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Properties of the Dirac Matrices

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CONTENTS

I. INTRODUCTION

THE Dirac matrices arise in Dirac's theory of electrons and positrons.[1] The elementary properties of these matrices are very well known. However, in order to understand some of the features of Dirac's theory, a person needs to be fluent with more than these elementary properties. Also some of the current applications of Dirac's theory are based on some of the less well-known theorems. The purpose of this paper is to give a consistent and reasonably complete account of the properties of these matrices.

The discussion below is restricted to four-by-four matrices since only these are needed in Dirac's theory. Most of the algebraic properties of the matrices were originally developed by Pauli[2]; the treatment given here follows his work closely, especially in the sections on the fundamental theorem and on the identities between scalars formed from four wave functions.

In the next section an outline of the derivation of the Dirac equation is given in order to introduce the matrices. Section III contains Pauli's proof of the fundamental theorem—that any two sets of four-by-four Dirac matrices are connected by a similarity transformation. Sections IV to VI contain discussions of the Lorentz transformation properties of the Dirac wave functions, the charge-conjugation operation, and the formation of covariants. Some special examples of sets of Dirac matrices are given in Sec. VII, especially the set which is used to show the nonrelativistic limit of Dirac's theory, the set used in developing the spinor point of view, and a set which makes charge conjugation identical with complex conjugation. Finally in Sec. VIII the quadratic identities connecting scalars formed from four wave functions are derived and some of the consequences of the identities are discussed.

The paper is written entirely in terms of ordinary matrix theory even though some of the proofs could be shortened by using theorems from abstract algebra. (The matrices form a set of hypercomplex numbers of the type first studied by Clifford.[3]) Especially the fundamental theorem can be proven very neatly by using group theoretical ideas; this was done by van der Waerden.[4] Matrix theory is used exclusively and Pauli's proof of the fundamental theorem is given rather than van der Waerden's in order to emphasize the matrices themselves rather than their abstract algebraic properties. This point of view makes the arguments closer to the applications of the Dirac theory and also makes them readable to people not well versed in modern algebra.

II. THE DIRAC EQUATION

The purpose of this section is to review the derivation of the Dirac equation for a charged particle in an electromagnetic field. The starting point is the Lagrangian for the classical relativistic motion of the particle:

$$L(x,\dot{x}) = -mc^2(1-c^{-2}\dot{x}_j\dot{x}_j)^{\frac{1}{2}} - e\Phi + ec^{-1}\dot{x}_jA_j, \tag{1}$$

where m and e are the mass and charge of the particle, x_j and $\dot{x}_j$ are its position and velocity,[5] and c is the speed of light. For an electron, e is negative. The functions $\Phi(x,t)$ and $A_j(x,t)$ are the scalar and vector potentials of the external field. The actual classical equations of motion are not needed below but, as a matter of completeness, are

$$\frac{d}{dt}\frac{m\dot{x}_i}{(1-c^{-2}\dot{x}_j\dot{x}_j)^{\frac{1}{2}}} = e\left(-\frac{\partial\Phi}{\partial x_i} - \frac{1}{c}\frac{\partial A_i}{\partial t}\right) + \frac{e}{c}\dot{x}_j\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right) = eE_i + ec^{-1}\epsilon_{ijk}\dot{x}_jB_k, \tag{2}$$

where

$$E_i = -(\partial\Phi/\partial x_i) - c^{-1}(\partial A_i/\partial t),$$
$$B_k = \epsilon_{klm}(\partial A_m/\partial x_l)$$

are the electromagnetic-field vectors and ϵ_{ijk} is the usual Levi-Civita three-index symbol, zero if any two of the

[1] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, London, England, 1947), third edition, Chap. XI.
[2] W. Pauli, Ann. inst. Henri Poincaré **6**, 109 (1936).
[3] W. K. Clifford, Am. J. Math. **1**, 350 (1878).
[4] B. L. van der Waerden, *Die Gruppentheoretische Methode in der Quantenmechanik* (Verlag Julius Springer, Berlin, Germany, 1932), p. 55.
[5] Lower case Latin indices range from 1 to 3; lower case Greek from 1 to 4. Indices repeated in a product are to be summed.

indices are equal and plus or minus one, according to whether ijk is an even or an odd permutation of 123. These equations can be cast into a Hamiltonian form in the usual way. The momenta conjugate to the coordinates are

$$\begin{aligned} p_i &= \partial L/\partial \dot{x}_i \\ &= (1-c^{-2}\dot{x}_j\dot{x}_j)^{-\frac{1}{2}} m\dot{x}_i + ec^{-1}A_i, \end{aligned} \tag{3}$$

and the Hamiltonian is

$$\begin{aligned} H(x,p) &= p_i\dot{x}_i - L \\ &= c[m^2c^2 + (p_j - ec^{-1}A_j)(p_j - ec^{-1}A_j)]^{\frac{1}{2}} + e\Phi. \end{aligned} \tag{4}$$

The nonrelativistic approximation is valid when the second term in the brackets is small compared to the first. If an expansion of the square root is made, and only the first two terms are retained the result is

$$H(x,p) = mc^2 + (2m)^{-1}(p_j - ec^{-1}A_j)(p_j - ec^{-1}A_j) + e\Phi, \tag{5}$$

as expected.

The usual procedure for setting up the quantum mechanics of a particle is to introduce a wave function $\psi(x,t)$ satisfying the equation,

$$H(x, -i\hbar\partial/\partial x)\psi = i\hbar\partial\psi/\partial t, \tag{6}$$

where $\hbar$ is Planck's constant divided by 2π. This procedure leads to

$$\left\{\left[m^2c^2 + \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - \frac{e}{c}A_j\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - \frac{e}{c}A_j\right)\right]^{\frac{1}{2}} + \frac{\hbar}{ic}\frac{\partial}{\partial t} + \frac{e}{c}\Phi\right\}\psi = 0. \tag{7}$$

Dirac[1] regards this equation as unsatisfactory from a relativistic point of view because of the dissymmetry between the time and space coordinates. However, he points out that in the field-free case

$$\left\{\left[m^2c^2 - \hbar^2\frac{\partial^2}{\partial x_j\partial x_j}\right]^{\frac{1}{2}} + \frac{\hbar}{ic}\frac{\partial}{\partial t}\right\}\psi = 0, \tag{8}$$

and solutions of this equation also satisfy the equation,

$$\left\{-m^2c^2 + \hbar^2\frac{\partial^2}{\partial x_j\partial x_j} - \frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}\right\}\psi = 0, \tag{9}$$

as is seen by operating on Eq. (8) with

$$-\left[m^2c^2 - \hbar^2\frac{\partial^2}{\partial x_j\partial x_j}\right]^{\frac{1}{2}} + \frac{\hbar}{ic}\frac{\partial}{\partial t}.$$

Equation (9) is of a relativistically invariant form although Eq. (8) is not. The solutions of the equation,

$$[\beta mc + \alpha_j(-i\hbar\partial/\partial x_j) + c^{-1}(i\hbar\partial/\partial t)]\psi = 0, \tag{10}$$

also satisfy Eq. (9) if β and the α_j are operators which are independent of the coordinates and the time and which satisfy

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}, \tag{11}$$

$$\alpha_j\beta + \beta\alpha_j = 0, \tag{12}$$

$$\beta^2 = 1. \tag{13}$$

(Here δ_{jk} is the usual Kronecker delta, one when the indices are the same, otherwise zero.) This statement is easily verified by operating on Eq. (10) with

$$-\beta mc - \alpha_j(-i\hbar\partial/\partial x_j) + c^{-1}(i\hbar\partial/\partial t).$$

The end result of these considerations is Eq. (10), again of first order in the time-differentiation but now with the space and time coordinates on an equal footing. However the wave function must be enlarged to depend on more than space and time in order to accommodate the operators α_j, β. In Eq. (10) and in the rest of the paper ψ is given the meaning of a column matrix, each element of which is a function of space and time, and α_j, β are square matrices which operate on ψ to produce a new column matrix. In the next section it is shown that four rows in the column are needed to make the above equations sensible. Dirac thus proposes Eq. (10) to govern the motion of a free particle. Written in the form of Eq. (6) it is

$$[-\beta mc^2 - c\alpha_j(-i\hbar\partial/\partial x_j)]\psi = i\hbar\partial\psi/\partial t. \tag{14}$$

The generalization to the case when fields are present is made, as suggested by Eqs. (3) and (4), by decreasing the momenta by $ec^{-1}A_j$ and the Hamiltonian by $e\Phi$. The result is that

$$H_D\psi = i\hbar\partial\psi/\partial t, \tag{15}$$

where the Dirac Hamiltonian is given by

$$H_D = -\beta mc^2 - c\alpha_j[(-i\hbar\partial/\partial x_j) - ec^{-1}A_j] + e\Phi. \tag{16}$$

The Dirac equation can be written very neatly by using a relativistic notation. Combining Eqs. (15) and (16) and introducing

$$x_4 = ict,$$

$$A_4 = i\Phi,$$

one finds that

$$\left[i\alpha_j\left(\frac{\partial}{\partial x_j} - \frac{ie}{\hbar c}A_j\right) + \left(\frac{\partial}{\partial x_4} - \frac{ie}{\hbar c}A_4\right) - \beta\frac{mc}{\hbar}\right]\psi = 0. \tag{17}$$

This is evidently to be simplified by multiplying on the left by $-\beta$ and by introducing

$$\gamma_j = -i\beta\alpha_j, \tag{18}$$

$$\gamma_4 = -\beta \tag{19}$$

so that the equation becomes

$$\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}A_\mu\right)\gamma_\mu\psi + \frac{mc}{\hbar}\psi = 0. \tag{20}$$

As a consequence of Eqs. (11)–(13), the γ_μ satisfy the equations

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}. \tag{21}$$

Matrices which satisfy these relations are called Dirac matrices.

The matrices γ_μ in Eq. (20) will always be chosen Hermitian, in the sense that the complex conjugate is equal to the transpose. It is seen from Eqs. (18) and (19) that if α_j, β are Hermitian, then the γ_μ are also, for[6]

$$\gamma_j{}^H=(-i\beta\alpha_j)^H=i\alpha_j{}^H\beta^H=i\alpha_j\beta=-i\beta\alpha_j=\gamma_j, \tag{22}$$

$$\gamma_4{}^H=(-\beta)^H=-\beta=\gamma_4. \tag{23}$$

The converse of this statement can be verified easily in the same way. As a consequence of choosing the γ_μ Hermitian, one can set up a row-matrix equation complementary to the column matrix Eq. (20). The Hermitian conjugate of Eq. (20) is

$$\left(\frac{\partial}{\partial x_j}+\frac{ie}{\hbar c}A_j\right)\psi^H\gamma_j-\left(\frac{\partial}{\partial x_4}+\frac{ie}{\hbar c}A_4\right)\psi^H\gamma_4+\frac{mc}{\hbar}\psi^H=0, \tag{24}$$

where the nonuniformity in the space and time parts comes about because x_4 and A_4 are pure imaginary. From Eq. (21) it is seen that γ_4 anticommutes with each of the γ_j. Accordingly, multiplying on the right with $-\gamma_4$ and introducing what is called the adjoint wave function by

$$\psi^A=\psi^H\gamma_4, \tag{25}$$

one obtains

$$\left(\frac{\partial}{\partial x_\mu}+\frac{ie}{\hbar c}A_\mu\right)\psi^A\gamma_\mu-\frac{mc}{\hbar}\psi^A=0, \tag{26}$$

ordinarily called the adjoint equation. The conservation of probability equation follows immediately by multiplying Eq. (20) on the left by $ic\psi^A$, multiplying Eq. (26) on the right by $ic\psi$, and adding:

$$\partial(ic\psi^A\gamma_\mu\psi)/\partial x_\mu=0. \tag{27}$$

This equation, together with the fact (discussed in Sec. VI) that $ic\psi^A\gamma_\mu\psi$ is a vector with respect to Lorentz transformations, permits

$$\psi^A\gamma_4\psi=\psi^H\psi$$

to be interpreted as the probability density and

$$ic\psi^A\gamma_j\psi=-c\psi^H\alpha_j\psi$$

to be interpreted as the probability current.

As an example of Dirac matrices, one set can be built up from the two-by-two Pauli spin matrices,

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{28}$$

which are Hermitian and satisfy the equations

$$\sigma_j\sigma_k=i\epsilon_{jkl}\sigma_l+\delta_{jk}, \tag{29}$$

so that

$$\sigma_j\sigma_k+\sigma_k\sigma_j=2\delta_{jk}. \tag{30}$$

As a consequence of Eq. (30) it is seen that the four-by-four matrices,

$$\alpha_j=\begin{pmatrix}0 & \sigma_j\\ \sigma_j & 0\end{pmatrix},\quad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{31}$$

where here the 0's and 1's represent the zero and unit two-by-two matrices, fulfil Eqs. (11) to (13). Also they are evidently Hermitian since the σ_j are Hermitian.

For some of the arguments in the following sections it is useful to keep the nonrelativistic limit of the Dirac equation in mind and so an outline of how the limit may be obtained is given next. In discussions involving this limit the matrices in Eq. (31) are especially appropriate. As suggested by the form of the matrices, one may write the four-rowed column matrix ψ in terms of two two-rowed column matrices ψ_S, ψ_L so that

$$\psi=\begin{pmatrix}\psi_S\\ \psi_L\end{pmatrix}\exp(mc^2t/i\hbar). \tag{32}$$

The exponential factor is added also to remove the rest-mass contribution to the wave function. Then Eq. (15) may be rewritten as two equations:

$$-2mc^2\psi_S-c\sigma_j[(-i\hbar\partial/\partial x_j)-ec^{-1}A_j]\psi_L+e\Phi\psi_S-i\hbar\partial\psi_S/\partial t=0, \tag{33}$$

$$-c\sigma_j[(-i\hbar\partial/\partial x_j)-ec^{-1}A_j]\psi_S+e\Phi\psi_L-i\hbar\partial\psi_L/\partial t=0. \tag{34}$$

The nonrelativistic approximation applies when the last two terms in Eq. (33) can be neglected compared to the $-2mc^2\psi_S$ term. Then, introducing the abbreviation

$$\pi_j=(-i\hbar\partial/\partial x_j)-ec^{-1}A_j, \tag{35}$$

one can solve Eq. (33) for ψ_S:

$$\psi_S=-(2mc)^{-1}\sigma_j\pi_j\psi_L. \tag{36}$$

Since the ψ_S are of order c^{-1} compared to the ψ_L, the ψ_S are called the small components of ψ and the ψ_L the large. This value of ψ_S can now be inserted into Eq. (34) to obtain an equation involving the large components only:

$$(2m)^{-1}\sigma_j\pi_j\sigma_k\pi_k\psi_L+e\Phi\psi_L=i\hbar\partial\psi_L/\partial t. \tag{37}$$

The properties of the Pauli matrices, Eqs. (29), lead to the following simplification of the operator in the first

[6] Superscripts ^{T}, ^{C}, ^{H} are used to designate the transposed, complex conjugated, and Hermitian conjugated matrices. The Dirac wave functions ψ are treated strictly as matrices of many rows and one column. For example, if the elements of ψ are ψ_1, ψ_2, ψ_3, ψ_4 reading from the top down, then ψ^H represents the row matrix $(\psi_1{}^C\ \psi_2{}^C\ \psi_3{}^C\ \psi_4{}^C)$.

term:

$$\sigma_j\sigma_k\pi_j\pi_k = i\epsilon_{jkl}\sigma_l\pi_j\pi_k + \delta_{jk}\pi_j\pi_k$$
$$= \tfrac{1}{2}i\epsilon_{jkl}\pi_j\pi_k\sigma_l + \tfrac{1}{2}i\epsilon_{kjl}\pi_k\pi_j\sigma_l + \pi_j\pi_j$$
$$= \tfrac{1}{2}i\epsilon_{jkl}[\pi_j,\pi_k]\sigma_l + \pi_j\pi_j,$$

where $[\pi_j,\pi_k]$ indicates the commutator of π_j and π_k, $\pi_j\pi_k - \pi_k\pi_j$. From the definition, Eq. (35), this is readily found to be

$$[\pi_j,\pi_k] = ie\hbar c^{-1}[(\partial A_k/\partial x_j) - (\partial A_j/\partial x_k)]. \quad (38)$$

The result for the operator is

$$\sigma_j\sigma_k\pi_j\pi_k = -\tfrac{1}{2}e\hbar c^{-1}\sigma_l\epsilon_{ljk}[(\partial A_k/\partial x_j) - (\partial A_j/\partial x_k)] + \pi_j\pi$$
$$= -e\hbar c^{-1}\sigma_l\epsilon_{ljk}(\partial A_k/\partial x_j) + \pi_j\pi_j$$
$$= -e\hbar c^{-1}\sigma_l B_l + \pi_j\pi_j. \quad (39)$$

The final equation for the large component is found by substituting Eq. (39) into Eq. (37) and by using Eq. (35) to remove the π_j abbreviations:

$$\left[\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - \frac{e}{c}A_j\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - \frac{e}{c}A_j\right) - \left(\frac{e\hbar}{2mc}\right)\sigma_j B_j + e\Phi\right]\psi_L = i\hbar\frac{\partial\psi_L}{\partial t}. \quad (40)$$

This is of course the Schrödinger equation increased by the Pauli spin contribution. The nonrelativistic expression for the probability density results when the products of the small components are discarded compared to the products of the large components:

$$\psi^H\psi = \psi_S{}^H\psi_S + \psi_L{}^H\psi_L$$
$$= \psi_L{}^H\psi_L. \quad (41)$$

The first step in finding the corresponding expression for the probability current is to use Eq. (36) to eliminate the small components,

$$-c\psi^H\alpha_j\psi = -c\psi_S{}^H\sigma_j\psi_L - c\psi_L{}^H\sigma_j\psi_S$$
$$= (2m)^{-1}[(\pi_k{}^C\psi_L{}^H)\sigma_k\sigma_j\psi_L + \psi_L{}^H\sigma_j\sigma_k\pi_k\psi_L],$$

and Eq. (35) to remove the π_k abbreviations,

$$-c\psi^H\alpha_j\psi = (2m)^{-1}[i\hbar(\partial\psi_L{}^H/\partial x_k)\sigma_k\sigma_j\psi_L - ec^{-1}A_k\psi_L{}^H\sigma_k\sigma_j\psi_L - i\hbar\psi_L{}^H\sigma_j\sigma_k(\partial\psi_L/\partial x_k) - ec^{-1}A_k\psi_L{}^H\sigma_j\sigma_k\psi_L].$$

The terms involving A_k are easily simplified by Eq. (30); the other terms are reduced with the help of Eq. (29). The result is evidently

$$-c\psi^H\alpha_j\psi = \frac{i\hbar}{2m}\frac{\partial\psi_L{}^H}{\partial x_j}\psi_L - \frac{i\hbar}{2m}\psi_L{}^H\frac{\partial\psi_L}{\partial x_j} - \frac{e}{mc}A_j\psi_L{}^H\psi_L + \frac{\hbar}{2m}\epsilon_{jkl}\frac{\partial}{\partial x_k}(\psi_L{}^H\sigma_l\psi_L). \quad (42)$$

The first three are the usual nonrelativistic current terms, and the last is a contribution from the spin. A more complete discussion of this transition to the nonrelativistic limit is given by Pauli.[7] A different way of studying the limiting process, which has certain definite advantages, has been developed by Foldy and Wouthuysen.[8]

III. THE FUNDAMENTAL THEOREM

As outlined in the previous section, Dirac's theory of electrons and positrons leads to the consideration of sets of four Hermitian matrices γ_μ satisfying Eqs. (21). The Hermitian property provides the adjoint equation and the conservation of probability; also it assures the reality of the probability density and current. However, in some of the subsequent arguments, it is convenient to consider non-Hermitian sets of matrices satisfying Eqs. (21); accordingly in this section the matrices will not be assumed Hermitian. A basic property of these matrices is that any two sets of four-by-four matrices, each of which satisfy Eqs. (21), are connected by a similarity transformation. In other words, if γ_μ and $\gamma_\mu{}'$ are two sets of four four-by-four matrices such that

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu} \quad (43)$$

and

$$\gamma_\mu{}'\gamma_\nu{}' + \gamma_\nu{}'\gamma_\mu{}' = 2\delta_{\mu\nu}, \quad (44)$$

then there exists a nonsingular four-by-four matrix S, such that

$$\gamma_\mu{}' = S\gamma_\mu S^{-1}. \quad (45)$$

Pauli[2] has called this the fundamental theorem of the Dirac matrices and has given the simple algebraic proof which is reviewed below.

