

in detail,<sup>29</sup> the dependence of the velocity on the wavelength (dispersion) of the electron wave gives rise to important effects. It is still not obvious how these patterns will be experimentally obtained. Neither of the instruments so far constructed, even in a more developed form, appears to be suitable. The wave-front splitter of the German group, judging from the results of Haine and Mulvay,<sup>30</sup> can be expected to run into the geometrical coherence limit of the source before path lengths of ten thousand wavelengths are obtained, provided that the "aberrations" of their biprism do not present a prior limitation. This path difference is still far below the fundamental coherence limit of the electron wave train. The NBS instrument, on the other hand, although in principle avoiding any fundamental limit set by the geometrical coherence of the source, has the disadvantage for this application of being achromatic and hence since its coherence pattern is more or less independent of the initial wave train or, at least, is not simply related to the more usual visibility patterns.

As additional detailed theoretical studies of the electron wave train are made, perhaps it will become obvious that the coherence length of the electron wave train is so long as to constitute no experimental limi-

<sup>29</sup> D. Gabor, *Revs. Modern Phys.* **28**, 206 (1956), this issue.

<sup>30</sup> M. E. Haine and T. Mulvay, *J. Opt. Soc. Am.* **42**, 763 (1952).

tation to interference experiments. In the work presented here by Dr. Gabor and some unpublished calculations of Professor Uyeda, it was indicated that this may be the case.

Development work is continuing on electron interferometers at NBS under Dr. Marton, and in the laboratories of Professor Mollenstedt in Tubingen. At Nagoya, Japan, under the direction of Professor Uyeda, a group is about to start another effort to develop a useful interferometer. The German group is continuing the development of several different interferometer geometries, having in common division of wave front by the use of the Fresnel biprism. The groups in America and Japan are using the wave amplitude dividing properties of crystalline lamella in different arrangements. There are theoretical and practical advantages of both types of dividers, and only time and further experience will tell which will prove to have the greater utility. One thing does seem certain; the electron interferometer will continue to develop. By its use we will continue to expand our empirical knowledge of the wave properties of the free electron and in time be able to place our knowledge on the same firm footing that has been achieved for light.

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## Theory of Electron Interference Experiments

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### INTRODUCTION

THE last years have seen the successful realization of the first electron interferometers by Marton, and by Mollenstedt and Düker.<sup>1-9\*</sup> In these instruments a primary electron beam is split into at least two beams, of which at least two are reunited at the photographic plate or other type of receptor. The essential feature is, that these interference phenomena arise by the intersection of at least two beams, in different directions. In this they differ fundamentally from electron diffraction experiments, now known for over 30 years, in which one beam only is observed at any point of the receptor. Here the interference is between the elementary wavelets, which are split off the primary beam by the atomic scattering centers, and extinguish one another outside a very small angle.<sup>†10</sup>

\* References are listed at the end of the article.

† Thomson and Cochrane<sup>10</sup> have proposed "interfraction" as a more fitting name for this phenomenon.

On the other hand, experiments were known for some time in which *real* interferences were observed, that is to say interferences between two scattered beams, or between scattered beams and the primary. First of these were the observations of Fresnel diffraction fringes at the edges of electron microscope objects, by Boersch<sup>11</sup> and by Hillier,<sup>12</sup> in which the edge-wave and the wave which had penetrated through a thin foil with a certain change of phase was brought to interference with the primary beam. Interferences between two diffracted beams were first observed in the electron microscope by Mitsuishi, Nagasaki, and Uyeda<sup>13</sup> in 1951. See also Fues and Wagner,<sup>14</sup> Menzel-Kopp,<sup>15-16</sup> Pfister,<sup>17</sup> Rang,<sup>18-19</sup> Ito and Ito,<sup>20</sup> Niehrs,<sup>21</sup> Hibi, Kambe, and Honjo,<sup>22</sup> and Farrand and Rees.<sup>23</sup> These arise, for instance in lamellar crystals, in which the flakes form slight angles with one another, or blisters. We shall not deal in detail with these more or

less fortuitous interferometers, but the general theory to be developed will cover these too.

The purpose of this article is to provide a theory for the use of experimenters in the further development and utilization of interferometers, with special emphasis on the problem of coherence, that is to say, on the conditions in which interferences can be observed. Such a theory did not exist so far because it was not needed, except perhaps in connection with the electron diffraction microscope (Gabor,<sup>24</sup> Haine and Dyson<sup>25</sup>). Experimenters were mainly guided by light-optical analogs, without being quite sure how far this analogy could be extended. The need for an "adequately detailed wave-mechanical treatment of the interference of quasi-monoenergetic electrons" was voiced in particular by J. Arol Simpson.<sup>5</sup>

Though the theory will be very general, it will be useful to keep in mind the design of the only two electron interferometers which so far have proved successful, which are sketched out in Figs. 1 and 2. In the Marton-type interferometer the beam traverses in succession three thin crystalline laminae, at nearly equal spacings. This is a rough equivalent of an optical interferometer of the Mach-Zehnder type, and as it turned out in the end, it happens to be a reinvention of the now forgotten optical interferometer by Barus,<sup>26</sup> 1911, who first used diffraction gratings as elements in an optical interferometer. Interferences up to the order of 6000 wavelengths path difference could be observed, at a spacing of the laminae of 3.5 cm, and an angle  $\theta$  of about 0.027 radian. The two interfering beams crossed one another at an angle of less than  $10^{-3}$ , else they could not have been taken in by the aperture of a conventional electron microscope lens, which was required for enlarging the fringe spacing to a size at which they could be resolved by a photographic plate. In fact, in successful experiments this angle could not be made more than  $2.5 \times 10^{-5}$ . With 60-keV electrons,  $\lambda = 0.048$  Å this gives a fringe spacing of 1650 Å. At larger interference angles the fringes could not be kept steady enough to be photographed. It is important to note that in the Marton-type interferometer the two inter-

FIG. 1. Marton-type electron interferometer. Coherent beam splitting by crystal diffraction.

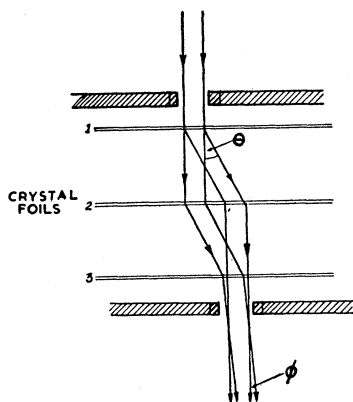
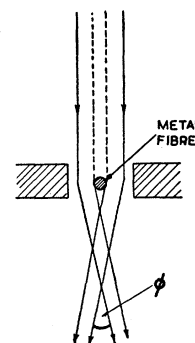


FIG. 2. Möllenstedt-Düker-type of electron interferometer. Coherent beam crossing by macroscopic fields.



fering beams are deflected by appreciable angles, but these are very nearly equal.

In the Möllenstedt-Düker interferometer, the interference angle is produced by macroscopic electric fields. This instrument, schematically illustrated in Fig. 2, is an electron-optical analog of Fresnel's biprism. Interferences up to the order of about 300 could be detected, with very nearly the same interference angles and fringe spacings as in the experiments of Marton, Simpson, and Suddeth. The fringes are very luminous, contrasty, and steady.

Up to the date of writing, neither of these instruments has been put to use in the sense as light-optical interferometers are used, that is to say, for the exploration of the refractive properties of materials. It is one of the chief purposes of this report to forecast, on general theoretical grounds, whether such a use of electron interferometers will ever be possible. Optical interferometers would be useless if we had no other objects than strongly fluorescent substances and frosted glass, and some justifiable doubts have been expressed whether all electron-microscopic objects do not belong to these categories. Of course the very existence of Marton's interferometer proves that thin crystalline lamellae can be made so perfect as to be considered as regular optical elements, but these are not typical electron-microscopic objects. It will be shown though, and it agrees with experience on Fresnel diffraction fringes in electron microscopes, that sufficiently thin noncrystalline foils need not diffuse so strongly as to wipe out all traces of interference fringes.

The main interest of interferometric methods lies in the fact that they enable us to determine the phase (not only the amplitude) of diffracted beams, by comparison with a coherent background. This, in principle, may open a new chapter in the determination of structures, in particular of small, somewhat irregular objects, which are of chief interest in organic chemistry and in biology. It will be shown that there exists indeed an important field for such applications of interferometric methods, though not an unlimited one. The limitation is in the fundamental fact, hitherto mostly ignored, that a material structure cannot deflect (diffract) a beam without slightly altering its energy. In other words, exactly elastic collisions do not exist. The small

losses of energy suffered in Bragg-reflections will, beyond certain angles, shift the energy of the diffracted beam outside the range of the primary beam, thus destroying coherence. This effect could be compensated by various measures, such as broadening the primary energy width, or making the phase comparison with a beam which has suffered a similar energy loss, with known phase. But these artifices will fail at a certain limit which is set by what may be called *de Broglie's principle*: It is impossible to explore a material structure beyond a certain detail without destroying it. More exactly, it is impossible to specify the position and shape of the *individual* atomic scatters beyond a certain limit; a small structure cannot yield more than a certain amount of information about itself without being destroyed. Roughly speaking this limit is at about 1 Å, about 5 times beyond the best resolution of present-day electron microscopes.† This is the fundamental limit for electron interferometers, as well as for any other exploring instrument, but there is a world of interest this side of the limit.

There are of course many practical limitations, chiefly of stability and of luminosity of the fringes, but it is hardly possible to discuss these on a general theoretical basis.

#### COHERENCE. GENERAL DISCUSSION

Consider two scalar wave fields  $W_1(\mathbf{x}, t)$  and  $W_2(\mathbf{x}, t)$  superimposed in space, and assume, as is the case both in light optics and in electron optics, that the only observable quantity is the absolute square of the resulting amplitude  $W_1 + W_2$ , summed over a time long compared with the duration of  $W_1, W_2$ . This quantity, which may be called the light sum is

$$\begin{aligned} S &= \int_{-\infty}^{\infty} (W_1 + W_2)(W_1^* + W_2^*) dt \\ &= \int W_1 W_1^* dt + \int W_2 W_2^* dt + \int (W_1 W_2^* + W_1^* W_2) dt \\ &= S_1 + S_2 + S_{12}. \quad (1) \end{aligned}$$

The last term,  $S_{12}$  may be called the mutual light sum. If  $W_1, W_2$  have the character of a stationary series of pulses, we can divide by the observation time (always assumed as very long relative to the pulse durations), and talk of a mutual intensity. Coherence is then synonymous with the condition that  $S_{12}$  is nonzero, and not everywhere very small relative to  $S_1 + S_2$ .

It is convenient to express this in spectral language. Let  $w_1(\mathbf{x}, f), w_2(\mathbf{x}, f)$  be the Fourier transforms of

† This does not of course contradict the fact that x-ray investigations have specified the details of structure in the elementary cell of large crystals to well below 0.1 Å. The information contributed by the individual atom is very small, and the high accuracies result from *averaging* over very large numbers.

