Theory of Wave Propagation in a Gyromagnetic Medium*

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1. Introduction

IN recent years certain ferromagnetic materials called *ferrites* have become commercially available and are gaining increasing importance in various applications. The chemical constitution of the ferrites is expressed by the formula $MO\cdot Fe_2O_3$, where M symbolizes a metal. Thus magnetite, $FeO\cdot Fe_2O_3$, is a natural ferrite known since antiquity, but the important step of replacing FeO by oxides of other metals was carried out but recently and led to the development of numerous artificial ferrites. The crystal structure of practically all of these new substances is the same as that of magnetite, namely, the body-centered cubic "spinel" lattice. In their electric characteristics the ferrites are semiconductors. It was just their low conductivity combined with a high magnetic permeability which originally attracted to them the attention of physicists and engineers since it made them a most valuable material for transformer cores.[1]

However, later investigations by F. F. Rogers[2] revealed in the ferrites another property of equal practical importance: they possess a remarkably high Faraday effect, a characteristic fitting them for various useful applications in the microwave field. It is this aspect of the electromagnetic field in ferrites which will form the subject of the present paper. In this relation these substances may be characterized as nearly *gyromagnetic*, a term that describes a hypothetical medium of the following properties.

The displacement **D** does not present anything unusual in the gyromagnetic medium and is related to the field strength **E** by the familiar equation

$$\mathbf{D}=\epsilon\mathbf{E}, \tag{1.1}$$

where the dielectric constant ϵ is an isotropic scalar either real or complex.

Let now a permanent magnetic field H_0 be produced in the medium by an external source. This field (called *exciting* or *permanent*) produces in the medium a magnetic anisotropy of such a nature that the induction **B** and the strength of field **H** of an additional periodic magnetic field, superposed upon H_0, stand in the relation

$$\mathbf{B}=(\mu)\mathbf{H}, \tag{1.2}$$

where (μ) denotes the tensor,

$$(\mu)=\begin{bmatrix}\mu_1, & i\kappa, & 0\\ -i\kappa, & \mu_1, & 0\\ 0, & 0, & \mu_3\end{bmatrix}. \tag{1.3}$$

In the usual way, Eq. (1.2) stands for the three component equations

$$B_x=\mu_1H_x+i\kappa H_y,\quad B_y=-i\kappa H_x+\mu_1H_y,\quad B_z=\mu_3H_z, \tag{1.4}$$

where the coordinate z is chosen so as to coincide with the direction of the permanent field H_0.

Usually the parameters μ_1, μ_3, κ of the tensor (1.3) are taken to be real and positive. It may be remarked, however, that no part of our theory will invoke the Hermitean character of the tensor. Hence, all the results will remain valid, if the diagonal elements μ_1, μ_3 should be considered complex.

It is true that the real ferrites of nature do not rigorously conform to the definitions (1.1) and (1.2). The approximation is, however, sufficiently close so that the investigation of a gyromagnetic medium has great practical importance. In general, the discussion will be formally mathematical and questions of the physical nature of ferrites will be left aside.[3] However, two things must be mentioned as relevant to the formal theory.

(1) The laws (1.1) and (1.2) hold only for periodic fields since ϵ and the components of (μ) are functions of the frequency ω. Hence, the field vectors will be supposed to contain a time factor of the form $\exp(-i\omega t)$. This factor will be generally omitted, but its existence permits of using interchangeably the notations

$$\dot{u}=\partial u/\partial t=-i\omega u=-ik_0u, \tag{1.5}$$

where u is any field quantity, t the time, c the velocity of light *in vacuo*, and k_0 the wave number,

$$k_0=\omega/c. \tag{1.6}$$

(2) The parameters ϵ, μ_1, μ_3, κ are also functions of the permanent field H_0. Hence, they can be treated as spatially constant only in the cases of such experimental arrangements in which the permanent field is homo-

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[1] For the properties of ferrites and the history of their development see J. L. Snoek, *New Development in Ferromagnetic Materials* (Elsevier Publishing Company, Inc., Amsterdam, 1947); C. A. Donencali, Phys. Rev. **78**, 458 (1950); Blewett, Plotkin, and Blewett, "The properties of ferromagnetic ferrites" (Brookhaven National Laboratory).

[2] F. F. Rogers, J. phys. radium **12**, 305 (1952).

[3] Investigations about the physical causes of gyromagnetism in ferrites are, among others, due to D. Polder, Phil. Mag. **40**, 99 (1949); C. L. Hogan, Bell System Tech. J. **31**, 1 (1952); C. Kittel, Phys. Rev. **73**, 155 (1948); C. Kittel, J. phys. radium **12**, 291 (1951).

geneous within each ferrite.[4] In this paper they shall be always regarded as spatially constant.

It was shown by C. H. Papas that in rarefied and ionized gases, such as exist in the upper atmosphere or in electric discharge tubes, a permanent magnetic field produces an anisotropy of the dielectric constant which assumes tensor character expressed by the scheme,

$$(\epsilon)=\begin{bmatrix}\epsilon_1, & i\epsilon', & 0\\ -i\epsilon', & \epsilon_1, & 0\\ 0, & 0, & \epsilon_3\end{bmatrix}. \tag{1.7}$$

At the same time the magnetic permeability μ remains scalar and close to unity. A medium of these properties is called *gyroelectric*.[5] In view of the symmetry of Maxwell's equations, the theory of gyromagnetism contained in the following sections can be readily adapted to this case by means of the simultaneous substitution $\epsilon\rightarrow\mu$, $(\mu)\rightarrow(\epsilon)$, $\mathbf{E}\rightarrow\mathbf{H}$, $\mathbf{H}\rightarrow-\mathbf{E}$.

Occasionally more general media than the gyromagnetic and the gyroelectric are mentioned in literature, but their discussion will lie outside the scope of this paper. This applies, in the first place, to the so-called *gyrotropic* medium in which both the dielectric constant and the magnetic permeability have tensor character, so that the Eqs. (1.3) and (1.7) are valid at the same time.[6] However, no substance of nature is known to possess such properties while the mathematics involved is necessarily cumbersome permitting only of treating comparatively simple special cases. In the second place, a medium of still greater generality was envisaged by B. D. H. Tellegen: the possibility is admitted that an electric field can produce magnetization and a magnetic field can cause polarization. Therefore, the components of the vectors **D**, **B** are related with those of **E**, **H** by a linear substitution with six rows and six columns.[7] Although such processes are not entirely unknown, the Hall effect being a case in point, the general investigation of the Tellegen medium is not sufficiently interesting to justify entering here into its formidable mathematical difficulties.

PART I. GENERAL THEORY

2. The Basic Equations

Wave propagation in a gyromagnetic medium was treated theoretically by several authors; the conditions in wave guides were considered by Kales[8] and Gamo[9] and, in greater detail, by Van Trier[10] and by Suhl and Walker[11]; some problems of the propagation in an open space were discussed by Gintzburg.[12] However the analysis of the following sections proceeds on somewhat different lines.

Under the conditions specified in Sec. 1, Maxwell's equations can be written as follows:

$$\left.\begin{aligned}-i\epsilon k_0\mathbf{E}=\nabla\times\mathbf{H},\quad & ik_0\mathbf{B}=\nabla\times\mathbf{E},\\ \nabla\cdot\mathbf{E}=0,\quad & \nabla\cdot\mathbf{B}=0.\end{aligned}\right\} \tag{2.1}$$

Since we shall be interested only in states of vanishing free charge, the simplest way of treating these equations is to regard the electric strength of field **E** as a kind of vector potential from which the magnetic field is derived by the relations,

$$\mathbf{H}=-(i/k_0)(\mu^{-1})\nabla\times\mathbf{E}, \tag{2.2}$$

where (μ^{-1}) is the tensor reciprocal to (μ),

$$(\mu^{-1})=\begin{bmatrix}M, & iK, & 0\\ -iK, & M, & 0\\ 0, & 0, & M_3\end{bmatrix}, \tag{2.3}$$

with

$$\begin{gathered}M=\mu_1/d,\quad K=-\kappa/d,\quad M_3=1/\mu_3,\\ d=\mu_1^2-\kappa^2.\end{gathered} \tag{2.4}$$

The remaining two equations of the system (2.1) can then be regarded as the basic field equations for the determination of the vector **E**. They take the form,

$$\epsilon k_0^2\mathbf{E}=\nabla\times[(\mu^{-1})\nabla\times\mathbf{E}], \tag{2.5}$$

$$\nabla\cdot\mathbf{E}=0, \tag{2.6}$$

while the equation $\nabla\cdot\mathfrak{B}=0$ is satisfied identically.

However, it is desirable to throw these equations into a more convenient form. For this purpose we first write out Eqs. (2.5) in Cartesian coordinates:

$$\left.\begin{aligned}&\left[\epsilon k_0^2+M_3\nabla^2-(M_3-M)\frac{\partial^2}{\partial z^2}\right]E_x+iK\frac{\partial^2 E_y}{\partial z^2}\\ &\qquad+\frac{\partial}{\partial z}\left[(M_3-M)\frac{\partial E_z}{\partial x}-iK\frac{\partial E_z}{\partial y}\right]=0,\\ &\left[\epsilon k_0^2+M_3\nabla^2-(M_3-M)\frac{\partial^2}{\partial z^2}\right]E_y-iK\frac{\partial^2 E_x}{\partial z^2}\\ &\qquad+\frac{\partial}{\partial z}\left[(M_3-M)\frac{\partial E_z}{\partial y}+iK\frac{\partial E_z}{\partial x}\right]=0,\\ &[\epsilon k_0^2+M\nabla^2]E_z-iK\frac{\partial}{\partial z}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)=0.\end{aligned}\right\} \tag{2.7}$$

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[4] Some considerations relating to nonhomogeneous constants in gyromagnetic media can be found in a paper by H. Suhl and L. R. Walker, Bell System Tech. J. **33**, 1133 (1954).

[5] C. H. Papas, A Note Concerning a Gyroelectric Medium (Office of Naval Research Report No. 4, California Institute of Technology, May, 1954). Even before Papas, large Faraday effects were observed in vacuum discharge tubes by Goldstein, Lampert, and Heney, Phys. Rev. **82**, 956, (1951). See also H. Suhl and R. C. Walker, Bell System Tech. J. **33**, 579 (1954). In earlier work by E. V. Appleton, Proc. V. R. S. I. Washington (1927), and D. R. Hartree, Proc. Cambridge Phil. Soc. **27**, 143 (1931), the influence of the magnetic field on wave propagation in the ionosphere was investigated without formulating it in terms of a tensor permeability.