In making the proof the number of matrices considered is increased from the four γ_μ to the following sixteen:

1

$\gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \gamma_4$

$i\gamma_2\gamma_3 \quad i\gamma_3\gamma_1 \quad i\gamma_1\gamma_2 \quad i\gamma_1\gamma_4 \quad i\gamma_2\gamma_4 \quad i\gamma_3\gamma_4$

$i\gamma_1\gamma_2\gamma_3 \quad i\gamma_1\gamma_2\gamma_4 \quad i\gamma_3\gamma_1\gamma_4 \quad i\gamma_2\gamma_3\gamma_4$

$\gamma_1\gamma_2\gamma_3\gamma_4$

where 1 refers to the unit four-by-four matrix. The factors of i in the third and fourth rows are chosen so that the square of each of these matrices is the unit matrix[9]:

$$\gamma_A{}^2 = 1. \quad (46)$$

This is easily verified in each case since, according to Eq. (43), the square of each γ_μ is one, and the γ_μ anticommute with each other. If the γ_μ happen to be Hermitian, it is seen that the γ_A are Hermitian also. Occasionally the matrices in the third and fourth rows

[7] W. Pauli, *Handbuch der Physik*, H. Geiger and K. Scheel, editors (Verlag Julius Springer, Berlin, Germany, 1933), second edition, Vol. 24, Part 1, p. 236.

[8] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[9] Capital Latin indices are used to range from 1 to 16 so that γ_A represents any of the sixteen matrices displayed above.

are represented by the symbols $\gamma_{[\mu\nu]}$ and $\gamma_{[\lambda\mu\nu]}$, where the brackets indicate that the indices are unequal and that their order is immaterial. An abbreviation often used is

$$\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4. \tag{47}$$

This is appropriate because γ_5 has the same properties as the four γ_μ:

$$\gamma_5^2=1, \tag{48}$$

$$\gamma_5\gamma_\mu+\gamma_\mu\gamma_5=0. \tag{49}$$

As a matter of convenience the proof of the theorem will be broken up into a series of eight steps. The first five steps deal with properties of just one set of matrices γ_μ and the last three with connections between two sets of Dirac matrices γ_μ and γ_μ'.

1. For each γ_A except the identity, one can choose a γ_B such that[10]

$$\gamma_B\gamma_A\gamma_B=-\gamma_A. \tag{50}$$

A proof of this statement can be made by considering each type of γ_A separately. It is easily seen that Eq. (50) is fulfilled when $\gamma_A=\gamma_\mu$ by $\gamma_B=\gamma_5$, when $\gamma_A=\gamma_{[\mu\nu]}$ by $\gamma_B=\gamma_\mu$, when $\gamma_A=\gamma_{[\lambda\mu\nu]}$ by $\gamma_B=\gamma_5$, and when $\gamma_A=\gamma_5$ by $\gamma_B=\gamma_\mu$.

2. The spurs of all the γ_A except the identity are zero:

$$\mathrm{Sp}(\gamma_A)=0 \quad \text{if} \quad \gamma_A\neq 1. \tag{51}$$

This is easily seen by taking the spur of Eq. (50). Since the spur of the product of two matrices is independent of the order of the factors, and since γ_B^2 is 1 the left side becomes simply $\mathrm{Sp}(\gamma_A)$ while the right is $-\mathrm{Sp}(\gamma_A)$.

3. The γ_A are linearly independent. In other words the equation,

$$\sum_{A=1}^{16} a_A\gamma_A=0, \tag{52}$$

where the a_A are complex numbers, only holds when the a_A are all zero. To show that, for example, a_B is zero, one multiplies Eq. (52) through by γ_B to obtain

$$a_B+\sum\nolimits_{A(\neq B)}a_A\gamma_A\gamma_B=0. \tag{53}$$

With the help of Eqs. (43) a product $\gamma_A\gamma_B$ can always be reduced to the form $b\gamma_C$ where b is some complex number. By considering the various types of products it is easily seen that if $A\neq B$, γ_C will not be the identity. Accordingly, all the matrices in the sum in Eq. (53) have zero spur and, by taking the spur of the equation, one concludes that a_B is zero. A consequence of this proof is that the sixteen γ_A are all distinct. In this respect the properties of the Dirac matrices γ_μ are different from those of the Pauli matrices σ_j even though they satisfy similar sets of equations, Eqs. (43) and Eqs. (30), for there are connections like

$$\sigma_1\sigma_2=i\sigma_3$$

between the Pauli matrices. Also at this point it is seen that a set of matrices satisfying Eqs. (43) must be at least four-by-four in order to generate a set of sixteen linearly independent matrices, for the above arguments hold regardless of the size of the matrices.

4. An arbitrary four-by-four matrix X can be written as a linear combination of the sixteen γ_A matrices:

$$X=\sum_{A=1}^{16} x_A\gamma_A. \tag{54}$$

This statement is obviously true since a set of sixteen linearly independent vectors form a basis for a space of sixteen dimensions. In any special case the expansion coefficients x_B will be given by the formula

$$x_B=\tfrac{1}{4}\,\mathrm{Sp}(\gamma_B X), \tag{55}$$

as is seen by multiplying Eq. (54) by γ_B and taking the spur.

5. A four-by-four matrix which commutes with each of the γ_μ is a multiple of the unit matrix. Let X be the matrix in question; as argued in step 4, one may write it as a linear combination of the γ_A. Equation (54) may be written in the form

$$X=x_B\gamma_B+\sum\nolimits_{A(\neq B)}x_A\gamma_A, \tag{56}$$

where γ_B is one of the sixteen matrices and is chosen arbitrarily except that it is not the identity. The assertion will be proven if it can be demonstrated that x_B is 0. According to step 1, a matrix γ_C can be chosen so that

$$\gamma_C\gamma_B\gamma_C=-\gamma_B. \tag{57}$$

By hypothesis X commutes with all of the γ_μ and therefore with γ_C so that

$$X=\gamma_C X\gamma_C. \tag{58}$$

In terms of the expansion coefficients of Eq. (56), Eq. (58) reads

$$x_B\gamma_B+\sum\nolimits_{A(\neq B)}x_A\gamma_A=x_B\gamma_C\gamma_B\gamma_C+\sum\nolimits_{A(\neq B)}x_A\gamma_C\gamma_A\gamma_C,$$

and this can be written in the form

$$x_B\gamma_B+\sum\nolimits_{A(\neq B)}x_A\gamma_A=-x_B\gamma_B+\sum\nolimits_{A(\neq B)}(\pm 1)x_A\gamma_A,$$

as a consequence of Eq. (57) and the fact that γ_C and γ_A either commute or anticommute. If finally this equation is multiplied by γ_B and the spur is taken, the result is

$$x_B=-x_B$$

so that x_B is zero as required. In summary, if

$$X\gamma_\mu=\gamma_\mu X$$

then

$$X=k,$$

where k is a complex number times the unit matrix. This result, a special consequence of Schur's lemma,[11] will often be used below.

[10] The convention of summing on repeated indices is not used when capital Latin indices are involved.

[11] See, for example, Hermann Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, New York, 1931), p. 153.

6. If γ_μ and γ_μ' are two sets of matrices satisfying Eqs. (43) and (44) and if γ_A and γ_A' are sets of sixteen matrices formed in parallel from them according to the definitions at the beginning of this section, then

$$\gamma_A'S=S\gamma_A, \tag{59}$$

where

$$S=\sum_{B=1}^{16}\gamma_B'F\gamma_B, \tag{60}$$

and F is any four-by-four matrix. (The last two steps in the proof of the theorem consist of showing that F can be chosen so that S is nonsingular. Pauli attributes this method of proof to Schur.[12]) This assertion is easily proven from a consideration of the matrix

$$\gamma_A'S\gamma_A=\sum_{B=1}^{16}\gamma_A'\gamma_B'F\gamma_B\gamma_A. \tag{61}$$

The product $\gamma_B\gamma_A$ can always be simplified so that

$$\gamma_B\gamma_A=\epsilon_C\gamma_C, \tag{62}$$

where ϵ_C has one of the four values ± 1, $\pm i$. For arbitrary γ_A, as B ranges from 1 to 16, C must range also from 1 to 16 since if both

$$\gamma_B\gamma_A=\epsilon_C\gamma_C$$

and

$$\gamma_D\gamma_A=\delta_C\gamma_C$$

for two different matrices γ_B and γ_D, then

$$\gamma_B=\epsilon_C\gamma_C\gamma_A=(\epsilon_C/\delta_C)\gamma_D,$$

and this would contradict the linear independence demonstrated in step 3. Furthermore, the ϵ_C in Eq. (62) are determined by the definitions of the γ_A in terms of the γ_μ and by the anticommutation rules, Eqs. (43). Since these definitions are made in parallel for the primed matrices and since the primed matrices satisfy the same anticommutation rules, Eqs. (44), the same numbers will arise in the primed system:

$$\gamma_B'\gamma_A'=\epsilon_C\gamma_C'.$$

The product $\gamma_A'\gamma_B'$ is conveniently found by taking the inverse of this last equation:

$$\gamma_A'\gamma_B'=(1/\epsilon_C)\gamma_C'. \tag{63}$$

Introducing Eqs. (62) and (63) into Eq. (61) and changing the sum-index from B to C one finds

$$\gamma_A'S\gamma_A=\sum_{C=1}^{16}(1/\epsilon_C)\gamma_C'F\epsilon_C\gamma_C$$

$$=\sum_{C=1}^{16}\gamma_C'F\gamma_C$$

$$=S,$$

and obtains Eq. (59) by multiplying from the right with γ_A.

7. The matrix F can be chosen so that S is not zero. If S were zero for all F, the equations

$$\sum_{B=1}^{16}(\gamma_B')_{\mu\nu}(\gamma_B)_{\rho\sigma}=0 \tag{64}$$

could be constructed from Eq. (60) by choosing F successively to be the various matrices which have one element unity and the other fifteen zero. Here, $(\gamma_B')_{\mu\nu}$ are the elements of the matrix γ_B'. Equations (64) imply the matrix equation

$$\sum_{B=1}^{16}(\gamma_B')_{\mu\nu}\gamma_B=0.$$

The $(\gamma_B')_{\mu\nu}$ are not all zero since $\gamma_B'^2=1$, so this equation is incompatible with the linear independence of the γ_B. It must be concluded that some matrices F exist for which S, defined by Eq. (60), is not zero.

8. The matrix F can be chosen so that S is nonsingular. A proof of this statement can be made easily with the help of a matrix T defined by

$$T=\sum_{B=1}^{16}\gamma_B G\gamma_B', \tag{65}$$

where G is a four-by-four matrix to be chosen below. The matrix F in the definition of S,

$$S=\sum_{B=1}^{16}\gamma_B'F\gamma_B, \tag{66}$$

will also be left arbitrary for the moment. The argument of step 6 with the primed and unprimed matrices interchanged can be applied to the T matrix so that

$$\gamma_A T=T\gamma_A', \tag{67}$$

and step 6 applied directly to S gives

$$\gamma_A'S=S\gamma_A. \tag{68}$$

However, from Eqs. (67) and (68) together it may be concluded that

$$\gamma_A TS=T\gamma_A'S=TS\gamma_A$$

so that, as a consequence of step 5,

$$TS=k, \tag{69}$$

where k is a complex number times the unit matrix. At this point the matrix G will be chosen so that T is not zero; this was demonstrated to be possible in step 7. Then the matrix F will be chosen so that k is not zero. To see that this also is possible one notices that, if k is zero for every choice of F, Eqs. (69) and (66) imply that

$$\sum_{B=1}^{16}T\gamma_B'F\gamma_B=0.$$

Therefore, if F is chosen successively to be the matrices

[12] Reference 2, pp. 110, 115; I. Schur, Berliner Sitzber., 406 (1905).

with one element unity and the rest zero, one finds

$$\sum_{B=1}^{16} (T\gamma_B')_{\mu\nu}(\gamma_B)_{\rho\sigma}=0.$$

Written as a matrix equation in the γ_B this becomes

$$\sum_{B=1}^{16} (T\gamma_B')_{\mu\nu}\gamma_B=0.$$

Not all of the $(T\gamma_B')_{\mu\nu}$ are zero since T was chosen previously not zero and γ_B' includes the identity in its range. The assumption that k is zero for all F is then in contradiction with the proven linear independence of the γ_B. It must then be possible to choose F and G so that Eqs. (68) and (69) hold, where k is not zero. Accordingly, there must exist a matrix S, nonsingular, such that

$$\gamma_\mu'=S\gamma_\mu S^{-1}. \tag{70}$$

This completes the proof of the theorem.

The matrix S which connects two sets of Dirac matrices according to Eq. (70) is uniquely determined by the two sets except for an arbitrary numerical factor. In order to prove this suppose that γ_μ and γ_μ' are two definite sets of Dirac matrices and suppose also that there are two matrices S_1 and S_2 such that

$$\gamma_\mu'=S_1\gamma_\mu S_1^{-1},$$
$$\gamma_\mu'=S_2\gamma_\mu S_2^{-1}.$$

If γ_μ' is eliminated from these two equations, it is seen that

$$S_1\gamma_\mu S_1^{-1}=S_2\gamma_\mu S_2^{-1},$$
$$S_2^{-1}S_1\gamma_\mu=\gamma_\mu S_2^{-1}S_1.$$

Here again, step 5 can be applied; since $S_2^{-1}S_1$ commutes with each of the γ_μ, it must be a number, say k, times the unit matrix so that

$$S_1=kS_2,$$

and this proves the assertion. Occasionally it is convenient to impose the extra condition

$$\det S=1.$$

Evidently this condition can always be fulfilled by an appropriate choice of the numerical factor k. The effect of this condition is that S is determined except for an arbitrary factor of ± 1, $\pm i$.