$W_1, W_2$ , i.e.,

$$W_1(\mathbf{x}, t) = \int_{-\infty}^{\infty} w_1(\mathbf{x}, f) e^{2\pi i f t} df$$

$$W_2^*(\mathbf{x}, t) = \int_{-\infty}^{\infty} w_2^*(\mathbf{x}, f) e^{-2\pi i f t} df.$$

We have then

$$\begin{aligned} S_{12} &= \int \int_{-\infty}^{\infty} w_1(\mathbf{x}, f_1) w_2^*(\mathbf{x}, f_2) e^{2\pi i (f_1 - f_2) t} df_1 df_2 dt + \text{conj.} \\ &= \lim_{T \rightarrow \infty} \int \int w_1 w_2 2 \frac{\sin 2\pi (f_1 - f_2) T}{2\pi (f_1 - f_2)} df_1 df_2 + \text{conj.} \\ &= \int_{-\infty}^{\infty} [w_1(\mathbf{x}, f) w_2^*(\mathbf{x}, f) + w_1^* w_2] df. \quad (2) \end{aligned}$$

This form appears mathematically identical with (1) but it is more physical, because it can be given a detailed physical interpretation in cases in which the time functions are unobservable. The absolute values of  $w_1, w_2$  are the spectral amplitudes, which can be directly measured, and it may be noted that only the *relative phase* has remained in the expressions  $w_1 w_2^*$ , which is also a physical observable, because it can be either calculated from the geometrical data, or directly observed by comparison with a standard coherent background. In brief, the transition from (1) to (2) can be expressed by the simple rule that each frequency interferes only with itself; as it were *with a product of coherent band width and effective time which is just unity*.

In the case of light and of electrons the frequencies  $f$  have of course no meaning as such; they must be interpreted as energies, by Planck's rule  $E = hf$ . Equation (2) means therefore that only monoenergetic components can interfere with one another. The overlapping of the spectra of the two beams is an obvious necessary condition of coherence. The sufficient criteria are also contained in (2), but in a less obvious form. They are revealed by the following discussion.

Consider two beams in an interferometer, originally issuing from a point source  $S$ , interfering at a receptor point  $R$  (Fig. 3). In the neighborhood of  $R$ , we can consider each monoenergetic component as a plane wave with wave number  $k_r = 2\pi/\lambda_r$ . Let  $\mathbf{n}_1, \mathbf{n}_2$  be unit vectors in the direction of the wave normals at  $R$ . (These directions will be in general color-dependent.) We can write, therefore, in the neighborhood of  $R$ , if we take  $R$  as the origin of the vector  $\mathbf{x}$

$$w_1(\mathbf{x}, f) = \text{amplitude factor} \times x \exp(ik_r \mathbf{n}_1 \cdot \mathbf{x} + ix \text{ phase } S - R).$$

The amplitude factors  $a_1(f), a_2(f)$  are real, and functions of  $f$  only. The phase change from  $S$  to  $R$  is deter-

mined by tracing the two rays between the source and the receptor point, by the rules of geometrical optics, and counting the wavelengths on them. It will be convenient to write, e.g., for the first beam,

$$\text{phase } S-R = k_r s_1 \quad \text{where} \quad s_1 = \frac{1}{k_r} \int_S^R (k ds)_1,$$

and the suffix means "along the ray 1." Therefore,  $s_1, s_2$  are the optical path lengths, reduced to the optical space of the receptor point  $R$ . They are illustrated in Fig. 3 as straight tangents to the rays at  $R$ . In general they will be color-dependent, both in direction and in length.

We have now for the interference product,

$$w_1(x, f) w_2^*(x, f) = a_1(f) a_2(f) \cdot \exp[ik_r [(\mathbf{n}_1 - \mathbf{n}_2) \mathbf{x} + s_1 - s_2]].$$

In order to obtain convenient expressions we imagine  $a_1, a_2$  expanded in series of orthogonal Hermite functions (parabolic cylindrical functions). It will be sufficient to consider the first term only for the discussion. Let us put, for the first beam,

$$a_1(f) = a_{10} \exp[-(f - f_0)^2 / 4(\Delta f)^2] + \dots$$

We now assume that the second beam has the same spectral distribution, but shifted towards lower frequencies by a loss frequency  $f_L$ ,

$$a_2(f) = a_{20} \exp[-(f - f_0 + f_L)^2 / 4(\Delta f)^2] + \dots$$

It may be seen that  $(\Delta f)^2$  has the meaning of the mean square width of the intensity distributions,  $a_1^2$  or  $a_2^2$ , measured from the central frequencies, which are  $f_0$  in one case,  $f_0 - f_L$  in the other.

We now expand the phase difference  $s_1 - s_2$  to the first order in the frequency differences

$$k_r(s_1 - s_2) = k_0(s_1 - s_2)_0$$

$$+ \frac{dk}{df} \bigg|_{k=k_0} \left[ (s_1 - s_2)_0 + k_0 \frac{d}{dk} (s_1 - s_2) \right] (s_1 - s_2)_0.$$

Substituting  $dk/df = 2\pi/v$ , where  $v$  is the group velocity gives, after a little rearrangement and dropping of suffixes,

$$k_r(s_1 - s_2) = k_0(s_1 - s_2) + \frac{2\pi}{v} (s_1 - s_2) \left[ 1 + \frac{k_0}{s_1 - s_2} \frac{d}{dk} (s_1 - s_2) \right] (f - f_0).$$

The second term under the square bracket

$$C = \frac{k_0}{s_1 - s_2} \frac{d}{dk} (s_1 - s_2) \quad (3)$$

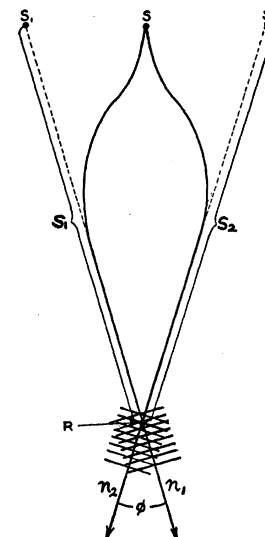


FIG. 3. Schema of general electron interferometer.

may be called the chromatic coefficient. It is the relative change of the optical path-length difference, reduced to the last medium, pro relative change of the wave number. In the Möllenstedt and Düker interferometer this is zero, because the deflection shifts the image of the source only sidewise, leaving the length unaltered in this approximation. In Marton's interferometer on the other hand  $C = -1$ , because by the remarkable achromatic properties of this instrument the difference  $s_1 - s_2$  is proportional to the wavelength, hence inversely proportional to the wave-number  $k$ .

We have now, for the interference product

$$w_1 w_2 = a_{10} a_{20} \exp i k_0 (s_1 - s_2) \cdot \exp [i k_r (\mathbf{n}_1 - \mathbf{n}_2) \mathbf{x}] \times \exp \left\{ - \frac{(f - f_0)^2 + (f - f_0 + f_L)^2}{4(\Delta f)^2} + i \frac{2\pi}{v} (1 + C) (f - f_0) \right\}.$$

The terms under the curly bracket in the last exponential can be rearranged in the form

$$- \frac{1}{2(\Delta f)^2} \left[ f - f_0 + \frac{1}{2} f_L - 2(\Delta f)^2 \frac{2\pi}{v} (1 + C) (s_1 - s_2) \right]^2 - \frac{f_L^2}{8(\Delta f)^2} - 2(\Delta f)^2 \left[ \frac{2\pi}{v} (1 + C) (s_1 - s_2) \right]^2.$$

The frequency  $f$  occurs now only in the first square bracket, and the integral between infinite limits gives a constant, which is of no interest. We thus obtain finally

for the mutual light sum (or intensity) near  $R$

$$S_{12} = \text{const} \cdot \underbrace{\exp[ik_r(\mathbf{n}_1 - \mathbf{n}_2)\mathbf{x}]}_{\text{Geometrical factor}} \cdot \underbrace{\exp\left[-2\left[\Delta f \frac{2\pi}{v}(1+C)(s_1 - s_2)\right]^2\right]}_{\text{Loss of coherence by path differences}} \cdot \underbrace{\exp\left[-\frac{fL^2}{8(\Delta f)^2}\right]}_{\text{Loss of coherence by frequency shift}} \quad (4)$$

The expansion which we have chosen for the spectral distributions has enabled us to separate the three factors in the somewhat complex notion of coherence, which can now be discussed one by one.

### Geometrical Coherence

The first factor in Eq. (4) is not complete, because so far we have assumed a point source, which assures geometrical coherence. But we can easily complete it, by introducing displacements  $\delta\mathbf{x}_s$  of the source, reduced to the optical space of  $R$ , so that the factor becomes

$$\exp[ik_r(\mathbf{n}_1 - \mathbf{n}_2)(\delta\mathbf{x}_r - \delta\mathbf{x}_s)]. \quad (5)$$

For symmetry we have written here  $\delta\mathbf{x}_r$  instead of  $\mathbf{x}$  for the displacement of the observation point.

If the wave normals intersect at a small angle  $\phi$ , we have from (5) for the fringe spacing, or period  $\phi$  the well-known formula

$$k_r \phi \phi = 2\pi.$$

In general,  $\phi$  is color-dependent, except in Marton's interferometer, where it is a constant, because  $k_r$  and  $\phi$  vary inversely.

Equation (5) now gives also immediately the geometrical condition of coherence. If we imagine the point source shifted sideways by  $\frac{1}{2}\phi$ , the interference fringes will be wiped out completely, and this is also the case

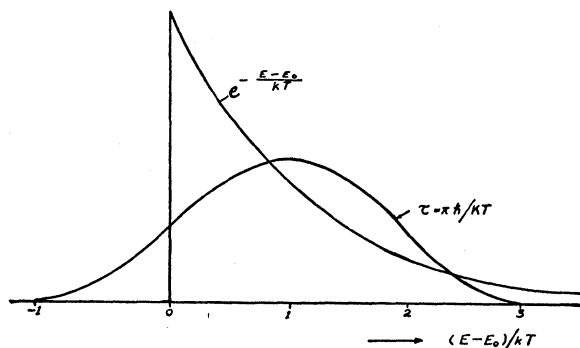


FIG. 4. Energy distribution in an electron beam from a hot cathode, (classical), and its closest imitation by a chopped wave of constant amplitude and duration  $\tau$ .

if it is replaced by a uniform line source of width  $\phi$ . This too is a well-known result; the width of the source image, reduced to the optical space in which the fringes are observed must not exceed the fringe spacing.

There is no need to go into further discussion of geometrical coherence, as in this respect there is no difference between electrons and light, and in optics the problem has received very full treatment. Michelson and Rayleigh were probably the first to understand it practically, and the theory has been worked out in recent years very completely by Zernike,<sup>27</sup> Hopkins,<sup>28</sup> Blanc-Lapierre and Dumontet,<sup>29</sup> Wolf,<sup>30</sup> and the author.<sup>31</sup>

### Loss of Coherence by the Path Difference Exceeding the Train Length

It was considered almost as axiomatic in optics, that waves could not be shifted relatively by more than their "train lengths" without losing coherence. This is also suggested by the form (1) for the mutual light sum. It was a considerable surprise when it was realized that the Barus-Marton type of interferometer distorts the wave trains in such a fashion as to make the notion of train length rather meaningless.