[6] W. L. Ginzburg, Theory of Propagation of Radio-Waves in the Ionosphere. (Russian) 1949.

[7] B. D. H. Tellegen, Phillips Research Repts. **3**, 81 (1948).

[8] M. L. Kales, N. B. L. Report No. 4027, August 8, 1952.

[9] H. J. Gamo, Phys. Soc. Japan **8**, 176 (1953).

[10] A. A. Th. M. Van Trier, Appl. Sci. Research **B3**, 305 (1953).

[11] H. Suhl and R. L. Walker, Phys. Rev. **86**, 1922 (1952); Bell System Tech. J. **33**, 987 (1954).

[12] M. A. Gintzburg, Repts. (Doklady) Acad. Sci. U.S.S.R. **95**, 489, 753 (1954).

At this point it is helpful to introduce the two linear combinations,

$$Q_{1,2}=E_x\pm E_y, \tag{2.8}$$

where the subscripts 1, 2, respectively, refer to the upper and lower signs.

Now the second equation (2.7) is multiplied by $\pm i$ and added to the first. To the third equation is added

$$\mp K\frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)=0.$$

The result is

$$\left.\begin{aligned}\left[k_0{}^2\epsilon+M_3\nabla^2+(M-M_3\pm K)\frac{\partial^2}{\partial z^2}\right]Q\\ =(M-M_3\pm K)\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\pm i\frac{\partial}{\partial y}\right)E_z,\\ \left[k_0{}^2\epsilon+M\nabla^2\mp K\frac{\partial^2}{\partial z^2}\right]E_z=\pm K\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\mp i\frac{\partial}{\partial y}\right)Q\end{aligned}\right\} \tag{2.9}$$

where the upper sign refers to Q_1 and the lower to Q_2.

It was pointed out in Sec. 1 that the coordinate z represents a preferred direction imposed by the permanent field H_0. It is well to bring out the different footing on which z stands by writing,

$$\nabla^2=\nabla_p{}^2+\partial^2/\partial z^2, \tag{2.10}$$

where

$$\nabla_p{}^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}, \tag{2.11}$$

denotes the Riemannian operator in the (x,y)-plane. In a similar way,

$$\nabla_p\rightarrow\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right), \tag{2.12}$$

will represent the two dimensional gradient in the (x,y)-plane.

Equations (2.9) acquire now the more concise form

$$\left.\begin{aligned}\left[\epsilon k_0{}^2+M_3\nabla_p{}^2+(M\pm K)\frac{\partial^2}{\partial z^2}\right]Q\\ =(M-M_3\pm K)\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\pm i\frac{\partial}{\partial y}\right)E_z,\\ \left[\epsilon k_0{}^2+M\nabla_p{}^2+(M\mp K)\frac{\partial^2}{\partial z^2}\right]E_z\\ =\pm K\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\mp i\frac{\partial}{\partial y}\right)Q.\end{aligned}\right\} \tag{2.13}$$

These expressions, together with the Eq. (2.6), can be taken as an alternative form of the basic equations.

3. Final Form of the Basic Equations

Equations (2.13) are suitable for eliminating, respectively, E_z and Q. Operating upon the second equation with $(M-M_3\pm K)(\partial/\partial z)(\partial/\partial x\pm i\partial/\partial y)$ we find

$$L=0,$$

where the symbol L denotes the operator,

$$L\equiv\left\{\epsilon^2k_0{}^4+\epsilon k_0{}^2\left[(M+M_3)\nabla_p{}^2+2M\frac{\partial^2}{\partial z^2}\right]+MM_3\nabla_p{}^4\right.$$
$$\left.+[M(M+M_3)-K^2]\nabla_p{}^2\frac{\partial^2}{\partial z^2}+(M^2-K^2)\frac{\partial^4}{\partial z^4}\right\}. \tag{3.1}$$

On the other hand, operating with $\mp K(\partial/\partial z)\times(\partial/\partial x\mp i\partial/\partial y)$ on the first equation leads to an expression of exactly the same form for E_z,

$$L(E_z)=0.$$

It should be noticed that, while Eqs. (2.15) are different for Q_1 and Q_2 ($\pm$ sign), the operator (3.1) is the same for both, so that Q_1 and Q_2 satisfy the same equation of the fourth order. Inasmuch as

$$E_x=\tfrac{1}{2}(Q_1+Q_2),\quad E_y=-\tfrac{1}{2}i(Q_1-Q_2),$$

the components E_x, E_y must also satisfy the same equation. Therefore, we can summarize our results in the vector equations,

$$L(\mathbf{E})=0, \tag{3.2}$$

$$\nabla\cdot\mathbf{E}=0. \tag{3.3}$$

This new system contains the same number of equations as the system (2.5), (2.6) from which we started, namely, four equations. Nevertheless, it would be a mistake to think that the two systems are completely equivalent. The difference lies in the fact that the Eq. (2.5) is of the second order, while the Eq. (3.2) is of the fourth order. Therefore, the new system (3.2), (3.3) has far more integrals: it contains all the solutions of the system (2.5), (2.6) and in addition a comparable number of spurious solutions. Hence, only solutions may be used which also satisfy Eqs. (2.5), (2.6) or the Eqs. (2.13) equivalent to them. It is sufficient for them to fulfill one of the Eqs. (2.13) because the other can then be deduced from (3.2). We select, therefore the second equation (2.13) as a *subsidiary condition* which must be satisfied in order that the solutions of the system (3.2), (3.3) have physical reality.

It is convenient, however, to simplify that equation by an appropriate transformation. This can be done by eliminating from it the component E_z by means of Eq. (3.3) or in component form,

$$\frac{\partial E_z}{\partial x}=-\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}\right). \tag{3.4}$$

The subsidiary condition is first differentiated with respect to z, and then (3.4) is substituted, with the result

$$\left.\begin{aligned}&\frac{\partial}{\partial x}\left[(\epsilon k_0^2+M\nabla^2)E_x+iK\frac{\partial^2}{\partial z^2}E_y\right]\\&\qquad+\frac{\partial}{\partial y}\left[-iK\frac{\partial^2}{\partial z^2}E_x+(\epsilon k_0^2+M\nabla^2)E_y\right]=0.\end{aligned}\right\}\quad(3.5)$$

This is the final form of the subsidiary condition. It has the advantage that in the process of transformation the double sign has disappeared so that it represents a single equation.

4. Theory of the Fourth-Order Equation

Let Φ denote a function satisfying the equation

$$L(\Phi)=0,\quad(4.1)$$

where L is the operator (3.1) of the preceding section.

It was explained above that the coordinate z represents a preferred rectilinear direction fixed by the nature of the gyromagnetic medium, while the coordinates in the plane normal to z can be chosen arbitrarily. Under these circumstances it is no loss of generality if we restrict ourselves *prima facie* to cylindric systems of coordinates whose generatrix has the direction z. Then the operator ∇_p^2 in (3.1) is independent of the coordinate z which therefore becomes *cyclic*, that is, it enters into the equation only as a differential. This has the consequence that the function can be built up of partial solutions of the form

$$\Phi(x,y,z)=\Phi'(x,y)\exp(i\gamma z),\quad(4.2)$$

where γ is a constant. Hence, it is permissible to write

$$\partial/\partial z=i\gamma,\quad(4.3)$$

and to represent L in the simplified form

$$L\equiv\{MM_3\nabla_p^4+[(M+M_3)(\epsilon k_0^2-M\gamma)+K^2\gamma^2]\nabla_p^2+[(\epsilon k_0^2-M\gamma^2)^2-K^2\gamma^4]\}.$$

It is obvious that the operator can be split into two factors

$$L\equiv MM_3[\nabla_p^2+k_1^2]\cdot[\nabla_p^2+k_2^2],\quad(4.4)$$

where

$$k_1^2+k_2^2=[(\epsilon k_0^2-M\gamma^2)(M+M_3)+K^2\gamma^2]/MM_3,$$
$$k_1^2k_2^2=[(\epsilon k_0^2-M\gamma^2)^2-K^2\gamma^4]/MM_3.$$

Introducing the abbreviation,

$$f=[(M-M_3)^2(\epsilon k_0^2-M\gamma^2)^2+2(M+M_3)\times(\epsilon k_0^2-M\gamma^2)K^2\gamma^2+K^2(K^2+4MM_3)\gamma^4]^{\frac{1}{2}},$$

the explicit expressions become

$$\left.\begin{matrix}k_1^2\\k_2^2\end{matrix}\right\}=[(\epsilon k_0^2-M\gamma^2)(M+M_3)+K^2\gamma^2\pm f]/2MM_3.\quad(4.5)$$

Inasmuch as k_1 and k_2 are never equal, the general solution (4.2) is

$$\Phi=\Phi_1+\Phi_2,\quad(4.6)$$

where the two terms satisfy two ordinary wave equations,

$$(\nabla_p^2+k_1^2)\Phi_1=0,\quad(\nabla_p^2+k_2^2)\Phi_2=0.\quad(4.7)$$

In application to the vectors $\mathbf{E}$, which satisfy Eq. (4.1) in all three of their components, some additional remarks must be made. In view of the result (4.6) the potential $\mathbf{E}$ is divided into two vector parts,

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2.$$

For a given frequency ω the two part vectors have different wave numbers (k_1 and k_2) and therefore different velocities of propagation. Thus, it is hardly conceivable that Eq. (3.3) could be satisfied in any other way but by each part separately,

$$\nabla\cdot\mathbf{E}_1=0,\quad\nabla\cdot\mathbf{E}_2=0.$$

The same is true with respect to the subsidiary condition (3.5). Therefore, the two partial wave potentials, $\mathbf{E}_1$ and $\mathbf{E}_2$, are entirely independent as long as they propagate in the open space, but the independence ceases as soon as they hit a boundary: the boundary conditions, in general, establish a linkage between the two partial potentials.

