\Gamma_{jk}{}^i=\tfrac{1}{2}(\Gamma_{jk}{}^i-\Gamma_{kj}{}^i) \tag{3.5b}$$

is a tensor. Accordingly, the symmetric and antisymmetric parts transform separately. In a Riemannian space an affinity is uniquely defined by the requirement that it be symmetric and that the length of a vector remain unchanged by parallel displacement [see Eq. (2.4)].

The first who based a unified field theory on the affinity—instead of the metric—was Eddington. (The affinity was assumed *symmetric.*) Both the metric and the electromagnetic field appeared as derived quantities. Eddington did not postulate specific field equations. This was done, in 1923, by Einstein, who derived the field equations from a Lagrangian depending only on the components R_{ik} of the contracted curvature tensor. Soon after, however, Einstein abandoned the theory because the electromagnetic field equations appeared to lead to inadmissible consequences (for example, the vector potential was proportional to the current).

For many years this theory lay dormant until it was revived, about ten years ago, by Einstein and his co-workers and by Schrödinger. The most significant change was the introduction of a nonsymmetric affinity and of a nonsymmetric metric tensor. Schrödinger introduced only the Γ's as primary field variables (obtaining the g_{ij} as derived quantities); Einstein, on the other hand, used both the $\Gamma_{jk}{}^i$ and the nonsymmetric $\mathfrak{g}^{ij}$ as building stones for the theory.

For a nonsymmetric g_{ij} the equations

$$g_{ij}g^{kj}=\delta_i{}^k; \quad \mathfrak{g}^{ij}=|g|^{\frac{1}{2}}g^{ij}$$

define g^{ij} and $\mathfrak{g}^{ij}$. The symmetric and antisymmetric parts of $\mathfrak{g}^{ij}$ are given by

$${}^s\mathfrak{g}^{ij}=\tfrac{1}{2}(\mathfrak{g}^{ij}+\mathfrak{g}^{ji}); \quad {}^a\mathfrak{g}^{ij}=\tfrac{1}{2}(\mathfrak{g}^{ij}-\mathfrak{g}^{ji}). \tag{3.6}$$

In the following, I shall mainly report on Einstein's form of the theory. Since the symmetric and antisymmetric parts of the Γ and of the $\mathfrak{g}^{ij}$ transform separately under coordinate transformations the objection might be raised that no genuine unification is achieved, and that the field equations will not be unambiguously defined by the covariance group of the theory. To the extent that it applies to the affinity Einstein has met this objection in the following way.

Invariance under λ transformations.—It is postulated

[35] The feasibility of such an experiment was discussed by G. M. Clemence and R. H. Dicke at an astrophysical seminar at Princeton (April, 1955).

[36] See reference 32 (bibliography) and Klein's report to the Berne conference.

[37] For the final version of Einstein's theory see reference 33 (bibliography), and "Meaning of Relativity," Appendix II. For Schrödinger's theory see reference 34 (bibliography) and his book, reference 9 (bibliography). An excellent account of the nonsymmetric theories is given in M. A. Tonnelat's book [reference 10 (bibliography)], which contains also an extensive bibliography.

that two affinities Γ and Γ' describe the same field if

$$\Gamma_{jk}{}^{i\prime}=\Gamma_{jk}{}^{i}+\delta_j{}^i\frac{\partial\lambda}{\partial x^k}, \tag{3.7}$$

where λ is an arbitrary coordinate function. This λ transformation, which resembles the gauge transformation of electromagnetism, mixes the symmetric and antisymmetric parts of $\Gamma_{jk}{}^i$; these lose therefore their independent character if the λ transformations are combined with ordinary coordinate transformations to form the full covariance group of the theory. The symmetric and antisymmetric parts of $\mathfrak{g}^{ij}$, however, remain independent.

