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Statistical Theory of Multiple Meson Production*

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The Fermi statistical theory of multiple meson production is examined, and its application to medium-energy pion-nucleon and nucleon-nucleon collisions is discussed. Various approximate calculational procedures are described and their results compared critically. It is found, by the use of a precise integration procedure, that a rigorous relativistic inclusion of the meson rest mass is necessary if one is to draw accurate conclusions from the postulates of the theory. The consequences of isotopic spin conservation are described and tables are given for the expected distribution of charged products from both pion-nucleon and nucleon-nucleon collisions. The theory is applied to the case of pion-nucleon collisions at 1.4 Bev, and is found to predict only poorly the tentative results of recent experiments. The failure of the theory to predict nucleon-nucleon collisions is also described, and possible modifications and improvements are discussed.

I. PREFACE

SINCE the original introduction by Fermi of a statistical theory of multiple meson production,[1] numerous workers have sought to apply such a theory to various sorts of high-energy interactions. Recent experiments with the Cosmotron at the Brookhaven National Laboratory, and impending experiments at the Bevatron in Berkeley have stimulated interest in the application of the theory to nucleon-nucleon and meson-nucleon events occurring at energies of the order of one to ten billion electron volts. Except for two basic papers by Fermi,[1,2] and a paper by Lepore and Stuart,[3] much of this work has appeared only in the form of internal or project reports of limited circulation.[3–5] It seems desirable, therefore, to present a summary describing the present state of the Fermi theory for the use, primarily, of experimental workers. One must observe in addition, as we shall illustrate, that a strict application of Fermi's simple basic postulates requires meticulous attention to detail. Most existing calculations have, of practical necessity, been based upon approximations of a somewhat unsatisfactory sort, and in general have been directed toward the prediction of events occurring at one or another specific primary energy. To facilitate and stimulate more exact calculations, and applications to a wider variety of problems, there is included in the following a discussion of the various methods of computation which may be utilized, and a comparison of the several sorts of approximation which may be made. We shall not discuss, however, the thermodynamic approximation, which applies to events of extremely high energy, and which involves theoretical methods of a different nature from those useful at the lower energies of interest here. This theory has been described in detail elsewhere.[1,6–8]

II. INTRODUCTION TO THE THEORY

We wish to predict the outcome of nucleon-nucleon and meson-nucleon collisions occurring at energies sufficiently high for meson production and other possible inelastic processes to become significant. We assume, with Fermi,[1,9] that the two incident particles in each of such collisions coalesce and that the energy brought in by them is released within a common region of interaction. After a time this region of interaction is then considered to disintegrate into one or another of the

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[1] E. Fermi, Progr. Theoret. Phys. (Japan) **5**, 570 (1950).

[2] E. Fermi, Phys. Rev. **92**, 452 (1953); Phys. Rev. **93**, 1434 (1954).

[3] J. V. Lepore and R. N. Stuart, UCRL-2386, Nov. 17, 1953; Phys. Rev. **94**, 1724 (1954). *Note:* In Eq. (33) of UCRL-2386 and Eq. (20) of the Phys. Rev. article, the factor $(4n-3)!$ should read $(4n-4)!$

[4] J. V. Lepore, UCRL-2396, UCRL-2398.

[5] C. Yang and R. Christian, Brookhaven Cosmotron Internal Report.

[6] E. Fermi, Phys. Rev. **81**, 683 (1951).

[7] R. Marshak, *Meson Physics* (McGraw-Hill Book Company, Inc., New York, 1952), p. 290 ff.

[8] W. Kraushaar and L. Marks, Phys. Rev. **93**, 326 (1954).

[9] E. Fermi, *Elementary Particles* (Yale University Press, New Haven, 1951), p. 84 ff.

possible final states of the system. We further assume that the interactions involved are sufficiently strong, and that the interaction region survives sufficiently long, for all of the possible final states to have become equally excited. By thus assuming the attainment of a statistical equilibrium among all possible final states we may prorate the probability for a given outcome according to the number of states embraced by that outcome. The ensemble of possible final states is to be selected on the basis of those conservation laws which are expected to apply during the interaction. Besides the usual laws of energy, momentum, and charge conservation, one may also require that the total isotopic spin of the system, the number of nucleonic particles, and similar variables be subject to appropriate constraints.

