REVIEWS OF MODERN PHYSICS VOLUME 27, NUMBER 1 JANUARY, 1955

Symmetry of Physical Laws Part II. Q-Number Theory of Space-Time Inversions and Charge Conjugation

SATOSI WATANABE

U. S. Naval Postgraduate School, Monterey, California

Symmetry properties of physical laws with respect to space-inversion, time-reversal and charge-conjugation are investigated in detail in the framework of the quantized field theory. In most cases, the requirements of invariance for these transformations are automatically satisfied; in a few other cases they can impose certain conditions on the ways in which different types of interaction are to be mixed. The ideas of space-parity, charge-parity and superselection rules are coherently derived from the general formulation of field theory.

1. INTRODUCTION

THE aim of this paper is to give a systematic exposition of the q-number theory of space-inversion (mirage), time-reversal (reversion), and charge conjugation. As such, the present article is a review report on this field of problems which, in recent years, has attracted increased attention from theoreticians. This paper is also the first attempt of exposition which, from the outset, is based on full recognition of the fact that Hermitian or unitary operators exist which, without being c-numbers, can commute with all the known physical quantities. The existence of such operators was previously pointed out by the author, and the operator of double reversion was shown to be an example of such operators.[1] This fact was used by Wick, Sightman, and Wigner to introduce what they called the superselection rule.[1] In this paper, the superselection rule is not only given thorough investigation and natural generalization, but its basic idea is invoked at various stages of the paper. Although the idea of superselection is indispensable for a clear understanding of symmetry problems, and its discovery originated from a consideration of symmetry problems, the superselection rule itself can be proven without the help of space-time or charge symmetry.

In Part I, reversibility (invariance for reversion), reflectibility (invariance for mirage), and inversibility (invariance for space-and-time inversion) have been formally defined and assumed to hold whenever necessary. This Part II provides the proofs of these invariance principles, together with the proof of charge-conjugation-invariance. In some particular cases, these invariance principles are conversely used to determine the correct prescription for mixing interaction types. Combinations of two kinds of symmetry, such as reversion and charge-conjugation, are also given due consideration. The so-called "charge-symmetry," which consists of interchange of neutrons with protons is not considered in this paper.

In Sec. 2 of this part, the spinor is introduced according to Cartan's method, since it is certainly the best suited for the mathematical analysis of the congruent group of the Minkowski space. This short introduction regarding the spinor is intended to fill theoretical lacunae left in the current expositions on spinors. The ten different "kinds" of spinors previously introduced by the author are also explained systematically. It will presently become clear that this classification of spinors is no longer of great importance in the field theoretical application of spinors. This is essentially due to the fact that the physical content of the theory is left unchanged by a gauge transformation of all the spinors involved. Specification of the "kind" of a spinor has sometimes been associated with the law of conservation of heavy particles.[2] In this paper, however, we take the viewpoint that the "kind" of a spinor does not have much physical importance and that the physical laws such as conservation of heavy particles should rather be associated with the superselection rules.

As a result of our conformity with Cartan's method, our formulas will be written with the help of the E-matrices instead of the customary γ-matrices, which may be a little obnoxious to some readers. However, is it aesthetically more acceptable, to have $E_4\psi$ and $E_1E_2E_3\psi$ for time-reversal and 3-dimensional space-inversion, respectively, or to have $\gamma_1\gamma_2\gamma_3\psi$ and $\gamma_4\psi$ for these transformations? In any event, a dictionary of translation from one language to another is attached in Sec. 2.

In Sec. 3, the unitary operators R, M, I, and C, corresponding to reversion, mirage, space-and-time inversion, and charge-conjugation, are introduced and their properties are investigated in great detail. Section 4 shows how the c-number field theory has the correct

[1] S. Watanabe, Phys. Rev. **81**, 1008 (1951). This paper will hereafter be referred to as (R). See, in particular, the discussion in connection with Eqs. (4.25) and (8.15) of (R). See also, Wick, Wightman, and Wigner, Phys. Rev. **88**, 101 (1952).

[2] As many pertinent references as are known to the author are cited in each individual section in the following dealing with a specific problem. However no bibliographical completeness has been contrived.

behavior for space-mirage but the wrong behavior for time-reversion, and how the q-number theory can correct the reversion property retaining the mirage property, and at the same time correctly formulate charge-conjugation.

Sections 5, 6, 7, and 8 demonstrate the existence of the unitary operators for the various transformations, thus proving the invariance of the theory for these transformations. These operators will be given explicit operatorial expressions. Section 9 discusses the transformation properties of various tensorial quantities formed with spinors, thus substantiating some of the results anticipated, without proof, in Part I. Sections 10 and 11 deal with the various types of nucleon-pion interaction and nucleon-lepton interaction. It will become clear through these discussions that, except in the case of a photon or a neutral pion, it is meaningless to speak of space-parity or of charge parity of a single elementary particle.

For the nucleon-pion interaction, the mixture of the scalar and vector types and the mixture of the pseudovector and pseudotensor types are forbidden by both reversibility and charge-conjugation-invariance. The mixture of regular tensorial quantities and pseudotensorial quantities is also refuted from a general ground. For the nucleon-lepton interaction, there are two general families of types. The first family is characterized by coupling of the same tensorial or pseudotensorial quantities originating from nucleons and from leptons. The other family is characterized by the coupling of tensorial quantities of nucleons with pseudotensorial quantities of leptons, and vice versa.[2] Both families are equally justifiable, but it is forbidden to mix them. Moreover, charge-conjugation and reversibility require certain phase-factor relations among the interaction constants in each of the families.

Sections 5–9 will show how the concepts of reversion, mirage, and charge-conjugation are useful in determining the eigenfunctions of each field and also in establishing relationship among eigenfunctions. The angular momentum representations for photons (Sec. 6) and for electrons (Sec. 8) may prove to be instrumental for problems other than the formal ones of symmetry.

As an illustration of the selection rules originating from space-symmetry and charge-conjugation, Sec. 13 will discuss the two-photon decay of a positronium and of a neutral meson.

It was pointed out in a previous paper by the author (paper R in reference 1) that reversibility as such does not determine the commutation rules of charged fields, but, rather, it is charge-conjugation that does this determination. It is not intended to emphasize this point any more in this paper, but Secs. 4 and 5 will give a sketch of the underlying facts. If one had the impression that reversibility had the power of determining the statistics-type, it is only because the combination of what we call reversion in this paper and charge conjugation was used for time reversal. This fact should not be interpreted as meaning that one view point is "correct" and the other is "incorrect." What we should call time-reversal is more or less a matter of taste.* The important fact is that we have as many laws of invariance as we have modes of transformation allowed by the theory, although some of the invariance laws can be derived from other invariance laws. It is also important to recognize this fact in regard to space parity. The author showed in a separate paper that we can define many different mirage operators leading to many different parity values of a given state. There is no room for argument as regards which one of the different definitions is the "correct" one. Even after we have imposed upon the mirage operator M the conditions that $M^2=1$ and $M\Psi_{\text{vac}}=\Psi_{\text{vac}}$, there still remains ambiguity of parity value of a state in which the number of charged particles is odd. [See Eq. (8.28).] Any conclusion that can be drawn from either value of parity is "correct."

2. MATHEMATICAL PRELIMINARIES

A. Rotations and Reflections

The mathematical entity now called spinor was first introduced by Élie Cartan[3] in 1913, i.e., some fifteen years before the discovery of the relativistic wave equation of the electron. Cartan's method is based on the fact that any congruent transformation can be considered as a product of simple reflections, although the way of this decomposition involves certain arbitrariness. For this reason, Cartan's method is best suited for discussions pertaining to inversions, while it brings about the same results pertaining to rotations as in the ordinary method.

By the simple reflection with respect to a plane passing through the origin and having the normal vector a^μ ($\mu=1, 2, 3, 0$), an arbitrary, regular vector x^μ is transformed into

$$x'^\mu=x^\mu-2a^\mu(a_\nu x^\nu)/(a_\kappa a^\kappa). \tag{2.1}$$

The metric tensor $g_{\mu\nu}$ of the Minkowsky space is given in (I.2.2).[4] We assume hereafter that the normal vector a^μ is always normalized to 1 or -1, and we speak of a space-like reflection or time-like reflection according as

$$a_\mu a^\mu=+1 \quad \text{or} \quad a_\mu a^\mu=-1. \tag{2.2}$$

The sign of a^μ itself is arbitrary, but the sign of $a_\mu a^\mu$ has a vital meaning.

Let the number of space-like reflections, the number of time-like reflections, and the total number of reflections of both kinds in a decomposition of a given congruent transformation be denoted, respectively, by ν_s, ν_t, and ν. The parities of these numbers are determined

* The definition used in this paper is more convenient only in the sense that it has a direct correspondence with the time-reversal in classical physics.

[3] É. Cartan, Bull. Soc. Math. de France, 41, 53 (1913). See also É. Cartan, *La théorie des spineurs* (Hermann et Cie, Paris, 1938).

[4] (I.2.2) mean formula (2.2) of Part I of this paper.

when the congruent transformation is given, and we have

$$\sigma_s=(-1)^{\nu_s},\quad \sigma_t=(-1)^{\nu_t},\quad \sigma=(-1)^{\nu}, \tag{2.3}$$

where σ_s, σ_t, and σ are given in (I.2.8), (I.2.9), and (I.2.10). It can be shown that, by a suitable choice of simple reflections, ν can be made equal to or less than 4.

The general congruent transformations can be classified into four categories[5]: ($\mathfrak{A}$) $\sigma_s=+1$, $\sigma_t=+1$; ($\mathfrak{B}$) $\sigma_s=-1$, $\sigma_t=-1$; ($\mathfrak{C}$) $\sigma_s=-1$, $\sigma_t=+1$; ($\mathfrak{D}$) $\sigma_s=+1$, $\sigma_t=-1$. The invariance of the theory for ($\mathfrak{A}$), Lorentz transformations and space-rotations, is well established. All the transformations belonging to class ($\mathfrak{C}$) can be obtained by multiplying all the transformations of class ($\mathfrak{A}$) to any single transformation of class ($\mathfrak{C}$), say,

$$(x, y, z, t)\rightarrow(-x, -y, -z, t). \tag{2.4}$$

All the transformations of class ($\mathfrak{D}$) can be obtained by multiplying all the transformations of class ($\mathfrak{A}$) to any single transformation of class ($\mathfrak{D}$), say,

$$(x, y, z, t)\rightarrow(x, y, z, -t). \tag{2.5}$$

All the transformations of class ($\mathfrak{B}$) can be obtained by multiplying all the transformations of class ($\mathfrak{A}$) to any single transformation of class ($\mathfrak{B}$), say, the product of (2.4) and (2.5). Thus the study of invariance for the general congruent transformations reduces to that of the transformations given in (2.4) and (2.5).

Here we shall insert illustrations to show how a spatial rotation and a Lorentz transformation can be decomposed into simple reflections. Take two purely spatial unit vectors, $\mathbf{a}$ and $\mathbf{b}$, where $a^0=b^0=0$. We perform reflections first with regard to $\mathbf{a}$ and then with regard to $\mathbf{b}$:

$$\left.\begin{aligned}\mathbf{x}'&=\mathbf{x}-2\mathbf{a}(\mathbf{a}\cdot\mathbf{x}),\\ \mathbf{x}''&=\mathbf{x}'-2\mathbf{b}(\mathbf{b}\cdot\mathbf{x}').\end{aligned}\right\} \tag{2.6}$$

Putting

$$[\mathbf{a}\times\mathbf{b}]=\sin\frac{\varphi}{2}\cdot\mathbf{c},\quad (\mathbf{a}\cdot\mathbf{b})=\cos\frac{\varphi}{2}, \tag{2.7}$$

where $\mathbf{c}$ is a unit vector, we can easily deduce from Eq. (2.6)

$$\mathbf{x}''=\mathbf{x}+(1-\cos\varphi)[\mathbf{c}\times[\mathbf{c}\times\mathbf{x}]]+\sin\varphi[\mathbf{c}\times\mathbf{x}]. \tag{2.8}$$

As a simple geometrical consideration will show, this transformation Eq. (2.8) is nothing but a rotation by angle φ about the axis $\mathbf{c}$. Suppose we perform furthermore a simple reflection with regard to $\mathbf{c}$. Then we obtain

$$\begin{aligned}\mathbf{x}'''&=\mathbf{x}''-2\mathbf{c}(\mathbf{c}\cdot\mathbf{x}'')\\ &=-\mathbf{x}-(1+\cos\varphi)[\mathbf{c}\times[\mathbf{c}\times\mathbf{x}]]+\sin\varphi[\mathbf{c}\times\mathbf{x}].\end{aligned} \tag{2.9}$$

This reduces to (2.4) for $\varphi=\pi$, agreeing with the fact that the total mirage is the product of a simple reflection with regard to any direction in space and a rotation by 180° about this direction.

In a similar way, let us take two space-like unit vectors,

$$\left.\begin{aligned}a^\mu&=(\cosh\alpha, 0, 0, \sinh\alpha),\\ b^\mu&=(\cosh\beta, 0, 0, \sinh\beta),\end{aligned}\right\} \tag{2.10}$$

and perform simply a reflection first with regard to a^μ and then with regard to b^μ. Then, the product of these two simple reflections becomes

$$\left.\begin{aligned}x'^1&=\cosh\varphi x^1+\sinh\varphi x^0,\\ x'^0&=\sinh\varphi x^1+\cosh\varphi x^0,\end{aligned}\right\} \tag{2.11}$$

with

$$\varphi=2(\beta-\alpha). \tag{2.12}$$

Equation (2.11) is merely an ordinary Lorentz transformation with a relative velocity in x'-direction of magnitude v:

$$\sinh\varphi=v/(1-v^2)^{\frac{1}{2}} \tag{2.13}$$

The interchange of a^μ with b^μ changes the sign of the relative velocity.

B. Spinors

The basic matrices E_μ ($\mu=1, 2, 3, 0$) are defined by

$$E_\mu E_\nu+E_\nu E_\mu=2g_{\mu\nu}. \tag{2.14}$$

They must have at least four columns and four rows to satisfy Eq. (2.14). The "contravariant" E's are defined by

$$E^\mu g_{\mu\nu}=E_\nu. \tag{2.15}$$

The matrix E_4, defined by

$$E_4=iE^0=-iE_0, \tag{2.16}$$

will occasionally be used, but usually the index μ will be supposed to run over (1,2,3,0) instead of (1,2,3,4). It should also be noted that E_5 defined by

$$E_1E_2E_3E_0=iE_5 \tag{2.17}$$

anticommutes with the four E's and $E_5^2=1$. The change of factor i into $-i$ in Eq. (2.17) will change what Eddington would call chirality, but it does not affect the ordinary four-dimensional theory.

There exist two important matrices, J and K which transform the basic E's respectively into their Hermitian conjugates and their transposes.[6] The matrix J is a necessary instrument to construct the "adjoint" spinors, while K is closely connected with the charge-conjugation. Their properties are, more precisely,

$$J^{-1}E_\mu J=-\bar{E}_\mu,\quad J^{-1}E_5J=-\bar{E}_5,\quad \bar{J}=J; \tag{2.18}$$

$$K^{-1}E_\mu K=-E_\mu{}^T,\quad K^{-1}E_5K=+E_5{}^T,\quad K^T=-K, \tag{2.19}$$

[5] See the Appendix in (R), reference 1 of this paper.

[6] The J used here is i-times or minus i-times the J used in (R). For the derivation of the theorems concerning K and J, see W. Pauli, Ann. Inst. Henri Poincaré, **6**, 137 (1936).

where the bar means the Hermitian conjugate, and T means the transpose. Between J and K there exists a relation:

$$\bar{K}=-J^TK^{-1}J. \tag{2.20}$$

There are sets of E's such that E_1, E_2, E_3, $E_4(=-iE_0)$ and E_5 are all Hermitian (E_0: anti-Hermitian):

$$E_\mu=\bar{E}^\mu, \quad E_5=\bar{E}_5. \quad \text{(Hermitian system)} \tag{2.21}$$

In a Hermitian system, we can take

$$J=J^{-1}=\bar{J}=E_4=-iE_0, \quad \text{(Hermitian system)} \tag{2.22}$$

and we have

$$\bar{K}=K^{-1}. \quad \text{(Hermitian system)} \tag{2.23}$$

Furthermore, among the Hermitian systems, there exist such sets of E's, that E_0 is antisymmetric, three out of E_1, E_2, E_3, E_5 are symmetric, and the remaining one is antisymmetric. If, for instance, we take E_2 as this antisymmetric one, then we have the following rule:

$$\left.\begin{array}{l} E_1,\ E_3,\ E_0,\ E_5 \text{ have real elements,} \\ E_2 \qquad \text{has imaginary elements.} \end{array}\right\} \text{(special)} \tag{2.24}$$

In such a system we can take

$$J=-iE_0; \quad K=E_1E_3. \quad \text{(special)} \tag{2.25}$$

If we take E_5 as the antisymmetric one, we have the following rule:

$$\left.\begin{array}{l} E_1,\ E_2,\ E_3,\ E_0 \text{ have real elements,} \\ E_5 \qquad \text{has imaginary elements.} \end{array}\right\} \text{(special)} \tag{2.26}$$

In such a system, we can take

$$J=K=-iE_0. \tag{2.27}$$

As will be seen presently, we must adopt a Hermitian system in order to make the normalization of spinors in the ordinary 3-dimensional space feasible. We shall not use particular representations of J and K, except in the last stage of calculations, in order to keep the physical implications of J and K clear. As to the special systems in (2.24) and (2.26), they are mentioned only to compare the E-system to the usually adopted representations of the α's and the γ's, and will not be used in this paper.

We now introduce a 4–4-matrix representation of the entire congruent group. Any congruent transformation can be expressed as the result of a series of simple reflections characterized by normal vectors: $a_{(1)}{}^\mu$, $a_{(2)}{}^\mu$, $\cdots$, $a_{(\nu)}{}^\mu$, whereby the order in which the reflections appear must be respected. Corresponding to this transformation, we introduce a matrix:

$$S=A_\nu\cdots A_2A_1, \tag{2.28}$$

with

$$A_\tau=a_{(\tau)}{}^\mu E_\mu. \tag{2.29}$$

It should be noted that

$$AA=a^\mu a^\nu\tfrac{1}{2}(E_\mu E_\nu+E_\nu E_\mu)=a_\mu a^\mu. \tag{2.30}$$

Therefore,

$$S^{-1}=\sigma_t A_1A_2\cdots A_\nu. \tag{2.31}$$

It is easy[3] to show that the S's defined by Eq. (2.28) are faithful representations of the congruent group, but they are by definition two-valued since the sign of each a^μ is arbitrary. From the defining properties of J and K follow

$$J^{-1}SJ=\sigma_s\bar{S}^{-1}, \tag{2.32}$$

$$K^{-1}SK=\sigma_sS^{T-1}. \tag{2.33}$$

The spinor ξ is defined as the representation vector of the S's, i.e.,

$$\xi'=S\xi. \tag{2.34}$$

The adjoint spinor has to be defined by a transformation rule which contains S^{-1}, so that it may cancel with the transformation of ξ. This could be done, in view of Eqs. (2.32) and (2.33), either by the use of J or K. However, in order to provide the possibility of positive definite normalization of spinors, we have to choose the former alternative. Thus, the adjoint spinor $\xi^\times$ is defined by

$$\xi^\times=\bar{\xi}J^{-1}. \tag{2.35}$$

where the bar means the complex conjugate in the c-number theory and the Hermitian conjugate in the q-number theory. The star will be reserved to designate exclusively the complex conjugate. In virtue of Eqs. (2.32) and (2.34), the transformation rule of $\xi^\times$ becomes

$$\xi'^\times=\sigma_s\xi^\times S^{-1}. \tag{2.36}$$

Now, the relation in Eq. (2.33) can be exploited to derive a physically important theorem—namely, if ξ_1 and $\xi_1{}^\times$ transform according to Eqs. (2.34) and (2.36), then ξ_2 and $\xi_2{}^\times$ defined by

$$\xi_2=\xi_1{}^\times K, \quad \xi_2{}^\times=\xi_1K^{-1} \tag{2.37}$$

transform, because of Eq. (2.33), again according to the same rules as seen in Eqs. (2.34) and (2.36). That ξ_2 and $\xi_2{}^\times$, defined by Eq. (2.37), are connected by Eq. (2.35) can be shown by Eq. (2.20). We shall see that transformation in Eq. (2.37) provides the possibility of charge-conjugation.

The "normalization" of spinors is usually done by requiring

$$\int\xi^\times iE^0\xi dxdydz=+1, \tag{2.38}$$

which is possible in a Hermitian E-system, because, on account of Eq. (2.22), Eq. (2.38) becomes

$$\int\bar{\xi}\xi dxdydz=+1.$$

We shall have to use quantities of the type $\xi^{\times}O\xi$ where O is some operator with Dirac indices. It should always be kept in mind that if

$$Q=\xi^{\times}O\xi=\xi O'\xi, \tag{2.39}$$

this means that

$$O'=-iE_0O \quad \text{or} \quad O=-iE_0O'. \tag{2.40}$$

If O' is Hermitian, then Q will be real in the c-number theory.

It may be well to introduce here a set of simplifying notations:

$$\left.\begin{aligned} E_{\mu\nu}&=(1/2)\sum P(\alpha,\beta)E_\alpha E_\beta=(E_\mu E_\nu-E_\nu E_\mu)/2,\\ E_{\mu\nu\kappa}&=(1/6)\sum P(\alpha,\beta,\gamma)E_\alpha E_\beta E_\gamma,\\ E_{\mu\nu\kappa\lambda}&=(1/24)\sum P(\alpha,\beta,\gamma,\delta)E_\alpha E_\beta E_\gamma E_\delta, \end{aligned}\right\} \tag{2.41}$$

where the P's are sign-functions such that, for instance, $P(\alpha,\beta,\gamma)$ is $+1$ when (α,β,γ) is an even permutation of (μ,ν,κ), and -1 when (α,β,γ) is an odd permutation of (μ,ν,κ), and otherwise zero.

We shall now give the transformation properties of various tensorial quantities that can be built with the two spinors, ψ and φ, obeying the same transformation rules of Eqs. (2.34) and (2.36):

$$\left.\begin{aligned} \Lambda&=i\psi^{\times}E_5\varphi, & \Gamma&=\psi^{\times}\varphi,\\ \Lambda_\mu&=i\psi^{\times}E_\mu\varphi, & \Gamma_\mu&=i\psi^{\times}E_5E_\mu\varphi,\\ \Lambda_{\mu\nu}&=\psi^{\times}E_5E_{\mu\nu}\varphi, & \Gamma_{\mu\nu}&=i\psi^{\times}E_{\mu\nu}\varphi. \end{aligned}\right\} \tag{2.42}$$

The factor i is inserted in four of those quantities so that the complex conjugate of each quantity becomes, in c-number theory, identical with the original quantity with ψ and φ interchanged. It is easy to show, with the help of Eqs. (2.34) and (2.36), that the "kinds" of tensors, defined in Part I, of these quantities are as follows:

$$\left.\begin{aligned} &\Lambda,\ \Lambda_\mu,\ \Lambda_{\mu\nu}, && \text{second kind}\\ &\Gamma,\ \Gamma_\mu,\ \Gamma_{\mu\nu}. && \text{third kind} \end{aligned}\right\} \tag{2.43}$$

Because of Eq. (2.17), $\Lambda_{\mu\nu}$ and $\Gamma_{\mu\nu}$ are complementary to each other. See (I.2.13). The assignment of kinds given in (2.43) will no longer be valid in the q-number theory. See Table III, Sec. 9.

To avoid repetition, let us prove only that Λ_μ is a second kind pseudovector. Taking an arbitrary regular vector, x^μ, we built a matrix

$$X=x^\mu E_\mu.$$

If x^μ undergoes the transformation of Eq. (2.1), X will undergo the transformation,

$$\begin{aligned} X'=x'^\mu E_\mu&=x^\mu E_\mu-2\sigma_t a^\mu E_\mu(a_\kappa x^\kappa)\\ &=X-\sigma_t A(AX+XA)=-\sigma_t AXA, \end{aligned}$$

where A is given by Eq. (2.29). More generally, if x^μ undergoes a series of simple reflections, X will obviously transform according to

$$\begin{aligned} X'&=(-1)^\nu\sigma_t A_\nu\cdots A_1XA_1\cdots A_\nu\\ &=\sigma SXS^{-1}. \end{aligned} \tag{2.44}$$

Now take the product of Λ_μ with x^μ and call it C:

$$C=\Lambda_\mu x^\mu, \tag{2.45}$$

then this will transform, as a result of Eqs. (2.34), (2.36), and (2.44), so that

$$C'=i\psi'^{\times}E_\mu\varphi'x'^\mu=\sigma_s\sigma\psi^{\times}E_\mu\varphi x^\mu, \tag{2.46}$$

showing that C is a second kind scalar. Then, in virtue of the quotient rule of tensor calculus and of the product rule of Part I, we conclude that Λ_μ is a second kind pseudovector.

Equation (2.44) can also be written, with the help of $a^\mu{}_{.\nu}$ (I.2.1), as

$$a^\mu{}_{.\nu}x^\nu E_\mu=\sigma Sx^\mu E_\mu S^{-1}. \tag{2.47}$$

Since x^ν is arbitrary, we obtain

$$E_\mu a^\mu{}_{.\nu}=\sigma SE_\nu S^{-1}. \tag{2.48}$$

The relation in Eq. (2.48) enables us to compare the E-system with the customary α system and γ system. More precisely, the transformation rules for class $\mathfrak{A}$ alone are not sufficient to determine the unique correspondence between the E-system and the γ system. If we assume, for the γ system, the transformation rule (including inversions),

$$\gamma_\mu a^\mu{}_{.\nu}=S\gamma_\nu S^{-1}, \tag{2.49}$$

then we obtain the following correspondence:

$$\left.\begin{aligned} E_1&=i\gamma_5\gamma_1=-i\alpha_2\alpha_3\beta=\sigma_1\rho_3\\ E_2&=i\gamma_5\gamma_2=-i\alpha_3\alpha_1\beta=\sigma_2\rho_3\\ E_3&=i\gamma_5\gamma_3=-i\alpha_1\alpha_2\beta=\sigma_3\rho_3\\ E_0&=iE_4=-\gamma_5\gamma_4=-i\alpha_1\alpha_2\alpha_3\beta=-i\rho_2\\ E_5&=\gamma_5=i\alpha_1\alpha_2\alpha_3=-\rho_1=i\rho_2\rho_3 \end{aligned}\right\} \tag{2.50}$$

$$\left.\begin{aligned} &\gamma_1=iE_1E_5,\quad \gamma_2=iE_2E_5,\quad \gamma_3=iE_3E_5,\\ &\qquad\qquad\qquad\qquad \gamma_4=E_0E_5,\quad \gamma_5=E_5\\ &\alpha_1=E_0E_1,\quad \alpha_2=E_0E_2,\quad \alpha_3=E_0E_3,\quad \beta=E_0E_5 \end{aligned}\right\} \tag{2.51}$$

$$\xi^{\times}=-i\xi E_0=-i\xi^\dagger\gamma_5,\quad \xi^\dagger=\xi\gamma_4=i\xi^{\times}E_5 \tag{2.52}$$

$$\left.\begin{aligned} \Lambda&=\psi^\dagger\varphi, & \Gamma&=-i\psi^\dagger\gamma_5\varphi\\ \Lambda_\mu&=i\psi^\dagger\gamma_\mu\varphi, & \Gamma_\mu&=i\psi^\dagger\gamma_\mu\gamma_5\varphi\\ \Lambda_{\mu\nu}&=-i\psi^\dagger\gamma_{\mu\nu}\varphi, & \Gamma_{\mu\nu}&=i\psi^\dagger\gamma_{\mu\nu}\gamma_5\varphi \end{aligned}\right\} (\mu,\nu=1,2,3,4). \tag{2.53}$$

From this correspondence, it can be seen that the Hermitian E's result in Hermitian α's and Hermitian γ's.