The second factor in Eq. (4),

$$\exp\left\{-2\left[\Delta f \frac{2\pi}{v}(1+C)(s_1 - s_2)\right]^2\right\},$$

shows that the mutual intensity falls to  $e^{-2}$  of its maximum if the path difference is

$$s_1 - s_2 = \frac{v}{2\pi\Delta f(1+C)}.$$

Apart from the factor  $1/(1+C)$  the expression at the right represents what is commonly called the train length. This is indeed a limiting factor in the Möllendstedt-Düker interferometer, but in the Marton instrument the second factor is infinity; hence, interferences up to any order appear theoretically possible.

It may be noted that this does not involve any fundamental impossibility. Let us define the rms duration of the wave train in the same way as we have defined the rms band width  $\Delta f$ , and call it  $\Delta t$ . There exists between these two the "indeterminacy relation," (Schwarz inequality)

$$\Delta f \cdot \Delta t \geq 1/4\pi$$

which sets, at given frequency spread, a certain *lower limit* to the train length, but there is no reason why it should not exceed any limit upwards.

There are of course good practical reasons which prevent obtaining interferences at very large path differences; these are fully discussed in the article by Marton, Simpson, and Suddeth.<sup>4</sup> Carrying out the mathematical process which led to Eq. (4) to higher

terms would also doubtlessly reveal limitations. But this is a rather academic question, because, as will be shown in the next section, the train length of electrons as used in electron optics is enormous in terms of wavelengths.

**Loss of Coherence by Frequency Shifts**

This is contained in the last factor in Eq. (4) and is as expected. This factor can be precalculated without knowledge of the geometry of the interferometer, from the properties of its components and of the object; its discussion will be left to the end of this article.

**ELECTRON WAVE TRAINS**

In electron-optical experiments the primary electron beam almost always issues from a hot cathode, with temperature  $T$ , and is accelerated to energies very much larger than  $kT$ . At the cathode surface,  $x=0$  (more exactly in the plane of the potential minimum), the distribution is semi-Maxwellian, that is to say Maxwellian in the solid angle  $2\pi$ . Let  $v_0$  be the electron velocity at this surface, in the  $x$  direction. The current in the limits  $v_0, dv_0$  is then proportional to

$$\exp(-\frac{1}{2}mv_0^2/kT)v_0dv_0.$$

Let these be accelerated by a voltage drop  $E_0/e$ , which adds the energy  $E_0$  to the energy of each electron. At this point the beam current in the limits  $E, dE$  is then proportional to

$$\exp[-(E-E_0)/kT]dE. \tag{6}$$

This is illustrated in Fig. 4. The mean forward energy of the electrons in the beam is  $E_0+kT$ , to which must be added  $kT$  for the transversal energy, which remains unaltered by the acceleration. Note that we have operated here with currents, not with particle densities. The functions  $W, w$  of the previous section must be also interpreted as *stream densities*, which for all practical purposes means local densities multiplied by  $v$ .

So far we have followed the classical description, in which each electron has a definite velocity. In the quantum description, a hot cathode is not considered as essentially different from a radioactive preparate, say a  $\beta$  emitter, and the description is complete if we know the law (6) which is interpreted as the probability distribution of energies, together with the mean densities in the stream. We can formally interpret the energies as frequencies by Planck's formal, but we cannot ask for details of time-space description beyond what is contained in Eq. (6). This then appears as the "energy spectrum" of a mixed oscillation, and if we want to visualize the electron stream as a stochastic sequence of wave trains, without knowledge of the phases, we can legitimately go only as far as to specify the *autocorrelation function* of these trains. By the theorem of N. Wiener, this is the Fourier transform of the energy spectrum (6), and if we define the frequencies as

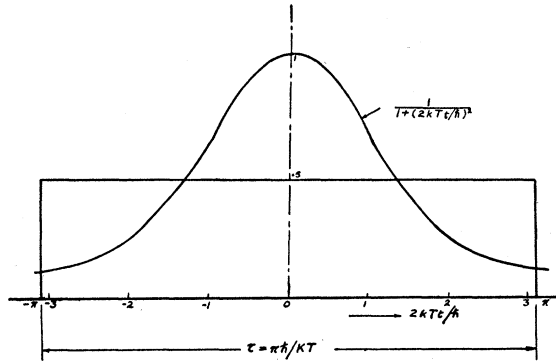


FIG. 5. Representing the beam emitted by a hot cathode by the *shortest* equivalent wavetrain, and by the chopped wave of closest fit. The ordinates are probability *amplitudes*, not probabilities.

$f = E/2\pi\hbar$ , this is found to be, apart from an unimportant constant

$$\int_{-\infty}^{\infty} \Psi_s(t)\Psi_s^*(t+\tau)dt = \text{const}/[1+(kT\tau/\hbar)^2]. \tag{7}$$

(The suffix "s" stands for stream densities.) Physically the expression means the mutual intensity of two part streams, one of which is delayed by  $\tau$  against the other. This is exactly what is required in the theory of optical interferometers, such as Mach's or Jamin's, in which one beam is delayed as a whole by mirrors or other achromatic elements, and also for the theory of the Möllenstedt-Düker interferometer. It is not sufficient, as we have seen, for the Barus-Marton instruments.

Though legitimately we cannot ask for any closer description of wave trains, the concept is so helpful to the imagination that one wishes to go a step further, if it is possible with impunity. We can in fact go a step further but not by describing the "real" wave train, which has no meaning, but the *shortest* wave train which is compatible with Eq. (7). One obtains this by making all the phases coincide at the middle point of the wave, which proceeds with the mean velocity  $\bar{v}$ , corresponding to the mean energy  $E_0+kT$ . This is

$$\Psi_s(x,t) = \text{const}/[1+(2kT(t-x/\bar{v})/\hbar)^2] \tag{8}$$

and is illustrated in Fig. 5. It has mathematically the same form as the correlation function (7), but is twice as narrow. Its physical meaning is, that a hot cathode could not be distinguished from a source which emits probability-amplitude pulses of this shape (i.e., probability pulses of the shape of the *square* of  $\Psi_s$ ).

It is of some interest to know the length of this shortest equivalent train, which may be conveniently defined as the *chopped* monochromatic wave train of closest fit. It has been shown elsewhere, (Gabor<sup>32</sup>) that this is a wave train of a frequency corresponding to the

mean energy  $E_0 + kT$ , and of a duration

$$\tau = \pi\hbar/kT = 2.4 \times 10^{-11}/T \text{ sec.} \quad (9)$$

Its energy distribution is of the form

$$\left( \frac{\sin[\frac{1}{2}\pi(E - E_0 - kT)/kT]}{\frac{1}{2}\pi(E - E_0 - kT)/kT} \right)^2 dE, \quad (10)$$

which, as shown in Fig. 4, is a not unreasonably close approximation to (6). The length of the equivalent chopped wave train is

$$\tau v = (\pi\hbar/kT)v = (\pi\hbar/c)\beta/T = 0.72\beta/T \text{ cm.} \quad (11)$$

Thus the duration of the equivalent pulse is for a tungsten cathode,  $T = 2500^\circ\text{K}$ , about  $10^{-14}$  sec, and for 60-kev electrons,  $\beta = 0.45$ , the train length is about  $1.3 \times 10^{-4}$  cm or 13 000 Å. As the wavelength at 60 kev is 0.048 Å, almost 300 000 fringes could be observed. Marton and his collaborators have in fact achieved observations at the order 5800. The delicacy of the instruments is such that substantial progress beyond this appears rather unlikely.

One question which is often asked must be answered at this point. We have assumed the theoretical energy spread, calculated from the temperature of the hot cathode, but it is well known that the actual spread is often much more, sometimes of the order of a score of ev, owing to high-frequency oscillations in the beam, originating in the cathode region (Boersch<sup>33</sup>). But though these oscillations may well affect the achromatic properties of interferometers, they will not affect the coherence. The frequency of these oscillations can be estimated as at most  $10^{10}$  cycles/sec; hence, they are very slow compared with the frequency associated with the thermal energy spread, which is of the order of  $10^{14}$ , and can be left out of account in the theoretical treatment.

#### BORN APPROXIMATION FOR TIME-DEPENDENT SCATTERERS

One-electron wave mechanics was so far found appropriate for all electron optical problems, and it is sufficient also for the theory of interferometers, as it breaks down only for inelastic losses so large that they destroy the coherence altogether.

The most convenient mathematical approach for these purposes is the Born approximation (see Mott and Massey<sup>34</sup>) because it operates with concepts which have simple optical interpretations. It will be sufficient to use the first Born approximation in order to reveal the physical principles; the need for higher approximations arises mostly in cases in which coherence is practically destroyed, and which are, therefore, of little interest for the present purpose.

It will be necessary, however, to extend the Born theory in two directions; to time-dependent scattering potentials, and to nonuniform primary beams. Time-independent fields produce only completely elastic

scattering, as if the electron had collided with an object of infinite mass. Strictly elastic collisions do not exist; there is always a transfer, however slight, of energy as well as of momentum. Small energy losses can be treated in a one-electron theory by introducing the change of the scattering field caused by the reaction of the electron on the scatterer, as will now be shown.

Consider the time-dependent, nonrelativistic wave equation, neglecting magnetic effects

$$\nabla^2\Psi - \frac{2m}{i\hbar} \frac{\partial\Psi}{\partial t} = -\frac{2m}{\hbar^2} V(\mathbf{x}, t)\Psi. \quad (12)$$

Here  $\Psi(\mathbf{x}, t)$  is the wave function of the electron in space-time language,  $V(\mathbf{x}, t)$  the scattering potential. We shall use throughout capital letters for space-time functions, lower case letters for space-energy functions, i.e., for their Fourier transforms. With this convention we define

$$V(\mathbf{x}, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} v(\mathbf{x}, \epsilon) \exp(i\epsilon t/\hbar) d\epsilon. \quad (13)$$

$\epsilon$  has the physical meaning of an energy loss component. Similarly we represent the primary wave, with energy components  $E$  by

$$\Psi_0(\mathbf{x}, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \psi_0(\mathbf{x}, E) \exp(-iEt/\hbar) dE \quad (14)$$

and the scattered wave conveniently by

$$\Psi_s(\mathbf{x}, t) = \frac{1}{(2\pi\hbar)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_s(\mathbf{x}, \epsilon, E) \cdot \exp(-i\frac{E-\epsilon}{\hbar}t) dE d\epsilon. \quad (15)$$

$\psi_s$  is the component which has suffered an energy loss  $\epsilon$ , and is scattered with the energy  $E - \epsilon$ .

We now pass from the exact equation (12) to the first Born approximation, in which  $\Psi$  in the perturbation term at the right-hand side is replaced by the primary wave  $\Psi_0$ . Substituting the definitions (13)–(15) we obtain the Fourier transform of the wave equation in the Born approximation

$$\nabla^2\psi_s + \frac{2m}{\hbar^2}(E - \epsilon)\psi_s = \frac{2m}{\hbar^2}v(\mathbf{x}, \epsilon)\psi_0(\mathbf{x}, E). \quad (16)$$

We have now at both sides of the equation, which has become time-independent, components with the energy  $E - \epsilon$ . Write

$$(2m/\hbar^2)E_0 = k_0^2, \quad (2m/\hbar^2)(E - \epsilon) = k^2. \quad (17)$$

$k$  is the wave number of the scattered beam com-

ponents;  $k_0$  corresponds to the *mean* energy  $E_0$  of the primary electrons.

