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Introduction to Some Recent Work in Meson Theory

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INTRODUCTION

THE aim of this article is to present a unified and relatively simple discussion of meson theory, which takes into account some of the most recent and promising developments, without requiring from the reader any previous knowledge of special field theoretic methods. Familiarity with ordinary nonrelativistic quantum mechanics (and its application to scattering), and with the main experimental facts of meson physics,[1] should suffice to read this article. Some field theoretic ideas and methods, such as the Feynman-Dyson graphs and renormalization, will be explained as we go along, in the very limited and simple form, in which they occur here. On the other hand, I have relegated to Appendix A such more familiar topics as charge independence and isotopic spin. An easy access to these things is already provided by current textbooks.[2]

My aim could only be achieved at the cost of a drastic restriction in the subject matter. This is no review of the present status of the theory, and the picture I shall present is deliberately one-sided: I have selected ideas and results according as they fitted into the chosen pattern. For example a discussion of completely relativistic schemes was out of the question from the start; but even such more closely related topics as the important "strong" and "intermediate" couplings[3] approximations were omitted in favor of a fuller presentation of the main subject.

To put it briefly, then, the form of the theory developed here is the so-called symmetric pseudoscalar theory with the Yukawa (i.e. linear) coupling of the meson field to infinitely heavy nucleons. This implies a form of the Hamiltonian [see Eqs. (1.10) and (1.14)], that was already extensively studied,[4] among several alternative forms, in the early 1940's. The detailed experimental information available today allows us to discard many of the alternative forms on mere inspection. They assume either the wrong spin, or the wrong parity or the wrong number of charge states for the meson. Some of these theories, of course, still have an interest for the specialist; they provide him with guinea pigs,[5] on which he can try out mathematical tricks, etc. From general pedagogical considerations I felt, however, that it was better in this report to stick to the one theory which has at least a fighting chance at the beginning.

I am happy to acknowledge here my indebtedness to the work of G. F. Chew.[6] Thanks to Chew's efforts, the Yukawa theory has at last achieved some contact with the quantitative aspects of meson physics. I am well aware that a wide range of opinions is possible, as to how deep that contact goes. For the purposes of this article, however, Chew's work also contained another useful lesson. It showed, namely, that it was possible to apply to this theory ideas and methods derived from modern field theoretic work, without getting involved into too abstruse formalisms. In order to achieve

[1] For a recent review see M. Gell-Mann and K. M. Watson, Ann. Rev. Nuclear Sci. 4 (1954); also Henley, Ruderman, and Steinberger, Ann. Rev. Nuclear Sci. 3 (1953).

[2] See for example J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., 1952), Chap. 3; G. Wentzel, *Quantum Theory of Wave Fields* (Interscience Publishers, Inc., 1949), Chap. 3; H. A. Bethe and F. de Hoffman, *Mesons* (Row Peterson, 1955).

[3] See G. Wentzel, Revs. Modern Phys. **19**, 1 (1947); for more recent literature see T. D. Lee and R. Christian, Phys. Rev. **94**, 1760 (1954); the "weak-coupling" viewpoint is discussed amongst other things in R. E. Marshak's *Meson Physics* (McGraw-Hill Book Company, Inc., 1952).

[4] See W. Pauli, *Meson Theory of Nuclear Forces* (Interscience Publishers, Inc., 1946).

[5] An especially simple and instructive example has been expressly constructed for this purpose by T. D. Lee [Phys. Rev. **95**, 1329 (1954)]; see also, G. Källen and W. Pauli, Dan. Math. Fys. Medd. (to be published).

[6] G. F. Chew, Phys. Rev. **94**, 1748, 1755 (1954); Phys. Rev. **95**, 1669 (1954).

my purpose, therefore, I only had to carry the process a little further, and give an elementary proof of some statements which in Chew's paper are simply taken over from Dyson's well-known, but rather difficult, discussion of the renormalization of the S-matrix.

The key to this proof is Eq. (5.39) which I derived for this purpose. The same equation, it turned out, had been obtained a little earlier by Low from a less-elementary argument. Low, however, built on this equation a new and promising approach[7] to the scattering problem which has been further developed by Chew and Low.[8] I have been able to give an account of this more recent work, insofar as it was available to me from communications at the 1955 Rochester Conference.

It is mostly because of all this recent work being in the process of publication, that I have given up any attempt to include a detailed comparison between experiment and the results of the theory. I am afraid many readers will find that the theory as presented here sort of stops in mid-air. I felt, however, that to present in detail the calculations based on the older approach of Chew, now being rapidly superseded, would not have made much sense. As to the newer results, it was better to leave the word to the original authors.

I feel, however, that I shall have attained my purpose if this article were accepted as a useful introduction to the study of some of these new and interesting developments.

1. BASIC ASSUMPTIONS: HAMILTONIAN

The mesons we are interested in here are the π mesons or pions. We shall refer the reader to some excellent summaries already quoted,[1] for the experimental evidence indicating that the spin of a pion is zero, and that an "intrinsic" parity change is associated with the creation (destruction) of a meson. By that one means that the parity of a state containing one meson with the orbital angular momentum l, when compared with the parity of the state with no meson (all other things being the same), is *not*, as one might expect, $(-1)^l$ but rather $(-1)^{l+1}$, the additional factor -1 being the "intrinsic" parity of the meson.

Free mesons are then described by the Klein-Gordon equation; the possible states of a meson are given by the eigensolutions of the wave equation

$$\Delta u+k^2u=0 \tag{1.1}$$

the energy ω of the meson being given by

$$\omega=\omega_k=(\mu^2+k^2)^{\frac{1}{2}}, \tag{1.2}$$

where μ is the meson mass. (We use units $\hbar=1$, $c=1$.) We shall assume for simplicity a finite quantization volume. For most purposes it is convenient to assume a cubic volume with periodic boundary condition so that the solutions of (1.1) are of the form

$$u_{\mathbf{k}}=Ne^{i\mathbf{k}\cdot\mathbf{x}} \tag{1.3}$$

where $N=(\text{quantization volume})^{-\frac{1}{2}}$ is a normalization factor. The number of eigenstates in the volume element $d\mathbf{k}$ of momentum space is then

$$N^{-2}(2\pi)^{-3}d\mathbf{k}. \tag{1.4}$$

The orthogonality condition has the form

$$\int u_{\mathbf{k}}^*(x)u_{\mathbf{k}'}(\mathbf{x})d\mathbf{x}=\delta_{\mathbf{k}\mathbf{k}'} \tag{1.5}$$

where $\delta_{\mathbf{k}\mathbf{k}'}$ is the Kronecker symbol, $=1$ if $\mathbf{k}=\mathbf{k}'$ and $=0$ otherwise. The essential properties of this symbol can also be expressed by the rule

$$\textstyle\sum_{\mathbf{k}}f(\mathbf{k})\delta_{\mathbf{k}\mathbf{k}'}=f(\mathbf{k}'). \tag{1.6}$$

When $f(\mathbf{k})$ is a continuous function, the sum in (1.6) may be replaced by an integral, according to (1.4) provided $\delta_{\mathbf{k}\mathbf{k}'}$ is replaced by a Dirac delta function according to the prescription

$$\delta_{kk'}\rightarrow N^2(2\pi)^3\delta(\mathbf{k}-\mathbf{k}'). \tag{1.7}$$

In order to specify completely the state of a meson it is necessary to give, besides $\mathbf{k}$, also an index λ, which has three possible values, corresponding to the three possibilities $\pi^{\pm}$ and π^0. We shall use a lightface index k to summarize $\mathbf{k}$ and λ. Thus $\sum_k$ means $\sum_\lambda$ combined with the integration (1.4).

In addition to mesons, we may have one or more nucleons, photons, etc. For the moment let us consider only nucleons. We shall treat the nucleons as infinitely heavy and at rest; thus we shall not need any variables (say, momenta) to describe their transitional motion. We shall, however, need variables to describe their spin and charge states (see later).

The theory we are considering takes over from the original Yukawa theory the assumption that mesons are emitted or absorbed by nucleons singly, like photons by electrons.[9] This refers of course to the elementary acts implied by the Hamiltonian: that is the Hamiltonian has matrix elements only between states differing by the emission or absorption of one meson.

More precisely the Hamiltonian consists of two parts

$$H=H_0+\mathcal{H}, \tag{1.8}$$

where H_0, the unperturbed Hamiltonian, is simply the sum of the kinetic and rest energy of the mesons present

$$H_0=\textstyle\sum_k n_k\omega_k \tag{1.9}$$

where n_k is the number of mesons in the state $k=(\mathbf{k},\lambda)$ and ω_k is given by (1.2). We do not have to include in H_0 a term for the nucleon or nucleons; since we

[7] F. Low, Phys. Rev. **97**, 1392 (1955). I wish to thank Dr. Low for advance communication of his results and for an illuminating conversation.

[8] G. F. Chew and F. Low (to be published).

[9] A simple and lucid introduction to these concepts is provided by E. Fermi's beautiful booklet *Elementary particles* (Yale University Press, New Haven, 1951).

neglect recoil as well as the neutron-proton mass difference, this term would be a constant of the motion and can be ignored. We neglect the mass difference between charged and neutral mesons, so that $\omega_k=\omega(\mathbf{k})$ independent of λ. Since pions have spin zero, we assume that they obey Bose statistics, so that n can take any value 0, 1, 2, ... As to $\mathcal{H}$, the "interaction" Hamiltonian, it consists of a sum of terms, one for each nucleon present. The number of nucleons is a constant of the motion; if there are no nucleons $\mathcal{H}=0$ and the mesons are completely free. In this respect the theory is fundamentally different from the relativistic theories, in which the possibility of the creation of nucleon-antinucleon pairs leads to some interaction in all cases.

We then need to write $\mathcal{H}$ only for the case of a single nucleon; if there are more nucleons, each will be represented by a similar term. We may also assume for simplicity that the nucleon is placed at the origin. Let us introduce absorption operators a_k and creation operators $a_k{}^*$ defined in the usual way for bosons.[9] That is, the matrix element of a_k is equal to one when it is taken between a state with no meson in the state k and a state with one meson in the state k (all other occupation numbers being the same for the two states). More generally if the state at the right contains n_k mesons in the state k, and the state at the left contains $n_k{}'=n_k=-1$ mesons in k (all other occupation numbers being unchanged) the matrix element of a_k is equal to $nk^{\frac{1}{2}}$. This has the effect that the probability of absorption from a state containing n mesons is proportional to n, as one might expect. If $n_k-n_k{}'\neq 1$ or if $n_p-n_p{}'\neq 0$ for any of the other states p, then the matrix element of a_k is zero. The creation operator $a_k{}^*$ is simply the Hermitean conjugate of a_k.

The assumption that mesons are absorbed or emitted singly is then simply expressed by saying that $\mathcal{H}$ is linear in the a_k's and $a_k{}^*$'s. That is

$$\mathcal{H}=\sum_k(a_kV_k+a_k{}^*V_k{}^*). \tag{1.10}$$

In determining further the form of V_k and $V_k{}^*$ one makes use of the additional requirements of charge conservation, charge independence (see Appendix A) and conservation of angular momentum. It will be most simple to write down at once the form of V_k, and justify it piece by piece afterwards. One has

$$V_k=N(4\pi)^{\frac{1}{2}}(f/\mu)\tau_\lambda i\mathbf{k}\cdot\boldsymbol{\sigma}v(k)(2\omega_k)^{-\frac{1}{2}}. \tag{1.11}$$

Beginning from the left we have: $(4\pi)^{\frac{1}{2}}f$ is a proportionality constant which characterizes the strength of the interaction; the $(4\pi)^{\frac{1}{2}}$ is sometimes omitted, the value of f must then be correspondingly larger and is then called a "rationalized" coupling constant. The μ^{-1} is introduced to make f dimensionless. To make the absorption probability proportional to the meson *density* (i.e., inversely proportional to the quantization volume) N is necessary. Next come the most significant parts of the expression. We have so far only spoken of the changes that occur in the meson occupation numbers. Changes must occur, however, also in the nucleon. Owing to charge conservation, for example, a π^+ can only be absorbed by a neutron becoming a proton, and a π^- by a proton becoming a neutron; a π^0 can be absorbed by either one, without change. This may be described by treating neutron and proton as two different states of the same particle, the nucleon, characterized by an "isotopic spin" variable, $=+\frac{1}{2}$ for the proton and $-\frac{1}{2}$ for the neutron. Then all we have to do is to insert in V_k a factor τ_λ representing an operator on the isotopic spin variable which has nonzero matrix elements only for the appropriate transitions. The precise form of these operators is discussed in Appendix A, under the more stringent assumption of charge independence.

Just as τ_λ represents the possibility of changes in the charge of the nucleon, accompanying the emission or absorption of a meson, in a similar way the next term $i\mathbf{k}\cdot\boldsymbol{\sigma}$ takes into account the possibility of ordinary spin changes. We may notice here that the most general form of an operator acting on the spin variable is $a_x\sigma_x+a_y\sigma_y+a_z\sigma_z$, where on grounds of invariance $a_xa_\lambda a_z$ must transform under rotations like the components of a vector. The only vector available, however, is $\mathbf{k}$ the momentum of the meson, hence the form $\mathbf{k}\cdot\boldsymbol{\sigma}$ in (1.11). One must further point out that since $\mathbf{k}$ is a polar but $\boldsymbol{\sigma}$ an axial vector, their product is a pseudoscalar. Thus under an inversion at the origin V_k is not invariant, but instead changes sign. This, however, is just as it should be on the assumption that the meson has negative intrinsic parity. This remark also explains why $\mathbf{k}\cdot\boldsymbol{\sigma}$ cannot be accompanied by a spin-independent term, in the form $A+\mathbf{k}\cdot\boldsymbol{\sigma}$ where A is a scalar quantity. The matrix element cannot be part scalar, part pseudoscalar; it has to be one or the other, depending on what "intrinsic" parity is assumed for the meson.

Finally (1.11) contains a factor $v(k)(2\omega_k)^{-\frac{1}{2}}$ which depends only on the magnitude of k, not on the direction. About the precise form of this factor it is not necessary to make any assumption, except that $v(k)\rightarrow 0$ when k becomes very large. If this assumption is not made, virtual emission and reabsorption of mesons of large momentum gives rise to divergent expressions, much in the same way as virtual emission and reabsorption of quanta gives rise to divergencies in quantum electrodynamics.

The reason why the last factor in (1.11) is written in the composite form $v(k)(2\omega)^{-\frac{1}{2}}$ rather than simply $v(k)$ is that (1.11) is often obtained from field theory as follows. Writing explicitly $\lambda\mathbf{k}$ for k, $a_{\lambda\mathbf{k}}$ for a_k we consider the expression (see below)

$$\phi_\lambda(\mathbf{x})=N\sum_k(2\omega_k)^{-\frac{1}{2}}\{a_{\lambda\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}+a_{\lambda\mathbf{k}}{}^*e^{-i\mathbf{k}\mathbf{x}}\} \tag{1.12}$$

and write $v(k)$ as the Fourier transform of a function $\rho(x)$

$$v(k)=\int e^{i\mathbf{k}.\mathbf{x}}\rho(\mathbf{x})d\mathbf{x}. \tag{1.13}$$

This does not involve any really restrictive assumption about $v(k)$. Besides, since v depends only on $k=|\mathbf{k}|$, ρ will similarly depend only on $r=|\mathbf{x}|$. With (1.12) and (1.13) it is now possible to write (1.11) as follows:

$$\mathcal{H}=(4\pi)^{\frac{1}{2}}\frac{f}{\mu}\sum_\lambda \tau_\lambda \int \rho(\mathbf{x})\boldsymbol{\sigma}\cdot\nabla\phi_\lambda(\mathbf{x})d\mathbf{x} \qquad (1.14)$$

which is the interaction suggested by field theoretic considerations. In this case $\rho(\mathbf{x})$ plays the role of "extended source distribution" for the nucleon.

In field theory, (1.12) is just the meson field operator; the most obvious assumption would be that the interaction involves only the value of ϕ, or respectively $\nabla\phi$ exactly *at* the position of the nucleon. That is like saying that the nucleon is point-like, $\rho(x)$ is the delta function $\delta(\mathbf{x})$ if the nucleon is at the origin; this, however, leads to $v(k)=1$ for all values of k, which leads to divergencies. It is necessary to attribute a finite extension to the nucleon in order to get a convergent theory.

Two more remarks: if the nucleon is not located at the origin, but say at x_n, then Eq. (14) suggests immediately that all we have to do is to replace $\rho(x)$ by $\rho(x-x_n)$. The effect of this is that the value of V_k, Eq. (1.11) is multiplied by $\exp(i\mathbf{k}\cdot\mathbf{x}_n)$, the amplitude of the meson wave function at the nucleon (as one might expect). Such factors become important when one has more than one nucleon, as in the nuclear force problem.