5. (TEM)-Modes. Faraday Effect

Before treating the problem of wave propagation in its generality it is well to dispose of a few solutions relating to special or singular conditions. An interesting case arises when the component E_z of the vector potential is identically zero ($E_z=0$). Inasmuch as our final equations (3.2), (3.5) were obtained by elimination of E_z and of $\partial E_z/\partial_z$, it is not obvious that they still hold in this case. At any event, it is more convenient to fall back on the form (2.6), (2.13) of the basic equations which now become

$$[\epsilon k_0^2+M_3\nabla_p^2+(M\pm K)\partial^2/\partial z^2]Q=0,\quad(5.1)$$

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\mp i\frac{\partial}{\partial y}\right)Q=0,\quad(5.2)$$

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=0.\quad(5.3)$$

Taking into account the definition (2.8) of Q and the relation (5.3), one can write Eq. (5.2) as

$$\frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}\right)=0.\quad(5.4)$$

The variable z being cyclic (see preceding section), the solutions are of the type

$$\mathbf{E}=\mathbf{C}(x,y)\exp i\gamma z.\quad(5.5)$$

Equation (5.4) then becomes

$$\partial E_x/\partial y - \partial E_y/\partial x = 0. \tag{5.6}$$

From this, together with (5.3), follows,

$$\nabla_p^2 E_x = \nabla_p^2 E_y = \nabla_p^2 Q = 0. \tag{5.7}$$

The two equations contained in (5.1) take now the form

$$[\epsilon k_0^2 - (M+K)\gamma^2](E_x + iE_y) = 0,$$
$$[\epsilon k_0^2 - (M-K)\gamma^2](E_x - iE_y) = 0.$$

Since they must be satisfied simultaneously, there are two possibilities:

(a) $$\left.\begin{aligned} \epsilon k_0^2 - (M+K)\gamma^2 = 0, \\ E_x - iE_y = 0; \end{aligned}\right\} \tag{5.8}$$

(b) $$\left.\begin{aligned} \epsilon k_0^2 - (M-K)\gamma^2 = 0, \\ E_x + iE_y = 0. \end{aligned}\right\} \tag{5.9}$$

We first consider the case that the components E_x and E_y are independent of x and y, that is, the coefficient $\mathbf{C}$ in Eq. (5.5) is a constant vector. The vector $\mathbf{E}$ represents then a *plane wave* propagating in the preferred direction z. Our results show that the gyromagnetic medium can support two such plane waves with the respective propagation constants

$$\gamma_a = [\epsilon k_0^2/(M+K)]^{\frac{1}{2}}, \quad \gamma_b = [\epsilon k_0^2/(M-K)]^{\frac{1}{2}}. \tag{5.10}$$

According to Eqs. (5.7), (5.8) the components E_y and E_x are of the same magnitude but in case (a) the phase of E_y is advanced by $\pi/2$ compared with the phase of E_x; in case (b) it is retarded by the same amount. This means that the two waves are circularly polarized in opposite senses (Faraday effect).

In the ferrites the parameters ϵ and μ_1 (Sec. 1) are both large and κ approaches μ_1 as to the order of its magnitude, whence M and K are small as defined by Eqs. (2.3). Conversely, the propagation constants γ_a and γ_b are both large and considerably different making for a large difference in the optical paths of the two waves. The large Faraday effect is one of the most striking and important characteristics of the ferrites, and its value for practical applications was first pointed out by Hogan.[3] It offers the possibility of building into a wave guide a device called a *circulator* which is an analog to the quarter-wave and half-wave plates of optics. As is apparent from Eqs. (5.8) and (5.9), the sense of rotation of vector the $\mathbf{E}$ in space is the same for positive and negative propagation of the wave. This property makes it possible to obtain unilateral propagation by combining the circulator with suitable polarizers.[13]

In view of Maxwell's equations (2.1) it is a matter of course that in a plane wave the electric vector and the magnetic induction are transverse to the direction of propagation. In the particular case when this direction is z, the vectors $\mathbf{B}$ and $\mathbf{H}$ lie both in the plane normal to z, according to the fundamental relations (1.2) and (1.3). In the terminology of the wave-guide theory our plane waves represent thus (TEM)-modes, that is, modes in which both the electric and the magnetic field strengths are transverse to the propagation. Returning to the question of (TE)-modes outlined in the beginning of this section, i.e., of modes in which the vector $\mathbf{E}$ is transverse to the z direction, it is easy to see that there do not exist any others than the plane waves just discussed. Indeed, when the coefficient $\mathbf{C}$ of Eq. (5.5) is not a constant vector, the components E_x and E_y satisfy the Eqs. (5.7) and represent harmonic functions of the variables x, y. This fact combined with the validity of the relations (5.8) and (5.9) cannot be reconciled with the physical requirements of a mode. The harmonic functions are not appropriate for the unlimited medium because they necessarily possess singularities in some part of the field. Neither are they suitable for wave guides having the z-direction as their axis. In this case, the tangential component of $\mathbf{E}$ must vanish at the boundary and, according to Eqs. (5.8) and (5.9) the normal component will also vanish. But it is known from the theory of potentials that such boundary conditions cannot be satisfied. We shall see in Sec. 7 that (TM)-modes are also impossible.

Thus the (TEM)-modes in air filled coaxial guides have no analog in wave guides filled with a gyromagnetic medium.

6. Two-Dimensional Solutions

The fourth-order equation is also inconvenient when the field is independent of the coordinate z. This implies $\gamma = 0$, so that such solutions represent cylindric waves propagating in planes normal to the preferred direction. Equations (2.13) are now reduced to

$$(M\nabla_p^2 + \epsilon k_0^2)E_z = 0, \quad (M_3\nabla_p^2 + \epsilon k_0^2)Q = 0. \tag{6.1}$$

The last equation is equivalent to the two

$$(M_3\nabla_p^2 + \epsilon k_0^2)E_x = 0, \quad (M_3\nabla_p^2 + \epsilon k_0^2)E_y = 0, \tag{6.2}$$

to these must be added the transversality condition (2.6) in the form

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0. \tag{6.3}$$

It is apparent that the field defined by these equations can be represented as the superposition of two independent systems of modes.

(1) *Longitudinal Modes* ($E_x = E_y = 0$)

$$E_z \neq 0, \tag{6.4}$$

$$(M\nabla_p^2 + \epsilon k_0^2)E_z = 0. \tag{6.5}$$

[13] An up-to-date review of the experimental possibilities of ferrite devices is due to Fox, Miller, and Weiss, Bell System Tech. J. 34, 5 (1955).

Since in this case

$$\nabla_p\times\equiv(S_p)\nabla_p,\quad (S_p)=\begin{pmatrix}0, & 1\\ -1, & 0\end{pmatrix}, \tag{6.6}$$

the magnetic field, according to Eq. (2.2), lies in the (x,y)-plane and is expressed by

$$\mathbf{H}=(P_L)\nabla_p E_z, \tag{6.7}$$

$$(P_L)=-(i/k_0)(\mu^{-1})(S_p)=(i/k_0)\begin{pmatrix}iK, & -M\\ M, & iK\end{pmatrix}. \tag{6.8}$$

We shall see in Secs. 11, 12, and 13 that these modes have considerable practical interest.

(2) *Transverse modes* ($E_z=0$)

Equation (6.3) shows that the components E_x, E_y can be derived from a Hertzian function ϕ by the operation $(S_p)\nabla_p$ or

$$E_x=\partial\phi/\partial y,\quad E_y=-\partial\phi/\partial x, \tag{6.9}$$

provided that this function satisfies the equation

$$(M_3\nabla_p^2+\epsilon k_0^2)\phi=0. \tag{6.10}$$

The magnetic field is longitudinal and equal to

$$H_x=H_y=0,\quad H_z=-ik_0\epsilon M_3\phi. \tag{6.11}$$

7. General Field Expressions

We shall now develop the consequences of Eqs. (3.2) and (3.3)

$$L(\mathbf{E})=0,\quad \nabla\cdot\mathbf{E}=0, \tag{7.1}$$

when the subsidiary condition (3.5) is used in its general form. Taking into account Eqs. (4.3) and (4.7), it can be written as

$$\frac{\partial}{\partial x}[\tau E_x+i\sigma E_y]+\frac{\partial}{\partial y}[-i\sigma E_x+\tau E_y]=0, \tag{7.2}$$

with the abbreviations

$$\tau=M(k^2+\gamma^2)-\epsilon k_0^2,\quad \sigma=K\gamma^2. \tag{7.3}$$

As was brought out in the preceding sections this implies that the component E_z neither vanishes identically nor is independent of z.

The relation (7.2) points to the existence of a Hertzian function from which the two-bracket expressions may be derived by differentiation. Since this function is indeterminate to the extent of an arbitrary constant factor, it may be denoted by $(\tau^2-\sigma^2)\Pi$,

$$\left.\begin{aligned}\tau E_x+i\sigma E_y&=(\tau^2-\sigma^2)\partial\Pi/\partial y,\\ -i\sigma E_x+\tau E_y&=-(\tau^2-\sigma^2)\partial\Pi/\partial x.\end{aligned}\right\} \tag{7.4}$$

If we resolve with respect to E_x, E_y,

$$\left.\begin{aligned}E_x&=i\sigma\frac{\partial\Pi}{\partial x}+\tau\frac{\partial\Pi}{\partial y},\\ E_y&=-\tau\frac{\partial\Pi}{\partial x}+i\sigma\frac{\partial\Pi}{\partial y}.\end{aligned}\right\} \tag{7.5}$$

By substituting into the divergence equation (7.1), we obtain

$$\partial E_z/\partial z=i\gamma E_z=-i\sigma\nabla_p^2\Pi.$$

Inasmuch as Π must satisfy the same equation as its derivatives, namely, $L(\Pi)=0$, and the operator L has the form (4.4), it follows that

$$\nabla_p^2\Pi=k^2\Pi, \tag{7.6}$$

where k^2 is either k_1^2 or k_2^2, as defined by Eq. (4.5). Hence,

$$E_z=-(\sigma k^2/\gamma)\Pi=-i(\sigma k^2/\gamma^2)\partial\Pi/\partial z. \tag{7.7}$$

The results (7.5) and (7.7) can be summarized in the tensor equation,

$$\mathbf{E}=(S_E)\nabla\Pi, \tag{7.8}$$

$$(S_E)=\begin{bmatrix}i\sigma, & \tau, & 0\\ -\tau, & i\sigma, & 0\\ 0, & 0, & -i\sigma k^2/\gamma^2\end{bmatrix}. \tag{7.9}$$

Although this result was obtained under the assumption that the factor depending on z is exponential (namely, $\exp i\gamma z$), the fact that the tensor is even in γ shows that Eqs. (7.8) and (7.9) remain valid also when the factor is of the form $\cos z$ or $\sin z$.

The magnetic field is obtained with the help of the relations (2.2) which lead to another tensor equation

$$\mathbf{H}=\gamma(S_H)\nabla\Pi, \tag{7.10}$$

$$(S_H)=\begin{bmatrix}a, & ib, & 0\\ -ib, & a, & 0\\ 0, & 0, & g\end{bmatrix}, \tag{7.11}$$

with the abbreviations,

$$\begin{gathered}a=[M\tau-K^2(\gamma^2+k^2)]/k_0,\quad b=-K\epsilon k_0,\\ g=-M_3k^2\tau/k_0\gamma^2.\end{gathered} \tag{7.12}$$

With the help of Eqs. (7.3) for τ and (4.5) for k^2 it can be further shown that

$$a-g=\tau\epsilon k_0. \tag{7.13}$$

This relation is helpful for testing that all of Maxwell's equations are satisfied.

