Equations of Motion of the Ocean and Atmosphere*

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The density of the air in the atmosphere, or of the water in the ocean, is a function of altitude or depth. Linear equations for the motion of such stratified fluids have been derived by many authors. Many have introduced simplifying assumptions into their derivations, with the result that it is difficult to compare one paper with another. The present review is a collection of the major systems of equations, in a uniform notation. The derivations given are those which bring out the interrelations of the various systems, and are sometimes quite different from those of the original papers. The solution of the equations is not discussed, so that the relative merits of the various systems cannot be evaluated.

INTRODUCTION

THE most obvious fact about the earth's atmosphere is the variation of its density and temperature with altitude. This stratification gives it a stability that strongly influences the motion of the air. Although it is not so obvious, the earth's oceans and lakes are also stratified, and it is known that their stability (at least in the thermocline layer) is comparable to, or even greater than, that of the atmosphere.

Through the years, a very considerable but scattered literature on the motion of stratified fluids has accumulated. This includes the theory of convection as well as the theory of waves in the geophysical media. There has been little effort to systematize these theories, so that a typical paper will begin with a derivation of the equations of motion, at the same time introducing simplifying assumptions, and proceed to the discussion of special approximate solutions. The comparison of different papers is rendered difficult both by differences in the basic equations and by differences in notation.

The objective of this review is to collect the major systems of equations that have been derived, in a uniform notation, and to discuss the relations between them. It is clear that this requires the formulation of the physical assumptions from which they follow. No attempt is made to compare these assumptions in terms of validity, nor is any attempt made to preserve the arguments advanced by their original proponents. Neither is there any attempt to describe the solutions of the equations. Such attempts would have caused this review to grow beyond reasonable bounds; the interested reader must consult the original papers on these matters.

In Sec. III, the most general system of differential equations is discussed in somewhat more detail than the others. A transformation of dependent variables is found which exhibits their essential mathematical structure. This is rather simpler than has been supposed.

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I. THE MOST GENERAL EQUATIONS

A. Thermodynamics

The thermodynamic properties of a pure substance are completely determined when its internal energy ϵ (erg/g) is given as a function of the specific volume v (cm^3/g) and the entropy η (erg/g deg). Then the pressure p (dyne/cm^2) and temperature θ are calculable from the equations

$$p=-\partial\epsilon/\partial v,\quad \theta=\partial\epsilon/\partial\eta. \tag{1}$$

In differential form, these are equivalent to

$$\delta p=-X\delta v+Y\delta\eta, \tag{2}$$

$$\delta\theta=-Y\delta v+Z\delta\eta, \tag{3}$$

X, Y, Z, being the second derivatives of ϵ, taken with appropriate signs.

These coefficients are also expressible in terms of the following four quantities:

c = velocity of sound,

a = coefficient of thermal expansion,

s = specific heat at constant volume,

γ = ratio of the specific heats;

with $\rho=1/v$, the expressions are

$$X=\rho^2c^2,\quad Y=\rho(\gamma-1)/a,\quad Z=\theta/s. \tag{4}$$

The four quantities are not independent, but are related by the equation

$$\gamma(\gamma-1)s=a^2c^2\theta, \tag{5}$$

which is equivalent to

$$XZ-Y^2=XZ/\gamma. \tag{6}$$

B. Hydrodynamics

The hydrodynamic equations are[1]

$$D\mathbf{u}/Dt + v\nabla p + g\nabla\chi = \mathbf{f}, \tag{7}$$

$$Dv/Dt - v\nabla\cdot\mathbf{u} = 0, \tag{8}$$

$$D\eta/Dt = q/\theta, \tag{9}$$

where

$\mathbf{u}$ = velocity (cm/sec),
$g\chi$ = gravitational potential $(\text{cm/sec})^2$,
$g = 980$ cm/sec^2 (by international convention),
$\mathbf{f}$ = resultant of all nongravitational forces (dyne/g),
q = net accession of heat, (erg/g sec),
$D/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla$.

In laboratory applications, the forces $\mathbf{f}$ are primarily viscous; in large scale geophysical applications, they are primarily the Coriolis force due to the earth's rotation, and the tide producing forces of the sun and moon. If the transfer of heat occurs only by conduction,

$$q = v\nabla\cdot(k\nabla\theta),$$

k being the thermal conductivity of the fluid. In meteorological applications, the transfer of heat is primarily by radiation, and the expression for q is more elaborate Many of the references discussed below either ignore $\mathbf{f}$ and q, or adopt specific forms for them. Such specializations will be mentioned only when there is a definite reason to do so.