The foregoing assumptions should be phrased in formal terms before proceeding with calculations based upon the Fermi statistical theory. The transition rate, w, from the initial state, $|i)$, of a quantum mechanical system to a final state, $|f)$, is given by the customary formula

$$w=\frac{2\pi}{\hbar}|(f|\mathcal{H}|i)|^2\cdot\rho(W), \qquad (2.1)$$

where $(f|\mathcal{H}|i)$ is the matrix element of the Hamiltonian connecting the two states and where the factor $\rho(W)$ represents the density of final states of the system evaluated for a total energy W. In general we shall specify the final states in terms of the numbers and charges of the nucleons, mesons, and perhaps other particles which emerge from the interaction region.

The matrix element $(f|\mathcal{H}|i)$ vanishes, of course, for interactions which do not connect states satisfying the laws of energy, momentum, angular momentum, and charge conservation. It may also be assumed to vanish for other interactions, such as those which, *a priori*, do not conserve isotopic spin. To apply the Fermi assumption that equilibrium is attained, we consider that the interaction $\mathcal{H}$ projects the initial state vector $|i)$ uniformly over the space of all state vectors which are compatible with the conservation laws and which represent the existence of virtual particles of all possible momenta confined to the interaction volume Ω. The final state vector $|f)$ represents a state containing free particles, and will consist of the direct product of independent free-particle wave functions. Let us assume that these individual wave functions are normalized with respect to a large spatial volume V. For a total of N particles in the final state the function $|f)$ will thus contain the normalizing factor $(V)^{-(1/2)N}$. On the other hand, each of the N-particle virtual states contained initially within the volume Ω, will have a normalizing factor of the order of $(\Omega)^{-(1/2)N}$. Upon computing the overlap integrals for each of the N particles the matrix element becomes

$$(f|\mathcal{H}|i)=\text{const}(\Omega/V)^{(1/2)N}, \qquad (2.2)$$

where the constant, by the assumption of statistical equilibrium, is independent of the final state. Put more physically, the transition rate w to a given final state is proportional to $(\Omega/V)^N$, which is the probability that N particles confined to a box of volume V will simultaneously be observed within a region of volume Ω. The constant appearing above is useful only in determining the over-all transition rate for all types of final states. Since the purpose here is to establish the relative abundance of different final states for a given sort of interaction, this constant will be neglected in calculations. The only remaining parameter in the theory is the interaction volume Ω. Fermi assumes that Ω will be of nucleonic dimensions, say $\Omega=(4/3)\pi r_0^3$, where r_0 is the approximate radius of the meson cloud surrounding a nucleon. Also Ω may be made energy dependent, if desired. However, for the moment we shall consider Ω to be a free parameter of the system.

Refinements may be made in the above estimate of the matrix element. If n of the total of N particles are considered to be physically indistinguishable, then only those wave functions for the final state, $|f)$, are admissible which are completely symmetric (or, appropriately, antisymmetric) in the coordinates of the identical particles. The degeneracy so introduced is reflected by the replacement of $|f)$ with $(n!)^{-1/2}|f)$, wherever the former function occurs. Hence the matrix element becomes, in this case,

$$(f|\mathcal{H}|i)=\text{const}\left(\frac{\Omega}{V}\right)^{N/2}\cdot\frac{1}{(n!)^{1/2}}. \qquad (2.3)$$

Generalizations of this procedure are evident. It should be remarked that the concept of indistinguishability must be carefully applied. Consider the case of pions. Were one to treat positive pions as being distinguishable from negative pions, that is, as being essentially different particles, then one would allow for possible degeneracies through dividing the matrix element by

$$[(n^+)!(n^-)!]^{1/2},$$

where n^+ and n^- represent the number of positive and negative pions present in the final state. In this scheme, say, a state of one positive pion and one negative pion would be considered to have twice the weight—from the (2!)— of a state containing two positive pions. If, on the other hand, we wish to impose the requirement that the total isotopic spin of the system is conserved during the interaction, we must then consider that positive, neutral, and negative pions are three states of a *single* particle. The matrix element for the process must then be divided by $(n!)^{1/2}$, where n represents the total number of pions present. In this case the weights for the various combinations of charges must be established by a more complicated process which will be described in Sec. IV of this paper.