Secondly, we must notice that (1.11) implies that mesons are absorbed or emitted in p-states only.[10] The transformation which exhibits this fact explicitly is often useful, and we shall therefore mention it here. Instead of using for the free mesons the plane-wave states Eq. (1.3), one can introduce states of definite angular momentum

$$v_{klm}(\mathbf{x})=N'j_l(kr)Y_{lm}(\mathbf{x}) \qquad (1.15)$$

where j_l is the "spherical Bessel function" of order l which behaves asymptotically as $(kr)^{-1}\sin(kr-\frac{1}{2}l\pi)$ and Y_{lm} is a normalized spherical harmonic depending only on the *direction* of $\mathbf{x}(\int|Y|^2d\Omega=1)$. Finally N' is a normalization factor which like N occurs also in the expression for the number of states in the interval dk

$$(N')^{-2}(2/\pi)k^2dk. \qquad (1.16)$$

It is clear that the transformation from plane to spherical waves only makes sense in the limit of infinite volume (spherical waves require a spherical box), so we shall carry out the calculation in this limit. Now V_k, Eq. (1.11), represents the matrix element for the absorption of a meson from a state (1.3) and according to the general rules of transformation theory, in order to calculate the corresponding matrix element for a state v_{plm}, Eq. (1.15), we must multiply (1.11) by the scalar product

$$(u_\mathbf{k},v_{plm})=\int u_\mathbf{k}^*(\mathbf{x})v_{plm}(\mathbf{x})d\mathbf{x} \qquad (1.17)$$

and sum over k. The integral (1.17) can be easily carried out in polar coordinates by means of the well-known expansion of a plane wave in spherical harmonics; for the reason just mentioned, the integration over $r=|\mathbf{x}|$ must extend from 0 to infinity. One easily finds

$$(u_\mathbf{k},v_{plm})=2NN'(-i)^lY_{lm}(\mathbf{k})(\pi/k)^2\delta(p-k). \qquad (1.18)$$

After multiplying (1.11) by (1.18) we carry out the sum over k in the limit of infinite volume, that is by means of (1.4), whereupon N cancels out of the results, as of course it should. The absorption operator V_{plm} (which replaces V_k) is then seen to depend on the integral of $(\boldsymbol{\sigma}\cdot\mathbf{k})Y_{lm}(\mathbf{k})$ over the direction of $\mathbf{k}$, which is zero if $l\neq 1$.

Having to do with $l=1$ only we may simplify the notation by dropping l from the indices of v_{plm} altogether. Furthermore, we may choose as basic spherical harmonics $Y_{lm}(\mathbf{k})$ simply the components of $\mathbf{k}$

$$Y_{lm}(\mathbf{k})\to(3/4\pi)^{\frac{1}{2}}k^{-1}k_i \quad (i=1,2,3). \qquad (1.19)$$

The function v_{klm} may be designated by v_{ki} and the absorption operator by $V_{ki\lambda}$ (where k now is the magnitude of the momentum and the indices i and λ both run from 1 to 3. Finally one finds

$$V_{ki\lambda}=N'3^{-\frac{1}{2}}(f/\mu)kv(k)(2\omega)^{-\frac{1}{2}}\sigma_i\tau_\lambda. \qquad (1.20)$$

In this way one has the advantage that only p-states for the mesons appear in the calculation; owing to the similarity of σ and τ matrices one sees furthermore that the isotopic index λ and the angular momentum index i play completely similar and interchangeable roles in the calculation.[11]

2. THE "REAL NUCLEON" STATES

All physical problems we want to discuss next can be reduced to a study of the eigenstates of the Hamiltonian (1.8). For example, all the information on meson scattering is contained in certain eigenstates described in Sec. 5.

We shall be concerned, in this section, with the lowest eigenstate of the Hamiltonian (1.8), which, if the theory has anything to do with reality, must represent a free nucleon. To be sure, this nucleon will be surrounded by

[10] This is, of course, an immediate consequence of angular momentum and parity conservation in the elementary act. A nucleon has angular momentum $\frac{1}{2}$; after a meson of orbital momentum l is emitted the total angular momentum of the system can only take the two values $l\pm\frac{1}{2}$, and this can only be equal to $\frac{1}{2}$ for either $l=0$ or $l=1$. That is, this excludes emission into other than s of p states. The final decision rests with parity. If the meson were a particle of + intrinsic parity it would be emitted in an s state, the opposite assumption leads to p-wave emission.

[11] As remarked by F. H. Harlow and B. A. Jacobsohn, Phys. Rev. 93, 333 (1954). This immediately leads to identities such as $\delta_{31}=\delta_{13}$. See Sec. 5.

a virtual meson cloud, but in the state in question there should be no real (incident or outgoing) mesons.

One possible aim of studying this eigenstate might be, of course, the calculation of the level shift due to the perturbation (1.10), i.e., the so-called "nucleon self-energy," which in a cut-off theory is a finite and well-defined quantity. So far nobody has dared suggest, however, that the self-energy effect due to (1.10) is connected to any observable mass or mass difference. In the work of Chew, as well as in the more sophisticated relativistic field theories, one manipulates the formulas in such a way that the self-energy (and other quantities, which are similarly regarded as unobservable) do not appear in the physically significant results. This gives rise to the following paradoxical situation: While the self-energy is an unobservable quantity which one does not have to evaluate, it is nevertheless necessary to go into a detailed formal discussion of it, as a preliminary to its elimination from the results.

This detailed discussion will be carried out presently, and will also give us an opportunity to prove certain relations, which are useful in the elimination of other unobservable quantities from the theory.

As explained in the Appendix, the Hamiltonian (1.8) allows four "good" quantum numbers J, J_z, T, and T_3. A state with $J=\frac{1}{2}$ and $T=\frac{1}{2}$ is fourfold degenerate, corresponding to $J_z=\pm\frac{1}{2}$ and $T_3=\pm\frac{1}{2}$. The nucleon is, of course, assumed to be in such a state; this leads to the assignment of total angular momentum $J=\frac{1}{2}$, and total isotopic spin $T=\frac{1}{2}$ to the lowest level of the Hamiltonian.

One assumes that as f is decreased continuously from its actual value to $f=0$, the aforementioned fourfold degenerate level remains the lowest level of the system. That is why the lowest level of H_0, i.e., the "bare nucleon" is also assumed to have the same fourfold degeneracy. We shall use a four-valued index α as an abbreviation for a pair of values (J_z, T_3) to designate the four possible nucleon states, and we can obviously use the same index for the corresponding bare nucleon states.

As we already see on this example we shall have to do with two types of states: the eigenstates $\phi_1, \phi_2, \cdots, \phi_n, \cdots$ of the unperturbed Hamiltonian H_0, and the eigenstates of the exact Hamiltonian H, for which we shall employ Dirac bras and kets. In particular we shall use the notation ϕ_α with a Greek index ($\alpha=1, \cdots, 4$) for the four bare-nucleon states, and $|\alpha>$ for the corresponding real nucleon states.

Our somewhat hybrid notation has the advantage of making the two kinds of states more clearly distinguishable.[12] Matrix elements of the type $<\beta|\cdots|\alpha>$ will be called "nucleon expectation values"; they play a similar role as "vacuum-expectation values" do in other work. The corresponding bare-nucleon expectation values will be written: $\phi_\beta^*\cdots\phi_\alpha$ [see for example Eq. (2.3)].

Starting from ϕ_α one can construct all other unperturbed states ϕ_n containing one and only one nucleon, by applying to ϕ_α one, two, $\cdots$ meson creation operators, thus obtaining the complete orthogonal set

$$\begin{aligned} &\phi_\alpha && \text{(no mesons),} \\ &\phi_{k\alpha}=a_k^*\phi_\alpha && \text{(one meson)} \\ &\phi_{kq\alpha}=a_k^*a_q^*\phi_\alpha && \text{(two mesons).} \end{aligned} \tag{2.1}$$

These states satisfy the unperturbed Schrödinger equations

$$H_0\phi_\alpha=0, \tag{2.2a}$$

$$H_0\phi_{k\alpha}=\omega_k\phi_{k\alpha}, \tag{2.2b}$$

$$\cdots\cdots \text{ etc.}$$

and are normalized to unity.[13]

We shall designate with $\mathcal{H}_{nm}$ the matrix element of the perturbation operator between any two states ϕ_n and ϕ_m of the sequence (2.2)

$$\mathcal{H}_{nm}=\phi_n^*\mathcal{H}\phi_m. \tag{2.3}$$

We notice in particular that (2.3) is zero if both ϕ_n and ϕ_m are bare-nucleon states

$$\mathcal{H}_{\alpha\beta}=0. \tag{2.3$'$}$$

We are now in a position to discuss the application of perturbation methods to the complete Schrödinger equation

$$(H_0+\mathcal{H})|>=E|> \tag{2.4}$$

where in particular we designate with E_s the eigenvalue for a state $|\alpha>$. Comparing with (2.2a) we see that E_s is at the same time the level shift, i.e., the nucleon self-energy. The state $|\alpha>$ can be expanded in the orthogonal system (2.1), the coefficients being the so-called zero-meson, one-meson, $\cdots$ amplitudes. The zero-meson (or bare-nucleon) part of the expansion must have the same J_z and T_3, i.e., the same Greek index, as $|\alpha>$, hence we can write

$$|\alpha>=c_0\phi_\alpha+\textstyle\sum'_n c_n\phi_n \tag{2.5}$$

where $\sum'$ (here and henceforward as well) is used to denote a summation *excluding the bare-nucleon states*. One can, as usual, rewrite (2.4) for the state (2.5) as an infinite system of linear equations in the amplitudes $c_0, \cdots, c_n \cdots$. The first of these equations is for example

$$Ec_0=\textstyle\sum'_n\mathcal{H}_{\alpha n}c_n, \tag{2.4a}$$

where E is now, of course, just E_s. The remaining set of equations, which the reader can easily write down for himself, can be divided by c_0 and solved for the

[12] This notation becomes rather clumsy if one has to consider mixed scalar products between the two types of states, as in Eq. (2.17). However, we shall usually avoid this.

[13] This requires the addition of numerical factors on the right-hand side of (2.1) when some of the indices $k, q \cdots$ are equal. E.g.

$$\phi_{kk\alpha}=(1/\sqrt{2})a_k^*a_k^*\phi_\alpha, \cdots, \text{etc.}$$

ratios c_n/c_0 by an obvious perturbation expansion,[14] yielding

$$c_n/c_0=\mathcal{H}_{n\alpha}/(E-E_n)+\sum'_m\mathcal{H}_{nm}\mathcal{H}_{m\alpha}/(E-E_n)(E-E_m)$$
$$+\sum'_{lm}\mathcal{H}_{nm}\mathcal{H}_{ml}\mathcal{H}_{l\alpha}/(E-E_n)(E-E_m)(E-E_l)+\cdots. \quad (2.6)$$

Inserting this back into (2.4a), we find for the eigenvalue E the transcendent equation

$$E=\sum(E)\equiv\sum'_n\frac{\mathcal{H}_{\alpha n}\mathcal{H}_{n\alpha}}{E-E_n}+\sum'_{nm}\frac{\mathcal{H}_{\alpha n}\mathcal{H}_{nm}\mathcal{H}_{m\alpha}}{(E-E_n)(E-E_m)}+\cdots \quad (2.7)$$

[we may notice, incidentally, that in our case the terms of odd order in (2.7) drop out]. It is to be understood that E_s is, in particular, that root of (2.7) which is of order of f^2 when $f\rightarrow 0$. We shall see in Sec. 3 that $\sum(E)$ has the same meaning here as in the literature.[6]

For the sake of later applications, it is desirable to rewrite (2.5), (2.6), and (2.7) in a more compact notation. Let us introduce a projection operator Λ defined by

$$\Lambda\chi=\sum'_n\phi_n(\phi_n{}^*\chi), \quad (2.8)$$

where Λ projects an arbitrary state vector χ onto the subspace orthogonal to the bare-nucleon states ϕ_α. We may also say that the matrix representing Λ in the system (2.1) is diagonal, with diagonal elements $\Lambda_{\alpha\alpha}=0$, $\Lambda_{nn}=1\,(n\neq\alpha)$. The restriction expressed by the apex in Eqs. (2.6) and (2.7) can then be replaced by the insertion of factors Λ_{nn} between matrix elements of $\mathcal{H}$. Furthermore, one can write

$$|\alpha\rangle=c_0\phi_\alpha+\Lambda|\alpha\rangle \quad (2.5')$$

and replace (2.6) by

$$\Lambda|\alpha\rangle=c_0[(E-H_0)^{-1}+(E-H_0)^{-1}\Lambda\mathcal{H}(E-H_0)^{-1}+(E-H_0)^{-1}\Lambda\mathcal{H}(E-H_0)^{-1}\Lambda\mathcal{H}(E-H_0)^{-1}+\cdots]\Lambda\mathcal{H}\phi_\alpha$$

which is an expansion of

$$\Lambda|\alpha\rangle=c_0(E-H_0-\Lambda\mathcal{H})^{-1}\Lambda\mathcal{H}\phi_\alpha \quad (2.6')$$

where one can further save the last factor Λ since owing to (2.3′),

$$\Lambda\mathcal{H}\phi_\alpha=\mathcal{H}\phi_\alpha. \quad (2.9)$$

Similarly, one can abbreviate (2.7) as follows:

$$\sum(E)=\phi_\alpha{}^*\mathcal{H}(E-H_0-\Lambda\mathcal{H})^{-1}\mathcal{H}\phi_\alpha \quad (2.7')$$

which at first sight may not seem a real quantity because $\Lambda\mathcal{H}$ is non-Hermitean. Notice, however, that

$$H_0\Lambda=\Lambda H_0 \quad\text{and:}\quad \Lambda^2=\Lambda. \quad (2.10)$$

It is then easy to see that the value of (2.7′) does not change if $\Lambda\mathcal{H}$ is replaced by the Hermitean $\Lambda\mathcal{H}\Lambda$. [To prove this, make the replacement in the expanded form, and use (2.9).]

Let us now assume that $|\alpha\rangle$ is normalized, so that

$$1=\langle\alpha|\alpha\rangle=|c_0|^2\{1+\phi_\alpha{}^*\mathcal{H}(E-H_0-\Lambda\mathcal{H})^{-2}\mathcal{H}\phi_\alpha\}$$

where use has been made of (2.5′), (2.6′), and (2.9), and where, of course, E must be set equal to E_s.

Comparing with (2.7′) we see that [15] $|c_0|^2$ is equal to Z_2, the "renormalization constant" which (in analogy with the procedure described by Dyson, Ward *et al.* for the relativistic case) is defined in Chew's paper by

$$Z_2^{-1}=1-(d\sum/dE)_{E=E_s}. \quad (2.11)$$

Notice that $|c_0|^2$ represents the probability of finding no mesons around the nucleus; in the following, we shall dispose of the arbitrary phase factor in the state (2.5′) by choosing c_0 real and positive so that

$$c_0=Z_2^{\frac{1}{2}}. \quad (2.12)$$

With Eqs. (2.11) and (2.12), we have proved a relationship which plays an important role in the elimination of unobservable quantities from the theory.

We wish to conclude this section with another relationship, which plays a similar role. Consider the matrix element

$$\mathbf{M}_{\beta\alpha}=\langle\beta|\boldsymbol{\sigma}\tau_\lambda|\alpha\rangle \quad (2.13)$$

which, as we shall see, plays an important role in the theory of meson scattering (and also in photomeson production). Apart from trivial factors it is the matrix element of the meson absorption, or emission, operators (1.11) and (1.20); it is, however, the matrix element between *real nucleon states*, and not the ordinary matrix element

$$\mathbf{M}_{\beta\alpha}{}^0=\phi_\beta{}^*\boldsymbol{\sigma}\tau_\lambda\phi_\alpha, \quad (2.14)$$

which plays the central role in a more primitive approach. Since, however, σ_i and τ_λ obey simple commutation rules with the components of $\mathbf{J}$ and $\mathbf{T}$, and since $|\alpha\rangle$ and ϕ_α have the same quantum numbers J, J_z, T, T_3 it is possible to prove by the customary methods[16] that the ratio of (2.13) to (2.14) is a proportionality constant independent of α, β, and λ. The fact that this ratio is $\neq 1$ is, of course, an effect of the meson cloud surrounding the nucleon and is described graphically as a *change in the strength of coupling*. Thus one writes

$$f\mathbf{M}_{\beta\alpha}=f_r\mathbf{M}_{\beta\alpha}{}^0 \quad (2.15)$$

[14] This method is known as the Brillouin-Wigner form of perturbation theory, and is characterized by the fact that the eigenvalue itself is not expanded in powers of f. The result (2.7) may look superficially like the customary Schrödinger expansion, but of course it is not since it gives E only implicitly. The explicit expansion of E in powers of f can be obtained from the implicit formula (2.7) by known methods; a little thought will show that the structure of the higher-order terms in the explicit expansion is far less simple than that in Eq. (2.7). This is why the latter is much more suitable for the discussion of the following section.