Unlike the relation (7.8) the validity of the formula (7.10) is limited to the exponential form of the factor depending on z in the function Π. However, this limitation disappears when it is written in the alternative form

$$\mathbf{H}=-i(S_H)\nabla(\partial\Pi/\partial z). \tag{7.14}$$

This equation possesses the same generality as (7.8). It must be borne in mind that the Eqs. (7.8) and (7.14) represent two systems of solutions since the parameter k in them can assume either the value k_1 or k_2.

8. Completeness of the Solution

An important question may obtrude itself on the reader. Do the solutions obtained in the preceding section represent all the modes possible under the specified limitations ($E_z \neq 0$, $\partial E_z/\partial z \neq 0$)? An answer to this question can be found by going to the limit $\kappa=0$ (or $K=0$): if the system (7.8) and (7.10) is complete the transition to the limit should produce from it all solutions possible in a medium without magnetic activity.

When κ is assumed to vanish, the medium described in Sec. 1 becomes a uniaxial crystal with the principal permeabilities μ_1, μ_1, μ_3. The electromagnetic field in them is described by equations resulting from Eqs. (2.5) and (2.6) when $\kappa=K=0$. Since the properties of such a medium have been much studied and are well known, it will be sufficient briefly to summarize their characteristics. The basic equations in them have then two solutions.

(1) The *system of* (TE)-*modes* is derivable from a Hertzian function subject to the equation

$$\nabla_p{}^2\Phi_1+\bar{k}_1{}^2\Phi_1=0, \quad \bar{k}_1{}^2=(\epsilon k_0{}^2-M\gamma^2)/M_3 \tag{8.1}$$

so that

$$\left.\begin{aligned} &E_x=-\frac{\partial\Phi_1}{\partial y}, && E_z=\frac{\partial\Phi_1}{\partial x}, && E_z=0,\\ &H_x=-(\gamma/\mu k_0)\frac{\partial\Phi_1}{\partial x}, && H_y=-(\gamma/\mu k_0)\frac{\partial\Phi_1}{\partial y}, && H_z=(i\bar{k}_1{}^2/\mu_3 k_0)\Phi_1.\end{aligned}\right\} \tag{8.2}$$

(2) The *system of* (TM)-*modes* results from a function Φ_2 satisfying the equation

$$\nabla_p{}^2\Phi_2+\bar{k}_2{}^2\Phi_2=0, \quad \bar{k}_2{}^2=(\epsilon k_0{}^2-M\gamma^2)/M, \tag{8.3}$$

with

$$\left.\begin{aligned} &E_x=\frac{\partial\Phi_2}{\partial x}, && E_y=\frac{\partial\Phi_2}{\partial y}, && E_z=-(i\bar{k}_2{}^2/\gamma)\Phi_2,\\ &H_x=-(k_0\epsilon/\gamma)\frac{\partial\Phi_2}{\partial y}, && H_y=(k_0\epsilon/\gamma)\frac{\partial\Phi_2}{\partial x}, && H_z=0.\end{aligned}\right\} \tag{8.4}$$

Equations (7.8) and (7.10), valid in a gyromagnetic medium, are more unified in form since they are represented by a single set of expressions and not by two different sets that appear in the magnetically inactive medium. However, it must be recalled that the set (7.8) and (7.10) is double because the wave number k in it can assume either of the two root values k_1 or k_2 of Eq. (4.5).

For very small values of K the parameter f of Eq. (4.5) assumes the expression

$$f=(M-M_3)(\epsilon k_0{}^2-M\gamma^2) +K^2\gamma^2(M+M_3+2M_3\gamma^2/\bar{k}_2{}^2)(M-M_3)^{-1}.$$

There follows for k^2,

$$\left.\begin{aligned} k_1{}^2&=\bar{k}_1{}^2+K^2\gamma^2(1+\gamma^2/\bar{k}_1{}^2)/(M-M_3)M_3,\\ k_2{}^2&=\bar{k}_2{}^2-K^2\gamma^2(1+\gamma^2/\bar{k}_2{}^2)/(M-M_3)M.\end{aligned}\right\} \tag{8.5}$$

At the same time the constant τ defined by Eq. (7.3) becomes

$$\tau=M(k^2-\bar{k}_2{}^2). \tag{8.6}$$

Now it is possible to carry out the transition to the limit of extremely small K. It is sufficient to consider the "vector potential" $\mathbf{E}$ since the magnetic field is determined by it through Eq. (2.2). Thus we shall be concerned only with Eqs. (7.8).

(1) $k=k_1$, in this case the limiting values for $K\to 0$ are

$$\tau_1=M(\bar{k}_1{}^2-\bar{k}_2{}^2). \tag{8.7}$$

This is finite, and $\sigma=K\gamma^2$ is negligible by comparison so that we must let $\sigma=0$, whence

$$E_x=ik_0\tau_1\frac{\partial\Pi_1}{\partial y}, \quad E_y=-ik_0\tau_1\frac{\partial\Pi_1}{\partial x}, \quad E_z=0.$$

This becomes identical with (8.2) when

$$\Pi_1=(i/\tau_1 k_0)\Phi_1,$$

because Π_1 obviously satisfies the Eq. (8.1).

(2) $k=k_2$, the limiting value is in this case

$$\tau_2=K^2\gamma^2(1+\gamma^2/\bar{k}_2{}^2)/(M-M_3). \tag{8.8}$$

Here τ_2 is negligible compared with σ so that we must let $\tau_2=0$, whence Eqs. (7.8) reduce to

$$E_x=ik_0\frac{\partial(\sigma\Pi_2)}{\partial x}, \quad E_y=ik_0\frac{\partial(\sigma\Pi_2)}{\partial y}, \quad E_z=(\sigma k_0\bar{k}_2{}^2/\gamma)\Pi_2.$$

Of course, the transition must be carried out in such a way that, while σ goes to zero the product $\sigma\Pi_2$ remains finite. Comparing these equations with (8.4) we see that the two sets are identical when

$$\sigma\Pi_2=-ik_0\Phi_2,$$

since the function Π_2 satisfies Eq. (8.3).

These considerations constitute the proof that our equations (7.8) and (7.10) contain indeed all possible modes (apart from some of the degenerate cases discussed in the Secs. 5 and 6). It is interesting to notice that the components E_z and H_z are always different from zero. While in a magnetically inactive medium every vibrational state can be represented as a superposition of a (TE)-mode and a (TM)-mode, this is not the case in the gyromagnetic medium. As was shown in Sec. 5 the only state in which **E** and **H** are transverse to z is the plane wave in the unlimited space [which is a (TEM)-mode].

PART II. SELECTED APPLICATIONS

9. Boundary Conditions

In practice the gyromagnetic medium (ferrite) is necessarily restricted, it is bounded by surfaces adjacent either to a metallic conductor or to a more or less perfect insulator, which may be the vacuum, a dielectric or another ferrite. In the microwave region metals can be safely regarded as perfect conductors, so that the boundary conditions at conducting surfaces are

$$E_t=0, \tag{I}$$

where the subscript t indicates a *tangential* component. On the other hand, at an insulating or partially conducting boundary the tangential components of **E** and **H** must be continuous in passing from the gyromagnetic medium (unprimed) into the adjacent (primed) one,

$$E_t=E_t', \quad H_t=H_t'. \tag{II}$$

Our equations (7.6), (7.8), and (7.10) represent the solution of the field problem in general cylindric coordinates, in which the choice of the mesh system in the plane normal to the preferred direction z is entirely unrestricted. However, the necessity of satisfying boundary conditions in the form (I) or (II) introduces restrictions. In the first place the principal method of satisfying these conditions—we may even say, the only practicable method—is the method of separation of variables. Let the parameters u, v of the mesh system in the (x,y)-plane be orthogonal, and the element of length dl be defined by the usual form,

$$dl^2=U^2du^2+V^2dv^2.$$

By separation of variables is meant that the Hertzian function satisfying Eq. (7.6) can be represented as the sum of partial solution each being the product of factors of only one of the parameters u, v, z, respectively, namely,

$$\Pi=\Phi(u)\Psi(v)\exp i\gamma z.$$

It is known that separation is possible only in elliptic coordinates[14] and their degenerations (Cartesian, polar, parabolic). It is then possible to satisfy the boundary conditions if the boundary surfaces are of the forms $u=\text{const}$, $v=\text{const}$, or $z=\text{const}$. Even in the case of isotropic media the method is straightforward for all types of elliptic coordinates only when the boundary conditions are of the form (I). In the case of the type (II) only plane and circularly cylindric boundaries are convenient to handle, while eliptic and parabolic cylinders lead to an infinity of simultaneous equations for the coefficients of the partial solutions, which seriously impair the usefulness of the method.

In the second place, the properties of the gyromagnetic medium, as expressed in Eqs. (7.8) and (7.10) introduce additional limitations in satisfying the boundary conditions. Indeed, the transverse components of **E** and **H** are according to these equations, of the general type of

$$E_v=\left[(C_1/U)\frac{\partial\Phi(u)}{\partial u}\Psi(v)\right.$$
$$\left.+(C_2/V)\Phi(u)\frac{\partial\Psi(v)}{\partial v}\right]\exp i\gamma z, \tag{9.1}$$

etc. where C_1, C_2 are two constants. Let us consider any cylindrical boundary, for instance $u=u_0$, at which the components E_v of the incident, reflected and transmitted waves must satisfy the conditions (I) or (II). A necessary prerequisite for fulfilling these conditions in a simple and direct way is that the factors depending on the variable v, changing along the boundary, be the same in both terms of the expression (9.1). This means that $\partial\Psi/\partial v$ and Ψ must have the same analytic structure, which is only possible when $\Psi(v)$ is an exponential. For any other function $\Psi(v)$, the boundary conditions of either form will result in an infinite system of simultaneous equations.

There exist two systems of coordinates in the elliptic family, in which the separation factors are exponential. (1) Cartesian coordinates where the factors depending on x and y may assume the form $\exp i\alpha x$, $\exp i\beta y$; (2) circularly cylindric coordinates where the azimuthal factor may be $\exp(in\varphi)$, n being a number and φ the azimuth. Thus the number of physical systems amenable to straightforward treatment is very limited and restricted to: (1) plane waves with their reflections from plane discontinuities; (2) rectangular wave guides; (3) circular wave guides with their axes in the preferred direction. Besides the conditions must be such as to make the relevant separation factors exponential (and not trigonometric).