As an example, the operation of taking the transpose of a matrix provides a direct application of the fundamental theorem. Let γ_μ be a set of Dirac matrices so that

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}.$$

The transpose of this equation is

$$\gamma_\nu{}^T\gamma_\mu{}^T+\gamma_\mu{}^T\gamma_\nu{}^T=2\delta_{\mu\nu},$$

so that the $\gamma_\mu{}^T$ also form a set of Dirac matrices. According to the fundamental theorem there must exist a nonsingular matrix B such that

$$\gamma_\mu{}^T=B\gamma_\mu B^{-1}. \tag{71}$$

This matrix is of special interest in some of the sections below. An interesting property of it is that it is antisymmetric. Still following Pauli,[2] one can demonstrate this by first of all taking the transpose of Eq. (71) and then substituting Eq. (71) into the result:

$$\begin{aligned}\gamma_\mu&=(B^{-1})^T\gamma_\mu{}^TB^T\\&=(B^{-1})^TB\gamma_\mu B^{-1}B^T\\&=(B^{-1}B^T)^{-1}\gamma_\mu(B^{-1}B^T).\end{aligned}$$

This equation can then be multiplied from the left by $(B^{-1}B^T)$; the result is that $(B^{-1}B^T)$ commutes with the γ_μ and so, according to step 5 of the proof of the fundamental theorem, is a constant times the identity:

$$B^{-1}B^T=k.$$

However, the equation

$$B^T=kB \tag{72}$$

implies that

$$B=kB^T=k^2B$$

so that k is ± 1; this establishes that B is either symmetric or antisymmetric. The possibility that B is symmetric can be ruled out by means of an argument which Pauli credits to Haantjes.[13] In this argument one counts which of the $B\gamma_A$ are symmetric and which are antisymmetric. On rewriting Eq. (71) in the form

$$\gamma_\mu{}^TB=B\gamma_\mu, \tag{73}$$

it is easily seen that

$$(B1)^T=k(B1), \tag{74}$$

$$(B\gamma_\mu)^T=k\gamma_\mu{}^TB=k(B\gamma_\mu), \tag{75}$$

$$(iB\gamma_\mu\gamma_\nu)^T=ki\gamma_\nu{}^T\gamma_\mu{}^TB=-k(iB\gamma_\mu\gamma_\nu), \tag{76}$$

$$(iB\gamma_\lambda\gamma_\mu\gamma_\nu)^T=ki\gamma_\nu{}^T\gamma_\mu{}^T\gamma_\lambda{}^TB=-k(iB\gamma_\lambda\gamma_\mu\gamma_\nu), \tag{77}$$

$$(B\gamma_1\gamma_2\gamma_3\gamma_4)^T=k\gamma_4{}^T\gamma_3{}^T\gamma_2{}^T\gamma_1{}^TB=k(B\gamma_1\gamma_2\gamma_3\gamma_4), \tag{78}$$

where Eq. (72) has been used to eliminate B^T, and λ, μ, ν are all unequal. If B is symmetric so that k is $+1$ it is seen that there are ten antisymmetric matrices $B\gamma_{[\mu\nu]}$ and $B\gamma_{[\lambda\mu\nu]}$. These ten matrices must be linearly independent since the $\gamma_{[\mu\nu]}$ and $\gamma_{[\lambda\mu\nu]}$ are linearly independent and B has an inverse. This is an impossibility; there can only be six linearly-independent antisymmetric four-by-four matrices. It must be concluded that k is -1 and the B matrix is antisymmetric. It is also pertinent to consider the connection between the B matrices defined for different sets of γ_μ. That is, if

$$\gamma_\mu{}^T=B\gamma_\mu B^{-1}, \tag{79}$$

$$\gamma_\mu'{}^T=B'\gamma_\mu'B'^{-1} \tag{80}$$

define the matrices B, B' while

$$\gamma_\mu'=S\gamma_\mu S^{-1} \tag{45}$$

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[13] Reference 2, p. 121.

gives the connection between the γ_μ' and the γ_μ, one may ask for the corresponding connection between B' and B. This connection can be found by substituting Eq. (45) into Eq. (80) to eliminate the γ_μ',

$$(S^{-1})^T\gamma_\mu{}^T S^T = B'S\gamma_\mu S^{-1}B'^{-1},$$

then substituting Eq. (79) on the left to eliminate the $\gamma_\mu{}^T$,

$$(S^{-1})^T B\gamma_\mu B^{-1}S^T = B'S\gamma_\mu S^{-1}B'^{-1},$$

and then multiplying from the left and right by the proper quantities to cast the equation into the form

$$S^{-1}B'^{-1}(S^{-1})^T B\gamma_\mu = \gamma_\mu S^{-1}B'^{-1}(S^{-1})^T B.$$

The matrix which here commutes with the γ_μ must be a number, say k, times the unit matrix so that

$$S^{-1}B'^{-1}(S^{-1})^T B = k,$$

or

$$B = kS^T B'S, \tag{81}$$

which is the required connection. A final property of the B matrix is that, if the γ_μ are Hermitian, B can be chosen unitary. This is easily seen by taking the Hermitian conjugate of the defining equation, Eq. (71), to obtain, when $\gamma_\mu{}^H = \gamma_\mu$,

$$\gamma_\mu{}^T = (B^{-1})^H \gamma_\mu B^H,$$

and by recombining this equation with Eq. (71) so that

$$B^H B\gamma_\mu = \gamma_\mu B^H B,$$

which implies that

$$B^H B = k,$$

where k is some number. The number k is easily shown to be real and positive by writing in detail one of the diagonal elements of $B^H B$. Accordingly, by proper choice of the arbitrary factor in B, $B^H B$ can be adjusted until it is the unit matrix and B is unitary. In the applications below it is always assumed that this adjustment has been made so that B is unitary; it is clear that then B is determined uniquely except for an arbitrary factor whose absolute value is one. As a concrete example, for the set of Dirac matrices γ_μ defined by Eqs. (18), (19), and (31), the antisymmetric unitary matrix

$$B = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tag{82}$$

satisfies the defining equation, Eq. (71). This can be seen by expressing the γ_μ explicitly in terms of the σ_j,

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and noticing that, as a consequence of the definitions of the Pauli matrices Eq. (28), γ_1 and γ_3 are antisymmetric while γ_2 and γ_4 are symmetric. The matrix of Eq. (82) evidently anticommutes with γ_1 and γ_3 but commutes with γ_2 and γ_4 and so it does give a solution of Eq. (71). In Sec. VII examples of the form of the B matrix for other special sets of Dirac matrices are given.

IV. LORENTZ TRANSFORMATIONS OF DIRAC WAVE FUNCTIONS

In this section the Lorentz transformation properties of the four-component Dirac wave functions introduced in Sec. II are reviewed. A Lorentz transformation is defined as a coordinate transformation of the type

$$x_\mu' = a_{\mu\nu}x_\nu, \tag{83}$$

where the transformation coefficients satisfy the orthogonality relations

$$a_{\mu\nu}a_{\mu\xi} = \delta_{\nu\xi} \tag{84}$$

or, equivalently,

$$a_{\nu\mu}a_{\xi\mu} = \delta_{\nu\xi}, \tag{85}$$

and also satisfy the conditions of reality: a_{jk}, ia_{4j}, ia_{j4}, a_{44} are real. It is well known that the group of Lorentz transformations consists of two types of reflections as well as the transformations which can be formed continuously from the identity. The first type arises in considering the determinant of the transformation coefficients $a_{\mu\nu}$. If A is the square four-by-four matrix formed from these coefficients, Eq. (84) can be written in the form

$$A^T A = 1,$$

and the determinant of this equation gives

$$\begin{aligned}\det(A^T A) &= (\det A^T)(\det A) \\ &= (\det A)^2 \\ &= 1,\end{aligned}$$

so that

$$\det A = \pm 1. \tag{86}$$

Evidently, the plus sign will apply for transformations continuous with the identity. However, as well as these, transformations in which the determinant is -1 also must be taken into account. The other type of reflection arises in considering the sign of the transformation coefficient a_{44}. When both ν and ξ are 4, Eq. (84) reads

$$a_{14}{}^2 + a_{24}{}^2 + a_{34}{}^2 + a_{44}{}^2 = 1.$$

Since, according to the conditions of reality, the a_{j4} are pure imaginary,

$$a_{j4}{}^2 = -|a_{j4}|^2,$$

where the vertical bars indicate the absolute value, and therefore

$$a_{44}{}^2 = 1 + |a_{14}|^2 + |a_{24}|^2 + |a_{34}|^2.$$

In consequence there are two possibilities for a_{44}: either

$$a_{44} \geq 1 \quad \text{or} \quad a_{44} \leq -1. \tag{87}$$

The first alternative applies to transformations continuous with the identity, but transformations in which a_{44} is negative also must be taken into account. It is not convenient below to treat the Lorentz transformations entirely uniformly; the Lorentz group will on occasion be regarded as consisting of the transformations continuous with the identity, the space reflection

$$x_j'=-x_j, \quad x_4'=x_4, \tag{88}$$

which has $\det A=-1$, $a_{44}=1$, and the time reflection

$$x_j'=x_j, \quad x_4'=-x_4, \tag{89}$$

which has $\det A=-1$, $a_{44}=-1$.

There are three considerations involved in assigning the transformation properties of the Dirac wave function ψ. First of all the Dirac equation, Eq. (20), should be covariant with respect to Lorentz transformations; this requirement is discussed in the present section. The other considerations arise when the subjects of charge conjugation and the formation of covariants quadratically from wave functions are treated; they are discussed in Secs. V and VI. The Dirac equation is to be covariant in the sense that, when the Lorentz transformation of Eq. (83) is applied, the equation

$$\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}A_\mu\right)\gamma_\mu\psi+\frac{mc}{\hbar}\psi=0 \tag{90}$$

leads to the equation

$$\left(\frac{\partial}{\partial x_\mu'}-\frac{ie}{\hbar c}A_\mu'\right)\gamma_\mu\psi'+\frac{mc}{\hbar}\psi'=0. \tag{91}$$

The operator $(\partial/\partial x_\mu)$ is a vector,

$$(\partial/\partial x_\mu')=a_{\mu\nu}(\partial/\partial x_\nu), \tag{92}$$

but, following Watanabe's assignments,[14] the four-vector potential transformation rule has an extra sign change whenever there is a time-reflection:

$$A_\mu'(x')=(a_{44}/|a_{44}|)a_{\mu\nu}A_\nu(x). \tag{93}$$

Because of this nonuniformity it is convenient to postpone the discussion of time-reflections and give them separate consideration later. For transformations in which a_{44} is positive the covariance can be obtained by the transformation

$$\psi'(x')=\Lambda\psi(x), \tag{94}$$

provided the matrix Λ is appropriately chosen. It is assumed that Λ has an inverse. In view of the arbitrariness of the Lorentz transformation under discussion, the covariance can be demonstrated by showing that Eq. (90) follows from Eq. (91) rather than vice versa. Substituting Eqs. (92), (93), (94) into Eq. (91) and operating from the left with Λ^{-1}, one finds

$$\left(\frac{\partial}{\partial x_\nu}-\frac{ie}{\hbar c}A_\nu\right)a_{\mu\nu}\Lambda^{-1}\gamma_\mu\Lambda\psi+\frac{mc}{\hbar}\psi=0,$$

so the covariance is established if Λ can be chosen so that

$$a_{\mu\nu}\Lambda^{-1}\gamma_\mu\Lambda=\gamma_\nu. \tag{95}$$

[14] Satosi Watanabe, Revs. Modern Phys. **27**, 26 (1955). See also Satosi Watanabe, Phys. Rev. **84**, 1008 (1951).

The existence of a matrix Λ satisfying this condition is easily shown in the following way. As a consequence of the orthogonality relations, Eq. (85), the matrices γ_μ' defined by

$$\gamma_\mu'=a_{\mu\rho}\gamma_\rho \tag{96}$$

satisfy the anticommutation rules

$$\begin{aligned}\gamma_\mu'\gamma_\nu'+\gamma_\nu'\gamma_\mu'&=a_{\mu\rho}\gamma_\rho a_{\nu\sigma}\gamma_\sigma+a_{\nu\sigma}\gamma_\sigma a_{\mu\rho}\gamma_\rho\\&=a_{\mu\rho}a_{\nu\sigma}(\gamma_\rho\gamma_\sigma+\gamma_\sigma\gamma_\rho)\\&=2a_{\mu\rho}a_{\nu\sigma}\delta_{\rho\sigma}\\&=2a_{\mu\rho}a_{\nu\rho}\\&=2\delta_{\mu\nu}\end{aligned}$$

and so, according to the fundamental theorem, a matrix Λ such that

$$\gamma_\mu'=\Lambda^{-1}\gamma_\mu\Lambda \tag{97}$$

can be found. This is just the matrix required to satisfy Eq. (95) since Eqs. (96) and (97), together with the orthogonality conditions Eqs. (84), yield

$$\begin{aligned}a_{\mu\nu}\Lambda^{-1}\gamma_\mu\Lambda&=a_{\mu\nu}\gamma_\mu'\\&=a_{\mu\nu}a_{\mu\rho}\gamma_\rho\\&=\delta_{\nu\rho}\gamma_\rho\\&=\gamma_\nu.\end{aligned}$$

The γ_μ' defined in Eq. (96) are not necessarily Hermitian even though the γ_μ are Hermitian because the $a_{\mu\rho}$ are not necessarily real; it is important here that the fundamental theorem holds between sets of matrices which need not be Hermitian. In summary, the covariance of Eq. (90) with respect to Lorentz transformations which preserve the direction of the time is obtained if the wave function transforms according to

$$\psi'(x')=\Lambda\psi(x), \tag{98}$$

where the matrix Λ is connected with the Lorentz transformation coefficients by

$$a_{\mu\rho}\gamma_\rho=\Lambda^{-1}\gamma_\mu\Lambda. \tag{99}$$

According to the discussion in the paragraph following Eq. (70), this defines Λ except for an arbitrary numerical factor.

When there is a time-reflection, $a_{44}\leq -1$, the covariance requires a different treatment than in the previous paragraph because of the extra sign change of the four-potential in Eq. (93). As a first step in compensating for the extra sign change, one takes the complex conjugate of Eq. (91) because this operation also changes the sign of the four-vector potential relative to the gradient operator:

$$\left(\frac{\partial}{\partial x_j'}+\frac{ie}{\hbar c}A_j'\right)\gamma_j^C\psi'^C-\left(\frac{\partial}{\partial x_4'}+\frac{ie}{\hbar c}A_4'\right)\gamma_4^C\psi'^C+\frac{mc}{\hbar}\psi'^C=0. \tag{100}$$

Here the difference in sign between the first two terms arises because x_4' and A_4' are pure imaginary. From the form of this equation one is led to introduce a matrix C such that

$$\gamma_j{}^C = C\gamma_j C^{-1}, \tag{101}$$

$$\gamma_4{}^C = -C\gamma_4 C^{-1} \tag{102}$$

because then, in terms of ϕ' defined by

$$\psi'^C(x') = C\phi'(x'), \tag{103}$$

and after multiplying by C^{-1}, Eq. (100) becomes

$$\left(\frac{\partial}{\partial x_\mu'} + \frac{ie}{\hbar c}A_\mu'\right)\gamma_\mu\phi' + \frac{mc}{\hbar}\phi' = 0. \tag{104}$$

The fundamental theorem ensures the existence of such a matrix C, for the γ_μ are a set of Dirac matrices satisfying the anticommutation rules Eq. (43) and, by taking the complex conjugate of Eq. (43), it is seen that $\gamma_j{}^C$, $-\gamma_4{}^C$ satisfy a similar set of equations. In fact it is easily verified that the matrix

$$C = B\gamma_4\gamma_5 \tag{105}$$

satisfies Eqs. (101) and (102), as a consequence of the defining equation for B, Eq. (71), and of the fact that the γ_μ are Hermitian. The final step is to transform Eq. (104) to the unprimed quantities according to Eqs. (92) and (93), keeping in mind that $a_{44} \leq -1$. By comparing this with the non-time-reflection case discussed above it is obvious that if

$$\phi'(x') = \Lambda\psi(x), \tag{106}$$

where

$$a_{\mu\rho}\gamma_\rho = \Lambda^{-1}\gamma_\mu\Lambda, \tag{99}$$

then Eq. (104) becomes

$$\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}A_\mu\right)\gamma_\mu\psi + \frac{mc}{\hbar}\psi = 0,$$

as required. The complete wave-function transformation is found by combining Eqs. (103) and (106). The Dirac equation is covariant with respect to time reflections if the wave function transforms according to

$$\psi'^C(x') = C\Lambda\psi(x), \tag{107}$$

where Λ is chosen to satisfy Eq. (99), the same as for non-time-reflections. (One could, of course, perform the two steps above in the opposite order, first transforming to the unprimed quantities and then making the transformation with the C matrix. This leads to the alternate transformation rule

$$\psi'^C(x') = \Lambda^C C\psi(x). \tag{108}$$

In the next section it is argued that only transformations such that

$$C\Lambda = \Lambda^C C$$

should be considered; in that case Eqs. (107) and (108) are equivalent.)

Regardless of whether it is a time reflection or not, when the Lorentz transformation

$$x_\mu' = a_{\mu\nu}x_\nu$$

is applied, a matrix Λ such that

$$a_{\mu\rho}\gamma_\rho = \Lambda^{-1}\gamma_\mu\Lambda \tag{109}$$

must be found before the transformation of the wave function is known. It would be a complicated affair to write down Λ explicitly in the general case, but for the special cases of two-dimensional rotations and the space and time reflections of Eqs. (88) and (89) the matrix Λ has a simple form, as exhibited below. In a sense the formulas below give Λ in the general case because any Lorentz transformation can be expressed as a sequence of reflections and two-dimensional rotations. To begin with, the matrix

$$\Lambda = \cos(\omega/2) + \gamma_\mu\gamma_\nu \sin(\omega/2) \tag{110}$$

(where $\mu \neq \nu$) corresponds to a rotation in the $\mu\nu$ plane; the parameter ω measures the rotation angle. The inverse of this matrix is

$$\Lambda^{-1} = \cos(\omega/2) - \gamma_\mu\gamma_\nu \sin(\omega/2),$$

as is easily verified by direct multiplication. When $\mu = 1$, $\nu = 2$ for example, it is easily seen that

$$\begin{aligned} \Lambda^{-1}\gamma_1\Lambda &= \cos\omega\ \gamma_1 + \sin\omega\ \gamma_2, \\ \Lambda^{-1}\gamma_2\Lambda &= -\sin\omega\ \gamma_1 + \cos\omega\ \gamma_2, \\ \Lambda^{-1}\gamma_3\Lambda &= \gamma_3, \\ \Lambda^{-1}\gamma_4\Lambda &= \gamma_4. \end{aligned}$$

By comparison with Eq. (109) this is found to correspond to the Lorentz transformation

$$\begin{aligned} x_1' &= \cos\omega\ x_1 + \sin\omega\ x_2, \\ x_2' &= -\sin\omega\ x_1 + \cos\omega\ x_2, \\ x_3' &= x_3, \\ x_4' &= x_4, \end{aligned}$$

which is a space rotation through an angle ω about the x_3 axis. A parallel result is obtained for the case $\mu = 3$, $\nu = 4$ with the difference that ω must be pure imaginary to satisfy the conditions of reality. With the change to the parameter v such that

$$\left.\begin{aligned} \sin\omega &= ic^{-1}v(1-c^{-2}v^2)^{-\frac{1}{2}}, \\ \cos\omega &= (1-c^{-2}v^2)^{-\frac{1}{2}}, \end{aligned}\right\} \tag{111}$$

the transformation can be written in the form

$$\left.\begin{aligned} x_1' &= x_1, \\ x_2' &= x_2, \\ x_3' &= (x_3 - vt)(1-c^{-2}v^2)^{-\frac{1}{2}}, \\ t' &= (t - c^{-2}vx_3)(1-c^{-2}v^2)^{-\frac{1}{2}}, \end{aligned}\right\} \tag{112}$$

which is the usual form for the Lorentz transformation between two parallel coordinate systems with relative velocity v in the x_3 direction. A more complete discussion of the matrix in Eq. (110) is given by Pauli.[15] For a space reflection, the assignment

$$\Lambda = i\gamma_4 \tag{113}$$

gives, on substitution into Eq. (109), the Lorentz transformation coefficients required in Eq. (88). Finally it is easily verified that the matrix

$$\Lambda = \gamma_5\gamma_4 \tag{114}$$

corresponds to the time reflection of Eq. (89). From Eqs. (107), (105), and (114) combined, it is seen that the net wave function transformation, corresponding to the time reflection of Eq. (89), is

$$\begin{aligned}\psi'^C(x') &= C\Lambda\psi(x)\\ &= B\gamma_4\gamma_5\gamma_5\gamma_4\psi(x)\\ &= B\psi(x).\end{aligned} \tag{115}$$

The arbitrary numerical factors in the matrices Λ of Eqs. (110), (113), and (114) have been chosen in advance to satisfy conditions which will be imposed in the next two sections. It will be seen that these conditions determine these matrices except for an arbitrary factor of -1. In the case of the transformations continuous with the identity this indeterminacy is in keeping with the dependence of the matrices in Eq. (110) on the half-angle, because if ω is increased by 2π, the matrix in Eq. (110) changes sign but the corresponding Lorentz transformation is unaffected.