[15] T. D. Lee, Phys. Rev. **95**, 1329 (1954). This proves incidentally that $0<Z_2<1$. For the case of relativistic theories see H. Lehmann, Nuovo cimento **11**, 342 (1954); M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

[16] E.g., Condon and Shortley, *The Theory of Atomic Spectra*, (Cambridge University Press, New York, 1935), Chap. III.

where f_r is a "renormalized coupling constant," which for our purposes may be defined by Eq. (2.15). Since the definition usually given is another one,

$$f_r = fZ_2Z_1^{-1} \tag{2.16}$$

where Z_1 is another renormalization constant, let us briefly mention the relationship between the two definitions.[17] Evaluating (2.14) by means of (2.5′) and (2.12), we see that

$$\mathbf{M}_{\beta\alpha} = Z_2\mathbf{M}_{\beta\alpha}{}^0 + c_0 \langle\beta|\Lambda\boldsymbol{\sigma}\tau_\lambda\phi_\alpha) + c_0(\phi_\beta{}^*\boldsymbol{\sigma}\tau_\lambda\Lambda|\alpha\rangle + \langle\beta|\Lambda\sigma\tau_\lambda\Lambda|\alpha\rangle. \tag{2.17}$$

Of the four terms on the right, the second and third are zero (Λ commutes with $\sigma\tau$, and $\Lambda\phi_\alpha = 0$), the fourth may be transformed by means of (2.6′) and (2.9). If now we define the renormalization constant Z_1 by means of

$$Z_1^{-1}\mathbf{M}_{\beta\alpha}{}^0 = \mathbf{M}_{\beta\alpha}{}^0 + \phi_\beta\mathcal{H}(E_s - H_0 - \Lambda\mathcal{H})^{-1}\boldsymbol{\sigma}\tau_\lambda(E_s - H_0 - \Lambda\mathcal{H})^{-1}\mathcal{H}\phi_\alpha, \tag{2.18}$$

we see that (2.17) is equivalent to (2.15) and (2.16). Now it is easy to see, by a discussion of the kind developed in the next section, that (2.18) is only a shorthand for the definition of Z_1 ordinarily given. We shall not go into this, however, in any more detail here.

3. GRAPHS

Perturbation expansions like Eq. (2.7) can be discussed most easily in terms of "graphs." The graphs which occur in the theory we are discussing are simpler and less general than those occurring in the full-fledged relativistic theories. Just for this reason, the present theory is a very good example with which to begin the study of graphs, for the reader who is unfamiliar with them (other readers may skip this section altogether).

The idea is to represent each nucleon by a straight *horizontal* line, and each meson by a dashed line. A dashed line terminating on a nucleon line quite naturally represents the absorption or emission of a meson. If one wishes, one can append to a meson line an index k, (or p, $\cdots$, etc.) specifying the momentum $\mathbf{k}$ and charge state λ of the meson.

Let us now see how this works in practice. Consider the first term on the right-hand side of (2.7). Omitting the summation sign, the quantity to be evaluated is of the type

$$\mathcal{H}_{\gamma n}\mathcal{H}_{n\alpha}/(E - E_n), \tag{3.1}$$

where α (which in our case equals γ) is a bare-nucleon state, and the intermediate state n is clearly of the type $\phi_{k\beta}$ in the sequence (2.1). One associates conventionally $\mathcal{H}_{n\alpha}$ to the transition from α to n, $\mathcal{H}_{\alpha n}$ to the inverse transition; the process to be described is clearly the

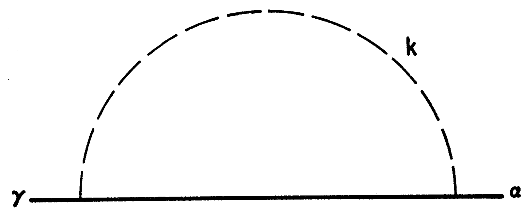

FIG. 1. Simplest self-energy graph representing the emission and reabsorption of a meson in a state k.

emission and reabsorption of a virtual meson, the corresponding graph is Fig. 1. For completeness we have also added indices α and γ to the bare-nucleon "stumps" at the ends; the reader is asked to notice that α is written to the right and γ to the left, just as they appear in (3.1). In our case, of course, this does not matter, since $\alpha = \gamma$, but in the *transition processes* to be considered later the end states are different. It is then somewhat more convenient to have the indices appear in the same order on the graph as in the formula. Since α will then be the initial, γ the final state, one also says that, according to our convention, "time runs from right to left" on the graph; events appearing at the right are "earlier," those at the left are "later."

Now in the actual calculation of (3.1) one will notice that for a given state n (i.e., given $k\beta$) the only terms of $\mathcal{H}$, Eq. (1.10), contributing to $\mathcal{H}_{n\alpha}$ and $\mathcal{H}_{\gamma n}$ are, respectively, $a_k{}^*V_k{}^*$ and a_kV_k. Since moreover the matrix elements of $a_k{}^*$ and a_k are equal to $\sqrt{1}$, we have

$$\begin{aligned} \mathcal{H}_{n\alpha} &= \phi_\beta{}^*V_k{}^*\phi_\alpha = (V_k{}^*)_{\beta\alpha}, \\ \mathcal{H}_{\gamma n} &= \phi_\gamma{}^*V_k\phi_\beta = (V_k)_{\gamma\beta}. \end{aligned} \tag{3.2}$$

Here an important simplification occurs, owing to the fact that the energy of the intermediate state $E_n = \omega_k$ is independent of β; the same simplification will be seen to occur in the general case. Namely, in the summation $\sum_n = \sum_k\sum_\beta$, the first step can be carried out symbolically

$$\textstyle\sum_\beta (V_k)_{\gamma\beta}(V_k{}^*)_{\beta\alpha} = (V_kV_k{}^*)_{\gamma\alpha}. \tag{3.3}$$

In so doing, V_k and $V_k{}^*$ are regarded, of course, as 4×4 matrices (operating on the nucleon spin and isotopic spin variables). One then has

$$V_kV_k{}^* = 4\pi N^2(f/\mu)^2|v(k)|^2(2\omega_k)^{-1}\tau_\lambda{}^2(\mathbf{k}\cdot\boldsymbol{\sigma})^2, \tag{3.4}$$

where one further has $\tau_\lambda{}^2 = 1$, $(\mathbf{k}\cdot\boldsymbol{\sigma})^2 = k^2$. Remembering (1.4),

$$\textstyle\sum_k \cdots = \sum_{\lambda,\mathbf{k}} \cdots = 3N^{-2}(2\pi)^{-3}\displaystyle\int d\mathbf{k}\cdots,$$

so that finally we have

$$\sum_n \frac{\mathcal{H}_{\alpha n}\mathcal{H}_{n\alpha}}{E - E_n} = (3/2\pi^2)(f/\mu)^2 \int \frac{|v(k)|^2k^2dk}{2\omega_k(E - \omega_k)}. \tag{3.5}$$

We have presented all the steps in this pedantic detail,

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[17] T. D. Lee (reference 15). Equivalent relationships in the relativistic theories have been known for some time (J. Schwinger, unpublished); see also H. Umezawa and S. Kamefuchi, Progr. Theoret. Phys. Japan **6**, 543 (1951); G. Källen, Helv. Phys. Acta. **25**, 417 (1952) and the papers previously quoted.

FIG. 2. An "improper" self-energy graph which does not occur in the evaluation of Eq. (2.7).

so that we can proceed more speedily in the subsequent cases.

Let us now go over to the fourth-order term in the expansion of $\sum(E)$ (the third-order term is zero). Various possibilities now arise, and the advantage of graphs in quickly visualizing all possibilities will appear. We have a product $\mathcal{H}_{\alpha l}\mathcal{H}_{lm}\mathcal{H}_{mn}\mathcal{H}_{n\alpha}$ with three intermediate states l, m, and n. Beginning from the right, we see that $\mathcal{H}_{n\alpha}$ can lead only from the bare-nucleon state α to a one-meson state, say, $\phi_{k\beta}$. The subsequent transition, however, can be either the reabsorption of the meson already emitted or the emission of a new meson; *in the first alternative the state m would be a bare-nucleon state, and this is forbidden by the apex.* If it were not so, the process could go on with the emission of a new meson ($\mathcal{H}_{lm}$) and reabsorption of the same ($\mathcal{H}_{\alpha l}$); the graph, which as we said is forbidden by the apex symbol would be that of Fig. 2. The other alternative means that the state "in the middle," i.e., m, is a two-meson state, $\phi_{kp\gamma}$, say. From that one can proceed in two ways. Clearly the two subsequent transitions must both be absorptions (since we must return to a bare nucleon at the end), but we can absorb meson k with $\mathcal{H}_{lm}$, and p with $\mathcal{H}_{\alpha l}$ or vice versa. This gives the two graphs, Fig. 3(a) and Fig. 3(b). We *could* write near the nucleon lines the symbols for the states (β, γ, δ) of the nucleon in the states n, m, l but this is hardly necessary [it is done in Fig. 3(b) as an example], since the summation over β, γ, and δ is carried out implicitly as before, that is, the symbols β, γ, and δ do not even appear in the calculation. Let us also write down the unperturbed energies of the intermediate states for the two sequences above:

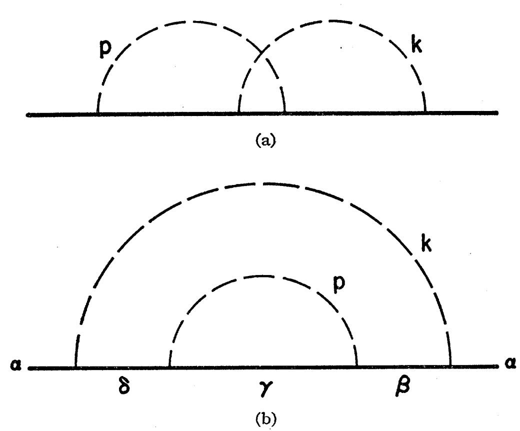


FIG. 3. The two possible fourth-order graphs for Eq. (2.7).

[Fig. 3(a)]

$$E_n=\omega_k, \quad E_m=\omega_k+\omega_p, \quad E_l=\omega_p, \tag{3.6a}$$

[Fig. 3(b)]

$$E_n=\omega_k, \quad E_m=\omega_k+\omega_p, \quad E_l=\omega_k. \tag{3.6b}$$

These values can, of course, be read off the graphs very quickly. An intermediate state is the state of affairs between two successive transitions; if we draw a vertical line intersecting the nucleon line between two meson endpoints, the meson lines it intersects correspond to the mesons present in the field at that particular virtual stage of the transition; in Fig. 3(b), for example, a vertical line between the last two meson endpoints gives $E_l=\omega_k$.

We are now in a position to write down immediately the fourth-order term of (23) as the sum of two terms corresponding, respectively, to Figs. 3(a) and 3(b):

$$\frac{V_pV_kV_p^*V_k^*}{(E-\omega_p)(E-\omega_k-\omega_p)(E-\omega_k)} + \frac{V_kV_pV_p^*V_k^*}{(E-\omega_k)(E-\omega_k-\omega_p)(E-\omega_k)}. \tag{3.7}$$

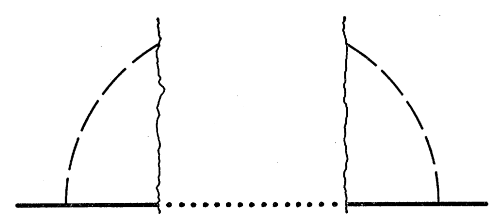


FIG. 4. General self-energy graphs.

We shall not carry on this calculation any further now. We shall instead make some remarks on the structure of the general term in the expansion (2.7).

In the term of order $2n$ there are $2n$ factors $\mathcal{H}_{lm}$, half of which must be used to create virtual mesons, and the other half to destroy them. The number of choices, however, becomes very large if n is large. The reader may try as an example to draw all the graphs for $2n=6$. The general character of these graphs may be symbolized as in Fig. 4; the graph begins with an emission act at the right and closes with an absorption act at the left; in the region between the wavy lines, anything may happen, with one exception: there must not be any intermediate state with no virtual mesons. In other words, it must not be possible to separate the graph in the way indicated in Fig. 5. To be more specific, graphs such as this do not occur in formula (2.7), although they may be considered in other contexts. They were called by Dyson "improper" self-energy graphs. A graph like Fig. 4, which cannot be split as in Fig. 5, is called a proper self-energy graph. The expression (2.7) is a sum over proper self-energy graphs only.

Once a graph is plotted, the corresponding term in the sum can be written down immediately. To this end it is desirable to develop good bookkeeping habits. As Chew suggests, it is best in this to adhere strictly to Feynman. This has the advantage that the habits acquired here can be transferred to relativistic theories without effort. In the bookkeeping of "classical" perturbation theory it was customary to say that in the virtual intermediate states "energy is not conserved." By that one meant that the energies E_n (that is, the unperturbed energy values) of the intermediate states can differ by arbitrarily large amounts from the true energy E of the stationary state under consideration. The energy denominators $E-E_n$ measure just this difference.

In the Feynman bookkeeping, instead, one says that energy is conserved at every emission or absorption act, and the energy of the nucleon changes accordingly. Thus in a graph like Figs. 3 and 4 the nucleon enters at the right with energy E, and emits a meson of energy ω_k; it then has an energy $E-\omega_k$. If now it emits a new meson ω_p, it will have an energy $E-\omega_k-\omega_p$ and so on. Clearly E_n (the energy of the virtual mesons present in the virtual state n) represents the sum total of the energy of the mesons previously emitted minus those

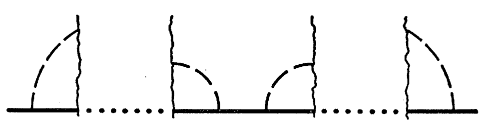


FIG. 5. Improper S.-E. graph: a bare-nucleon state occurs in the middle.

reabsorbed by the nucleon. Hence, the energy denominators are now simply the "energy of the nucleon," as defined previously, in the subsequent segments of the nucleon line, between two successive elementary acts. The most natural thing is then to write operators V_k and V_k^* and the energy denominators in the same order as they appear on the graph, thus, for Fig. 6,

$$V_q\frac{1}{E-\omega_q}V_k\frac{1}{E-\omega_k-\omega_q}V_q^*\frac{1}{E-\omega_k}V_p \times\frac{1}{E-\omega_k-\omega_p}V_p^*\frac{1}{E-\omega_k}V_k^*.$$

These conventions apply equally well to the more general graphs to be encountered later, in the discussion of scattering processes. In that case, of course, the nucleon will not be assumed to enter the graph with an energy E. Since E is used to denote the total energy, the initial energy of the nucleon will be reckoned as E minus the energy of the incident meson or mesons. If then one proceeds as we have said, assuming energy conservation at each step, it is easy to see again that energy denominators and "nucleon energy" are one and the same thing.

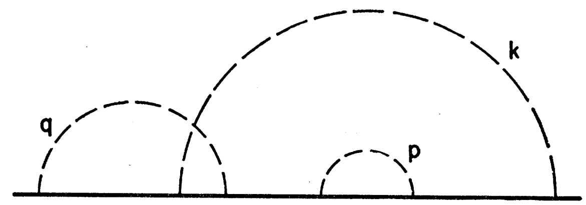


FIG. 6. A possible sixth-order graph.

To conclude this discussion, we go back to Fig. 3(a), Fig. 3(b), and Eq. (3.7) and notice that a refinement is necessary. Consider that in summing over k and p, the case will also occur that $k=p$. But in this case it has no meaning to draw two graphs; what happens is that after the second emission act we have two mesons in state k, and since they are indistinguishable we cannot ask the question whether it is the first-emitted or the second-emitted meson which is absorbed first. Thus, we should not have two terms in Eq. (3.7) but only one. We have, however, also made a compensating mistake! When $k=p$, the emission amplitude for the second meson is not V_k^* but $\sqrt{2}V_k^*$; similarly the subsequent reabsorption act contains another $\sqrt{2}$. The omission of two factors $\sqrt{2}$ just compensates for the mistake of writing two terms instead of one. This is not accidental. The reader is invited to think through for himself the more complicated case where the occupation number of a given meson state goes up to a maximum value n. There occur then in the correct expression factors $\sqrt{2}$, $\sqrt{3}$, $\cdots$ $\sqrt{n}$ each an even number of times, and these just make up for the reduction in the number of truly distinct processes, as compared to the fictitious multiple choices indicated by the graph method.

One might point out, that this remark is really unnecessary, for the following reason. What we want to evaluate is, for example, the sum of (3.7) over k and p. Now owing to expression (1.4), the part of the sum with $k=p$ is negligibly small (when the quantization volume is large) compared to the rest of the sum.

We have preferred not to rely on this argument, not merely because it is somewhat neater to show that the graph method of calculation is valid irrespective of the size of the quantization volume, but also because there are cases, not encountered in this section, where the "volume" argument fails.

4. MESON SCATTERING I. BORN APPROXIMATION

With the Hamiltonian (1.10), mesons interact with a nucleon only when in a p-state. Thus only p-wave scattering occurs. Obviously this cannot be the whole story in reality, but it is a fact that p-wave scattering is the dominant feature at low energies (say below 250 Mev); s-wave scattering is surprisingly weak, while, not so surprisingly, scattering of waves with $l\geq 2$ becomes important only at higher energies.[1]

It is Chew's great merit, that he showed that not merely the qualitative fact of the predominance of

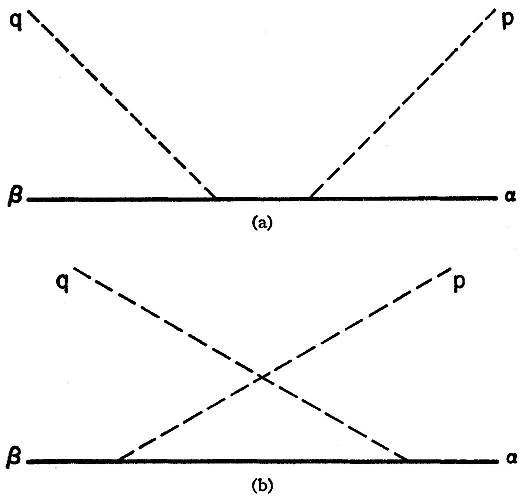


FIG. 7. (a) Second-order graph for meson scattering with uncrossed lines. (b) The same with crossed lines.

p-wave scattering, but also the main detailed features of it are satisfactorily explained by this theory.[6] This fact had been previously obscured by an excessive reliance on the predictions of lowest-order perturbation calculations.