[14] H. Weber, Math. Ann. **1**, 1 (1869); see also P. S. Epstein, Enzyklop. der Math. Wiss. V **24**, 505 (1915).

We repeat that the preceding conclusions apply only to cylindric boundaries parallel to the principal direction z. Remarks about satisfying the boundary conditions at the plane surfaces $z=\text{const}$ will be made in discussing the individual examples.

10. The Plane Wave

A special case of a plane wave was mentioned in Section 5. We wish now to treat the general case when the plane wave has the arbitrary direction of propagation z' under an angle ϑ to the preferred direction z. If the x-coordinate of the Cartesian system x,y,z is is chosen normal to the (z,z')-plane, the propagation vector lies in the (y,z)-plane and the Hertzian vector Π describing the wave according to the Eqs. (7.6), (7.8) becomes independent of x and equal to

$$\Pi = \exp i(ky+\gamma z) = \exp ik'z'. \tag{10.1}$$

From $z'=y\sin\vartheta+z\cos\vartheta$ there follows,

$$k=k'\sin\vartheta, \quad \gamma=k'\cos\vartheta, \tag{10.2}$$

or

$$k=\gamma tg\vartheta, \qquad k'=\gamma/\cos\vartheta. \tag{10.3}$$

The wavelength in the direction of propagation is

$$\lambda = 2\pi/k' = 2\pi\cos\vartheta/\gamma, \tag{10.4}$$

and the parameter τ of Eq. (7.3) is

$$\tau = (M\gamma^2/\cos^2\vartheta) - \epsilon k_0^2. \tag{10.5}$$

Equation (4.5) determining k, permits calculating γ in terms of the angle ϑ, with the result,

$$\gamma^2 = \epsilon k_0^2 \frac{2M-(M-M_3)\sin^2\vartheta \pm [4K^2\cos^2\vartheta + (M-M_3)^2\sin^4\vartheta]^{\frac{1}{2}}}{2[M^2-K^2+MM_3tg^2\vartheta]}. \tag{10.6}$$

The double sign shows that the medium can support in any direction two different plane waves. The limiting cases of this formula, $\vartheta=0$ and $\vartheta=\pi/2$, are already familiar: the case $\vartheta=0$ leads to Eq. (5.10) and the case $\vartheta=\pi/2$ yields $\gamma=0$, $k'^2=\epsilon k_0^2/M$ or $k'^2=\epsilon k_0^2/M_3$, in conformity with Eqs. (6.1) and (6.2).

The components of the electric vector $\mathbf{E}$ are readily obtainable from Eq. (7.8);

$$E_x = i\tau\gamma\Pi tg\vartheta, \quad E_y = -\sigma\gamma\Pi tg\vartheta, \quad E_z = \sigma\gamma\Pi tg^2\vartheta. \tag{10.7}$$

Of particular interest are the components in the directions x', y', z' associated with the direction of propagation z' so that y' lies in the (y,z)-plane and x' coincides with x,

$$E_{y'} = E_y\cos\vartheta - E_z\sin\vartheta, \quad E_{z'} = E_y\sin\vartheta + E_z\cos\vartheta,$$

whence

$$E_{x'} = E_x, \quad E_{y'} = -(\sigma/\sin\vartheta)\gamma\Pi, \quad E_{z'} = 0. \tag{10.8}$$

The components $E_{x'}$ and $E_{y'}$ are, in general, not equal in absolute value and possess a phase difference of $\pi/2$. Therefore, the waves they represent are elliptically polarized, and the sense of the polarization is opposite in the two waves because it can be shown that the parameter τ is positive in one and negative in the other in conformity with the double sign in Eq. (10.6). As is well known, it is an immediate consequence of Maxwell's equations (2.1) that in a plane wave the electric vector $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ are both transverse to the direction of propagation and orthogonal to each other. Hence, in relation to its elliptic polarization the induction $\mathbf{B}$ has the same characteristics as $\mathbf{E}$. However, in view of the tensor relation (1.2), this is not true of the magnetic strength of field $\mathbf{H}$. As a consequence of this relation $\mathbf{H}$ is not transverse but has a component in z'-direction. Besides, in addition to the rotation of $\mathbf{B}$ around this direction, $\mathbf{H}$ possesses another rotation about the preferred axis z imposed by the tensor (μ^{-1}).

The most significant difference between the elliptically polarized plane waves just discussed and the familiar analogous waves of an isotropic medium is as follows. In the isotropic medium the elliptically polarized plane wave can be always decomposed into a superposition of two independent linearly polarized waves. This is not possible in a gyromagnetic medium where the expressions (10.1) and (10.7) represent the irreducibly simplest type of wave motion. This fact has an important bearing on the problem of the *reflection of plane waves* from an infinite conducting plane or from an infinite plane of discontinuity separating another medium. If the media are isotropic the problem can be reduced to the reflection of linearly polarized waves with fixed directions of the vectors $\mathbf{E}$ and $\mathbf{H}$. In the boundary plane only one component E_t and one component H_t need be considered, and therefore, the boundary conditions reduce to one equation, in the case (I), and to two equations, in the case (II). The conditions are then satisfied by one appropriately reflected wave (I) or by one reflected and one transmitted wave (II). It is different when the media are gyromagnetic, the elliptic polarization of the plane waves is then irreducible and entails constantly changing directions of $\mathbf{E}$ and $\mathbf{H}$. Hence, it is necessary to consider two components E_t (at right angles to each other) and two components H_t. The boundary conditions amount to two (case I) or to four equations (case II). To satisfy them, respectively, two or four new waves are needed. These waves are available because the medium can support plane waves of two kinds ($k=k_1$ and $k=k_2$). Thus an incident wave produces two reflected waves (of the kinds k_1 and k_2) in its own medium and two transmitted waves in the adjacent medium.

Apart from this complication the mathematics of the problem is straightforward because the Hertzian vector in Cartesian coordinates has here the simple form $\Pi = \exp i(\alpha x + \beta y + \gamma z)$ which remains covariant in transformations to other systems of Cartesian coordinates. This expression is thus always a product of three exponential functions and permits of easily satisfying the boundary conditions at any plane surface, as was brought out in the preceding sections for the analogous case of side surfaces. Though cumbersome, the calculation of the reflected and transmitted fields is in all cases entirely elementary. However, the practical interest of the reflection coefficients of plane waves is so limited that it would not justify the space necessary for the detailed analysis.[15]

Suffice it to treat here a very simple case of reflection which has some theoretical interest in connection with the contents of the later Sec. 13; it is a case of a two-dimensional field in the sense of Sec. 6. Let us consider a plane polarized incident wave propagating in air and possessing the field

$$\left.\begin{aligned} E_z^{(i)} &= \exp ik_0(x\cos\zeta + y\sin\zeta), \\ H_x^{(i)} &= \sin\zeta E_z^{(i)}, \quad H_y^{(i)} = -\cos\zeta E_z^{(i)}. \end{aligned}\right\} \tag{10.9}$$

The negative half-space ($x<0$) is filled with air, the positive ($x>0$) with a gyromagnetic medium having z as its preferred direction. The wave (10.9) falls on the surface of discontinuity, $x=0$, under the angle ζ to the normal. The transmitted field is then described by Eqs. (6.4), (6.5), and (6.7) and is, therefore, also linearly polarized. Hence, the conditions are not much different from those in an isotropic medium: there is only one transmitted and one reflected wave. The field of the reflected wave is

$$\left.\begin{aligned} E_z^{(R)} &= R\exp ik_0(-x\cos\zeta + i\sin\zeta), \\ H_x^{(R)} &= \sin\zeta E_z^{(R)}, \quad H_y^{(R)} = \cos\zeta E_z^{(R)}. \end{aligned}\right\} \tag{10.10}$$

Introducing the abbreviation,

$$k_3 = (\epsilon/M)^{\frac{1}{2}} k_0 = n_3 k_0, \tag{10.11}$$

the transmitted wave follows from Eqs. (6.6) and (6.7)

$$\left.\begin{aligned} E_z^{(T)} &= T\exp ik_3(x\cos\zeta' + y\sin\zeta') \\ H_x^{(T)} &= (-iK\cos\zeta' + M\sin\zeta')n_3 E_z^{(T)}, \\ H_y^{(T)} &= -(M\cos\zeta' + iK\sin\zeta')n_3 E_z^{(T)}. \end{aligned}\right\} \tag{10.12}$$

The boundary conditions (II) are in this case: $E_z^{(i)} + E_z^{(R)} = E_z^{(T)}$; $H_y^{(i)} + H_y^{(R)} = H_y^{(T)}$, for $x=0$. In addition to

$$n_3 \sin\zeta' = \sin\zeta \tag{10.13}$$

they lead to the equations,

$$\left.\begin{aligned} 1+R &= T, \\ \cos\zeta(1-R) &= (M\cos\zeta' + iK\sin\zeta')n_3 T. \end{aligned}\right\} \tag{10.14}$$

Thus, the expressions (10.10) and (10.12) for the reflected and transmitted fields are completed by the explicit values of the coefficients,

$$\left.\begin{aligned} R &= [\cos\zeta - n_3(M\cos\zeta' + iK\sin\zeta')]/ \\ &\qquad [\cos\zeta + n_3(M\cos\zeta' + iK\sin\zeta')], \\ T &= 2\cos\zeta/[\cos\zeta + n_3(M\cos\zeta' + iK\sin\zeta')]. \end{aligned}\right\} \tag{10.15}$$

Worthy of attention is the asymmetry of R and T with respect to $\pm\zeta$ (and therefore, to $\pm\zeta'$) which has its origin in the helical character of the preferred ferrite axis.