The matrix Λ of Eq. (109) has an especially simple form in the special case of a space rotation or reflection,

$$x_j' = a_{jk}x_k, \quad x_4' = x_4,$$

and when the matrices introduced in Sec. II, Eqs. (18), (19), and (31), are used for the γ_μ:

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{116}$$

First of all, from the fourth component of Eq. (109), it is seen that

$$\gamma_4 = \Lambda^{-1}\gamma_4\Lambda.$$

If the equation

$$\Lambda\gamma_4 = \gamma_4\Lambda$$

is written in terms of two-by-two matrices using γ_4 as given in Eq. (116), it is seen that Λ has the form

$$\Lambda = \begin{pmatrix} \Lambda_S & 0 \\ 0 & \Lambda_L \end{pmatrix},$$

where Λ_S and Λ_L are two-by-two matrices which have yet to be determined and 0 indicates the zero two-by-two matrix. The subscripts S and L are used because Λ_S and Λ_L are the transformation matrices for the small and large components of the wave function in the nonrelativistic limit. The next step is obviously to restate the equation for Λ, Eq. (109), in terms of Λ_S and Λ_L. The first three of Eqs. (109) become

$$a_{jk}\begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix} = \begin{pmatrix} \Lambda_S^{-1} & 0 \\ 0 & \Lambda_L^{-1} \end{pmatrix}\begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix}\begin{pmatrix} \Lambda_S & 0 \\ 0 & \Lambda_L \end{pmatrix},$$

where the γ_j have been replaced according to Eq. (116). On multiplying out the right-hand side, one finds that the equation will be satisfied provided the equations

$$a_{jk}\sigma_k = \Lambda_S^{-1}\sigma_j\Lambda_L, \tag{117}$$

$$a_{jk}\sigma_k = \Lambda_L^{-1}\sigma_j\Lambda_S, \tag{118}$$

are valid. These two equations can be uncoupled by using the properties of the Pauli matrices. If Eq. (29) is operated on from the left by Λ_L^{-1} and from the right by Λ_L, the result is

$$\Lambda_L^{-1}\sigma_j\Lambda_S\Lambda_S^{-1}\sigma_k\Lambda_L = i\epsilon_{jkl}\Lambda_L^{-1}\sigma_l\Lambda_L + \delta_{jk},$$

where the factor $\Lambda_S\Lambda_S^{-1}$ has been added in the middle of the term on the left to facilitate the substitution of Eqs. (117) and (118):

$$a_{jl}\sigma_l a_{km}\sigma_m = i\epsilon_{jkl}\Lambda_L^{-1}\sigma_l\Lambda_L + \delta_{jk}.$$

This equation can be considerably simplified by applying Eq. (29) again on the left to make the equation linear in the σ_j. It is seen that

$$a_{jl}a_{km}(i\epsilon_{lmn}\sigma_n + \delta_{lm}) = i\epsilon_{jkl}\Lambda_L^{-1}\sigma_l\Lambda_L + \delta_{jk}.$$

The terms independent of the σ_j cancel as a consequence of the orthogonality relations. These relations can also be introduced into the first term on the left so that the equation becomes

$$a_{jl}a_{km}(a_{pr}a_{pn})\epsilon_{lmr}\sigma_n = \epsilon_{jkl}\Lambda_L^{-1}\sigma_l\Lambda_L.$$

The reason for this introduction is that now the well-known fact

$$a_{jl}a_{km}a_{pr}\epsilon_{lmr} = (\det a)\epsilon_{jkp}$$

brings the equation to the form

$$(\det a)\epsilon_{jkp}a_{pn}\sigma_n = \epsilon_{jkl}\Lambda_L^{-1}\sigma_l\Lambda_L$$

so that, finally,

$$(\det a)a_{ln}\sigma_n = \Lambda_L^{-1}\sigma_l\Lambda_L. \tag{119}$$

This equation is to be compared with Eq. (109); it is seen that the Pauli matrices play a part in the theory of three-dimensional orthogonal transformations parallel to the Dirac matrices in the theory of four-dimensional orthogonal transformations. The connection between Λ_S and Λ_L is easily found by combining Eqs. (118) and (119):

$$\begin{aligned}\sigma_j\Lambda_S &= \Lambda_L a_{jk}\sigma_k\\ &= (\det a)^{-1}\sigma_j\Lambda_L,\end{aligned}$$

[15] Reference 7, pp. 222–224.

and therefore, since the square of the determinant is one and σ_j^2 is 1,

$$\Lambda_S = (\det a)\Lambda_L. \tag{120}$$

The problem of finding the four-by-four matrix Λ is thus reduced to the problem of solving Eq. (119) for the two-by-two matrix Λ_L. Solutions of Eq. (119) in the case of a rotation about one of the coordinate axes are easily obtained by substituting the γ_j from Eq. (116) into Eq. (110) and using Eq. (29) to simplify the products of the Pauli matrices. For a rotation through an angle ω in the right-hand sense about the k-axis the result is

$$\Lambda_L = \cos(\omega/2) + i\sigma_k \sin(\omega/2). \tag{121}$$

For rotations the matrix Λ_L is just the matrix of the Cayley-Klein parameters.[16] A matrix Λ_L for the space reflection of Eq. (88) is

$$\Lambda_L = i, \tag{122}$$

as may be found by using the γ_4 of Eq. (116) in Eq. (113).

The covariance of the Dirac equation with respect to Lorentz transformations leads uniformly in the nonrelativistic limit to the covariance of the Schrödinger-Pauli equation, Eq. (40), with respect to space rotations and reflections, Galilean transformations, and time reflection. As discussed in Sec. II the nonrelativistic limit of the Dirac equation may be taken by introducing the matrices of Eq. (116) and discarding the small components of the wave function compared to the large. For a space rotation or reflection the results of the previous paragraph may be used. The small components do not even enter in to the transformation rule for the large components. It is easily verified that Eq. (40) is covariant with respect to the transformation

$$\begin{aligned} x_j' &= a_{jk}x_k, \\ t' &= t, \\ A_j'(x') &= a_{jk}A_k(x), \\ \Phi'(x') &= \Phi(x), \\ B_j'(x') &= (\det a)a_{jk}B_k(x), \\ \psi_L'(x') &= \Lambda_L\psi_L(x), \end{aligned}$$

where

$$\begin{aligned} a_{jk}a_{jl} &= \delta_{kl}, \\ (\det a)a_{jk}\sigma_k &= \Lambda_L^{-1}\sigma_j\Lambda_L. \end{aligned}$$

For the Galilean transformation it is convenient to refer to the Lorentz transformation of Eq. (112) which, when the terms involving (v/c) can be neglected, reduces to

$$\left.\begin{aligned} x_1' &= x_1, \\ x_2' &= x_2, \\ x_3' &= x_3 - vt, \\ t' &= t. \end{aligned}\right\} \tag{123}$$

[16] See, for example, Herbert Goldstein, *Classical Mechanics* (Addison-Wesley Press, Inc., Cambridge, Massachusetts, 1950), p. 116.

The corresponding transformation of the space and time derivatives is easily seen to be

$$\left.\begin{aligned} (\partial/\partial x_j') &= (\partial/\partial x_j), \\ (\partial/\partial t') &= (\partial/\partial t) + v(\partial/\partial x_3). \end{aligned}\right\} \tag{124}$$

The potentials A_j, $i\Phi$ form a four-vector in parallel with the coordinates x_j, ict; by comparison with Eq. (112) one has

$$\begin{aligned} A_3' &= (A_3 - c^{-1}v\Phi)(1 - c^{-2}v^2)^{-\frac{1}{2}}, \\ \Phi' &= (\Phi - c^{-1}vA_3)(1 - c^{-2}v^2)^{-\frac{1}{2}}, \end{aligned}$$

where the nonrelativistic limit has yet to be taken. In Eq. (40) it is seen that an effect of the limiting process was to multiply the scalar potential by a factor of c relative to the vector potential. Therefore in the nonrelativistic limit one must carry the transformation rule for the scalar potential to one higher order than for the vector potential and write

$$\left.\begin{aligned} A_j' &= A_j, \\ \Phi' &= \Phi - (v/c)A_3. \end{aligned}\right\} \tag{125}$$

A minor consequence of the first of these is that

$$B_j' = B_j. \tag{126}$$

The matrix Λ corresponding to the Lorentz transformation of Eq. (112) is given by Eq. (110) when $\mu = 3$, $\nu = 4$, and when the parameter ω is given by Eqs. (111). When (v/c) is small compared to 1, it is seen that ω is just $i(v/c)$, Λ is the identity, and

$$\psi'(x') = \psi(x).$$

In view of the defining equation for the large components, Eq. (32), they transform according to

$$\begin{aligned} \psi_L'(x') &= \psi_L(x)\exp[mc^2(t-t')/i\hbar] \\ &= \psi_L(x)\exp[(mvx_3 - \tfrac{1}{2}mv^2t)/i\hbar], \end{aligned} \tag{127}$$

where the last of Eqs. (112) has been used to find the limiting value of the exponent. One can then easily verify by direct differentiation that Eq. (40) is covariant with respect to the transformation of Eqs. (124) to (127) so the covariance with respect to Galilean transformations does come out as a limit of the covariance with respect to Lorentz transformations. The final transformation to be considered is the time reflection

$$x_j' = x_j, \quad t' = -t. \tag{128}$$

As required by Eq. (93) the potentials transform according to

$$A_j' = -A_j, \quad \Phi' = \Phi, \tag{129}$$

and so the rule for the magnetic field is

$$B_j' = -B_j. \tag{130}$$

Substituting for the B matrix from Eq. (82) into the wave-function transformation rule, Eq. (115), and using the definition of the large components, Eq. (32), one finds that

$$\psi_L'^C(x')=\sigma_2\psi_L(x). \tag{131}$$

In order to demonstrate the covariance, it is convenient to start with the complex conjugate of Eq. (40) in the primed coordinate system:

$$\left[\frac{1}{2m}\left(-\frac{\hbar}{i}\frac{\partial}{\partial x_j'}-\frac{e}{c}A_j'\right)\left(-\frac{\hbar}{i}\frac{\partial}{\partial x_j'}-\frac{e}{c}A_j'\right)\right.$$
$$\left.-\left(\frac{e\hbar}{2mc}\right)\sigma_j{}^C B_j'+e\Phi'\right]\psi_L'^C=-i\hbar\frac{\partial\psi_L'^C}{\partial t'}.$$

If here the primed quantities are eliminated in favor of the unprimed by Eqs. (128) to (131) and if then the equation is multiplied through by σ_2, the result is

$$\left[\frac{1}{2m}\left(-\frac{\hbar}{i}\frac{\partial}{\partial x_j}+\frac{e}{c}A_j\right)\left(-\frac{\hbar}{i}\frac{\partial}{\partial x_j}+\frac{e}{c}A_j\right)\right.$$
$$\left.+\left(\frac{e\hbar}{2mc}\right)\sigma_2\sigma_j{}^C\sigma_2 B_j+e\Phi\right]\psi_L=i\hbar\frac{\partial\psi_L}{\partial t}.$$

The remark that, as a consequence of the definition of the Pauli matrices Eq. (28),

$$\sigma_2\sigma_j{}^C\sigma_2=-\sigma_j$$

completes the proof of the covariance with respect to time reflection. Time reflection in the nonrelativistic limit has been discussed especially by Wigner.[17]

In the treatment of time reflections given above, the vector-potential transformation rule was assumed to be

$$A_\mu'(x')=(a_{44}/|a_{44}|)a_{\mu\nu}A_\nu(x), \tag{93}$$

and this led to the wave-function transformation

$$\psi'^C(x')=C\Lambda\psi(x), \tag{107}$$

where

$$a_{\mu\rho}\gamma_\rho=\Lambda^{-1}\gamma_\mu\Lambda. \tag{109}$$

If, in contrast to this approach, the vector potential is assumed to be a regular vector even with respect to time reflections,

$$A_\mu'(x')=a_{\mu\nu}A_\nu(x),$$

then the time reflections will not require a separate treatment and

$$\psi'(x')=\Lambda\psi(x) \tag{132}$$

will hold for all Lorentz transformations. This point of view has the advantage that it permits a uniform treatment for all Lorentz transformations, and it has the disadvantage that it does not lead to the nonrelativistic time-reflection of the preceding paragraph.

V. CHARGE CONJUGATION

In this section the basic idea of charge conjugation is outlined, and some of the properties of the operation are discussed. The effect on the transformation matrix Λ of requiring that charge conjugation be a covariant operation is discussed in detail.

The idea of charge conjugation arises in a further consideration of the C-matrix substitution made in the treatment of time reflections in the preceding section. Referring back to Eqs. (100) to (105), one sees that the substitution

$$\psi^C(x)=C\phi(x) \tag{133}$$

carries the equation

$$\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}A_\mu\right)\gamma_\mu\psi+\frac{mc}{\hbar}\psi=0 \tag{134}$$

into the equation

$$\left(\frac{\partial}{\partial x_\mu}+\frac{ie}{\hbar c}A_\mu\right)\gamma_\mu\phi+\frac{mc}{\hbar}\phi=0, \tag{135}$$

where the C matrix may be defined by

$$C=B\gamma_4\gamma_5, \tag{136}$$

and has the properties

$$\gamma_j{}^C=C\gamma_jC^{-1}, \tag{137}$$

$$\gamma_4{}^C=-C\gamma_4C^{-1}. \tag{138}$$

Before discussing the significance of this substitution, some properties of the C matrix and of Eq. (133) will be pointed out. Two properties of C are that it is symmetric and unitary. These are consequences of the properties of the B matrix discussed at the end of Sec. III. The symmetry of C follows from the antisymmetry of B:

$$\begin{aligned}C^T&=(B\gamma_4\gamma_5)^T\\&=\gamma_5{}^T\gamma_4{}^TB^T\\&=-\gamma_5{}^T\gamma_4{}^TB\\&=-B\gamma_5\gamma_4\\&=B\gamma_4\gamma_5\\&=C,\end{aligned} \tag{139}$$

where Eq. (73) was used to eliminate the transposed matrices. The fact that C is unitary follows from the fact that B is unitary:

$$\begin{aligned}C^HC&=(B\gamma_4\gamma_5)^H(B\gamma_4\gamma_5)\\&=\gamma_5{}^H\gamma_4{}^HB^HB\gamma_4\gamma_5\\&=\gamma_5\gamma_4\gamma_4\gamma_5\\&=1.\end{aligned} \tag{140}$$

The function $\phi(x)$ connected with $\psi(x)$ according to Eq. (133) is called the charge conjugate wave function to $\psi(x)$. Charge conjugation is reciprocal in the sense that if ϕ is charge conjugate to ψ then ψ is charge conjugate to ϕ. This can be seen by multiplying Eq. (133) by C^H to obtain

$$C^H\psi^C(x)=C^HC\phi(x).$$

[17] E. Wigner, Göttinger Nachrichten **31**, 546 (1932).