C. The Zero-Order Solution

Following the standard scheme of perturbation theory,[2] we shall derive the linearized equations by considering the first-order perturbations of a zero-order state. This zero-order state will be one of rest in all that follows. Some of the references cited use other zero-order states[2–5] and the present discussion assumes that the equations of these references have been appropriately simplified.

The zero-order state of rest is specified when the pressure is given as a positive, monotonically decreasing function of the gravitational potential χ:

$$p = p_0(\chi). \tag{10}$$

Then the density is

$$\rho = \rho_0(\chi) = -p_0'/g, \tag{11}$$

where the accent indicates differentiation with respect to χ. Equations (1), (2), and (3) combine with these to determine θ_0 and η_0, also as functions of χ only.

It then becomes possible to express all other zero-order quantities, such as X, Y, Z, as functions of χ. Strictly, these functions should be distinguished by a subscript zero, as in the case of p_0 etc., but no confusion will result if this subscript is omitted. With this understanding, Eqs. (2) and (3) yield

$$\begin{aligned} p_0' &= -Xv_0' + Y\eta_0', \\ \theta_0' &= -Yv_0' + Z\eta_0'. \end{aligned} \tag{12}$$

D. The First-Order Equations

In the perturbed state, $v = v_0 + v_1 + \cdots$, $p = p_0 + p_1 + \cdots$, $\mathbf{u} = \mathbf{u}_1 + \cdots$, and the linearized equations are to be obtained by neglecting all terms indicated by dots, as well as squares and products of quantities carrying the subscript one. Equations (1) and (2) then yield

$$\begin{aligned} p_1 &= -Xv_1 + Y\eta_1, \\ \theta_1 &= -Yv_1 + Z\eta_1. \end{aligned} \tag{13}$$

With

$$\boldsymbol{\zeta} = \nabla\chi, \tag{14}$$

Eqs. (7), (8), and (9) yield

$$(\partial\mathbf{u}_1/\partial t) + p_0'v_1\boldsymbol{\zeta} + v_0\nabla p_1 = \mathbf{f}_1, \tag{15}$$

$$(\partial v_1/\partial t) + v_0'\mathbf{u}_1\cdot\boldsymbol{\zeta} - v_0\nabla\cdot\mathbf{u}_1 = 0, \tag{16}$$

$$(\partial\eta_1/\partial t) + \eta_0'\mathbf{u}_1\cdot\boldsymbol{\zeta} = q_1/\theta_0. \tag{17}$$

These equations suffice to determine all first-order quantities. For many applications, it is desirable to consider two other equations, obtainable from Eqs. (16) and (17) by using Eqs. (12) and (13):

$$(\partial p_1/\partial t) + p_0'u_1\cdot\zeta + Xv_0\nabla\cdot u_1 = Yq_1/\theta_0, \tag{18}$$

$$(\partial\theta_1/\partial t) + \theta_0'u_1\cdot\zeta + Yv_0\nabla\cdot u_1 = Zq_1/\theta_0. \tag{19}$$

Except for notation, these equations are identical with those derived by Bjerknes, Bjerknes, Solberg, and Bergeron[6] and Blokhintzev.[7] If the fluid is a perfect gas, γ is a constant and

$$\begin{aligned} c^2 &= \gamma R\theta_0 = \gamma p_0/\rho_0, \\ a &= 1/\theta_0, \\ s &= R/(\gamma - 1) \end{aligned} \tag{20}$$

Using these special values of the thermodynamic parameters, we find Eqs. (15), (16), and (17) are essentially identical with equations derived by Lamb.[8,9] They have also been used by Pekeris,[10–12] Wilkes,[13]

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[1] C. Eckart, Phys. Rev. **58**, 207 (1940).
[2] V. Bjerknes and Colaborators, *Physikalische Hydrodynamik* (Verlag Julius Springer, Berlin, 1933), Chapter VII.
[3] D. Blokhintzev, *The Acoustics of an Inhomogeneous and Moving Medium*, Translated from the Russian by R. T. Beyer and D. Mintzer, Providence, Research and Analysis Group, Physics Department, Brown University (1952).
[4] B. Haurwitz, *Perturbation Equations in Meterology* (Compendium of Meterology, American Meteorology Society, Boston, 1951), p. 401.
[5] P. Queney, *Theory of Perturbations in Stratified Currents with Applications to Air Flow Over Mountain Barriers*, University of Chicago, Department of Meteorology, Misc. Rept. No. 23, (1946).