In fact, although, as we shall see, the value of the coupling constant f is fairly small, higher-order effects play an important role in meson scattering. Nevertheless, we shall discuss in this section the lowest-order calculation, purely as an occasion for certain remarks to be used later.

We wish, then, to calculate by means of perturbation theory the transition amplitude from an initial state a with a nucleon in a state ϕ_α and a meson in a state $p(=\mathbf{p}\lambda)$, to a final state b with nucleon in a state ϕ_β and a meson in a state $q(=\mathbf{q}\mu)$. To achieve this we must absorb meson p and emit meson q; in addition any number of virtual mesons may be emitted and reabsorbed. To lowest order, however, there are no virtual mesons in the process, and the matrix element for the transition has the familiar second-order form

$$M=\sum_n \frac{\mathcal{H}_{bn}\mathcal{H}_{na}}{E-E_n}, \tag{4.1}$$

where E is the energy, which may be identified with either the energy of the initial state E_a or with E_b (since $E_a=E_b$, necessarily). Finally we equate E_a with the energy ω_p of the incident meson, since the nucleon energy has been set equal to zero (see, however, Sec. 5).

Now the transition may proceed in two ways; namely, $\mathcal{H}_{na}$ may be used to absorb meson p, and $\mathcal{H}_{bn}$ to emit meson q, see Fig. 7(a), or vice versa Fig. 7(b). In the first case $E_n=0$ and $E-E_n=\omega_p$ in the second case $E_n=\omega_p+\omega_q$. Since $\omega_p=\omega_q$ and $p=q$, in magnitude, we write simply ω and p in the following. Then if we proceed as we did in the previous section, we see that

$$\begin{aligned} M&=M_a+M_b,\\ M_a&=V_q{}^*\omega_p{}^{-1}V_p=X\tau_\mu\tau_\lambda(\mathbf{q}\cdot\boldsymbol{\sigma})(\mathbf{p}\cdot\boldsymbol{\sigma}),\\ M_b&=V_p(-\omega_q)^{-1}V_q{}^*=-X\tau_\lambda\tau_\mu(\mathbf{p}\cdot\boldsymbol{\sigma})(\mathbf{q}\cdot\boldsymbol{\sigma}),\\ X&=2\pi(f/\mu)^2|v(p)|^2\omega^{-2}. \end{aligned} \tag{4.2}$$

The differential cross section, of course, is given by $|M|^2$ multiplied by the number of final states per unit energy interval and divided by the velocity p/ω of the incident meson. If $d\Omega$ is the solid angle, $d\mathbf{q}=q^2dqd\Omega$, and dividing by the energy interval $d\omega$ and using $d\omega/dq=q/\omega$, the number of states becomes $d\mathbf{q}/(2\pi)^3d\omega=(2\pi)^{-3}q\omega\cdot d\Omega$. Hence, the differential cross section is

$$d\sigma/d\Omega=(2\pi)^{-3}\omega^2|M|^2. \tag{4.3}$$

We have omitted the "volume" factor N throughout, but, if we had not, it would have cancelled out of the final result.

Let us now go into the calculation of M in some more detail. We may consider separately the dependence of M on the variables specifying the direction of motion of the meson and the ordinary spin of the nucleon, and the dependence on the "charge states" of meson and nucleon. As regards the latter we have six possible states ($\lambda=1$, 2, 3 together with neutron or proton) so that M is a 6 by 6 matrix. In Eq. (4.2) this part of the dependence is contained in the operators $\tau_\mu\tau_\lambda$ and $\tau_\lambda\tau_\mu$. Instead of $\lambda=1$, 2, 3 we may use the physically significant states $\pi^\pm$, π^0; as explained in the Appendix this may be achieved without altering the formalism by using τ_+, τ_-, and τ_3 instead of τ_1, τ_2, τ_3. This only requires the minor modification that since τ_+ and τ_- are not Hermitean, it is necessary to pay more attention to the "star" in an emission operators $V_q{}^*$. In (4.2) this means replacing τ_μ by $\tau_\mu{}^*$ when using $\tau_\pm$.

One gets easily for the relevant transitions the matrix elements specified in Table I. As regards the remaining variables, we can use the well-known formula

$$(\mathbf{A}\cdot\boldsymbol{\sigma})(\mathbf{B}\cdot\boldsymbol{\sigma})=\mathbf{A}\cdot\mathbf{B}+i\boldsymbol{\sigma}\cdot\mathbf{A}\times\mathbf{B}. \tag{4.4}$$

Let us write

$$\mathbf{q}\cdot\mathbf{p}=qp\cos\theta;\quad (qp)^{-1}\mathbf{p}\times\mathbf{q}=\sin\theta\cdot\mathbf{n} \tag{4.5}$$

then

$$\begin{aligned} M_a&=Xp^2\tau_\mu\tau_\lambda(\cos\theta-i\sin\theta\mathbf{n}\cdot\boldsymbol{\sigma}),\\ M_b&=Xp^2\tau_\lambda\tau_\mu(\cos\theta+i\sin\theta\mathbf{n}\cdot\boldsymbol{\sigma}). \end{aligned} \tag{4.6}$$

TABLE I.

Transition \ Operator	$\tau_\mu{}^*\tau_\lambda$	$\tau_\lambda\tau_\mu{}^*$
$\pi^+p\rightarrow\pi^+p$	0	2
$\pi^+n\rightarrow\pi^+n$	2	0
$\pi^+n\rightleftarrows\pi^0p$	$\sqrt{2}$	$-\sqrt{2}$
$\pi^0p\rightarrow\pi^0p$	1	1
$\pi^0n\rightarrow\pi^0n$	1	1
$\pi^0n\rightleftarrows\pi^-p$	$-\sqrt{2}$	$\sqrt{2}$
$\pi^-p\rightarrow\pi^-p$	2	0
$\pi^-n\rightarrow\pi^-n$	0	2

Thus as an example we find, from Table I, the following values for M:

Reaction	Value of M/Xp^2
$\pi^- p \to \pi^- p$	$2\,(\cos\theta - i\sin\theta\,\mathbf{n}\cdot\boldsymbol{\sigma})$
$\pi^- p \to \pi^0 n$	$-2\sqrt{2}\cos\theta$
$\pi^+ p \to \pi^+ p$	$-2\,(\cos\theta + i\sin\theta\,\mathbf{n}\cdot\boldsymbol{\sigma})$

(4.7)

Thus for the exchange reaction $|M|^2 \sim 8\cos^2\theta$ with an average of 8/3 (over the direction of $\mathbf{q}$). The elastic cross sections contain a so-called "spin-flip" term $\mathbf{n}\cdot\boldsymbol{\sigma}$. This expression refers to the fact that if one uses the incident direction as axis of quantization, then $\mathbf{n}\cdot\boldsymbol{\sigma}$ has matrix elements only between states of opposite spin.

On the other hand, one often uses $\mathbf{n}$ (the normal to the scattering plane) as axis of quantization. Then the nucleon spin does *not* change, but the scattering amplitude depends on the spin direction; for example, in the first reaction (4.7) the amplitude is $2e^{-i\theta}$ for spin down. As it happens, in this approximation the difference between the two is only a phase factor, and $|M|^2$ does not depend on the spin direction, in fact

$$|M|^2 = 4; \quad (\pi^- p \to \pi^- p) \quad \text{or} \quad (\pi^+ p \to \pi^+ p).$$

In this approximation, therefore, we find that the elastic σ^+ and σ^- cross section on hydrogen are isotropic and equal to one another. Furthermore, the ratio of elastic to exchange cross section for π^- in hydrogen is 3/2. These results are in complete disagreement with experiment; as we shall see, this is only because of the inadequacy of the Born approximation.

Nevertheless, let us pursue the analysis further. The raw experimental data appear, of course, as data for the reactions of Table I, but they are usually analyzed in terms of other reactions, in which the initial and final states have a well defined total isotopic spin quantum number T (see Appendix B). Only the first and last reactions in Table I are of this type; states such as $\pi^+ n$, $\pi^0 p$, etc., are not eigenstates of $\mathbf{T}^2$. The reason for using eigenstates of $\mathbf{T}^2$ is that the results appear in a more condensed form. Let us say that initial and final states are characterized by quantum numbers T, T_3 and T', T_3', respectively ($T=\frac{1}{2}$ or $\frac{3}{2}$; $T_3 = T, \cdots, -T$). Then in a charge-independent theory the results for the matrix M must be of the form[18]

$$(T'T_3'|M|TT_3) = M(T)\delta_{TT'}\delta_{T_3T_3'}. \tag{4.8}$$

That is, as a result of reshuffling the basic states, the 6×6 matrix is now diagonal, four of the elements on the diagonal being equal to $M(\frac{3}{2})$ and two equal to $M(\frac{1}{2})$.

Let us now have another look at our notation. We have previously regarded $\tau_\mu{}^*\tau_\lambda$ and $\tau_\lambda\tau_\mu{}^*$ as 6×6 matrices with the matrix elements specified in Table I. We may also regard them as 3×3 matrices Q and Q' with matrix elements

$$Q_{\mu\lambda} = \tau_\mu{}^*\tau_\lambda; \quad Q_{\mu\lambda}' = \tau_\lambda\tau_\mu{}^*, \tag{4.9}$$

which are not numbers but 2×2 submatrices. This simply takes account of the fact that in our notation the meson index $\lambda(\mu)$ for the initial (final) state is explicit, while the nucleon indices are implicit. Thus reverting for simplicity to $\lambda = 1, 2, 3$ instead of $+, 0, -$, and remembering that $\tau_1{}^2 = 1$, $\tau_1\tau_2 = i\tau_3$, etc., we see that the matrix Q, for example, has the form

$\mu \diagdown \lambda$	1	2	3
1	1	$i\tau_3$	$-i\tau_2$
2	$-i\tau_3$	1	$i\tau_1$
3	$i\tau_2$	$-i\tau_1$	1

where the 1's on the diagonal are, of course, 2×2 unit matrices. Remembering the form of the $t_{\mu\lambda}$ matrices [Eq. (A22)] for the meson spin we see that

$$\begin{aligned} Q &= 1 - (\mathbf{t}\cdot\boldsymbol{\tau}) \\ Q' &= 2 - Q = 1 + (\mathbf{t}\cdot\boldsymbol{\tau}) \end{aligned} \tag{4.10}$$

where $(\mathbf{t}\cdot\boldsymbol{\tau}) = t_1\tau_1 + t_2\tau_2 + t_3\tau_3$. We now see that Q and Q' are of the form (4.8). Namely, since

$$\begin{aligned} \mathbf{T}^2 \equiv (\mathbf{t} + \tfrac{1}{2}\boldsymbol{\tau})^2 &= \mathbf{t}^2 + \tfrac{1}{4}\boldsymbol{\tau}^2 + \mathbf{t}\cdot\boldsymbol{\tau} \\ &= 2 + \tfrac{3}{4} + \mathbf{t}\cdot\boldsymbol{\tau} \end{aligned} \tag{4.11}$$

we see that $\mathbf{t}\cdot\boldsymbol{\tau}$ is diagonal together with $\mathbf{T}^2$ and has the eigenvalues

$$\mathbf{t}\cdot\boldsymbol{\tau} = \begin{cases} 1 \\ -2 \end{cases} \quad \text{if} \quad T = \begin{cases} 3/2 \\ 1/2 \end{cases}. \tag{4.12}$$

Thus we may replace Table I by the simpler Table II.[19]

Now to the σ dependence of the matrix element. Just as a great simplification was achieved by remembering that the total isotopic spin vector $\mathbf{T}$ is conserved

TABLE II.

State \ Operator	$\tau_\mu{}^*\tau_\lambda$	$\tau_\lambda\tau_\mu{}^*$
$T=\frac{1}{2}$	3	-1
$T=\frac{3}{2}$	0	2

[18] W. Heitler, Proc. Roy. Irish Acad. 51, 33 (1946). Interest in Heitler's analysis was revived by the work of K. M. Watson [Phys. Rev. 85, 852 (1952)] and its importance became apparent when the first detailed experimental data became available; see Proc. Rochester Conference, 1952, contributions by E. Fermi and K. A. Brueckner.

[19] A fast way of deriving these eigenvalues (see Dyson et al.[31]) is to notice that the matrix Q^2 has the elements

$$(Q^2)_{\mu\lambda} = \sum_{\nu=1}^{3} Q_{\mu\nu}Q_{\nu\lambda} = \tau_\mu\tau_\nu\tau_\nu\tau_\lambda = 3\tau_\mu\tau_\lambda = 3Q_{\mu\lambda}.$$

Hence $Q^2 = 3Q$, showing that the eigenvalues of Q are 3 and 0. It is easy to recognize that the eigenvalue 0 belongs to $T=\frac{3}{2}$, because the operator Q appears in connection with the graph in Fig. 7(a) and it is clear that for the state $\pi^+ p$, which is a $T=\frac{3}{2}$ state, the graph must contribute 0, since a proton cannot absorb π^+!

in the collision, similarly we can benefit from the fact that the total angular momentum **J** is conserved.

The latter vector is the resultant of the meson orbital momentum and of the nucleon spin. As pointed out in an earlier section, mesons are only scattered, in the present theory, when their orbital quantum number l is unity. This is, of course, immediately apparent in the fact that the matrix element (4.2) is a bilinear function of the momenta p and q. In order to diagonalize J it is desirable to use meson states of definite orbital momentum rather than plane waves. This can be achieved very easily either by transforming (4.2) by means of (1.18), or, what is the same thing, replacing V_p and V_q with the modified V's of Eq. (1.20). All that amounts to (apart from a factor) is writing (4.2) as a bilinear function of p and q and picking the coefficient B_{ji} of $q_j p_i$.

One finds then that B_{ji} is a combination of two operators

$$\sigma_j\sigma_i \quad \text{and} \quad \sigma_i\sigma_j, \tag{4.13}$$

which obviously have the same properties as $\tau_\mu\tau_\lambda$ and $\tau_\lambda\tau_\mu$. Owing to the symmetry between isotopic spin and angular momentum indices (see Sec. 1 at the end), one can replace in Eqs. (4.8) to (4.12) and in Table II T, T_3, $\mathbf{t}$, $\boldsymbol{\tau}$, τ_μ, τ_λ by J, J_z, $\mathbf{l}$, $\boldsymbol{\sigma}$, σ_j, and σ_i, respectively.

The matrix element Eq. (4.2), is in this notation, and apart from a factor, of the form

$$\tau_\mu\tau_\lambda\sigma_j\sigma_i - \tau_\lambda\tau_\mu\sigma_i\sigma_j. \tag{4.14}$$

The eigenvalues of this operator may be easily calculated, according to what we have just said, from Table II. Following accepted habit, we designate the possible states of the system with the symbol $(2T)(2J)$, i.e. for example 31 means $T=\frac{3}{2}$, $J=\frac{1}{2}$. Then the eigenvalues of (4.14) are

state:	33	31 or 13	11	
eigenvalue:	-4	2	8.	(4.15)

That is, in a scheme in which T and J are diagonal, the matrix B is also diagonal and its diagonal elements are given by (4.15) times a common positive factor.

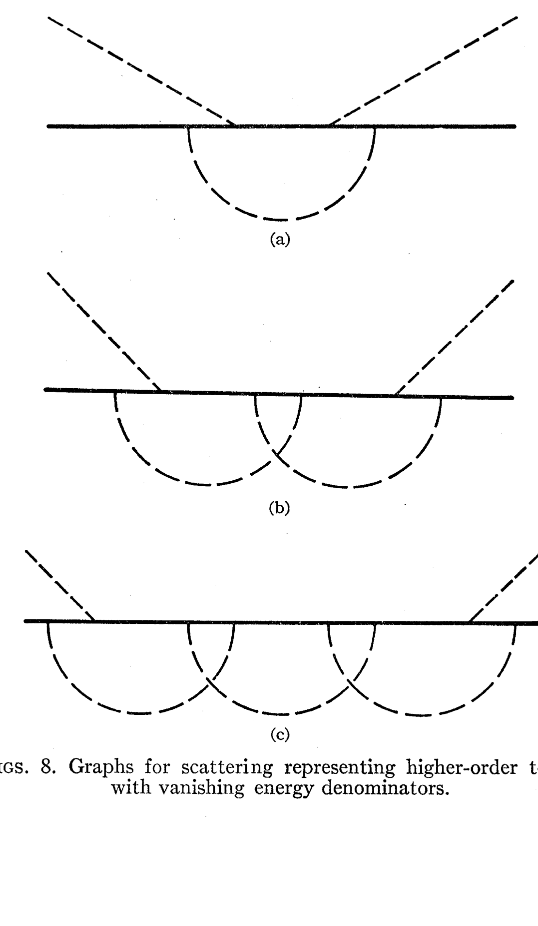
The result (4.15) is, of course, only a different presentation of Table I and Eqs. (4.6), and is, therefore, in equally bad disagreement with the facts. It points out, however, one qualitative feature which is in agreement. Namely (4.15) corresponds to "attraction" in the 33 state and repulsion in the other two states.[20] Experiments indicate indeed a strong attractive effect in the 33 state. They do not, however, indicate an even stronger repulsive effect in the 11 state; in fact (4.15) would be in much better agreement with the facts if the 2 and 8 were replaced by zero.