On the right Eq. (140) may be applied, and on the left C^H can be replaced by C^C in view of Eq. (139). Then, on taking the complex conjugate, one finds

$$\phi^C(x)=C\psi(x),$$

which proves the assertion. Now that these properties have been established, the significance of the charge-conjugation operation can be discussed. From Eqs. (134) and (135) it is seen that, for a certain set of potentials $A_\mu(x)$, if $\psi(x)$ is a solution of the Dirac equation for a particle with charge e then $\phi(x)$ is a solution for a particle with charge $-e$. In view of the reciprocal property of the charge conjugation this correspondence between solutions is one-to-one. The corresponding solutions have opposite signs of energy for if

$$i\hbar\partial\psi/\partial t=W\psi$$

then it is seen that

$$\begin{aligned} i\hbar\partial\phi/\partial t &= i\hbar\partial(C^C\psi^C)/\partial t \\ &= -C^C(i\hbar\partial\psi/\partial t)^C \\ &= -WC^C\psi^C \\ &= -W\phi. \end{aligned}$$

This is the basis of Dirac's theory of the positron[18]: the correspondence between negative-energy solutions of the electron equation and positive-energy solutions of the positron equation led him to propose a single theory which includes both particles. In it he makes the assumption that the negative-energy electron states are nearly all filled and that a positron is the absence of an electron from one of the negative energy states. This theory has the advantage that it explains why electrons in positive-energy states do not ordinarily fall to negative-energy states; such transitions are forbidden by the Pauli exclusion principle as long as the negative-energy states are already filled. In many of the modern theories the postulate of the infinite number of electrons in negative-energy states is avoided by quantizing the particle field in such a way that electrons and positrons appear on an equal footing from the start.[19]

Especially if Dirac's theory of the positron is used, it is to be assumed that the correspondence between positive-energy positron states and negative-energy electron states is independent of a particular Lorentz frame of reference, so that the charge conjugation operation is covariant with respect to Lorentz transformations. This has been pointed out by Pauli.[20] To put this requirement in symbols, if

$$\psi^C(x)=C\phi(x), \tag{141}$$

and if a Lorentz transformation is made so that

$$\left.\begin{aligned} \psi'(x') &= \Lambda\psi(x), \\ \phi'(x') &= \Lambda\phi(x) \end{aligned}\right\} \tag{142}$$

when the time is not reflected or so that

$$\left.\begin{aligned} \psi'^C(x') &= C\Lambda\psi(x), \\ \phi'^C(x') &= C\Lambda\phi(x) \end{aligned}\right\} \tag{143}$$

otherwise, then it is to be assumed in consequence that

$$\psi'^C(x')=C\phi'(x'). \tag{144}$$

This assumption imposes a condition on the transformation matrix Λ. Considering a non-time-reflection, if one substitutes Eqs. (142) into Eq. (144) the result is

$$\Lambda^C\psi^C(x)=C\Lambda\phi(x),$$

and if then Eq. (141) is used on the left,

$$\Lambda^C C\phi(x)=C\Lambda\phi(x),$$

so Λ should satisfy

$$\Lambda^C C=C\Lambda. \tag{145}$$

The same condition is obtained in a similar way when a time-reflection is considered. Given the Lorentz transformation coefficients $a_{\mu\nu}$, Λ is determined except for an arbitrary numerical factor by Eq. (109):

$$a_{\mu\rho}\gamma_\rho=\Lambda^{-1}\gamma_\mu\Lambda. \tag{146}$$

It is easily seen that Λ must satisfy an equation like Eq. (145) as a consequence of the conditions of reality on the $a_{\mu\rho}$. The complex conjugate of Eqs. (146) is

$$\begin{aligned} a_{jk}\gamma_k{}^C-a_{j4}\gamma_4{}^C &= (\Lambda^{-1})^C\gamma_j{}^C\Lambda^C, \\ -a_{4k}\gamma_k{}^C+a_{44}\gamma_4{}^C &= (\Lambda^{-1})^C\gamma_4{}^C\Lambda^C, \end{aligned}$$

and these equations can be combined with the help of Eqs. (137) and (138) into the form

$$a_{\mu\rho}C\gamma_\rho C^{-1}=(\Lambda^{-1})^C C\gamma_\mu C^{-1}\Lambda^C.$$

Here Eqs. (146) may be reapplied to eliminate the $a_{\mu\rho}$ on the left

$$C\Lambda^{-1}\gamma_\mu\Lambda C^{-1}=(\Lambda^{-1})^C C\gamma_\mu C^{-1}\Lambda^C,$$

and if this equation is multiplied from the left by $C^{-1}\Lambda^C$ and from the right by $C\Lambda^{-1}$ the result is

$$C^{-1}\Lambda^C C\Lambda^{-1}\gamma_\mu=\gamma_\mu C^{-1}\Lambda^C C\Lambda^{-1}.$$

Thus the matrix $C^{-1}\Lambda^C C\Lambda^{-1}$ commutes with the γ_μ and so, according to step 5 of the proof of the fundamental theorem, it is a number, say k, times the identity. This means that

$$\Lambda^C C=kC\Lambda. \tag{147}$$

Furthermore by taking the determinant of this last equation one sees that the absolute value of k is unity. Then from Eq. (147) it is clear that k can be adjusted to unity by proper choice of the argument of the arbitrary complex number in Λ. One can thus obtain the covariance of the charge conjugation operation in this simple way. In what follows it will be supposed that this adjustment has been made so that Eq. (145) does apply

[18] Reference 1, p. 272.

[19] See, for example, W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, England, 1954), third edition, Chap. III.

[20] Reference 2, p. 129.

to Λ. It is clear that Eqs. (145) and (146) determine Λ except for a real (positive or negative) numerical factor.

A consequence of Eq. (145) is that when a series of Lorentz transformations is applied, the net wave-function transformation can be found by multiplying all the Λ matrices together and using

$$\psi'(x')=\Lambda\psi(x)$$

or

$$\psi'^C(x')=C\Lambda\psi(x)$$

depending on whether the over-all transformation is a time-reflection or not. For example suppose the time-reflection,

$$x_\mu'=b_{\mu\nu}x_\nu,$$
$$\psi'^C(x')=C\mathrm{M}\psi(x),$$
$$b_{\mu\nu}\gamma_\nu=\mathrm{M}^{-1}\gamma_\mu\mathrm{M},$$

is followed by the non-time-reflection,

$$x_\lambda''=a_{\lambda\mu}x_\mu',$$
$$\psi''(x'')=\Lambda\psi'(x'),$$
$$a_{\lambda\mu}\gamma_\mu=\Lambda^{-1}\gamma_\lambda\Lambda.$$

Then the net transformation is found by direct substitution to be

$$\begin{aligned}x_\lambda''&=a_{\lambda\mu}b_{\mu\nu}x_\nu,\\ \psi''(x'')&=\Lambda C^C\mathrm{M}^C\psi^C(x),\\ a_{\lambda\mu}b_{\mu\nu}\gamma_\nu&=a_{\lambda\mu}\mathrm{M}^{-1}\gamma_\mu\mathrm{M}\\ &=\mathrm{M}^{-1}\Lambda^{-1}\gamma_\lambda\Lambda\mathrm{M}\\ &=(\Lambda\mathrm{M})^{-1}\gamma_\lambda(\Lambda\mathrm{M}).\end{aligned}$$

Since Eq. (145) applies to Λ, the wave-function transformation can be rewritten as

$$\begin{aligned}\psi''^C(x'')&=\Lambda^C C\mathrm{M}\psi(x)\\ &=C(\Lambda\mathrm{M})\psi(x),\end{aligned}$$

which proves the assertion in this particular case. The assertion in the other cases can be verified in a similar way. The special values of Λ given in Eqs. (110), (113), and (114) have been chosen so that they satisfy Eq. (145). That charge conjugation should be covariant with respect to those two reflections was first pointed out by Racah.[21] Any matrix compounded from those given in Eqs. (110), (113), and (114) will still satisfy Eq. (145) since if both

$$\Lambda^C C=C\Lambda$$

and

$$\mathrm{M}^C C=C\mathrm{M},$$

then for the product $\Lambda\mathrm{M}$,

$$\begin{aligned}(\Lambda\mathrm{M})^C C&=\Lambda^C\mathrm{M}^C C\\ &=\Lambda^C C\mathrm{M}\\ &=C(\Lambda\mathrm{M}),\end{aligned}$$

as required.

[21] Giulio Racah, Nuovo cimento **14**, 322 (1937).

VI. COVARIANTS FORMED QUADRATICALLY FROM DIRAC WAVE FUNCTIONS

In this section it is shown how tensors can be formed quadratically from Dirac wave functions. These tensors are of physical interest because they can be chosen so that their components are real or pure imaginary according to whether their indices contain an even or an odd number of fours—with such a choice they can represent observable quantities. The discussion of these tensors leads to an additional condition which is imposed on the wave function transformation matrix Λ.

In forming tensors the basic idea is to consider the quantities $\psi^A\gamma_B\psi$ where ψ is a Dirac wave function and ψ^A is the adjoint wave function of Eq. (25). Therefore the first thing to be discussed is the transformation rule for the adjoint wave function. In an unprimed coordinate system the adjoint is defined by

$$\psi^A=\psi^H\gamma_4 \tag{148}$$

and in a primed system by

$$\psi'^A=\psi'^H\gamma_4. \tag{149}$$

If the two coordinate systems are connected by a non-time-reflection,

$$\psi'=\Lambda\psi, \tag{150}$$

then, combining these three equations, it is seen that

$$\begin{aligned}\psi'^A&=\psi'^H\gamma_4\\ &=\psi^H\Lambda^H\gamma_4\\ &=\psi^A\gamma_4\Lambda^H\gamma_4.\end{aligned} \tag{151}$$

On the other hand, if the two are connected by a time-reflection

$$\psi'^C=C\Lambda\psi, \tag{152}$$

it is found that

$$\begin{aligned}(\psi'^A)^C&=(\psi'^H\gamma_4)^C\\ &=(\psi'^C)^H\gamma_4{}^C\\ &=(C\Lambda\psi)^H\gamma_4{}^C\\ &=\psi^H\Lambda^H C^H\gamma_4{}^C\\ &=\psi^A\gamma_4\Lambda^H C^{-1}\gamma_4{}^C\\ &=-\psi^A\gamma_4\Lambda^H\gamma_4 C^{-1},\end{aligned} \tag{153}$$

where properties of the C matrix, Eqs. (140) and (138) have been used in the last two steps. One can see that if ψ^A were to transform with a factor of Λ^{-1}, the tensor transformation rules for the quantities $\psi^A\gamma_B\psi$ would result in a direct way from the fact that

$$a_{\mu\rho}\gamma_\rho=\Lambda^{-1}\gamma_\mu\Lambda. \tag{154}$$

It does happen that the matrix $\gamma_4\Lambda^H\gamma_4$ is related to Λ^{-1} as long as the $a_{\mu\rho}$ satisfy the conditions of reality. This can easily be seen by taking the Hermitian conjugate of Eq. (154),

$$\begin{aligned}a_{jk}\gamma_k-a_{j4}\gamma_4&=\Lambda^H\gamma_j(\Lambda^{-1})^H,\\ -a_{4k}\gamma_k+a_{44}\gamma_4&=\Lambda^H\gamma_4(\Lambda^{-1})^H,\end{aligned}$$

noticing that these equations can be combined into the form

$$a_{\mu\rho}\gamma_4\gamma_\rho\gamma_4=\Lambda^H\gamma_4\gamma_\mu\gamma_4(\Lambda^{-1})^H,$$

substituting Eq. (154) on the left to eliminate the $a_{\mu\rho}$,

$$\gamma_4\Lambda^{-1}\gamma_\mu\Lambda\gamma_4=\Lambda^H\gamma_4\gamma_\mu\gamma_4(\Lambda^{-1})^H,$$

and multiplying from the left by $\Lambda\gamma_4$ and from the right by $\Lambda^H\gamma_4$ to obtain

$$\gamma_\mu\Lambda\gamma_4\Lambda^H\gamma_4=\Lambda\gamma_4\Lambda^H\gamma_4\gamma_\mu.$$

Thus the matrix $\Lambda\gamma_4\Lambda^H\gamma_4$ commutes with the γ_μ and so, according to step 5 of the proof of the fundamental theorem, is a number, say k, times the identity

$$\Lambda\gamma_4\Lambda^H\gamma_4=k. \tag{155}$$

It can further be seen that k here is real by taking the Hermitian conjugate of this equation,

$$\gamma_4\Lambda\gamma_4\Lambda^H=k^C,$$

and multiplying from the left and right by γ_4:

$$\Lambda\gamma_4\Lambda^H\gamma_4=k^C. \tag{156}$$

Equations (155) and (156) show that k is equal to its conjugate and so is real. Furthermore from Eq. (155) it is seen that by proper choice of the absolute value of the arbitrary factor in Λ one can adjust the absolute value of k to be one. This can be done independently of the assignment made in the preceding section to obtain Eq. (145) since that required a choice of the phase of the arbitrary factor only. In what follows it will be supposed that the arbitrary factor in Λ has been chosen so that both Eq. (145) and

$$\Lambda\gamma_4\Lambda^H\gamma_4=\pm 1 \tag{157}$$

are satisfied by Λ. It happens that the sign in Eq. (157) is just the sign of a_{44}. One can see this by multiplying the fourth of Eqs. (154),

$$a_{4\rho}\gamma_\rho=\Lambda^{-1}\gamma_4\Lambda,$$

from the left and from the right by γ_4 and adding to obtain

$$a_{4\rho}(\gamma_4\gamma_\rho+\gamma_\rho\gamma_4)=\gamma_4\Lambda^{-1}\gamma_4\Lambda+\Lambda^{-1}\gamma_4\Lambda\gamma_4.$$

The term on the left can be simplified by using the anticommutation relations of the γ_ρ. Also according to Eq. (157), $\gamma_4\Lambda^{-1}$ can be replaced by $\pm\Lambda^H\gamma_4$ in the first term on the right and $\Lambda\gamma_4$ by $\pm\gamma_4(\Lambda^{-1})^H$ in the second. The equation then becomes

$$2a_{44}=\pm[\Lambda^H\Lambda+\Lambda^{-1}(\Lambda^{-1})^H].$$

The sign can now be decided by considering one of the diagonal elements of this matrix equation. A diagonal element of the product of a matrix and its Hermitian conjugate must be real and positive so the term in the square brackets gives a positive contribution. The sign must accordingly be identical with the sign of a_{44} and Eq. (157) can be written

$$\Lambda\gamma_4\Lambda^H\gamma_4=(a_{44}/|a_{44}|). \tag{158}$$

In view of this equation the transformation rules for the adjoint wave function, Eqs. (151) and (153), are as follows:

$$\psi'^A=\psi^A\Lambda^{-1} \tag{159}$$

for a non-time-reflection, and

$$(\psi'^A)^C=\psi^A\Lambda^{-1}C^{-1} \tag{160}$$

for a time-reflection.

Further discussion of the formation of tensors will be postponed for a paragraph in order to make a few more remarks about the wave-function transformation matrix Λ. Given a set of Lorentz transformation coefficients $a_{\mu\rho}$, Λ is to be chosen so that it satisfies Eqs. (109), (145), and (158):

$$a_{\mu\nu}\gamma_\nu=\Lambda^{-1}\gamma_\mu\Lambda, \tag{161}$$

$$\Lambda^C C=C\Lambda, \tag{162}$$

$$\Lambda\gamma_4\Lambda^H\gamma_4=(a_{44}/|a_{44}|). \tag{163}$$

Equation (161) determines Λ as far as a numerical factor, Eq. (162) fixes the argument of the factor as far as a multiple of π, and Eq. (163) specifies the absolute value of the factor. These three equations then determine Λ except for a factor of ± 1. For each Lorentz transformation there are two matrices Λ satisfying these conditions and these two differ only in sign. The special solutions for Λ given in Eqs. (110), (113), (114) have already been chosen to satisfy Eq. (163) as well as the other two. Products of these matrices will also satisfy Eq. (163) properly, as well as the other two equations, for if also

$$M\gamma_4M^H\gamma_4=(b_{44}/|b_{44}|)$$

then, for the product ΛM,

$$\begin{aligned}(\Lambda M)\gamma_4(\Lambda M)^H\gamma_4&=\Lambda M\gamma_4M^H\Lambda^H\gamma_4\\&=(b_{44}/|b_{44}|)\Lambda\gamma_4\Lambda^H\gamma_4\\&=(b_{44}/|b_{44}|)(a_{44}/|a_{44}|).\end{aligned}$$

It can easily be seen that the matrix Λ is unimodular:

$$\det\Lambda=1. \tag{164}$$

To prove this one takes the determinant of Eq. (162) to see that $\det\Lambda$ is real and the determinant of Eq. (163) to see that $\det\Lambda$ has absolute value unity. The last step is to decide between the values plus and minus one. For the identity Lorentz transformation evidently Λ is plus or minus the identity so the sign of the determinant must be positive for Lorentz transformations continuous with the identity. For the space and time reflections whose matrices Λ are $i\gamma_4$ and $\gamma_5\gamma_4$, Eqs. (113) and (114), the determinants of the matrices γ_4 and $i\gamma_5\gamma_4$ may be considered since they have the same value. Each of the matrices γ_4 and $i\gamma_5\gamma_4$ is Hermitian and so has real eigenvalues; each squared is one and so its eigenvalues are either plus or minus one; each has zero spur (step 2 of the proof of the fundamental theorem) and so its eigenvalues are $+1$, $+1$, -1, -1; therefore, each has determinant plus one. The determinant then is positive for transformations continuous with the identity and for these two special space and time reflections. It will still be positive for arbitrary products of transformation matrices so Eq. (164) must hold in general. Another property of Λ is that, when a_{j4} and a_{4j} are zero so that the $a_{\mu\nu}$ are all real, Λ is unitary. This is easily demonstrated by taking the Hermitian conjugate of Eq. (161)

$$a_{\mu\nu}\gamma_\nu=\Lambda^H\gamma_\mu(\Lambda^{-1})^H,$$

recombining the result with Eq. (161)

$$\Lambda^{-1}\gamma_\mu\Lambda=\Lambda^H\gamma_\mu(\Lambda^{-1})^H,$$

and rearranging the equation into the form

$$\gamma_\mu\Lambda\Lambda^H=\Lambda\Lambda^H\gamma_\mu,$$

which implies that

$$\Lambda\Lambda^H=k,$$

where k is a positive number. It can be concluded that k is 1 by taking the determinant of this last equation; this completes the proof that

$$\Lambda\Lambda^H=1 \tag{165}$$

when the $a_{\mu\nu}$ are all real.