[6] See reference 2, Sec. 84.
[7] See reference 3, p. 17.
[8] H. Lamb, *Hydrodynamics* (Cambridge University Press, New York, 1932), sixth edition, p. 547.
[9] H. Lamb, Proc. Roy. Soc. (London) **A84**, 551 (1890).
[10] C. L. Pekeris, Proc. Roy. Soc. (London) **A158**, 650 (1937).
[11] C. L. Pekeris, Proc. Roy. Soc. (London) **A171**, 434 (1939).
[12] C. L. Pekeris, Phys. Rev. **73**, 145 (1948).
[13] M. V. Wilkes, *Oscillations of the Earth's Atmosphere* (Cambridge University Press, New York, 1949).

Haurwitz,[4] Gossard and Munk,[14] and others. Some of these writers introduce the assumption that the vertical acceleration is zero, before proceeding to the solution.

The Laplace-Rayleigh theory of sound is obtained from Eqs. (15) and (18) by neglecting p_0' and setting $\eta_1=0$ in Eq. (13).[15] Bergmann[16] and Blokhintzev[3] have considered the effect of the terms in p_0' and v_0' on the propagation of sound.

The older Newtonian theory of sound is obtained from Eqs. (15) and (18) by setting $\theta_1=0$ in Eq. (13) and eliminating η_1. This theory has been used by Laplace[17] and Rayleigh[18] for the discussion of atmospheric and oceanic tides, with the further assumption that $\theta_0'=0$. These classic researches have strongly influenced later work.[19]

Margules[20] attempted a theory of the thermal atmospheric tide, based on Eqs. (15) and (16) with $f_1=0$ and the assumption of θ_1 as a specified function of time and position. Most later work on this problem is based either on equations to be developed in Sec. II or on the unsimplified equations (15) to (19).

The simplification $v_1=0$ does not seem to have been investigated in connection with stratified fluids. It is always combined with $\rho_0'=0$, and then leads to the classical theory of the homogeneous incompressible fluid.

Neither is the simplification $p_1=0$ discussed in the literature. Instead, there is a series of papers whose apparent objective is the avoidance of this assumption while achieving some of its consequences. These will be discussed in Sec. II.

II. LOVE'S EQUATIONS AND THEIR VARIANTS

A. The Assumption of Incompressibility

Equation (2) implies

$$\begin{aligned} Y(D\eta/Dt) &= (Dp/Dt)+\rho^2c^2(Dv/Dt), \\ &= (Dp/Dt)-c^2(D\rho/Dt). \end{aligned} \tag{21}$$

Since c^2 is a large number, many writers neglect the first term on the right. With this approximation, Eqs. (4) and (5) lead to

$$\theta(D\eta/Dt)=-(\gamma s/\rho a)(D\rho/Dt). \tag{22}$$

Equation (9) then becomes

$$(D\rho/Dt)=-\rho aq/\gamma s, \tag{23}$$

and when this is combined with Eq. (8), the latter becomes

$$\nabla\cdot\mathbf{u}=aq/\gamma s. \tag{24}$$

[14] E. Gossard and W. Munk, J. Meteorol. 11, 259 (1954).
[15] Lord Rayleigh, *Theory of Sound* (Dover Publications, New York, 1945), Vol. II, p. 18.
[16] P. Bergmann, J. Acoust. Soc. Am. 17, 329 (1946).
[17] Laplace, *Mécanique Céleste* (English Translation by N. Bowditch, Boston, 1832), Book 4, Chapter 5.
[18] Lord Rayleigh, Phil. Mag. 29, 173 (1890).
[19] G. I. Taylor, Proc. Roy. Soc. (London) A156, 318 (1936).
[20] M. Margules, Wien. Sitzber. Ak. Wiss. 99, 204 (1890).