Furthermore, the customary representation of α's in which only α_2 has imaginary elements corresponds to the special system as seen in (2.24) and Eq. (2.25). The γ system, in which only γ_4 and γ_5 have imaginary elements, corresponds to the special system of (2.26) and Eq. (2.27). If we translate the charge-conjugation of Eq. (2.37) into the language of the γ system with the help of Eqs. (2.50) and (2.52), we get

$$\xi_2=-i\xi_1^\dagger E_5 K.$$

Comparing this with Schwinger's definition[7] of charge-conjugation matrix C:

$$\xi_2=-\xi_1^\dagger C \tag{2.54}$$

we get $C=i\gamma_5 K$ and, in particular, in the system of (2.26) and Eq. (2.27),

$$C=-\gamma_4, \quad \xi_2=\bar{\xi}_1. \tag{2.55}$$

We shall now discuss briefly the spinor transformations corresponding to the illustrations given in Eqs. (2.8) and (2.11). The transformation matrices S corresponding to Eq. (2.6) are, according to Eq. (2.29),

$$A_a=a^\alpha E_\alpha, \quad A_b=b^\alpha E_\alpha \quad (\alpha=1,2,3). \tag{2.56}$$

Then the S for Eq. (2.8) will be

$$\left.\begin{aligned} S=A_bA_a=-\sum_\gamma \sin\frac{\varphi}{2}c^\gamma E_{\alpha\beta}+\cos\frac{\varphi}{2} \\ (\alpha,\beta,\gamma: \text{cyclic in } 1,2,3). \end{aligned}\right\} \tag{2.57}$$

In particular, when $\varphi=\pi$, Eq. (2.57) becomes

$$S_1=-(E_{12}c_3+E_{23}c_1+E_{31}c_2). \tag{2.58}$$

The simple reflection with regard to $\mathbf{c}$ is

$$S_2=E_1c_1+E_2c_2+E_3c_3. \tag{2.59}$$

The product of these two, which is the mirage, is given by

$$S_3=S_2S_1=-E_1E_2E_3. \tag{2.60}$$

These S_1, S_2, S_3 will play a certain role in the angular momentum representation of a spinor field. It should be noted with regard to Eq. (2.57) that if we continuously increase φ up to $\varphi=2\pi$, S becomes -1 instead of $+1$, which is another manifestation of the two-valuedness of S.

Next for the transformations given in Eq. (2.10), we have

$$\begin{aligned} A_1&=E_1\cosh\alpha+E_0\sinh\alpha, \\ A_2&=E_1\cosh\beta+E_0\sinh\beta. \end{aligned} \tag{2.61}$$

The result of these, which is a Lorentz transformation, is

$$S=A_2A_1=\cosh(\beta-\alpha)-E_1E_0\sinh(\beta-\alpha). \tag{2.62}$$

[7] J. Schwinger, Phys. Rev. 74, 1439 (1948).

The transformations considered in Eqs. (2.56)–(2.62) pertain to the spinors which do not depend on the position in the space-time. Usually the spinor represents a certain field, and therefore should undergo the transformation of the argument (x,y,z,t) of ξ in addition to the spinor transformation considered above. As is always the case in the problems of this kind, two "pictures" are possible: either the vectors are fixed while the coordinates are shifted or the vectors are shifted while the coordinates are fixed. Taking the first picture, let us denote the transformation in question by an abstract symbol $\mathfrak{S}$. The same space-time point will be given coordinates x^μ and x'^μ in the old and the new frames of reference, in such a way that $x'=\mathfrak{S}x$. At the same time the field quantity F will become $\mathfrak{S}F=F'$. Therefore the field quantity F in the new description is

$$F'(x')=\mathfrak{S}F(x)=\mathfrak{S}F(\mathfrak{S}^{-1}x'). \tag{2.63}$$

As an illustration, let us apply this formula to the transformation of Eqs. (2.8) and (2.57):

$$\begin{aligned} \xi'(\mathbf{x}')=&\left(-\sum_\gamma \sin\frac{\varphi}{2}c^\gamma E_{\alpha\beta}+\cos\frac{\varphi}{2}\right) \\ &\times\xi(\mathbf{x}'+(1-\cos\varphi)[\mathbf{c}\times[\mathbf{c}\times\mathbf{x}']] \\ &-\sin\varphi[\mathbf{c}\times\mathbf{x}']). \end{aligned} \tag{2.64}$$

The sign of φ is opposite to that used in Eq. (2.8) since we have to take $\mathfrak{S}^{-1}$ instead of $\mathfrak{S}$ in the argument of ξ. In particular, if $\mathbf{c}$ is oriented in the z-direction,

$$\begin{aligned} \xi'(x',y',z')=&\left(-\sin\frac{\varphi}{2}E_{12}+\cos\frac{\varphi}{2}\right) \\ &\times\xi(\cos\varphi x'+\sin\varphi y', -\sin\varphi x'+\cos\varphi y', z'). \end{aligned}$$

For the infinitesimal rotation $d\varphi$, we then obtain

$$\begin{aligned} [d\xi'(x',y',z')/d\varphi]_{\varphi=0} &=y'(d\xi/dx')-x'(d\xi/dy')-\tfrac{1}{2}E_{12}\xi \\ &=-i(L_3+\tfrac{1}{2}\sigma_3)\xi, \end{aligned} \tag{2.65}$$

where L_3 and σ_3 are the z-components of the familiar operators of orbital angular momentum and spin angular momentum, i.e.,

$$L_3=-i(x\partial/\partial y-y\partial/\partial x), \quad \sigma_3=-iE_1E_2. \tag{2.66}$$

C. Various Kinds of Spinors

Just as the four kinds of tensors (Part I, Sec. 2) are equally justified representations, the four kinds of spinors,[8]

$$\begin{aligned} \xi'&=S\xi, \qquad \eta'=\sigma S\eta, \\ \zeta'&=\sigma_t S\zeta, \quad \omega'=\sigma_s S\omega, \end{aligned} \tag{2.67}$$

[8] See the Appendix of reference (R).

are equally justified spinors. The tensorial quantities, Eq. (2.42), made out of two spinors belonging to the same kind have the same transformation properties as have been given with regard to the ξ spinor. If ψ and φ of Eq. (2.42) belong to different kinds, then the transformation properties of the tensorial quantities are modified by the σ's involved in ψ and φ, just like the product rule of Part I, Sec. 2.

However, Eq. (2.67) is not the only possible analog of pseudotensors for spinors. Actually, the three groups, each with two elements, ($\sigma=1$, $\sigma=-1$), ($\sigma_t=1$, $\sigma_t=-1$), and ($\sigma_s=1$, $\sigma_s=-1$), are respectively isomorphic to the three factor groups, ($\mathfrak{A}+\mathfrak{B}$, $\mathfrak{C}+\mathfrak{D}$), ($\mathfrak{A}+\mathfrak{C}$, $\mathfrak{B}+\mathfrak{D}$), and ($\mathfrak{A}+\mathfrak{D}$, $\mathfrak{B}+\mathfrak{C}$) of the congruent group since $\sigma^2=\sigma_t{}^2=\sigma_s{}^2=1$. The spinor representation is basically two-valued, therefore, -1 as well as $+1$ can serve as the identity transformation. Thus, we can equally well take $((\sigma)^{\frac{1}{2}}=\pm 1$, $(\sigma)^{\frac{1}{2}}=\pm i)$, $((\sigma_t)^{\frac{1}{2}}=\pm 1$, $(\sigma_t)^{\frac{1}{2}}=\pm i)$, and $((\sigma_s)^{\frac{1}{2}}=\pm 1$, $(\sigma_s)^{\frac{1}{2}}=\pm i)$, as the representations of the above factor groups. This consideration leads to the following new kinds of spinors:

$$\mu'=(\sigma)^{\frac{1}{2}}S\mu, \quad \nu'=(\sigma_t)^{\frac{1}{2}}S\nu, \quad \kappa'=(\sigma_s)^{\frac{1}{2}}S\kappa, \tag{2.68}$$

where the square root of -1 can be either $+i$ or $-i$. Thus, Eq. (2.68) actually contains six different kinds of spinors. The possibilities of Eq. (2.68) were first indicated by the author[9] in connection with a 5-dimensional theory, but it obviously applies to the 4-dimensional theory. It can easily be seen that Yang-Tiomno's A- and B-types[10] belong to our Eq. (2.68) and C- and D-types belong to our Eq. (2.67). Our classification is more detailed than Yang-Tiomno's because classes $\mathfrak{B}$ and $\mathfrak{D}$ are taken into consideration.

We should not spend more space here for these different kinds of spinors, because, in field theory, we shall be able to deal with the possible phase changes in reversion and mirage without specifying, in advance, the kinds of involved spinors.

D. Transpose Operators

In the main body of this paper, we shall have to deal constantly with transpose operators. The transpose O^T of an operator O can of course be defined in the matrix form,

$$O^T{}_{ij}=O_{ji}=(\bar{O}_{ij})^*. \tag{2.69}$$

The operation of transposition itself is not an invariant operation for an arbitrary unitary transformation. We shall see in the next section that, in spite of this, a certain operation involving transposition acquires an invariant meaning.

For instance, the operator $O=\partial/\partial x$ has the matrix elements,

$$(x'|O|x'')=\delta'(x'-x''). \tag{2.70}$$

Its transpose is given by

$$(x'|O^T|x'')=\delta'(x''-x')=-\delta'(x'-x''). \tag{2.71}$$

Therefore, $O=\partial/\partial x$ is an antisymmetric operator in the x-representation. Equation (2.71) shows that here O^T means the differentiation of a function standing to the left of this operator. This agrees with the more elementary definition of the transpose operator,

$$Of=fO^T. \tag{2.72}$$

In the same way, $p_x=-i\partial/\partial x$ is also an antisymmetric operator in the x-representation, because it is Hermitian and has imaginary matrix elements. We shall often use an operator Ω_x which is defined by

$$\Omega_x f(x)=f(-x). \tag{2.73}$$

For this operator, we have

$$(x'|\Omega_x|x'')=(x'|\Omega_x{}^T|x'')=\delta(x'+x''), \tag{2.74}$$

i.e., in the x-representation,

$$\Omega_x=\Omega_x{}^T(=\bar{\Omega}_x=\Omega_x{}^{-1}). \tag{2.75}$$

For an operator which involves both space-time coordinates and the Dirac indices, we have to take the transpose with respect to both of these variables. For instance, if O is given by[11]

$$I=L_\alpha E_\alpha+E_0E_5 \quad (\alpha=1, 2, 3), \tag{2.76}$$

with

$$L_\alpha=-L_\alpha{}^T=-i[x_\beta(\partial/\partial x_\gamma)-x_\gamma(\partial/\partial x_\beta)]$$
$$(\alpha, \beta, \gamma: \text{cyclic in } 1, 2, 3), \tag{2.77}$$

then we can write, with the help of Eq. (2.19),

$$I^T=K^{-1}IK. \tag{2.78}$$

E. Redundancy of State Functions

It is well known that the state function Ψ has an inherent arbitrariness of phase factor, i.e., Ψ and

$$e^{i\alpha}\Psi \tag{2.79}$$

represent physically the same state, where α is an arbitrary real constant. However, it seems to be generally overlooked[12] that α can be a particular function of the occupation number of spinor particles involved in Ψ. For instance, we can take as $\exp(i\alpha)$ the following function:

$$\left.\begin{aligned} e^{i\alpha}=\Delta^{-1}=\bar{\Delta}=\Delta^T=\Delta\equiv\prod_i(-1)^{N_i}=\prod_i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \\ \text{with} \qquad \Delta^2=1. \end{aligned}\right\} \tag{2.80}$$

[9] S. Watanabe, Sci. Pap. Inst. Phys. Chem. Research (Tokyo) **39**, 157 (1941).

[10] C. N. Yang and J. Tiomno, Phys. Rev. **79**, 495 (1950).

[11] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, London, 1947), third edition, p. 267, Eq. (40).

[12] This fact was noticed by the author in relation to Eqs. (4.25) and (9.15) in reference (R).

The index i should run all over the spinor eigenstates. If Ψ is known to represent a state with an even (odd) number of spinor particles, then $\Delta\Psi$ will be $+\Psi$ $(-\Psi)$. But, in general, we have to leave Δ as an operator. If Δ commutes with any arbitrary physical quantity, Ψ and $\Delta\Psi$ will represent the same state because these two state functions will have the same expectation values for all the physical quantities.

The commutability of Δ with any physical quantity is a consequence of the fact that any physical quantity must involve an even number of spinors. In Sec. 3 of Part I of this paper, we adopted the viewpoint that all physically observable quantities are tensorial quantities of some "kind." If this postulate is maintained, the spinor analysis will tell us that any physically observable quantity must be an expression quadratic or bilinear in spinors or a product of such expressions. It usually has the form $\psi^{\times}O\varphi$, but in some cases (e.g., pair creation interaction energy) $\psi^{\times}O\varphi^{\times}$ may also happen, where $\psi^{\times}$ is linear in creation operators $\bar{g}$ and φ is linear in annihilation operators g. The forms of g and $\bar{g}$ are:

$$\left.\begin{aligned} g&=\left[\Pi\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]\begin{pmatrix}0&1\\0&0\end{pmatrix}\\ \bar{g}&=\left[\Pi\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]\begin{pmatrix}0&0\\1&0\end{pmatrix}\end{aligned}\right\} \tag{2.81}$$

or their transforms by an unitary transformation. The factor represented by Π is the well-known Wigner-Jordan factor. We now have

$$\Delta g\Delta=-g, \quad \Delta\bar{g}\Delta=-\bar{g}. \tag{2.82}$$

As a result any expression which contains an even number of spinors commutes with Δ.

Later we shall see that Δ is not the only unitary operator that commutes with all the known physical quantities. In general we shall write W for any operator that has this property. A more general consideration on this line will be given in the section on the superselection rule.

3. OPERATORS FOR REVERSION, MIRAGE, INVERSION AND CHARGE CONJUGATION

We use the interaction "picture" throughout this paper, in order to have at hand a formula applicable for both the Heisenberg and Schrödinger pictures rather than for the purpose of actual calculations of specific problems in this picture. We can obtain the formulas for the Heisenberg and Schrödinger pictures from those of the interaction picture only by including the entire Hamiltonian either in the "free" Hamiltonian or in the interaction Hamiltonian of the interaction picture.

The time development of the state function, $\Psi(t)$, is governed by the interaction Hamiltonian $\mathbf{H}(t)$ and expressed by

$$\Psi(t_2)=U(t_2,t_1)\Psi(t_1), \tag{3.1}$$

with

$$\left.\begin{aligned} dU(t_2,t_1)/dt_2&=-i\mathbf{H}(t_2)U(t_2,t_1),\\ dU(t_2,t_1)/dt_1&=+iU(t_2,t_1)\mathbf{H}(t_1),\end{aligned}\right\} \tag{3.2}$$

and

$$U(t_1,t_1)=\mathbf{1}, \quad U^{-1}(t_2,t_1)=\bar{U}(t_2,t_1)=U(t_1,t_2). \tag{3.3}$$

The time-development of the physical quantity, $Q(t)$, is governed by the free Hamiltonian $\mathbf{H}_0$ and expressed by

$$Q(t_2)=U_0^{-1}(t_2,t_1)Q(t_1)U_0(t_2,t_1), \tag{3.4}$$

with

$$dU_0(t_2,t_1)/dt_2=-i\mathbf{H}_0U(t_2,t_1), \quad U(t_1,t_1)=1. \tag{3.5}$$

According to the definitions of reversed phenomenon, miraged phenomenon and inverted phenomenon given in Part I, Sec. 3, they are characterized by

$$Q'(-t)=\rho_RQ(t), \quad Q'(-\mathbf{x})=\rho_MQ(\mathbf{x}),$$
$$Q'(-\mathbf{x},-t)=\rho_IQ(\mathbf{x},t). \tag{3.6}$$

See (I.2.18), (I.2.19), and (I.2.21). In quantum physics, we have to reinterpret Eq. (3.6) in terms of expectation values. Thus, the reversed phenomenon $\Psi_R(t)$, miraged phenomenon $\Psi_M(t)$, and inverted phenomenon $\Psi_I(t)$ of an original phenomenon $\Psi(t)$ should be defined by

$$(\Psi_R(-t), Q(\mathbf{x},-t)\Psi_R(-t))$$
$$=\rho_R(\Psi(t), Q(\mathbf{x},t)\Psi(t)), \tag{3.7}$$

$$(\Psi_M(t), Q(-\mathbf{x},t)\Psi_M(t))$$
$$=\rho_M(\Psi(t), Q(\mathbf{x},t)\Psi(t)), \tag{3.8}$$

$$(\Psi_I(-t), Q(-\mathbf{x},-t)\Psi_I(-t))$$
$$=\rho_I(\Psi(t), Q(\mathbf{x},t)\Psi(t)), \tag{3.9}$$

where ρ_R, ρ_M, and ρ_I are given in Part I, Tables II, III, IV, and V.

We add to these sign functions another sign function ρ_C for the charge conjugation. For all the "mechanical" quantities ρ_C is $+1$ and is -1 for all the "electromagnetic" quantities. Among the various quantities mentioned in Part I, Table V, the following are electromagnetic": electric charge, magnetic charge, current, electromagnetic potentials, electromagnetic field strengths, electromagnetic moment. All the rest are "mechanical." Using ρ_C, thus defined, the charge-conjugate phenomenon $\Psi_C(t)$ should be defined by

$$(\Psi_C(t), Q(\mathbf{x},t)\Psi_C(t))=\rho_C(\Psi(t), Q(\mathbf{x},t)\Psi(t)). \tag{3.10}$$

We can now define reversibility, reflectibility, etc. as follows: If $\Psi_R(t)$, $\Psi_M(t)$, $\Psi_I(t)$, or $\Psi_C(t)$, defined in Eq. (3.7) through Eq. (3.10), is a solution of Eqs. (3.1) and (3.2) on condition that $\Psi(t)$ is a solution, then we

speak of reversibility, reflectibility, inversibility, or charge-invariance. Actually, we gave, in Part I, a slightly different definition of reversibility, etc. According to this definition, reversibility holds if the transition probability from an arbitrary Θ to an arbitrary Ω, say, is equal to the transition probability from the reversed state of Ω to the reversed state of Θ. We shall presently see that the definition given above automatically entails this last definition.

We now introduce four time-independent, unitary operators which are useful tools to prove the four invariances in question. Reversion operator R, mirage operator M, inversion operator I, and charge-conjugation operator C are defined by

$$Q(\mathbf{x}, -t)=\rho_R(R^{-1}Q(\mathbf{x},t)R)^T, \tag{3.11}$$

$$Q(-\mathbf{x}, t)=\rho_M(MQ(\mathbf{x},t)M^{-1}), \tag{3.12}$$

$$Q(-\mathbf{x}, -t)=\rho_I(I^{-1}Q(\mathbf{x},t)I)^T, \tag{3.13}$$

$$Q(\mathbf{x}, t)=\rho_C(CQ(\mathbf{x},t)C^{-1}). \tag{3.14}$$

In the quantized field theory, the field strengths appearing in Q are q-numbers, and the transposition T refers to the Q as a q-number. For instance,

$$(R^{-1}\psi^{\times}O\psi R)^T=R^T\psi^TO^T\psi^{\times T}R^{-1T}, \tag{3.15}$$

where T on O has the meaning of transposition discussed in Sec. 2, D. The left-hand side of Eq. (3.15) should not be equated to the negative of the right-hand side on the ground that ψ and $\psi^{\times}$ anticommute. The anticommutability of ψ and $\psi^{\times}$ is implicitly taken care of by the matrix representations of ψ and $\psi^{\times}$.

It is true that the operation of transposition is not invariant for a unitary transformation. For this reason, one may think that the definition in Eq. (3.11) has no physical meaning. However, this is not the case, for we have not yet determined the transformation rule of R for a unitary transformation. Take a time-independent[13] unitary transformation V, by which Q and Ψ are transformed into

$$Q'=V^{-1}QV, \quad \Psi'=V^{-1}\Psi. \tag{3.16}$$

Then we shall have again

$$Q'(\mathbf{x}, -t)=\rho_R(R'^{-1}Q'(\mathbf{x},t)R')^T, \tag{3.17}$$

in the primed system if we take

$$R'=V^{-1}RV^*, \tag{3.18}$$

where $V^*=\bar{V}^T$. Thus, we assume Eq. (3.18) to be the transformation rule of R for the time-independent unitary transformation V. We have the same rule for I.

We shall now show that R, M, I, and C, if they exist, enable us to build Ψ_R, Ψ_M, Ψ_I, and Ψ_C, defined in Eqs. (3.7)–(3.10), from a given Ψ:

$$\left.\begin{aligned}\Psi_R(-t)&=\Psi^*(t)R, \quad \Psi_M(t)=M\Psi(t),\\ \Psi_I(-t)&=\Psi^*(t)I, \quad \Psi_C(t)=C\Psi(t).\end{aligned}\right\} \tag{3.19}$$

We can, of course, insert arbitrary phase factors in these equations, but, for the moment, we assume them to be included in R, etc. To avoid repetitions, let us prove only that Ψ_R defined by Eq. (3.19) satisfies Eq. (3.7). By the use of Eq. (3.19), the left-hand side of Eq. (3.7) becomes

$$(R^T\Psi^*(t), Q(-t)R^T\Psi^*)=(\Psi^*(t), \bar{R}^TQ(-t)R^T\Psi^*(t)).$$

Because of Eq. (3.11), this is equal to

$$\rho_R(\Psi^*(t), \bar{Q}^*(t)\Psi^*(t))=\rho_R(\Psi(t), \bar{Q}(t)\Psi(t))^*.$$

This is equal to the right-hand side of Eq. (3.7). The proof runs in a similar way for M, I, and C.

It should be noted that if Ψ and Ψ_R undergo the transformation of Eq. (3.16) and R undergoes the transformation of Eq. (3.18), then Eq. (3.19) again holds in the primed system.

It is now clear that reversibility, reflectibility, inversibility, and charge invariance will be established if we can prove that Ψ_R, Ψ_M, Ψ_I, and Ψ_C defined in Eq. (3.19) are solutions of Eqs. (3.1) and (3.2) on condition that Ψ is a solution. We shall now show that this is in fact the case. Take two transformation functions $U(t,0)$ and $U(0, -t)$ which are, according to Eq. (3.2), given by

$$\left.\begin{aligned}dU(t,0)/dt&=-i\mathbf{H}(t)U(t,0),\\ dU(0,-t)/d(-t)&=+iU(0,-t)\mathbf{H}(-t),\\ \text{with}&\\ U(0,0)&=1.\end{aligned}\right\} \tag{3.20}$$

Their transposes are then determined by

$$\left.\begin{aligned}dU^T(t,0)/dt&=-iU^T(t,0)\mathbf{H}^T(t),\\ dU^T(0,-t)/d(-t)&=+i\mathbf{H}^T(-t)U^T(0,-t),\\ \text{with}&\\ U^T(0,0)&=1.\end{aligned}\right\} \tag{3.21}$$

Applying R to Eq. (3.20), we get (on account of the time-independence of R)

$$\left.\begin{aligned}dR^{-1}U(t,0)R/dt&=-i\mathbf{H}^T(-t)R^{-1}U(t,0)R,\\ dR^{-1}U(0,-t)R/d(-t)&=+iR^{-1}U(0,-t)R\mathbf{H}^T(t),\\ \text{with}&\\ R^{-1}U(0,0)R&=1,\end{aligned}\right\} \tag{3.22}$$

because ($\rho_R=+1$ for energy)

$$\mathbf{H}(-t)=(R^{-1}\mathbf{H}(t)R)^T. \tag{3.23}$$

[13] A time-dependent unitary transformation will be described later in this section.

Comparing Eq. (3.22) with Eq. (3.21), we obtain

$$R^{-1}U(t,0)R=U^T(0,-t), \quad R^{-1}U(0,-t)R=U^T(t,0). \tag{3.24}$$

Combining the two equations of (3.24), we get[14] for $U(t,-t)=U(t,0)U(0,-t)$,

$$U^T(t,-t)=R^{-1}U(t,-t)R. \tag{3.25}$$

In a similar way

$$\left.\begin{aligned} U(t,-t)&=MU(t,-t)M^{-1},\\ U^T(t,-t)&=I^{-1}U(t,-t)I,\\ U(t,-t)&=CU(t,-t)C^{-1}. \end{aligned}\right\} \tag{3.26}$$

It should be noted that the two equations of (3.24) are equivalent to each other if R^TR^{-1} commutes with U. We shall discuss this condition in more detail later. Once Eq. (3.25) is established, it is an easy matter to show that if

$$\Psi(t)=U(t,-t)\Psi(-t), \tag{3.27}$$

then

$$\Psi_R(t)=U(t,-t)\Psi_R(-t). \tag{3.28}$$

In fact, because of Eqs. (3.19) and (3.27), we have

$$\Psi_R(t)=\Psi^*(-t)R=\Psi^*(t)U(t,-t)R, \tag{3.29}$$

which, in virtue of Eq. (3.25), equals

$$\Psi^*(t)RU^T(t,-t)=U(t,-t)\Psi_R(-t). \tag{3.30}$$

Q.E.D. The proof runs the same way for the other operators.

It has been shown that if R exists to satisfy Eq. (3.11), reversibility is guaranteed. We shall show in the following sections that such an R in fact exists.

Incidentally, by the use of

$$R^{-1}U_0(t,0)R=U_0{}^T(0,-t), \quad \text{etc.}, \tag{3.31}$$

which is a consequence of

$$\mathbf{H}_0=(R^{-1}\mathbf{H}_0R)^T, \tag{3.32}$$

we can rewrite Eq. (3.11) in a simpler way:

$$Q(\mathbf{x},0)=\rho_R(R^{-1}Q(\mathbf{x},0)R)^T. \tag{3.33}$$

The derivation is similar to our argument regarding Eqs. (3.20)–(3.24).

Another important consequence of Eq. (3.25) is the alternative definition of reversibility used in Part I. Suppose there are given two arbitrary states, Θ and Ω. In one process, we consider the transition probability from Θ at $t=-t_1$, to Ω at $t=+t_1$. In the other process we consider the transition probability from the reversed state Ω_R of Ω at $t=-t_1$ to the reversed state Θ_R of Θ at $t=+t_1$:

$$\Theta_R=\Theta^*R, \quad \Omega_R=\Omega^*R. \tag{3.34}$$

[14] There is a misprint in the corresponding Eq. (4.27) in (R).

The first transition probability is determined by the transition matrix

$$(\Omega, U(t_1,-t_1)\Theta)$$

and the second by

$$(\Theta_R, U(t_1,-t_1)\Omega_R)=(\Theta^*, R^{T-1}U(t_1,-t_1)R^T\Omega^*).$$

On account of Eq. (3.25), we can rewrite this last expression as

$$(\Theta, \bar{U}(t_1,-t_1)\Omega)^*=(\Omega, U(t_1,-t_1)\Theta).$$

Thus,

$$(\Omega, U(t,-t)\Theta)=(\Theta_R, U(t,-t)\Omega_R). \tag{3.35}$$

From this it follows, to use the notation of Part I, that

$$P(\Theta\to\Omega, 2t)=P(\Omega_R\to\Theta_R, 2t). \tag{3.36}$$

In a similar way, we obtain

$$\left.\begin{aligned} (\Omega, U(t,-t)\Theta)&=(\Omega_M, U(t,-t)\Theta_M)\\ &=(\Theta_I, U(t,-t)\Omega_I)\\ &=(\Omega_C, U(t,-t)\Theta_C). \end{aligned}\right\} \tag{3.37}$$

It should now be noted that the four operators defined by Eqs. (3.11)–(3.14) have two types of basic arbitrariness:

$$S\to e^{i\alpha}S, \tag{3.38}$$

and

$$S\to WS, \tag{3.39}$$

where S stands for any one of the four operators, and W is supposed to satisfy

$$WQW^{-1}=Q \tag{3.40}$$

for any known physical quantity Q, as has been explained towards the end of the last section. For our study of symmetry properties of physical laws, however, we need to pick any one S that satisfies the defining equation; the arbitrariness expressed by (3.39) can be relegated to a separate study of operators satisfying Eq. (3.40). We shall see in Sec. 12 that we can assume, without loss of generality, that Ψ under consideration belongs to one or another of the Hilbert subspaces, within each of which the effect of W on Ψ is equivalent to multiplication by a phase factor.

Suppose then that one such R satisfying Eq. (3.11) is obtained and that its phase factor in the sense of (3.38) is fixed arbitrarily. Then Eq. (3.19) will be written more generally as

$$\Psi_R(-t)=e^{i\alpha}\Psi^*(t)R, \tag{3.41}$$

with an arbitrary real α for the reversed state, satisfying Eqs. (3.7) and (3.28). If we take two different values of α in Eq. (3.34), we get an arbitrary phase factor appearing in Eq. (3.35), but Eq. (3.36) is still true. The situation is the same for M, I, and C, in Eq. (3.37).

Now, if $\Psi_R(-t)$ is the reversed state of $\Psi(t)$, then $\Psi(-t)$ must be the reversed state of $\Psi_R(t)$. Therefore, we shall have

$$\Psi(-t)=e^{i\beta}\Psi_R{}^*(t)R. \tag{3.42}$$

Combining Eqs. (3.41) and (3.42), we obtain

$$\Psi(t)=e^{i(\beta-\alpha)}R^TR^{-1}\Psi(t) \tag{3.43}$$

for any Ψ. Equation (3.43) represents the effect of a "double reversion."