11. Rectangular Wave Guide with Transverse Permanent Magnetization

Let the longitudinal direction of a wave guide filled with a gyromagnetic medium be denoted by x, and let the coordinates y, z of its cross section lie between the limits,

$$0<y<l, \quad 0<z<h. \tag{11.1}$$

If z is the direction of the permanent magnetic field, it is possible to have in the wave guide a two-dimensional wave in the sense of Sec. 6, and to apply Eqs. (6.5) and (6.7). Indeed, modes of the type

$$E_x = E_y = 0, \quad E_z = \sin\beta y \exp i\alpha x, \tag{11.2}$$

with

$$\alpha^2 + \beta^2 = k_3^2 = (\epsilon/M)k_0^2, \tag{11.3}$$

can be made to satisfy all boundary conditions at the walls of the guide. The conditions at the walls $z=0$, and $z=h$ are $E_x=E_y=0$, which are satisfied identically; at the walls $y=0$, $y=l$, they are $E_x=E_z=0$, which require

$$\beta = n\pi/l, \tag{11.4}$$

where n is an integer. Thus, the final expression of the field is

$$E_z = \sin(n\pi y/l)\exp i\alpha_n x, \tag{11.5}$$

$$\alpha_n = \pm(k_3^2 - n^2\pi^2/l^2)^{\frac{1}{2}}, \tag{11.6}$$

and according to Eq. (6.7), the magnetic field is

$$\left.\begin{aligned} H_x &= -ik_0^{-1}[K\alpha\sin\beta y + M\beta\cos\beta y]\exp i\alpha x, \\ H_y &= -k_0^{-1}[M\alpha\sin\beta y + K\beta\cos\beta y]\exp i\alpha x. \end{aligned}\right\} \tag{11.7}$$

The propagation constants α_n are, thus, discrete with the spectrum (11.6). For a given value of k_3, the constant α_n is real as long as $n<\pi k_3/l$, beyond this "cutoff," the propagation constants are imaginary and the modes (11.5) become *inhomogeneous*, or *evanescent*. In microwave practice it is desirable to have only one mode in the wave guide. This is accomplished

[15] Coefficients of reflection of plane waves were calculated by M. A. Gintzburg[12] for several cases, including even the more general problem of the reflection from a plane-parallel slab of a gyrotropic medium of special orientation.

by choosing the frequency ω so that $\pi/l<k_3<2\pi/l$; then all the modes except the fundamental ($n=1$) are inhomogeneous. If a higher mode should get excited, it dies out in a short length of the wave guide and only the fundamental mode remains. Thus, for practical applications, only the case $n=1$ is of interest.

Since the sine factor in Eq. (11.5) is independent of α_n, it is possible to solve the problem of reflection of the wave from a plane shorting plate, say, at $x=0$. The boundary conditions at the shorting plate are $E_y=0$, $E_z+E_z^{(R)}=0$, which are, obviously, satisfied by a reflected wave differing from the incident only in that α_n and $E_z^{(R)}$ are negative. The combined field of the incident and reflected waves is then

$$E_z^{(\mathrm{tot})}=2\sin(n\pi y/l)\sin\alpha x.$$

This leads, finally, to the solution of the problem of parallelepipedal cavity resonator of the length L in the x-direction. The boundary conditions at both ends are satisfied when $\alpha=m\pi/L$, m being another integer. Hence the characteristic frequency of the cavity resonator becomes (because of $\omega=ck_3M^{\frac{1}{2}}/\epsilon^{\frac{1}{2}}$),

$$\omega^2=(\pi c)^2(M/\epsilon)[(n/l)^2+(m/L)^2]. \qquad (11.8)$$

It must be borne in mind, however, that the constants M and ϵ are both functions of ω (Sec. 1), so that we have here an implicit equation for the determination of ω.

The other two-dimensional solution of Sec. 6 given by Eqs. (6.9) finds no application in rectangular wave guides because the conditions $E_x=E_y=0$ at the walls, $z=0$ and $z=h$, would be compatible only with the trivial case of $\mathbf{E}$ vanishing everywhere.

12. Rectangular Wave Guide with Transversely Magnetized Ferrite Lining

The wave guide which we wish to consider now is of the same shape and size as that of the preceding section but it is only partially filled with ferrite. The ferrite is restricted to a plane parallel layer along the wall $y=0$, extending to the interface at $y=d$, while the remaining space, $d<y<l$, is air filled.

Within the ferrite Eqs. (11.2), (11.3), and (11.7) of the preceding section remain valid. In the air-filled part the corresponding expressions are, obviously, omitting the factor $\exp i\alpha x$,

$$\begin{aligned} E_z^{(0)}&=C\sin\beta_0(y-l),\\ H_x^{(0)}&=ik_0^{-1}C\beta_0\cos\beta_0(y-l), \qquad (12.1)\\ H_y^{(0)}&=-k_0^{-1}C\alpha\sin\beta_0(y-l),\end{aligned}$$

$$\alpha^2+\beta_0^2=k_0^2. \qquad (12.2)$$

By this choice the conditions at the walls of the wave guide are satisfied. There only remains to fulfill the boundary conditions at the interface, $y=d$, namely, $E_z=E_z^{(0)}$, $H_x=H_x^{(0)}$, or explicitly,

$$\left.\begin{aligned} \sin\beta d&=C\sin\beta_0(d-l),\\ K\alpha\sin\beta d+M\beta\cos\beta d&=C\beta_0\cos\beta_0(d-l).\end{aligned}\right\} \qquad (12.3)$$

These equations are compatible when

$$tg\beta_0(d-l)=\beta_0[M\beta\cot g\beta d+K\alpha]^{-1}. \qquad (12.4)$$

Inasmuch as both β and β_0 are functions of α through the relations (11.3) and (12.2), this is the equation for the determination of the propagation constant α. The equation is of a transcendental type which possesses an infinity of roots accounting for the α spectrum. We shall not enter here into the numerical calculation of the roots.

It should be pointed out, however, that in this system the problem of reflection from a shorting plate is by no means simple. Equation (12.4) is not symmetric in $\pm\alpha$ so that waves with positive and negative directions of propagation have different propagation constants and consequently also different values of β and β_0. This causes the failure of the method of finding the reflected field, successfully used in the preceding section.

Further problems capable of solution with transversely magnetized ferrite in a rectangular wave guide are: (a) longitudinal ferrite slabs touching two walls of the guide and parallel to the other two (with magnetization parallel to their free surface).[11,16] (b) circular ferrite post, provided that its radius is small compared to the distance from either wall parallel to its axis.[17]

13. Reflection from Ferrite Filling a Rectangular Wave Guide

We consider an infinite wave guide of the same shape and size as in Sec. 11. However, only its positive half ($x>0$) is filled with a gyromagnetic medium (magnetized in the transverse z-direction), while the negative half ($x<0$) is air filled. Let an incident wave be moving in the air-filled half in the positive x-direction, we wish to discuss the problem of its reflection and transmission upon hitting the interface at $x=0$. As a matter of fact, our purpose is not to find workable expressions for the coefficient of reflection and transmission but merely to give an example of the type of mathematical difficulties which are involved in all but the simplest cases in the theory of gyromagnetic media.

It is convenient to choose the unit of length in such a way that the width of the wave guide, l, becomes equal to π, so that the range of variability of y is $0<y<\pi$. The incident wave in the air-filled half shall then be represented by the mode

$$\left.\begin{aligned} E_z^{(i)}&=\sin my\exp i\alpha_{0m}x,\\ H_y^{(i)}&=-k_0^{-1}\alpha_{0m}\sin my\exp i\alpha_{0m}x,\end{aligned}\right\} \qquad (13.1)$$

[16] Lax, Button, and Roth, J. Appl. Phys. **25**, 1413 (1954).
[17] P. S. Epstein and A. D. Berk (to be published).

where

$$\alpha_{0m}=(k_0^2-m^2)^{\frac{1}{2}}.$$

On the other hand, the following reflected modes are possible

$$\left.\begin{aligned}E_{nz}^{(R)}&=\sin ny\exp(-i\alpha_{0n}x),\\ H_{ny}^{(R)}&=k_0^{-1}\alpha_{0n}\sin ny\exp(-i\alpha_{0n}x),\end{aligned}\right\}\tag{13.2}$$

and according to Eqs. (11.5) and (11.6), the transmitted modes,

$$\left.\begin{aligned}E_{nz}^{(T)}&=\sin ny\exp i\alpha_n x,\\ H_{ny}^{(T)}&=-k_0^{-1}[M\alpha_n\sin ny+Kn\cos ny]\exp i\alpha_n x.\end{aligned}\right\}\tag{13.3}$$

The boundary conditions at the interface ($x=0$) are

$$E_z^{(i)}+E_z^{(R)}=E_z^{(T)},\quad H_y^{(i)}+H_y^{(R)}=H_y^{(T)}.\tag{13.4}$$

These conditions cannot be satisfied by choosing the reflected and transmitted waves as single modes. Mathematically this is apparent from the structure of the factor depending on y in $H_{ny}^{(T)}$ which contains two terms with $\sin ny$ and $\cos ny$; thus it cannot be matched for all values of y with $H_{ny}^{(i)}$ and $H_{ny}^{(R)}$. The underlying physical cause can be seen by the following consideration. As far as the electric vector is concerned, the waves (13.1), (13.2), and (13.3) can be regarded as superpositions of two plane waves, for instance,

$$E_z^{(i)}=-\tfrac{1}{2}i\exp i(my+\alpha_{0m}x)+\tfrac{1}{2}i\exp i(-my+\alpha_{0m}x).$$

Each of these modes consists of two plane waves of equal amplitudes and opposite phases, including oppositely equal angles with the interface. However, it was shown in Sec. 11 that the reflection of a plane wave of this sort is asymmetric, a consequence of the gyroidal properties of the ferrite. Thus, the fields in the two media cannot be joined together if they are expressed by single modes.

In view of this we represent the reflected and transmitted waves in the most general form available, namely, as the superposition of all possible modes,

$$E_z^{(R)}=\sum_{n=1}^{\infty}R_nE_{nz}^{(R)},\quad E_z^{(T)}=\sum_{n=1}^{\infty}T_nE_{nz}^{(T)},\tag{13.5}$$

with similar expressions for $H_y^{(R)}$, $H_y^{(T)}$. We shall consider the only case interesting in practice, when the incident wave is the fundamental mode and the frequency is such that all overtones are inhomogeneous,

$$m=1,\quad \alpha=i|\alpha|,\quad (\text{for } n>1).\tag{13.6}$$

Then the boundary conditions (13.4) lead to the following explicit equations,

$$\left.\begin{aligned}&\sin y+\sum_{n=1}^{\infty}(R_n-T_n)\sin ny=0,\\ &\alpha_{01}\sin y-\sum_{n=1}^{\infty}[(\alpha_{0n}R_n+M\alpha_nT_n)\sin ny\\ &\qquad\qquad+Kn\cos ny]=0,\end{aligned}\right\}\tag{13.7}$$

for all values of y between 0 and π. The following relations will be useful

$$\int_0^\pi\sin my\sin ny\,dy=\int_0^\pi\cos my\cos ny\,dy=\begin{cases}0, & \text{for } m\neq n,\\ \frac{1}{2}\pi, & \text{for } m=n.\end{cases}$$

$$\int_0^\pi\sin my\cos ny\,dy=\begin{cases}0, & \text{for } (m-n) \text{ even},\\ 2m/(m^2-n^2), & \text{for } (m-n) \text{ odd}.\end{cases}$$

Multiplying the first equation (13.7) by $\sin my$ and integrating from 0 to π gives

$$T_1=R_1+1,\quad T_n=R_n\quad (\text{for } n>1),$$

so that the second equation can be transformed into

$$\left.\begin{aligned}&(\alpha_{01}-M\alpha_1)\sin y-H\cos y\\ &\quad-\sum_{n=1}^{\infty}R_n[(\alpha_{0n}+M\alpha_n)\sin ny+Kn\cos ny]=0.\end{aligned}\right\}\tag{13.8}$$

When K is small ($K\ll1$), this system can be resolved with respect to R_n by successive approximations. Letting

$$R_n=R_n^0+KR_n'+K^2R_n''+\cdots,$$

with the abbreviations

$$A=\alpha_{01}-M\alpha_1,\quad B_n=\alpha_{0n}+M\alpha_n,$$

one finds,

$$R_1^0=A/B_1,\quad R_1'=0,$$

$$R_1''=-\left(\frac{8}{\pi}\right)^2\frac{A+B_1}{B_1^2}\sum_{n=1}^{\infty}\frac{1}{B_{2n}}\cdot\frac{n^2}{(4n^2-1)^2}.\tag{13.9}$$

Only the homogeneous part of the reflected wave ($n=1$) is of practical interest because the inhomogeneous modes ($n>1$) do not carry any energy.