Returning to the subject of the construction of tensors, one can find the transformation properties of the quantities $\psi^A\gamma_B\psi$ from the transformation properties of the wave function and its adjoint. The results of arguments to be given below are the following transformation rules, corresponding to the Lorentz transformation

$$x_\mu'=a_{\mu\nu}x_\nu$$

of the coordinates:

$$\psi'^A\psi'=\psi^A\psi, \tag{166}$$

$$i\psi'^A\gamma_\mu\psi'=(a_{44}/|a_{44}|)a_{\mu\rho}i\psi^A\gamma_\rho\psi, \tag{167}$$

$$i\psi'^A(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\psi'=(a_{44}/|a_{44}|)a_{\mu\rho}a_{\nu\sigma}\times i\psi^A(\gamma_\rho\gamma_\sigma-\gamma_\sigma\gamma_\rho)\psi, \tag{168}$$

$$i\psi'^A\gamma_5\gamma_\mu\psi'=(\det a)a_{\mu\rho}i\psi^A\gamma_5\gamma_\rho\psi, \tag{169}$$

$$i\psi'^A\gamma_5\psi'=(\det a)i\psi^A\gamma_5\psi. \tag{170}$$

The five quantities on the left are often called the Dirac covariants; individually they are known as the scalar, vector, tensor, axial or pseudovector, and pseudoscalar because of the way they transform with respect to non-time-reflections. Evidently the tensor is antisymmetric. The factors of i have been chosen so that a component is real if its indices contain an even number of fours and so that otherwise it is pure imaginary. For example, for the space parts of the vector one finds

$$\begin{aligned}(i\psi^A\gamma_j\psi)^C&=-i(\psi^H\gamma_4\gamma_j\psi)^H\\&=-i\psi^H\gamma_j\gamma_4\psi\\&=i\psi^H\gamma_4\gamma_j\psi\\&=i\psi^A\gamma_j\psi\end{aligned}$$

($\psi^A\gamma_j\psi$ is a one-by-one matrix and so is trivially equal to its transpose), and similarly finds for its time part

$$\begin{aligned}(i\psi^A\gamma_4\psi)^C&=-i\psi^H\gamma_4\gamma_4\psi\\&=-i\psi^A\gamma_4\psi.\end{aligned}$$

The proof of the first three transformation rules, Eqs. (166) to (168), for non-time-reflections follows directly from the transformation rules for the wave function, Eqs. (150) and (159), and the first property of the Λ matrix, Eq. (161). For example the proof of the tensor rule is as follows:

$$\begin{aligned}i\psi'^A(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\psi'&=i\psi^A\Lambda^{-1}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\Lambda\psi\\&=i\psi^A\Lambda^{-1}\gamma_\mu\gamma_\nu\Lambda\psi-i\psi^A\Lambda^{-1}\gamma_\nu\gamma_\mu\Lambda\psi\\&=a_{\mu\rho}a_{\nu\sigma}i\psi^A\gamma_\rho\gamma_\sigma\psi-a_{\nu\sigma}a_{\mu\rho}i\psi^A\gamma_\sigma\gamma_\rho\psi\\&=a_{\mu\rho}a_{\nu\sigma}i\psi^A(\gamma_\rho\gamma_\sigma-\gamma_\sigma\gamma_\rho)\psi\end{aligned}$$

for non-time-reflections. The proofs for the axial vector and pseudoscalar follow directly in the same way once it is established that

$$\Lambda^{-1}\gamma_5\Lambda=(\det a)\gamma_5. \tag{171}$$

This can easily be proven by writing γ_5 in the form $(4!)^{-1}\epsilon_{\mu\nu\xi o}\gamma_\mu\gamma_\nu\gamma_\xi\gamma_o$. This form is allowable because there are 4! nonzero terms in the sum, in each of which the indices are all different so that both $\epsilon_{\mu\nu\xi o}$ and $\gamma_\mu\gamma_\nu\gamma_\xi\gamma_o$ are completely antisymmetric—this means that each such term can be reduced to γ_5. The proof is then as follows:

$$\begin{aligned}\Lambda^{-1}\gamma_5\Lambda&=(4!)^{-1}\epsilon_{\mu\nu\xi o}\Lambda^{-1}\gamma_\mu\gamma_\nu\gamma_\xi\gamma_o\Lambda\\&=(4!)^{-1}\epsilon_{\mu\nu\xi o}a_{\mu\rho}a_{\nu\sigma}a_{\xi\tau}a_{o\upsilon}\gamma_\rho\gamma_\sigma\gamma_\tau\gamma_\upsilon\\&=(4!)^{-1}(\det a)\epsilon_{\rho\sigma\tau\upsilon}\gamma_\rho\gamma_\sigma\gamma_\tau\gamma_\upsilon\\&=(\det a)\gamma_5.\end{aligned}$$

The next thing to discuss is the derivation of the transformation rules when there is a time-reflection. It is convenient to consider all the covariants at the same time. From Eqs. (152) and (160) it is found that

$$\begin{aligned}\psi'^A\gamma_B\psi'&=[(\psi'^A)^C\gamma_B{}^C\psi'^C]^C\\&=[\psi^A\Lambda^{-1}C^{-1}\gamma_B{}^CC\Lambda\psi]^C\\&=[\psi^H\gamma_4\Lambda^{-1}C^{-1}\gamma_B{}^CC\Lambda\psi]^H.\end{aligned}$$

It is a property of Λ, Eq. (163), that $\gamma_4\Lambda^{-1}$ is $-\Lambda^H\gamma_4$ so

$$\psi'^A\gamma_B\psi'=-[\psi^H\Lambda^H\gamma_4C^{-1}\gamma_B{}^CC\Lambda\psi]^H,$$

and it is a property of C, Eq. (138), that γ_4C^{-1} is $-C^{-1}\gamma_4{}^C$ so

$$\psi'^A\gamma_B\psi'=[\psi^H\Lambda^HC^{-1}\gamma_4{}^C\gamma_B{}^CC\Lambda\psi]^H. \tag{172}$$

The expression has been cast into this form in order to make use of the fact that

$$C\gamma_4\gamma_BC^{-1}=-\kappa(\gamma_4\gamma_B)^T, \tag{173}$$

where $\kappa=1$ when $\gamma_B=1$, $\gamma_5\gamma_\mu$, or γ_5,
$=-1$ when $\gamma_B=\gamma_\mu$ or $\gamma_\mu\gamma_\nu(\mu\neq\nu)$.

Equation (173) is just a convenient way of summarizing the symmetries considered by Haantjes in his proof of the antisymmetry of B, Eqs. (74) to (78). Equation (173) can be rewritten in the form

$$C\gamma_4\gamma_B = -\kappa(\gamma_4\gamma_B)^T C$$

or, since C is symmetric,

$$C\gamma_4\gamma_B = -\kappa(C\gamma_4\gamma_B)^T.$$

If C is replaced by $B\gamma_4\gamma_5$ according to Eq. (136) the statement becomes

$$B\gamma_5\gamma_B = -\kappa(B\gamma_5\gamma_B)^T,$$

and the truth of this is easily verified from Eqs. (74) to (78), in which k is -1. This establishes the validity of Eq. (173). Since C is symmetric and unitary its inverse is its conjugate, and the complex conjugate of Eq. (173) is

$$C^{-1}\gamma_4{}^C\gamma_B{}^C C = -\kappa(\gamma_4\gamma_B)^H.$$

This last equation can now be introduced into Eq. (172) so that

$$\begin{aligned}\psi'^A\gamma_B\psi' &= -\kappa[\psi^H\Lambda^H(\gamma_4\gamma_B)^H\Lambda\psi]^H \\ &= -\kappa\psi^H\Lambda^H\gamma_4\gamma_B\Lambda\psi \\ &= -\kappa\psi^A\gamma_4\Lambda^H\gamma_4\gamma_B\Lambda\psi \\ &= \kappa\psi^A\Lambda^{-1}\gamma_B\Lambda\psi,\end{aligned} \tag{174}$$

where Eq. (163) has been used again in the last step. This equation shows that the transformation rule for non-time-reflections differs from the rule for time-reflections only by the factor κ. This difference has been taken into account in the transformation rules for the covariants, Eqs. (166) to (170), by adding the factor $(a_{44}/|a_{44}|)$ in the vector and tensor equations. This completes the discussion of these transformation rules. If the time-reflection viewpoint discussed at the end of Sec. IV is adopted then evidently the adjoint wave function always transforms according to Eq. (151),

$$\psi'^A = \psi^A\gamma_4\Lambda^H\gamma_4$$

or, from Eq. (163)

$$\psi'^A = (a_{44}/|a_{44}|)\psi^A\Lambda^{-1}.$$

It is seen that when this point of view is used all of the above covariants transform with the factor $(a_{44}/|a_{44}|)$ rather than just the vector and tensor.

Since the charge conjugation operation is covariant, it is clear that five other covariants of the form $\phi^A\gamma_B\phi$ can be constructed, where ϕ is $(C\psi)^C$, the charge conjugate wave function to ψ. These, however, are proportional to the covariants above:

$$\begin{aligned}\phi^A\gamma_B\phi &= \phi^H\gamma_4\gamma_B\phi \\ &= [(C\psi)^T\gamma_4\gamma_B C^C\psi^C]^T \\ &= \psi^H C^{-1}(\gamma_4\gamma_B)^T C\psi \\ &= -\kappa\psi^H\gamma_4\gamma_B\psi \\ &= -\kappa\psi^A\gamma_B\psi,\end{aligned} \tag{175}$$

where Eq. (173) is used to obtain the fourth line. One may consider next quantities of the form

$$\begin{aligned}\phi^A\gamma_B\psi &= \phi^H\gamma_4\gamma_B\psi \\ &= \psi^T C\gamma_4\gamma_B\psi.\end{aligned} \tag{176}$$

First of all it is clear that these are identically zero unless γ_B is γ_μ or $\gamma_{[\mu\nu]}$, for, as another application of Eq. (173),

$$\begin{aligned}\psi^T C\gamma_4\gamma_B\psi &= -\kappa\psi^T(\gamma_4\gamma_B)^T C\psi \\ &= -\kappa[\psi^T(\gamma_4\gamma_B)^T C\psi]^T \\ &= -\kappa\psi^T C\gamma_4\gamma_B\psi,\end{aligned} \tag{177}$$

and κ is $+1$ except for those assignments of γ_B. Only the quantities $\phi^A\gamma_{VT}\psi$ need be considered, where γ_{VT} is any one of the γ_μ or $\gamma_{[\mu\nu]}$. With respect to non-time-reflections their transformation rule is

$$\phi'^A\gamma_{VT}\psi' = \phi^A\Lambda^{-1}\gamma_{VT}\Lambda\psi \tag{178}$$

and, in parallel with the derivation of Eq. (174), with respect to time-reflections it is found that

$$\phi'^A\gamma_{VT}\psi' = -\psi^A\Lambda^{-1}\gamma_{VT}\Lambda\phi. \tag{179}$$

In view of Eq. (179) these quantities are not covariants. However, one may consider also the quantities $\psi^A\gamma_B\phi$ which are plus or minus the complex conjugates of the $\phi^A\gamma_B\psi$:

$$\begin{aligned}(\phi^A\gamma_B\psi)^C &= (\phi^H\gamma_4\gamma_B\psi)^C \\ &= \phi^T(\gamma_4\gamma_B)^C\psi^C \\ &= \psi^H C^{-1}(\gamma_4\gamma_B)^C C\phi \\ &= -\kappa\psi^H(\gamma_4\gamma_B)^H\phi \\ &= \pm\psi^A\gamma_B\phi,\end{aligned} \tag{180}$$

where Eq. (173) has been used and the last step is justified since $\gamma_4\gamma_B$ is either Hermitian or anti-Hermitian. In view of Eq. (180) only the $\psi^A\gamma_{VT}\phi$ are not zero; evidently they transform according to

$$\psi'^A\gamma_{VT}\phi' = \psi^A\Lambda^{-1}\gamma_{VT}\Lambda\phi \tag{181}$$

for non-time-reflections and according to

$$\psi'^A\gamma_{VT}\phi' = -\phi^A\Lambda^{-1}\gamma_{VT}\Lambda\psi \tag{182}$$

for time-reflections. From Eqs. (178), (179), (181), (182) it is seen that the sum and difference of $\phi^A\gamma_{VT}\psi$ and $\psi^A\gamma_{VT}\phi$ are covariant quantities. In detail in terms of the wave function ψ one finds, corresponding to the Lorentz transformation

$$x_\mu' = a_{\mu\nu}x_\nu,$$

the following transformation rules:

$$\begin{aligned}&i[\psi'^T C\gamma_4\gamma_\mu\psi' + \psi'^H\gamma_4\gamma_\mu C^{-1}\psi'^C] \\ &\quad = (a_{44}/|a_{44}|)a_{\mu\rho}i[\psi^T C\gamma_4\gamma_\rho\psi + \psi^H\gamma_4\gamma_\rho C^{-1}\psi^C],\end{aligned} \tag{183}$$

$$\begin{aligned}&i[\psi'^T C\gamma_4(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\psi' + \psi'^H\gamma_4(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)C^{-1}\psi'^C] \\ &\quad = (a_{44}/|a_{44}|)a_{\mu\rho}a_{\nu\sigma}i[\psi^T C\gamma_4(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho)\psi \\ &\qquad + \psi^H\gamma_4(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho)C^{-1}\psi^C],\end{aligned} \tag{184}$$

$$'^{T}C\gamma_4\gamma_\mu\psi' - \psi'^{H}\gamma_4\gamma_\mu C^{-1}\psi'^{C}]$$
$$\imath_{\mu\rho}[\psi^T C\gamma_4\gamma_\rho\psi - \psi^H\gamma_4\gamma_\rho C^{-1}\psi^C], \tag{185}$$

$$\gamma_4(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\psi' - \psi'^{H}\gamma_4(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)C^{-1}\psi'^{C}]$$
$$\imath_{\mu\rho}a_{\nu\sigma}[\psi^T C\gamma_4(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho)\psi - \psi^H\gamma_4(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho)C^{-1}\psi^C]. \tag{186}$$

Here also the factors of i have been added so that the j components of the vectors and the jk components of the tensors are real and the other components are pure imaginary. Covariants can be constructed quadratically from two different wave functions in a similar way. The ambiguity in the sign of Λ has no effect on any of the transformation rules above because Λ and Λ^{-1} always enter together into any specific term.

VII. SPECIAL EXAMPLES OF DIRAC MATRICES

The effect on the Dirac equation and on the covariants of introducing a different set of Dirac matrices is the first subject treated in this section. Following that, three special sets of Dirac matrices are discussed: the set already considered above in discussions of the non-relativistic limit, a set in which the matrix C is the identity so that charge conjugation is identical with complex conjugation, and a set which leads to the spinor terminology.

In order to discuss the effect of using different sets of matrices, suppose γ_μ and $\bar{\gamma}_\mu$ are two sets of Hermitian Dirac matrices:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu},$$
$$\bar{\gamma}_\mu\bar{\gamma}_\nu + \bar{\gamma}_\nu\bar{\gamma}_\mu = 2\delta_{\mu\nu}.$$

According to the fundamental theorem of Sec. III these are connected by a similarity transformation:

$$\bar{\gamma}_\mu = S\gamma_\mu S^{-1}. \tag{187}$$

Since both γ_μ and $\bar{\gamma}_\mu$ are Hermitian, S can be chosen unitary,

$$S^H S = 1. \tag{188}$$

This is easily proven by substituting Eq. (187) into

$$\bar{\gamma}_\mu{}^H = \bar{\gamma}_\mu$$

to obtain

$$(S^{-1})^H\gamma_\mu S^H = S\gamma_\mu S^{-1},$$

which can be rewritten in the form

$$\gamma_\mu S^H S = S^H S\gamma_\mu.$$

However, if $S^H S$ commutes with each of the γ_μ it must, according to the fifth step in the proof of the fundamental theorem, be a multiple, say k, of the identity

$$S^H S = k.$$

The diagonal element of the product of a matrix and its Hermitian conjugate is real and positive so k is real and positive. It is then possible to choose the arbitrary factor in S so that Eq. (188) is fulfilled. In what follows it is assumed that this choice has always been made at the beginning. One can see next that the substitution

$$\bar{\psi}(x) = S\psi(x) \tag{189}$$

carries the Dirac equation with matrices γ_μ

$$\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}A_\mu\right)\gamma_\mu\psi + \frac{mc}{\hbar}\psi = 0$$

into the Dirac equation with matrices $\bar{\gamma}_\mu$

$$\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}A_\mu\right)\bar{\gamma}_\mu\bar{\psi} + \frac{mc}{\hbar}\bar{\psi} = 0.$$

Therefore the introduction of a different set of Hermitian-Dirac matrices only amounts to a unitary transformation among the four components of the wave function. It will next be shown that Eq. (189) also applies to the charge conjugate wave function. To make this proof one may begin with the connection between B and $\bar{B}$, Eq. (81),

$$B = kS^T\bar{B}S.$$

As argued following Eq. (81), B and $\bar{B}$ are both unitary and are undetermined as far as a numerical factor which has unit absolute value. This means that the absolute value of k is also one, for

$$\begin{aligned} B^H B &= kk^C S^H\bar{B}^H(S^T)^H S^T\bar{B}S \\ &= |k|^2 S^H\bar{B}^H(SS^H)^T\bar{B}S \end{aligned}$$

and, since S also is unitary, this reduces to

$$1 = |k|^2.$$

Suppose the relative phase of the arbitrary factors in B and $\bar{B}$ or the argument of the arbitrary factor in S is always chosen so that

$$B = S^T\bar{B}S$$

or

$$\bar{B} = S^C B S^{-1}. \tag{190}$$

The corresponding connection between C and $\bar{C}$ is then found directly to be

$$\begin{aligned} \bar{C} &= \bar{B}\bar{\gamma}_4\bar{\gamma}_5 \\ &= S^C B S^{-1} S\gamma_4\gamma_5 S^{-1} \\ &= S^C C S^{-1}. \end{aligned} \tag{191}$$

Finally then, for the charge conjugate wave function,

$$\begin{aligned} \bar{\phi} &= \bar{C}^C\bar{\psi}^C \\ &= SC^C(S^{-1})^C S^C\psi^C \\ &= SC^C\psi^C \\ &= S\phi, \end{aligned}$$

as required. However this result holds only if some of the arbitrary factors are adjusted until Eq. (190) is valid. Since

$$\begin{aligned} \bar{\psi}^A\bar{\gamma}_B\bar{\psi} &= \bar{\psi}^H\bar{\gamma}_4\bar{\gamma}_B\bar{\psi} \\ &= \psi^H S^H S\gamma_4\gamma_B S^{-1}S\psi \\ &= \psi^A\gamma_B\psi, \end{aligned}$$

it is immaterial whether the barred or unbarred quantities are used to evaluate the covariants. Next it will be shown that the connection between the two wave functions, Eq. (189), is covariant with respect to Lorentz transformations. Assume it holds in the unprimed system. By operating from the left and right appropriately with factors of S, S^C, and S^{-1} one may convert Eqs. (161) to (163) into the form

$$a_{\mu\nu}\bar{\gamma}_\nu = (\Lambda^-)^{-1}\bar{\gamma}_\mu\Lambda^-,$$
$$(\Lambda^-)^C\bar{C} = \bar{C}\Lambda^-,$$
$$\Lambda^-\bar{\gamma}_4(\Lambda^-)^H\bar{\gamma}_4 = (a_{44}/|a_{44}|),$$

where

$$\Lambda^- = S\Lambda S^{-1}.$$

Therefore, Λ^- is the transformation matrix for the barred wave functions and, for non-time-reflections

$$\begin{aligned}\bar{\psi}'(x') &= \Lambda^-\bar{\psi}(x)\\ &= S\Lambda S^{-1}S\psi(x)\\ &= S\Lambda\psi(x)\\ &= S\psi'(x'),\end{aligned}$$

as required. Using Eq. (191), one can easily construct a similar proof for time-reflections.