One may, of course, modify the matrix element in still other manners in order to refine the theory further.

A few such refinements will be discussed at the end of Sec. III.

The density of final states, $\rho(W)$, depends both upon the phase space available to the particles in the final state of energy W, and upon the number of possible charge and angular momentum states consistent with the constraints on the system. Henceforth we shall consider only situations in which pions and nucleons are produced. We shall specify the final state according to the number and charge of these particles, irrespective of their individual energies, momenta, and angular momenta. Since one state for each particle produced corresponds to a volume $(2\pi\hbar)^3$ in the phase space of that particle, N independent particles, of total energy W, will possess the density of states per unit energy:

$$(d/dW)(2\pi\hbar)^{-3N}\int_V\int_W\prod_{i=1}^{N}d^3\mathbf{x}_id^3\mathbf{p}_i. \qquad (2.4)$$

The spatial integral for each particle is taken over the large normalization volume V, while each momentum integral is required to be consistent with a total energy W. The relative probability that n pions and s nucleons will emerge from a high-energy interaction will then be given by

$$S'(n,s)=(\Omega/V)^N\sum(d/dW)(2\pi\hbar)^{-3N}\times\int_V\int_W\prod_{i=1}^{N}d^3\mathbf{x}_id^3\mathbf{p}_i, \qquad (2.5)$$

with $N=(n+s)$, where it has been assumed for the moment that the pions are all distinguishable. The integrations above must be restricted to conform to the various constraints upon the system. The conservation of linear momentum will, for example, reduce the number of dynamically independent particles from N to $(N-1)$. Not only will the integrations above be taken over the phase space of only $(N-1)$ particles, but also the matrix element factor $(\Omega/V)^{N/2}$ will become $(\Omega/V)^{(N-1)/2}$, for the same reason. The relative probabilities then become

$$S(n,s)=(\Omega/V)^{N-1}\sum(d/dW)(2\pi\hbar)^{-3(N-1)}\times\int_V\int_W\prod_{i=1}^{N-1}d^3\mathbf{x}_id^3\mathbf{p}_i. \qquad (2.6)$$

A summation symbol has been included in the above equations to indicate that a suitable sum must be taken over the various possible charge and angular momentum states.

From this point on, the determination of the relative probabilities, $S(n,s)$, is largely one of mathematical calculation. We may simplify the problem considerably by neglecting entirely the conservation of angular momentum. The effects of such a neglect have been estimated by Fermi.[1] The spatial integration in (2.6) becomes $(V)^{N-1}$, cancelling a similar factor in the matrix element. Thus we find

$$S(n,s)=\Omega^{N-1}(2\pi\hbar)^{-3(N-1)}\sum(d/dW)\int_W\prod_{i=1}^{N}d^3\mathbf{p}_i$$
$$=\Omega^{N-1}(2\pi\hbar)^{-3(N-1)}\sum(d/dW)\mathcal{V}_W(n,s). \qquad (2.7)$$

Here $\mathcal{V}_W$ is the volume of the $3(N-1)$ dimensional momentum space corresponding to the $(N-1)$ independent particles, calculated to be consistent with the law of momentum conservation, and such that the total energy of all N particles is less than or equal to W.

In Sec. III we shall discuss various procedures and results for the calculation of $(d/dW)\mathcal{V}_W$. In Sec. IV, on the other hand, we shall consider the calculation of the sum over charge states which is required if we are to assume that the total isotopic spin is conserved during the interaction. We shall then illustrate the application of the theory to experimental situations and shall also describe certain modifications which have been proposed.

III. THE PHASE SPACE CALCULATIONS

The problem is to calculate the quantity $\mathcal{V}_W$ as seen in Eq. (2.7), and from it $d\mathcal{V}_W/dW$, where $\mathcal{V}_W$ is the total volume of momentum space available to all the independent particles of the system, subject to the law of conservation of momentum, and to the requirement that the total energy of all particles, independent or not, is less than or equal to W. The calculation of this quantity becomes very difficult if one tries to treat all particles concerned in a proper relativistic manner. To date, numerical results have been yielded only by treatments in which one or another drastic approximation has been made. In the following sections we shall summarize and illustrate some of these results.