Equation (3.43) shows that R^TR^{-1} is either a c-number, $e^{i(\alpha-\beta)}$, or a q-number for which Ψ is an eigenfunction corresponding to eigenvalue, $e^{i(\alpha-\beta)}$. Such a q-number must obviously be of the type as seen in Eq. (3.40), exemplified by Δ in Eq. (2.80), since otherwise Ψ and $R^TR^{-1}\Psi$ would be physically distinguishable from each other.

Whatever R^TR^{-1} may be, we cannot exclude the possible existence of self-reversed processes in Nature. The simplest example is the vacuum state. For such a state α and β must be the same, and we obtain

$$\Psi_{\text{self}R}(t)=R^TR^{-1}\Psi_{\text{self}R}(t). \tag{3.44}$$

If R^TR^{-1} is a c-number, then (3.44) gives

$$R^TR^{-1}=1 \quad \text{or} \quad R^T=R. \tag{3.45}$$

If R^TR^{-1} is, for instance, a multiple of Δ, i.e., if $R^TR^{-1}=a\Delta$, then Eq. (3.44) shows the existence of an eigenstate of R^TR^{-1} for which the eigenvalue is $+1$. Since Δ has eigenvalues ±1, a must be ±1. In particular, since Δ operated on the vacuum-state Ψ_{vac} must give $+1$, due to the definition in Eq. (2.80):

$$\Delta\Psi_{\text{vac}}=\Psi_{\text{vac}} \tag{3.46}$$

and since Ψ_{vac} is a self-reversed state, we have to choose $a=1$, or

$$R^TR^{-1}=\Delta. \tag{3.47}$$

It should be noted that the transformation in Eq. (3.18) leaves the condition in Eq. (3.45) unchanged. It also leaves the condition in Eq. (3.47) unchanged provided V commutes with Δ. This commutability may be assumed as quite a general rule since Δ must be unobservable before as well as after the transformation. The transformation in (3.38) can be considered as an especially simple case of Eq. (3.18). The transformation of (3.39) transforms R^TR^{-1} as follows:

$$R^TR^{-1}\rightarrow R^TR^{-1}(RW^TR^{-1}W^{-1}) \tag{3.48}$$

i.e., if $R=WRW^*$, then R^TR^{-1} remains unchanged.

We shall see later that for the spinor fields we actually get Eq. (3.47) and not Eq. (3.45), at least insofar as we remain in the framework of the accepted theory of quantized spinor fields. No transformation W is discovered that makes $RW^TR^{-1}W^{-1}=\Delta$, so that the transformation in (3.48) would result in Eq. (3.45). For the boson fields, we obtain Eq. (3.45) in a quite natural fashion.

A by-product of this consideration is that Eq. (3.44) is satisfied only if $\Delta=+1$, i.e., the self-reversed states must involve even numbers of spinor particles. It should, however, be kept in mind that this conclusion is derived in a purely field theoretical point of view, i.e., the entire physical system is inclusively represented by Ψ.

As for the operators M and C, if we perform a double-mirage and a double-charge-conjugation, in a similar way to Eqs. (3.41) and (3.42), we obtain

$$\Psi(t)=e^{i(\alpha+\beta)}M^2\Psi(t), \quad \Psi(t)=e^{i(\alpha+\beta)}C^2\Psi. \tag{3.49}$$

This shows that M^2 and C^2 must be a c-number or a W, in the sense of Eq. (3.40). If the former is the case, we have to put $M^2=a$, and $C^2=b$, where a and b are some constants of absolute value unity. But this a and b can be transformed easily into unity by the transformation in (3.38), which does not change anything essential except that the eigenvalues are renamed. Therefore we can, in this case, write

$$M^2=1, \quad C^2=1, \tag{3.50}$$

i.e.,

$$M=M^{-1}=\bar{M}, \quad C=C^{-1}=\bar{C}. \tag{3.51}$$

In contrast to the case of R^TR^{-1}, the conditions of Eq. (3.50) on M^2 and C^2 determine the numerical factors of M and C. We shall see that we can find in a quite natural way M and C which satisfy Eq. (3.50) in both spinor and boson cases. However, it should be noted that the transformation in (3.39) leads to

$$M^2\rightarrow(WMWM^{-1})M^2, \tag{3.52}$$

$$C^2\rightarrow(WCWC^{-1})C^2, \tag{3.53}$$

the right sides of which are not necessarily c-numbers even if the original M and C obey Eq. (3.50). As a matter of fact, we can easily find a W which commutes with M and C and whose square is not a c-number. Under these conditions, the right-hand sides of (3.52) and (3.53) become W^2 which is again an operator of the type of Eq. (3.40). Actually, if we use the general expression of M satisfying Eq. (3.12), we are led to the expression of M^2 which is not a c-number, and this fact was used to discover a new W.[15] However such M can be brought back to an operator satisfying Eq. (3.50) by a transformation of the type of (3.39). Therefore, we can always assume Eq. (3.50) and study the W's of Eq. (3.40) as a separate problem.

Coming back to Eq. (3.49), if we take as Ψ a self-miraged state or a self-charge-conjugate state, we obtain, under the assumption of Eq. (3.50),

$$e^{2i\alpha}=1, \quad e^{i\alpha}=\pm1. \tag{3.54}$$

[15] S. Watanabe, Proc. Internatl. Nuc. Phys. Conf. Glasgow (1954) (to be published).

In other words, an operator satisfying Eq. (3.50) will have eigenvalues $+1$ and -1, corresponding to which there will be two groups of eigenfunctions, dividing self-miraged or self-charge-conjugate states into two groups. This is the origin of the even and odd parity with regard to space-symmetry and charge-interchange.

We will now briefly discuss the combinations of two operators introduced in Eqs. (3.11)–(3.14). As a representative of the various combinations, let us first study the time-reversal considered by some authors,[16] which is actually the combination of the reversion in our sense and the charge-conjugation. According to this point of view, the reversed state Ψ_R' is defined by

$$(\Psi_R'(-t), Q(\mathbf{x}, -t)\Psi_R'(-t)) = \rho_R\rho_C(\Psi(t), Q(\mathbf{x},t)\Psi(t)), \quad (3.55)$$

instead of Eq. (3.7). Such a state can be considered either as the reversed state in our sense of the charge-conjugate state or the charge-conjugate state of the reversed state in our sense. Accordingly, the operator R' playing the role of R in this point of view, i.e., the operator such that $\Psi_R'(-t)=\Psi^*(t)R'$, will be given by either

$$R'=C^{-1}R \quad \text{or} \quad R'=RC^T. \quad (3.56)$$

Since these two operators must physically have the same effect, we have to write

$$C^{-1}R=WRC^T, \quad (3.57)$$

where W may be a c-number or an operator of the type in Eq. (3.40). If we adopt $R'=C^{-1}R$ and $C^2=1$, the double time-reversal in this modified point of view will be represented by

$$R'^TR'^{-1}=R^TR^{-1}W^{-1}, \quad (3.58)$$

where W is the one used in Eq. (3.57).

As a matter of fact, exploiting the arbitrariness still disposable within the conditions of Eqs. (3.47) and (3.50), we can adjust C and R in such a way that W in Eq. (3.57) becomes an operator with a wide range of arbitrariness, including $W=$ c-number as well as $W=\Delta$. As a result, R', in contrast to R, can satisfy $R'^TR'^{-1}=\Delta$ as well as $R'^TR'^{-1}=1$, among other possibilities.[17]

Another important combination is one of reversion operator and mirage operator, which should result in the total inversion operator I:

$$I=M^{-1}R \quad \text{or} \quad I=RM^T. \quad (3.59)$$

Since R' and I can thus be obtained from the basic three operators, R, C, and M, we shall not discuss R' and I any further in this paper. If reversibility and charge-invariance hold, then the invariance for the modified time-reversal will hold. If reversibility and space-invariance hold, then inversibility will hold.

Before closing this section, let us briefly consider the effect of a time-dependent unitary transformation R:

$$Q'(t)=V^{-1}(t)Q(t)V(t), \quad \Psi'(t)=V^{-1}(t)\Psi(t). \quad (3.60)$$

We can re-establish Eqs. (3.11) and (3.19) in the primed system by taking

$$R'=V^{-1}(t)RV^{T-1}(-t). \quad (3.61)$$

We can pass from the interaction picture to the Schrödinger picture by putting

$$V^{-1}(t)=U_0(t,0). \quad (3.62)$$

The reversion operator in the Schrödinger picture then becomes

$$R'=U_0(t,0)RU_0{}^{T-1}(0,-t)=R, \quad (3.63)$$

on account of Eq. (3.31). This shows that we can use the same R in the Schrödinger picture.

In a similar way, we can show by the use of Eq. (3.24) that the reversion operator in the Heisenberg picture is the same as in the interaction picture. By the same token, M, C, and I remain unchanged in all the three pictures.

4. PASSAGE FROM C-NUMBER THEORY TO Q-NUMBER THEORY

Before discussing the symmetry properties of physical quantities in the q-number theory, it may be well to survey the situation in the c-number theory and to see in what respects the c-number theory encounters difficulties. As is well known, the c-number theory is analogous to the Heisenberg picture in the sense that the time-development is attributed solely to the time-dependence of the physical quantities.

The problem in the c-number theory lies in the following situation. The field variables u (which may be tensorial or spinorial) are supposed to have certain transformation properties. The physical quantities Q, such as spin-density, electric current, etc., are expressed in terms of field variables u. By the transformation properties attributed to the u's, we can therefore determine the sign function ρ' (for reversion, mirage, etc.) of each Q. On the other hand, due to the physical consideration given in Part I, each Q has a definite ρ. The problem is then to see whether the ρ' determined by the transformation properties of the field variables u

[16] Called standpoint (I) in paper (R). See, for instance, J. Schwinger, Phys. Rev. **82**, 914 (1951). Our standpoint in this present paper, which was labeled (II) in paper (R), is in accordance with the standpoint adopted originally by Wigner and the author's older papers. See. E. P. Wigner, Göttinger Nachr. 546 (1932); S. Watanabe, *Le deuxième Théorème de la Thermodynamique et la Mécanique Ondulatoire* (Hermann et Cie, Paris, 1935); S. Watanabe, Sci. Pap. Inst. Phys. Chem. Research (Tokyo) **31**, 109 (1937). Dr. R. H. Good showed, in a private communication, that this standpoint applied to the Dirac theory of electrons can be smoothly connected through Pauli's 2-component theory to the time-reversal of the Schrödinger theory of electrons.

[17] See discussions connected with Eqs. (6.5) and (6.19) in paper (R).

actually coincide with the ρ determined by physical requirements. We shall presently see that this is not always the case.

In reference to Eq. (2.63) of Sec. 2.B, let us consider $\mathfrak{S}$ as representing reversion $\mathfrak{R}$, mirage $\mathfrak{M}$ or inversion $\mathfrak{I}$. In these cases, we have: $\mathfrak{S}^{-1}=\mathfrak{S}$, or $\mathfrak{S}^2=1$. Of course, for spinors the effect of $\mathfrak{S}$ has an ambiguity of sign; as a result, $\mathfrak{S}^2=\pm 1$. But, the physical quantities being of even order in spinors, this ambiguity does not affect our argument and we can take $\mathfrak{S}^2=1$ without loss of generality. For the field variable u, Eq. (2.63) becomes

$$u'(x')=\mathfrak{S}u(\mathfrak{S}^{-1}x'). \tag{4.1}$$

This can be interpreted as follows: Let $u(x)$ and $u'(x)$ represent the original process and its transformed (i.e., reversed, etc.) process. Then $u(x)$ and $u'(x)$ are connected by Eq. (4.1).

Now each physical quantity $Q(x)$ has an expression in terms of the $u(x)$ and some operator which may also depend on x. Therefore we can write

$$Q(x)=Q[u(x);x]. \tag{4.2}$$

For the transformed process, we have to take u' at $\mathfrak{S}x$ and transform the explicit x in Eq. (4.2) into $\mathfrak{S}x$. Thus we have to compare with Eq. (4.2) the following quantity:

$$Q[u'(\mathfrak{S}x);\mathfrak{S}x],$$

which, in virtue of Eq. (4.1), becomes

$$Q[\mathfrak{S}u(x);\mathfrak{S}x].$$

The operation $\mathfrak{S}$ has been shifted from the argument x to the tensorial or spinorial function u. The ρ' which was in question in the foregoing must then be given by

$$\left.\begin{aligned} &Q[\mathfrak{S}u(x);\mathfrak{S}x]=\rho' Q[u(x);x],\\ \text{or}\quad &Q[\mathfrak{S}u(\mathfrak{S}^{-1}x),x]=\rho' Q[u(\mathfrak{S}^{-1}x);\mathfrak{S}^{-1}x].\end{aligned}\right\} \tag{4.3}$$

The second equation is obtained from the first just renaming $\mathfrak{S}x$ as x. The first one shows that ρ' can be determined simply considering the tensorial or spinorial transformation of u and the transformation of the coordinates explicitly involved in Q.

We shall soon see that the ρ' thus determined is not necessarily the same as the ρ given in the tables of Part I. The physical quantities can be classified into "mechanical" and "electromagnetic." In the field theory, mechanical quantities are primarily derived from the "free" Lagrangean density, by the well-known procedure. For instance, if the Lagrangean density is a regular scalar, the energy-momentum tensor $T^{\mu\nu}$ and the angular momentum tensor $M^{\mu\nu\kappa}$ are regular tensors according to this procedure. We have shown in Part I that the energy momentum tensor should belong to the regular kind. As to the angular momentum tensor, we notice that the complementary vector, in the sense of (I.2.12): $M_\mu=\frac{1}{6}\epsilon_{\mu\nu\kappa\lambda}M^{\nu\kappa\lambda}$ becomes a first kind pseudovector. We have explained in Part I, Sec. 3, that the angular momentum (including spin), if represented as a vector, should behave like a first kind pseudovector. This shows that if the Lagrangean density is a regular scalar, all the rest of the mechanical quantities will become what they should be. In a similar way, if the current-density appearing in the interaction Lagrangean density is a second kind pseudovector, all the rest of electromagnetic quantities will behave as they should. For this reason, we shall limit our discussion to the free Lagrangean and the current-density. This will be the case also in our q-number discussion. If it is desired, one can check the results for individual physical quantities, but we shall ignore such a discussion in this paper. See, however, (*R*) and Sec. 9 of this paper.

Let us first consider the free Lagrangean of tensorial fields v, i.e., pion-field and electromagnetic field. This term depends on v through a combination of the type $\bar{v}v$, or $\bar{v}_\mu v^\mu$, and it contains an operator which is a regular scalar, such as $(\partial/\partial x_\nu)(\partial/\partial x^\nu)$. Now, whatever the kind of the field v may be, the combination of the type $\bar{v}v$ or $\bar{v}_\mu v^\mu$ is a regular scalar on account of the product rule, Part I, Sec. 2. Therefore, the free Lagrangean of tensorial fields is a regular scalar. The current-density of a tensorial field is typified by

$$s^\mu(x)=ie[(\partial\bar{v}(x)/\partial x_\mu)v(x)-\bar{v}(x)(\partial v(x)/\partial x_\mu)], \tag{4.4}$$

which refers to a scalar or pseudoscalar pion field. Since $\bar{v}v$ is a regular scalar, quantity (4.4) is a regular vector. This is, however, not the desired transformation of the current-density, which should be a second kind pseudovector. In other words ρ_M' is all right, but ρ_R' is wrong. This is also true for the current due to a vector or pseudovector pion field.

For a spinor field, the free Lagrangean has the form:

$$\mathfrak{L}(x)=-\tfrac{1}{2}\psi^\times(x)[E_\mu(\partial/\partial x_\mu)-E_\mu(\overleftarrow{\partial}/\partial x_\mu)+2miE_5]\psi(x). \tag{4.5}$$

Now, since $\psi^\times E_\mu\psi$ and $\psi^\times E_5\psi$ belong to the second kind, Eq. (2.43), and $\partial/\partial x_\mu$ to the regular kind, $\mathfrak{L}(x)$ (4.5) is a second kind scalar. In other words, $\rho_M'=\rho_M$ but $\rho_R'=-\rho_R$, in the c-number theory. On the other hand, the current density due to a charged spinor field:

$$s^\mu(x)=ei\psi^\times(x)E^\mu\psi(x) \tag{4.6}$$

is already a second kind pseudovector, (2.43).

As a conclusion, we can say the following in the c-number theory. As far as mirage is concerned, ρ_M' is exactly what it should be, $\rho_M'=\rho_M$, for all the quantities. For reversion, the mechanical quantities of tensorial fields and the electromagnetic quantities of spinorial fields have the right signs, $\rho_R'=\rho_R$, but the electrical quantities of tensorial fields and the mechanical quantities of spinorial fields have the wrong signs, $\rho_R'=-\rho_R$.

Regarding the charge-conjugation, we can see from

Eq. (4.4) that the interchange of v and $\bar{v}$ will result in a reversal of sign of the current. Thus, we can define the c-number transformation for charge conjugation by

$$v\rightarrow e^{i\alpha}\bar{v}, \quad \bar{v}\rightarrow e^{-i\alpha}v. \tag{4.7}$$

It should be noted already at this state that by (4.7) a representative term in the free Lagrangean will be transformed like

$$(\partial\bar{v}/\partial x_\mu)(\partial v/\partial x^\mu)\rightarrow(\partial v/\partial x_\mu)(\partial\bar{v}/\partial x^\mu),$$

showing that, if we adopt the Fermi statistics in q-number theory, the transformation of Eq. (4.7) will result in an undesired sign-change of energy.

For the charged spinor field it is suggested by Eq. (4.7) that ψ may probably be transformed to a multiple of $\psi^\times$. Since we have seen that ψ and $\psi^\times K$ have the same transformation rule for the congruent group, it would be a natural choice to take

$$\psi\rightarrow e^{i\alpha}\psi^\times K, \quad \psi^\times\rightarrow e^{-i\alpha}\psi K^{-1}, \tag{4.8}$$

as the c-number transformation rule for charge conjugation. However (4.8) would transform Eq. (4.6) into

$$-ei\psi K^{-1}E^\mu K\psi^\times = ei\psi E^{\mu T}\psi^\times \tag{4.9}$$

on account of Eq. (2.19). Equation (4.9) is, in the c-number theory, equal to Eq. (4.6). Thus Eq. (4.9) does not change the sign of the current-vector. However, if we adopt the Fermi statistics in the q-number theory, Eq. (4.9) will become the negative of Eq. (4.6), "except for a c-number additional term." We shall later see that we can drop this last restrictive clause by adopting the so-called Heisenberg prescription.

Now, in anticipation, we shall sketch what will be done in the q-number theory to remedy what was wrong in the c-number theory and to retain what was right in the c-number theory. Take, for instance, Eq. (3.12), where the matrix M is supposed to operate on the field variables. The left-hand side of Eq. (3.12) means, to use the notation of Eq. (4.2),

$$Q(\mathfrak{M}x)=Q[u(\mathfrak{M}x);\mathfrak{M}x],$$

and the right-hand side means

$$Q[Mu(x)M^{-1};x],$$

i.e., Eq. (3.12) is equivalent to

$$Q[Mu(x)M^{-1};x]=\rho_M Q[u(\mathfrak{M}x);\mathfrak{M}x]. \tag{4.10}$$

Since $\rho_M'=\rho_M$, the second equation of Eq. (4.3) becomes

$$Q[\mathfrak{M}u(\mathfrak{M}x);x]=\rho_M Q[u(\mathfrak{M}x),\mathfrak{M}x], \tag{4.11}$$

having the same right side as Eq. (4.10). Therefore all we need do in the q-number theory is to put

$$Mu(x)M^{-1}=\mathfrak{M}u(\mathfrak{M}x). \tag{4.12}$$

Thus, for instance, since we know that the effect of $\mathfrak{M}$ on a spinor is essentially multiplication by E_{123}, we get, from Eq. (4.12),

$$M\psi(\mathbf{x},t)M^{-1}=e^{i\alpha}E_{123}\psi(-\mathbf{x},t). \tag{4.13}$$

If we want to be more general, we could also put

$$M\psi(\mathbf{x},t)M^{-1}=e^{i\alpha}E_{123}W\psi(-\mathbf{x},t).$$

However, this is nothing but the combination of Eq. (4.13) with a transformation of the type

$$W'\psi W'^{-1}=W\psi, \quad W'QW'^{-1}=Q, \tag{4.14}$$

where W' is again an operator of the type in Eq. (3.40), which can be studied separately from space-symmetry.

As for reversion, we have to note that the right side of Eq. (3.11) involves the transposition. As a result, the effect of R on u must contain the transposition ($u\rightarrow u^T$) besides the transformation that could be inferred from a comparison with the c-number theory. First, regarding the tensorial field, ρ_R' is wrong only for the electromagnetic quantities. We can remedy this by assuming that the effect of R is the combination of the c-number transformation for $\mathfrak{R}$ and the c-number transformation for $\mathfrak{C}$ as in Eq. (4.7):

$$R^{-1}u(x)R=\mathfrak{R}\mathfrak{C}u^T(\mathfrak{R}x). \tag{4.15}$$

As $\mathfrak{C}$ does not change the sign of mechanical quantities, the correct sign ρ_R' of the free Lagrangean will be retained by Eq. (4.15). Moreover, since $\mathfrak{C}$ in (4.7) interchanges once v and $\bar{v}$ and, since Eq. (3.11) also interchanges v and $\bar{v}$, as can be seen from Eq. (3.15), the transformation in Eq. (4.15) will serve the purpose without any specific assumption as to the statistics.

Suppose we apply the same procedure as that of Eq. (4.15) to the spinor field, *viz.*, let us assume that the effect of R is essentially the combination of the c-number transformation of ψ for reversion and the charge-conjugation of (4.8). Then, on account of the c-number reversion, $\mathfrak{L}(x)$ changes the sign, but the effect of (4.8) will be such that $(\partial/\partial x_\mu)$ is interchanged to $-(\partial/\partial x_\mu)$. Thus, it will change the sign of the first two terms of Eq. (4.5). Since $K^{-1}E_5K=E_5^T$, while $K^{-1}E_\mu K=-E_\mu^T$, the effect of (4.8) will be such that $\psi^\times E_5\psi$ changes the sign, in contrast to Eq. (4.9). Thus, such a combination will correct the sign ρ_R' of Eq. (4.5). On the other hand, (4.8) does not change the sign of Eq. (4.6); therefore, the right sign of the current will be retained. In this argument, it is understood that, since Eq. (3.11) involves the interchange of the factors ψ and $\psi^\times$ and (4.8) also interchanges these factors, we need not invoke a specific assumption regarding the statistics.[18] This shows that the desired effect of R

[18] For this reason, we could adopt the transformations given in Eqs. (4.15) and (4.16) also in the c-number theory, though not in (4.8). The terminology "c-number transformation" in this paper means the one which directly follows from the mathemati-

would be

$$R^{-1}\psi(t)R=e^{i\alpha}E_0K\psi^{\times T}(-t), \tag{4.16}$$

where E_0 stands for the c-number transformation for $\mathfrak{R}$.

As regards charge-conjugation, we have already seen that (4.7) and (4.8) give the correct signs provided we adopt Bose statistics for tensorial fields and Fermi statistics for spinorial fields. Thus, we should take

$$CuC^{-1}=\mathfrak{C}u, \tag{4.17}$$

where $\mathfrak{C}$ is (4.7) or (4.8).

The pion nucleon interaction plays a unique role in determining the kind of pion field. Tensorial fields in general appear in a quadratic or bilinear form in the free Lagrangean. Their sign in no way affects the mechanical behavior of their fields. However, as has been seen in Part I, there seems to be a sound physical reason to believe that the electromagnetic field is "observable," in the sense that its sign-change actually changes the physical situation. Thus we could meaningfully assign the second kind to the electromagnetic quantities. This determination, in essence, has been done through their interaction with the charged field, by assuming that, in a reversed state, a particle of the same charge is performing a kinematically reversed motion.

In contrast to this, there is room for doubt as to whether there is any physical meaning in assigning a kind to the pion field. If this assignment is to be done, it must be possible only through the pion-nucleon interaction, which is the only term where the mesic field strength is standing in the first power. Indeed, we could determine the interaction type by experiments, and require the regular invariance of the interaction Lagrangean. The sign-change of the source involved in this interaction is not necessarily uniquely determined. For the neutral pion, the source is represented by a quantum jump from a proton state to another proton state, or from a neutron state to another neutron state. Therefore, the source has a definite sign-change for $\mathfrak{M}$ and $\mathfrak{R}$ for the neutral meson. For a charged pion field, the source represents a transition from a proton state to a neutron state, or vice versa. Therefore the sign-change of the source depends on the *relative* phase-change of proton spinor and neutron spinor for $\mathfrak{M}$ and $\mathfrak{R}$. So far there is no reason to believe that there should be any physical meaning in such a relative phase-change of proton and neutron. Therefore, in conclusion, we could say something regarding the neutral pion field, but we cannot say anything definite regarding the changed pion field. See Sec. 10 for more details.

cally assigned transformations of the field variables, and the "q-number transformation" means the one consistently corrected according to the consideration of this section. Thanks are due Dr. R. H. Good who has pointed out this rather misleading use of terminology.

In passing, it should be noted that we have to include in Eqs. (4.15) and (4.17) the arbitrary phase factor $\exp(i\alpha)$ of (4.7) and (4.8). Even if we obtain Eqs. (4.15) and (4.17) with a fixed phase factor in a certain gauge reference, an arbitrary phase factor $\exp(i\alpha)$ will reappear by an arbitrary gauge transformation, for these transformations connect a field variable to its Hermitian conjugate. The situation is different for M, since a gauge transformation will leave Eq. (4.12) or (4.13) invariant. This situation is connected with the fact that the condition, $M^2=$c-number, actually determines the phase factor involved in Eq. (4.12), e.g., $\exp(i\alpha)$ in Eq. (4.13), while the conditions on C^2 and R^TR^{-1} have no bearings on the phase factors in (4.7) and (4.8). This situation will become more clear when we discuss the concrete cases.

5. PION FIELDS

We can describe the pion field alternatively by three Hermitian field components $u^{(\rho)}$ $(\rho=1, 2, 3)$ or by one complex component v and a Hermitian component $u^{(3)}$, with

$$u^{(1)}=(v+\bar{v})/\sqrt{2}, \quad u^{(2)}=(v-\bar{v})/\sqrt{2}i. \tag{5.1}$$

However, the gauge transformation (with constant phase),

$$v\rightarrow e^{i\varphi}v, \quad \bar{v}\rightarrow e^{-i\varphi}\bar{v}, \tag{5.2}$$

is equivalent to

$$\begin{aligned} u^{(1)}&\rightarrow\cos\varphi u^{(1)}-\sin\varphi u^{(2)},\\ u^{(2)}&\rightarrow\sin\varphi u^{(1)}+\cos\varphi u^{(2)} \end{aligned} \tag{5.3}$$

resulting in mixing of $u^{(1)}$ and $u^{(2)}$. There is no observational ground for separating the charged pion-fields into $u^{(1)}$-field and $u^{(2)}$-field. The effect of a transformation $\mathfrak{S}$, expressed in terms of independents $u^{(1)}$ and $u^{(2)}$ will then involve the danger of losing from sight the physically pertinent general features.

We shall first discuss a scalar u (of unspecified kind) and later briefly survey the case of a vector u (of unspecified kind). The free Lagrangean density and the Lagrangean density of electromagnetic interaction are, in the case of a scalar u,

$$\begin{aligned} \mathfrak{L}_u=&-(\partial\bar{v}/\partial x_\mu)(\partial v/\partial x^\mu)-\kappa^2\bar{v}v\\ &-\tfrac{1}{2}(\partial u^{(3)}/\partial x_\mu)(\partial u^{(3)}/\partial x^\mu)-\tfrac{1}{2}\kappa^2u^{(3)}u^{(3)}, \end{aligned} \tag{5.4}$$

$$\mathfrak{L}_{Au}=A_\mu s^\mu-A_\mu A^\mu s, \tag{5.5}$$

where s^μ is given by Eq. (4.4) and s by

$$s=e^2\bar{v}v. \tag{5.6}$$

The pion-nucleon interaction will be discussed separately in Sec. 10.