We now apply to Eq. (16) Born's elementary solution for the wave scattered by the volume element  $\mathbf{x}'$ ,  $d\mathbf{x}'$

$$d\psi_s(\mathbf{x}, \epsilon, E) = -\frac{1}{4\pi} \frac{2m}{\hbar^2} v(\mathbf{x}', \epsilon) \psi_0(\mathbf{x}', E) \frac{e^{ikr}}{r} d\mathbf{x}' \quad (18)$$

where  $r = |\mathbf{x} - \mathbf{x}'|$ , the radius vector from the scatterer to the observer. Substituting into Eq. (15) we obtain the scattered wave in space-time language

$$\Psi_s(\mathbf{x}, t) = -\frac{2\pi m}{(2\pi\hbar)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_0(\mathbf{x}', E) v(\mathbf{x}', \epsilon) \frac{e^{ikr}}{r} \times \exp\left(-i\frac{E-\epsilon}{\hbar}t\right) d\mathbf{x}' d\epsilon dE. \quad (19)$$

This can be put into a more convenient form, taking account of the fact that the primary wave has a narrow energy spread, centering on  $E_0$ . We rewrite the exponential factors under the integral in (19) as follows

$$kr - (E - \epsilon)t/\hbar = -(Et/\hbar - k_0 r) + (\epsilon t/\hbar + (k - k_0)r).$$

In the last term we use the expansion

$$k - k_0 = \frac{dk}{d\omega} \Big|_{k=k_0} \frac{E - \epsilon - E_0}{\hbar} + \frac{1}{2} \frac{d^2k}{d\omega^2} \Big|_{k=k_0} \frac{(E - \epsilon - E_0)^2}{\hbar^2} + \dots \quad (20)$$

We have used here the radian frequency  $\omega$  as expansion variable, as this leads to familiar expressions. Using the nonrelativistic formulas

$$k = (2m\omega_0/\hbar)^{1/2} \quad \text{and} \quad \frac{1}{2}mv_0^2 = \omega_0\hbar$$

we obtain

$$\left(\frac{dk}{d\omega}\right)_{k=k_0} = 1/v_0, \quad \left(\frac{d^2k}{d\omega^2}\right)_{k=k_0} = -(1/v_0)(\hbar/2E_0)$$

where  $v_0$  is the mean velocity of the primaries. Substituting this into the exponent we obtain

$$kr - (E - \epsilon)t/\hbar = -\frac{E - \epsilon}{\hbar} \left(t - \frac{r}{v'}\right) + \frac{1}{2}k'r \quad (21)$$

where

$$v' = v_0 \left/ \left(1 + \frac{E_0 - (E - \epsilon)}{2E_0}\right)\right. \quad k' = k_0 \frac{E_0 + E - \epsilon}{2E_0}. \quad (22)$$

It can be seen that up to terms of the order  $(\epsilon/E_0)^2$  the velocity  $v'$  is equal to the velocity of an electron with energy  $E - \epsilon$ , that is to say  $v'$  can be identified with the velocity of the scattered electron.  $k'$  on the other hand

is not the wave number of the scattered electron, but the mean of this and of the wave number of the (mean) primary.

Substituting Eq. (21) into Eq. (19), we now obtain the more convenient formula

$$\Psi_s(\mathbf{x}, t) = -\frac{m}{2\pi\hbar^2} \int \int \int_{\mathbf{x}'} \Psi_0\left(\mathbf{x}', t - \frac{r}{v'}\right) \times V\left(\mathbf{x}', t - \frac{r}{v'}\right) \frac{e^{ik'r}}{r} d\mathbf{x}' d\epsilon dE. \quad (23)$$

This formula suggests at first sight a simple physical interpretation. The first two factors under the integral are, as if the primary wave at the retarded time  $t' = t - r/v'$  had interacted with the potential  $V$  at the point  $\mathbf{x}'$ , at the same time, and if this scattered wave, in the form of a spherical wave had spread from the scattering center with a phase velocity equal to the velocity of the scattered electron. But we are reminded by the factor  $\exp(\frac{1}{2}k'r)$  that this interpretation is too realistic. It appears as if this wavelet had approximately half the wave number which it ought to have, i.e. twice the wavelength. The reason is that the time-dependent wave equation (12) is not a true wave equation; it contains the first time derivative instead of the second. This results in the well-known dispersion properties of electron waves, in particular, in the fact that (in the nonrelativistic Schrödinger equation) the wave velocity  $\omega/k$  is one-half of the group velocity  $d\omega/dk$ . § In Eq. (23) it appears as if the phase velocity of the scattered signal had been twice what it is; this error is compensated by the last term. In fact no explanation of electron scattering by retarded signals can have a real physical meaning, as the instant of interaction of monochromatic wave components cannot be exactly defined. Thus Eq. (23) remains a useful formula, only one must not put a too realistic interpretation on it.

#### COHERENCE IN ELECTRON INTERFERENCES

Equation (23) is sufficient for the discussion of all types of interferences which have been so far observed in electron microscopes and in interferometers. It is not required, of course, for the discussion of the Möllenstedt and Düker interferometer without an object, as in this instrument the beams are brought to crossing by steady macroscopic fields, without any observable energy loss. The only limitation here is the primary energy spread, which we have already discussed. Equation (23) is required, however, for all interferometers which contain an object, and for the Marton interferometer even without an object, as the beams are crossed here by atomic scatterers, which may well restrict the coherence.

In order to understand the essentials of the process, it will be sufficient to consider the interference of a

§ This, of course, has no physical significance whatever; the zero level of  $\omega$  is completely arbitrary.



scattered beam with the primary beam  $\Psi_0(\mathbf{x}, t)$ . The modifications which have to be made in other cases will be easy to see. It will be also convenient for the start to neglect the small differences between  $v_0$  and  $v'$ ,  $k_0$  and  $k'$ , i.e., to consider first the case of elastic scattering.

At the receiver, such as a photographic plate, we observe the electron flux density

$$j = \frac{\hbar}{2im} (\Psi^* \text{grad} \Psi - \Psi \text{grad} \Psi^*).$$

At a plate approximately at right angles to the electron beams, this is only insignificantly different from

$$\frac{\hbar k_0}{2m} (\psi^* \psi + \psi \psi^*) = v_0 \Psi \Psi^*.$$

Hence, the "mutual electron flux" of the primary and of the scattered wave is, using Eq. (23), integrated over all times

$$S_{0s} = -\frac{mv_0}{2\pi\hbar^2} \int_{x'} \frac{e^{\frac{1}{2}ik_0r}}{r} dx' \int_{-\infty}^{\infty} \Psi_0^*(\mathbf{x}, t) \Psi_0(\mathbf{x}', t') \times V(\mathbf{x}', t') dt + \text{conj.}, \quad (24)$$

where, for brevity, we have written  $t' = t - r/v_0$  for the retarded time.

To make the meaning of this equation clearer, consider the example of illumination by a plane-wave group, proceeding in the  $z$  direction

$$\Psi_0(\mathbf{x}, t) = A(t - z/v_0) \exp[-i(E_0 t/\hbar - k_0 z)]. \quad (25)$$

Hence

$$\Psi_0(\mathbf{x}', t') = \Psi_0\left(\mathbf{x}', t - \frac{r}{v_0}\right) = A\left(t - \frac{z' + r}{v_0}\right) \times \exp\left[-i\left(\frac{E_0}{\hbar}t - \frac{E_0 r}{\hbar v_0} - k_0 z'\right)\right]. \quad (26)$$

It can be seen that the delay has introduced a new phase factor,

$$\exp(iE_0 r/\hbar v_0) = \exp(\frac{1}{2}ik_0 r),$$

which, in conjunction with the factor  $\exp(\frac{1}{2}ik_0 r)$  under the first integral, will restore the wavelength which is effective in the interference to  $2\pi/k_0$ , instead of twice this value, as might appear from Eqs. (23) or (24). Substitution in Eq. (24) gives

$$S_{0s} = -\frac{mv_0}{2\pi\hbar^2} \int_{x'} \frac{\exp[ik_0(r - (z - z'))]}{r} dx' \times \int_{-\infty}^{\infty} A^*\left(t - \frac{z}{v_0}\right) A\left(t - \frac{z' + r}{v_0}\right) V\left(\mathbf{x}', t - \frac{r}{v_0}\right) dt. \quad (27)$$

Let us now write

$$r = (z - z')/\cos\theta,$$

where  $\theta$  is the angle of the deflected ray with the  $z$  axis, and shift the time scale, writing  $\tau$  for  $t - z/v_0$ . We obtain the mutual flux in the form

$$S_{0s} = -\frac{mv_0}{2\pi\hbar^2} \int_{x'} \frac{\exp ik_0 r (1 - \cos\theta)}{r} dx' \times \int_{-\infty}^{\infty} A^*(\tau) A\left(\tau - \frac{r(1 - \cos\theta)}{v_0}\right) \times V\left(\mathbf{x}', \tau - \frac{r(1 - \cos\theta)}{v_0}\right) d\tau. \quad (28)$$

In this expression the first integral is a geometrical phase factor, which expresses the fact that the phase shift between the direct and the scattered wave is the difference between the lengths of the direct and of the deflected ray between the object plane and the observation plane, divided by the wavelength. The second integral, which is real, is the correlation function between the envelope function  $A$  and the product  $AV$ , shifted relatively to one another by the same phase difference.