A. The General Formula of Lepore and Stuart

First we shall describe a general formula developed by Lepore and Stuart.[3] This formula, although resistant to numerical calculation in the general case, still represents a powerful approach to the problem. The quantity of interest in Eq. (2.7) may be written

$$\frac{d\mathcal{V}_W}{dW}=\frac{d}{dW}\left[\prod_{i=1}^{N}\int d^3\mathbf{p}_i\right]\delta\left(\sum_{i=1}^{N}\mathbf{p}_i\right)\times U\left[W-\sum_{i=1}^{N}(p_i^2+M_i^2)^{1/2}\right], \qquad (3.1)$$

where the integrals are taken over the momentum space of all N particles, with masses M_i. The units are such that $c=1$. The delta function in the momentum insures that momentum will be conserved (effectively reducing the number of independent integrations by one). Also, $U(x)$ is a step function equalling unity for positive values of its argument and zero for other values. The momentum integrals may now be taken to plus-or-minus infinity. The discontinuous functions have the Fourier representations

$$\delta(\mathbf{x})=(2\pi)^{-3}\int\int\int_{-\infty}^{\infty} d^3\boldsymbol{\lambda}\,\exp(i\boldsymbol{\lambda}\cdot\mathbf{x}), \qquad (3.2)$$

and

$$U(x)=(2\pi i)^{-1}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} d\alpha\cdot\alpha^{-1}\cdot e^{i\alpha x}, \quad (\epsilon>0). \qquad (3.3)$$

Thus Eq. (3.1) breaks down into a product of integrals of similar form,

$$\frac{d\mathcal{V}_W}{dW}=\frac{1}{(2\pi)^4}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} d\alpha\cdot e^{i\alpha W}\int_{-\infty}^{\infty} d^3\boldsymbol{\lambda}\prod_{i=1}^{N} I_i, \qquad (3.4)$$

where

$$I_i=\int d^3\mathbf{p}\,\exp[i\boldsymbol{\lambda}\cdot\mathbf{p}-i\alpha(p^2+M_i^2)^{\frac{1}{2}}]. \qquad (3.5)$$

The angular integration may be conducted by taking $\boldsymbol{\lambda}$ to be the polar axis. If in addition one substitutes $p=M\sinh\theta$, $\lambda_1=M\lambda$, and $\alpha_1=M\alpha$, one finds that

$$\begin{aligned} I_i&=\frac{2\pi}{i\lambda}\int_{-\infty}^{\infty} dp\cdot p\,\exp\{i[\lambda p-\alpha(p^2+M_i^2)^{1/2}]\} \\ &=-\frac{2\pi M_i^3}{\lambda_1}\cdot\frac{d}{d\lambda_1}\int_{-\infty}^{\infty} d\theta\cdot\cosh\theta \\ &\qquad\cdot\exp[i(\lambda_1\sinh\theta-\alpha_1\cosh\theta)]. \end{aligned} \qquad (3.6)$$

We may write

$$\lambda_1\sinh\theta-\alpha_1\cosh\theta=-(\alpha_1^2-\lambda_1^2)^{1/2}\cosh(\theta-\varphi), \qquad (3.7)$$

with $\cosh\varphi=\alpha_1(\alpha_1^2-\lambda_1^2)^{-1/2}$. Considering λ_1 to be fixed and positive, we observe $(\alpha_1^2-\lambda_1^2)^{1/2}$ to be an analytic function of the complex variable α_1, having branch points at $\pm\lambda_1$ on the real axis. The function is specified completely by requiring that $(\alpha_1^2-\lambda_1^2)^{1/2}$ be positive for large positive values of α_1. In particular $(\alpha_1^2-\lambda_1^2)^{1/2}$ is analytic in a neighborhood of the line $\mathrm{Im}\alpha=-\epsilon$, along which α_1 will later be integrated. Letting $z=(\alpha_1^2-\lambda_1^2)^{1/2}$, and requiring that ϵ be infinitesimal, we see that the phase of z will be 0, for $\alpha_1>\lambda_1>0$; $(-i\pi/2)$, for $\lambda_1>\alpha_1>-\lambda_1$; and $(-i\pi)$, for $\alpha_1<-\lambda_1$. We may then write, putting $(\theta-\varphi)=\theta'$,