The defining equations of R, M, and C are Eqs. (3.11), (3.12), and (3.14) applied to $\mathfrak{L}_u$, s^μ and s. Since $\mathfrak{L}_u$ and s must belong to the regular kind and s^μ to the

second kind, we should have

$$\left.\begin{aligned} &\rho_R=\rho_M=\rho_C=1 && \text{for } \mathfrak{L}_u,\\ -\rho_R=-\rho_M=-\rho_C=1 &\quad \text{for } s^\alpha, && (\alpha=1,2,3)\\ \rho_R=\rho_M=-\rho_C=1 &\quad \text{for } s^0,\\ \rho_R=\rho_M=\rho_C=1 &\quad \text{for } s. \end{aligned}\right\} \quad (5.7)$$

From (4.7) and Eq. (4.17), we get

$$CvC^{-1}=e^{i\alpha}\bar{v}, \quad C\bar{v}C^{-1}=e^{-i\alpha}v, \quad Cu^{(3)}C^{-1}=\pm u^{(3)}. \quad (5.8)$$

It is obvious that Eq. (5.8) satisfies the requirements in Eq. (5.7) for $\mathfrak{L}_u$ only if v obeys the Bose statistics. As for s^μ, if it is written in the form given in Eq. (4.4), the assumption regarding statistics is necessary in order to satisfy Eq. (5.7). However, the c-number theory can just as well lead to the expression of s^μ:

$$s^\mu=ie[(\partial\bar{v}/\partial x_u)v-(\partial v/\partial x_u)\bar{v}]. \quad (5.9)$$

If we use this expression for s^μ, the transformation in Eq. (5.8) satisfies the requirement in Eq. (5.7) for s^μ without an assumption regarding statistics. It is to be noted that Eq. (5.8) is compatible with the supposed properties of C: $C^2=1$, $\tilde{C}C=1$ [see Eq. (3.51)]. The transformation in Eq. (5.8) also leaves the usual commutation rules unchanged.

Since the interchange of a factor in v and a factor in $\bar{v}$ results, because of the D-function in the commutation relation, in an additional c-number term, it is desirable to replace an expression of Q by

$$Q\rightarrow\tfrac{1}{2}[Q+\rho_C CQC^{-1}], \quad (5.10)$$

or at least to require that the expression of Q must be such that it remains invariant by the transformation in (5.10). Then we no longer need be concerned about the additional c-number term. If the quantities are written according to (5.10), charge-invariance is automatically satisfied. If we apply (5.10), to $\mathfrak{L}_u$, then the expression will become such that it will vanish for the Fermi statistics.

A similar prescription to (5.10) was first introduced by Heisenberg in relation to positron theory to avoid the zero-point charge, and (5.10) has the same effect for the v-field.[19] For instance, the expression of Eq. (4.4) for the current does not obey the Heisenberg prescription, but Eq. (5.9) does. It is well known that Eq. (5.9) leads to the vanishing zero-point charge. It is easily understandable that if the theory is exactly charge-invariant, then no zero-point charge should appear. For, if there would be any zero-point charge, it should as well be positive as negative.

Next, following Eq. (4.15), we can write down the relations that R should satisfy:

$$\left.\begin{aligned} (R^{-1}v(t)R)^T&=e^{i\beta}\bar{v}(-t),\\ (R^{-1}\bar{v}(t)R)^T&=e^{-i\beta}v(-t),\\ (R^{-1}u^{(3)}(t)R)^T&=\pm u^{(3)}(-t). \end{aligned}\right\} \quad (5.11)$$

This automatically satisfies Eq. (5.7) for $\mathfrak{L}_u$. If s^μ is expressed as Eq. (4.4), then Eq. (5.11) automatically satisfies Eq. (5.7) for s^μ. If it is written as Eq. (5.9), we need an assumption regarding statistics to satisfy Eq. (5.7) for s^μ. Since the c-number theory provides no reason for preference between Eqs. (4.4) and (5.9), we cannot claim that reversibility has the power to determine the statistics.

It should be noted that Eq. (5.11) is compatible with the anticipated properties of R: $R^T=R$ [Eq. (3.45)], $\tilde{R}=R^{-1}$. The transformation in Eq. (5.11) also leaves the commutation rule of v and $u^{(3)}$ unchanged.

The combination of C and R, considered in connection with Eq. (3.56), has two expressions:

$$\left.\begin{aligned} C(R^{-1}v(t)R)^TC^{-1}&=e^{-i(\alpha-\beta)}v(-t),\\ (R^{-1}Cv(t)C^{-1}R)^T&=e^{i(\alpha-\beta)}v(-t). \end{aligned}\right\} \quad (5.12)$$

The difference between these two, expressed by the unitary transformation W in Eq. (3.57), is essentially a gauge transformation of v, which will be discussed separately.

Analogously to (5.10), in order to have the "exact" reversibility, it is desirable to require

$$Q(t)\rightarrow\tfrac{1}{2}[Q(t)+\rho_R(R^{-1}Q(-t)R)^T]. \quad (5.13)$$

For instance, the expression *formally* desirable for the current would then be

$$s^\mu=\tfrac{1}{2}ie[(\partial\bar{v}/\partial x_\mu)v-\bar{v}(\partial v/\partial x_\mu) +v(\partial\bar{v}/\partial x_\mu)-(\partial v/\partial x_\mu)\bar{v}]. \quad (5.14)$$

Obviously this does not bring about anything new.

The above argument shows that the requirement of charge invariance determines the statistics-type of the field[20] but the requirement of reversibility does not. It is of some interest to note that we can restate this situation saying that the compatibility of (5.10) (charge-invariance) and (5.13) (reversibility) determines the statistics. For instance, the compatibility of Eqs. (4.4) and (5.9) requires the Bose statistics. We can generalize this statement for other quantities and fields. However, the underlying fact is that if we suitably express the physical quantities in the forms allowed by the c-number theory, reversibility alone can always be satisfied without a specific assumption regarding statistics.

If we translate the transformations in Eqs. (5.8) and

[19] W. Heisenberg, Z. Physik **90**, 209 and **92**, 692 (1934). See also W. Pauli, Revs. Modern Phys. **13**, 203 (1941), in particular, pp. 208, 224.

[20] W. Pauli and F. J. Belinfante, Physica **7**, 177 (1940).

(5.11) into the language of the Hermitian $u^{(\rho)}$, we obtain

$$\left.\begin{aligned} Cu^{(1)}C^{-1}&=\cos\alpha u^{(1)}+\sin\alpha u^{(2)},\\ Cu^{(2)}C^{-1}&=\sin\alpha u^{(1)}-\cos\alpha u^{(2)},\\ (R^{-1}u^{(1)}(t)R)^T&=\cos\beta u^{(1)}(-t)+\sin\beta u^{(2)}(-t),\\ (R^{-1}u^{(2)}(t)R)^T&=\sin\beta u^{(1)}(-t)-\cos\beta u^{(2)}(-t),\end{aligned}\right\}\quad(5.15)$$

showing that it is meaningless to speak of a transformation rule for $u^{(1)}$ alone or $u^{(2)}$ alone. We can write $\mathfrak{L}_u$ in two terms referring separately to $u^{(1)}$ and $u^{(2)}$, but the equivalence of such an expression to Eq. (5.4) requires the Bose assumption.

Finally the mirage operator is given, in accordance with Eq. (4.12), by

$$\left.\begin{aligned} Mv(\mathbf{x})M^{-1}&=e^{i\gamma}v(-\mathbf{x}),\\ M\bar{v}(\mathbf{x})M^{-1}&=e^{-i\gamma}\bar{v}(-\mathbf{x}),\\ Mu^{(3)}(\mathbf{x})M^{-1}&=\pm u^{(3)}(-\mathbf{x}).\end{aligned}\right\}\quad(5.16)$$

These relations are compatible with the relation $M^{-1}=\bar{M}$ and the commutation rules. However, in order to satisfy $M^2=$c-number, we have to adopt γ such that

$$e^{i\gamma}=\pm 1=e^{-i\gamma}.\quad(5.17)$$

This situation makes a contrast to the case of charge-conjugation in which $C^2=$c-number does not determine the phase factor in Eq. (5.8). The double-sign in Eq. (5.17) and the one in the third relation of Eq. (5.16) are independent of each other.

Expressing the u-field by its Fourier expression,

$$v(\mathbf{x},t)=\sum_{\mathbf{k}}(2V\omega)^{-\frac{1}{2}}[g_+(\mathbf{k})\exp(+-)+\bar{g}_-(\mathbf{k})\exp(-+)],\quad(5.18)$$

$$u^{(3)}(\mathbf{x},t)=\sum_{\mathbf{k}}(2V\omega)^{-\frac{1}{2}}[g_3(\mathbf{k})\exp(+-)+\bar{g}_3(\mathbf{k})\exp(-+)],\quad(5.19)$$

with

$$\left.\begin{aligned} \exp(+-)&=\exp(+i\mathbf{k}\cdot\mathbf{x}-i\omega t),\\ \exp(-+)&=\exp(-i\mathbf{k}\cdot\mathbf{x}+i\omega t),\\ \omega&=|(\mathbf{k}^2+\kappa^2)^{\frac{1}{2}}|,\end{aligned}\right\}\quad(5.20)$$

we get from Eqs. (5.8), (5.11), and (5.16)

$$\left.\begin{aligned} Cg_+(\mathbf{k})C^{-1}&=e^{i\alpha}g_-(\mathbf{k}),\\ Cg_-(\mathbf{k})C^{-1}&=e^{-i\alpha}g_+(\mathbf{k}),\\ Cg_3(\mathbf{k})C^{-1}&=\pm g_3(\mathbf{k});\end{aligned}\right\}\quad(5.21)$$

$$\left.\begin{aligned} (R^{-1}g_+(\mathbf{k})R)^T&=e^{i\beta}\bar{g}_+(-\mathbf{k}),\\ (R^{-1}g_-(\mathbf{k})R)^T&=e^{i\beta}\bar{g}_-(-\mathbf{k}),\\ (R^{-1}g_3(\mathbf{k})R)^T&=\pm\bar{g}_3(-\mathbf{k});\end{aligned}\right\}\quad(5.22)$$

$$\left.\begin{aligned} Mg_+(\mathbf{k})M^{-1}&=(\pm)g_+(-\mathbf{k}),\\ Mg_-(\mathbf{k})M^{-1}&=(\pm)g_-(-\mathbf{k}),\\ Mg_3(\mathbf{k})M^{-1}&=(\pm)'g_3(-\mathbf{k}).\end{aligned}\right\}\quad(5.23)$$

It is easy to show that C, R, and M can be expressed in the operator forms as follows[21]:

$$\left.\begin{aligned} C&=(-1)^{\Phi}(\pm 1)^{\Theta},\\ \Phi&=\tfrac{1}{2}\sum_{\mathbf{k}}[N_+(\mathbf{k})+N_-(\mathbf{k})-e^{+i\alpha}\bar{g}_+(\mathbf{k})g_-(\mathbf{k})-e^{-i\alpha}\bar{g}_-(\mathbf{k})g_+(\mathbf{k})],\\ \Theta&=\sum_{\mathbf{k}}N_3(\mathbf{k}),\end{aligned}\right\}\quad(5.24)$$

$$R=e^{\Phi}(-1)^{\Psi}(-1)^{\Theta}$$

with

$$\left.\begin{aligned} \Phi&=\sum_{\pm}\sum_{\mathbf{k}}(\pm i\beta N_{\pm}(\mathbf{k})),\\ \Psi&=\tfrac{1}{2}\sum_{\pm}\sum_{\mathbf{k}}{}'[N_{\pm}(\mathbf{k})+N_{\pm}(-\mathbf{k})-\bar{g}_{\pm}(\mathbf{k})g_{\pm}(-\mathbf{k})-\bar{g}_{\pm}(-\mathbf{k})g_{\pm}(\mathbf{k})],\\ \Theta&=\tfrac{1}{2}\sum_{\mathbf{k}}{}'[N_3(\mathbf{k})+N_3(-\mathbf{k})-(\pm)\bar{g}_3(\mathbf{k})g_3(-\mathbf{k})-(\pm)\bar{g}_3(-\mathbf{k})g_3(\mathbf{k})];\end{aligned}\right\}\quad(5.25)$$

$$M=(-1)^{\Phi}(-1)^{\Theta},$$

with

$$\left.\begin{aligned} \Phi&=\tfrac{1}{2}\sum_{\mathbf{k}}{}'[N_+(\mathbf{k})+N_+(-\mathbf{k})-(\pm)\bar{g}_+(\mathbf{k})g_+(-\mathbf{k})-(\pm)\bar{g}_+(-\mathbf{k})g_+(\mathbf{k})+N_-(\mathbf{k})+N_-(-\mathbf{k})-(\pm)\bar{g}_-(\mathbf{k})g_-(-\mathbf{k})-(\pm)\bar{g}_-(-\mathbf{k})g_-(\mathbf{k})],\\ \Theta&=\tfrac{1}{2}\sum_{\mathbf{k}}{}'[N_3(\mathbf{k})+N_3(-\mathbf{k})-(\pm)'\bar{g}_3(\mathbf{k})g_3(-\mathbf{k})-(\pm)'\bar{g}_3(-\mathbf{k})g_3(\mathbf{k})].\end{aligned}\right\}\quad(5.26)$$

The arbitrary signs of the entire C and M are so chosen that $C\Psi_{\text{vac}}=M\Psi_{\text{vac}}=\Psi_{\text{vac}}$.

The summation symbol with a prime, $\sum'$, over $\mathbf{k}$ means a summation such that $\mathbf{k}$ and $-\mathbf{k}$ would cover the entire momentum space. For Eq. (5.25), it is assumed that g and $\bar{g}$ are expressed in the usual representation in which they have real matrix elements. In Eq. (5.26) $(\pm)$ and $(\pm)'$ refer to the same double-signs in Eq. (5.23).

The vector pion field (of unspecified kind) can be described by a complex vector $v^{\mu}(\mathbf{x},t)$ and a Hermitian vector $u^{(3)\mu}(\mathbf{x},t)$ with $\mu=1, 2, 3, 0$. The transformation rules corresponding to Eqs. (5.8), (5.11), and (5.16) will become

$$Cv_{\mu}C^{-1}=e^{i\alpha}\bar{v}_{\mu},\quad Cu_{\mu}^{(3)}C^{-1}=\pm u_{\mu}^{(3)},\quad(5.27)$$

$$\left.\begin{aligned} (R^{-1}v_{\mu}(t)R)^T&=e^{i\beta}\bar{v}^{\mu}(-t),\\ (R^{-1}u_{\mu}^{(3)}(t)R)^T&=\pm u^{(3)\mu}(-t),\end{aligned}\right\}\quad(5.28)$$

and

$$\left.\begin{aligned} Mv_{\mu}(\mathbf{x})M^{-1}&=(\pm)v^{\mu}(-\mathbf{x}),\\ Mu_{\mu}^{(3)}(\mathbf{x})M^{-1}&=(\pm)'u^{(3)\mu}(-\mathbf{x}).\end{aligned}\right\}\quad(5.29)$$

[21] The method used here is taken from L. Wolfenstein and D. G. Ravenhall, Phys. Rev. 88, 279 (1952); see, in particular, reference 7, p. 280.

It should be noticed that in Eqs. (5.28) and (5.29) we have covariant components on the left sides and contravariant components on the right sides. This will take care of the vectorial transformations of the field variables for $\mathfrak{R}$ and $\mathfrak{M}$.

The plane wave expansion of the vectorial field needs a little caution. The waves will be decomposed into transversal waves ($r=1, 2$) and longitudinal waves ($r=3$). The transformation rules for $\mathfrak{R}$ and $\mathfrak{M}$ of the annihilation operators are different for the two groups of waves. On one hand this will take care of the fact that the longitudinal polarization is defined by the propagation vector $\mathbf{k}$ which changes its sign for $\mathfrak{R}$ and $\mathfrak{M}$, while, on the other hand, it will take care of the fact that the component $\mu=0$ has different transformation rules in Eqs. (5.28) and (5.29). The relations corresponding to Eqs. (5.21), (5.22), and (5.23) will be as follows:

$$\left.\begin{aligned} Cg_{\pm r}(\mathbf{k})C^{-1}&=e^{\pm i\alpha}g_{\mp r}(\mathbf{k}), \quad r=1,2,3,\\ Cg_{3r}(\mathbf{k})C^{-1}&=\pm g_{3r}(\mathbf{k}), \quad r=1,2,3,\end{aligned}\right\}\tag{5.30}$$

$$\left.\begin{aligned} (R^{-1}g_{\pm r}(\mathbf{k})R)^T&=e^{\pm i\beta}\bar{g}_{\pm r}(-\mathbf{k}), \quad r=1,2,\\ &=-e^{\pm i\beta}\bar{g}_{\pm r}(-\mathbf{k}), \quad r=3.\\ (R^{-1}g_{3r}(\mathbf{k})R)^T&=\pm g_{3r}(-\mathbf{k}), \quad r=1,2,\\ &=\mp g_{3r}(-\mathbf{k}), \quad r=3.\end{aligned}\right\}\tag{5.31}$$

$$\left.\begin{aligned} Mg_{\pm r}(\mathbf{k})M^{-1}&=e^{\pm i\gamma}g_{\pm r}(-\mathbf{k}), \quad r=1,2;\\ &\qquad\qquad \gamma=0 \text{ or } \pi,\\ &=-e^{\pm i\gamma}g_{\pm r}(-\mathbf{k}), \quad r=3,\\ Mg_{3r}(\mathbf{k})M^{-1}&=(\pm)'g_{3r}(-\mathbf{k}), \quad r=1,2,\\ &=-(\pm)'g_{3r}(-\mathbf{k}), \quad r=3.\end{aligned}\right\}\tag{5.32}$$

It is assumed in these expressions that the positive directions of the transversal polarization, $r=1, 2$, are defined to be the same for both $\mathbf{k}$ and $-\mathbf{k}$.

6. ELECTROMAGNETIC FIELD

A. Linear Momentum Representation

In the case of electromagnetic field, the sign-functions ρ_C, ρ_R, and ρ_M of the field variables are well-defined. Therefore, all we need is to show that the C, R, and M defined by these sign-functions do not change the sign of the free Lagrangean and that they leave the Lorentz condition unchanged. The invariance of the interaction Lagrangean will be guaranteed, if the other fields are so adjusted that the electric current generated by these fields becomes a second kind pseudovector.

By the direct application of Eqs. (3.11), (3.12), and (3.14) to the electromagnetic potential, we obtain

$$\left.\begin{aligned} CA_\mu C^{-1}&=-A_\mu,\\ (R^{-1}A_\mu(t)R)^T&=-A^\mu(-t),\\ MA_\mu(\mathbf{x})M^{-1}&=-A^\mu(-\mathbf{x}).\end{aligned}\right\}\tag{6.1}$$

The transformation of a covariant component into the contravariant component takes care of the required transformation of a second kind pseudovector. We see that Eq. (6.1) is compatible with Eqs. (3.45), (3.50), and (3.57) with $W=1$. The transformations in Eq. (6.1) leave the commutation relations of A unchanged.

The transformations in Eq. (6.1) certainly make the free Lagrangean a regular scalar, and transform the Lorentz condition,

$$(\partial A^\mu/\partial x^\mu)\Psi(t)=0,\tag{6.2}$$

into the corresponding equation for the charge-conjugate, reversed, and miraged states.

In this subsection, we shall use the linear momentum representation for simplicity's sake. In the next subsection, we shall use the angular momentum representation, because this representation is "compatible" with the parity operator. In the last subsection, we shall use a mixed representation mainly for the purpose of applications.

If we expand $A^\mu(x)$ by the plane waves:

$$\begin{aligned} A^\mu(\mathbf{x},t)=\sum(2V\omega)^{-\frac{1}{2}}[&g^\mu(\mathbf{k})\exp(+-)\\ &+\bar{g}^\mu(\mathbf{k})\exp(-+)],\\ |\mathbf{k}|=\omega,\quad &\exp(+-)=\exp(+i\mathbf{k}\cdot\mathbf{x}-i\omega t),\end{aligned}\tag{6.3}$$

the defining equations in Eq. (6.1) become

$$\left.\begin{aligned} Cg^\mu(\mathbf{k})C^{-1}&=-g^\mu(\mathbf{k}),\\ (R^{-1}g^\mu(\mathbf{k})R)^T&=-\bar{g}_\mu(-\mathbf{k}),\\ Mg^\mu(\mathbf{k})M^{-1}&=-g_\mu(-\mathbf{k}).\end{aligned}\right\}\tag{6.4}$$

Equation (6.4) shows that in a representation in which g and $\bar{g}$ have real matrix elements, R and M become identical. The explicit expressions of C, R, and M are then (the sign being so determined that $C\Psi_{\text{vac}}=M\Psi_{\text{vac}}=\Psi_{\text{vac}}$)

$$C=\prod_\mu\prod_{\mathbf{k}}(-1)^{N^\mu(\mathbf{k})};\tag{6.5}$$

$$\left.\begin{aligned} R=M&=(-1)^\Phi,\\ \Phi=\tfrac{1}{2}\sum_{\mathbf{k}}{}'\sum_{\alpha=1}^{3}&[N^\alpha(\mathbf{k})+N^\alpha(-\mathbf{k})+\bar{g}^\alpha(\mathbf{k})g^\alpha(-\mathbf{k})\\ &+\bar{g}^\alpha(-\mathbf{k})g^\alpha(\mathbf{k})]+\tfrac{1}{2}\sum_{\mathbf{k}}{}'[N^0(\mathbf{k})+N^0(-\mathbf{k})\\ &-\bar{g}^0(\mathbf{k})g^0(-\mathbf{k})-\bar{g}^0(-\mathbf{k})g^0(\mathbf{k})].\end{aligned}\right\}\tag{6.6}$$

B. Angular Momentum Representation

We only consider a pure radiation field

$$A^0=0,\quad \text{div}\mathbf{A}=0.\tag{6.7}$$

A physical quantity Q derivable from $\mathbf{A}$ can be written in terms of a Hermitian operator O operating on $\mathbf{A}$:

$$Q=-i[(\partial\mathbf{A}/\partial t)\cdot O\mathbf{A}],\tag{6.8}$$

where the brackets mean the scalar product of two

spatial vectors $(\partial\mathbf{A}/\partial t)$ and $O\mathbf{A}$, and O is in general a tensor in the 3-dimensional space. We have to consider energy $\mathfrak{W}$, orbital angular momentum $\mathbf{L}$, spin angular momentum $\mathbf{S}$, and total angular momentum $\mathbf{J}$, which are respectively given by

$$\left.\begin{aligned}&\mathfrak{W}=i\frac{\partial}{\partial t}\mathbf{1};\quad L_x=-i[y(\partial/\partial z)-z(\partial/\partial y)]\mathbf{1},\quad \text{etc.};\\&S_x=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\ \text{etc.};\quad \mathbf{1}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix};\mathbf{J}=\mathbf{L}+\mathbf{S}.\end{aligned}\right\}\tag{6.9}$$

Except for

$$[S_x,S_y]_-=iS_z,\quad [L_x,L_y]_-=iL_z,\quad [J_x,J_y]_-=iJ_z,\ \text{etc.},$$

these quantities commute among themselves, with Eq. (6.7) and with

$$\left(\Delta-\frac{\partial^2}{\partial t^2}\right)\mathbf{A}=0.\tag{6.10}$$

Calling the eigenvalues of $\mathfrak{W}$, $\mathbf{J}^2$, $\mathbf{L}^2$, J_z, respectively, ω, $j(j+1)$, $l(l+1)$, and M, we denote the corresponding eigenfunctions by

$$\mathbf{u}[\omega,j,l,M](x,y,z,t),\tag{6.11}$$

where the brackets indicate their eigenvalues and the parentheses their dependence on space-time. To give an explicit form to $\mathbf{u}$, we introduce the three eigenvectors $\mathbf{e}[\mu]$, $(\mu=-1,\ 0,\ +1)$ of S_z:

$$\left.\begin{aligned}\mathbf{e}[1]&=-(1/\sqrt{2})(\mathbf{e}_x+i\mathbf{e}_y),\quad \mathbf{e}[0]=\mathbf{e}_z,\\\mathbf{e}[-1]&=(1/\sqrt{2})(\mathbf{e}_x-i\mathbf{e}_y),\end{aligned}\right\}\tag{6.12}$$

which satisfy

$$(\mathbf{e}^*[\mu]\cdot\mathbf{e}[\mu'])=\delta(\mu,\mu').\tag{6.13}$$

Next, we introduce the normalized eigenfunctions $Y_l{}^m(\theta,\varphi)$ of L_z and $\mathbf{L}^2$ which satisfy

$$\int Y_l{}^{m*}Y_{l'}{}^{m'}\sin\theta d\theta d\varphi=\delta(m,m')\delta(l,l').\tag{6.14}$$

Now we can write $\mathbf{u}$ with the help of the Clebsch-Gordon coefficients[22]:

$$\mathbf{u}[\omega,j,l,M](x,y,z,t)=b[l,\omega](r,t)\sum_{m,\mu}C_{l1}(j,M;m,\mu)\times Y_l{}^m(\theta,\varphi)\mathbf{e}[\mu],\tag{6.15}$$

where b should be determined by Eq. (6.10) and can be expressed in terms of Bessel functions of half-odd-integer order, $B_{l+\frac{1}{2}}(\omega r)$:

$$b[l,\omega](r,t)=(1/r^{\frac{1}{2}})B_{l+\frac{1}{2}}(\omega r)e^{-i\omega t}\tag{6.16}$$

which can be normalized by

$$\lim_{R\to\infty}(1/R)\int_0^R(1/r^{\frac{1}{2}})B_{l+\frac{1}{2}}(\omega r)(1/r^{\frac{1}{2}})B_{l+\frac{1}{2}}(\omega' r)r^2dr=\delta(\omega,\omega').\tag{6.17}$$

The left-hand side of Eq. (6.17) can be expressed again in terms of Bessel functions,[23] and assuming asymptotic expressions of the Bessel functions for large R, we can satisfy Eq. (6.17). In particular, since we are interested in the solutions in the empty space, we have to take a superposition of incoming and outgoing waves, which is free from singularity at the origin:

$$B_{l+\frac{1}{2}}(\omega r)=(\pi\omega)^{-\frac{1}{2}}J_{l+\frac{1}{2}}(\omega r).\tag{6.18}$$

Using Eq. (6.15), we can expand $\mathbf{A}$ as

$$\mathbf{A}(\mathbf{x},t)=(2R)^{-\frac{1}{2}}\sum_\omega\sum_j\sum_l\sum_M\omega^{-\frac{1}{2}}\times[g(\omega,j,l,M)\mathbf{u}[\omega,j,l,M](\mathbf{x},t)+\text{Hermitian conj.}]\tag{6.19}$$

From the completeness of the $\mathbf{u}$'s, we have the inverse orthogonality:

$$\lim_{R\to\infty}\sum_\omega\sum_j\sum_l\sum_M(1/R)u^{\alpha*}[\omega,j,l,M](\mathbf{x},t)\times u^\beta[\omega,j,l,M](\mathbf{x}',t)=\delta(\mathbf{x}-\mathbf{x}')\delta(\alpha,\beta).\tag{6.20}$$

With the help of Eq. (6.20), we can easily show that the commutation rules:

$$[g(\omega,j,l,M),\ \bar{g}(\omega',j',l',M')]_-=\delta(\omega,\omega')\delta(j,j')\delta(l',l')\delta(M,M'),\ \text{etc.}\tag{6.21}$$

lead to the ordinary commutation rules of $\mathbf{A}$.

Next, in order to determine the effect of R and M on the g's, we first have to determine the sign-change of the $\mathbf{u}$'s for $\mathfrak{R}$ and $\mathfrak{M}$ applied to their argument $(\mathbf{x},t)$. Using the symbol introduced in Eq. (2.73), we get from Eq. (6.15)

$$\left.\begin{aligned}\Omega_{xyz}\mathbf{u}[\omega,j,l,M]&=(-1)^l\mathbf{u}[\omega,j,l,M],\\\Omega_t\mathbf{u}[\omega,j,l,M]&=(-1)^{j+l+M+1}\mathbf{u}^*[\omega,j,l,-M].\end{aligned}\right\}\tag{6.22}$$

The first relation represents the well-known transformation rule of $Y_l{}^m$ while the second relation can be derived by noticing:

$$\left.\begin{aligned}C_{l1}(j,-M;-m,-\mu)&=(-1)^{j+l+1}C_{l1}(j,M,m,\mu),\\Y_l{}^{-m}&=(-1)^mY_l{}^{*m},\quad \mathbf{e}[-\mu]=(-1)^\mu\mathbf{e}^*[\mu],\\b[l,\omega](r,-t)&=b^*[l,\omega](r,t)\end{aligned}\right\}\tag{6.23}$$

[22] See, for instance, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), pp. 789 *ff*.

[23] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1922), Eqs. (8) and (11), pp. 134–135.

Then, substituting Eq. (6.19) in Eq. (6.1), we obtain

$$\left.\begin{aligned} &Cg(\omega,j,l,M)C^{-1}=-g(\omega,j,l,M),\\ &(R^{-1}g(\omega,j,l,M)R)^T=(-1)^{j+l+M}\bar{g}(\omega,j,l,-M),\\ &Mg(\omega,j,l,M)M^{-1}=(-1)^{l+1}g(\omega,j,l,M). \end{aligned}\right\}(6.24)$$

The effect of C in this representation is essentially the same as in Eq. (6.4). The important feature of this representation is that, except for the factor $(-1)^{l+1}$, the effect of M is to transform a g into itself with the same (ω,j,l,M). This is of course a consequence of the fact that $\mathfrak{M}$ commutes with the quantities defining the eigenstates in this representation.