The λ transformation has a simple geometrical significance. Under two affinities, Γ and Γ', the ratios $a^i:a^j$ of the components of a vector change in exactly the same way if and only if

$$\Gamma_{jk}{}^{i\prime}=\Gamma_{jk}{}^{i}+\delta_j{}^i\zeta_k,$$

where ζ_k is an arbitrary covariant vector. If we require, in addition, that under Γ and Γ' parallel displacement along *closed paths* is the same for the vector components themselves (not only for their ratios) we find as a necessary and sufficient condition that ζ_k is a gradient, i.e., $\zeta_k=\partial\lambda/\partial x^k$, as in (3.7).

Since the curvature tensor measures the change of a vector by parallel displacement along infinitesimal closed paths it follows immediately that the curvature tensor as well as its contractions are unaffected by λ transformations.

In order to facilitate the formulation of the theory Einstein and Kaufman replaced the $\Gamma_{jk}{}^i$ by the combinations

$$U_{jk}{}^i=\Gamma_{jk}{}^i-\delta_k{}^i\Gamma_{jt}{}^t$$

which, by a λ transformation, change into

$$U_{jk}{}^{i\prime}=U_{jk}{}^i+\delta_j{}^i\frac{\partial\lambda}{\partial x_k}-\delta_k{}^i\frac{\partial\lambda}{\partial x^j}.$$

No direct geometrical interpretation has been given for them.

Transposition invariance.—The field equations are to be derived from a variational principle

$$\delta\int\mathfrak{L}d^4x=0.$$

The Lagrangian is a scalar density *invariant under* λ *transformations*, which is a function of the $\mathfrak{g}^{ij}$ and the $U_{jk}{}^i$. The $\mathfrak{g}^{ij}$ and $U_{jk}{}^i$ are to be varied independently.

In order to narrow down the choice of the Lagrangian, Einstein introduced the postulate of transposition invariance: $\mathfrak{L}$ remains unchanged if $\mathfrak{g}^{ij}$ and $U_{jk}{}^i$ are replaced, respectively, by their transposed

$$\tilde{\mathfrak{g}}^{ij}=\mathfrak{g}^{ji};\quad \tilde{U}_{jk}{}^i=U_{kj}{}^i.$$

Presumably, this expresses the symmetry of the theory in positive and negative electric charges. The transposition invariance of $\mathfrak{L}$ implies that the field equations admit, together with a field $\mathfrak{g},U$ also the transposed field $\tilde{\mathfrak{g}}$, $\tilde{U}$ as a solution.

Einstein's choice for the Lagrangian is

$$\mathfrak{L}=\mathfrak{g}^{ij}R_{ij}.$$

It satisfies the conditions of transposition and of λ invariance, and it is a natural generalization of the Lagrangian of general relativity to the nonsymmetric case.

The field equations.—Variation of the $\mathfrak{g}^{ij}$ and of the $U_{jk}{}^i$ yields the two sets of field equations

(A) $R_{ij}=0$

(B) $\dfrac{\partial\mathfrak{g}^{ij}}{\partial x^k}+\Gamma_{sk}{}^i\mathfrak{g}^{sj}+\Gamma_{ks}{}^j\mathfrak{g}^{is}-\Gamma_{ks}{}^s\mathfrak{g}^{ij}+\frac{2}{3}\delta_k{}^j\Gamma_{st}{}^t\mathfrak{g}^{is}=0.$

(A) consists of 16 equations. (B) consists of 64 equations, which immediately imply

$$\partial^a\mathfrak{g}^{ik}/\partial x^k=0. \tag{3.8}$$

Einstein's original equations for a pure gravitational field ($T_{ij}=0$) follow from (A) and (B) for symmetric $\mathfrak{g}$ and Γ. The set (B) is then equivalent to Eq. (2.4), and the set (A) reduces to ten equations, because R_{ij} is then symmetric.