This expression gives us nothing new beyond the elementary optical analog, and we cannot expect it otherwise, having neglected the energy losses. In order to establish the general formula we must go back to Eq. (24) and substitute in it the "spectral," i.e., space-energy representations of the direct wave, of the scattered wave, and of the scattering potential. We use Eq. (13) for representing the scattering potential, Eq. (23) for the scattered wave, and we write for the primary wave

$$\psi_0^*(\mathbf{x}, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \psi_0^*(\mathbf{x}, E') \exp(iE't/\hbar) dE'. \quad (29)$$

This gives the mutual electron flux in the form of the fivefold integral

$$S_{0s} = -\frac{mv_0}{2\pi\hbar^2} \frac{1}{(2\pi\hbar)^3} \int \int \int \int \int \psi_0^*(\mathbf{x}, E') \times \psi_0(\mathbf{x}', E) v(\mathbf{x}', \epsilon) \exp\left[i\left(\frac{1}{2}k'r + \frac{E - \epsilon}{\hbar} \frac{r}{v'}\right)\right] \times \exp[i(E' - E + \epsilon)t/\hbar] dx' dE d\epsilon dE' dt + \text{conj.} \quad (30)$$

This can be reduced to a threefold integral, by carrying out the integrations with respect to  $t$  and  $E'$ , in the same way as Eq. (2) was obtained from Eq. (1). It is found that only that component of the primary wave contributes to the integral for which

$$E' = E - \epsilon,$$

i.e., only components of the same energy interfere with one another. Writing for brevity

$$E - \epsilon = e$$

we obtain

$$S_{0s} = -\frac{k_0}{(2\pi\hbar)^3} \int_{x'} \frac{e^{ik_0 r}}{r} dx' \int_{-\infty}^{\infty} v(\mathbf{x}', \epsilon) d\epsilon \times \int_{-\infty}^{\infty} \psi_0^*(\mathbf{x}, e) \psi_0(\mathbf{x}', e + \epsilon) \exp\left(\frac{e r}{i\hbar v'}\right) de. \quad (31)$$

In this useful form the first factor and the last are phase factors; it is seen that they combine approximately to a phase factor  $\exp(ik_0 r)$ , as expected. These two are the "geometrical" factors, and we need not discuss them further, as we have already discussed the geometrical conditions of coherence (with more general assumptions regarding the phase shift between the interfering beams) in the second section of this review. The energetic conditions of coherence are very clearly exhibited in Eq. (31). It is seen that the mutual electron flux is, other things being equal, proportional to the integral over the energy loss of the spectral representation of the scattering potential, multiplied by the autocorrelation function of the spectral representation of the primary beam, for a shift equal to the loss  $\epsilon$ . Thus, if we knew the loss spectrum  $v(\epsilon)$ , we could calculate the coherence. In fact, the loss spectrum of solid objects in the range of perhaps 0.1 eV is not known; these are usually considered as "elastic" collisions, and the only method for exploring them experimentally is just by electron-interferometry. Later on we will make conjectures regarding these losses, but first it will be useful to consider the limit of validity of the Born approximation on which this and the previous section were based.

#### VALIDITY OF THE BORN APPROXIMATION

The limits of validity of the Born approximation in atomic collision processes have been very clearly discussed by E. J. Williams (Compare Mott and Massey,<sup>34</sup> p. 124, also p. 119 and 195). We are less interested here in collisions in the strong field of single atom cores, than in the processes which occur in the weaker interatomic fields, which can be approximately represented by an inner potential, or refractive index. These are likely to be more important for the very small scattering angles which are of interest in electron microscopy and interferometry.

The first Born approximation consists in replacing  $\Psi$  by the primary wave  $\Psi_0$  in the wave equation (12), at the right-hand side, while at the left-hand side one puts  $\Psi = \Psi_0 + \Psi_s$ . An obvious first check on this approximation is to investigate whether it violates the conservation laws of particle number, energy, and momentum. This has been carried out in Appendix I, only the results will be given here.

It is found that the first Born approximation tends to infringe the conservation law of particles, in the sense that electrons appear to be lost in the collision process. The rate of loss is, during the interaction time, approximately

$$\frac{\pi^3 \langle V^2 \rangle}{3 \hbar \cdot m v^2}, \quad (32)$$

where  $\langle V^2 \rangle$  is the mean square potential energy in the volume occupied by one electron.

In order to estimate this error, consider the following example. Electrons of an energy 50 keV,  $\beta = 0.41$  bombard a foil 200 Å thick, in which the mean square potential amplitude is  $(10 \text{ eV})^2$ . The train length of these electrons is about 12 000 Å by Eq. (11), about 240 times the foil thickness, and the interaction time is about  $10^{-14}$  sec. The rate of loss is found to be about  $10^{13}$ /sec; hence, about 10% of the electrons get lost. For thicker foils or stronger perturbing fields, the Born approximation rapidly loses its validity.

On the other hand, it is found that the conservation laws of energy and momentum are satisfied exactly in the Born approximation, not only for time-independent but also for time-dependent perturbing potentials, if these laws are reinterpreted in the following form:

The work done (or the momentum transferred to) the time-dependent potential field  $V$  by the primary wave, (*not* by the resulting wave!) is equal to the energy (or momentum) deficit in the scattered wave.

In other words, Born's first approximation violates the conservation laws of energy and momentum only insofar that, for obtaining a balance, one must omit the reaction of the scattered wave on the scattering field  $V$ .

Another question which had better be cleared up from the start is this. The first Born approximation represents *single* collisions. How far can we expect a single-collision theory to explain the coherence-destroying properties of random objects of the "frosted glass" type? The pioneer investigation by Booker, Ratcliffe, and Shinn<sup>35</sup> on diffraction at irregular screens, though it contains all the relevant mathematical methods, does not give a direct answer to this question, which can be formulated more exactly as follows:

Consider a rectangular foil with an irregularly distributed refractive index (or perturbing potential). A coherent plane wave is incident on the foil. At a distance at which the foil subtends the (small) angles  $\theta_1, \theta_2$ , this wave has practically uniform intensity, and a random phase  $P(x, y)$  if  $x, y$  are the coordinates in the observation plane. If one knows  $P$  at  $x, y$ , one can calculate its probable distribution at  $x + \Delta x, y + \Delta y$ , and if the mean spread of  $P$  is of the order  $2\pi$ , interference fringes of the half-period  $\Delta x, \Delta y$  will have become invisible.

The investigation gives the simple result that a

single-collision mechanism, such as embodied in the Born formula, can at most destroy the visibility of fringes down to a minimum fringe period of  $\lambda/\theta_1, \lambda/\theta_2$ . This is indeed as must be expected. By superposing plane waves in an angular range  $\theta_1, \theta_2$ , with phases however arbitrary, one cannot produce a phase-pattern with finer spatial detail than  $\lambda/\theta_1, \lambda/\theta_2$ . Hence, a foil thin enough to ensure the applicability of single-collision theory, however arbitrarily constructed, cannot destroy *all* interference fringes, though it may very well destroy the broad fringes which are most easily observed.

It is a well-founded fact of electron microscopy that, with electron energies of 30–100 kev, single-collision theory applies reliably to foils of 100–200 Å thickness, of organic materials or metals of not more than about  $Z=16$ . Hence, we can predict with some confidence that electron-interferometry of thin foils, though probably experimentally difficult, will not run up against a fundamental barrier right from the start.]]

#### USEFUL FORMS OF BORN'S FORMULA

It is often useful to operate with equivalents of the formulas (19) and (23), in which the charges and their motions figure in the integrals, instead of their potentials. In particular, it is of interest to describe these charges in a solid either by their densities  $\rho$  and velocities  $\mathbf{u}$ , or, as in macroscopic theory, by a polarization vector  $\mathbf{P}$ .

It is convenient to start not from Eq. (23), but from

$$\Psi_s(\mathbf{x}, t) = -\frac{m}{2\pi\hbar^2} \int \Psi_0(\mathbf{x}', t) V(\mathbf{x}', t') \frac{e^{ik_0 r}}{r} d\mathbf{x}'. \quad (33)$$

Here the receiver or observation time  $t$  figures in  $\Psi_0$ , while the retarded time  $t' = t - r/v$  figures only in the perturbation potential  $V$ . This has the advantage that it avoids the somewhat confusing factor  $\exp\frac{1}{2}ik_0 r$ . For simplicity,  $\Psi_0$  may be monoenergetic, and we neglect the small difference between  $v'$  and the primary velocity  $v$ .

We apply Green's theorem

$$\int (U\nabla^2 V - V\nabla^2 U) d\mathbf{x}' = \int_{\Sigma} \left( U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n} \right) d\sigma$$

to the case in which  $U=AB$ , where  $A$  and  $B$  are both solutions of the homogenous wave equation

$$\nabla^2 \Psi + k_0^2 \Psi = 0$$

and we let the integration volume comprise the whole region in which  $V \neq 0$ , so that the surface integral

|| *Note added in proof.*—This has been verified in the meantime by experiments carried out in Professor Möllenstedt's laboratory (private communication).

vanishes. This gives

$$\begin{aligned} \int AB\nabla^2 V d\mathbf{x}' &= \int V\nabla^2(AB) d\mathbf{x}' \\ &= -2 \int V(k_0^2 AB - \text{grad} A \cdot \text{grad} B) d\mathbf{x}'. \end{aligned}$$

Putting  $A = -(m/2\pi\hbar^2)\Psi_0$ ,  $B = \exp(ik_0 r)/r$  gives

$$\begin{aligned} \Psi_s(\mathbf{x}, t) &= -\frac{m}{2\pi(k\hbar)^2} \int V \text{grad} \Psi_0 \cdot \text{grad} \left( \frac{e^{ik_0 r}}{r} \right) d\mathbf{x}' \\ &\quad + \frac{m}{4\pi(\hbar k)^2} \int \Psi_0 \frac{e^{ik_0 r}}{r} \nabla^2 V d\mathbf{x}'. \quad (34) \end{aligned}$$

The gradients under the integral sign are taken with respect to  $\mathbf{x}'$ . We have

$$\frac{\partial}{\partial r} \frac{e^{ik_0 r}}{r} = \left( ik_0 - \frac{1}{r} \right) \frac{e^{ik_0 r}}{r} \approx ik_0 \frac{e^{ik_0 r}}{r}$$

and

$$\text{grad}_{(\mathbf{x}')} \frac{e^{ik_0 r}}{r} = -\text{grad}_{(\mathbf{x}')} \frac{e^{ik_0 r}}{r} = -ik_0 \mathbf{r} \frac{e^{ik_0 r}}{r^2}. \quad (35)$$

If  $\Psi_0$  is a (not too strongly distorted) wave with wave normal  $\mathbf{n}$ , we can put

$$\text{grad} \Psi_0 = ik_0 \mathbf{n} \Psi_0. \quad (36)$$

Substituting these into Eq. (34), it is seen that the first integral differs from  $\Psi_s$ , as given by Eq. (33), only by a factor  $\cos(\mathbf{n}, \mathbf{r})$  under the integral sign. If now we assume, as is always the case, that the observation point is so far away that we can consider this practically as a constant,  $\cos\theta$ , over the whole object, we obtain

$$\Psi_s(\mathbf{x}, t) = \frac{1}{1 - \cos\theta} \frac{1}{4\pi m v^2} \int \Psi_0 \nabla^2 V_{(t)} \frac{e^{ik_0 r}}{r} d\mathbf{x}'. \quad (37)$$

This is essentially Rutherford's formula, because for stationary charges

$$\nabla^2 V = -4\pi e \rho. \quad (38)$$

There is, however, a difference in the case of moving charges, because the operations which led from Eq. (33) to (37) were carried out at constant observation time  $t$ , whereas Poisson's formula must be applied at constant local time  $t'$ . The relation between the two is given by

$$\nabla_{(t)}^2 V = \nabla_{(t')}^2 V - \frac{2}{v} \frac{\partial^2 V}{\partial r \partial t'} + \frac{1}{v^2} \frac{\partial^2 V}{\partial t'^2}. \quad (39)$$

Substitution in Eq. (37) gives, after some transformations of the middle term in Eq. (39), by partial integration,

$$\begin{aligned} \Psi_s(\mathbf{x}, t) &= \frac{1}{4\pi m v^2} \frac{1}{1 - \cos\theta} \int \Psi_0 \frac{e^{ik_0 r}}{r} \\ &\quad \times \left[ \nabla_{(t')}^2 V - \frac{2ik_0}{v} (1 - \cos\theta) \frac{\partial V}{\partial t'} + \frac{\partial^2 V / \partial t'^2}{v^2} \right] d\mathbf{x}'. \quad (40) \end{aligned}$$