We can satisfy the third relation of Eq. (6.24) by

$$M=\pm\prod_{\substack{l\\ \text{even}}}(-1)^{N(\omega,j,l,M)}, \tag{6.25}$$

where the double-sign is not determined even by the condition $M^2=1$. However, if we want to have

$$M\Psi_{\text{vac}}=+\Psi_{\text{vac}}, \tag{6.26}$$

we have to take the upper sign. Any state in this representation may be expressed by

$$\Psi=\sum_i a_i\bar{g}_1\bar{g}_2\cdots\Psi_{\text{vac}}, \tag{6.27}$$

with the g's used in Eq. (6.19). If the number of g's with $l=$even in each term of Eq. (6.27) is even (odd), the state of Eq. (6.27) is an eigenstate of M corresponding to the eigenvalue $+1(-1)$:

$$M\Psi=\pi_s\Psi,\quad \pi_s=\pm1. \tag{6.28}$$

The value of π_s is the "space-parity" of the state Ψ. The eigenstates in Eq. (6.28) of M are "self-miraged" states in the sense of Eq. (3.54).

In particular, if a single quantum is excited,

$$\Psi=\bar{g}\Psi_{\text{vac}}, \tag{6.29}$$

$(-1)^{l+1}$ gives the parity of the state, where l is the orbital angular momentum of $\bar{g}$. Comparing this result with Eq. (6.22), we see that the parity of the state in Eq. (6.29) is the opposite to the parity of the corresponding $\mathbf{u}$ in the c-number theory. This is, however, in agreement with the usual definition of parity of electromagnetic field, since the parity of $\mathbf{u}$ is the same as the parity of the electric field and opposite to the parity of the magnetic field, and the parity is defined usually by the magnetic field $\mathbf{h}$[24]:

$$\mathbf{h}(\mathbf{x})=\pm\mathbf{h}(-\mathbf{x}). \tag{6.30}$$

The only caution we have to take is that the c-number theoretical definition of parity in Eq. (6.30) should be reinterpreted in the q-number theory through Eq. (6.22) and not through

$$(\Psi,\mathbf{h}(\mathbf{x})\Psi)=\pm(\Psi,\mathbf{h}(-\mathbf{x})\Psi).$$

For, Ψ being a self-miraged state, we have always the positive sign in this equation (see Part I, Tables III and V). If Ψ is an eigenfunction of photon numbers, then the phase of the field is completely undetermined, and its expectation values vanishes any way.

If we introduce a distinction between the "electric" radiation and "magnetic" radiation by

$$\left.\begin{aligned} &\text{"electric"}:\quad j=l+1,\\ &\text{"magnetic"}:\quad j=l, \end{aligned}\right\}(6.31)$$

we have the rule that the parity of an electric radiation is determined by $(-1)^j$, and that of a magnetic radiation by $(-1)^{j+1}$.

The charge-parity can be defined by

$$\left.\begin{aligned} &C=\Pi(-1)^N,\quad C\Psi_{\text{vac}}=\Psi_{\text{vac}},\\ &C\Psi=\pi_c\Psi,\qquad\quad \pi_c=\pm1. \end{aligned}\right\}(6.32)$$

The reversion operator can be written

$$\left.\begin{aligned} &R=(-1)^{\Theta},\\ &\Theta=\tfrac{1}{2}\sum_\omega\sum_j\sum_l\sum_M{}'[N(\omega,j,l,M)+N(\omega,j,l,-M)\\ &\quad-(-1)^{j+l+M}\bar{g}(\omega,j,l,M)g(\omega,j,l,-M)\\ &\quad-(-1)^{j+l+M}\bar{g}(\omega,j,l,-M)g(\omega,j,l,M)]. \end{aligned}\right\}(6.33)$$

C. Mixed Representation

In this representation, linear momentum $\mathbf{p}$ and circular polarization, i.e., spin angular momentum $\mathbf{S}$, in the direction of propagation $\mathbf{p}$ are used. For instance, for the waves propagating in the z-direction, the circular polarization can be expressed by S_z. Obviously, S_z does not commute with p_x and p_y, but if $p_x=0$ and $p_y=0$, we can still use S_z to characterize eigenfunctions. Thus, we can write for such waves

$$\mathbf{A}(x,t)=(2V)^{-\frac{1}{2}}\sum\omega^{-\frac{1}{2}}[g(k_z,\mu)\mathbf{u}[k_z,\mu](\mathbf{x},t)+\text{Hermitian conj.}] \tag{6.34}$$

with

$$\mathbf{u}[k_z,\mu](\mathbf{x},t)=\exp(ik_zZ-i\omega t)\mathbf{e}[\mu],\qquad |k_z|=\omega. \tag{6.35}$$

where $\mathbf{e}[0]$, [see Eq. (6.12)], should be absent on account of transversality as seen in Eq. (6.7). From this follows

$$\left.\begin{aligned} &Cg(k_z,\mu)C^{-1}=-g(k_z,\mu),\\ &(R^{-1}g(k_z,\mu)R)^T=\bar{g}(-k_z,-\mu),\\ &Mg(k_z,\mu)M^{-1}=-g(-k_z,\mu). \end{aligned}\right\}(6.36)$$

For the purpose of a later application, let us also consider the simple reflection $\mathfrak{M}_x$ of the x-axis: $(x,y,z,t)\to(-x,y,z,t)$. The operator M_x corresponding

[24] J. M. Blatt and V. F. Weisskopf, reference 23, p. 585.

to $\mathfrak{M}_x$ should then be defined by

$$\begin{aligned} M_x A_x(x,y,z,t) M_x^{-1} &= -A_x(-x,y,z,t), \\ M_x A_{y,z}(x,y,z,t) M_x^{-1} &= +A_{y,z}(-x,y,z,t). \end{aligned} \tag{6.37}$$

Since $\mathbf{e}[1]$ and $\mathbf{e}[-1]$ differ only in the sign of $\mathbf{e}_x$, [see Eq. (6.12)], we obtain

$$M_x g(k_z,\mu) M_x^{-1} = g(k_z,-\mu). \tag{6.38}$$

One of the advantages of this mixed representation is that we can easily discuss the connection between the spin and the linear polarization of a photon. The operators, in the sense of O in Eq. (6.8), representing the linear polarization are

$$\pi_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \pi_y \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \pi_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6.39}$$

and the corresponding physical quantities Q are

$$\Pi_\alpha = -i \int [(\partial \mathbf{A}/\partial t) \cdot \pi_\alpha \mathbf{A}] dV, \quad (\alpha = x,y,z). \tag{6.40}$$

We can give more explicit expressions to Π_α by the use of Eqs. (6.34) and (6.12). In particular, for a given value of k_z, we get

$$\begin{aligned} \Pi_z(k_z) &= \tfrac{1}{2}[N(k_z,+1)+N(k_z,-1)-\bar{g}(k_z,-1) \\ &\quad \times g(k_z,+1)-\bar{g}(k_z,+1)g(k_z,-1)], \\ \Pi_y(k_z) &= \tfrac{1}{2}[N(k_z,+1)+N(k_z,-1)+\bar{g}(k_z,-1) \\ &\quad \times g(k_z,+1)+\bar{g}(k_z,+1)g(k_z,-1)]. \end{aligned} \tag{6.41}$$

These quantities will be profitably used in our discussion of the positronium decay in Sec. 13.

7. CHARGED SPINOR FIELD—LINEAR MOMENTUM REPRESENTATION

In the case of a charged spinor field, we have to consider the free Lagrangean given in Eq. (4.5) and the current-density given in Eq. (4.6). When these quantities are substituted in Eqs. (3.11), (3.12), and (3.14), we should have

$$\left.\begin{aligned} &\rho_C = \rho_R = \rho_M = +1 \quad \text{for } \mathfrak{L}, \\ &-\rho_C = -\rho_R = -\rho_M = +1 \text{ for } s^\alpha, \quad (\alpha = 1,2,3), \\ &-\rho_C = \rho_R = \rho_M = +1 \text{ for } s^0. \end{aligned}\right\} \tag{7.1}$$

These requirements can be satisfied by

$$C\psi C^{-1} = -e^{i\alpha} K \psi^\times, \quad C\psi^\times C^{-1} = e^{-i\alpha}\psi K^{-1}, \tag{7.2}$$

$$\begin{aligned} (R^{-1}\psi(t)R)^T &= e^{i\beta} E_0 K \psi^\times(-t), \\ (R^{-1}\psi^\times(t)R)^T &= e^{-i\beta}\psi(-t) K^{-1} E_0, \end{aligned} \tag{7.3}$$

$$\begin{aligned} M\psi(\mathbf{x})M^{-1} &= e^{i\gamma} E_{123}\psi(-\mathbf{x}), \\ M\psi^\times(\mathbf{x})M^{-1} &= -e^{-i\gamma}\psi^\times(-\mathbf{x}) E_{321}. \end{aligned} \tag{7.4}$$

It can easily be seen, with the help of Eqs. (2.18) and (2.20), that these transformations are compatible with the unitarity of C, R, and M. The usual commutation rules of ψ are left unchanged by these transformations. The combination of the two relations of Eq. (7.2) gives $(C)^2\psi(C^{-1})^2 = \psi$, agreeing with $C^2 = 1$ irrespectively of α. The reiteration of the transformations in Eq. (7.3) gives

$$R^T R^{-1}\psi(t) R R^{T-1} = -\psi(t),$$

implying

$$R^T R^{-1} = \Delta. \tag{7.5}$$

The double mirage, according to Eq. (7.4), yields

$$M^2\psi(M^{-1})^2 = -e^{2i\gamma}\psi, \tag{7.6}$$

which means, among other possibilities,[15] that

$$\left.\begin{aligned} M^2 &= 1 \quad \text{for} \quad e^{i\gamma} = \pm i, \\ M^2 &= \Delta \quad \text{for} \quad e^{i\gamma} = \pm 1. \end{aligned}\right\} \tag{7.7}$$

According to Eq. (3.50), we shall adopt the former alternative.

We can easily see that the double-time-reversal in the sense of Eq. (3.56) can be a c-number or Δ, or a more general W. If we apply Eq. (7.2) first and then Eq. (7.3) we get

$$(R^{-1}C\psi(t)C^{-1}R)^T = -e^{i(\alpha-\beta)} E_0 \psi(-t). \tag{7.8}$$

If we invert the order of these two, we get

$$(C^{-1T}R^{-1}\psi(t)RC^T)^T = -e^{-i(\alpha-\beta)} E_0 \psi(-t). \tag{7.9}$$

By using Eq. (3.57), we obtain from Eqs. (7.8) and (7.9)

$$W^{-1}\psi W = e^{2i(\alpha-\beta)}\psi, \tag{7.10}$$

which means that if $\alpha = \beta$ or $\alpha = \beta \pm \pi$ then $W =$ c-number, and if $\alpha = \beta \pm \pi/2$ then $W = \Delta$. In the latter case, the double-time-reversal in the sense of Eq. (3.56) will become a c-number because of Eq. (3.58). The more general case of W where α and β are arbitrary will be included in our discussion of the superselection rules.

The relation in Eq. (7.2) satisfies the requirements of Eqs. (7.1) for Eqs. (4.5) and (4.6) only on the assumption that ψ and $\psi^\times$ anticommute. As a result of the application of the commutation relations, there appears a c-number additional term. In order to avoid such an additional term, it is desirable to write $\mathfrak{L}$ and s^μ in the form

$$\left.\begin{aligned} \mathfrak{L} &\to \tfrac{1}{2}(\mathfrak{L} + \rho_C C \mathfrak{L} C^{-1}) \\ &= -\tfrac{1}{4}\psi^\times[E_\mu(\partial/\partial x_\mu) - E_\mu(\overleftarrow{\partial}/\partial x_\mu) + 2imE_5]\psi \\ &\quad -\tfrac{1}{4}\psi[E_\mu{}^T(\partial/\partial x_\mu) - E_\mu{}^T(\overleftarrow{\partial}/\partial x_\mu) - 2imE_5{}^T]\psi^\times, \\ s^\mu &\to \tfrac{1}{2}(s^\mu + \rho_C C s^\mu C^{-1}) = \frac{i}{2}(\psi^\times E^\mu \psi - \psi E^{\mu T}\psi^\times). \end{aligned}\right\} \tag{7.11}$$

Naturally, Eq. (7.11) is written so that the requirements in Eq. (7.1) are automatically satisfied. If we assume the Bose statistics, Eq. (7.11) becomes essentially a c-number.

We shall now proceed to determine the effect of C, R, and M in the linear momentum representation. This actually corresponds to what we have called mixed representation in the case of photons. In this representation, we use the following mutually commuting Hermitian operators:

$$\left.\begin{aligned} &p_\alpha=-i(\partial/\partial x_\alpha), \quad (\alpha=1,2,3), \quad \mathfrak{d}=i(\partial/\partial t),\\ &H_0=E_0E_\alpha p_\alpha+E_0E_5 m, \quad \sigma=E_0E_5E_\alpha p_\alpha/2p, \end{aligned}\right\} \tag{7.12}$$

with

$$p=|(\sum_\alpha p_\alpha^2)^{\frac{1}{2}}|, \tag{7.13}$$

where σ means the spin in the direction of propagation p_α. Denoting the eigenvalues of p_α, $H_0=\mathfrak{d}$, σ respectively by $\mathbf{k}$, ω, μ, we use the expression:

$$\psi[\mathbf{k},\omega,\mu](\mathbf{x},t), \quad \omega=\pm(\mathbf{k}^2+m^2)^{\frac{1}{2}}, \tag{7.14}$$

for the corresponding eigenfunctions. The absolute value of ω is determined by $\mathbf{k}$, but its sign is an independent quantum number. Thus we can expand any solution of the wave equation as

$$\psi(\mathbf{x},t)=(V)^{-\frac{1}{2}}\sum_\lambda a_\lambda\psi[\lambda](\mathbf{x},t) \tag{7.15}$$

where λ stands collectively for $(\mathbf{k},\omega,\mu)$.

Next we investigate the effect of the transformations appearing on the right sides of the relations in Eqs. (7.2), (7.3), and (7.4) on the eigenfunctions in Eq. (7.14). In other words, we shall study the behaviors of the functions:

$$\begin{gathered} K\psi^\times[\lambda](\mathbf{x},t), \quad E_0K\psi^\times[\lambda](\mathbf{x},-t),\\ E_{123}\psi[\lambda](-\mathbf{x},t), \end{gathered} \tag{7.16}$$

towards the operators in Eq. (7.12), as compared with the behavior of the original function of Eq. (7.14).

The transformations from Eq. (7.14) to Eq. (7.16) can be decomposed into: multiplication by K, Hermitian conjugation, multiplication by J (to obtain $\psi^\times$), and multiplication by

$$T\equiv iE_0\Omega_t \quad \text{and} \quad P\equiv E_{05}\Omega_{xyz}, \quad (E_{05}=-iE_{123}), \tag{7.17}$$

where the meaning of Ω symbol is given by Eq. (2.73). The factor i is inserted to make T and P Hermitian, but does not have much importance.

The defining equations for Eq. (7.14) have the form

$$O\psi=O'\psi, \tag{7.18}$$

where O stands for any one of the operators in Eq. (7.12) and O' stands for the eigenvalue. From Eq. (7.18) it follows, by the application of the above-mentioned transformations, that

$$\left.\begin{aligned} &(K(J^{-1}OJ)^TK^{-1})K\psi^\times(\mathbf{x},t)=O'K\psi^\times(\mathbf{x},t),\\ &(TK(J^{-1}OJ)^TK^{-1}T)E_0K\psi^\times(\mathbf{x},-t)\\ &\qquad\qquad =O'E_0K\psi^\times(\mathbf{x},-t),\\ &(POP^{-1})E_{123}\psi(-\mathbf{x},t)=O'E_{123}\psi(-\mathbf{x},t). \end{aligned}\right\} \tag{7.19}$$

The functions mentioned in Eq. (7.16) have the eigenvalues O' for the operators appearing on the left sides of Eq. (7.19), instead of the operator O for which $\psi(\mathbf{x},t)$ has the eigenvalue O'. To characterize these operators, we introduce the following sign-functions:

$$\left.\begin{aligned} J^{-1}OJ&=\rho_J\bar{O}=\rho_JO,\\ K^{-1}OK&=\rho_KO^T,\\ T^{-1}OT&=\rho_TO,\\ P^{-1}OP&=\rho_PO, \end{aligned}\right\} \tag{7.20}$$

where O^T should be understood as the transpose of O, as explained at the end of Sec. 2D. With the help of these ρ's, we can write Eq. (7.19) more compactly as

$$\left.\begin{aligned} O\psi(\mathbf{x},t)&=O'\psi(\mathbf{x},t),\\ OK\psi^\times(\mathbf{x},t)&=\rho_J\rho_KO'K\psi^\times(\mathbf{x},t),\\ OE_0K\psi^\times(\mathbf{x},-t)&=\rho_J\rho_K\rho_TO'E_0K\psi^\times(\mathbf{x},-t),\\ OE_{123}\psi(-\mathbf{x},t)&=\rho_PO'E_{123}\psi(-\mathbf{x},t). \end{aligned}\right\} \tag{7.21}$$

The ρ's for the operators mentioned in Eq. (7.12) are listed in Table I.

TABLE I. The sign functions ρ's defined in Eq. (7.20) and used in Eq. (7.21) for various operators introduced in Eq. (7.12).

	ρ_J	ρ_K	ρ_T	ρ_P	$\rho_J\rho_K$	$\rho_J\rho_K\rho_T$
p_α	+	−	+	−	−	−
$\mathfrak{d}$	+	−	−	+	−	+
H_0	−	+	−	+	−	+
σ	+	+	+	−	+	+

Reading the column for $\rho_J\rho_K$ in Table I, we see on account of Eq. (7.21), that $K\psi^\times(\mathbf{x},t)$ has the opposite signs for momentum, and energy and the same sign for spin σ as compared with $\psi(\mathbf{x},t)$. Since σ is the spin in the direction of $\mathbf{k}$, the same sign of σ here actually means the opposite spin. In any event, if $\psi[\lambda]$ represents a positive energy, $K\psi^\times[\lambda]$ will represent a negative energy. Therefore, we can write, instead of Eq. (7.15),

$$\psi(\mathbf{x},t)=V^{-\frac{1}{2}}\sum_\lambda[a_\lambda\psi[\lambda](\mathbf{x},t)+\bar{b}_\lambda\psi^\times[\lambda](\mathbf{x},t)K]. \tag{7.22}$$

An important difference between Eqs. (7.15) and (7.22) is that λ in Eq. (7.22) stands only for $(\mathbf{k},\mu)$ while λ in Eq. (7.21) stood for $(\mathbf{k},\omega,\mu)$. Hereafter $\omega=+|(\mathbf{k}^2+m^2)^{\frac{1}{2}}|$ The spinor adjoint of Eq. (7.22) is, due to $(\overline{\psi^\times K})J^-$

$=\psi K^{-1}$,

$$\psi^{\times}(\mathbf{x},t)=V^{-\frac{1}{2}}\sum_{\lambda}[\bar{a}_{\lambda}\psi^{\times}[\lambda](\mathbf{x},t)+b_{\lambda}\psi[\lambda](\mathbf{x},t)K^{-1}]. \quad (7.23)$$

The fact that $K\psi^{\times}(=-\psi^{\times}K)$ has the opposite values of $\mathbf{p}$ and spin is just compensated in the q-number theory by the fact that, in an expression like $\psi^{\times}J O\psi=\bar{\psi}O\psi$, the amplitudes a_{λ} and $\bar{a}_{\lambda}$ stand in the order $\bar{a}_{\lambda}a_{\lambda}$ while the amplitudes b_{λ} and $\bar{b}_{\lambda}$ stand in the order $b_{\lambda}\bar{b}_{\lambda}=1-\bar{b}_{\lambda}b_{\lambda}$. Therefore, $\bar{b}_{\lambda}b_{\lambda}$ will represent the number of particles having the same values of λ as $\bar{a}_{\lambda}a_{\lambda}$, only with opposite charge.

Reading the column for $\rho_J\rho_K\rho_T$, we see that $E_0K\psi^{\times}(\mathbf{x},-t)$ has the opposite $\mathbf{k}$ and the same value of $\mathfrak{d}$, H, and σ as $\psi(\mathbf{x},t)$. Since $\mathbf{k}$ is opposite, the same value of σ actually means the flipping of spin. Therefore we can write

$$E_0K\psi^{\times}[\mathbf{k},\mu](\mathbf{x},t)=e^{i\alpha}\psi[-\mathbf{k},\mu](\mathbf{x},-t), \quad (7.24)$$

where μ on the left side refers to $\mathbf{k}$ and μ on the right refers to $-\mathbf{k}$. If we solve for $\psi[-\mathbf{k},\mu]$, keeping the same values of μ, $\mathbf{k}$, $\mathbf{x}$, and t, we obtain, by the use of Eq. (2.20),

$$E_0K\psi^{\times}[-\mathbf{k},\mu](\mathbf{x},-t)=-e^{i\alpha}\psi[\mathbf{k},\mu](\mathbf{x},t). \quad (7.25)$$

If we rename $-\mathbf{k}$ and $-t$ as $\mathbf{k}$ and t, Eqs. (7.24) and (7.25) become contradictory. This contradiction is of course only apparent because the same value of μ referring to $-\mathbf{k}$ and to $\mathbf{k}$ means opposite spins. The simplest way to avoid the confusion would be to divide the space into two hemispaces $\mathfrak{S}_1$ and $\mathfrak{S}_2$ such that $\mathbf{x}$ and $-\mathbf{x}$ cannot belong to the same hemispace, and to introduce a new spin-index τ such that

$$\left.\begin{aligned}&\tau=+1 \quad \text{if the actual spin direction is in } \mathfrak{S}_1,\\ &\tau=-1 \quad \text{if the actual spin direction is in } \mathfrak{S}_2.\end{aligned}\right\} \quad (7.26)$$

Then we can write consistently

$$E_0K\psi^{\times}[\mathbf{k},\tau](\mathbf{x},t)=\tau\psi[-\mathbf{k},-\tau](\mathbf{x},-t) \quad (7.27)$$

suitably choosing $e^{i\alpha}$. Writing this way, we can now freely change the sign of $\mathbf{k}$ and t. Equation (7.26) is just a matter of convention, and we still consistently use the relativistic eigenfunctions.

Next, according to the column for ρ_P in Table I, we see that $E_{123}\psi(-\mathbf{x},t)$ belongs to the same energy and the opposite $\mathbf{p}$ and μ (i.e., the same spin). Therefore we can write

$$iE_{123}\psi[\mathbf{k},\tau](\mathbf{x},t)=\psi[-\mathbf{k},\tau](-\mathbf{x},t). \quad (7.28)$$

The factor i is added to secure the symmetry with regard to the signs of $\mathbf{k}$ and $\mathbf{x}$. It can be easily shown that there is no conflict between Eqs. (7.27) and (7.28), in the sense that the application of the transformation in Eq. (7.27) first and Eq. (7.28) next gives the same result as the application of Eq. (7.28) first and Eq. (7.27) next.

Finally using the explicit expressions of ψ as plane waves:

$$\psi[\mathbf{k},\tau]=\alpha[\mathbf{k},\tau]\exp(+i\mathbf{k}\cdot\mathbf{x},-i\omega t), \quad (7.29)$$

we can rewrite Eqs. (7.22) and (7.23) as

$$\left.\begin{aligned}\psi(\mathbf{x},t)&=V^{-\frac{1}{2}}\sum_{\mathbf{k}}\sum_{\tau}[g_+(\mathbf{k},\tau)\alpha[\mathbf{k},\tau]\exp(+-)\\ &\qquad+\bar{g}_-(\mathbf{k},\tau)\alpha^{\times}[\mathbf{k},\tau]K\exp(-+)],\\ \psi^{\times}(\mathbf{x},t)&=V^{-\frac{1}{2}}\sum_{\mathbf{k}}\sum_{\tau}[\bar{g}_+(\mathbf{k},\tau)\alpha^{\times}[\mathbf{k},\tau]\exp(-+)\\ &\qquad+g_-(\mathbf{k},\tau)\alpha[\mathbf{k},\tau]K^{-1}\exp(+-)].\end{aligned}\right\} \quad (7.30)$$

Using Eq. (7.27), we can also write for $\psi^{\times}$

$$\begin{aligned}\psi^{\times}(\mathbf{x},t)=V^{-\frac{1}{2}}\sum_{\mathbf{k}}\sum_{\tau}(-\tau)[&\bar{g}_+(\mathbf{k},\tau)\\ &\times\alpha(-\mathbf{k},-\tau)K^{-1}E_0\exp(-+)\\ &+g_-(\mathbf{k},\tau)\alpha^{\times}(-\mathbf{k},-\tau)E_0\exp(+-)]. \quad (7.31)\end{aligned}$$

and using Eq. (7.28), we can write for ψ

$$\begin{aligned}\psi(\mathbf{x},t)=V^{-\frac{1}{2}}\sum_{\mathbf{k}}\sum_{\tau}[&\bar{g}_+(\mathbf{k},\tau)(iE_{123})\alpha(-\mathbf{k},\tau)\exp(+-)\\ &+\bar{g}_-(\mathbf{k},\tau)\alpha^{\times}(-\mathbf{k},\tau)(iE_{321})K\exp(-+)]. \quad (7.32)\end{aligned}$$

Substituting first Eq. (7.30) into Eq. (7.2), we get

$$\left.\begin{aligned}Cg_+(\mathbf{k},\tau)C^{-1}&=e^{i\alpha}g_-(\mathbf{k},\tau),\\ Cg_-(\mathbf{k},\tau)C^{-1}&=e^{-i\alpha}g_+(\mathbf{k},\tau).\end{aligned}\right\} \quad (7.33)$$

Substituting Eqs. (7.30) and (7.31) into Eq. (7.3), we obtain

$$\left.\begin{aligned}(R^{-1}g_+(\mathbf{k},\tau)R)^T&=-e^{i\beta}\tau\bar{g}_+(-\mathbf{k},-\tau),\\ (R^{-1}g_-(\mathbf{k},\tau)R)^T&=-e^{-i\beta}\tau\bar{g}_-(-\mathbf{k},-\tau).\end{aligned}\right\} \quad (7.34)$$

Substituting Eqs. (7.30) and (7.32) into Eq. (7.4), we obtain

$$\left.\begin{aligned}Mg_+(\mathbf{k},\tau)M^{-1}&=-ie^{i\gamma}g_+(-\mathbf{k},\tau),\\ Mg_-(\mathbf{k},\tau)M^{-1}&=-ie^{-i\gamma}g_-(-\mathbf{k},\tau),\end{aligned}\right\} \quad (7.35)$$

which should lead to $M^2=1$ for $\gamma=\pm\pi/2$. The charge-conjugation operator C as defined by Eq. (7.33) can easily be obtained. Thus,

$$\begin{aligned}&C=C^{-1}=\bar{C}=\prod_{\mathbf{k}}\prod_{\tau}C(\mathbf{k},\tau),\\ &C(\mathbf{k},\tau)=1-N_+(\mathbf{k},\tau)-N_-(\mathbf{k},\tau)+e^{i\alpha}\bar{g}_+(\mathbf{k},\tau)\\ &\qquad\times g_-(\mathbf{k},\tau)+e^{-i\alpha}\bar{g}_-(\mathbf{k},\tau)g_+(\mathbf{k},\tau). \quad (7.36)\end{aligned}$$

Next for the reversion operator R, we have to satisfy

$$\left.\begin{aligned}R^{-1}g_{\pm}(\mathbf{k},+)R&=-e^{\pm i\beta}g_{\pm}(-\mathbf{k},-),\\ R^{-1}g_{\pm}(\mathbf{k},-)R&=+e^{\pm i\beta}g_{\pm}(-\mathbf{k},+),\end{aligned}\right\} \quad (7.37)$$

where the usual representation of g and $\bar{g}$ is assumed in which g and $\bar{g}$ have real matrix elements. We can first

get rid of numerical factors by putting

$$R=R_0R',$$

$$R_0=\prod_{\pm}\prod_{\mathbf{k}}(-e^{\pm i\beta})^{N_\pm(\mathbf{k},+)}(+e^{\pm i\beta})^{N_\pm(\mathbf{k},-)}. \quad (7.38)$$

Then Eq. (7.37) becomes

$$R'^{-1}g_\pm(\mathbf{k},\tau)R'=g_\pm(-\mathbf{k},-\tau) \quad (7.39)$$

which is obviously satisfied by

$$R'=\prod_{\mathbf{k}}{}'\prod_{\pm}R'_\pm(\mathbf{k}),$$

$$R_\pm'(\mathbf{k})=1-N_\pm(\mathbf{k},+)-N_\pm(-\mathbf{k},-)$$
$$+\bar{g}_\pm(\mathbf{k},+)g_\pm(-\mathbf{k},-)+\bar{g}_\pm(-\mathbf{k},-)g_\pm(\mathbf{k},+), \quad (7.40)$$

where the multiplication symbol Π' with a prime means that $\mathbf{k}$ should be taken from one hemispace. Now we can confirm that the R, thus determined, satisfies $R^{-1}R^T=\Delta$. Noticing that $R_0{}^T=R_0$ and $R'^T=R'$ we get

$$R^{-1}R^T=R'^{-1}R_0{}^{-1}R'R_0.$$

From Eq. (7.39), we see that the effect of the transformation by R' is such that $N_\pm(\mathbf{k},+)$ and $N_\pm(-\mathbf{k},-)$ are interchanged. This amounts to an interchange of the factor of the type $(-e^{\pm i\beta})^N$ and the factor of the type $(+e^{\pm i\beta})^N$ in the expression of R_0 or of $R_0{}^{-1}$, which results in a multiplication by $(-1)^N$ for all oscillators. Thus we have

$$R'^{-1}R_0{}^{-1}R'=\Delta R_0{}^{-1},$$

and

$$R^{-1}R^T=\Delta. \quad (7.41)$$

Finally the mirage operator M satisfying Eq. (7.35) can be written as

$$M=M'M_0, \quad (7.42)$$

with

$$M_0=\prod_{\mathbf{k}}\prod_{\tau}(+ie^{-i\gamma})^{N_+(\mathbf{k},\tau)}(+ie^{+i\gamma})^{N_-(\mathbf{k},\tau)}. \quad (7.43)$$

and

$$M'=\prod_{\mathbf{k}}{}'\prod_{\tau}\prod_{\pm}[1-N_\pm(\mathbf{k},\tau)-N_\pm(-\mathbf{k},\tau)$$
$$+\bar{g}_\pm(\mathbf{k},\tau)g_\pm(-\mathbf{k},\tau)+\bar{g}_\pm(-\mathbf{k},\tau)g_\pm(\mathbf{k},\tau)]. \quad (7.44)$$

If we make the square of M, we get

$$M^2=M_0{}^2=\prod_{\mathbf{k}}\prod_{\tau}(-e^{-2i\gamma})^{N_+(\mathbf{k},\tau)}(-e^{+2i\gamma})^{N_-(\mathbf{k},\tau)}, \quad (7.45)$$

which is unity for $\gamma=\pm\pi/2$, and Δ for $\gamma=0$ or $\pm\pi$, in agreement with Eq. (7.7). To satisfy $M^2=1$, we should take

$$M_0=\prod_{\mathbf{k}}\prod_{\tau}(-1)^{N_+(\mathbf{k},\tau)} \text{ or } M_0=\prod_{\mathbf{k}}\prod_{\tau}(-1)^{N_-(\mathbf{k},\tau)}. \quad (7.46)$$

The mirage operator thus defined satisfies

$$M\Psi_{\text{vac}}=\Psi_{\text{vac}}. \quad (7.47)$$

Evidently this is a matter of convention for we could as well take the negative of Eq. (7.42) as M.