When Eqs. (7), (24), and (23) are linearized as in Sec. I.D the result is

$$(\partial\mathbf{u}_1/\partial t)+v_0g\rho_1\boldsymbol{\zeta}+v_0\nabla p_1=\mathbf{f}, \tag{25}$$

$$\nabla\cdot\mathbf{u}_1=aq_1/\gamma s, \tag{26}$$

$$(\partial\rho_1/\partial t)+\rho_0'\mathbf{u}_1\cdot\boldsymbol{\zeta}=-\rho_0aq_1/\gamma s, \tag{27}$$

use having been made of the equations $p_0'=-\rho_0g$, $v_1\rho_0=-\rho_1v_0$.

These equations were derived by Love[21] and have been used by Lamb,[22] Groen,[23] Fjeldstad,[24] and others.

B. Oberbeck's Equations

Equations (2) and (3) also imply

$$-Z\delta p+Y\delta\theta=XZ\delta v/\gamma, \tag{28}$$

or

$$\rho_0v_1=a\theta_1-\gamma p_1/\rho_0c^2, \tag{29}$$

$$\rho_0v_0'=a\theta_0'-\gamma p_0'/\rho_0c^2. \tag{30}$$

Following the idea of Love's approximation, many writers neglect the second terms on the right of the last two equations, obtaining

$$\rho_0v_1=-v_0\rho_1=a\theta_1, \tag{31}$$

$$\rho_0v_0'=-v_0\rho_0'=a\theta_0'. \tag{32}$$

When these approximations are substituted into Eqs. (23) to (27) they become

$$(\partial\mathbf{u}_1/\partial t)-ga\theta_1\boldsymbol{\zeta}+v_0\nabla p_1=\mathbf{f}_1, \tag{33}$$

$$\nabla\cdot\mathbf{u}_1=aq_1/\gamma s, \tag{34}$$

$$(\partial\theta_1/\partial t)+\theta_0'\mathbf{u}_1\cdot\boldsymbol{\zeta}=q_1/\gamma s. \tag{35}$$

Overbeck[25] introduced equations similar to these into the theory of convection. However, he simplified them with two further approximations:

(1) Terms proportional to ga are to be retained, but terms proportional to a alone are to be neglected.
(2) In Eq. (35), $\theta_0'\mathbf{u}_1\cdot\boldsymbol{\zeta}$ is to be neglected.

Babcock[26] and Pekeris[27] have also used these simplified equations. Rayleigh[28] has used equations that differ from Oberbeck's only in the retention of the term in θ_0', but he refers them to Boussinesque.[29] These equations are still widely quoted, e.g., by Jeffreys,[30] Pellew and Southwell,[31] and Chandrasekhar.[32]

[21] A. E. H. Love, Proc. Math. Soc. (London) 22, 307 (1891).
[22] See reference 8, p. 378.
[23] P. Groen, *Contribution to Theory of Internal Waves* (Köninklijk Nederlands Meteorlogisch Institut De Bild, 1948), No. 125.
[24] J. E. Fjeldstad, Geofys. Publikasjoner 10, No. 6 (1933).
[25] A. Oberbeck, Ann. Physik 7, 271 (1879).
[26] R. W. Babcock, Phys. Rev. 35, 1008 (1930).
[27] C. L. Pekeris, Monthly Notices Roy. Astron. Soc. Geophys. Supplement 3, 343 (1935).
[28] Lord Rayleigh, Phil. Mag. 32, 529 (1916).
[29] J. Boussinesque, *Théorie Analytique De La Chaleur* (Paris, 1903).
[30] H. Jeffreys, Phil. Mag. 2, 833 (1926).
[31] Anne Pellew and R. V. Southwell. Proc. Roy. Soc. (London) A176, 312 (1940).
[32] S. Chandrasekhar, Proc. Roy. Soc. (London) A217, 306 (1953).

C. Jeffreys' Equations

Jeffreys[33] has proposed a modification of Oberbeck's equations. The derivation starts from the general equations (15) to (19) rather than from Love's equations (23) to (27); moreover, Eqs. (30) and (31) are used instead of Eqs. (31) and (32). Equation (15) then reduces to

$$(\partial \mathbf{u}_1/\partial t)-ga\theta_1\boldsymbol{\zeta}+v_0\nabla p_1=\mathbf{f}_1, \quad (36)$$

which is identical with Eq. (33). Equation (16) becomes

$$\nabla\cdot\mathbf{u}_1=a(\partial\theta_1/\partial t)+(a\theta_0'+\gamma g/c^2)\mathbf{u}_1\cdot\boldsymbol{\zeta}.$$