One special set of Hermitian-Dirac matrices is as follows:

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (192)$$

where the σ_j are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (193)$$

For this set $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, a B chosen unitary, and $C = B\gamma_4\gamma_5$ are:

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}. \qquad (194)$$

In Sec. II it is shown how, when these matrices are used, the Schrödinger-Pauli limit of the Dirac theory is obtained. At the end of Sec. III this choice of the B matrix is discussed. In Sec. IV these matrices are used to discuss the matrix Λ for a space rotation and to show the connection between Λ and the Cayley-Klein parameters.

Another special set of Hermitian-Dirac matrices is

$$\gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}. \qquad (195)$$

This is a natural type to consider because, as is seen from the definitions of the Pauli matrices above, the γ_j here are real, and γ_4 is pure imaginary. The matrices γ_5, B, C as previously defined are for this set

$$\gamma_5 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since the charge conjugation matrix C is the id[illegible] when this set of matrices is used charge conju[illegible]n, Eq. (133), reduces to complex conjugation and, as is seen from Eq. (162), the wave-function transformation matrix Λ is always real. It is clear from Eqs. (137) and (138) that C can be the identity whenever γ_j, $i\gamma_4$ are all real.

Since γ_5 is Hermitian, anticommutes with each of the γ_μ, and gives unity when squared, another set of Dirac matrices can be formed by interchanging the definitions of γ_4 and γ_5 in Eqs. (192) and (194). Such a set is of interest because the diagonal elements of all of the γ_μ, when written in terms of two-by-two matrices, are then zero. This property permits the Dirac equation for the four-component wave function ψ to be written neatly as two coupled equations involving two-component functions called spinors which, it develops, transform separately with respect to Lorentz transformations continuous with the identity. Spinors were originally discovered by Cartan[22] but had no application in physics before Dirac's theory. The spinor point of view has recently been reviewed by Bade and Jehle.[23] The basic properties of spinors are developed below in order to show how they arise from the general treatment of the Dirac equation; for further information the work of Bade and Jehle can be consulted. Besides the permutation of the matrices in Eqs. (192) and (194) mentioned above, there are other sets of matrices which also are used to divide the four-component wave function into two spinors. To conform with the general practice the discussion here will be based on the two-by-two matrices

$$\rho_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$
$$\rho_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad (197)$$

which are an arrangement of the Pauli matrices and the identity with various factors added. It is easily verified that these matrices satisfy the equations

$$\left.\begin{aligned}\rho_j{}^C\rho_k + \rho_k{}^C\rho_j &= 2\delta_{jk},\\ \rho_4{}^C\rho_j - \rho_j{}^C\rho_4 &= 0,\\ \rho_4{}^C\rho_4 &= -1,\end{aligned}\right\} \qquad (198)$$

and that, as a consequence, the matrices

$$\gamma_j = \begin{pmatrix} 0 & \rho_j \\ \rho_j{}^C & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & \rho_4 \\ -\rho_4{}^C & 0 \end{pmatrix}, \qquad (199)$$

[22] E. Cartan, Bull. Soc. Math. France 41, 53 (1913).
[23] W. L. Bade and Herbert Jehle, Revs. Modern Phys. 25, 714 (1953).

satisfy the equations

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu},$$

and so are a set of Dirac matrices. The γ_μ are evidently also Hermitian. This starting point is similar to Bade and Jehle's—they use a different metric. The matrices γ_5, B, C defined as above are easily found to be

$$\gamma_5=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad B=\begin{pmatrix}\rho_4{}^C & 0\\ 0 & \rho_4\end{pmatrix},\quad C=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{200}$$

Next the four-component Dirac equation

$$\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}A_\mu\right)\gamma_\mu\psi+\frac{mc}{\hbar}\psi=0 \tag{201}$$

will be broken up into two coupled two-component equations. The four-component wave function ψ will be written as

$$\psi(x)=\begin{pmatrix}\zeta^C(x)\\ \eta(x)\end{pmatrix}, \tag{202}$$

where ζ and η are two-rowed column matrices. Two-component functions like ζ and η are called spinors. Substituting Eqs. (199) and (202) into Eq. (201), one finds

$$\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}A_\mu\right)\rho_\mu\eta+\frac{mc}{\hbar}\zeta^C=0, \tag{203}$$

$$\left(\frac{\partial}{\partial x_\mu}+\frac{ie}{\hbar c}A_\mu\right)\rho_\mu\zeta+\frac{mc}{\hbar}\eta^C=0, \tag{204}$$

where the complex conjugate of the second equation was taken to bring the j- and 4-terms together. The charge conjugate wave function is

$$\phi(x)=C^C\psi^C(x)=\begin{pmatrix}\eta^C(x)\\ \zeta(x)\end{pmatrix},$$

so charge conjugation is equivalent to interchanging ζ and η. This can be verified directly in Eqs. (203) and (204) for if ζ and η are interchanged there the equations are reproduced except with e replaced by $-e$. The transformation properties of the spinors ζ and η can be found from the transformation properties of the wave function ψ; it is assumed that the same decomposition, Eq. (202), is made in each coordinate system. With respect to the space reflection

$$x_j'=-x_j,\quad x_4'=x_4, \tag{88}$$

it is seen from Eq. (113) that

$$\left.\begin{aligned}\zeta'^C(x')&=i\rho_4\eta(x),\\ \eta'^C(x')&=i\rho_4\zeta(x).\end{aligned}\right\} \tag{205}$$

With respect to the time-reflection

$$x_j'=x_j,\quad x_4'=-x_4, \tag{89}$$

it is seen from Eq. (115) that

$$\left.\begin{aligned}\zeta'^C(x')&=\rho_4\zeta(x),\\ \eta'^C(x')&=\rho_4\eta(x).\end{aligned}\right\} \tag{206}$$

The remainder of the discussion will be restricted to the transformations continuous with the identity. The matrix Λ is of course determined by Eqs. (161), (162), (163). However, before applying them, one may notice that Eq. (171)

$$\Lambda^{-1}\gamma_5\Lambda=(\det a)\gamma_5$$

means, when γ_5 is as defined in Eq. (200), that Λ is of the form

$$\Lambda=\begin{pmatrix}\Lambda_s{}^C & 0\\ 0 & \Lambda_t\end{pmatrix}, \tag{207}$$

where Λ_s and Λ_t are two-by-two matrices. Then, from Eq. (162),

$$\Lambda^C C=C\Lambda,$$

and C as given in Eq. (200), it is seen that

$$\Lambda_s=\Lambda_t, \tag{208}$$

so that the two spinors transform identically:

$$\left.\begin{aligned}\zeta'(x')&=\Lambda_s\zeta(x),\\ \eta'(x')&=\Lambda_s\eta(x).\end{aligned}\right\} \tag{209}$$

Next it is found that Eq. (163) for non-time-reflections

$$\Lambda\gamma_4\Lambda^H\gamma_4=1,$$

rewritten in terms of Λ_s, becomes

$$\Lambda_s\rho_4{}^C\Lambda_s{}^T\rho_4=-1. \tag{210}$$

From this equation it can be concluded that the spinor transformation is unimodular. By taking the determinant of the equation one can see that $(\det\Lambda_s)^2$ is unity since, according to the definition of ρ_4 in Eq. (197), $(\det\rho_4)$ is minus one. However, Λ_s is continuous with the identity and therefore

$$\det\Lambda_s=1. \tag{211}$$

Finally it is easily seen that Eq. (161)

$$a_{\mu\nu}\gamma_\nu=\Lambda^{-1}\gamma_\mu\Lambda \tag{161}$$

is equivalent to

$$a_{\mu\nu}\rho_\nu=(\Lambda_s{}^{-1})^C\rho_\mu\Lambda_s, \tag{212}$$

which is somewhat in parallel with Eqs. (119) and (161). The covariance of Eqs. (203) and (204) with respect to Lorentz transformations continuous with the identity can be seen directly as a consequence of Eqs. (209) and (212). Independent of the choice of the γ_μ, it was shown that Λ is unitary when a_{4j}, a_{j4} are zero, Eq. (165). This property applies also to Λ_s in view of Eq. (207); for a space rotation Λ_s is unitary. For any two-dimensional rotation Λ_s can be found explicitly by specializing Eq. (110) to the matrices above, Eq. (199). If the adjoint spinor is introduced according to

$$\zeta^A=\zeta^H\rho_4=-\zeta^H\rho_4{}^C, \quad (213)$$

then it is found to transform according to

$$\begin{aligned}\zeta'^A&=-\zeta'^H\rho_4{}^C\\&=-\zeta^H\Lambda_s{}^H\rho_4{}^C\\&=-\zeta^A\rho_4\Lambda_s{}^H\rho_4{}^C\\&=\zeta^A(\Lambda_s{}^{-1})^C,\end{aligned} \quad (214)$$

where the conjugate of Eq. (210) was used in the last step. As a consequence of this and Eq. (212) one sees that $i\zeta^A\rho_\mu\zeta$ is a vector with respect to transformations continuous with the identity:

$$\begin{aligned}i\zeta'^A\rho_\mu\zeta'&=i\zeta^A(\Lambda_s{}^{-1})^C\rho_\mu\Lambda_s\zeta\\&=a_{\mu\nu}i\zeta^A\rho_\nu\zeta.\end{aligned} \quad (215)$$

The factor of i was added here so that the j-components would be real and the fourth component pure imaginary. To demonstrate that this result has been obtained, one takes the complex conjugate of each component and manipulates it in the following way:

$$\begin{aligned}(i\zeta^A\rho_\mu\zeta)^C&=i(\zeta^H\rho_4{}^C\rho_\mu\zeta)^H\\&=i\zeta^H\rho_\mu{}^H\rho_4{}^T\zeta\\&=-i\zeta^A\rho_4{}^C\rho_\mu{}^H\rho_4{}^T\zeta.\end{aligned}$$

From Eqs. (197) and (198) it is easily found that $\rho_4{}^C\rho_j{}^H\rho_4{}^T$ is $-\rho_j$ and that $\rho_4{}^C\rho_4{}^H\rho_4{}^T$ is ρ_4. Therefore the equation

$$(i\zeta^A\rho_\mu\zeta)^C=\pm i\zeta^A\rho_\mu\zeta$$

is valid, where the plus sign applies for the j components and the minus for the fourth component as required. Also it can be shown that $-i\zeta_1{}^T\rho_4\zeta_2$ is a scalar with respect to Lorentz transformations continuous with the identity, where ζ_1 and ζ_2 are two spinors:

$$\begin{aligned}-i\zeta_1'^T\rho_4\zeta_2'&=-i\zeta_1{}^T\Lambda_s{}^T\rho_4\Lambda_s\zeta_2\\&=-i\zeta_1{}^T\rho_4\Lambda_s{}^{-1}\Lambda_s\zeta_2\\&=-i\zeta_1{}^T\rho_4\zeta_2;\end{aligned} \quad (216)$$

here Eq. (210) was used in the first step. By writing it explicitly in terms of the components of the spinors, one finds that this scalar can be expressed as the determinant of the two-by-two matrix whose columns are ζ_1 and ζ_2:

$$-i\zeta_1{}^T\rho_4\zeta_2=\det(\zeta_1\ \zeta_2). \quad (217)$$

Evidently this scalar is zero when ζ_1 and ζ_2 are identical. An alternative proof that this quantity is a scalar rests on the fact that the transformation is unimodular. It is easily seen that

$$\begin{aligned}\det(\zeta_1'\ \zeta_2')&=\det(\Lambda_s\zeta_1\ \Lambda_s\zeta_2)\\&=\det[(\Lambda_s)(\zeta_1\ \zeta_2)]\\&=\det(\Lambda_s)\det(\zeta_1\ \zeta_2)\\&=\det(\zeta_1\ \zeta_2),\end{aligned} \quad (218)$$

which proves the assertion.

VIII. IDENTITIES BETWEEN SCALARS FORMED FROM FOUR WAVE FUNCTIONS

In theories of reactions involving four particles which have Dirac four-component wave functions, the problem of forming scalars proportional to each of the four wave functions arises.[24] In this section connections between some of the possible scalars will be given. The prototype of these theories is Fermi's theory of beta-decay[25]; the arguments below are put in the terms of this theory although they have somewhat greater applicability.

For emission of electrons the reaction considered is

$$N\rightleftarrows P+e+\bar{\nu}, \quad (219)$$

where N, P, e, $\bar{\nu}$ indicate the neutron, proton, electron, and antineutrino. Dirac's theory of holes is assumed to apply to both the electron and neutrino so the creation of an electron and antineutrino is identical with the destruction of a positron and a neutrino. Therefore the reaction of Eq. (219) can also be written in the form

$$P\rightleftarrows N+\bar{e}+\nu, \quad (220)$$

where $\bar{e}$, ν indicate a positron and a neutrino, and thus the theory of positron emission is included in the theory of electron emission. Various arguments in the theory[24] require the interaction term to be real, a scalar with respect to Lorentz transformations, proportional to the wave function of a particle when that particle is destroyed, and proportional to the complex conjugate of the wave function when the corresponding particle is created. Actually in the theory the wave functions are second quantization operators. However they commute with each other and so, for the present purposes, the operator properties can be disregarded. The reactions of Eqs. (219) and (220) can be written in the form

$$N+\nu\rightleftarrows P+e, \quad (221)$$

and then it is easily seen that the interaction terms

$$J_1=[\psi_P{}^H\gamma_4\psi_N][\psi_e{}^H\gamma_4\psi_\nu]+[\psi_N{}^H\gamma_4\psi_P][\psi_\nu{}^H\gamma_4\psi_e], \quad (222)$$

$$\begin{aligned}J_2=&[\psi_P{}^H\gamma_4\gamma_\rho\psi_N][\psi_e{}^H\gamma_4\gamma_\rho\psi_\nu]\\&+[\psi_N{}^H\gamma_4\gamma_\rho\psi_P][\psi_\nu{}^H\gamma_4\gamma_\rho\psi_e],\end{aligned} \quad (223)$$

$$\begin{aligned}J_3=&-\tfrac{1}{8}[\psi_P{}^H\gamma_4(\gamma_\rho\gamma_\sigma-\gamma_\sigma\gamma_\rho)\psi_N]\\&\times[\psi_e{}^H\gamma_4(\gamma_\rho\gamma_\sigma-\gamma_\sigma\gamma_\rho)\psi_\nu]\\&-\tfrac{1}{8}[\psi_N{}^H\gamma_4(\gamma_\rho\gamma_\sigma-\gamma_\sigma\gamma_\rho)\psi_P]\\&\times[\psi_\nu{}^H\gamma_4(\gamma_\rho\gamma_\sigma-\gamma_\sigma\gamma_\rho)\psi_e],\end{aligned} \quad (224)$$

$$\begin{aligned}J_4=&-[\psi_P{}^H\gamma_4\gamma_5\gamma_\mu\psi_N][\psi_e{}^H\gamma_4\gamma_5\gamma_\mu\psi_\nu]\\&-[\psi_N{}^H\gamma_4\gamma_5\gamma_\mu\psi_P][\psi_\nu{}^H\gamma_4\gamma_5\gamma_\mu\psi_e],\end{aligned} \quad (225)$$

$$\begin{aligned}J_5=&[\psi_P{}^H\gamma_4\gamma_5\psi_N][\psi_e{}^H\gamma_4\gamma_5\psi_\nu]\\&+[\psi_N{}^H\gamma_4\gamma_5\psi_P][\psi_\nu{}^H\gamma_4\gamma_5\psi_e],\end{aligned} \quad (226)$$

contain the wave functions and their complex conjugates in the proper way. In each case the first term applies

[24] See, for example, Enrico Fermi, *Elementary Particles* (Yale University Press, New Haven, Connecticut, 1951).