Comparison with Schrödinger's theory.—In Schrödinger's theory the primary field variables are the $\Gamma_{jk}{}^i$, the Lagrangian $\mathfrak{L}$ is a function of the R_{ij}, and the $\mathfrak{g}^{ij}$ are defined as

$$\mathfrak{g}^{ij}=\partial\mathfrak{L}/\partial R_{ij}.$$

Schrödinger chooses $\mathfrak{L}=(2/\Lambda)(-\Delta)^{\frac{1}{2}}$. Λ is a constant, and Δ is the determinant of the R_{ij}. The resulting field equations are similar to Einstein's. The set (B) is unchanged, and (A) is replaced by

(A′) $R_{ij}=\Lambda g_{ij}$

so that Λ plays the role of a cosmological constant.

Content of the theory.—While the mathematical basis of the theory has been perfected, and may now be presented in a simple and transparent form, very little can as yet be said about the content of the theory. Even the interpretation of the field variables is difficult; it cannot be totally separated from the analysis of the field equations and of their solutions.[38]

A few clues, however, are available regarding the symmetric and antisymmetric parts of the tensor $\mathfrak{g}^{ij}$. The first is furnished by the equations (3.8). These coincide with the second set of Maxwell's equations if we put

$$\mathbf{E}=\alpha({}^a\mathfrak{g}^{23},{}^a\mathfrak{g}^{31},{}^a\mathfrak{g}^{12});\quad \mathbf{B}=\alpha({}^a\mathfrak{g}^{14},{}^a\mathfrak{g}^{24},{}^a\mathfrak{g}^{34})$$

[38] The assumption II of Sec. 2(a) is no longer applicable, and presumably the gravitational and electromagnetic fields, for example, can only be clearly separated in the limit of very weak fields.

with some constant α. Hence, Einstein suggests identifying ${}^a\mathfrak{g}^{ij}$ with the electromagnetic field.

Secondly, Lichnérowicz has shown what corresponds to the local light cone of relativity, as far as the propagation of waves (or disturbances) is concerned.[39] By an analysis of the continuation properties of the field equations he finds

$$l_{ij}dx^idx^j=0,$$

where ${}^s\mathfrak{g}^{ki}l_{ij}=\delta_i{}^k$. Note that, in general, l_{ij} is not proportional to the symmetric part of the covariant tensor g_{ij}. Note also that the light cone defines only the ratios of the l_{ij} and hence only the ratios of the ${}^s\mathfrak{g}^{ij}$.

Being derived from an invariant variational principle the field equations satisfy the necessary identities and lead to conservation laws [see Sec. 2(c)]. Because of prohibitive mathematical difficulties, it has not been possible to derive much more than that from the field equations.

A number of explicit solutions (with special symmetries) have been found, but all of them are singular. According to Einstein's program, however, the field equations (A) and (B) must be considered complete, and their solutions must be *free of singularities*. So far no such solution (for reasonable boundary conditions) has been found, nor has it been shown that no such solution exists.

Einstein has made it quite clear that he regarded the variables of this theory as classical field variables which were not to be quantized.

4. REMARKS ON THE QUANTIZATION OF COVARIANT THEORIES

There have been, of late, a number of investigations on the quantization of covariant field theories. The most extensive work has probably been done by P. G. Bergmann and his students.[40] For details I refer to his report to the Berne conference.

Since the mathematical analysis is rather complex, I shall be quite brief, and shall confine myself to pointing out the specific difficulties which have their root in the covariance of the theory to be quantized.

In applying the Heisenberg-Pauli method of field quantization the first step—still on the c-number level—consists in transforming a theory from its Lagrangian to a Hamiltonian form. This leads to difficulties whenever the momenta are not independent. The best known example is Maxwell's theory, where π^4, the momentum conjugate to the scalar potential, vanishes identically.

Any covariant theory leads to four independent relations between the momenta, the "primary" constraints in Bergmann's terminology. These are obtained by the same method which furnishes the identities and conservation laws for a covariant field theory and which has been sketched in 2(c).[41] The condition that the primary constraints, if satisfied at some initial time, remain valid in virtue of the Hamiltonian equations of motion yields four additional, the "secondary" constraints.