8. CHARGED SPINOR FIELD—ANGULAR MOMENTUM REPRESENTATION

In this representation, the operators used to characterize eigenstates are, besides H and $\mathfrak{d}$ as in Eq. (7.12), the following three operators which commute among themselves and with H and $\mathfrak{d}$:

$$I=L_\alpha E_\alpha+E_0E_5, \quad (\alpha=1,2,3), \quad (8.1)$$

$$J_3=L_3+\tfrac{1}{2}E_0E_5E_3, \quad (8.2)$$

$$P=E_0E_5\Omega_{\mathbf{x}}, \quad (8.3)$$

where

$$L_1=-i[x_2(\partial/\partial x_3)-x_3(\partial/\partial x_2)], \text{ etc.}, \quad (8.4)$$

and

$$\Omega_{\mathbf{x}}f(\mathbf{x})=f(-\mathbf{x}), \quad (8.5)$$

or

$$(\mathbf{x}'|\Omega_{\mathbf{x}}|\mathbf{x}'')=\delta(\mathbf{x}'+\mathbf{x}''). \quad (8.6)$$

It is convenient, in this representation, to introduce the following operators, X, Y:

$$X=(i/r)E_\alpha x_\alpha\Omega_{\mathbf{x}}, \quad (\alpha=1,2,3), \quad (8.7)$$

$$Y=(i/r)(E_{12}x_3+E_{23}x_1+E_{31}x_2). \quad (8.8)$$

It is easy to see that X, Y, and P satisfy the same commutation rules as the spin matrices:

$$\left.\begin{aligned} &X^2=Y^2=P^2=1,\\ XY=-YX=iP, \quad &YP=-PY=iX,\\ &PX=-XP=iY. \end{aligned}\right\} \quad (8.9)$$

These operators, X, Y, and P, are analogs of S_1, S_2 and S_3, as seen in Eqs. (2.58), (2.59), and (2.60).

Now noticing that

$$p_1=-i(\partial/\partial x_1)=(x_1/r)(p_r+i/r)$$
$$-(x_2/r^2)L_3+(x_3/r^2)L_2, \text{ etc.}, \quad (8.10)$$

with

$$p_r=-i[(\partial/\partial r)+(1/r)]$$
$$=(1/r)(x_1p_1+x_2p_2+x_3p_3-i) \quad (8.11)$$

we can rewrite the Hamiltonian by a straightforward substitution in the form[25]

$$H_0=-iE_0X\Omega_{\mathbf{x}}p_r-iE_0Y(1/r)L+mP\Omega_{\mathbf{x}}. \quad (8.12)$$

The eigenfunctions in this representation will be characterized by ω, κ, m, and ρ, which are, respectively, eigenvalues of $H=\mathfrak{d}$, I, J_3, and P. Thus,

$$\psi[\omega,\kappa,m,\rho](\mathbf{x},t). \quad (8.13)$$

It is obvious that if ω is an eigenvalue then $-\omega$ is also an eigenvalue. However, in contrast to the case of the

[25] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, London, 1947), third edition, p. 268.

TABLE II. The sign functions ρ's defined in Eqs. (7.20) and (8.32) for various operators used in the angular momentum representation.

	ρ_J	ρ_K	ρ_T	ρ_P	$\rho_J\rho_K$	$\rho_J\rho_K\rho_T$	ρ_{P_x}
H_0	−	+	−	+	−	+	+
$\mathfrak{d}$	+	−	−	+	−	+	+
P	−	+	−	+	−	+	+
I	−	+	−	+	−	+	+
J_3	+	−	+	+	−	−	−

linear momentum representation, $|\omega|$ is not determined by the other quantum numbers. Similarly to the case of the linear momentum representation, here too, we have four quantum numbers.

To determine the relationship among eigenfunctions, we calculate the sign functions ρ_J, ρ_K, ρ_T, and ρ_P, defined by Eq. (7.20), for the various operators used in this representation. The results are listed in Table II.

We first notice that $\rho_J\rho_K=-1$ for all the operators, which means, in virtue of Eq. (7.21), that $K\psi^\times(\mathbf{x},t)$, as compared with $\psi(\mathbf{x},t)$, has the opposite sign for all the quantities involved, including energy. Therefore we can expand $\psi(\mathbf{x},t)$ with the help of Eq. (8.13) and

$$K\psi^\times[\omega,\kappa,m,\rho](\mathbf{x},t), \tag{8.14}$$

while limiting the values of ω to positive values. Thus

$$\psi(\mathbf{x},t)=R^{-\frac{1}{2}}\sum_\lambda[g_+(\lambda)\psi[\lambda](\mathbf{x},t) +\bar{g}_-(\lambda)\psi^\times[\lambda](\mathbf{x},t)K], \tag{8.15}$$

where λ stands for $[\omega,\kappa,m,\rho]$. As before, the fact that Eq. (8.14) has the opposite values of λ is compensated by the fact that the emission operator $\bar{g}$, instead of the absorption operator g, is standing as its coefficient, where g and $\bar{g}$ anticommute. The commutation rules here should read

$$[g_\pm(\omega,\kappa,m,\rho),\bar{g}_\pm(\omega',\kappa',m',\rho')]_+ =\delta(\omega,\omega')\delta(\kappa,\kappa')\delta(m,m')\delta(\rho,\rho')\delta(\pm,\pm), \quad \text{etc.} \tag{8.16}$$

Next, we see from Table II that $\rho_J\rho_K\rho_T$ is negative only for J_3. This means, in the light of Eq. (7.21), that $E_0K\psi^\times[\lambda](-t)$, as compared with $\psi[\lambda](t)$, has the opposite value of J_3 and the same values of $H=\mathfrak{d}$, I, and P. Therefore we can write

$$E_0K\psi^\times[\omega,\kappa,m,\rho](\mathbf{x},-t)=e^{i\alpha}\psi[\omega,\kappa,-m,\rho](\mathbf{x},t). \tag{8.17}$$

Solving for $\psi[-m]$, and taking the spinor adjoint by the help of Eq. (2.20), we obtain

$$E_0K\psi^\times[\omega,\kappa,-m,\rho](\mathbf{x},t) =-e^{i\alpha}\psi[\omega,\kappa,m,\rho](\mathbf{x},-t). \tag{8.18}$$

The apparent dissymmetry between Eqs. (8.17) and (8.18) can be overcome by noticing that α can depend on m. Thus we can adjust α so that both Eq. (8.17) and Eq. (8.18) are incorporated in

$$E_0K\psi[\omega,\kappa,m,\rho](\mathbf{x},-t) =(m/|m|)\psi[\omega,\kappa,-m,\rho](\mathbf{x},t). \tag{8.19}$$

One of the advantageous features of the angular momentum representation is that P commutes with the other operators defining eigenfunctions. This fact is reflected in Table II by ρ_P being positive for all the operators. Thus we should have, according to Eq. (7.21),

$$E_{123}\psi[\omega,\kappa,m,\rho](-\mathbf{x},t)=e^{i\alpha}\psi[\omega,\kappa,m,\rho](\mathbf{x},t),$$

or more directly, because of the fact that $\psi[\lambda]$ is an eigenfunction of P,

$$E_{05}\psi[\omega,\kappa,m,\rho](-\mathbf{x},t)=\rho\psi[\omega,\kappa,m,\rho](\mathbf{x},t). \tag{8.20}$$

Now we can determine the effect of C, R, and M on the q-number amplitudes g's. Applying Eq. (7.2) to Eq. (8.15), we obtain

$$Cg_\pm(\omega,\kappa,m,\rho)C^{-1}=e^{\pm i\alpha}g_\mp(\omega,\kappa,m,\rho), \tag{8.21}$$

which is satisfied by

$$C=C^{-1}=\bar{C}=\prod_\lambda[1-N_+(\lambda)-N_-(\lambda)+e^{i\alpha}\bar{g}_+(\lambda)g_-(\lambda) +e^{-i\alpha}\bar{g}_-(\lambda)g_+(\lambda)]. \tag{8.22}$$

The application of Eq. (7.3) to Eq. (8.15) yields, in virtue of Eq. (8.19),

$$(R^{-1}g_\pm(\omega,\kappa,m,\rho)R)^T=-\frac{m}{|m|}e^{\pm i\beta}\bar{g}_\pm(\omega,\kappa,-m,\rho), \tag{8.23}$$

which is satisfied by

$$\begin{aligned} &R=R_0R',\\ &R_0=\prod_\pm\prod_{\lambda'}\prod_m\left(-\frac{m}{|m|}e^{\pm i\beta}\right)^{N_\pm(\lambda',m)},\\ &R'=\prod_\pm\prod_{\lambda'}\prod_{|m|}[1-N_\pm(\lambda',m)-N_\pm(\lambda',-m) +\bar{g}_\pm(\lambda',m)g_\pm(\lambda',-m)+\bar{g}_\pm(\lambda',-m)g(\lambda',m)], \end{aligned} \tag{8.24}$$

where λ' stands for (ω,κ,ρ). This R obviously satisfies

$$R^TR^{-1}=\Delta. \tag{8.25}$$

Finally, the effect of M on the g's is obtained by applying Eq. (7.4) to Eq. (8.15) with the help of Eq. (8.20). Thus,

$$\left.\begin{aligned} &Mg_+(\lambda',\rho)M^{-1}=-ie^{i\gamma\rho}g_+(\lambda'\rho),\\ &Mg_-(\lambda',\rho)M^{-1}=-ie^{-i\gamma\rho}g_-(\lambda',\rho) \end{aligned}\right\} \tag{8.26}$$

and are satisfied by

$$M=\prod_{\lambda'}\prod_\rho(+i\rho e^{-i\gamma})^{N+(\lambda',\rho)}(+i\rho e^{+i\gamma})^{N-(\lambda',\rho)}, \tag{8.27}$$

where λ' stands for (ω,κ,m). We can satisfy the condition $M^2=1$, by either one of the following two alternatives:

$$\text{or}\qquad \begin{aligned} M_1&=\prod_{\lambda'}\prod_{\rho}\rho^{N+(\lambda',\rho)}(-\rho)^{N-(\lambda',\rho)},\\ M_2&=\prod_{\lambda'}\prod_{\rho}(-\rho)^{N+(\lambda',\rho)}\rho^{N-(\lambda',\rho)}. \end{aligned}\tag{8.28}$$

This ambiguity is not surprising since the determination of ρ itself is a matter of convention; we could as well use $-P$ for P. In spite of this, Eq. (8.28) can give a definite value of parity when the total number of spinor particles is even. Moreover, we can speak of a definite "relative" parity of two states, each having an odd number of spinor particles, since the "relative" parity (i.e., equality or inequality of parity) has a definite meaning no matter whether we use M_1 or M_2. Both M_1 and M_2 satisfy $M\Psi_{\text{vac}}=\Psi_{\text{vac}}$.

For the purpose of a later application, let us briefly consider the reflection with respect to the x-axis: $(x,y,z,t)\rightarrow(-x,y,z,t)$. The unitary operator M_x is obviously defined by

$$M_x\psi(x,y,z,t)M_x^{-1}=e^{i\alpha}E_1\psi(-x,y,z,t),\tag{8.29}$$

whose adjoint is, on account of Eq. (2.18),

$$M_x\psi^\times(x,y,z,t)M_x^{-1}=-e^{i\alpha}\psi^\times(-x,y,z,t)E_1,\tag{8.30}$$

where α has to be $=0$ or $\pm\pi$ in order that $M_x^2=1$. The right-hand side of Eq. (8.29) is obtained from $\psi(x,y,z,t)$ by applying the operator

$$P_x=E_1\Omega_x.\tag{8.31}$$

The comparison of an eigenfunction $\psi[\lambda]$ and $P_x\psi[\lambda]$ can easily be done by examining

$$P_x^{-1}OP_x=\rho_{Px}O,\tag{8.32}$$

for various operators. The results are listed in Table II, which shows that the only quantity that changes its sign is J_3. Therefore we can write

$$\begin{aligned}E_1\psi[\omega,\kappa,m,\rho](-x,y,z,t)\\ =e^{i\alpha}\psi[\omega,\kappa,-m,\rho](x,y,z,t),\end{aligned}\tag{8.33}$$

where $e^{i\alpha}$ can be put $=1$ without loss of generality. Applying Eq. (8.29) to Eq. (8.15), we obtain, with the help of Eq. (8.33).

$$M_xg_\pm[\lambda',m]M_x^{-1}=e^{i\alpha}g_\pm[\lambda',-m],\tag{8.34}$$

where λ' stands for (ω,κ,ρ) and $e^{i\alpha}$ must be ±1, in order that $M_x^2=1$. M_x can be given an explicit expression:

$$\begin{aligned}M_x=\prod_{\pm}\prod_{\lambda'}\prod_{|m|}[&1-N_\pm(\lambda',m)-N_\pm(\lambda',-m)\\ &+(\pm)'\bar g_\pm(\lambda',m)g_\pm(\lambda',-m)\\ &+(\pm)'\bar g_\pm(\lambda',-m)g_\pm(\lambda',m)],\end{aligned}\tag{8.35}$$

where $(\pm)'$ is an arbitrary double sign independent of the double sign of the charge. This M_x satisfies $M_x\Psi_{\text{vac}}=\Psi_{\text{vac}}$.

9. TENSORIAL QUANTITIES BUILT WITH SPINORS

We have established that charge-conjugation, reversion, and mirage with regard to spinor fields are engendered by C, R, and M defined by

$$\left.\begin{aligned} C\psi(\mathbf{x},t)C^{-1}&=e^{i\alpha}\psi^\times(\mathbf{x},t)K,\\ (R^{-1}\psi(\mathbf{x},t)R)^T&=e^{i\beta}\psi^\times(\mathbf{x},-t)E_0K,\\ M\psi(\mathbf{x},t)M^{-1}&=e^{i\gamma}E_{123}\psi(-\mathbf{x},t),\quad \gamma=\pm\pi/2,\end{aligned}\right\}\tag{9.1}$$

and their adjoint equations. We have also shown that such operators C, R, and M actually exist. These relations have been derived from the requirement that the free Lagrangean and the current have the correct transformation for charge-conjugation $\mathfrak{C}$, reversion $\mathfrak{R}$, and mirage $\mathfrak{M}$. Now, we can conversely use these relations in Eq. (9.1) to find the transformation rules of various tensorial quantities $Q(x,t)$ built with spinors for $\mathfrak{C}$, $\mathfrak{R}$, and $\mathfrak{M}$. Their transformation properties are determined by

$$\left.\begin{aligned} \rho_C&=(CQ(\mathbf{x},t)C^{-1})/Q(\mathbf{x},t),\\ \rho_R&=(R^{-1}Q(\mathbf{x},t)R)^T/Q(\mathbf{x},-t),\\ \rho_M&=(MQ(\mathbf{x},t)M^{-1})/Q(-\mathbf{x},t).\end{aligned}\right\}\tag{9.2}$$

The quantities to be considered have the general form

$$Q(\mathbf{x},t)=\psi^\times(\mathbf{x},t)O(\mathbf{x},t)\psi(\mathbf{x},t).\tag{9.3}$$

In the next section we shall investigate the case where the two spinors involved in Q are not the same. In Eq. (9.3), however, $\psi^\times$ is just $\bar\psi J^{-1}$, where $\bar\psi$ is the Hermitian conjugate of ψ,

$$Q=\bar\psi J^{-1}O\psi.\tag{9.4}$$

The requirement that Q be Hermitian is equivalent to the requirement that $J^{-1}O$ be Hermitian:

$$(J^{-1}O)^-=J^{-1}O,\quad\text{or}\quad \rho_J=(J^{-1}OJ)/\bar O=+1.\tag{9.5}$$

In the following we shall assume that the operators O are written so that Eq. (9.5) is the case. The operators considered in the preceding sections are $J^{-1}O$ rather than O. See Eq. (2.40) in this connection.

To determine ρ_C, ρ_R, and ρ_M, we use the auxiliary sign functions ρ_K, ρ_T, and ρ_P defined by

$$\left.\begin{aligned} K^{-1}OK&=\rho_KO^T,\quad & T^{-1}OT&=\rho_TO,\\ P^{-1}OP&=\rho_PO,\quad & J^{-1}OJ&=\rho_J\bar O.\end{aligned}\right\}\tag{9.6}$$

See Eqs. (7.17) and (7.20). It is easy to see, by applying Eq. (9.1) on Eq. (9.2), that

$$\rho_C=\rho_K,\quad \rho_R=-\rho_K\rho_T,\quad \rho_M=-\rho_P.\tag{9.7}$$

TABLE III. The various ρ's for the tensorial quantities built with the same spinors.

$\rho \backslash O$	1	iE_5	iE_α	iE_0	$iE_{5\alpha}$	iE_{50}	$iE_{\alpha\beta}$	$iE_{\alpha 0}$
ρ_J	+	+	+	+	+	+	+	+
ρ_K	+	+	−	−	+	+	−	−
ρ_T	+	−	−	+	+	−	+	−
ρ_P	+	−	+	−	−	+	+	−
ρ_C	+	+	−	−	+	+	−	−
ρ_R	−	+	−	+	−	+	+	−
ρ_M	−	+	−	+	+	−	−	+
Kind	1	reg.	2	2	1	1	3	3

Note that $\rho_C=\rho_K$ is based on the anticommutability of ψ and $\psi^\times$.

As an illustration, let us consider the free Hamiltonian H_0:

$$H_0=E_0E_\alpha p_\alpha+E_0E_5m, \quad (\alpha=1, 2, 3).$$

The corresponding operator $O=JH_0$ is then

$$O=iE_\alpha p_\alpha+iE_5m.$$

Since $J^{-1}E_iJ=-\bar{E}_i$, as in Eq. (2.18), and $\bar{p}_\alpha=p_\alpha$, we have $\rho_J=+1$ due to the imaginary unit i. Compare herewith that we had $\rho_J=-1$ for H_0. See Table I. Since O anticommutes with E_0 and since O does not contain the time variable, we have $\rho_T=-1$. Next, $K^{-1}E_\mu K=-E_\mu{}^T$, $K^{-1}E_5K=E_5{}^T$, as seen in Eq. (2.19). And, furthermore, since $p_\alpha{}^T=-p_\alpha$, from Eq. (2.71), we have $\rho_K=1$ for O. Since $\Omega_x p_x=-p_x\Omega_x$, in Eqs. (2.70) and (2.74), and since E_{123} commutes with E_α and anticommutes with E_5, we have $\rho_P=-1$. Therefore, by Eq. (9.7), $\rho_C=\rho_R=\rho_M=1$.

We are mainly interested in the O's which are the basic E's or their products. There are, as is well known, six types of quantities up to the second rank:

$$\left.\begin{array}{llll} (S) & O=iE_5, & (PS) & O=1, \\ (V) & O=iE^\mu, & (PV) & O=iE^{5\mu}, \\ (T) & O=E^{5\mu\nu}, & (PT) & O=iE^{\mu\nu}. \end{array}\right\} \quad (9.8)$$

The imaginary unit i is added wherever it is necessary to have $\rho_J=+1$. As we can write $E^{512}=iE_{30}$, etc., due to Eq. (2.17), the relation between (T) and (PT) is just that of complementary tensors as defined in Part I. Therefore, we need not investigate them separately. The various ρ's for the O's are listed in Table III, in which the indexes α and β refer to space-coordinates, $\alpha,\beta=1,2,3$.

From the line for ρ_C, we observe that iE_μ, $iE_{\mu\nu}$ and $E_{5\mu\nu}$ are "electromagnetic" quantities ($\rho_C=-1$) while all the others are "mechanical" ($\rho_C=1$). From the lines for ρ_R and ρ_M, we can determine the "kinds" of the quantities which are listed in the last line of the table. Comparing this result with (2.43), we observe that 1, iE_5, and $iE_{5\mu}$ change their "kinds" by the passage from the c-number theory to the q-number theory. This change is such that ρ_M remains the same while ρ_R changes its sign. It should also be noted that these quantities are "mechanical" quantities. In other words, the q-number theory changes the behavior of the mechanical quantities for reversion, as was required in Sec. 4. These results were mentioned in Part I, Table VI.

It should also be recalled that the various alternative expressions for C, R, and M pertain to the arbitrariness that is allowed without altering the transformation of the bilinear expression of the type in Eq. (9.3). Therefore, the above-mentioned results are not affected by the choices of the operators C, R, and M.

The transformation rules in Eq. (9.1) are determined by the consideration regarding $\mathfrak{L}_\psi$ and s^μ. But, for the neutral spinor particles, s^μ does not have the meaning of electric current, and one may argue that particles and antiparticles are physically undistinguishable. If we take this point of view, we could define C, R, and M by a combination of those C, R, and M given in Eq. (9.1) with charge-conjugation $\mathfrak{C}$: $\psi\to\psi^\times K$, $\psi^\times\to\psi K^{-1}$, i.e., by

$$\left.\begin{array}{l} C\psi C^{-1}=e^{i\alpha}\psi, \\ (R^{-1}\psi(t)R)^T=e^{i\beta}E_0\psi(-t), \\ M\psi(\mathbf{x})M^{-1}=e^{i\gamma}E_{123}\psi^\times(-\mathbf{x})K. \end{array}\right\} \quad (9.9)$$

But, such an assumption will immediately encounter a serious difficulty. Consider the pion-nucleon interaction:

$$N+\pi^+\to P, \quad (9.10)$$

to which corresponds an interaction Hamiltonian of the type

$$\psi_P{}^\times\psi_N v, \quad (9.11)$$

where the Dirac matrices are omitted for simplicity. Assuming Eq. (9.1) for protons and Eq. (9.9) for neutrons, we get as the result of mirage, for instance,

$$\psi_P{}^\times\psi_N{}^\times v, \quad (9.12)$$

which certainly should not exist in the original interaction Hamiltonian, since it implies a violation of "conservation of heavy particles":

$$\pi^+\to P+N. \quad (9.13)$$

The relations in Eq. (9.9) for C and R encounter the same difficulties. We therefore have to assume Eq. (9.1) and not Eq. (9.9) for neutrons.

We can further argue that the neutrino also has to obey Eq. (9.1) and not Eq. (9.9). If the neutrino would obey Eq. (9.9), the interaction Hamiltonian of the type

$$P\to N+e^+\bar{\nu}: \quad (\psi_N{}^\times\psi_P)(\psi_e{}^\times\psi_\nu), \quad (9.14)$$

would go over, by mirage, to

$$P\to N+e^++\nu: \quad (\psi_N{}^\times\psi_P)(\psi_e{}^\times\psi_\nu{}^\times), \quad (9.15)$$

where the neutron, proton, and electron are assumed to

obey Eq. (9.1). The combination of (9.14) and (9.15) would mean

$$\nu\rightarrow\bar{\nu} \quad \text{or} \quad \nu+\nu\rightarrow 0, \tag{9.16}$$

which is hard to accept.

For these reasons, we shall assume in the following sections that the neutron as well as neutrino obeys Eq. (9.1).

10. NUCLEON-PION INTERACTION

Let us consider the nucleon-pion interaction of the type

$$\mathfrak{L}=f\psi^{\times}O_\alpha\vec{\tau}\psi\vec{F}^\alpha, \tag{10.1}$$

where the index α designates the tensorial components and the arrow means a vector in the 3-dimensional Euclidean isotopic spin space. Here, ψ is actually an 8-component spinor composed of neutron spinor ψ_N and proton spinor ψ_P, which we define by

$$\tau_3\psi_N=+\psi_N, \quad \tau_3\psi_P=-\psi_P. \tag{10.2}$$

The factor $\vec{F}^\alpha$ is a Hermitian quantity derived from the pion field and the τ's are the well-known isotopic spin matrices which are also Hermitian. Then, if we take O, such that

$$J^{-1}OJ=\bar{O}, \quad \text{i.e.} \quad \rho_J=1, \tag{10.3}$$

$\mathfrak{L}$ will become Hermitian with real f.

Although we take the "symmetric" formulation, in Eq. (10.1), as the standard expression, it will be easy to infer the situation with the "charged" or "neutral" formulation from our results. As O in Eq. (10.1), we can take any one of the operators introduced in Eq. (9.8). The $\vec{F}^\alpha$ for (S) and (PS) is the scalar pion field strength itself $\vec{u}$. The $\vec{F}^\alpha$ for (V) and (PV) can be the vectorial field strength $\vec{u}^\mu$ or the 4-dimensional gradient of the scalar field $\vec{u}$. The $\vec{F}^\alpha$ for (T) and (PT) must be the 4-dimensional curl of a vectorial field $\vec{u}^\mu$.

In this section, we take the viewpoint that ψ_N and ψ_P obey the transformation rules in Eq. (9.1) and that the transformation rules of the u-field are to be determined by the requirement that $\mathfrak{L}$, as in Eq. (10.1), be a regular scalar. There is no reason why the phase angles α, β, and γ should be the same for protons and neutrons. Therefore, we shall write α_P, β_P, and γ_P for protons and α_N, β_N, and γ_N for neutrons.