This equation can be transformed by gradually converting the terms in  $\partial V/\partial t'$ ,  $\partial^2 V/\partial t'^2/v^2$  into terms in the time derivatives of  $\nabla^2 V$ , in the same way as this has been achieved for the first term. Using the operator

$$L = \frac{i}{k_0 v} \frac{\partial}{\partial t'} - \frac{1}{2k_0^2 v^2 (1 - \cos\theta)} \frac{\partial^2}{\partial t'^2} \quad (41)$$

and making use of Poisson's equation (38), we obtain the result

$$\Psi_s(\mathbf{x}, t) = -\frac{e}{mv^2} \frac{1}{1 - \cos\theta} \int \Psi_0(x', t) \frac{e^{ik_0 r}}{r} \times (1 + L + L^2 + \dots) \rho(\mathbf{x}', t') d\mathbf{x}'. \quad (42)$$

Here we have the scattered wave expressed by the charge density and its time derivatives. We now express it in terms of the charges and their motions, using the conservation equation

$$\dot{\rho} = -\text{div}(\rho \mathbf{u}). \quad (43)$$

(The dot stands for differentiation with respect to local time,  $t'$ .) Applying this to the term in  $\dot{\rho}$  in Eq. (42) and making use of the vector identity

$$\text{div}(\phi \mathbf{A}) = \phi \text{div} \mathbf{A} + \mathbf{A} \cdot \text{grad} \phi$$

we obtain, by partial integration

$$\begin{aligned} \int \psi_0 \frac{e^{ik_0 r}}{r} \dot{\rho} d\mathbf{x}' &= - \int \Psi_0 \frac{e^{ik_0 r}}{r} \text{div}(\rho \mathbf{u}) d\mathbf{x}' \\ &= \int \rho \mathbf{u} \cdot \text{grad}_{(x')} \left( \Psi_0 \frac{e^{ik_0 r}}{r} \right) d\mathbf{x}'. \end{aligned}$$

But

$$\text{grad}_{(x')} \left( \Psi_0 \frac{e^{ik_0 r}}{r} \right) = ik_0 (\mathbf{n} - \mathbf{r}/r) \Psi_0 e^{ik_0 r}/r$$

so that this first term now turns into

$$ik_0 \int \rho (u_n - u_r) \Psi_0 \frac{e^{ik_0 r}}{r} d\mathbf{x}'$$

where  $u_n$  is the velocity component in the direction of the wave normal, and  $u_r$  the component in the direction of  $\mathbf{r}$ , i.e., towards the observer. Applying the same transformations to the second term, one obtains the expansion up to terms of the order  $(u/v)^2$

$$\begin{aligned} \Psi_s(\mathbf{x}, t) &= -\frac{e}{mv^2} \frac{1}{1 - \cos\theta} \int \Psi_0 \frac{e^{ik_0 r}}{r} \\ &\times \left\{ 1 - \frac{u_n - u_r}{v} + \frac{(u_n - u_r)^2 - (i/k_0)(\dot{u}_n - \dot{u}_r)}{2v_0^2(1 - \cos\theta)} \right\} \rho d\mathbf{x}'. \quad (44) \end{aligned}$$

It is seen that the terms converge rapidly, but there is an indication that there is a sharp peak developing

at  $\theta=0$ , which represents the drop-out in the primary wave. It is interesting that in the case of the motion  $\mathbf{u}$  caused by the wave being parallel to the wave normal, the second and also the third term will be isotropic *spherical* waves.

Equation (44) is somewhat similar to the Lorentz retarded potentials of electromagnetic theory, insofar as it expresses the scattered wave in terms of the charges and their motions at the retarded time. It can be converted without difficulty into the Liénard-Wiechert form, expressing the scattering in terms of the motion of point charges. The formal change is only that a Doppler-factor  $1/(1 - u_r/v)$  must be introduced for each point charge. But this procedure is of little use when dealing with the scattering by solids, and it is of doubtful validity in atomic collisions.

In the problems arising in electron microscopy and in electron interferometry it is more appropriate to express the charge configuration by a *polarization vector*  $\mathbf{P}$ , such that

$$\rho = -\text{div} \mathbf{P}. \quad (45)$$

(It is unlikely that the quadrupole tensor  $\mathfrak{Q}$  need be called in.) This substitution in Eq. (42) gives, extended to the first three terms the simple result

$$\begin{aligned} \Psi_s(x, t) &= \frac{-ik_0}{1 - \cos\theta} \frac{e}{mv^2} \int \Psi_0 \frac{e^{ik_0 r}}{r} \left\{ (P_n - P_r) \right. \\ &\quad \left. + \frac{i}{k_0 v} (\dot{P}_n - \dot{P}_r) - \frac{1}{2(k_0 v)^2} \frac{\ddot{P}_n - \ddot{P}_r}{1 - \cos\theta} \right\} d\mathbf{x}'. \quad (46) \end{aligned}$$

Here  $P_n$  and  $P_r$  are again the components of the polarization vector in the direction of the wave normal and of  $\mathbf{r}$ . Formally the result is very similar to Eq. (44), but it is really simpler and also more rapidly convergent. As will be shown later, the discussion is particularly convenient in terms of harmonic *polarization waves*, such that  $\partial \mathbf{P}/\partial t'^2 = -\omega^2 \mathbf{P}$ . It is then seen that successive terms in the expansion (46) are in the ratio

$$\frac{\omega}{k_0 v} = \frac{\omega h}{mv^2} = \frac{\text{vibrational quantum}}{\text{twice primary energy}},$$

which in most cases of interest is well below  $10^{-4}$ . Hence the first term in the expansion (46), ought to be sufficient for most discussions.

#### LOSS OF COHERENCE BY INTERACTION WITH SOLIDS

Equations such as (44) and (46) provide only one-half of the theory. The other half must be provided by the knowledge of the solid itself, and must enable us to specify the motion of the charges in the solid, or of the transient changes in polarization arising as a consequence of the passage of fast electrons. We can do this at present only in a very cursory way, but the simple discussion will at least give the order of magnitudes of the losses to be expected.

Electron interferometers are devices for collecting information, and it will be useful to keep in mind two special embodiments of the general principle. The first is a *phase-marking electron diffraction camera*, in which the phases are ascertained by interference with a coherent wide-angle beam, issuing for instance from a very reduced image of the source, produced by a strong lens. The second instrument is an *electron interference microscope*, in which a strong, more or less plane wave is superimposed on the image carrying beam. This is rather similar in principle to the electron diffraction microscope, but with the important difference that the coherent background need not traverse the object; hence, there is far less restriction regarding the nature of the object. One can say, very broadly, that the diffraction camera collects Fourier information, the microscope point information.

Louis de Broglie<sup>36</sup> was the first to point out the unavoidable close connection between energy losses and information. An electron, or any other exploring agency will give information to the detail  $d$  of the position of a scattering center, such as an atom of mass  $M$  only if by a collision it is deflected from its original direction by  $\theta$ , which is connected with  $d$  by

$$d = \frac{1}{2} \lambda / \theta.$$

Using the conservation theorems of energy and momentum, one finds that the energy transferred to the atom is

$$\epsilon = \frac{h^2}{8M} \frac{1}{d^2} = 0.0207 \left( \frac{M_H}{M} \right) \frac{1}{d^2} \text{ ev/angstroms}^2. \quad (47)$$

This is not very large if one is satisfied with, e.g., 1 Å resolution, only 0.021 ev even for a hydrogen atom. The crux is that there is no way avoiding a few electron going very near the atom, if one wants to make sure of its position to say 1 Å, which means bombarding it with something like 1000 electrons. (It may be noted that it makes no difference whatever whether the bombarding agency is electrons, protons, or x-rays.) Thus during an observation to 1 Å, hydrogen atoms will be almost certainly thrown out of the compound or crystal lattice.

The author<sup>37</sup> has investigated the question whether collecting Fourier-information instead of point-information gives more favorable conditions. This means non-localized collisions, of which Bragg reflections in crystals are well-known examples. The basic process of this type is the excitation of one phonon, which deflects the electron by carrying away some momentum and one quantum of vibrational energy. The result is that here again information to the detail  $d$  can be obtained only at the cost of an energy loss, which is

$$\epsilon = 0.02 \frac{V(\text{km/sec})}{d(\text{Å})} \text{ ev}. \quad (48)$$

$V$  is the velocity of sound, of the order of a few km/sec in most solids. Again we see that one can explore a structure to a detail of about 1 Å without total loss of coherence in electron beams issuing from tungsten cathodes. The conclusion is, therefore, that interferometric methods ought to be applicable right to the fundamental limit at which the object is destroyed.

In the interferometer of Marton, Simpson, and Suddeth the two interfering beams *both* suffer deflections of the order  $\theta = 0.023$  and energy losses of the order 0.1 ev. As the crystal foils are all made of the same material, their coherence thereby is not at all impaired. Moreover, it may be noted that metal foils do not appear to suffer cumulative action of long electron bombardment, if it is not over intense. All difficulties experienced in this instrument must be ascribed to mechanical instability and creep, and in view of the extraordinarily exacting geometrical requirements it is an admirable achievement that interference fringes have been observed at all.

The outlook appears to be satisfactory for interferometers and electron interference microscopes; devices which deal with *individual* details, and therefore cannot be expected to yield data beyond the destruction limit. It is less satisfactory for the phase-marking diffraction camera, which deals with *averaged* details, and might employ deflection angles too large for coherence with the primary beam. Whether this difficulty can be overcome by a coherent multiple-beam background, such as is provided in the Marton interferometer by the crystal foils, remains to be seen.

## CONCLUSIONS

1. There is no difference in kind but only in degree between light and electron interferences.
2. The train length is not, as used to be believed, a fundamental limitation of coherence. It is meaningless in the optical interferometer of Barus and in the electron interferometer of Marton.
3. Electrons (and x-ray quanta) suffer non-negligible energy losses in solids by deflections which can supply information of detail of the order of 1 Å but these are still too small to destroy coherence with a primary beam which has issued from a hot tungsten cathode.
4. Born's first approximation is a valid tool for non-monoenergetic electron beams and for (slightly) lossy interactions, up to the limit to which it is legitimate to use single-collision theory. Foils thin enough to satisfy this condition cannot entirely destroy interference fringes, whatever the distribution of matter in them.
5. The possibility of two new research instruments emerges. The first is a phase-marking diffraction camera, the second an electron interference microscope with strong, uniform coherent background.
6. All coherence limitations so far observed must be ascribed to the extraordinarily delicate mechanical stability conditions in electron interferometers.

7. The fundamental limitation of electron interferometers, as of all information-collecting devices is the destruction of the object by the exploring agent, and in this respect interferometers appear to be neither better nor worse than other instruments.