Before we turn to a consideration of higher-order effects, let us finally rewrite (4.14) with all the factors in. The matrix element may be designated symbolically with $B_p(q)$ (which is a matrix in the four-dimensional vector space of the spin and charge states of the nucleon) or more explicitly with $B_{p\alpha}(q\beta)$. Remember also that p summarizes the magnitude $|\mathbf{p}|$ of the initial momentum, the index $i(=1,2,3)$ for the initial Y_{1i} spherical harmonic and the initial charge state λ; and similarly q summarizes $|\mathbf{q}|, j, \mu$. Using (1.20) with N' omitted, since it cancels out of the cross section when one uses (1.16) for the density of final states, one finds

$$B_p(q)=\tfrac{1}{6}(f/\mu)^2|v(p)|^2(p/\omega)^2[\sigma_j\sigma_i\tau_\mu\tau_\lambda-\sigma_i\sigma_j\tau_\lambda\tau_\mu]. \tag{4.16}$$

5. MESON SCATTERING II

One believes (see later) that the value of the coupling constant f^2 is probably ~ 0.08, which at first sight may seem small enough to justify use of the Born approximation. As in all such questions, however, one must be a little careful in estimating the order of magnitude of the successive terms of the Born expansion. In particular, Chew has pointed out that "higher-order" terms corresponding to graphs such as those in Figs. 8(a), 8(b), and 8(c) are enhanced owing to resonance effects (vanishing energy denominators) occurring when the energy of one of the virtual mesons is equal to that of the incident meson. In fact the graph of Fig. 8(a), as compared to the graphs of Fig. 7, gives an extra power of f^2 but also one vanishing denominator, the graph in Fig. 8(b) gives two extra powers of f^2, but



FIGS. 8. Graphs for scattering representing higher-order terms with vanishing energy denominators.

[20] This language originates in a comparison with scattering due to a potential. In the first Born approximation an attractive (i.e. negative) potential gives a negative matrix element (or a positive phase shift).

also two vanishing denominators,[21] the graph of Fig. 8(c) gives three extra powers and three vanishing denominators, etc. Pursuing this idea further, Chew showed that scattering is strongly enhanced in the state $T=\frac{3}{2}$, $J=\frac{3}{2}$, but not in the other states.

A satisfactory study of higher-order effects requires, however, as pointed out later by Chew, a systematic study of self-energy and "renormalization" effects. We begin, therefore, with a general formulation of the scattering problem.[22]

For the sake of comparison we must have at hand the main results of the ordinary formulation of scattering theory. It will suffice to consider the scattering of a particle by a static potential V. The Hamiltonian operator is then $H=H_0+V$, where H_0 is the kinetic-energy operator. Let ϕ_a, ϕ_b, $\cdots$ be a complete orthonormal set of eigenstates of H_0 (plane waves) with energy E_a, E_b, $\cdots$. Let ψ_a, ψ_b, $\cdots$ be eigenstates of H with the same energy values; specifically in the study of the scattering problems one considers solutions ψ_a^+ and ψ_a^- defined by the boundary condition at infinity

$$\psi_a^\pm\sim\phi_a+\left\{\begin{matrix}\text{outgoing}\\ \text{ingoing}\end{matrix}\right\}\text{waves only.} \tag{5.1}$$

A solution ψ_a satisfies the inhomogeneous equation

$$(E_a-H)(\psi_a-\phi_a)=V\phi_a \tag{5.2}$$

or alternatively

$$(E_a-H_0)(\psi_a-\phi_a)=V\psi_a. \tag{5.3}$$

From these equations one obtains in a well-known manner[23]

$$\psi_a^\pm-\phi_a=(E_a\pm i\eta-H)^{-1}V\phi_a \tag{5.4}$$

and

$$\psi_a^\pm-\phi_a=(E_a\pm i\eta-H_0)^{-1}V\psi_a^\pm \tag{5.5}$$

where $i\eta$ is an infinitesimal positive imaginary quantity which is inserted to define the operators $(E_a-H)^{-1}$ and $(E_a-H_0)^{-1}$ unambiguously and in agreement with the boundary condition (5.1).[24] The most familiar proof of the outgoing wave nature of $\psi_a^+-\phi_a$ is that which consists in calculating explicitly the Green function which represents the operator $(E_a+i\eta-H_0)^{-1}$ in configuration space. It is important for us to recognize, however, that the $i\eta$ device is of completely general validity. For example, it is correct in Eq. (5.4), although it is not possible to evaluate the corresponding Green function explicitly. That this is so, follows most simply from time-dependent arguments, such as those employed by Lippman and Schwinger and Gell-Mann and Goldberger.[23] To put it very briefly, these arguments boil down to this. The requirements to be satisfied by a scattering solution ψ_a^+ can be stated also in the form of a condition on the time dependence of a wave packet

$$\psi(t)=\sum_a c_a e^{-iE_at}\psi_a^+. \tag{5.6}$$

We assume that c_a is a continuous function of the energy E_a, and is different from zero in a finite (and, if one wishes, very narrow) interval ΔE. If ϕ_a represents the incident part of ψ_a^+, then the wave packet

$$\phi(t)=\sum_a c_a e^{-iE_at}\phi_a \tag{5.7}$$

should represent the incident part of the full wave packet (5.6). The latter is a solution of the complete time-dependent Schrödinger equation $\partial\psi/\partial t=H\psi$, while $\phi(t)$ is a solution of the corresponding free wave equation. The requirement then, is, that at sufficiently remote times in the past the wave packet $\psi(t)$ must reduce to $\phi(t)$,

$$\psi(t)-\phi(t)\rightarrow 0 \quad\text{if}\quad t\rightarrow-\infty. \tag{5.8}$$

This replaces (5.1) for ψ_a^+ (the two requirements are obviously equivalent on physical grounds). Now it is equally easy to verify that (5.8) is satisfied by either (5.4) or (5.5). For example, expanding (5.4) in a complete set[25] of eigenstates ψ_n of the operator H we get

$$\psi_a^+-\phi_a=\sum_n(E_a-E_n+i\eta)^{-1}\psi_nR_{na}, \tag{5.9}$$

where

$$R_{na}=(\psi_n,V\phi_a). \tag{5.10}$$

Inserting into (5.6)

$$\psi(t)-\phi(t)=\sum_n\psi_nf_n(t), \tag{5.11}$$

$$f_n(t)=\sum_aR_{na}c_a(E_a-E_n+i\eta)^{-1}e^{-iE_nt}. \tag{5.12}$$

Now we may designate by A a set of variables which *together* with E_a defines the state a and write

$$\sum_a=\sum_A\int dE_a. \tag{5.13}$$

It is then easy to evaluate the integral over E_a in the limit $t=\pm\infty$. The integral is of the form

$$I=\int g(E)(E-E_n+i\eta)^{-1}e^{-iEt}dE, \tag{5.14}$$

[21] The statements as to the number of vanishing denominators refer to the assumption that the energy available is insufficient to give two outgoing mesons.

[22] Our only excuse for indulging in this familiar divertisement is, that otherwise we would have to refer the reader to more specialized papers.

[23] The "explicit" form (5.4) has been used by Chew and Goldberger [Phys. Rev. **87**, 778 (1952)] and Gell-Mann and Goldberger [Phys. Rev. **91**, 70 (1953)]. The "implicit" form (5.5) has been used, with a different notation, since Born's classical paper [Z. Physik **37**, 863; **38**, 803 (1926)]. The concise notation (5.5) is that of B. Lippmann and J. Schwinger [Phys. Rev. **79**, 481 (1951)].

[24] A method to construct Green's functions satisfying the outgoing wave condition by integration along a deformed path (which is essentially the $i\eta$ trick) was described by A. Sommerfeld in 1912 ["Die Greensche Funktion der Schwingungsgleichung," Jahresber. der Deutschen Math. Verein. **21**, 309 (1912)].

[25] It is well known that the functions Ψ_a^+ (or alternatively the functions Ψ_a^-) constitute such a set, if the potential V has no bound states. Otherwise, one must add to the set $\Psi_a^+(\Psi_a^-)$ a complete set of bound states. For our present proof it is sufficient to assume that the set Ψ_n exists.

where $g(E)$ is a smoothly variable function of E which vanishes outside of a finite interval. It is well known (it can be shown by deformation of the path of integration into a semicircle) that

$$I \rightarrow 0 \quad \text{as} \quad t \rightarrow -\infty. \tag{5.15}$$

This then proves that (5.6) satisfies condition (5.8); the nature of the argument should make it clear that the $i\eta$ device is absolutely general.

We may, however, push the calculation a little further and obtain from it another well-known result. First of all, let us remark that if we build a wave packet such as (5.6) but using solution ψ_a^-, it will enjoy in a similar manner the property (5.8) but for $t \rightarrow +\infty$. We also note that the same procedure which leads to (5.15) shows that

$$I \sim -2\pi i g(E_n) e^{-iE_n t} \quad \text{as} \quad t \rightarrow +\infty \tag{5.16}$$

in which we have omitted terms in $i\eta$, since the limit $\eta \rightarrow 0$ was always understood. Then

$$f_n(t) = -2\pi i \sum_A R_{na} c_a e^{-iE_n t} \tag{5.17}$$

where the states a are now "on the energy shell," i.e., subject to $E_n = E_a$. We may notice that if E_n does not belong to the internal ΔE in which $c_a \neq 0$, $f_n(t)$ will vanish. In particular E_n must belong to the continuum.

We now specialize the set ψ_n by the assumption[26] that it consists of the bound states (if any) and of the states ψ_a^- and calculate the right-hand side of (5.11) in the limit $t \rightarrow +\infty$. The contribution of the bound state vanishes as explained before. Hence

$$\sum_n \psi_n f_n(t) \sim \sum_b c_b' e^{-iE_b t} \psi_b^- \tag{5.18}$$

with

$$c_b' = -2\pi i \sum_A R_{ba} c_a = -2\pi i \sum_a \delta(E_b - E_a) R_{ba} c_a, \tag{5.19}$$
$$R_{ba} = (\psi_b^-, V\phi_a).$$

And now the decisive point: (5.18) is a wave packet of the type discussed before, hence for $t \rightarrow +\infty$ it can be evaluated by replacing ψ_b^- by its plane wave part ϕ_b. Hence (5.18) tends to

$$\phi_{sc}(t) = \sum_b c_b' e^{-iE_b t} \phi_b. \tag{5.20}$$

The result of or calculation is that a wave process $\psi(t)$ which begins (at $t \approx -\infty$) with the free wave packet (5.7) will become after the scattering process is completed ($t \rightarrow +\infty$)

$$\psi(t) = \phi(t) + \phi_{sc}(t) \tag{5.21}$$

where ϕ_{sc} is the free wave packet described by Eqs. (5.19) and (5.20). Clearly ϕ and ϕ_{sc} contain all the information about incident and scattered states that we may wish, and once the coefficients R_{ba} (for $E_b = E_a$) are known, the calculation of the cross section is merely a matter of wave kinematics. We may save ourselves all further calculations, however, by pointing out that in the first Born approximation the kinematical part of the calculation must obviously be the same, the only difference being that in the expression for R_{ba} the exact solution ψ_b^- is replaced by the zero-order solution ϕ_b:

$$R_{ba} \rightarrow (\phi_b, V\phi_a). \tag{5.22}$$

Any cross section can therefore be obtained from the corresponding expression in the Born approximation by replacing the matrix element on the right of (5.22) by the exact expression (5.19).

And now, back to meson-field theory!

We must point out, first of all, that the standard discussion just presented cannot be applied, strictly speaking, to our problem. The whole discussion is based in fact on the assumption that the perturbation part V of the Hamiltonian *becomes negligible when the colliding particles are far away from each other*. Only then can we assume that the incident state is an eigenstate of H_0, and that the exact energy of the state is equal to the eigenstate of H_0 for the incident state.

As is well known, these assumptions fail when field theoretic interactions are involved. In particular, our perturbation operator $\mathcal{H}$ cannot be treated like V; even when the incident meson is far from the nucleon, $\mathcal{H}$ cannot obviously be neglected. This would be tantamount to neglecting the difference between a "real" and a "bare" nucleon.

One known way to deal with this difficulty is to use nonstationary perturbation theory, assuming in addition an "adiabatic switching-on" of the interaction constant f. For our present purpose, however, it is more instructive to rely on the stationary treatment, and we shall see that, for a theory having a relatively simple structure such as ours, this can be done quite simply.

Various authors[27] have shown how to take properly into account, in the stationary treatment, the correction of the energy of the incident state due to self-energy effects. This is not, however, the only correction that is required in the standard discussion; we must also correct *the wave function* describing the incident state.[28]

[26] We shall prove in Appendix C, that the states $|p\alpha+\rangle$, which are the field theoretic analog of our present ψ_a^+ form an orthonormal system. The proof for the present case is even simpler and well known, see for example Gell-Mann and Goldberger.[23] Furthermore, it is obvious that the states ψ_a^- satisfy a similar orthogonality relation as the states ψ_a^+.

[27] J. Pirenne, Helv. Phys. Acta **21**, 226 (1948); Phys. Rev. **86**, 395 (1952); M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 70 (1953). The existence of such questions was, of course, recognized much earlier, see for example the literature quoted in W. Heitler, *Theory of Radiation* (Clarendon Press, Oxford, England), third edition.

[28] Again, this question is already discussed in the classical papers on the infrared catastrophe of bremsstrahlung [F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937); W. Pauli and M. Fierz, Nuovo cimento **15**, 167 (1938)]. In that case, however, the Hamiltonian contains two terms H_{rad} and $V(x)$ of which the latter can be neglected when the electron is far from the scattering center; the above mentioned authors therefore proceed first to

This we do simply by writing the incident state Φ_a as follows:

$$\Phi_a = a_p^* |\alpha\rangle, \tag{5.23}$$

that is, by applying a meson creation operator to the state vector describing a real nucleon. Simultaneously, we assume for the incident energy the value

$$E_p = \omega_p + E_s. \tag{5.24}$$

We now show that (5.23) satisfies the necessary condition of being a good solution of the Schrödinger equation (2.4) *when the meson is far away* from the nucleon. In order to justify this statement, and in fact even to give it a definite meaning, it is necessary to use a formalism which somehow emphasizes the meson's position, rather than its momentum. This may be done, for instance, by using a wave packet as in (5.7). Consider a wave packet for the meson

$$g(x,t) = \sum_p c_p u_p(x) e^{-it\omega_p}, \tag{5.25}$$

where $u_p(x)$ is a plane wave [Eq. (1.3)], and suppose the c_i's are chosen so that for $t<0$ the wave packet is completely negligible at the nucleon's position, $g(0,t) = 0$. To describe now the same meson state in the presence of the nucleon we build.

$$\Phi(t) = \sum_p c_p e^{-it(E_s+\omega_p)} a_p^* |\alpha\rangle \tag{5.26}$$

and verify that this satisfies the time dependent Schrödinger equation $i\partial\Phi/\partial t = H\Phi$ as long as the meson wave packet does not overlap the nucleon. In fact, remembering that

$$\begin{aligned} [H_0, a_p] &= -\omega_p a_p, & [\mathcal{H}, a_p] &= -V_p^*, \\ [H_0, a_p^*] &= \omega_p a_p^*, & [\mathcal{H}, a_p^*] &= V_p, \end{aligned} \tag{5.27}$$

we find

$$\begin{aligned} &H a_p^* |\alpha\rangle = \omega_p a_p^* |\alpha\rangle + V_p |\alpha\rangle + \\ &a_p^* H |\alpha\rangle = (E_s + \omega_p) a_p^* |\alpha\rangle + V_p |\alpha\rangle \end{aligned} \tag{5.28}$$

and multiplying this by $c_p(t) = c_p e^{-t(E_s+\omega_p)}$ we see that

$$i\partial\Phi/\partial t - H\Phi = \sum_p c_p(t) V_p |\alpha\rangle. \tag{5.29}$$

Apart from trivial factors, $\sum c_p(t) V_p$ is precisely $g(0,t)$, which proves the assertion.

Having thus shown that (5.23) is a correct incident state, we may ask what incident meson density it represents. Since $|\alpha\rangle$ is normalized to unity, however, it is clear that the incident density at large distances from the nucleon is the same as for the meson wave packet (5.25).

We now proceed to discuss the complete state vector $|p\alpha\rangle$ which, like ψ_a^+ [Eq. (5.1)], consists of the incident state Φ_a plus a "scattered wave" containing no *ingoing* mesons at large distances from the nucleon. In analogy to ψ_a^+ we write $|p\alpha+\rangle$ for this state. Just as in ordinary scattering theory we may also consider the kind of state in which, in addition to Φ_a, there are instead no *outgoing* mesons at large distances; such states will be indicated by $|p\alpha-\rangle$. We now write

$$|p\alpha\pm\rangle = a_p^* |\alpha\rangle + \chi_\pm \tag{5.30}$$

and try to set up, as usual, an equation for χ, that will bring out the difference in boundary condition for the $+$ and $-$ states.