The situation has been sufficiently well analyzed, so that it is known, at least in principle, how to proceed. The difficulty remains, however, to express the relations which have been derived—notably for the gravitational field—in a tractable form.

There still remains the possibility of by-passing the Hamiltonian formalism and of working, instead, directly with the Lagrangian, in analogy to Feynman's method of quantization. Attempts in this direction are being made.

The experience with quantum electrodynamics would seem to indicate that, in any event, a thorough knowledge of the solutions of the classical field equations is indispensable, which makes the investigation of nonstationary solutions (mentioned in 2e) all the more important.

Since the work, so far, has been preparatory to an actual quantization it is not known to what extent specifically quantum theoretical difficulties may arise.

REMARK BY H. P. ROBERTSON IN THE DISCUSSION

I would like to know if any of you or your experimental colleagues know of any attempts at the repetition of the Kennedy-Thorndike experiment. It seems to me that the time may be ripe for the retest of this experiment which is radically different in principle from the Michelson-Morley. I think the other two experiments are adequate.

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[39] See reference 3, p. 288.

[40] See reference 35 (bibliography).

[41] Rosenfeld [reference 36 (bibliography)] has already shown that relations of this kind exist if the Lagrangian of the theory admits a group of transformations which depend on arbitrary space-time functions [such as the four functions ξ^i in 2(c) which determine the infinitesimal transformations of the group]. In Maxwell's theory the corresponding group is the gauge group. It involves only one arbitrary space-time function and leads to only one relation, *viz.*, $\pi^4=0$.

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REVIEWS OF MODERN PHYSICS VOLUME 29, NUMBER 2 APRIL, 1957

Nuclear Physics

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I

WITHIN the framework of a short review it is impossible to discuss all outstanding problems in nuclear physics. I therefore restrict myself to the problem of nuclear structure and stability. The main problem can be formulated as follows: Given A nucleons, what can we conclude from our present incomplete knowledge of nucleon interactions, in regard to the structure and properties of the nucleus which they form?

Our knowledge of interactions is, in fact, extremely limited. The properties of the deuteron and the nucleon-nucleon scattering experiments contain much information about the interaction between nuclear pairs. We have deduced phase shifts as functions of energy from many scattering experiments, but so far it was not possible to explain these phase shifts as arising from a well-defined potential. Hence, it is hard to apply this knowledge to the situation within nuclei which is so different from the scattering of free and isolated nucleons. Because of the proximity of many scattering centers within one wavelength inside the nucleus, the conditions of scattering are fundamentally altered; any definite conclusions from the two-body scattering can be drawn only with a knowledge of the scattering potential and not from the mere phase shifts. As an example, let us look at a slow nucleon whose wavelength $\lambda\!\!\!^{-}$ is large compared to the radius r^* of the scatterer. The scattered wave ψ_{sc} will then be an S wave ($\psi_{sc}=[\sin(kr-\delta)]/r$). If it hits a second scatterer at a long distance $d\gg\lambda\!\!\!^{-}$ from the first scatterer, the wave will be essentially constant over the second scattering region, and there will again be mainly S scattering. However if the second scatterer is near the first ($d\ll\lambda\!\!\!^{-}$), the scattered wave ψ_{sc} is far from constant over the scattering region because of the preponderance of the $1/r$ dependence, if $d\ll\lambda\!\!\!^{-}$. In this case, p scattering will be quite important in spite of the low energy. In the usual mathematical terminology, this condition is expressed by saying that the elements of the scattering matrix between states of different energy are also important in our problem. This makes our problem rather difficult since the scattering phase shifts only determine the scattering matrix elements between states of equal energy.

Apart from the incomplete knowledge of the interaction between two free nucleons, our problem is complicated by the fact that we do not know whether this interaction is changed considerably when the two nucleons are surrounded by other nucleons. The