## REFERENCES

1. L. Marton, Introductory remarks to Electron Physics Symposium at the National Bureau of Standards, November, 1951, "Electron physics," National Bureau of Standards Circular 527, Washington, 1954.
2. L. Marton, "Electron interferometer," *Phys. Rev.* **85**, 1057 (1952).
3. L. Marton, J. A. Simpson, and J. A. Suddeth, "Electron beam interferometer," *Phys. Rev.* **90**, 490 (1953).
4. L. Marton, J. A. Simpson, and J. A. Suddeth, "An electron interferometer," *Rev. Sci. Instr.* **25**, 1099 (1954).
5. J. A. Simpson, "The theory of the three-crystal electron interferometer," *Rev. Sci. Instr.* **25**, 1105 (1954).
6. Marton, "Electron interferometry," *Science* **118**, 470 (1953).
7. G. Möllenstedt and H. Düker, "Fresnelscher Interferenzversuch mit einem Biprisma für Elektronenwellen," *Naturwissenschaften* **42** (2), 41 (1954).
8. H. Düker, "Lichtstarke Interferenzen mit einem Biprisma für Elektronenwellen," *Z. Naturforsch.* **10a** (3), 256 (1955).
9. H. Düker, "Beobachtungen und Messungen an Biprisma-Interferenzen mit Elektronenwellen," thesis, Tübingen (1955).
10. G. P. Thomson and W. Cochrane, *Theory and Practice of Electron Diffraction* (Macmillan and Company, Ltd., London, 1939).
11. H. Boersch, "Fresnelsche Beugungserscheinungen im Elektronenmikroskop," *Naturwissenschaften* **28**, 709 (1940); also *Physik. Z.* **44**, 202 (1943).
12. J. Hillier, "Fresnel diffraction of electrons as a contour phenomenon in electron supermicroscope images," *Phys. Rev.* **58**, 842 (1940).
13. T. Mitsuishi, H. Nagasaki, and R. Uyeda, "A new type of interference fringes observed in electron micrograph of crystalline substance," *Proc. Japan Acad.* **27** (2) 86 (1951).
14. E. Fues and E. H. Wagner, "Zur Streifenstruktur der Kossel-Möllenstedtschen Elektroneninterferenzen mit konvergentem Bündel, I," *Z. Naturforsch.* **6a** (1) (1951); II, *ibid.*, **6a** (2) (1951); III, *ibid.*, **6a** (3) (1951).
15. Chr. Menzel-Kopp, "Intensitäts und Lagen-Anomalien bei Elektroneninterferenzen im konvergenten Bündel," *Ann. Physik* (6) **9** (5) (1951).
16. Chr. Menzel-Kopp, "Die Kreuzung von zwei Extinktionslinien bei Elektroneninterferenzen im konvergenten Bündel," *Z. Naturforsch.* **8a** (1) (1951).
17. H. Pfister, "Elektroneninterferenzen an Bleijodid bei Durchstrahlung im konvergenten Bündel," *Ann. Physik* (6) **11** (4-7) (1953).
18. O. Rang, "Fern-Interferenzen von Elektronenwellen," *Z. Physik* **136**, 465-479 (1953).
19. O. Rang, "Elektronen-Interferometrie," *Physik. Bl.* **10** (10) (1954).
20. K. Ito and T. Ito, *J. Electronmicroscopy* **1** (1) (1953).
21. H. Niers, "Das Strahlungsfeld auf der Kristallrückseite bei Elektroneninterferenzen," *Z. Physik* **138**, 570-597 (1954).
22. T. Hibi, K. Kambe and G. Honjo, "Interference fringes in electron micrographs of magnesium oxide," *J. Phys. Soc. Japan* **10**, 35-46 (1955).
23. J. Farrand and A. L. G. Rees, Contribution to International Conference on Electron Microscopy, London, 1954, to appear in 1956.
24. D. Gabor, "Microscopy by reconstructed wavefronts, I," *Proc. Roy. Soc. (London)* **A197**, 454 (1949); II. *Proc. Phys. Soc. (London)* **B64**, 449 (1951).
25. E. M. Haine and J. Dyson, *Nature* **166**, 315, 399 (1950).
26. C. Barus, *Carnegie Inst. Wash. Publ.* **149**, I. (1911); II. (1922).
27. F. Zernike, *Physica* **5**, 785 (1938).
28. H. H. Hopkins, "The concept of partial coherence in optics," *Proc. Roy. Soc. (London)* **A208**, 263 (1951); **A217**, 408 (1953).
29. A. Blanc-Lapierre and P. Dumontet, "La notion de la cohérence en optique," *Rev. opt.* **34**, 1-21 (1955).
30. F. Wolf, "A macroscopic theory of interference and diffraction," *Proc. Roy. Soc. (London)* **A225**, 96 (1954); **A230**, 246 (1955).
31. D. Gabor, "Optical Transmission," *Proceedings of the Third London Conference on Information Theory* (Butterworth Scientific Publications, London, 1956).
32. D. Gabor, "Collective oscillations and characteristic energy losses," *Phil. Mag.* (8) **1**, 1-18 (1956).
33. H. Boersch, "Experimentelle Bestimmung der Energieverteilung in thermisch ausgelösten Elektronenstrahlen," *Z. Physik* **139**, 115-146 (1954).
34. N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, New York, 1948), second edition, Oxford.
35. H. G. Booker, J. A. Ratcliffe and D. H. Shinn, "Diffraction from an irregular screen with application to ionospheric problems," *Trans. Roy. Soc. No. 856* **A242**, 579-609 (1950).
36. L. de Broglie, *Optique Electronique et Corpusculaire* (Hermann & Cie, Paris, 1950), pp. 230-238.
37. D. Gabor, "Louis de Broglie et les limites du monde visible," *Louis de Broglie, Physicien et Penseur* (Albin Michel, Paris, 1953).

## APPENDIX I. THE CONSERVATION CONDITIONS IN THE FIRST BORN APPROXIMATION

## 1. Conservation of Particle Number

The first Born approximation is a solution of the equation

$$\nabla^2 \Psi - \frac{2m}{i\hbar} \frac{\partial \Psi}{\partial t} = \frac{2m}{\hbar^2} V(x,t) \Psi_0, \quad (12B)$$

which differs from the Schrödinger equation (12) by the fact that at the right-hand side the wave function  $\Psi$  is replaced by the incident wave  $\Psi_0$ .

Multiplying Eq. (12B) by  $\Psi^*$ , and subtracting the conjugate equation gives

$$\frac{\partial}{\partial t} \Psi^* \Psi + \operatorname{div} \frac{\hbar}{2im} (\Psi^* \operatorname{grad} \Psi - \Psi \operatorname{grad} \Psi^*) = -i V (\Psi \Psi_0^* - \Psi^* \Psi_0). \quad (I.1)$$

The left-hand side is the rate of increase of particle number during the collision process. The first term, the increase in density, and the second term,  $\operatorname{div} \mathbf{j}$ , where  $\mathbf{j}$  is the current density

$$\mathbf{j} = \frac{\hbar}{2im} (\Psi^* \operatorname{grad} \Psi - \Psi \operatorname{grad} \Psi^*) \quad (I.2)$$

ought to balance everywhere. It is seen however that

they do not necessarily balance in the first Born approximation.

In order to estimate the error, we put  $\Psi = \Psi_0 + \Psi_s$ , and integrate the r.h.s. of Eq. (I.2) over the whole interaction space where  $V \neq 0$ .

$$\frac{i}{\hbar} \int V(\Psi_s \Psi_0^* - \Psi_s^* \Psi_0) dx.$$

Here we substitute the scattered wave  $\Psi_s$  from Eq. (23) (in which for simplicity we put  $v' = v$ ), and obtain for the total rate of increase of particle number

$$-\frac{1}{\hbar} \frac{m}{2\pi\hbar^2} \iint V(x,t)V(x',t') \times \{\Psi_0(x,t)\Psi_0^*(x',t') + \text{conj.}\} \frac{\sin \frac{1}{2}k_0 r}{r} dx dx'. \quad (\text{I.3})$$

Each volume element figures twice in this expression, once as a scatterer (at the retarded time), once as a receiver. It is easy to see that this integral will tend to have a negative value for extended scatterers, i.e., the Born approximation will correspond to an apparent loss in particle number.

The factor  $\sin \frac{1}{2}k_0 r / r$  retains a value near to its maximum  $2/k_0$  in a radius of about  $\frac{1}{2}k_0 r = \frac{1}{2}\pi$  or  $r = \pi/k_0$  (half a wavelength). In this region we can also neglect the retardation and consider  $\Psi_0(x,t) = \Psi_0(x',t')$ . Hence the integral is, as regards order of magnitude,

$$-\frac{1}{\hbar} \frac{m}{2\pi\hbar^2} \frac{4\pi}{3} \left(\frac{\pi}{k_0}\right)^3 \int V^2 \Psi_0 \Psi_0^* dx. \quad (\text{I.4})$$

That is to say, the electrons appear to get lost at a rate of, roughly,

$$\frac{1}{3} \pi^3 \frac{\langle V^2 \rangle}{\hbar \cdot mv^2} \quad (\text{I.5})$$

where  $\langle V^2 \rangle$  is the mean square perturbation potential over the volume occupied by one electron.

An example in which this apparent loss is of the order of 10% is given in the text. Note that the loss would be always zero in the usual applications of the first Born approximations in which the primary beam is monochromatic, hence of infinite extent. This is therefore a new limitation, of importance in nonmonochromatic beams only, and hence additional to the limitations found by E. J. Williams and others.

### 3. Conservation of Energy

One obtains the energy conservation equation for the exact wave equation (12) by differentiating with respect to time, multiplying by  $\Psi^*$ , and adding the

conjugate. The result can be put in the form

$$\frac{\partial}{\partial t}(U+T) - \Psi^* \Psi \frac{\partial V}{\partial t} + \text{div} \mathbf{E} = 0. \quad (\text{I.6})$$

Here  $U$  and  $T$  are the potential and the kinetic energy densities in the beam

$$U = V\Psi^*\Psi, \quad T = -\frac{\hbar^2}{4m}(\Psi^*\nabla^2\Psi + \Psi\nabla^2\Psi^*). \quad (\text{I.7})$$

The third term at the left is evidently the loss of energy by the charge density  $\Psi^*\Psi$  in the time-variable potential field  $V(x,t)$ . The last term is the energy stream carried out of the volume element, with a density

$$\mathbf{E} = \frac{\hbar^2}{4m}(\Psi^* \text{grad} \frac{\partial \Psi}{\partial t'} - \frac{\partial \Psi}{\partial t'} \text{grad} \Psi^* + \text{conj.}). \quad (\text{I.8})$$

Evidently we cannot expect this conservation condition to be strictly satisfied if we use Born's approximation, i.e., if we substitute a solution of Eq. (12B) instead of (12). But it can be shown that another condition will be exactly satisfied; the total work done by the perturbation potential *on the primary beam* will exactly balance the energy carried away *by the scattered beam*. In other words, the only error is that the work done by the scatterer on the scattered beam does not figure in the balance equation as it ought to.

A somewhat lengthy calculation leads to the modified balance equation

$$\frac{\partial}{\partial t}(U+T) - \frac{\partial V}{\partial t} \Psi_0^* \Psi_0 + \text{Div} \mathbf{E} = \frac{1}{2} \frac{\partial}{\partial t} V(\Psi_s^* \Psi_0 + \Psi_s \Psi_0^*). \quad (\text{I.9})$$

This differs from the exact balance equation (I.6) only in the second term at the left, which represents the work done by the perturbing potential on the primary beam alone, and in the term at the right-hand side, which is a time differential, and therefore does not affect the balance as a whole.