Using (5.23) and (5.27) we see that

$$0 = (H-E)|p\alpha\rangle = (H-E)\chi + V_p|\alpha\rangle. \tag{5.31}$$

This replaces (5.2); the operators of the present problem are not simple partial differential operators as in (5.2) but owing to the general validity of the $i\eta$ device we can still write

$$|p\alpha\pm\rangle = a_p^*|\alpha\rangle + (E_p \pm i\eta - H)^{-1} V_p |\alpha\rangle. \tag{5.32}$$

We expect the solution (5.32) to share many of the properties of the functions $\psi_a^\pm$. For example, selecting a given sign, say $+$, the functions should form an orthonormal system, that is, the scalar product of $|p\alpha+\rangle$ and $|q\beta+\rangle$ should satisfy[29]

$$\langle q\beta+|p\alpha+\rangle = \delta_{pq}\delta_{\alpha\beta}. \tag{5.33}$$

This does not mean, of course, that the $|p\alpha+\rangle$'s form a complete system. In ordinary scattering theory the ψ_a^+'s do not form a complete system, if there are bound states. In our case there are obviously infinitely many states besides (5.32); namely, first of all, the four real nucleon states, then all the states with several ingoing and outgoing mesons, possibly also isobaric states of the nucleon, etc. Nevertheless, (5.33) plays the role of a completeness relation in the subspace of "one real-meson states."

Our next task is to find a prescription on how to extract from (5.32) the information about the scattering amplitude from the incident state $a_p^*|\alpha\rangle$ to a state say $a_q^*|\beta\rangle$. We may do this by expanding the scattered part (i.e., the second term) of (5.32) in suitable eigenstates of the exact Hamiltonian. The procedure follows closely that of Eqs. (5.9) to (5.19); the state ψ_b^- is replaced by $|q\beta-\rangle$. Just as in that case we have to consider other eigenstates, in order to have a complete expansion, the real nucleon states, for example, but these will certainly not appear in the scattered wave (at $t\to+\infty$) for the same reason why bound states did not contribute before (physically: because otherwise energy conservation would be violated). On the other hand, if the incident energy is, for example, above the threshold for meson production, states with two real mesons (in our symbolism states $|qk\beta-\rangle$) would not only appear in the expansion but would contribute asymptotically at $t\to+\infty$. We do not plan to calculate such processes, however, and will not consider them

give a solution of the problem with H_{rad}, but neglecting $V(x)$; this gives the wave function for the incident state. From then on they can follow the standard procedure. The present problem is a little different.

[29] For a proof of (5.33) see Appendix C.

any further. The presence of such terms in the expansion is indicated by the dots in Eq. (5.34) below.

The analog of (5.9) is then

$$|p\alpha+\rangle - a_p^*|\alpha\rangle = \omega_p^{-1}\sum_\beta|\beta\rangle\langle\beta|V_p|\alpha\rangle + \sum_{q\beta}(\omega_p-\omega_q+i\eta)^{-1}|q\beta-\rangle R_{p\alpha}(q\beta)+\cdots, \quad (5.34)$$

where

$$R_{p\alpha}(q\beta) = \langle q\beta-|V_p|\alpha\rangle. \quad (5.35)$$

The rest of the argument proceeds exactly as in the potential case, so that (5.35) replaces R_{ba} of Eq. (5.19) in the cross-section formula without any further change.

As is well known, formulas such as (5.19) can be used as the starting point of a perturbation expansion. The same applies to (5.35). For this as well as for other purposes it is, however, desirable to subject the formula to a small transformation. Using (5.32) to express $|q\beta-\rangle$ we find

$$R_{p\alpha}(q\beta) = \langle\beta|a_qV_p|\alpha\rangle + {} + \langle\beta|V_q^*(E_q+i\eta-H)^{-1}V_p|\alpha\rangle. \quad (5.36)$$

The first term on the right can be transformed further using $a_qV_p = V_pa_q$ and the identity

$$a_q|\alpha\rangle = (E_s-\omega_q-H)^{-1}V_q^*|\alpha\rangle, \quad (5.37)$$

which can be proved as follows. The operator $H+\omega_q-E_s$ is definite positive, and has a well-defined inverse. Hence (5.37) is proved if we show that

$$(E_s-\omega_q-H)a_q|\alpha\rangle = V_q^*|\alpha\rangle. \quad (5.38)$$

The latter, however, can be easily verified by interchanging H and a_q by means of (5.27).

Inserting now (5.37) into (5.36) we have finally[30]

$$R_{p\alpha}(q\beta) = \langle\beta|V_q^*(E_q+i\eta-H)^{-1}V_p|\alpha\rangle + {} + \langle\beta|V_p(E_s-\omega_q-H)^{-1}V_q^*|\alpha\rangle \quad (5.39)$$

where we may notice that the second term on the right can be obtained from the first by changing ω_q to $-\omega_q$ (notice that $E_q=E_s+\omega_q$) and interchanging V_p with V_q^*.

Equations (5.39) and (5.35) form the basis for the following discussion of meson scattering.

6. MESON SCATTERING III

We shall now discuss the methods that can be used to evaluate (5.39). An obvious idea is to expand in powers of f. We shall examine this first.

To lowest order, i.e., replacing $|\alpha\rangle$ by ϕ_α, H by H_0, and neglecting E_s, the expression (5.39) reduces, of course, to the Born approximation of Sec. 4, Eqs. (4.1) and (4.2). The first and second terms give, respectively, the contributions of the graphs in Figs. 7(a) and 7(b). Higher-order contributions arise when we expand $(E-i\eta-H)^{-1}$ and $(E_s-\omega_q-H)^{-1}$ as well as the eigenstates $|\alpha\rangle$ and $|\beta\rangle$ in powers of f. In doing these expansions it is not necessary to remember that E_s and $E(=E_s+\omega_q)$ are themselves functions of f; we may instead regard E_s as an arbitrary parameter independent of f. This allows us to use the same kind of expansion as in Sec. 2, namely

$$|\alpha\rangle = Z_2^{\frac{1}{2}}\{\phi_\alpha+\cdots+(E_s-H_0)^{-1}\Lambda\mathcal{H}(E_s-H_0)^{-1}\cdots \cdots\Lambda\mathcal{H}(E_s-H_0)^{-1}\mathcal{H}\phi_\alpha+\cdots\}. \quad (6.1)$$

Also Z_2, of course, can be expanded, but it is preferable to leave it as an explicit factor. Expanding similarly the inverse operators

$$(E+i\eta-H)^{-1} = (E+i\eta-H_0)^{-1} + (E+i\eta-H_0)^{-1}\mathcal{H}(E+i\eta-H_0)^{-1}+\cdots, \quad (6.2)$$

we see that the general term of the expansion contains both factors $\mathcal{H}$ and $\Lambda\mathcal{H}$. Representing for brevity energy denominators by dashes, the structure of the general term is more precisely

$$\phi_\beta^*\mathcal{H}-\Lambda\mathcal{H}-\Lambda\mathcal{H}-\cdots-\Lambda\mathcal{H}-V-\mathcal{H}-\mathcal{H}-\cdots -\mathcal{H}-V-\Lambda\mathcal{H}-\cdots-\cdots\Lambda\mathcal{H}-\mathcal{H}\phi_\alpha.$$

When this is expressed more explicitly as a sum over intermediate states similar to (26a) we notice a fundamental difference. *Bare-nucleon states are now allowed*, although only for the intermediate states *between* the two V operators. A description in terms of graphs similar to Sec. 3 is clearly possible; we must only add a convention to represent the V operators. Since $V_p(V_q^*)$ clearly symbolizes the absorption (emission) of the incoming (outgoing) meson, we can represent V_p by allowing a dashed line representing the incoming meson to terminate at the appropriate point on the nucleon line and similarly for V_q^*. This convention is clearly related to that used in Sec. 4.

To give two examples we plot in Figs. 9(a) and 9(b) two graphs contained in respectively the first and second term of (5.39). All graphs from the second term have of course "crossed real meson lines." It is easy to see that the graphs that occur are all those that might be suggested by naive perturbation theory, *except* those with self-energy parts on the end lines. We have already explained what we mean by a self-energy part, so the examples of excluded graphs in Fig. 10 will suffice to explain the statement. A self-energy part between the end points of the free mesons, such as it occurs in Figs. 9(a) and 9(b) is of course allowed.

The correct recipe, which was first formulated by Dyson for quantum electrodynamics is then the following:

[30] One can also arrive at the expression (5.39) for the scattering amplitude from the S-matrix formula: $S_{q\beta,p\alpha} = \langle q\beta-|p\alpha+\rangle$. This requires manipulations similar to those of Appendix C, and is in fact the way in which I found the expression. The proof adopted above includes various simplifications suggested to me by Dr. F. Low, see also reference 8. A remark is needed also about Eq. (5.37); if one expands the inverse operator and $|\alpha\rangle$ in powers of the interaction one gets an expression which has an obvious physical meaning in terms of graphs (we leave this to the reader); this is, in fact, how I first found the equation. This is an elementary illustration of the usefulness of graphs.

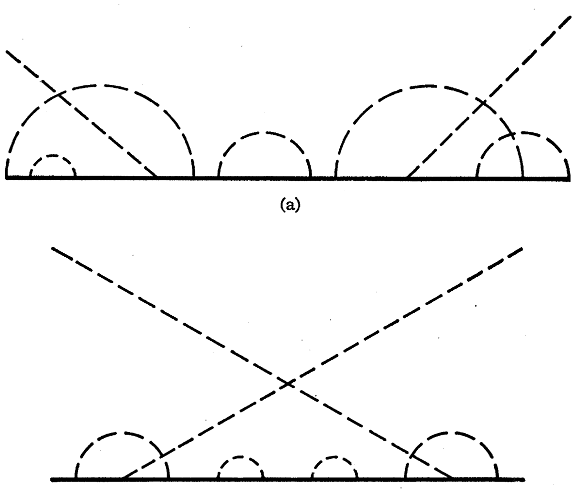
Omit all graphs with self-energy parts at the ends of nucleon lines, but multiply the sum over the remaining graphs by a factor $Z_2^{\frac{1}{2}}$ for each end of a nucleon line (this is the rule in a form valid for any number of nucleons present).

On the basis of this formulation of perturbation theory and of a further discussion of renormalization (which will not be necessary here), Chew has developed an approximate treatment of the meson scattering problem. The method is based on a classification of graphs in order of importance, suggested by the Tamm-Dancoff approximation,[31] and leads to a linear integral equation for the numerical discussion of which we must refer the reader to the original papers.[32] One can show that this linear integral equation also appears as an approximation to the nonlinear equation of Low,[7] to which we now turn our attention.*

Let us expand the inverse operators on the right-hand side of (5.39) by means of the theorem

$$(E-H)^{-1}=\sum_n |n\rangle (E-E_n)^{-1}\langle n|, \qquad (6.3)$$

where the sum is over a complete set of eigenvalues E_n and eigenstates $|n\rangle$ of the full Hamiltonian H. The eigenvalues (omitting the possibility of isobaric states) are E_s, $E_s+\omega_k$, $E_s+\omega_k+\omega_h$, $\cdots$. The contribution from



FIGS. 9. Two graphs representing higher-order terms contained in (5.39).

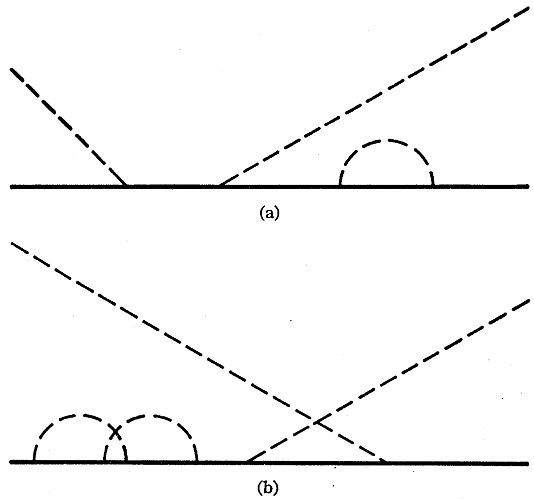


FIG. 10. Excluded graphs with self-energy parts on the end lines.

the fourfold level E_s is remarkably simple, namely

$$(1/\omega_q)\sum_\gamma\{\langle\beta|V_q^*|\gamma\rangle\langle\gamma|V_p|\alpha\rangle - \langle\beta|V_p|\gamma\rangle\langle\gamma|V_q^*|\alpha\rangle\}, \qquad (6.4)$$

which, owing to the proportionality relation (2.15), is nothing but the Born approximation (4.1) and (4.2), with f replaced by f_r.

In the remaining terms of the sum (6.3), Low chooses as states $|n\rangle$, with energy $E_s+\omega_k$, the scattering states Eq. (5.32) with the ingoing convention, i.e. $|k\gamma-\rangle$. For the higher states he would use similarly states $|kh\gamma-\rangle$, $\cdots$ etc. but no detailed calculations have really been made including these states.[33] Clearly it is possible to consider a first, second, $\cdots$, nth approximation in which one keeps only the amplitudes involving no more than one, two, $\cdots$, n mesons. The mesons counted, however, are real (incident or outgoing) meson not virtual mesons, as in the Tamm-Dancoff scheme. This seems a much better way to achieve a classification of terms in order of importance, better, that is, than the less well-defined scheme earlier suggested by Chew. It seems quite reasonable to assume, that for low-energy scattering the first approximation should be adequate (estimates of the next term made by Chew and Low confirm this).

Let us write, therefore, the Low integral equation in this approximation; to this end we notice that in expanding, for example the first term on the right of (5.39) there occurs the product of $\langle\beta|V_q^*|k\gamma-\rangle$ and of $\langle k\gamma-|V_p|\alpha\rangle$. The latter, and the complex conjugate of the former, are just the kind of expression considered in (5.35), except that the state k is not, in general, on the energy shell with the state p: $\omega_k\neq\omega_p$. Nothing prevents us, however, from assuming (5.35) as a definition of the symbol on the left, without restric-

[31] G. F. Chew.[6] For a discussion of the true Tamm-Dancoff method, as applied to meson scattering, see Dyson, Ross, Salpeter, Schweber, Sunderasan, Visscher, and Bethe, Phys. Rev. **95**, 1644 (1954).

[32] G. F. Chew, Phys. Rev. **95**, 285, 1669 (1954); F. F. Salzman and J. N. Snyder, Phys. Rev. **95**, 286 (1954); see also J. L. Gammel, Phys. Rev. **95**, 209 (1954).

* Castillejo, Dalitz, and Dyson (to be published) have shown that the Low Equation for a simplified model possesses an infinity of solutions. The question as to how the physically right solution is to be identified undoubtedly deserves further study. I am indebted to the aforementioned authors for advance communication of their results.

[33] This would involve setting up separate equations similar to (5.35) and (5.39) for higher-order processes. Although this seems to be feasible in principle, it gets quite involved.

tions. One then has, finally, using B for the Born approximation

$$R_{p\alpha}(q\beta)=B_{p\alpha}(q\beta) -\sum_{\gamma k}\{R_{q\beta}^*(k\gamma)R_{p\alpha}(k\gamma)/(\omega_k-\omega_q-i\eta) +R_{p\beta}^*(k\gamma)R_{q\alpha}(k\gamma)/(\omega_k+\omega_q)\}. \quad (6.5)$$

We are, of course, interested in the case $\omega_p=\omega_q$ but again, if we follow the steps of the proof carefully, we can see that (6.5) holds for the quantity defined by (5.35) without the foregoing restriction; one has only to be careful in writing $E=E_s+\omega_q$ (and not $E_s+\omega_p$) in expressions such as (5.39). As Low points out this is only possible because the dependence of (5.35) on the energy of the state p is trivial, namely just a factor $p\omega_p^{-\frac{1}{2}}v(p)$ present in the operator V_p; exactly the same dependence appears[34] on the right-hand side of (6.5). On the other hand the dependence on ω_q is nontrivial, as one sees from the right-hand side of both (5.35) and (6.5). (This is why in the energy denominators of (6.5) one *must* write ω_q and not ω_p.)

Thus (6.5) constitutes an inhomogeneous, nonlinear, integral equation for the scattering amplitude. In order to discuss its properties it is convenient to assume that the states p and q are not plane waves, but spherical waves like those considered at the end of Sec. 4. Nothing of what we have said is altered, except the meaning of the symbols p and q. Now $B_{p\alpha}(q\beta)$ is precisely the matrix element, between ϕ_β and ϕ_α, of (4.16), except that f is replaced by f_r. Low also saves some writing, by omitting the nucleon indices α, β, γ systematically; i.e., he treats $R_{p\alpha}(k\gamma)$ as the $\gamma\alpha$ element of a matrix $R_p(k)$; $R_{q\beta}^*(k\gamma)$ is then the $\beta\gamma$ element of the Hermitean conjugate of the matrix $R_q(k)$. This Hermitean conjugate is designated by $R_q^*(k)$. As a result one can simply rewrite (6.5) omitting the Greek indices throughout; the products $R_q^*(k)R_p(k)$ and $R_p^*(k)R_q(k)$ are now, of course, matrix products.