However, in ferrites K is not small and, therefore, the computation of the reflection meets with great difficulties. The multiplication of the expression (13.8) by $\sin my (m=0, 1, 2, \cdots\infty)$—or, alternatively, by $\cos my$—and integration from 0 to π leads to an infinite system of simultaneous linear equations of a rather slow convergence. We shall not write out this cumbersome set of equations because we do not know of any practicable method of resolving them.

14. Circularly Cylindric Coordinates

When the magnetization is longitudinal the boundary conditions at the walls of a rectangular wave guide cannot be satisfied by single modes. Indeed, the requirements that E_z must vanish both for $x=0$, $x=l_1$ and $y=0$, $y=l_2$ are then contradictory. According to the criterion at the end of Sec. 9 the condition at $x=0$, $x=l_1$ entails the exponential character of the factor depending on y, that is, $\exp i\beta y$. However, with such a factor it is impossible to fulfill the conditions at $y=0$, $y=l_2$. On the other hand there is no such difficulty in circular wave guides.

If a cylindric system r, φ, z is introduced with its origin in the axis z of the wave guide, Eq. (7.6) for the Hertzian function becomes

$$\frac{\partial^2\Pi}{\partial r^2}+\frac{1}{r}\frac{\partial\Pi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Pi}{\partial\varphi^2}+k^2\Pi=0, \tag{14.1}$$

so that it can be represented as

$$\Pi=C_n(kr)\exp i(n\varphi+\gamma z), \tag{14.2}$$

where $C_n(kr)$ denotes the general cylindric function of the order n, and k can assume the values k_1 or k_2 of Eq. (4.5).

The electric and magnetic fields follow from the Eqs. (7.8), (7.10). Leaving out the factor $\exp i(n\varphi+\gamma z)$ in all components, we have

$$\left.\begin{aligned} E_r&=i[k\sigma C_n'(kr)+(n\tau/r)C_n(kr)],\\ E_\varphi&=-[k\tau C_n'(kr)+(n\sigma/r)C_n(kr)],\\ E_z&=(k^2\sigma/\gamma)C_n(kr).\end{aligned}\right\} \tag{14.3}$$

$$\left.\begin{aligned} H_r&=\gamma[kaC_n'(kr)-(nb/r)C_n(kr)],\\ H_\varphi&=-i\gamma[kbC_n'(kr)-(na/r)C_n(kr)],\\ H_z&=i\gamma^2gC_n(kr).\end{aligned}\right\} \tag{14.4}$$

This type of wave is materially different from the plane wave propagating in the preferred direction discussed in Sec. 5. To make this clear it is best to consider the conditions for large values of the argument kr and to identify $C_n(kr)$ with the Hankel function of the first kind whose asymptotic expression for $x\ll 1$, is $H_n^{(1)}(x)\sim(\frac{1}{2}\pi x)^{-\frac{1}{2}}\exp ix$. Thus the Hertzian function gains the expression,

$$\Pi\sim(\tfrac{1}{2}\pi kr)^{-\frac{1}{2}}\exp i(kr+n\varphi+\gamma z).$$

At a given point it represents a wave possessing r and φ components of propagation, it may be called a *divergent helical wave.*

It is apparent that the phase difference between the two transverse components of $\mathbf{E}$ is $\frac{1}{2}\pi$ while their absolute values are different. Therefore, the waves represented by (14.3) and (14.4) are elliptically polarized and in general the degree of elipticity changes with the distance r from the axis since the ratio of the axes of the ellipse is equal to the absolute value of the ratio E_r/E_φ. It will be seen that this ratio is not the same for n and for $-n$, and this is not surprising in view of the remarks made above, since $+n$ and $-n$ characterize two quite distinct modes of the system. In comparison with the conditions in an isotropic medium there is the important difference that the two modes, $+n$ and $-n$, are asymmetric so that they cannot be combined to form a wave with linear polarization. Indeed,

$$\begin{aligned}\tfrac{1}{2}[E_r(n)+E_r(-n)]=[&ik\sigma C_n'(kr)\cos n\varphi\\ &-(n\tau/r)C_n(kr)\sin n\varphi]\exp i\gamma z,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}[E_\varphi(n)+E_\varphi(-n)]=-[&k\tau C_n'(kr)\cos n\varphi\\ &+i(n\sigma/r)C_n(kr)\sin n\varphi]\exp i\gamma z.\end{aligned}$$

For the same sign of n, the characteristics of the elliptic polarization are different for $k=k_1$ and $k=k_2$, which also constitute two different modes. It was shown in Sec. 9 that τ_1 and τ_2 have opposite signs.[18]

15. Circular Wave Guide with Longitudinal Magnetization

If the whole infinite wave guide is filled with a gyromagnetic medium, the function $C_n(kr)$ in Eq. (14.2) must be identified with the Bessel function $J_n(kr)$ which is the only cylindric harmonic without singularity at $r=0$. Hence,

$$\Pi=J_n(kr)\exp i(n\varphi+\gamma z). \tag{15.1}$$

It is convenient to choose the radius of the wave guide as the unit of length ($r=1$). The boundary conditions at the walls are then

$$E_\varphi(1)=E_z(1)=0. \tag{15.2}$$

It is clear that the expressions (14.3) cannot be made to satisfy these conditions; however, there are two such expressions available since the parameter k can assume the value k_1 or k_2. We shall call them *the first and the second system of solutions* and shall designate them, respectively, by the indices (1) and (2). Thus we can build up the Hertzian,

$$\Pi=A_1\Pi_1+A_2\Pi_2. \tag{15.3}$$

This combination can satisfy the conditions (15.2) for all values of z and φ only when the factor $\exp i(n\varphi+\gamma z)$ is the same in both terms so that it can be canceled out. Therefore,

$$\gamma_1=\gamma_2=\gamma,\quad n_1=n_2=n.$$

This leaves, however, $k_1\neq k_2$, because the two parameters are different functions of γ.

The boundary conditions become now

$$\begin{aligned}A_1E_{1\varphi}(1)+A_2E_{2\varphi}(1)&=0,\\ A_1E_{1z}(1)+A_2E_{2z}(1)&=0.\end{aligned} \tag{15.4}$$

They are compatible when,

$$E_{1\varphi}(1)E_{2z}(1)-E_{2\varphi}(1)E_{1z}(1)=0,$$

or according to (14.3)

$$\begin{aligned}&[k_1\tau_1J_n'(k_1)+n\sigma J_n(k_1)]k_2^2J_n(k_2)\\ &\quad-[k_2\tau_2J_n'(k_2)+n\sigma J_n(k_2)]k_1^2J_n(k_1)=0.\end{aligned} \tag{15.5}$$

Inasmuch as both k_1 and k_2 are functions of propagation constant γ, this relation serves to determine the permissible values of γ in the modes (15.3) of propagation for a given frequency ω. Indirectly it also determines the permissible values of k_1 and k_2 which shall be, therefore, called the roots of Eq. (15.5).

The ratio of the coefficients becomes

$$A_2/A_1=-k_1^2J_n(k_1)/k_2^2J_n(k_2), \tag{15.6}$$

[18] Curves relating to the magnitudes of the Faraday effect were calculated by Gamo.[9]

where k_1, k_2 satisfy Eq. (15.5), while the Hertzian can be written explicitly as

$$\Pi=[k_2^2J_n(k_2)J_n(k_1r)-k_1^2J_n(k_1)J_n(k_2r)]\times\exp i(n\varphi+\gamma z). \quad (15.7)$$

In view of the definitions (4.5) of k_1 and k_2 the bracket expression in Eq. (15.7) is independent of the sign of γ. This makes it possible to reflect the wave from a plane shorting plate. The total Hertzian, comprising the incident and the reflected waves, is then

$$\Pi=[A_1J_n(k_1r)+A_2J_n(k_2r)]\exp in\varphi\sin\gamma z. \quad (15.8)$$

When the propagation constant is chosen as

$$\gamma=\gamma m/l, \quad (15.9)$$

the expression (15.8) represents the field in a cylindric resonance cavity of the length l. Equations (15.5) and (15.8) jointly define the spectrum of characteristic frequencies of such a cavity.

16. Coaxial Wave Guide with Longitudinal Magnetization

Under more general conditions than in the preceding section the expressions become rather cumbersome. Therefore, in the remaining part of this paper we shall be satisfied with a schematic demonstration how solutions can be obtained without writing them out in detail.