Now, the adiabatic temperature gradient is defined as

$$\theta_a'=-(\gamma-1)g/ac^2, \quad (37)$$

so that this may be written

$$\nabla\cdot\mathbf{u}_1=a\{(\partial\theta_1/\partial t)+[\theta_0'-\gamma\theta_a'/(\gamma-1)]\mathbf{u}_1\cdot\boldsymbol{\zeta}\}. \quad (38)$$

If this is substituted into Eq. (15), the result is

$$(\partial\theta_1/\partial t)+(\theta_0'-\theta_a')\mathbf{u}_1\cdot\boldsymbol{\zeta}=q_1/\gamma s. \quad (39)$$

Jeffreys uses Oberbeck's principle (1) to simplify Eq. (38), but retains the term in $(\theta_0'-\theta_a')$. In many applications, the quantities θ_0' and θ_a' are of comparable magnitude. Other writers give Love's or Oberbeck's derivation, but substitute $\theta_0'-\theta_a'$ for θ_0' without attempting a formal justification. In meteorology, the potential temperature, θ_p, is often used. It may be defined by the equation

$$\theta_p'=\theta_0'-\theta_a',$$

and this notation is frequently employed in writing Eq. (39). The corresponding modification of the equations for oceanographic purposes is discussed by Hesselberg and Sverdrup.[34]

III. THE FIELD EQUATIONS

A. Energy and the Quadratic Integral

The energy of the fluid, in erg/g, is

$$W=\tfrac{1}{2}u^2+\epsilon+g\chi. \quad (40)$$

Equations (1), (7), (8), and (9) imply that

$$\rho(DW/Dt)+\nabla\cdot(p\mathbf{u})=\rho(\mathbf{f}\cdot\mathbf{u}+q), \quad (41)$$

which is known as the energy equation. If $\mathbf{u}=\mathbf{u}_1+\mathbf{u}_2+\cdots$, $p=p_0+p_1+p_2+\cdots$, etc., the energy $W=W_0+W_1+W_2+\cdots$, and Eq. (41) can be resolved into a set of equations, of zero, first, second . . . orders. The zero- and first-order equations, thus derived, are identically satisfied because of Eqs. (11) and (15) to (19). The expression for W_2 involves $\mathbf{u}_2$, p_2, $\cdots$ as well as squares and products of $\mathbf{u}_1$, p_1, $\cdots$. Therefore, the second-order energy equation cannot be derived from Eqs. (15) to (19), which do not involve the second-order quantities.

Equations (15), (16), and (17) do imply the equation

$$(\partial E/\partial t)+\nabla\cdot(p_1u_1)=\rho_0\left[\mathbf{u}_1\cdot\mathbf{f}_1+p_1q_1\frac{Y}{X\theta_0}-\eta_1q_1\frac{p_0'Y}{\eta_0'X\theta_0}\right], \quad (42)$$

with

$$E=\tfrac{1}{2}\rho_0u_1^2+\frac{p_1^2}{2\rho_0c^2}-\tfrac{1}{2}\rho_0\eta_1^2\frac{p_0'Y}{\eta_0'X}. \quad (43)$$

The quantities E and ρ_0W_2 are not identical, though they have some terms (e.g., $\frac{1}{2}\rho_0u_1^2$) in common; neither is Eq. (42) the second-order energy equation. Nevertheless, Eq. (42) is very important in the theory of Eqs. (15) to (19) and can often be used much as an energy equation would be used.

It would be desirable to have a simple terminology that preserves this distinction; in fact, this is essential if paradoxes are to be avoided. No such terminology has been invented, and E (or some similar expression) is usually called "energy." This misnomer is convenient in many applications, and is used in acoustics without causing confusion, but only because certain fundamental problems are ignored. The expression E may be called the quadratic integral of Eqs. (15) to (19), though this is clumsy, and does not indicate the physical importance of this quantity. Some adjectival phrase, like "external energy," would be desirable.