One is naturally tempted to go further, and treat also p and q in $R_p(q)$ as matrix indices; more specifically p stands for $|\mathbf{p}|$, i, λ and q for $|\mathbf{q}|$, j, μ where, for the sake of brevity, we shall replace the two indices i and λ by a single index λ $(=1,\cdots,9)$. This index specifies the angular momentum and charge state of the meson, just like α does for the nucleon. Similarly q will now be replaced by $|\mathbf{q}|\mu$, and k by $|\mathbf{k}|$, ν. This forces some changes in notation, like replacing $\sigma_i\tau_\lambda$ by $(\sigma\tau)_\lambda$, $\sigma_j\tau_\mu$ by $(\sigma\tau)_\mu$ but causes otherwise no trouble. We now write $R_p(q)$ as the $\mu\lambda$ element of a matrix r, namely

$$R_p(q)=-\tfrac{1}{2}[pv(p)v(q)/q^2(\omega_p\omega_q)^{\frac{1}{2}}]r_{\mu\lambda}. \quad (6.6)$$

Some of the factors introduced on the right will be convenient later; the p dependence, however, has been specifically brought out, in such a way that r depends on ω_q only. The matrix $r^\dagger$ with the matrix elements

$$(r^\dagger)_{\lambda\mu}=(r_{\mu\lambda})^* \quad (6.7)$$

is now the Hermitean conjugate of r, when regarded as a matrix in a space of $4\times 9=36$ dimensions (this being the number of possible angular momentum and charge states of the nucleon+one-meson system). In (6.5) we can now perform symbolically the summation over the intermediate index, ν, the first term between braces gives $(r^\dagger r)_{\mu\lambda}$ and the second $(r^\dagger r)_{\lambda\mu}$. We can relate $B_p(q)$, Eq. (4.16), to a matrix $b(\omega_q)$ just like $R_p(q)$ is related to $r(\omega_q)$:

$$b_{\mu\lambda}(\omega_q)=-\tfrac{1}{3}(f_r^2/\mu^2)(q^3/\omega_q) \times[(\sigma\tau)_\mu(\sigma\tau)_\lambda-(\sigma\tau)_\lambda(\sigma\tau)_\mu], \quad (6.8)$$

where a previous remark[34] has been duly considered. Equation (6.5) finally becomes [in summing over $|\mathbf{k}|$ use rule (1.16)]

$$r_{\mu\lambda}(\omega_q)=b_{\mu\lambda}(\omega_q) +\frac{q^3}{\pi}\int\frac{v^2(k)d\omega_k}{k^3}\left[\frac{(r^\dagger r)_{\mu\lambda}}{\omega_k-\omega_q-i\eta}+\frac{(r^\dagger r)_{\lambda\mu}}{\omega_k+\omega_q}\right]. \quad (6.9)$$

The $r^\dagger r$ under the integral sign is, of course, a function of ω_k; owing to the fact that in the last term the indices $\mu\lambda$ are crossed, we cannot write (6.9) as a matrix equation simply by dropping the indexes altogether. (One should also notice that while the meson indices are crossed, the nucleon indices are not.) We may just mention in passing that this pecular structure is related to the Gell-Mann–Goldberger crossing theorem[35] according to which

$$r_{\mu\lambda}(\omega)=r_{\lambda\mu}(-\omega). \quad (6.10)$$

Without discussing this relationship in general, we may see what it means in terms of Eq. (6.9). One sees at once that (6.8) satisfies the condition (6.10). In order for the whole expression (6.9) to satisfy (6.10), we must simply rewrite the second denominator as $\omega_k+\omega_q-i\eta$. This, of course, makes no difference as long as ω_q is positive ($\int d\omega_k$ runs from the meson mass μ to $+\infty$). It does, however, make a difference when we try to extend the value of $r(\omega_q)$ to negative values of ω_q by using the right-hand side of (6.9) as a definition. We now clearly see that (6.10) is a theorem about *analytic continuation* of the function $r(\omega)$. Integrals of the type appearing on the right-hand side of (6.9) are called Stieltjes transforms; they are known to define analytic functions of the variable ω_q, in the whole complex plane of this variable except for certain cuts. In the case of (6.9) the cuts run from $-\infty$ to $-\mu$ and from μ to $+\infty$. There is, therefore, a gap from $-\mu$ to $+\mu$, through which one can pass from the positive to the negative imaginary half-plane. If one thinks the foregoing relationships through, one sees that (6.10) is valid if the analytic continuation from the positive to the negative real axis is made along a path which starts just above the positive real axis and

[34] Provided B is written in the form $pqv(p)\,v(q)\omega_p^{-\frac{1}{2}}\omega_q^{-\frac{1}{2}}$.

[35] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).

goes, through the gap, to a point just below the negative real axis.

Special attention must be paid in this consideration to the q^3 factor in (6.8) and (6.9). In fact $q=(\omega_q^2-\mu^2)^{\frac{1}{2}}$ is also an analytic function of ω_q in the plane with the cuts defined previously; when continued along the path described earlier, q is real and positive on both the positive and the negative real axis, when $|\omega_q|>\mu$ and is positive imaginary in the interval $-\mu<\omega_q<+\mu$. The sign of q is, of course, essential in Eq. (6.10).

Equation (6.9) has many other remarkable properties, for which we must refer the reader to the work of Chew and Low.[8,36] We only wish to perform the transformation of (6.9) to Low's set of three equations[37] for the phase shifts.

As we know, one can pass by means of a unitary transformation, from the states defined by meson and nucleon indices λ and α, to states with quantum numbers J, J_z, T, T_3. In this representation, the matrix r must be diagonal, with eigenvalues depending only on J and T. Furthermore, owing to the symmetry already mentioned,[11] the eigenvalues for the states 13 and 31 must be equal. Let us designate with g_u ($u=1, 2, 3$) the eigenvalues of r for the states 11, 13 (or 31), and 33, respectively. Remembering Table II we see that the matrices ϵ and ζ defined by

$$\epsilon_{\mu\lambda}=\zeta_{\lambda\mu}=(\sigma\tau)_\mu(\sigma\tau)_\lambda, \tag{6.11}$$

have, respectively, the eigenvalues 9, 0, 0 and 1, -2, 4 for the same states. Hence introducing three new matrices Λ_u ($u=1, 2, 3$) such that

$$\begin{aligned}\epsilon&=9\Lambda_1,\\ \zeta&=\Lambda_1-2\Lambda_2+4\Lambda_3,\\ 1&=\Lambda_1+\Lambda_2+\Lambda_3,\end{aligned} \tag{6.12}$$

the eigenvalues of these Λ matrices will be 1, 0, 0 for Λ_1, 0, 1, 0 for Λ_2, and 0, 0, 1 for Λ . We then have

$$r=\sum_{u=1}^{3} g_u\Lambda_u \tag{6.13}$$

and since $\Lambda_u{}^\dagger=\Lambda_u=\Lambda_u{}^2$ we have also

$$r^\dagger r=\sum_{u=1}^{3}|g_u|^2\Lambda_u. \tag{6.14}$$

In order to cross the indices λ and μ, we may consider three other matrices Λ_u' such that $(\Lambda_u')_{\mu\lambda}=(\Lambda_u)_{\lambda\mu}$. Crossing indices, however, simply interchange ϵ and ζ; one finds then

$$\Lambda_u'=\sum_{v=1}^{3}\Lambda_v A_{vu} \tag{6.15}$$

where the matrix A is

$$A=1/9\begin{pmatrix}1 & -8 & 16\\ -2 & 7 & 4\\ 4 & 4 & 1\end{pmatrix}. \tag{6.16}$$

Equation (6.15) allows us to express the last term in (6.9) as a matrix element with uncrossed indices. The whole equation then becomes a matrix equation, of the form $\sum_u X_u\Lambda_u=0$ where X_u is not a matrix; clearly this gives three equations $X_u=0$.

In the foregoing statement, we have made use of the fact that the Born matrix (6.8) is itself of the form

$$\begin{aligned}b(\omega_q)&=(q^3/\mu^2\omega_q)\textstyle\sum_u\lambda_u\Lambda_u,\\ \lambda_u&=f_r{}^2(-8/3,\ -2/3,\ 4/3),\end{aligned} \tag{6.8'}$$

where one can point out, in addition, that since $b(\omega_q)$ satisfies, as it must, the Gell-Mann–Goldberger crossing theorem, one must have

$$(-\omega_q)^{-1}\textstyle\sum_u\lambda_u\Lambda_u'=\omega_q{}^{-1}\sum_u\lambda_u\Lambda_u,$$

which reduces to [38]

$$\textstyle\sum_v A_{uv}\lambda_v=-\lambda_u, \tag{6.17}$$

λ is therefore an eigenvector of the matrix A.

To sum up, one inserts (6.13), (6.14), and (6.8′) into (6.9), and taking into account Eq. (6.15) one finds Low's three equations:

$$g_u(\omega_q)=\lambda_u q^3/\mu^2\omega_q$$
$$+\frac{q^3}{\pi}\int\frac{v^2(k)d\omega_k}{k^3}\left[\frac{|g_u(\omega_k)|^2}{\omega_k-\omega_q-i\eta}+\frac{A_{uv}|g_v(\omega_k)|^2}{\omega_k+\omega_q}\right], \tag{6.18}$$

where a $\sum_v$ has been tacitly understood. To complete the discussion, we have only to establish the connection between the functions $g_u(\omega)$ and the phase shifts δ_{11}, $\delta_{13}=\delta_{31}$, and δ_{33}, which, with the abbreviation δ_u ($u=1, 2, 3$) for $\delta_{11}(\omega_q)$, etc., is

$$e^{i\delta_u}\sin\delta_u=|v(q)|^2 g_u(\omega_q) \tag{6.19}$$

i.e., apart from the cut-off factor, which Low omits, g is just $e^{i\delta}\sin\delta$. This comes about as follows:

Adapting Eq. (5.19) to the present situation, let us write the relation between incident and scattered wave packet in the form

$$c_{q\beta}'=-2\pi i\textstyle\sum_{p\alpha}\delta(\omega_q-\omega_p)R_{p\alpha}(q\beta)c_{p\alpha}. \tag{6.20}$$

We notice that (5.39) and (1.20) indicate that R is proportional to N'^2; this cancels against the inverse factor in the density of states (1.16), as expected. Transforming to states of definite J and T, $c_{p\alpha}$ becomes $c_u(\omega_p)$ and using (6.6) and (6.13) the relation (6.20)

[36] The author is also indebted to Dr. R. Serber and Dr. T. D. Lee for an instructive conversation about a paper they are preparing on this subject.

[37] Reference 7, Eq. (3.11).

[38] G. F. Low, Proceedings of the Fifth Annual Rochester Conference.

becomes a simple proportionality relation

$$c_u' = 2iv^2(q)g_u c_u \tag{6.21}$$

both sides are taken at the same energy ω_q. Now the wave packet existing after the process is equal to incident+scattered wave packet, i.e., $c+c'$; this must have the same norm as the incident packet, hence the ratio of c_u+c_u' to c_u must be merely a phase factor, which is usually written in the form $e^{2i\delta_u}$, where δ_u is called the phase shift. Hence

$$c_u' = 2ie^{i\delta_u}\sin\delta_u c_u$$

which together with (6.21) yields immediately (6.19).

7. SOME CONCLUSIONS AND REMARKS

We shall not attempt to discuss the problem of solving Eq. (6.18). If the "crossed" term is left out, the three functions g_u become uncoupled, and it is rather easy to find a solution.[39] This turns out to be identical with an approximate solution of the earlier formulation by Chew, in terms of a linear integral equation. This earlier formulation was able to reproduce the data for the large phase shift δ_{33} quite accurately. One has good reason to believe therefore, that also the newer and *a priori* better formulation will possess this feature. By means of an extrapolation to zero meson energy[8] it now seems that the best values for the coupling constant is

$$f^2 \approx 0.08 \tag{7.1}$$

in reasonable agreement with previous estimates. These earlier estimates also gave a cut-off momentum of the order

$$k_{\max} \sim M, \tag{7.2}$$

where M is the nucleon mass. As is well known, the phase shifts $\delta_{31}=\delta_{13}$ and δ_{11} are very small, and a comparison between calculated and measured values has not yet been possible. It would provide a very interesting test of the theory. It will also be quite interesting to see, how different these phase shifts turn out to be in the new as against the older formulation. A plausible guess is that they will be affected by the change much more than δ_{33}.

There are many other problems to which methods similar to those described here can be applied. One is, of course, photomeson production; this problem too has been formulated by Low in terms of an integral equation,[7] and work is being done on its solution.† This report, however, is already too long to allow us to include a discussion of this, except for one point, which is too essential to be overlooked.

Kroll and Ruderman[40] have proved an important theorem on photomeson production at threshold, in the relativistic theory. A similar relationship holds[6] in the cut-off theory. It may be explained as follows.

In order to discuss photomeson production we must add to the Hamiltonian (1.10) terms representing the interaction with an incident electromagnetic wave. Of these terms, the only one that matters near threshold, has the form

$$(4\pi)^{\frac{1}{2}}(f/\mu)e\int \boldsymbol{\sigma}\cdot\mathbf{A}(\tau_1\phi_2-\tau_2\phi_1)\rho(\mathbf{x})d\mathbf{x}, \tag{7.3}$$

where $\mathbf{A}$ is the vector potential of the electromagnetic wave. It is a term one must introduce in order to satisfy the continuity equation of charge and current[4] (the finite size of the source function gives rise to some complications which we cannot discuss here).

An interesting feature of (7.3) is that it contains the coupling constants f and e simultaneously. Owing to the smallness of e, it is much more reasonable to apply perturbation theory to (7.3) than to the pure meson term. We then calculate the matrix element of (7.3) between the initial state which is a *real* nucleon state $|\alpha>$ (we do not apply perturbation theory to f!) and a final state consisting of a real nucleon and a real meson. The interesting point arises, that in this case the meson is in an s state and can be treated as free [this is because the wavelength of the incident quantum is large compared to the size of the source function ρ, hence A may be pulled out of the integral, which then involves $\int\phi\rho d\mathbf{x}$ and expanding ϕ in spherical waves, only s states will contribute; for the sake of comparison (1.10) involves $\nabla\phi$ instead of ϕ !].

Owing to the fact that mesons in s states do not interact [except through (7.3)] the final state is not one of the complicated states of the previous paragraphs, but just of the form $a_{qs}^*|\beta>$ where a_{qs}^* is a creation operator for an s state. One then sees that (7.3) involves precisely a matrix element of the form (2.13). The result is that the cross section at threshold can be expressed directly in terms of the renormalized coupling constant, and gives another, completely independent determination of this constant.[6] Comparison with experiment[41] gives a value in tolerable agreement with (7.1)!

Other problems to which similar methods can be applied are the anomalous magnetic moments of the nucleon,[42] and probably nuclear forces and meson production in the nucleon-nucleon encounters.

ACKNOWLEDGMENTS

This article is, to a large extent, an improved version of lectures held at the Carnegie Institute of Technology in the Fall of 1954. I am indebted to several people for

[39] G. F. Chew (private communication).

† The reader will find a simple discussion of photomeson production in: E. Fermi, "Lectures on Pions and Nucleons," reproduced in Nuovo cimento Suppl. 17 (1955).

[40] N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1954).

[41] G. Bernardini and E. L. Goldwasser, Phys. Rev. **95**, 857 (1954).

[42] M. H. Friedman, Phys. Rev. **97**, 1123 (1955); G. Saltzman (private communication).

friendly comments or for advance information on their results. Amongst these I should mention Dr. F. Dyson, Dr. M. Goldberger, Dr. T. D. Lee, Dr. R. Serber, and especially Dr. G. F. Chew and Dr. F. Low, whom I must thank for several illuminating conversations.

APPENDIX A

We summarize here Kemmer's classical argument[43] for the isotopic-spin dependence of the Hamiltonian (1.10). We assume the following:

(a) Nuclear forces, as observed in low-energy nucleon phenomena, are due to virtual emissions and reabsorptions of π mesons, according to the Yukawa scheme.

(b) Nuclear forces (as distinct from Coulomb interactions and other minor electromagnetic interactions) are strictly charge independent, as first postulated by Breit and Feenberg.

Assumption (b), which is now much strengthened by the comparison of levels in "mirror nuclei," asserts, roughly speaking, that the energy of a nucleon level is not altered (apart from Coulomb corrections) if some of the neutrons are changed into protons or vice versa.

It is necessary, however, to formulate this more precisely. To this end we avail ourselves of the isotopic spin formalism. If we did not do this, the state of the nucleus would be described by a wave function

$$\psi(x_1x_2\cdots x_m x_{m+1}\cdots x_n), \tag{A.1}$$

antisymmetric in the proton coordinates x_1 to x_m, and again in the neutron coordinates x_{m+1} to x_n (x summarizes space and ordinary spin coordinates). A discussion of the relationship between states with the same number n of nucleons, but different neutron-proton ratios then involves rather complicated considerations about permutation operators, similar to those which are necessary when one deals with the n-electron problem *without introducing the spin coordinates explicitly*. As is well known, this procedure, although the most natural one when one neglects spin interactions, leads one to consider a wave function which is separately antisymmetric in the electrons with "spin up" and those with "spin down." It is then rather complicated to discuss the relationship between states with different up/down ratios.

If we introduce the spin coordinates explicitly, a change in the relative numbers of spin-up and spin-down electrons may be achieved very simply by a simultaneous rotation of all electron spins, or more precisely by repeated application of the operators $S_x \pm iS_y$, where S_x, S_y, and S_z are the components of total spin (which correspond to infinitesimal rotations of the aforementioned kind).