By decomposing Eq. (10.1) in three components in the isotopic spin space, we get

$$\left.\begin{aligned}\mathfrak{L}=&f(\psi_N{}^{\times}O_\alpha\psi_P+\psi_P{}^{\times}O_\alpha\psi_N)F^{(1)\alpha}\\&+f(-i\psi_N{}^{\times}O_\alpha\psi_P+i\psi_P{}^{\times}O_\alpha\psi_N)F^{(2)\alpha}\\&+f(\psi_N{}^{\times}O_\alpha\psi_N-\psi_P{}^{\times}O_\alpha\psi_P)F^{(3)\alpha}.\end{aligned}\right\}\tag{10.4}$$

Or, by introducing, in analogy to Eq. (5.1),

$$G^\alpha=(F^{(1)\alpha}+iF^{(2)\alpha})/\sqrt{2}, \quad \bar{G}^\alpha=(F^{(1)\alpha}-iF^{(2)\alpha})/\sqrt{2},$$

we can write

$$\mathfrak{L}=\sqrt{2}f(K_\alpha G^\alpha+\bar{K}_\alpha\bar{G}^\alpha)+fH_\alpha F^{(3)\alpha} \tag{10.5}$$

with

$$\left.\begin{aligned}K_\alpha&=\psi_P{}^{\times}O_\alpha\psi_N,\\\bar{K}_\alpha&=\psi_N{}^{\times}O_\alpha\psi_P,\\H_\alpha&=\psi_N{}^{\times}O_\alpha\psi_N-\psi_P{}^{\times}O_\alpha\psi_P.\end{aligned}\right\}\tag{10.6}$$

In the "charged" formulation, the third term with $F^{(3)}$ is absent. In the "neutral" formulation, the first two terms are absent and we have a plus sign instead of minus sign in the expression of H_α.

Our next step is to determine the transformation rules of K_α and H_α by the use of Eq. (9.1). The results are as follows:

$$\left.\begin{aligned}CK_\alpha C^{-1}&=[\pm]e^{i(\alpha_N-\alpha_P)}\rho_C\bar{K}_\alpha,\\CH_\alpha C^{-1}&=\rho_C H_\alpha.\end{aligned}\right\}\tag{10.7}$$

$$\left.\begin{aligned}(R^{-1}K_\alpha(t)R)^T&=e^{i(\beta_N-\beta_P)}\rho_R\bar{K}_\alpha(-t),\\(R^{-1}H_\alpha(t)R)^T&=\rho_R H_\alpha(-t),\end{aligned}\right\}\tag{10.8}$$

$$\left.\begin{aligned}MK_\alpha(\mathbf{x})M^{-1}&=e^{i(\gamma_N-\gamma_P)}\rho_M K_\alpha(-\mathbf{x}),\\MH_\alpha(\mathbf{x})M^{-1}&=\rho_M H_\alpha(-\mathbf{x}).\end{aligned}\right\}\tag{10.9}$$

The sign $[\pm]$ is positive if ψ_N and ψ_P anticommute and negative if they commute. ρ_C, ρ_R, and ρ_M are the ones determined in the last section for various O's. See Table III.

By the requirement that $\mathfrak{L}$ be a regular scalar, we can infer the transformation rules for $F^{(3)}$ from the results of Eqs. (10.7), (10.8), and (10.9):

$$CF^{(3)\alpha}C^{-1}=\rho_C F^{(3)\alpha}, \quad (R^{-1}F^{(3)\alpha}(t)R)^T=\rho_R F^{(3)\alpha}(-t),$$
$$MF^{(3)\alpha}(\mathbf{x})M^{-1}=\rho_M F^{(3)\alpha}(-\mathbf{x}), \tag{10.10}$$

where ρ_C, ρ_R, and ρ_M are those of the operator O used in (10.9). Since the $F^{(3)}$is the $u^{(3)}$ itself, or a quantity derived from $u^{(3)}$ by an operation of differentiation which is a regular tensor, the "kind" of $F^{(3)}$ is the same as the kind of $u^{(3)}$. Also, the charge-sign-function ρ_C must be the same for $F^{(3)}$ and $u^{(3)}$. These properties of $u^{(3)}$ are tabulated in Table IV.

From these properties, we can conclude an important rule concerning the mixing of different types of interaction. Mixing of (S) and (V) for scalar field and mixing of (PV) and (PT) for vector field are not allowed. This conclusion can be drawn from consideration of the "kind" as well as from the consideration of ρ_C. This conclusion is also valid in the "neutral formulation."

TABLE IV. The transformation properties of $u^{(3)}$ for various interaction types.

	(S)	(V)	(T)	(PS)	(PV)	(PT)
Kind	reg.	2	2	1	1	3
ρ_C	+	−	−	+	+	−

Contrary to the case of $u^{(3)}$, we cannot infer any definite conclusion regarding the "kind" of $u^{(1)}$ and $u^{(2)}$ because of the unknown factor such as $\exp(i\gamma_N - i\gamma_P)$ in Eq. (10.9). By the same argument as the one which led to Eq. (10.10) we can write

$$\left.\begin{aligned} CvC^{-1} &= e^{i(\alpha_P-\alpha_N)}\rho_C\bar{v}, \\ (R^{-1}v(t)R)^T &= e^{i(\beta_P-\beta_N)}\rho_R\bar{v}(-t), \\ Mv(\mathbf{x})M^{-1} &= e^{i(\gamma_P-\gamma_N)}\rho_M v(-\mathbf{x}), \end{aligned}\right\} \quad (10.11)$$

where ρ_C, ρ_R, and ρ_M are those of $u^{(3)}$ for the corresponding interaction type. These relations agree with Eqs. (5.8), (5.11), and (5.16). Since γ_P and γ_N are limited to $\pm\pi/2$, $\exp(i\gamma_P - i\gamma_N)$ as well as $\rho_M \exp(i\gamma_P - i\gamma_N)$ is limited to ± 1, which is in agreement with Eq. (5.17).

The prohibition rules of mixture, explained with regard to $u^{(3)}$, is still valid for the charged v. For, whatever the factors $\exp(i\alpha_P - i\alpha_N)$ and $\exp(i\beta_P - i\beta_N)$ may be, the phase factors in transformation rules for v must take the same values throughout the entire Lagrangean. It is true that due to the so-called equivalence theorem,[26] the (S)-interaction and (V)-interaction, for instance, are equivalent. Thus one may say that the mixture of (S) and (V) is unnecessary. However, these equivalence theorems are proved only in lower approximations and we cannot in general replace (S) by (V) or (V) by (S).

Before closing this section, a few lines may be spent in connection with the contention often raised[27] to the effect that the (S,V,T)-types can be mixed with (PS,PV,PT)-types. For simplicity, let us consider the mixture of (S) and (PS), in accordance with the usually proposed procedure:

$$\mathfrak{L} = f_1\psi^{\times}\vec{\tau}\psi\vec{u} + (1/24)f_2\psi^{\times}E_{\mu\nu\kappa\lambda}\vec{\tau}\psi\eta^{\mu\nu\kappa\lambda}\vec{u}, \quad (10.12)$$

where $E_{\mu\nu\kappa\lambda} = \pm iE_5$ according as (μ,ν,κ,λ) is an even or odd permutation of (1,2,3,0). See Eq. (2.41). According to the c-number theory, $\psi^{\times}E_{\mu\nu\kappa\lambda}\psi$ is a third kind pseudotensor of the fourth rank, corresponding to the fact that $\psi^{\times}E_5\psi$, which is the complementary tensor of $\psi^{\times}E_{\mu\nu\kappa\lambda}\psi$, is a second kind pseudoscalar. In the q-number theory, $\psi^{\times}E_{\mu\nu\kappa\lambda}\psi$ is first kind and $\psi^{\times}E_5\psi$ is regular.

Now referring to Eq. (10.12), the first term will be a regular scalar if $u^{(3)}$ is a third kind in the c-number theory and a first kind in the q-number theory and v obeys the corresponding relation in Eq. (10.11). Then, the second term in Eq. (10.12) can be made regular, if we assume $\eta^{\mu\nu\kappa\lambda}$ to be regular in both c-number and q-number theories. For, (third kind)$\times$(regular)$\times$(third kind) = regular, and (first kind)$\times$(regular)$\times$(first kind) = regular.

This regular tensor $\eta^{\mu\nu\kappa\lambda}$ is nothing but what we denoted by the same symbol in Part I, Sec. 2, as distinct from $\epsilon^{\mu\nu\kappa\lambda}$. The appearance of $\eta^{\mu\nu\kappa\lambda}$ in a physical law is highly questionable. For, η^{1230}, for instance, is $+1$ in one coordinate system, but -1 in another coordinate system which is connected to the first by mirage. This means that the physical law is different for the right-handed coordinate system and the left-handed system. How can one decide whether he should take $\eta^{1230} = +1$ or $= -1$ in his coordinate system? In other words, we can write the second term of Eq. (10.12) in the form

$$f_2 i\psi^{\times}E_5\vec{\tau}\psi\vec{u}, \quad (10.13)$$

assuming that f_2 is a first kind pseudoscalar. Is it permissible that a natural constant change its sign by mirage or reversion? Can it differ for a process and its mirage or reversed process?

Furthermore, the mixture of (S), (PS), or (PV) with one of $(V)(T)$ or (PT) cannot be made allowable by a similar artifice, because ρ_C is different for these two groups.

11. NUCLEON-LEPTON INTERACTION

The usual nucleon-lepton interaction Lagrangean can be written

$$\mathfrak{L} = \sum_\alpha f_\alpha(\psi_N{}^{\times}O_\alpha\psi_P)(\psi_e{}^{\times}O^\alpha\psi_\nu) + \sum_\alpha f_\alpha{}^*(\psi_P{}^{\times}O_\alpha\psi_N)(\psi_\nu{}^{\times}O^\alpha\psi_e), \quad (11.1)$$

where the "particle," as distinguished from the "anti-particle," of the ψ_e field is positron, and the "particle" of the ψ_ν field is the neutrino defined as the neutral partner of the negative beta-decay. The index α designates five independent types of operators mentioned in Eq. (9.8). Because of the existing condition in Eq. (9.5), $\mathfrak{L}$ is hermitian if $f_\alpha{}^*$ is the complex conjugate of f.

By applying the first transformation of Eq. (9.1) on the first term of Eq. (11.1) we obtain

$$f_\alpha C(\psi_N{}^{\times}O_\alpha\psi_P)(\psi_e{}^{\times}O^\alpha\psi_\nu)C^{-1} = f_\alpha(\psi_N O_\alpha{}^T\psi_P{}^{\times})(\psi_e O^{\alpha T}\psi_\nu{}^{\times}) \times e^{i(\alpha_P-\alpha_N+\alpha_\nu-\alpha_e)}(\rho_C)^2, \quad (11.2)$$

where $\rho_C(=\pm 1)$ refers to O_α, and naturally, $\rho_C{}^2 = 1$. It may be quite natural to assume that the commutation relations between ψ_N and ψ_P and the commutation relation between ψ_ν and ψ_e are of the same type. Therefore the q-number part of Eq. (11.2) is identical with the q-number part of the second term of Eq. (11.1). Therefore we get the relation

$$f_\alpha{}^* = f_\alpha e^{i(\alpha_P-\alpha_N+\alpha_\nu-\alpha_e)}, \quad (11.3)$$

or

$$f_\alpha = |f_\alpha| e^{i\theta_\alpha}, \quad (11.4)$$

with

$$\theta_\alpha = (-\alpha_P+\alpha_N-\alpha_\nu+\alpha_e+2n_\alpha\pi)/2, \quad (n_\alpha = 0, 1, 2\cdots). \quad (11.5)$$

[26] E. C. Nelson, Phys. Rev. **60**, 830 (1941); E. J. Dyson, Phys. Rev. **73**, 929 (1948); K. M. Case, Phys. Rev. **76**, 14 (1949).
[27] M. Schoenberg, Phys. Rev. **60**, 468 (1941).

By the same token, from reversion and mirage, we obtain (without any assumption regarding the types of commutation rules)

$$f_\alpha^* = f_\alpha e^{i(\beta_P-\beta_N+\beta_\nu-\beta_e)} \tag{11.6}$$

and

$$f_\alpha = f_\alpha e^{i(\gamma_P-\gamma_N+\gamma_\nu-\gamma_e)}. \tag{11.7}$$

The compatibility of Eq. (11.3) with Eq. (11.6) requires

$$\alpha_P-\alpha_N+\alpha_\nu-\alpha_e=\beta_P-\beta_N+\beta_\nu-\beta_e+2n\pi, \quad (n=0, 1, 2, \cdots), \tag{11.8}$$

and the self-consistency of Eq. (11.7) requires

$$\gamma_P-\gamma_N+\gamma_\nu-\gamma_e=2m\pi \quad (m=0, 1, 2, \cdots), \tag{11.9}$$

while individual γ's are $\pm\pi/2$.

Whether or not $\exp(i\theta_\alpha)$ separately for various α's have a physical meaning, the relative phases

$$\exp[i(\theta_\alpha-\theta_{\alpha'})]$$

have certainly a physically detectable effect.[28] From Eq. (11.5) follows

$$\theta_\alpha-\theta_{\alpha'}=(n_\alpha-n_{\alpha'})\pi \tag{11.10}$$

or

$$e^{i(\theta_\alpha-\theta_{\alpha'})}=\pm 1. \tag{11.11}$$

This means that for a mixture of the type in Eq. (11.1), the f's can be complex but the ratio $f_\alpha/f_{\alpha'}$ for all combinations of α and α' must be real.[28] Obviously the same conclusion can be drawn from the consideration of reversion, i.e., from Eq. (11.6).

Since the transformation of a quantity of the type $(\psi_1{}^\times O\psi_2)$ contains an undetermined factor, such as $\exp[i\beta_2-i\beta_1]$, we can no longer maintain the old prescription that a tensor (pseudotensor) of nucleon field should not be coupled with a pseudotensor (tensor) of lepton field. Therefore we can assume,[29] instead of Eq. (11.1),

$$\mathfrak{L}=\sum_\alpha h_\alpha(\psi_N{}^\times O_\alpha\psi_P)(\psi_e{}^\times O^{5\alpha}\psi_\nu) + \sum_\alpha h_\alpha^*(\psi_P{}^\times O_\alpha\psi_N)(\psi_\nu{}^\times O^{5\alpha}\psi_e), \tag{11.12}$$

where $O^{5\alpha}$ means E_5 times or iE_5 times O. Table V gives the exact correspondence.

By the same argument as before, we obtain from charge-conjugation, reversion, and mirage, the following

TABLE V. Correspondence of $O^{5\alpha}$ to E_α.

O_α	iE_5	1	iE_μ	$iE_{5\mu}$	$E_{5\mu\nu}$	$iE_{\mu\nu}$
$O^{5\alpha}$	1	iE^5	$iE^{5\mu}$	iE^μ	$iE^{\mu\nu}$	$E^{5\mu\nu}$

[28] L. C. Biedeharn and M. E. Rose, Phys. Rev. **83**, 459 (1951).
[29] C. N. Yang and J. Tiomno, Phys. Rev. **79**, 495 (1950).

TABLE VI. The values of $\rho_C(O_\alpha)\rho_C(O_{5\alpha})$, etc., as dependent on the type of O_α.

O	(S)	(PS)	(V)	(PV)	(T)	(PT)
$\rho_C(O_\alpha)\rho_C(O_{5\alpha})$	+	+	−	−	+	+
$\rho_R(O_\alpha)\rho_R(O_{5\alpha})$	−	−	+	+	−	−
$\rho_M(O_\alpha)\rho_M(O_{5\alpha})$	−	−	−	−	−	−
Class	I	I	II	II	I	I

results:

$$\left.\begin{aligned} h_\alpha^* &= e^{i(\alpha_P-\alpha_N+\alpha_\nu-\alpha_e)}\rho_C(O_\alpha)\rho_C(O_{5\alpha})h_\alpha, \\ h_\alpha^* &= e^{i(\beta_P-\beta_N+\beta_\nu-\beta_e)}\rho_R(O_\alpha)\rho_R(O_{5\alpha})h_\alpha, \\ h_\alpha &= e^{i(\gamma_P-\gamma_N+\gamma_\nu-\gamma_e)}\rho_M(O_\alpha)\rho_M(O_{5\alpha})h_\alpha, \end{aligned}\right\} \tag{11.13}$$

where ρ_C, ρ_R, and ρ_M are the ones given in Table III. The products such as $\rho_C(O_\alpha)\rho_C(O_{5\alpha})$ are listed in Table VI, in which the first line indicates the operator O_α.

We notice that $\rho_M(O_\alpha)\rho_M(O_{5\alpha})$ is always -1. Therefore from the third relation of Eq. (11.13), we have

$$\gamma_P-\gamma_N+\gamma_\nu-\gamma_e=(2n+1)\pi. \tag{11.14}$$

Regarding $\rho_C(O_\alpha)\rho_C(O_{5\alpha})$ and $\rho_R(O_\alpha)\rho_R(O_{5\alpha})$, we observe that the types of combinations can be divided into two classes:

$$\left.\begin{aligned} &\text{(I)} \quad (S)-(PS), \quad (PS)-(S), \quad (T)-(PT), \\ &\text{(II)} \quad (V)-(PV), \quad (PV)-(V). \end{aligned}\right\} \tag{11.15}$$

of course, $(T)-(PT)$ and $(PT)-(T)$ are actually the same interaction. For class (I), $\rho_C(O_\alpha)\rho_C(O_{5\alpha})$ is $+1$, and $\rho_R(O_\alpha)\rho_R(O_{5\alpha})$ is -1. For class (II), $\rho_C(O_\alpha)\rho_C(O_{5\alpha})$ is -1 and $\rho_R(O_\alpha)\rho_R(O_{5\alpha})$ is $+1$.

Writing

$$A=\alpha_P-\alpha_N+\alpha_\nu-\alpha_e, \quad B=\beta_P-\beta_N+\beta_\nu-\beta_e \tag{11.16}$$

we have for class (I),

$$h_\alpha=|h_\alpha|e^{i\theta_\alpha} \tag{11.17}$$

with

$$\theta_\alpha=-\tfrac{1}{2}A+n_\alpha\pi=-\tfrac{1}{2}B+(m_\alpha+\tfrac{1}{2})\pi. \tag{11.18}$$

And for class II

$$\theta_\alpha=-\tfrac{1}{2}A+(n_\alpha+\tfrac{1}{2})\pi=-\tfrac{1}{2}B+m_\alpha\pi. \tag{11.19}$$

From Eq. (11.18) it follows that if α and α' both belong to I or II

$$e^{i\theta_\alpha}/e^{i\theta_{\alpha'}}=\pm 1 \quad (\alpha, \alpha'\epsilon\text{I} \text{ or } \alpha, \alpha'\epsilon\text{II}) \tag{11.20}$$

and if one of α and α' belongs to I and the other belongs to II,

$$e^{i\theta_\alpha}/e^{i\theta_{\alpha'}}=\pm i \quad (\alpha\epsilon\text{I}, \alpha'\epsilon\text{II} \text{ or } \alpha\epsilon\text{II}, \alpha'\epsilon\text{I}). \tag{11.21}$$

In other words, within the same class, I or II, the

relative phase factors of h's are real, and the relative phase factors of the h's belonging to the different classes, I and II, are imaginary.

It should also be noted that a mixture of a type belonging to Eq. (11.1) and a type belonging to Eq. (11.12) is not allowed. For Eqs. (11.9) and (11.14) are in direct contradiction. The same conclusion can also be drawn from the comparison of Eq. (11.8) with Eq. (11.18) or (11.19).

12. SELECTION RULES AND SUPERSELECTION RULES

A self-miraged state Ψ is a state such that the expectation values of each physical quantity $Q(\mathbf{x})$ is ρ_M times that of $Q(-\mathbf{x})$. In particular, a quantity Q, which does not depend on $\mathbf{x}$ and whose ρ_M is negative, must have a vanishing expectation value. The total linear momentum is an example of such a quantity.

By definition of the mirage operator M, we should have for a self-miraged state

$$M\Psi=e^{i\alpha}\Psi;\quad M^2\Psi=e^{2i\alpha}\Psi.$$

If

$$M^2=1, \tag{12.1}$$

we have $\exp(2i\alpha)=1$. The self-miraged states are classified into states of even parity and odd parity such that

$$M\Psi_{\text{even}}=\Psi_{\text{even}},\quad M\Psi_{\text{odd}}=-\Psi_{\text{odd}}. \tag{12.2}$$

This definition is not yet unique, for we can still multiply M by (-1) without changing Eq. (12.1) but interchanging Ψ_{even} and Ψ_{odd}. Therefore, we usually impose an auxiliary definition,

$$M\Psi_{\text{vac}}=+\Psi_{\text{vac}}, \tag{12.3}$$

which is just a matter of convention.

In order that this classification in Eq. (12.2) is compatible with the other specification of Ψ, we have to use those physical quantities which commute with M or whose expectation values in Ψ are zero. For those quantities Q which do not depend on $\mathbf{x}$, the relation in Eq. (3.12) simply means that $\rho_M=1$ or -1 according as Q commutes or anticommutes with M. Therefore, Ψ must have vanishing expectation values for these Q's with $\rho_M=-1$. This is a condition that a self-miraged state satisfies.

This restriction to "self-miraged" states would seem to delimit the Hilbert space to its part or subspace. It is not so because any arbitrary state can be decomposed into two kinds of self-miraged states defined in Eq. (12.2). For any Ψ, we have

$$\Psi=\Psi_{\text{even}}+\Psi_{\text{odd}};\quad \Psi_{\text{even}}=(\Psi+M\Psi)/2;$$
$$\Psi_{\text{odd}}=(\Psi-M\Psi)/2, \tag{12.4}$$

where the resultant Ψ is not self-miraged.

In linear momentum representation, the self-miraged states must have the total linear momentum equal to zero. We have already noticed that the angular momentum representation is particularly suitable for the parity consideration. It should also be recalled that except for the case of photons and neutral mesons, there is actually no physical means of determining the parity of a single particle. This ambiguity is one which exists even after the restrictions are made in Eqs. (12.1) and (12.3). The situation has been explained in some detail with regard to the charged spinor fields in the angular momentum representation. A similar consideration applies also to the charged pion fields.

The selection rule regarding space-parity means that the transition between an even-parity state and an odd-parity state is prohibited if the transition operator $U(t,-t)$ commutes with the mirage operator M. This is quite obvious, because

$$(\Psi_{\text{odd}}, U\Psi_{\text{even}})=(\Psi_{\text{odd}}, UM\Psi_{\text{even}})$$
$$=(\bar{M}\Psi_{\text{odd}}, U\Psi_{\text{even}})=-(\Psi_{\text{odd}}, U\Psi_{\text{even}}). \tag{12.5}$$

We can do exactly the same thing with the charge conjugation operator C:

$$\left.\begin{aligned} &C^2=1;\quad C\Psi_{\text{vac}}=\Psi_{\text{vac}};\\ &C\Psi_{\text{even}}{}^{\text{ch}}=\Psi_{\text{even}}{}^{\text{ch}};\quad C\Psi_{\text{odd}}{}^{\text{ch}}=-\Psi_{\text{odd}}{}^{\text{ch}};\\ &(\Psi_{\text{odd}}{}^{\text{ch}}, U\Psi_{\text{even}}{}^{\text{ch}})=0,\quad \text{since}\quad UC-CU=0. \end{aligned}\right\} \tag{12.6}$$

Since we have the same $(\mathbf{x},t)$ on the left and right sides of Eq. (3.14), the self-change-conjugate states must have vanishing expectation values of all "electromagnetic" quantities ($\rho_C=-1$). By a superposition of a state with even charge-parity and a state with odd charge-parity, we can build a state in which "electromagnetic" quantities have non vanishing expectation values.

To have a concise view of the entire problem and also to prepare a step toward the consideration of superselection rules, let us consider a unitary or Hermitian operator W which has eigenvalues W_i and corresponding eigenfunctions Ψ_i. Here

$$W\Psi_i=W_i\Psi_i, \tag{12.7}$$

where we use the same value of i insofar as W_i is the same. In other words, Ψ_i determines in general a subspace in the Hilbert space. Now, take any operator X that commutes with W:

$$XW=WX, \tag{12.8}$$

then we have

$$(\Psi_j,XW\Psi_i)=(\Psi_j,WX\Psi_i)=(\bar{W}\Psi_j,X\Psi_i),$$

from which follows

$$\left.\begin{aligned} &W_i(\Psi_j, X\Psi_i)=(1/W_j{}^*)(\Psi_j, X\Psi_i)\quad (W:\text{unitary})\\ &\text{or } W_i(\Psi_j, X\Psi_i)=W_j{}^*(\Psi_j, X\Psi_i).\quad (W:\text{Hermitian}) \end{aligned}\right\} \tag{12.9}$$

Putting first $X=1$, we get the well-known theorems that the eigenvalues of a unitary operator are of absolute value unity and those of a Hermitian operator

are real, and that in either case Ψ_i and Ψ_j are orthogonal if $W_i \neq W_j$:

$$(\Psi_i,\Psi_j)=0, \quad i\neq j. \tag{12.10}$$

Further putting $X=U$, we get

$$(\Psi_i,U\Psi_j)=0, \quad i\neq j \tag{12.11}$$

on condition that

$$UW=WU. \tag{12.12}$$

This leads to the "conservation law" of the eigenvalue. The selection rules with regard to space-parity and charge-parity can be obtained by putting respectively $W=M$ and $W=C$.

The concept of a superselection rule is more restringent than that of a selection rule. If there is a unitary or Hermitian operator W that commutes not only with U but also all physical quantities Q, then we speak of a superselection rule. Under this condition, we shall have

$$QW=WQ. \tag{12.13}$$

Putting $X=Q$ we get from Eq. (12.9),

$$(\Psi_i,Q\Psi_j)=0, \quad i\neq j. \tag{12.14}$$

If W commutes with all Q, then it will commute with the Hamiltonian and also with the transition matrix. Therefore, if Eq. (12.13) is the case, it is not necessary to require Eq. (12.12) separately. Equation (12.14) means that the expectation value of Q in a state

$$\Psi=\sum a_i\Psi_i \tag{12.15}$$

is given by

$$(\Psi,Q\Psi)=\sum_i |a_i^2|\,(\Psi_i,Q\Psi_i). \tag{12.16}$$

This shows that the phase of a_i is deprived of any physical meaning, i.e., not only the phase of the total Ψ given in Eq. (12.15) but the relative phases of individual terms belonging to different subspaces lose their meaning. In other words, the knowledge of a state expressed by Eq. (12.15) is nothing better than a mixed density $\mathcal{G}$ such that

$$\mathcal{G}=\sum |a_i^2|\,\mathcal{P}[\Psi_i], \tag{12.17}$$

where $\mathcal{P}[\Psi_i]$ is the projection operator onto the direction of Ψ_i. We have

$$\mathcal{P}[\Psi_i]\Psi_i=\Psi_i, \quad \mathcal{P}[\Psi_i]\Phi=0, \tag{12.18}$$

where Φ is any state-function orthogonal to Ψ_i. Thus we need not consider any "superposition" of two state-functions arising from two separate subspaces. See Eq. (12.7).

Furthermore, it should be noted that a state Ψ and $W\Psi$ are physically indistinguishable if W is unitary, i.e.,

$$(W\Psi,QW\Psi)=(\Psi,\bar{W}QW\Psi)=(\Psi,Q\Psi). \tag{12.19}$$

The simplest way to discover superselection rules is to search for gauge transformations under which all the known physical quantities remain invariant. We have seen in the body of this paper that the sign of the electromagnetic field and the $u^{(3)}$-field seem to have physical meaning. All the rest of the field strengths are open to unobservable phase-change. The physical quantities which refer separately to a spinor field ψ and a charged boson field v have the form

$$\psi^{\times}\psi \quad \text{and} \quad \bar{v}v, \tag{12.20}$$

insofar as their dependence on ψ and v is concerned. The electromagnetic interaction has a term like $\psi_e{}^{\times}\psi_e A$ or $\psi_P{}^{\times}\psi_P A$, where A stands for electromagnetic potential and the suffices e and P refer to (positive) electron and proton. But, as A is not changed by a gauge transformation with a constant phase, these quantities can be included in Eq. (11.20). The interaction Hamiltonians between nucleon fields, lepton fields, and boson fields are of the type

$$\psi_N{}^{\times}\psi_P\bar{v}, \quad \psi_N{}^{\times}\psi_P\psi_e{}^{\times}\psi_\nu. \tag{12.21}$$

See Secs. 10 and 11.