$$\begin{aligned} I_i&=-\frac{2\pi M_i^3}{\lambda_1}\cdot\frac{d}{d\lambda_1}\left\{\frac{\alpha_1}{(\alpha_1^2-\lambda_1^2)^{1/2}}\int_{-\infty}^{\infty} d\theta'\cdot\cosh\theta'\right. \\ &\qquad\left.\cdot\exp[-i(\alpha_1^2-\lambda_1^2)^{1/2}\cosh\theta']\right\} \\ &=2\pi M_i^3\alpha_1\cdot\frac{1}{z}\cdot\frac{d}{dz}\left\{\frac{1}{z}\int_{-\infty}^{\infty} d\theta'\cdot\cosh\theta'\right. \\ &\qquad\left.\cdot\exp[-iz\cosh\theta']\right\}. \end{aligned} \qquad (3.8)$$

When the phase of z is either 0 or $-i\pi$ we may write the integral in terms of Hankel functions[10]

$$\int_{-\infty}^{\infty} d\theta'\cdot\cosh\theta'\cdot\exp(-iz\cosh\theta')=-\pi H_1^{(2)}(z), \qquad (3.9)$$

where the relation

$$H_1^{(1)}(Ze^{i\pi})=H_1^{(2)}(Z), \qquad (3.10)$$

has been used.[11] Hence, utilizing the recursion formula,[12]

$$\frac{1}{Z}\frac{d}{dZ}\left(\frac{H_1^{(2)}(Z)}{Z}\right)=-\frac{H_2^{(2)}(Z)}{Z^2}, \qquad (3.11)$$

we may write

$$I_i=2\pi^2 M_i^2\alpha\frac{H_2^{(2)}[M_i(\alpha^2-\lambda^2)^{1/2}]}{(\alpha^2-\lambda^2)}. \qquad (3.12)$$

This has been derived only for α lying on those parts of the real axis such that $|\alpha|>\lambda$. However, both Eq. (3.5) and Eq. (3.12) represent analytic functions of α. By an analytic continuation argument Eq. (3.12) can be stated to hold for all α's lying on $\mathrm{Im}\alpha=-\epsilon$. Indeed, the equation holds for the entire plane provided only that one does not cross the branch line between $\pm\lambda$. Finally we may write that

$$\begin{aligned} \frac{d\mathcal{V}_W}{dW}&=\frac{2}{(2\pi)^3}\cdot(2\pi^2)^N\prod_{i=1}^{N} M_i^2\int_{-\infty-i\epsilon}^{\infty-i\epsilon} d\alpha\cdot\alpha^N\cdot e^{i\alpha W} \\ &\times\int_0^{\infty}\frac{d\lambda\cdot\lambda^2}{(\alpha^2-\lambda^2)^N}\prod_{j=1}^{N}\{H_2^{(2)}[M_j(\alpha^2-\lambda^2)^{\frac{1}{2}}]\}, \end{aligned} \qquad (3.13)$$

which is the formula derived by Lepore and Stuart.[3]

B. Calculation for Zero-Mass Particles

The general formula just derived is such that the remaining integrations are difficult to perform, a problem arising from the fact that the points $\pm\lambda$ are essential singularities with respect to the α plane. Results may be obtained, however, for the case of zero-mass particles, that is, for the case in which all particles are treated as being extreme-relativistic. Following the procedure of Lepore and Stuart,[3] we expand the Hankel function about the origin and keep only the first term.[13] Thus,

$$H_2^{(2)}[M_i(\alpha^2-\lambda^2)^{1/2}]\doteq\frac{4i}{M_i^2\pi(\alpha^2-\lambda^2)}. \qquad (3.14)$$

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[10] G. Watson, *Bessel Functions* (Cambridge University Press, Cambridge, England, 1952), p. 180.
[11] See reference 10, p. 75.
[12] See reference 10, p. 74.
[13] See reference 10, p. 84.