A rotation of the spin of an electron, however, is described in quantum mechanics by a unitary transformation in the two-dimensional vector space subtended by the two basic spin states α and β. On this rests the possibility of an analogous treatment of our problem. We assume that the reader is familiar with the elements of the isotopic spin formalism.[2] The analogies noted previously will explain its usefulness in the present context. Regarding proton and neutron as two states of a single particle, two states, that is, of some internal degree of freedom, distinct from translational motion and ordinary spin, we may treat these states, P and N, on the same footing as α and β. One introduces then operators τ_1, τ_2, τ_3 defined by

$$\begin{aligned} &\tau_1 P = N, \quad \tau_1 N = P, \\ &\tau_2 P = iN, \quad \tau_2 N = -iP, \\ &\tau_3 P = P, \quad \tau_3 N = -N, \end{aligned} \tag{A.2}$$

in analogy with Pauli's spin operators σ_x, σ_y, σ_z. Of course, P and N may be regarded as functions $P(i)$ and $N(i)$ of an isotopic spin variable. The wave function of an n-nucleon system is now obtained multiplying (A1) by spin factors $P(i_1)\cdots P(i_m)N(i_{m+1})\cdots N(i_n)$, where i_k is the isotopic spin of the kth nucleon, and antisymmetizing the whole, so as to satisfy the Pauli principle for all nucleons.

All these formal manipulations are completely analogous to those normally carried out with electrons; the result at which we are aiming is the following. If we introduce the components of the total isotopic spin

$$T_\lambda = \tfrac{1}{2}\textstyle\sum_k \tau_\lambda^{(k)} \quad (\lambda = 1,2,3) \tag{A.3}$$

where $\tau_\lambda^{(k)}$ is the operator τ_λ, Eq. (A.2), for the kth nucleon, we notice that the operators T_1, T_2, T_3 represent simultaneous infinitesimal unitary transformations of the spin states of each nucleon and may thus be used to alter the neutron-proton ratio (in analogy to what we said about S_x, S_y, S_z). In fact noting that the linear combinations

$$\tau_\pm = (\tau_1 \pm i\tau_2)/\sqrt{2} \tag{A.4}$$

enjoy the properties

$$\tau_- P = \sqrt{2}N, \quad \tau_+ N = \sqrt{2}P, \quad \tau_- N = \tau_+ P = 0 \tag{A.5}$$

one sees at once that $(T_1 \pm iT_2)\psi$ is a state with one proton more (less) than ψ; it satisfies the Pauli principle if ψ does. The charge independence assumption then states that the foregoing state (if not identically zero) satisfies the Schrödinger equation if ψ does, and with the same energy eigenvalue. This shows that $T_1 \pm iT_2$ commutes with H, so that finally we can write

$$[T_\lambda, H] = 0 \tag{A.6}$$

for $\lambda = 1$, 2. This is the formulation of charge independence which lends itself to a discussion of meson theory. Before we proceed, let us take note of the commutation relations

$$[T_\lambda, T_\mu] = iT_\nu, \tag{A.7}$$

[43] N. Kemmer, Proc. Cambridge Phil. Soc. 34, 354 (1938).

where $(\lambda\mu\nu)$ is any cyclical permutation of (1 2 3). As a consequence, (A.6) is valid also for $\lambda=3$.

We now ask, what requirements a meson theory must satisfy, in order to yield (after elimination of the degrees of freedom of the meson field) a Hamiltonian possessing the three Hermitean integrals of the motion T_λ. It is hard to see how this can happen, unless the theory possesses quite generally three such hermitean integrals (in the same way as, say, conservation of energy, momentum, etc. in the interaction between charged particles is the result of the validity of more detailed conservation theorems in the interaction between each particle and the electromagnetic field). The quantity T_λ which is conserved in meson theory, however, need not be the expression on the right-hand side of (A.3), but may contain in addition to (A.3) a contribution θ_λ of the meson field. In fact the assumption that $\theta_\lambda\equiv 0$ leads, as one can see by an analysis similar to the following, to the conclusion that only neutral mesons are involved. In view of the facts now known, this is not an interesting possibility and we shall not pursue it further.

Let us then write, considering for simplicity the case of one nucleon only

$$T_\lambda=\tfrac{1}{2}\tau_\lambda+\theta_\lambda, \tag{A.8}$$

where θ_λ is a Hermitean operator acting on the states of a meson or more generally on the state of the meson field.[44] We may assume, for example, that π^+, π^-, and π^0 are three different states of the same particle, forming as it were a counterpart of the two states N and P of the nucleon. We might, however, also consider a theory in which only two states, say π^+ and π^-, play a role. For this reason we shall leave the number of charge states of the meson undetermined for the moment and designate an arbitrary orthogonal set of them by ω_m, where m may take 3 or possibly 2 values. For the sake of simplicity we shall forget all dynamical variables other than the i spin of the particles involved.

According to our basic assumption, the Hamiltonian $\mathcal{H}$ is a linear combination of absorption and emission operators b_m and b_m^*, where b_m absorbs mesons in the state ω_m. The coefficients are then operators on the nucleon only, i.e., linear combinations of τ_1, τ_2, τ_3, and the unit matrix 1. Hence the absorption part A of the Hamiltonian will be of the form

$$A=\sum_m b_m(A_{m0}\mathbf{1}+\sum_\lambda A_{m\lambda}\tau_\lambda) \tag{A.9}$$

with numerical coefficients A_{m0}, $A_{m\lambda}$. We set

$$fa_0=\sum_m b_m A_{m0};\quad fa_\lambda=\sum_m b_m A_{m\lambda}, \tag{A.10}$$

where f is a proportionality constant to be determined later. We shall see that the a_λ are themselves absorption operators from a set of orthogonal meson states. Now $\mathcal{H}=A+A^*$, and we shall satisfy (A.6) by the assumption that T_λ commutes with A and A^* separately:

$$[\theta_\lambda,a_0]\mathbf{1}+\sum_\mu[\theta_\lambda,a_\mu]\tau_\mu+\sum_\mu a_\mu\tfrac{1}{2}[\tau_\lambda,\tau_\mu]=0. \tag{A.11}$$

Using $\frac{1}{2}[\tau_1,\tau_2]=i\tau_3$, etc., one sees that (A.11) reduces to

$$\begin{aligned}&[\theta_1,a_0]=[\theta_1,a_1]=0,\\ &[\theta_1,a_2]=ia_3;\quad [\theta_1,a_3]=-ia_2\end{aligned} \tag{A.12}$$

plus the Hermitean conjugates thereof

$$[\theta_1,a_2^*]=ia_3^*;\quad [\theta_1,a_1^*]=0, \tag{A.13}$$

etc. We now notice that according to the well-known commutation laws

$$[b_m,b_n]=0;\quad [b_m,b_n^*]=\delta_{mn} \tag{A.14}$$

and Eq. (A.10) one has

$$[a_\lambda,a_\mu]=0;\quad f^2[a_\lambda,a_\mu^*]=\sum_m A_{m\lambda}A_{m\mu}^* \tag{A.15}$$

By using (A.12) to express a_2 and the identity

$$[[u,v],w]=[u,[v,w]]+[v,[w,u]]$$

we find

$$-i[a_2,a_1^*]=[[\theta_1,a_3],a_1^*]=[\theta_1,[a_3,a_1^*]]+[a_3,[a_1^*,\theta_1]].$$

Now using the fact that $[a_3,a_1^*]$ is a c number, see Eq. (A.15) and (A.13), we see that the foregoing expression vanishes. Similarly

$$\begin{aligned}-i[a_2,a_2^*]=[[\theta_1,a_3],a_2^*]=[\theta_1,[a_3,a_2^*]]\\ +[a_3,[a_2^*,\theta_1]]=-i[a_3,a_3^*].\end{aligned}$$

More generally one sees that, by an appropriate choice of the arbitrary constant f, we can set

$$[a_\lambda,a_\mu^*]=\delta_{\lambda\mu}. \tag{A.16}$$

We thus see that the a_λ's satisfy the canonical commutation laws. This has immediately several important consequences. First we see from (A.15) that the matrix $f^{-1}A_{m\lambda}$ is unitary. This implies that the states ω_m *cannot be less than three* (a charged-meson theory, with no π^0 is ruled out; in this way the existence of a neutral meson was correctly predicted before 1940!). If we take the states,

$$\Omega_\lambda=f^{-1}\sum_m A_{m\lambda}^*\omega_m, \tag{A.17}$$

as basic states for the meson, instead of ω_1, ω_2, ω_3, we see[45] that the variables a_λ, Eq. (A.10), represent absorption operators for these states. We also see that a_0, Eq. (A.10), can be expressed linearly in the a_λ's; no such linear combination can satisfy $[\theta_\lambda,a_0]=0$, however, unless it is identically zero. The unit matrix term in (A.9) therefore vanishes.[46] In conclusion we have

[44] It is customary, in discussing θ_λ, to rely heavily on the analogy with angular momenta. This allows one to short cut some of the mathematical detail. The following more abstract approach is mainly for those readers who do not find the analogy with angular momenta and rotations in ordinary space particularly convincing.

[45] It is actually even easier to see that a_λ^* when applied to the zero meson state creates a meson in a state (A.17). From that, however, the inverse statement follows.

[46] An alternative possibility exists: $f=0$, and only the 1 is present. This leads, however, to the already discarded theory with neutral mesons only.

reduced A to the form

$$A=f\sum_{\lambda=1}^{3} a_\lambda\tau_\lambda, \quad \text{(A.18)}$$

where a_1, a_2, a_3 are absorption operators from the states (A.17). This is, however, just the i-spin dependence postulated in (1.10); all one has to do is to add momentum indices and ordinary spin variables to the nucleon.

For the sake of completeness, however, let us still consider the structure of the operators θ_λ. As operators on the meson field they must be expressible in terms of the a_λ and $a_\lambda{}^*$ variables. The linearity of (A.12) in the a_λ variables, indicates that θ_λ is a bilinear form in a_λ and $a_\lambda{}^*$. One then easily finds by trial that

$$\theta_1=i(a_2a_3{}^*-a_3a_2{}^*) \quad \text{(A.19)}$$

etc. In order to get the general expression of θ_1 for the whole meson field we must append a momentum index to a_λ thus $a_{k\lambda}$ and write

$$\theta_1=i\textstyle\sum_k(a_{k2}a_{k3}{}^*-a_{k3}a_{k2}{}^*). \quad \text{(A.20)}$$

On the other hand we may wish to specialize (A.19) even further, namely to apply to a one-meson state only. Applying (A.19) to the one-meson states Ω_1, Ω_2, Ω_3 we find

$$\theta_1\Omega_1=0,\quad \theta_1\Omega_2=i\Omega_3,\quad \theta_1\Omega_3=-i\Omega_2. \quad \text{(A.21)}$$

Thus the matrix $(t_1)_{\mu\lambda}=(\Omega_\mu,\theta_1\Omega_\lambda)$ which represents θ_1 in the case of a single meson is of the form

$$t_1=\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{bmatrix}. \quad \text{(A.22)}$$

Similarly one finds

$$t_2=\begin{bmatrix} 0 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & 0\end{bmatrix};\quad t_3=\begin{bmatrix} 0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

One should also point out that these matrices, as well as the operators (A.19) or (A.20) satisfy the usual commutation rules

$$[\theta_1,\theta_2]=i\theta_3 \quad \text{(A.23)}$$

and that therefore the total operators (A.8) do also. All this boils down to the fact, then, that the Hamiltonian (A.18) or (1.10) is invariant against a unitary transformation in the nucleon (N,P) space, and a simultaneous unitary transformation of the meson states Ω_1, Ω_2, Ω_3 the infinitesimal unitary transformation being in one case of the form $1-(i/2)\varphi_\lambda\tau_\lambda$ (with infinitesimal angles φ_λ) and in the other of the form $1-i\varphi_\lambda t_\lambda$. In the latter case the matrix turns out to be a real orthogonal matrix in three dimensions, but the corresponding geometrical interpretation as a rotation in a mystical three-dimensional "isotopic-spin space" is of doubtful pedagogical value.

Finally we should explain the relation of the states Ω_1, Ω_2, Ω_3 to the physically more significant $\pi^\pm$ and π^0 states. By means of the inear combinations (A.4) we can write

$$\begin{aligned}\textstyle\sum_\lambda a_\lambda\tau_\lambda&=\tau_+a_++\tau_-a_-+\tau_3a_3,\\ a_\pm&=(a_1\mp ia_2)/\sqrt{2}.\end{aligned} \quad \text{(A.24)}$$

The passage from $a_1a_2a_3$ to a_+, a_-, a_3 is a unitary transformation similar to (A.10) and corresponds to using a new set of meson states [compare (A.17)].

$$\Omega_\pm=(\Omega_1\pm i\Omega_2)/\sqrt{2};\quad \Omega_3. \quad \text{(A.25)}$$

From (A.5) we see that destruction of a meson in a state Ω_+ (by means of a_+) accompanies a nucleon transition from neutron to proton, etc. Taking charge conservation into account, we are thus led to identify the states Ω_+, Ω_-, Ω_3 with π_+, π_-, π_0 states in the same order.

It is clear that in calculations we are free to use either the states Ω_1, Ω_2, Ω_3 or the states (A.25); as (A.24) shows even the symbol $\sum_\lambda a_\lambda\tau_\lambda$ can be interpreted to fit either scheme, one has only to remember that in the second scheme two matrices τ_λ are not Hermitean.

APPENDIX B

If the three operators (A.8) are constants of the motion, then another important integral is

$$\mathbf{T}^2=\textstyle\sum_\lambda T_\lambda{}^2=\frac{3}{4}+\sum_\lambda(\theta_\lambda{}^2+\tau_\lambda\theta_\lambda). \quad \text{(B.1)}$$

The eigenvalues of this operator are of the form $T(T+1)$ where T is a half-integer. This follows from the commutation laws (A.7) in the usual way. Here the analogy with angular momenta is really useful.

The value of T is used as a quantum number, the total isotopic-spin quantum number; the latter is, of course, well known from nuclear spectroscopy. If, in particular, we have only one meson, we see from (A.22) that

$$\textstyle\sum_\lambda\theta_\lambda{}^2=\sum_\lambda t_\lambda{}^2=2 \quad \text{(B.2)}$$

and (B.1) becomes

$$(T+\tfrac{1}{2})^2=3+(\mathbf{t}\cdot\boldsymbol{\tau}) \quad \text{(B.3)}$$

meaning, of course, that the number on the left is an eigenvalue of the operator on the right. By means of (A.22) it is an easy matter to find which linear combinations of the states

$$P\Omega_1,\ P\Omega_2,\ \cdots,\ N\Omega_3 \quad \text{(B.4)}$$

are eigenstates of $\mathbf{T}^2$. One finds for example that $P\Omega_+$ is such an eigenstate, with $T=\frac{3}{2}$. On the other hand

$N\Omega_+$ is not, etc. We need not go into details, as the subject is treated in detail elsewhere.[47]

APPENDIX C

One proves (5.33) as follows: Using (5.32) to represent $|q\beta+>$ we find

$$<q\beta+|p\alpha+> = <\beta|a_q|p\alpha+> \\ + <\beta|V_q^*(E_q-i\eta-H)^{-1}|p\alpha+>. \quad \text{(C.1)}$$

The second term on the right may be written as

$$(\omega_q-\omega_p-i\eta)^{-1}<\beta|V_q^*|p\alpha+>$$

which contains a quantity similar to (5.35) and may be similarly transformed to

$$(\omega_q-\omega_p-i\eta)^{-1}\{<\beta|V_p(E_s-\omega_p-H)^{-1}V_q^*|\alpha>+ \\ <\beta|V_q^*(E_p+i\eta-H)^{-1}V_p|\alpha>\}. \quad \text{(C.2)}$$

The first term on the right of (C.1) can be transformed as follows, by means of (5.32)

$$<\beta|a_q|p\alpha+> = <\beta|a_qa_p^*|\alpha>+ \\ <\beta|a_q(E_p+i\eta-H)^{-1}V_p|\alpha>. \quad \text{(C.3)}$$

In the first term we use $a_qa_p^*=\delta_{qp}+a_p^*a_q$ and then use (5.37) twice, getting

$$\delta_{qp}\delta_{\alpha\beta}+<\beta|a_p^*(E_s-\omega_q-H)^{-1}V_q^*|\alpha> = \\ \delta_{qp}\delta_{\alpha\beta}+<\beta|V_p(E_s-\omega_p-H)^{-1}(E_s-\omega_q-H)^{-1}V_q^*|\alpha>. \quad \text{(C.4)}$$

In order to evaluate the second term of (C.3) we notice that, owing to (5.27)

$$a_q(E_p+i\eta-H)^{-1} \\ = (E_p-\omega_q+i\eta-H)^{-1}\{a_q+V_q^*(E_p+i\eta-H)^{-1}\}.$$

The second term of (C.3) then becomes

$$(\omega_p-\omega_q+i\eta)^{-1}\{<\beta|a_qV_p|\alpha> \\ +<\beta|V_q^*(E_p+i\eta-H)^{-1}V_p|\alpha>$$

or, after interchanging a_q and V_p and using again (5.37),

$$(\omega_p-\omega_q+i\eta)^{-1}\{<\beta|V_p(E_s-\omega_q-H)^{-1}V_q^*|\alpha> \\ +<\beta|V_q^*(E_p+i\eta-H)^{-1}V_p|\alpha>. \quad \text{(C.5)}$$

We must now add (C.2), (C.4), and (C.5). This gives

$$<q\beta+|p\alpha+> = \delta_{qp}\delta_{\alpha\beta}+<\beta|V_pLV_q^*|\alpha>,$$

where

$$L=(\omega_q-\omega_p-i\eta)^{-1}\{(E_s-\omega_p-H)^{-1}-(E_s-\omega_q-H)^{-1}\} \\ +(E_s-\omega_p-H)^{-1}(E_s-\omega_q-H)^{-1}=0.$$

Hence (5.33)

[47] See for example Bethe and de Hoffman.[2]