One of the simplest gauge transformations that leave Eqs. (11.20) and (11.21) invariant is

$$\psi\rightarrow e^{i\alpha}\psi, \quad \psi^{\times}\rightarrow e^{-i\alpha}\psi^{\times} \tag{12.22}$$

for all the spinors. It is very tempting to try the same thing with respect to v, i.e., to assume $v\rightarrow e^{i\beta}v$. However, in this case, we have to take into consideration a term like $\psi_N{}^{\times}\psi_P\bar{v}$. If $\bar{v}\rightarrow e^{-i\beta}\bar{v}$, then $\psi_N{}^{\times}\psi_P$ must go over to $e^{i\beta}\psi_N{}^{\times}\psi_P$. Furthermore, if $\psi_N{}^{\times}\psi_P\rightarrow e^{i\beta}\psi_N{}^{\times}\psi_P$, then from a quantity like $\psi_N{}^{\times}\psi_P\psi_e{}^{\times}\psi_\nu$, we conclude that $\psi_e{}^{\times}\psi_\nu$ should pass to $e^{-i\beta}\psi_e{}^{\times}\psi_\nu$. All this chain of transformations can be written as follows

$$\left.\begin{aligned} &v\rightarrow e^{i\beta}, \quad \bar{v}\rightarrow e^{-i\beta}\bar{v}, \\ &\psi_N\rightarrow e^{i\gamma-i\beta/2}\psi_N, \quad \psi_P\rightarrow e^{i\gamma+i\beta/2}\psi_P, \\ &\psi_\nu\rightarrow e^{i\delta-i\beta/2}\psi_\nu, \quad \psi_e\rightarrow e^{i\delta+i\beta/2}\psi_e, \end{aligned}\right\} \tag{12.23}$$

where β, γ, and δ are entirely arbitrary except that they are real. If we put $\beta=0$, and $\gamma=\delta$, we come back to Eq. (11.22). If we put $\gamma=\delta=\beta/2$, then we obtain the familiar gauge transformation in which all the positive fields are multiplied by $e^{i\beta}$, the negative fields by $e^{-i\beta}$, and the neutral fields unchanged. If we put $\gamma=\delta=0$, we get a transformation which can be interpreted as a rotation by angle β about the third axis of the isotopic spin. This last transformation is obtained by considering ψ in $\psi^{\times}\vec{\tau}\psi\vec{u}$ as a spinor in the three-dimensional space in which $\vec{\tau}$ plays the role of spin matrices and by applying Eq. (2.8) to $\vec{u}$ and Eq. (2.57) to ψ.

In (12.23) the wave fields are not yet separated into "positive" waves and "negative" waves, or into "particle" waves and "antiparticle" waves. To avoid

confusion in doing this separation, let us make some agreement about symbols: We use (+) and (−), to designate positive and negative charge or to designate "particle" and "antiparticle." Thus, (+) will include proton, positive electron, neutron, neutrino, and positive pion, while (−) will include negative proton, negative electron, antineutron, antineutrino, and negative pion. We shall use (o) to indicate neutral fermions. Furthermore, (h) and (l) will differentiate the heavy particle (nucleon) family from the light particle (lepton) family. With this convention about the symbols, we can rewrite (12.23) as follows:

$$\left.\begin{array}{c} v_\pm \to e^{\pm i\beta} v_\pm, \\ \psi_\pm{}^h \to e^{\pm i(\gamma+\beta/2)}\psi_\pm{}^h, \quad \psi_{0\pm}{}^h \to e^{\pm i(\gamma-\beta/2)}\psi_{0\pm}{}^h, \\ \psi_\pm{}^l \to e^{\pm i(\delta+\beta/2)}\psi_\pm{}^l, \quad \psi_{0\pm}{}^l \to e^{\pm i(\delta-\beta/2)}\psi_{0\pm}{}^l. \end{array}\right\} \quad (12.24)$$

Now we reinterpret these relations in the quantized field theory by expanding the v's and the ψ's in the absorption operators and regarding the transformations in (12.24) as a unitary transformation, e.g.,

$$W v_+ W^{-1} = e^{i\beta} v_+, \quad \text{etc.} \quad (12.25)$$

Such a W is very easily found[15] to be

$$W = \prod \exp[-i\beta(M_+ - M_-) - i(\gamma+\tfrac{1}{2}\beta)(N_+{}^h - N_-{}^h) - i(\gamma-\tfrac{1}{2}\beta)(N_{0+}{}^h - N_{0-}{}^h) - i(\delta+\tfrac{1}{2}\beta)(N_+{}^l - N_-{}^l) - i(\delta-\tfrac{1}{2}\beta)(N_{0+}{}^l - N_{0-}{}^l)], \quad (12.26)$$

where the N's and the M's stand for occupation number operators respectively for fermion eigenstates and boson eigenstates. The multiplication Π should extend over all the possible eigenstates.

To make the implication of Eq. (12.26) more transparent, let us break it down to three independent cases: (1) $\gamma=\delta=0$, (2) $\beta=\delta=0$, and (3) $\beta=\gamma=0$. In the first case we have

$$W = \prod (e^{-i\beta/2})^{2M_+ - 2M_- + N_+ - N_- - N_{0+} + N_{0-}}, \quad (12.27)$$

where the N's stand for both (h) and (l). The eigenvalues of W in Eq. (12.27) are $(e^{-i\beta/2})^r$, $r=0, \pm1, \pm2, \cdots$, and the selection rule engendered by this W means the conservation of r, i.e., conservation of

$$\sum (2M_+ - 2M_- + N_+ - N_- - N_{0+} + N_{0-}) = \text{constant}. \quad (12.28)$$

In the second case we have

$$W = \prod \exp[-i\gamma(N_+{}^h - N_-{}^h + N_{0+}{}^h - N_{0-}{}^h)] \quad (12.29)$$

from which follows the conservation law of

$$\sum (N_+{}^h - N_-{}^h + N_{0+}{}^h - N_{0-}{}^h) = \text{constant}. \quad (12.30)$$

In the third case, we have

$$W = \prod \exp[-i\delta(N_+{}^l - N_-{}^l + N_{0+}{}^l - N_{0-}{}^l)], \quad (12.31)$$

and

$$\sum (N_+{}^l - N_-{}^l + N_{0+}{}^l - N_{0-}{}^l) = \text{constant}. \quad (12.32)$$

It is to be noted that Eqs. (12.30) and (12.32) can exclude the undesired transitions in nucleon-pion interaction and nucleon-lepton interaction,[30] such as

$$N + P \to \pi^+, \quad N + P \to \nu + e^+, \quad N + P \to \bar{\nu} + e^+.$$

By adding the three conservation laws in Eqs. (12.28), (12.30), and (12.32), we get

$$M_+ - M_- + N_+ - N_- = \text{constant}, \quad (12.33)$$

which is nothing but the law of conservation of electric charge. This law is obtained by putting $\beta=2\gamma=2\delta$ in Eq. (12.26).

By adding Eqs. (12.30) and (12.32) we get

$$\sum_{(l,h)} \sum (N_+ - N_- + N_{0+} - N_{0-}) = \text{constant}. \quad (12.34)$$

In particular, according as the quantity (12.34) is even or odd, we shall have also

$$N = \sum_{(l,h)} \sum (N_+ + N_- + N_{0+} + N_{0-}) = \text{even or odd}. \quad (12.35)$$

This means that the parity of the total number of spinor particles conserves. This is the original superselection rule first introduced by W.W.W.[31] The generating unitary operator W can be obtained by putting $\beta=0$, $\gamma=\delta=\pi$ in Eq. (12.26). Then Eq. (12.26) becomes Δ as introduced in Eq. (2.80).

The W considered in Eq. (7.10) is obtained from Eq. (12.26) by putting $\beta=0$ and equating $\gamma(=\delta)$ to $2(\alpha-\beta)$ of Eq. (7.10). The transformation considered in connection with Eq. (5.12) is essentially equivalent to the first line of (12.23). If we allow the value of γ in Eq. (5.16) or in Eq. (7.4) to be arbitrary, then the square of mirage operator becomes in general a W.[15]

Since W as seen in Eq. (12.26) commutes with all the known physical quantities and the transition matrix U, we may consider the selection rules engendered by it are actually also superselection rules.

Furthermore W also commutes with the mirage operator M and satisfies $W = W^T = R^{-1}WR$. This means that a state function Ψ and its miraged state Ψ_M and its reversed state Ψ_R all belong to the same subspace defined by W. The only exception is the charge-conjugation operator C, which has obviously the effect on W of interchanging the positive particle numbers with the negative particle numbers (and "particle" numbers with "antiparticle" numbers). Therefore, whether we

[30] C. N. Yang and J. Tiomno, Phys. Rev. **79**, 495 (1950).
[31] Wick, Wightman, and Wigner, Phys. Rev. **88**, 101 (1952).

can call the law engendered by W a superselection rule or just a selection rule depends on whether or not we should exclude C from the definition of "physically observable quantities." We can of course make W commutable with the C by putting either $\beta=0$, $\gamma=0,\pi$ and $\delta=0,\pi$, or $\beta=\pi$, $\gamma=\pm\pi/2$, and $\delta=\pm\pi/2$, which include eight possible combinations including $W=1$.

In this connection, it should be noticed that a self-charge-conjugate state and an eigenstate with $M_+-M_-+N_+-N_-=0$, Eq. (12.33), are two entirely different concepts, although both seem to mean "electrically neutral." For example, a "superposition" of a state with one positive particle and a state with one negative particle can be a self-charge-conjugate state but cannot be an eigenstate of the generating operator of Eq. (12.33).

The arbitrariness of the phase factor, which we have exploited in the foregoing, is not the only arbitrariness in the matrix representations of the emission and absorption operators. The absorption operator of the jth spinor eigenstate, for instance, has the expression:

$$g_j=\sum_{i<j}(1-2N_i)\cdot f_j\cdot\prod_{k>j}\mathbf{1}_k \tag{12.36}$$

with

$$(1-2N_i)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}_i,\quad f_j=\begin{pmatrix}0&1\\0&0\end{pmatrix}_j,$$

$$\mathbf{1}_k=\begin{pmatrix}1&0\\0&1\end{pmatrix}_k. \tag{12.37}$$

For brevity we shall write,

$$g_j=\Delta_j f_j,\quad \Delta_j=\prod_{i<j}(1-2N_i). \tag{12.38}$$

If we extend i in Eq. (12.38) to all the eigenstates, Δ_j becomes Δ in Eq. (2.80)

Now, it is obvious that this representation has a basic arbitrariness with regard to the ordering or labeling of the eigenfunctions. In particular, it is of some interest to investigate the total inversion of the ordering. This amounts to putting $(1-2N_k)$ where we had unity matrices $\mathbf{1}_k$ in Eq. (12.36) and putting unity matrices $\mathbf{1}_i$ where we had $(1-2N_i)$ in Eq. (12.36). This can be done by simply multiplying Eq. (12.36) by the Δ in Eq. (2.80) from the left:

$$Yg_iY^{-1}=\Delta g_i=-g_i\Delta,\quad Y\bar{g}_iY^{-1}=\bar{g}_i\Delta=-\Delta\bar{g}_i. \tag{12.39}$$

By a simple calculation, we can discover the explicit expression of Y:

$$Y=Y^{-1}=\prod_i^{\infty}(1-N_i+\Delta_iN_i). \tag{12.40}$$

However, this unitary transformation, as such, does not commute with all the physical quantities. If we get a unitary transformation W such that

$$W\psi W^{-1}=\Delta\psi,\quad W\psi^{\times}W^{-1}=\psi^{\times}\Delta \tag{12.41}$$

this W will certainly commute with all the physical quantities. This is the type of transformation considered in Eq. (4.14). Equation (12.41) will mean, in terms of "particles" and "antiparticles," that

$$\left.\begin{aligned}Wg_{+i}W^{-1}&=\Delta g_{+i},\quad W\bar{g}_{+i}W^{-1}=\bar{g}_{+i}\Delta,\\ Wg_{-i}W^{-1}&=g_{-i}\Delta,\quad W\bar{g}_{-i}W^{-1}=\Delta\bar{g}_{-i}.\end{aligned}\right\} \tag{12.42}$$

This W can be given the explicit form

$$W=W^{-1}=\prod_{i,j}(1-N_{+i}+\Delta_{+i}N_{+i})\times(1-N_{-j}-\Delta_{-j}N_{-j}). \tag{12.43}$$

Again this W is the generator of a superselection rule, but the physical conclusion that can be drawn from Eq. (12.43) brings nothing new. If we have a state with N_+ "particles" and N_- "antiparticles," the eigenvalue of W, in Eq. (12.43), for such a state is

$$(-1)^{(N_+-N_-)(N_+-N_--1)/2}=\pm1. \tag{12.44}$$

The upper sign applies when N_+-N_- is 0 or 1, modulo 4, while the lower sign applies when N_+-N_- is 2 or 3, modulo 4. In any case, if (N_+-N_-) is a constant of motion, as we have stated in relation to Eq. (12.34), then Eq. (12.44) is automatically conserved.

To avoid a possible misunderstanding, it should be emphasized that the superselection rule engendered by Eq. (12.26) is just an explicit formulation of the more or less known invariance laws underlying the present field theory and that it cannot claim any unconditional validity beyond the current field theory. However, when one tries to formulate a new theory, it will be a wise policy to be clearly aware of the existing invariance laws in the present theory, and to determine which laws would be discarded and which ones should be withheld or generalized. For instance, the conservation law of (N_+-N_-) with regard to spinor particles is deeply rooted in the spinor field theory and will not be so easily abandoned. Indeed, if the physical quantities can be written bilinear in a spinor and an adjoint spinor and if the decomposition

$$\psi\rightarrow\psi_++\psi_-^{\times}K$$

can be made, this type of conservation law is inevitable. Furthermore, the parity law of spinor particles (W.W.W.'s law) is more fundamental. This law is equivalent to the postulate that physical quantities are expressed in even powers in spinors. The charge conservation law, on the other hand, cannot be discarded. Therefore, we can believe that a superselection rule of

the type in Eq. (12.26) or one similar to it will survive until the basic postulates of the field theory are altered.

13. ILLUSTRATIONS—DISINTEGRATION OF POSITRONIUM AND NEUTRAL PION

The purpose of this section is to show by examples how the general principles developed in the foregoing sections can be applied to concrete problems. Therefore, it is not intended to give any exhaustive study of the decay modes of the positronium and the neutral pion, but only to derive some of the simplest facts about these decay processes. Except for a few among them, the results mentioned in this section have previously been discovered by other authors by more or less similar methods.[32]

Let us first determine the charge parity π_C, space parity π_S and x-parity π_x of positronium state Ψ_{pstr}:

$$C\Psi_{pstr}=\pi_C\Psi_{pstr};\quad M\Psi_{pstr}=\pi_S\Psi_{pstr};$$
$$M_x\Psi_{pstr}=\pi_x\Psi_{pstr},\quad (13.1)$$

where C, M, and M_x are given by Eqs. (8.21), (8.26), and (8.34) using the angular momentum representation. The lowest positronium state Ψ_{pstr} may be a singlet state or a triplet state, i.e., we shall have an expansion of the type

$$\Psi_{pstr}=\sum a\bar{g}_+(\omega,\kappa,m,\rho)\bar{g}_-(\omega',\kappa',m',\rho')\Psi_{\text{vac}}\quad (13.2)$$

corresponding to the resultant total angular momentum zero or one.

However, for our purpose of determining the π's, we do not need to handle the general expression in Eq. (13.2). From Eqs. (8.21), (8.26), and (8.34), we see that C, M, and M_x only change the charge and the magnetic quantum number and leave unchanged the other quantum numbers. Therefore, in an expression like Eq. (13.2), we have only to pick up, as representative terms, a small group of terms such that the effect of C, M, and M_x is limited within this group. Since C and M must commute with the resultant angular momentum and its z-component of the compound system, the value of π_C and π_S determined with respect to a small group of representative terms must be the same as for the entire expression. The operation M_x is commutable with the total angular momentum but not with its z-component, therefore π_x will have a meaning only for a state for which the z-component of the total angular momentum is zero.

The simplest representative terms in Eq. (13.2) would be those which correspond to a positron and a negatron, both in the same S-state: $\omega=\omega'$, $\kappa=\kappa'=-1$, $\rho=\rho'=1$. Thus, the singlet state must include at least

[32] Among others, see C. N. Yang, Phys. Rev. **77**, 242 (1950); L. Wolfenstein and D. G. Ravenhall, Phys. Rev. **88**, 279 (1952); A. Pais and R. Yost, Phys. Rev. **87**, 871 (1952).

TABLE VII. The charge parity π_C and the space parity π_S and the x-parity π_x of the lowest positronium states.

	Ψ_{sing}	$\Psi_{\text{trip}}{}^{(1)}$	$\Psi_{\text{trip}}{}^{(0)}$	$\Psi_{\text{trip}}{}^{(-1)}$
π_C	+	−	−	−
π_S	−	−	−	−
π_x	−	×	+	×

the following representative terms:

$$\Psi_{\text{sing}}=(1/\sqrt{2})[\bar{g}_+(\omega,-1,\tfrac{1}{2},1)\bar{g}_-(\psi,-1,-\tfrac{1}{2},1)$$
$$-\bar{g}_+(\omega,-1,-\tfrac{1}{2},1)\bar{g}_-(\omega,-1,\tfrac{1}{2},1)]\Psi_{\text{vac}}.\quad (13.3)$$

The triplet state will have the representative terms:

$$\Psi_{\text{trip}}{}^{(1)}=\bar{g}_+(\omega,-1,\tfrac{1}{2},1)\bar{g}_-(\omega,-1,\tfrac{1}{2},1)\Psi_{\text{vac}},\quad (13.4)$$

$$\Psi_{\text{trip}}{}^{(0)}=(1/\sqrt{2})[\bar{g}_+(\omega,-1,\tfrac{1}{2},1)\bar{g}_-(\omega,-1,-\tfrac{1}{2},1)$$
$$+\bar{g}_+(\omega,-1,-\tfrac{1}{2},1)\bar{g}_-(\omega,-1,\tfrac{1}{2},1)]\Psi_{\text{vac}},\quad (13.5)$$

$$\Psi_{\text{trip}}{}^{(-1)}=\bar{g}_+(\omega,-1,-\tfrac{1}{2},1)\bar{g}_-(\omega,-1,-\tfrac{1}{2},1)\Psi_{\text{vac}}.\quad (13.6)$$

The superscript on Ψ_{trip} indicates the z-component of the resultant angular momentum.

By applying Eqs. (8.21), (8.26), and (8.34) to Eqs. (13.3), (13.4), (13.5), and (13.6), we can determine the π's in the sense of Eq. (13.1). The results are listed in Table VII. However, $\Psi_{\text{trip}}{}^{(1)}$ and $\Psi_{\text{trip}}{}^{(-1)}$ are not eigenstates of M_x; they are interchanged by application of M_x.

Remembering that the charge parity of a photon state is just the parity of the number of photons present, as in Eq. (6.32), we conclude that Ψ_{trip} cannot decay into an even number of photons and Ψ_{sing} cannot decay into an odd number of photons.

Next we shall study two-photon states more closely using the representation used in Sec. 6C. If we fix the coordinate system to the center of mass of the decaying positronium, the two photons will have equal and opposite momenta, say k_z and $-k_z$. We can make four independent states:

$$\left.\begin{aligned}
&\Psi_1=(1/\sqrt{2})[\bar{g}(k_z,1)\bar{g}(-k_z,-1)\\
&\qquad\qquad+\bar{g}(k_z,-1)\bar{g}(-k_z,+1)]\Psi_{\text{vac}},\\
&\Psi_2=(1/\sqrt{2})[\bar{g}(k_z,1)\bar{g}(-k_z,-1)\\
&\qquad\qquad-\bar{g}(k_z,-1)\bar{g}(-k_z,+1)]\Psi_{\text{vac}},\\
&\Psi_3=\bar{g}(k_z,1)\bar{g}(-k_z,1)\Psi_{\text{vac}};\\
&\Psi_4=\bar{g}(k_z,-1)\bar{g}(-k_z,-1)\Psi_{\text{vac}}.
\end{aligned}\right\}\quad (13.7)$$

The states Ψ_3 and Ψ_4 do not interest us very much, because the z-component of the angular momentum is ± 2, which is certainly too large for comparison with the lowest positronium states or with the neutral pion at rest.

TABLE VIII. The charge parity π_C, space parity π_S and x-parity π_x of two-photon states. Here "pol" stands for planes of polarization of two photons.

	Ψ_1	Ψ_2	Ψ_3	Ψ_4
π_C	+	+	+	+
π_S	+	−	+	+
π_x	+	−	×	×
pol	∥	⊥	×	×

The charge parity π_C, space parity π_S, and x-parity π_x are determined with the help of Eqs. (6.36) and (6.38). The results are listed in Table VIII.

Since π_S of positronium is -1, the only possible two-photon state into which it can decay is Ψ_2. Since π_C of any two-photon state is $+1$, it can only originate from Ψ_{sing}. Thus, the two-photon decay of positronium must be

$$\Psi_{\text{sing}} \rightarrow \Psi_2. \tag{13.8}$$

Before the fruitful application of charge-parity was introduced, they excluded $\Psi_{\text{trip}}{}^{(0)} \rightarrow \Psi_2$ by comparing π_x.

It is interesting to characterize Ψ_1 and Ψ_2, in Eq. (13.7), by the correlation of planes of linear polarization of the two involved photons. In Eq. (6.41), an operator $\Pi_x(k_z)$ was introduced, which measures the probability of a photon having momentum k_z and linear polarization in the x-direction. Then the operator,

$$\Pi_{11} \equiv \Pi_x(k_z)\Pi_x(-k_z) + \Pi_y(k_z)\Pi_y(-k_z), \tag{13.9}$$

will measure the probability of two photons having momenta k_z and $-k_z$ and having parallel linear polarization. In the same way, for the probability of two photons being observed as having perpendicular planes of linear polarization, we get an operator

$$\Pi_\perp \equiv \Pi_x(k_z)\Pi_y(-k_z) + \Pi_y(k_z)\Pi_x(-k_z). \tag{13.10}$$

By applying Eqs. (13.9) and (13.10) to Eq. (13.7), we obtain, with the help of Eq. (6.41),

$$\left.\begin{aligned} &\Pi_{11}\Psi_1 = \Psi_1, \quad \Pi_\perp \Psi_1 = 0, \\ &\Pi_{11}\Psi_2 = 0, \quad \Pi_\perp \Psi_2 = \Psi_2. \end{aligned}\right\} \tag{13.11}$$

Although, Ψ_3 and Ψ_4 are not eigenfunctions of Π_{11} and $\Pi_\perp$, we can make eigenfunctions by taking $\Psi_3+\Psi_4$ and $\Psi_3-\Psi_4$. This analysis shows that if we observe the correlation of linear polarization of two decay-photons of the positronium, they will turn out to be perpendicular to each other.[33]

It may be worth noting also that, in the angular momentum representation, a two-photon state can have odd parity only if the values of l of the two photons are of different parity. See Eq. (6.24).

Next let us briefly review the decay of the neutral pion into photons. We have seen in Sec. 10 that the charge parity and the "kind" of the neutral pion are the same as their source, and they are given in Table IV. The "intrinsic" parity of the neutral pion is the space-parity of the state in which a single particle of this neutral field is present and at rest ($\mathbf{k}=0$). This sign corresponds to the symbol $(\pm)'$ in Eq. (5.23) or (5.32), and the double sign $(\pm)$ in Eq. (5.16), and is determined by ρ_M in Eq. (10.10). The connection between the "kind" and the intrinsic parity is given in Table IX. This is nothing but a rearrangement of a portion of Table III, in Part I. In the case of a vector, the space-component determines the parity.

TABLE IX. Connection between the "kind" and the intrinsic parity.

Kind	Scalar reg.	Scalar 2nd, 1st, 3rd	Vector reg., 2nd	Vector 1st, 3rd
Intrinsic parity	+	−	−	+

According to Sec. 10, there are actually six cases of $u^{(3)}$-field:

(1) scalar having (S)-interaction,
(2) scalar having (V)-interaction,
(3) vector having (V)-and (T)-interaction,
(4) pseudoscalar having (PS)-and (PV)-interaction,
(5) pseudovector having (PV)-interaction,
(6) pseudovector having (PT)-interaction.

The mixing of (1) and (2), and the mixing of (5) and (6) are forbidden, because of charge-symmetry and reversibility. Table X gives the intrinsic parity and charge parity of these six kinds of neutral pions, as can be derived from Tables IV and IX.

TABLE X. The charge parity π_C and the intrinsic space-parity π_S of a neutral pion classified according to its source.

	(1) $S(S)$	(2) $S(V)$	(3) $V(V,T)$	(4) PS (PS,PV)	(5) $PV(PV)$	(6) $PV(PT)$
π_C	+	−	−	+	+	−
Kind	reg.	2	2	1	1	3
π_S	+	+	−	−	+	+

If experiment shows that the neutral pion decays into two photons, it means that $\pi_C=+1$, and the possible type of pion is limited to (1), (4), (5).[34] If further

[33] J. A. Wheeler, Ann. N. Y. Acad. Sci. **48**, 219 (1946).

[34] More generally, we can see from Table X that a neutral pion of type (1), (4) or (5) cannot decay into an odd number of photons, and a neutral pion of type (2), (3) or (6) cannot decay into an even number of photons. This law is known as Fukuda-Miyamoto's rule. H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. **4**, 347 (1949). K. Nishijima, Prog. Theoret. Phys. **6**, 614 (1951). Since the charge parity of a photon is odd, the parity of the number of photons cannot change if photons are the only kind of particles present in the initial and final states, no matter what the intermediate process may be. This is Furry's rule. W. H. Furry, Phys. Rev. **51**, 135 (1937).

experiment shows that the planes of polarization of two photons are always perpendicular, it means $\pi_S = -1$, and the only possibility left is (4).

The author would like to thank all those who discussed with him one or the other of the subject matters dealt with in this work, thus indirectly helping him to prepare this survey article. In particular, he is grateful to Professor E. P. Wigner for many enlightening discussions. Apologies are extended to the authors whose contributions to the field of problems treated in this paper have not been quoted at the appropriate places.[35]

[35] Among many others: E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947); W. Pauli, lecture notes, Summer School of Theoretical Physics, Les Houches, 1952; F. Coester, Phys. Rev. **84**, 1259 (1951); F. Coester, Phys. Rev. **89**, 619 (1953); L. Michel, Nuovo Cimento, **10**, 320 (1953); G. Lüders, Kongel. Danske Vidensk. Selsk. Mat. fys. Medd. **28**, 1 (1954); Umezawa, Kamefuchi, and Tanaka, Soryushi-Ron-Kenkyu (in Japanese) **6**, 543, 1954; the same authors, Prog. Theoret. Phys. (to be printed); Y. Katayama, Soryushi-Ron-Kenkyu (in Japanese) **6**, 543 (1954); E. R. Caianiello, Physica **18**, 1020 (1952); and other papers of the same author; S. Oneda, Prog. Theoret. Phys. **9**, 327 (1953).

REVIEWS OF MODERN PHYSICS VOLUME 27, NUMBER 1 JANUARY, 1955

Erratum: The Energy Levels and the Structure of Light Nuclei

D. R. INGLIS
Argonne National Laboratory, Lemont, Illinois
[Revs. Modern Phys. **25**, 390 (1953)]

Figure 9 is in error in having the low ^2P plotted four units too low and is to be replaced by the accompanying figure. This removes what might have been considered a reason for doubting recent experimental suggestions[1] that there may be excited states of Be^9 near 1.8 and 3.1 Mev, indicated by dotted lines in the inserts of the figure, in addition to the well-established 2.43-Mev state recently observed[2] to have $J \geqslant \frac{3}{2}$ as now appears consistent with this rough exploratory treatment of the theory. The error was discovered and very kindly pointed out by Edith Halbert and Sudhir Pandya of the University of Rochester, who are calculating the intermediate-coupling transition of the lowest states of several (J,T).

[1] Kunz, Moak, and Good, Phys. Rev. **95**, 640 (1954); F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 91 (1955) which quotes K. W. Allen on $B^{10}(t,\alpha)Be^9$.

[2] F. L. Ribe and J. D. Seagrave, Phys. Rev. **94**, 934 (1954).

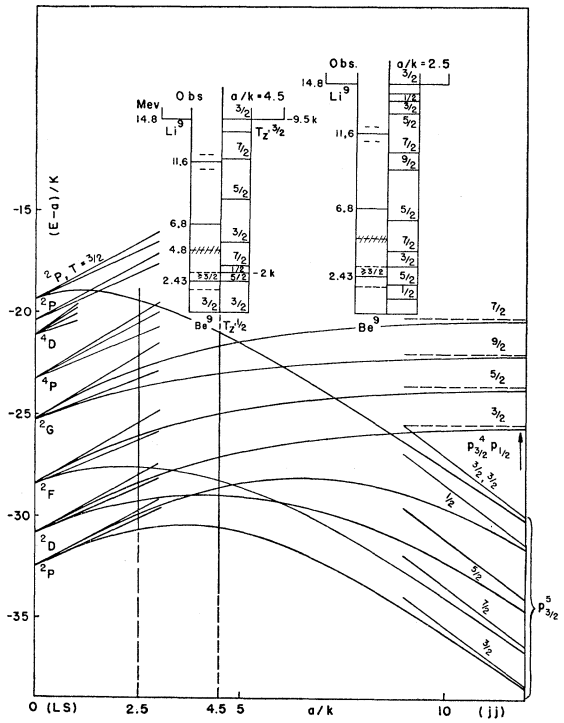

FIG. 9. Intermediate-coupling transition for the configuration p^5 applying to Py^9. The isobaric spin is $T=\frac{1}{2}$ except where indicated as $T=\frac{3}{2}$.