Let the coaxial wave guide have the outer radius 1 and the inner R. The boundary conditions are

$$E_\varphi=E_z=0, \quad \text{for} \quad r=1, \quad r=R. \quad (16.1)$$

Since the singular line $r=0$ of the partial differential equation (14.1) is now excluded from the range of the variables, in addition to Eq. (15.1) can be used the solution

$$\bar{\Pi}=N_n(kr)\exp i(n\varphi+\gamma z), \quad (16.2)$$

with the Neumann cylindric function as a factor. Solutions containing Neumann functions will be characterized by bars. The components of $\bar{\mathbf{E}}$ and $\bar{\mathbf{H}}$ are then obtained by identifying $C_n(kr)$ in Eqs. (14.3) and (14.4) with $N_n(kr)$. Two values of k being always available, the complete expressions of E_y and E_z become

$$\left.\begin{aligned} E_\varphi&=A_1E_{1\varphi}+A_2E_{2\varphi}+\bar{A}_1\bar{E}_{1\varphi}+\bar{A}_2\bar{E}_{2\varphi},\\ E_z&=A_1E_{1z}+A_2E_{2z}+\bar{A}_2\bar{E}_{1z}+\bar{A}_2\bar{E}_{2z}.\end{aligned}\right\} \quad (16.3)$$

Substituted into the boundary conditions (16.1) these expressions lead to a system of four simultaneous equations which are linear and homogeneous in the four coefficients A. The compatibility condition is

$$\begin{vmatrix} E_{1\varphi}(1), & E_{2\varphi}(1), & \bar{E}_{1\varphi}(1), & \bar{E}_{2\varphi}(1) \\ E_{1z}(1), & E_{2z}(1), & \bar{E}_{1z}(1), & \bar{E}_{2z}(1) \\ E_{1\varphi}(R), & E_{2\varphi}(R), & \bar{E}_{1\varphi}(R), & \bar{E}_{2\varphi}(R) \\ E_{1z}(R), & E_{2z}(R), & \bar{E}_{1z}(R), & \bar{E}_{2z}(R) \end{vmatrix}=0. \quad (16.4)$$

This is the equation which takes the place of (15.5) for the determination of the permissible propagation constants γ. With respect to the reflection from a plane shorting plate exactly the same considerations apply as in the case of the preceding section. To obtain the joint fields of the incident and the reflected waves it is only necessary to replace the factor $\exp i\gamma z$ in the potentials Π and $\bar{\Pi}$ by $\sin\gamma z$. The characteristic frequencies of a coaxial resonance cavity of the length l are then determined by the simultaneous equations (15.9) and (16.4).

17. Circular Wave Guide with Coaxial Ferrite Core

Let us consider an infinite circular wave guide inside which are two concentric and different gyromagnetic media. The inner fills a solid cylinder of the radius R, the outer the hollow coaxial cylinder outside up to the radius $r=1$. The inner is longitudinally magnetized; let the outer cylinder be characterized by unprimed letters k_1, k_2 the inner by primed k_1', k_2'.

In the outer cylinder the field can be represented in the same way as in Sec. 16 by Eqs. (16.3) to which must be added the expressions for the magnetic components of the same form. The inner (primed) region, $r<R$, contains the singularity, $r=0$, so that only Bessel functions may be used in its description. The representation of the field is, therefore, that given by Eqs. (15.3) in conjunction with (14.3). It can be written as

$$\mathbf{E}'=A_1'\mathbf{E}_1'+A_2'\mathbf{E}_2', \quad \mathbf{H}'=A_1'\mathbf{H}_1'+A_2'+\mathbf{H}_2'. \quad (17.1)$$

The accents at $\mathbf{E}'$, $\mathbf{H}'$ indicate that the wave numbers in them are k_1', k_2' characterizing the inner (primed) medium.

Altogether our expressions contain six coefficients A, and correspondingly there exist six boundary conditions: namely, two at the outer boundary, $r=1$,

$$E_\varphi(1)=E_z(1)=0, \quad (17.2)$$

and four at the interface, $r=R$,

$$\begin{aligned} E_\varphi(R)&=E_\varphi'(R), & E_z(R)&=E_z'(R),\\ H_\varphi(R)&=H_\varphi'(R), & H_z(R)&=H_z'(R).\end{aligned} \quad (17.3)$$

Thus the complete set of condition becomes

$$\begin{aligned} &A_1E_{1\varphi}(1)+A_2E_{2\varphi}(1)+\bar{A}_1\bar{E}_{1\varphi}(1)+\bar{A}_2\bar{E}_{2\varphi}(1)=0,\\ &A_1E_{1z}(1)+A_2E_{2z}(1)+\bar{A}_1\bar{E}_{1z}(1)+\bar{A}_2\bar{E}_{2z}(1)=0,\\ &A_1E_{1\varphi}(R)+A_2E_{2\varphi}(R)+\bar{A}_1\bar{E}_{1\varphi}(R)+\bar{A}_2\bar{E}_{2\varphi}(R)\\ &\qquad -A_1'E_{1\varphi}'(R)-A_2'E_{2\varphi}'(R)=0,\\ &A_1E_{1z}(R)+A_2E_{2z}(R)+\bar{A}_1\bar{E}_{1z}(R)+\bar{A}_2\bar{E}_{2z}(R)\\ &\qquad -A_1'E_{1z}'(R)-A_2'E_{2z}'(R)=0,\\ &A_1H_{1\varphi}(R)+A_2H_{2\varphi}(R)+\bar{A}_1\bar{H}_{1\varphi}(R)+\bar{A}_2\bar{H}_{2\varphi}(R)\\ &\qquad -A_1'H_{1\varphi}'(R)-A_2'H_{2\varphi}'(R)=0,\\ &A_1H_{1z}(R)+A_2H_{2z}(R)+\bar{A}_1\bar{H}_{1z}(R)+\bar{A}_2\bar{H}_{2z}(R)\\ &\qquad -A_1'H_{1z}'(R)-A_2'H_{2z}'(R)=0.\end{aligned} \quad (17.4)$$

The vanishing of the determinant of this system provides the relation determining the permissible values of the propagation constant.

When the outer medium is *isotropic*, for instance air, the equations are still valid but with a different definition of the unprimed components. In this case the first and the second system are, respectively, represented by the (TE) and (TM) modes of Sec. 8 (with $M=M_3$). In both systems the potential is

$$\Phi=C_n(kr)\exp i(n\varphi+\gamma z),$$

where

$$k_1=k_2=k=(\epsilon\mu k_0^2-\gamma^2)^{\frac{1}{2}}. \tag{17.5}$$

Consequently Eqs. (8.2) and (8.4) lead to the following expressions omitting the factor $\exp i(n\varphi+\gamma z)$:

$$\left.\begin{aligned} &E_{1r}=-i(n/r)C_n(kr), && E_{1\varphi}=kC_n'(kr), && E_{1z}=0,\\ &H_{1r}=-(k\gamma/\mu k_0)C_n'(kr), && H_{1\varphi}=-(n\gamma/\mu k_0 r)C_n(kr), && H_{1z}=i(k^2/\mu k_0)C_n(kr).\end{aligned}\right\} \tag{17.6}$$

$$\left.\begin{aligned} &E_{2r}=kC_n'(kr), && E_{2\varphi}=i(n/r)C_n(kr), && E_{2z}=-(k^2/\gamma)C_n(kr),\\ &H_{2r}=-i(nk_0\epsilon/\gamma r)C_n(kr), && H_{2\varphi}=(kk_0\epsilon/\gamma)C_n'(kr), && H_{2z}=0.\end{aligned}\right\} \tag{17.7}$$

The case of the isotropic outer medium is of great importance because it corresponds to the conditions in ferrite circulators. It is indeed an essential theoretical problem to investigate what influence the axial ferrite core has on the wave propagation in the isotropic space around it. Unfortunately Eqs. (17.4) are rather involved and, especially, the determination of the roots of their compatibility condition would require very cumbersome numerical calculations. However, it is possible to form, in a qualitative way, an idea about the nature of the resultant modes when the materials of both media are nonabsorbent. This can be done by the following successive steps. (1) From the principle of conservation of energy it may be concluded that the characteristic values of k, k_1', k_2' resulting from the compatability condition are always real. Hence, the real or imaginary character of all the components given by Eqs. (14.3), (14.4), (17.6), and (17.7) is determined solely by the occurrence in them of the explicit imaginary factor i. (2) When the coefficients A_2, $\bar{A}_2$ are replaced by $A_2=iD$, and $\bar{A}_2=i\bar{D}$ and Eqs. (17.4) are written out in terms of the cylindric functions, the factor i can be canceled from those equations in which it appears, and the whole system becomes entirely real. This means that all the ratios of the A and D coefficients are real, including $\bar{A}_1/A_1$, D/A_1, $\bar{D}/A_1$. (3) To bring out the influence of a reversal of the sign of n upon the coefficients it is helpful to compare the case under discussion with the case when the inner medium is isotropic (like the outer). Under the latter conditions the system (17.4) acquires a remarkable symmetry: when the sign of n is reversed and at the same time also the signs of D and $\bar{D}$, the equations remain the same. This means that (for constant A_1) the coefficient $\bar{A}_1$ is symmetric in $\pm n$ and the coefficients D, $\bar{D}$ are antisymmetric. When the inner medium is gyromagnetic no such symmetry exists, all three coefficients $\bar{A}_1$, D, $\bar{D}$ are entirely asymmetric with respect to $\pm n$.

Writing out the components E_r and E_φ in detail we find, omitting the factor $\exp i(n\varphi+\gamma z)$,

$$E_r=-i\{(n/r)[A_1J_n(kr)+\bar{A}_1N_n(kr)] + k[DJ_n'(kr)+\bar{D}N_n'(kr)]\},$$

$$E_\varphi=k[A_1J_n'(kr)+\bar{A}_1N_n'(kr)] - (n/r)[DJ_n(kr)+\bar{D}N_n(kr)].$$

When the inner core is absent, these expressions reduce each to a single term; for instance, the term with A_1, for (TE)-modes. It is clear that the modes $+n$ and $-n$ are in this case elliptically polarized in opposite senses but otherwise symmetric so that linearly polarized waves can be obtained by combining them. The presence of an inner cylinder of different material causes the appearance of the three additional terms. But, as long as the material of this core is isotropic, the symmetry of the $+n$ and $-n$ modes remains unchanged because of the symmetric properties of the coefficients mentioned above. It is different when the core material is gyromagnetic: the coefficients $\bar{A}$, D, $\bar{D}$ are then asymmetric and affect the respective ellipticities of the $+n$ and $-n$ modes to quite different extents. In consequence there does not exist any combination of the two modes having linear polarization.

As already mentioned, the quantitative evaluation of this effect requires laborious numerical calculations. In the case of an extremely thin ferrite core ($R\ll 1$) Van Trier gave a first approximation for the characteristic value of k, but the magnitudes of the coefficients A and D remain undetermined even for this case.[19]

Finally, it is interesting to note that the respective parity in $\pm\gamma$ of every component in Eqs. (17.6) and (17.7) is the same as in Eqs. (14.3) and (14.4). Therefore, the system (17.4) is invariant with respect to a reversal of the sign of γ, so that the modes moving in the positive and negative directions have the same spectrum of propagation constants. This makes it possible to solve the problems of reflection from a plane shorting plate and of the characteristic frequencies of a cavity resonator in the same way as in the two preceding sections.

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[19] The theoretical knowledge of the characteristic value of k opens a way for the experimental determination of the ferrite constants. Compare: A. A. Th. M. Van Trier, Appl. Sci. Research **B3**, 142 (1953).