[25] E. Fermi, Z. Physik 88, 161 (1934).

when the reaction of Eq. (221) is to proceed to the right—electron emission or positron capture. The second term applies when the reaction is to proceed to the left—positron emission or electron capture. The constant factors in the J's have been chosen so that each of them can be written in the form

$$\Sigma\{[\psi_P{}^H\gamma_4\gamma_B\psi_N][\psi_e{}^H\gamma_4\gamma_B\psi_\nu] + [\psi_N{}^H\gamma_4\gamma_B\psi_P][\psi_\nu{}^H\gamma_4\gamma_B\psi_e]\},$$

where the sum extends only over the appropriate type of γ_B. Referring to the definitions of the γ_B at the beginning of Sec. III, one sees that the negative signs in the definitions of J_3 and J_4 correspond to the factors of i in the definitions of $\gamma_{[\mu\nu]}$ and $\gamma_{[\lambda\mu\nu]}$ and that the factor of (1/8) in the definition of J_3 will reduce the sum of 48 terms to a sum of 6 terms, involving the six $\gamma_{[\mu\nu]}$. For brevity Eqs. (222) to (226) will be written collectively as

$$J_\Gamma = (\psi_P{}^H\gamma_4\Omega_\Gamma\psi_N)(\psi_e{}^H\gamma_4\Omega_\Gamma\psi_\nu) + (\psi_N{}^H\gamma_4\Omega_\Gamma\psi_P)(\psi_\nu{}^H\gamma_4\Omega_\Gamma\psi_e), \quad (227)$$

where the constant real factor and the sums on the right are not explicitly indicated and the capital Greek subscripts run from 1 to 5. These J_Γ are all real, for the second terms are the complex conjugates of the first:

$$\begin{aligned}[(\psi_P{}^H\gamma_4\Omega_\Gamma\psi_N)(\psi_e{}^H\gamma_4\Omega_\Gamma\psi_\nu)]^C &= (\psi_P{}^H\gamma_4\Omega_\Gamma\psi_N)^H(\psi_e{}^H\gamma_4\Omega_\Gamma\psi_\nu)^H \\ &= [\psi_N{}^H(\gamma_4\Omega_\Gamma)^H\psi_P][\psi_\nu{}^H(\gamma_4\Omega_\Gamma)^H\psi_e] \\ &= (\psi_N{}^H\gamma_4\Omega_\Gamma\psi_P)(\psi_\nu{}^H\gamma_4\Omega_\Gamma\psi_e).\end{aligned}$$

The last step is permitted since $\gamma_4\Omega_\Gamma$ is just some product of the γ_μ and so is either Hermitian or anti-Hermitian. With respect to Lorentz transformations which do not reflect the time the wave functions transform according to Eqs. (150) and (159), and it is evident that the J_Γ are scalars. When there is a time reflection, Eqs. (152) and (160) are to be used. One finds, in parallel with the derivation of Eq. (174), that

$$\psi_P{}'^H\gamma_4\Omega_\Gamma\psi_N{}' = \kappa\psi_N{}^H\gamma_4\Lambda^{-1}\Omega_\Gamma\Lambda\psi_P,$$

where κ, defined just below Eq. (173), is $+1$ when Γ is 1, 4, or 5 and is -1 when Γ is 2 or 3. In any case κ^2 is $+1$ and so the J_Γ are scalars with respect to time-reflections also, the first terms transforming into the second. These five interaction terms thus fulfil all the conditions imposed above, but of course there are several other possibilities. For example, a linear combination of these interactions,

$$J = \sum_{\Gamma=1}^{5} c_\Gamma J_\Gamma, \quad (228)$$

where the c_Γ are any real numbers, also fulfils the requirements. In beta-decay theory it is customary to take the five J_Γ as the primary interactions and to express other interactions linearly in terms of them when possible; this is what is done below. In terms of the matrices α_j, β defined by Eq. (31) and connected with the γ_μ according to Eqs. (18) and (19), the interactions have the form

$$\begin{aligned}J_1 &= (\psi_P{}^H\beta\psi_N)(\psi_e{}^H\beta\psi_\nu) + \text{c.c.},\\ J_2 &= (\psi_P{}^H\psi_N)(\psi_e{}^H\psi_\nu) - (\psi_P{}^H\alpha_j\psi_N)(\psi_e{}^H\alpha_j\psi_\nu) + \text{c.c.},\\ J_3 &= (\psi_P{}^H\beta\bar{\sigma}_j\psi_N)(\psi_e{}^H\beta\bar{\sigma}_j\psi_\nu) + (\psi_P{}^H\beta\alpha_j\psi_N)(\psi_e{}^H\beta\alpha_j\psi_\nu) + \text{c.c.},\\ J_4 &= (\psi_P{}^H\bar{\sigma}_j\psi_N)(\psi_e{}^H\bar{\sigma}_j\psi_\nu) - (\psi_P{}^H\gamma_5\psi_N)(\psi_e{}^H\gamma_5\psi_\nu) + \text{c.c.},\\ J_5 &= (\psi_P{}^H\beta\gamma_5\psi_N)(\psi_e{}^H\beta\gamma_5\psi_\nu) + \text{c.c.},\end{aligned}$$

where c.c. indicates the complex conjugate and

$$\bar{\sigma}_j = \begin{pmatrix}\sigma_j & 0\\ 0 & \sigma_j\end{pmatrix}, \quad \gamma_5 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

gives the matrices $\bar{\sigma}_j$, γ_5. These forms of the interactions are frequently used as the starting point of treatments of beta-decay theory.[26] Papers concerned with linear combinations of interactions have recently been published by Michel and Wightman[27] and by King and Peaslee.[28] Michel and Wightman give a résumé of the earlier work done on the subject.

One can easily construct other permissible interactions by forming scalars similar to the J_Γ but with the wave functions arranged differently within the products. The main purpose of this section is to show how these other interactions can be expressed as linear combinations of the J_Γ. One approach to the study of these other interactions is to rewrite the reaction equation in the form

$$\bar{P} + N + \bar{e} + \nu \rightleftarrows 0,$$

so that all the particles are uniformly created or annihilated. This corresponds to introducing the charge-conjugate wave functions for the proton and electron so that the interactions are written as

$$J_\Gamma = (\phi_P{}^TC\gamma_4\Omega_\Gamma\psi_N)(\phi_e{}^TC\gamma_4\Omega_\Gamma\psi_\nu) + \text{c.c.} \quad (229)$$

The other interactions are then easily formed by interchanging positions of the wave functions. The first type which will be discussed is formed by interchanging the wave functions of the heavy particles:

$$K_\Gamma = (\psi_N{}^TC\gamma_4\Omega_\Gamma\phi_P)(\phi_e{}^TC\gamma_4\Omega_\Gamma\psi_\nu) + \text{c.c.} \quad (230)$$

The K_Γ are simply related to the J_Γ, as a consequence of Eq. (173):

$$\begin{aligned}\psi_N{}^TC\gamma_4\Omega_\Gamma\phi_P &= -\kappa\psi_N{}^T(\gamma_4\Omega_\Gamma)^TC\phi_P\\ &= -\kappa[\psi_N{}^T(\gamma_4\Omega_\Gamma)^TC\phi_P]^T\\ &= -\kappa\phi_P{}^TC\gamma_4\Omega_\Gamma\psi_N,\end{aligned}$$

and this implies that

$$K_\Gamma = -\kappa J_\Gamma. \quad (231)$$

[26] See, for example, E. J. Konopinski, Revs. Modern Phys. **15**, 209 (1943).
[27] L. Michel and A. Wightman, Phys. Rev. **93**, 354 (1954).
[28] R. W. King and D. C. Peaslee, Phys. Rev. **94**, 1284 (1954).

Evidently a similar result applies when the light particle wave functions are interchanged so that

$$K_\Gamma = (\phi_P{}^T C\gamma_4\Omega_\Gamma\psi_N)(\psi_\nu{}^T C\gamma_4\Omega_\Gamma\phi_e) + \text{c.c.}$$

is an alternative expression for K_Γ.

The next set of interaction-terms which will be discussed is formed by interchanging the wave functions of the charged particles in Eq. (229):

$$\begin{aligned} L_\Gamma &= (\phi_e{}^T C\gamma_4\Omega_\Gamma\psi_N)(\phi_P{}^T C\gamma_4\Omega_\Gamma\psi_\nu) + \text{c.c.} \\ &= (\psi_e{}^H\gamma_4\Omega_\Gamma\psi_N)(\psi_P{}^H\gamma_4\Omega_\Gamma\psi_\nu) + \text{c.c.} \end{aligned} \tag{232}$$

These interactions also can be expressed linearly in terms of the J_Γ; the formulas were first published by Fierz.[29] The proof given below follows Pauli's proof of de Broglie's identities,[30] and is based on the fact that

$$\sum_{B=1}^{16} (\gamma_B)_{\mu\nu}(\gamma_B)_{\rho\sigma} = 4\delta_{\mu\sigma}\delta_{\rho\nu}, \tag{233}$$

where $(\gamma_B)_{\mu\nu}$ are the elements of the matrix γ_B. One can prove this identity by specializing step 6 of the proof of the fundamental theorem to the case when the two sets of matrices involved, γ_μ and γ_μ', are identical. Then, from Eqs. (59) and (60), for any four-by-four matrix F the matrix S defined by

$$S = \sum_{B=1}^{16} \gamma_B F \gamma_B$$

has the property

$$\gamma_A S = S\gamma_A.$$

However, from the fifth step in the proof of the fundamental theorem, a matrix which commutes with the γ_μ is a constant times the identity. It can be concluded that, for any matrix F,

$$\sum_{B=1}^{16} \gamma_B F \gamma_B = k,$$

where k is some number times the identity. One may now choose in succession for F the sixteen different matrices which have one element unity and the rest zero; this gives

$$\sum_{B=1}^{16} (\gamma_B)_{\mu\nu}(\gamma_B)_{\rho\sigma} = k_{\nu\rho}\delta_{\mu\sigma}.$$

The constants $k_{\nu\rho}$ can be found by contracting on μ and σ:

$$\begin{aligned} 4k_{\nu\rho} &= \sum_{B=1}^{16} (\gamma_B)_{\mu\nu}(\gamma_B)_{\rho\mu} \\ &= \sum_{B=1}^{16} (\gamma_B{}^2)_{\rho\nu} \\ &= 16\delta_{\rho\nu}. \end{aligned}$$

The last step here results from the fact that the square of each of the γ_B is the identity, Eq. (46). Substituting back for $k_{\nu\rho}$, one obtains

$$\sum_{B=1}^{16} (\gamma_B)_{\mu\nu}(\gamma_B)_{\rho\sigma} = 4\delta_{\mu\sigma}\delta_{\rho\nu}, \tag{233}$$

as required. If this equation is multiplied by $(\psi_1)_\mu(\psi_2)_\sigma(\psi_3)_\rho(\psi_4)_\nu$, where ψ_1, ψ_2, ψ_3, ψ_4 are any column matrices, and if the sum on B is expanded to show the five types of γ_B, the result is

$$\begin{aligned} 4(\psi_1{}^T\psi_2)(\psi_3{}^T\psi_4) &= (\psi_3{}^T\psi_2)(\psi_1{}^T\psi_4) + (\psi_3{}^T\gamma_\mu\psi_2)(\psi_1{}^T\gamma_\mu\psi_4) \\ &\quad + (\psi_3{}^T\gamma_{[\mu\nu]}\psi_2)(\psi_1{}^T\gamma_{[\mu\nu]}\psi_4) \\ &\quad + (\psi_3{}^T i\gamma_5\gamma_\mu\psi_2)(\psi_1{}^T i\gamma_5\gamma_\mu\psi_4) \\ &\quad + (\psi_3{}^T\gamma_5\psi_2)(\psi_1{}^T\gamma_5\psi_4). \end{aligned} \tag{234}$$

In the third term the sum is to be carried over the six $\gamma_{[\mu\nu]}$. Then, keeping in mind that each J_Γ involves a sum over one type of γ_B, it is easily seen that the five specializations

1. $\psi_1{}^T = \psi_e{}^H\gamma_4$, $\psi_2 = \psi_N$, $\psi_3{}^T = \psi_P{}^H\gamma_4$, $\psi_4 = \psi_\nu$
2. $\psi_1{}^T = \psi_e{}^H\gamma_4\gamma_\mu$, $\psi_2 = \psi_N$, $\psi_3{}^T = \psi_P{}^H\gamma_4\gamma_\mu$, $\psi_4 = \psi_\nu$
3. $\psi_1{}^T = \psi_e{}^H\gamma_4\gamma_{[\mu\nu]}$, $\psi_2 = \psi_N$, $\psi_3{}^T = \psi_P{}^H\gamma_4\gamma_{[\mu\nu]}$, $\psi_4 = \psi_\nu$
4. $\psi_1{}^T = \psi_e{}^H\gamma_4 i\gamma_5\gamma_\mu$, $\psi_2 = \psi_N$, $\psi_3{}^T = \psi_P{}^H\gamma_4 i\gamma_5\gamma_\mu$, $\psi_4 = \psi_\nu$
5. $\psi_1{}^T = \psi_e{}^H\gamma_4\gamma_5$, $\psi_2 = \psi_N$, $\psi_3{}^T = \psi_P{}^H\gamma_4\gamma_5$, $\psi_4 = \psi_\nu$

give directly these relations between the interaction terms:

$$\begin{aligned} 4L_1 &= J_1 + J_2 + J_3 + J_4 + J_5, \\ 4L_2 &= 4J_1 - 2J_2 + 2J_4 - 4J_5, \\ 4L_3 &= 6J_1 - 2J_3 + 6J_5, \\ 4L_4 &= 4J_1 + 2J_2 - 2J_4 - 4J_5, \\ 4L_5 &= J_1 - J_2 + J_3 - J_4 + J_5. \end{aligned}$$

These equations will be summarized in the form

$$L_\Gamma = \sum_{\Delta=1}^{5} A_{\Gamma\Delta} J_\Delta, \tag{235}$$

where

$$A_{\Gamma\Delta} = \tfrac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}. \tag{236}$$

Evidently

$$A^2 = 1 \tag{237}$$

because if the wave functions of the charged particles are interchanged twice, the original interaction is recovered. The interaction terms formed by interchanging in Eq. (229) the wave functions of the heavy particles, the light particles, and the charged particles have now been discussed, and all other possible interchanges can be expressed in terms of them.

[29] M. Fierz, Z. Physik **104**, 553 (1937). A general treatment of such identities has been given by K. M. Case, Phys. Rev. **97**, 810 (1955).

[30] Reference 2, p. 131.

Sometimes interactions which are symmetric or antisymmetric with respect to an interchange of the wave functions of two of the particles are considered. From Eq. (231) it is seen that the interaction

$$J=aJ_2+bJ_3 \tag{238}$$

is symmetric with respect to an interchange of the wave functions of either the heavy particles or the light particles and that the interaction

$$J=cJ_1+dJ_4+eJ_5 \tag{239}$$

is correspondingly antisymmetric, where a, b, c, d, e are any real numbers. It is evident that interchanging the charged particle wave functions is equivalent to interchanging the uncharged particle wave functions. If the interaction

$$J=\sum_{\Gamma=1}^{5} c_\Gamma J_\Gamma$$

is symmetric or antisymmetric with respect to such an interchange, then the c_Γ will satisfy the equation

$$\sum_{\Gamma=1}^{5} c_\Gamma L_\Gamma = \pm \sum_{\Gamma=1}^{5} c_\Gamma J_\Gamma$$

or, in view of Eq. (235),

$$\sum_{\Gamma=1}^{5}\sum_{\Delta=1}^{5} c_\Gamma A_{\Gamma\Delta} J_\Delta = \pm \sum_{\Delta=1}^{5} c_\Delta J_\Delta.$$

However, this implies the equation

$$\sum_{\Gamma=1}^{5} (A^T)_{\Delta\Gamma} c_\Gamma = \pm c_\Delta,$$

which may be used to determine the c_Δ. This is in fact the eigenvalue problem for the transpose of the matrix A, because its square is unity and so its eigenvalues are ± 1. The result is that the interaction

$$J=(12f+3g)J_1+3fJ_2+(2f+g)J_3+3fJ_4+3gJ_5 \tag{240}$$

is symmetric with respect to an interchange of the wave functions of either the charged or the uncharged particles, and that the interaction

$$J=(-2h-l)J_1+(h-k+l)J_2+hJ_3+kJ_4+lJ_5 \tag{241}$$

is correspondingly antisymmetric, where f, g, h, k, l are any real numbers.

It is easily verified from Eqs. (238) to (241) that the only interaction which is either symmetric or antisymmetric with respect to an interchange of any two wave functions is

$$J=m(-J_1+J_4+J_5), \tag{242}$$

where m is some real number. Since this is a special case of both Eqs. (239) and (241), this interaction is antisymmetric with respect to an interchange of any two wave functions. It was originally proposed by Wigner and Critchfield.[31,32] Since $(-J_1+J_4+J_5)$ is antisymmetric with respect to interchanges of any pair of the wave functions ϕ_P, ψ_N, ϕ_e, ψ_ν it must be proportional to the determinant formed from these four wave functions. The proportionality constant depends on what set of Dirac matrices are involved. The matrices of Eqs. (192) and (194) are used almost exclusively in beta-decay theory and the rest of the discussion is restricted to them. By calculating out some of the terms it can be verified that, with those matrices,

$$-J_1+J_4+J_5=-2\det(\phi_P\ \psi_N\ \phi_e\ \psi_\nu)+\text{c.c.} \tag{243}$$

In parallel with the treatment of the spinor-scalar, Eq. (218), the Wigner-Critchfield interaction can be shown directly to be a scalar as a consequence of the fact that the transformations of the wave function are unimodular, Eq. (164). For a non-time-reflection it is seen that

$$\begin{aligned}\det(\phi_P'\ \psi_N'\ \phi_e'\ \psi_\nu')&+\text{c.c.}\\ &=\det(\Lambda\phi_P\ \Lambda\psi_N\ \Lambda\phi_e\ \Lambda\psi_\nu)+\text{c.c.}\\ &=\det[\Lambda(\phi_P\ \psi_N\ \phi_e\ \psi_\nu)]+\text{c.c.}\\ &=\det(\phi_P\ \psi_N\ \phi_e\ \psi_\nu)+\text{c.c.},\end{aligned}$$

as required. For a time-reflection one must also note from Eq. (194) that

$$\det C=1,$$

and then the result is

$$\begin{aligned}\det(\phi_P'\ \psi_N'\ \phi_e'\ \psi_\nu')&+\text{c.c.}\\ &=\det(\phi_P'^C\ \psi_N'^C\ \phi_e'^C\ \psi_\nu'^C)+\text{c.c.}\\ &=\det(C\Lambda\phi_P\ C\Lambda\psi_N\ C\Lambda\phi_e\ C\Lambda\psi_\nu)+\text{c.c.}\\ &=\det[C\Lambda(\phi_P\ \psi_N\ \phi_e\ \psi_\nu)]+\text{c.c.}\\ &=\det(\phi_P\ \psi_N\ \phi_e\ \psi_\nu)+\text{c.c.},\end{aligned}$$

which proves the assertion.

[31] C. L. Critchfield and E. P. Wigner, Phys. Rev. **60**, 412 (1941).
[32] C. L. Critchfield, Phys. Rev. **63**, 417 (1943).