The conclusion is that there is no need to impose special limits of validity on the first Born approximation in lossy collision processes beyond those already mentioned.

### 3. Conservation of Momentum

Differentiating the Schrödinger equation (12) with respect to  $x$ , multiplying by  $\Psi^*$ , and adding the conjugate gives

$$-\frac{\hbar^2}{4m} \left( \Psi^* \frac{\partial}{\partial x} \nabla^2 \Psi + \Psi \frac{\partial}{\partial x} \nabla^2 \Psi^* \right) + \frac{\hbar}{2i} \left( \Psi^* \frac{\partial^2 \Psi}{\partial x \partial t} - \frac{\partial^2 \Psi^*}{\partial x \partial t} \right) + \frac{1}{2} V \left( \Psi^* \frac{\partial \Psi}{\partial x} + \Psi \frac{\partial \Psi^*}{\partial x} \right) = -\frac{\partial V}{\partial x} \Psi^* \Psi. \quad (\text{I.10})$$

The term at the right-hand side is easily identified as the momentum transferred by the perturbing potential on the electron. Using the definition of momentum density

$$\mathbf{p} = \frac{\hbar}{2i} (\Psi^* \text{grad} \Psi - \Psi \text{grad} \Psi^*)$$

we can identify the second term in Eq. (I.10) as

$$\frac{\partial p_x}{\partial t} - \frac{\hbar}{2i} \left( \dot{\Psi}^* \frac{\partial \Psi}{\partial x} - \dot{\Psi} \frac{\partial \Psi^*}{\partial x} \right).$$

We next transform the last term at the l.h.s. in (I.10,) using again the wave equation to eliminate  $V$

$$\begin{aligned} \frac{1}{2} V \left( \Psi^* \frac{\partial \Psi}{\partial x} + \Psi \frac{\partial \Psi^*}{\partial x} \right) &= \frac{\hbar^2}{4m} \left( \frac{\partial \Psi}{\partial x} \nabla^2 \Psi^* + \frac{\partial \Psi^*}{\partial x} \nabla^2 \Psi \right) \\ &+ \frac{\hbar}{2i} \left( \dot{\Psi}^* \frac{\partial \Psi}{\partial x} - \dot{\Psi} \frac{\partial \Psi^*}{\partial x} \right). \end{aligned}$$

This turns Eq. (I.10) into

$$\begin{aligned} \frac{\partial p_x}{\partial t} + \frac{\partial V}{\partial x} \Psi \Psi^* - \frac{\hbar^2}{4m} \left( \Psi^* \frac{\partial}{\partial x} \nabla^2 \Psi \right. \\ \left. + \Psi \frac{\partial}{\partial x} \nabla^2 \Psi^* - \frac{\partial \psi}{\partial x} \nabla^2 \Psi^* - \frac{\partial \Psi^*}{\partial x} \nabla^2 \Psi \right) &= 0. \end{aligned} \quad (\text{I.11})$$

It can be shown that the last term is the  $x$  component of the divergence of a tensor  $\mathfrak{T}$ , defined by

$$T_{ik} = -\frac{\hbar^2}{4m} \left( \frac{\partial \Psi}{\partial x_i} \frac{\partial \Psi^*}{\partial x_k} - \Psi^* \frac{\partial^2 \Psi}{\partial x_i \partial x_k} + \text{conj.} \right) \quad (\text{I.12})$$

so that finally the momentum conservation theorem appears in the vector form

$$\frac{\partial \mathbf{p}}{\partial t} + \Psi^* \Psi \text{grad} V + \text{Div} \mathfrak{T} = 0. \quad (\text{I.13})$$

Quite in parallel to what we have seen in the case of the energy conservation condition, this equation can be immediately adapted to solutions of the Born equation (12B) merely by writing  $\Psi_0^* \Psi_0$  in the second terms instead of  $\Psi^* \Psi$ . The other correction terms are again pure time derivatives, and do not affect the balance as a whole.

APPENDIX II. DIFFUSION BY A FOIL OF RANDOM REFRACTIVE PROPERTIES

Consider a foil with complex transmission coefficient  $t(x,y)$  illuminated by a plane wave. At a distance  $z$  large enough compared with the transverse dimensions

$W_1, W_2$  of the foil the wave amplitude will be

$$A e^{iP} = \int_{-\frac{1}{2}W_1}^{\frac{1}{2}W_1} \int_{-\frac{1}{2}W_2}^{\frac{1}{2}W_2} t(x,y) \exp[2\pi i(\xi x + \eta y)] dx dy. \quad (\text{II.1})$$

$\xi, \eta$  are Fourier coordinates, equal to the direction cosines of the rays in the  $x, y$  direction, divided by  $\lambda$ . We are interested only in the phase  $P$ , for which it is convenient to use the formula, derived from (II.1).

$$e^{2iP} = \frac{\iint t \exp[2\pi i(\xi x + \eta y)] dx dy}{\iint t^* \exp[2\pi i(\xi x + \eta y)] dx dy}. \quad (\text{II.2})$$

The knowledge of this phase function gives sufficient information on the diffusing properties of a partially turbid foil. For convenience we can write its transmission

$$t = t_r + t_s \quad (\text{II.3})$$

where “ $r$ ” means “regular” and “ $s$ ” means “stochastic.” We can assume  $t^* = 1$ , i.e., neglect the absorption and reflection by the foil.

In order to obtain an idea of how such a foil will interfere with the visibility of interference fringes, it will be sufficient to calculate the *mean square change of the phase  $P$  with the angles  $\xi, \eta$* , for  $\xi = \eta = 0$ . i.e., on the axis. One obtains immediately

$$\frac{\partial P}{\partial \xi} \Big|_{\xi=0, \eta=0} = \pi \left\{ \frac{\iint x t dx dy}{\iint t^* dx dy} + \text{conj.} \right\}$$

and

$$\begin{aligned} \left( \frac{\partial P}{\partial \xi} \right)_{\xi=0, \eta=0}^2 &= \pi^2 \left\{ \frac{\left[ \iint x t dx dy \right]^2}{\iint t^* dx dy} + \text{conj.} \right. \\ &\left. + 2 \frac{\iint x t dx dy \cdot \iint x t^* dx dy}{\iint t dx dy \cdot \iint t^* dx dy} \right\}. \end{aligned} \quad (\text{II.4})$$

We calculate first the mean value of the last term. (The first two give zero means, but this is easier to see afterwards.) Consider first the denominator of the



last term, which can be written

$$\int_{-\frac{1}{2}W_1}^{\frac{1}{2}W_1} \int_{-\frac{1}{2}W_2}^{\frac{1}{2}W_2} xx't(x,y)t^*(x',y')dxdydx'dy'.$$

We transform this, using the identity (derived by considering two ways of integrating over a rectangle)

$$\begin{aligned} &\int_{-\frac{1}{2}W_1}^{\frac{1}{2}W_1} \int_{-\frac{1}{2}W_2}^{\frac{1}{2}W_2} t(x,y)t^*(x',y')dxdx'dydy' \\ &= 4 \int_0^{W_1} \int_0^{W_2} d\xi d\eta \int_{-\frac{1}{2}W_1-\xi}^{\frac{1}{2}W_1-\xi} \int_{-\frac{1}{2}W_2-\eta}^{\frac{1}{2}W_2-\eta} \\ &\quad \times t(x,y)t^*(x+\xi, y+\eta)dxdy. \end{aligned} \quad (\text{II.5})$$

It is seen that this transformation has introduced the *autocorrelation function* of the transmission function under the integral sign. We define this by

$$\begin{aligned} &\lim_{(W_1=W_2=\infty)} \int_{-\frac{1}{2}W_1}^{\frac{1}{2}W_1} \int_{-\frac{1}{2}W_2}^{\frac{1}{2}W_2} t_s(x,y)t_s^*(x',y') \\ &\quad \times (x+\xi, y+\eta)dxdy = W_1W_2\phi(\xi,\eta) \end{aligned}$$

for the stochastic part  $t_s$  of the transmission function. The regular component of the transmission will now be assumed as constant, for convenience. This gives for the denominator

$$\begin{aligned} &(W_1W_2)^2 t_r t_r^* \\ &+ 4 \int_0^{W_1} \int_0^{W_2} (W_1-\xi)(W_2-\eta)\phi(\xi,\eta)d\xi d\eta \end{aligned} \quad (\text{II.6})$$

and we can write this

$$W_1W_2[W_1W_2 t_r t_r^* + \delta^2 \langle t_s t_s^* \rangle_{Av}]. \quad (\text{II.7})$$

$\delta^2$  is the *area of the correlated patch*, that is to say the integral of the correlation function which figures in (II.6), divided by the maximum, which is  $t_s t_s^*$ . We have assumed that  $\delta$  is small compared with  $W_1W_2$ .

We now consider the numerator of the last term in (II.4). The same transformation as used before gives

the value

$$\begin{aligned} &4 \int_0^{W_1} \int_0^{W_2} d\xi d\eta \int_{-\frac{1}{2}W_1-\xi}^{\frac{1}{2}W_1-\xi} \int_{-\frac{1}{2}W_2-\eta}^{\frac{1}{2}W_2-\eta} x(x+\xi) \\ &\quad \times t(x,y)t^*(x+\xi, y+\eta)dxdy \\ &= 4 \int_0^{W_1} \int_0^{W_2} d\xi d\eta \langle x(x+\xi)t(x,y) \\ &\quad \times t^*(x+\xi, y+\eta) \rangle \cdot (W_1-\xi)(W_2-\eta). \end{aligned}$$

We can consider the product

$$\langle t(x,y)t^*(x+\xi, y+\eta) \rangle$$

as a constant, independent of  $x, y$ , and taking this before the integral sign, a straightforward calculation gives for the whole numerator the value

$$\frac{1}{3}W_2W_1^3 \langle t_s t_s^* \rangle_{Av} \delta^2. \quad (\text{II.8})$$

It is now easy to see that the same calculation applied to the first two terms of (II.4) would have given the value zero. We obtain, therefore, for the mean square gradient of the phase, in terms of the angular coordinates

$$\left\langle \left( \frac{\partial P}{\partial \xi} \right)^2 \right\rangle_{Av} = \frac{2}{3}\pi^2 \frac{W_1 \delta^2 \langle t_s t_s^* \rangle_{Av}}{W_1 W_2 t_r t_r^* + \delta^2 \langle t_s t_s^* \rangle_{Av}}. \quad (\text{II.9})$$

This shows the interesting result that so long as a part of the transmission is regular, the phase change in the wave front remains slow, but once the regular transmission becomes negligible, it jumps to the limiting value

$$\left\langle \left( \frac{\partial P}{\partial \xi} \right)^2 \right\rangle_{Av \max} = \frac{2}{3}\pi^2 W_1^2.$$

It is easy to see that this is just sufficient to wipe out interference fringes produced by wave fronts crossing an the angle at which  $W_1$  appears in the observation plane, in other words to wipe out all interferences between beams which have gone through the foil, and not around it.