B. The Field Equations

Equations (42), (43), and (15) to (19) are much simplified by the following transformation of variables:

$$\begin{aligned} \mathbf{U}&=\mathbf{u}_1(\rho_0c)^{\frac{1}{2}}, & \mathbf{F}&=\mathbf{f}_1(\rho_0c)^{\frac{1}{2}},\\ P&=p_1(\rho_0c)^{-\frac{1}{2}}, & G&=(q_1Y/\theta_0)(\rho_0c)^{-\frac{1}{2}}, \\ Q&=(\eta_1/\eta_0')(\rho_0c)^{\frac{1}{2}}, & H&=(q_1/\eta_0'\theta_0)(\rho_0c)^{\frac{1}{2}}. \end{aligned} \quad (44)$$

Similar transformations are to be found in references 5, 14, and 16. Equations (44) are not uniquely determined, and modifications may be desirable for special purposes.

Equations (42) and (43) then become

$$(\partial E/\partial t)+\nabla\cdot(PU)=(\mathbf{F}\cdot\mathbf{U}+GP+N^2HQ)/c, \quad (45)$$

$$E=(U^2+P^2+N^2Q^2)/2c, \quad (46)$$

where

$$\begin{aligned} N^2&=\rho_0g\eta_0'Y/X\\ &=\eta_0'g(\gamma-1)/ac^2=-\eta_0'\theta_a'. \end{aligned} \quad (47)$$

Equations (15), (17), and (18) become

$$(\partial\mathbf{U}/\partial t)+c(\nabla+\Gamma\boldsymbol{\zeta})P-N^2Q\boldsymbol{\zeta}=\mathbf{F}, \quad (48)$$

$$(\partial P/\partial t)+c(\nabla-\Gamma\boldsymbol{\zeta})\cdot\mathbf{U}=G, \quad (49)$$

$$(\partial Q/\partial t)+\mathbf{U}\cdot\boldsymbol{\zeta}=H, \quad (50)$$

[33] H. Jeffreys, Proc. Cambridge Phil. Soc. **26**, 170 (1930).

[34] Th. Hesselberg and H. U. Sverdrup, *Die Stabilitätsverhältnisse des Seewassers bei vertikalen Verchiebungen* (Bergens Museums Aarbok, 1914–1915), No. 14.

where

$$\Gamma = \tfrac{1}{2}[(\rho_0 c)'/(\rho_0 c)] + (g/c^2). \tag{51}$$

These will be called the field equations, and have a considerably simpler form than Eqs. (15) to (19), to which they are equivalent. In particular, their left sides depend only on three parameters, c, N^2, and Γ. These are generally functions only of the vertical coordinate (i.e., of χ); in the particular case of an isothermal ideal gas, all three are constants.

Equations (48), (49), and (50) are essentially self-adjoint, and if $N^2>0$, their quadratic integral is positive definite. These are important physical facts. Most of the simplifying assumptions discussed above destroy one or both of these properties of the equations. An exception is the acoustic assumption, $\eta_1=0$.

The literature does not contain a systematic account of the solutions of these equations, although special or approximate solutions are frequently implied.

C. The Parameters of the Field Equations

The parameter, c (cm/sec), is the Laplacian velocity of sound, and requires no further discussion. The parameter Γ (radians/cm) has very small values, and does not affect the qualitative character of the solutions of the equations. In the case of the oceans, $2\pi/\Gamma$ is so much greater than their depth that its quantitative influence is completely negligible. In the case of the atmosphere, the quantitative influence of Γ becomes appreciable only at altitudes greater than about 50 km; however, it does determine the behavior of the solutions at infinite altitudes, so that the approximation $\Gamma=0$ may introduce paradoxes.

The parameter N (radians/sec) is sometimes called Väisälä's frequency.[35] The sign of N^2 is that of the entropy gradient, η_0'; when this is positive, the stratification of the fluid is stable, and small disturbances have an oscillatory character. When $\eta_0'<0$, N is imaginary, and the stratification is unstable. Väisälä established these facts by simple physical considerations; they can also be derived from a study of Eqs. (48), (49), and (50). They are frequently used by meteorologists; the limiting case $N=0$ is called convective equilibrium.

Using Eq. (12), one obtains the following alternative expressions for N^2:

$$N^2 = -g[(\rho_0'/\rho_0) + (g/c^2)], \tag{52}$$

$$= ga(\theta_0' - \theta_a'). \tag{53}$$

The last expression should be compared with the formulas of Sec. II.C; the appearance of the combination ga also relates it to Oberbeck's principle (1), but not in any consistent fashion.

[35] Vilho Väisälä, Über die Wirkung der Windschwankungen auf die Pilotbeobachtungen. Soc. Sci. Fennica, Commentationes Phys.-Math. **II 19**, 37 (1925).