Substituting this in Eq. (3.13) we have

$$\frac{d\mathcal{V}_W^{ER}}{dW}=\frac{2^{3N-3}i^N}{\pi^{3-N}}\cdot\int_{-\infty-i\epsilon}^{\infty-i\epsilon}d\alpha\cdot\alpha^N\cdot e^{i\alpha W}\int_{-\infty}^{\infty}d\lambda\cdot\frac{\lambda^2}{(\alpha^2-\lambda^2)^{2N}}$$

$$=\frac{i^{N-1}\pi^{N-2}}{2^N}\cdot\frac{(4N-4)!(2N-1)}{[(2N-1)!]^2}\times\int_{-\infty-i\epsilon}^{\infty-i\epsilon}d\alpha\frac{e^{i\alpha W}}{\alpha^{3(N-1)}},\quad(3.15)$$

or finally,

$$\frac{d\mathcal{V}_W^{ER}}{dW}=\frac{\pi^{N-1}}{2^{N-1}}\cdot\frac{(4N-4)!(2N-1)}{[(2N-1)!]^2(3N-4)!}W^{3N-4},\quad(3.16)$$

where both the λ and the α integrals are evaluated by the method of residues upon closing the contours above the real axis. It must be remembered that α has an infinitesimal negative imaginary part.

It is well to compare this result with that obtained originally by Fermi for the same quantity.[1] Fermi initially neglected momentum conservation and calculated the phase space available to N extreme-relativistic particles. This was subject only to the condition that the upper bound of the total energy be fixed. Therefore,

$$\sum_{i=1}^N p_i\leqslant W.$$

The integrals may be calculated directly, with the result

$$(\mathcal{V}_W^{ER})_{\text{Fermi}}=(4\pi)^N\int_0^W p_1^2dp_1\int_0^{W-p_1}p_2^2dp_2\cdots\times\int_0^{W-\sum_1^{N-1}p_i}p_N^2dp_N=\frac{2^{3N}\pi^N}{(3N)!}W^{3N}.\quad(3.17)$$

Fermi then assumed that momentum conservation could be taken into account simply by reducing the number of independent particles by unity, that is, by letting $N\to(N-1)$. Doing this and taking the energy derivative one obtains

$$\left(\frac{d\mathcal{V}_W^{ER}}{dW}\right)_{\text{Fermi}}=\frac{\pi^{N-1}2^{3(N-1)}}{(3N-4)!}\cdot W^{3N-4}.\quad(3.18)$$

This has the same energy dependence as the exact value in Eq. (3.16), but quite a different coefficient. Table I shows the numerical results for small values of N, and arbitrary W. It is evident from this table that the strict imposition of momentum conservation has a marked effect upon the relative statistical weights for different multiplicities.

C. Classical Particles

An exact calculation may also be made for the case of s classical particles of mass M. The integral in Eq. (3.5) may be written as

$$I_{NR}=-(2\pi/\lambda)(d/d\lambda)\int_{-\infty}^{\infty}dp\times\exp\{i[\lambda p-\alpha(M+p^2/2M)]\}$$

$$=-\pi^{3/2}e^{i\pi/4}(2M/\alpha)^{3/2}e^{-i\alpha M}e^{iM\lambda^2/2\alpha},\quad(3.19)$$

where the calculation is carried out by completing the square in the exponent. The phase space calculation then is reduced to the determination of two elementary integrals, and yields

$$\frac{d\mathcal{V}_W^{NR}}{dW}=(2\pi)^{-4}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}d\alpha e^{i\alpha W}\cdot4\pi\int_0^{\infty}d\lambda\cdot\lambda^2(I_{NR})^s$$

$$=\frac{2^{3(s-1)/2}\pi^{3(s-1)/2}M^{3s/2}}{(sM)^{3/2}}\cdot\frac{(W-sM)^{(3s/2-5/2)}}{\Gamma[3(s-1)/2]}.\quad(3.20)$$

When classical particles of different masses are involved the same formula holds, provided one makes the replacement

$$sM\to\sum_{i=1}^s M_i,\qquad M^{3s/2}\to\prod_{i=1}^s(M_i^{3/2}).\quad(3.21)$$

D. Approximate Calculations for Nucleon-Nucleon Collisions

It is of considerable practical interest to examine the case in which s nucleons and n pions are emitted. Lepore and Stuart[3] studied this case under the assumption that the nucleons are classical particles of mass M, and the pions are extreme-relativistic particles. Account was taken of the pion mass μ by assuming that the pion total energy has the expression $(p+\mu)$. This serves to limit pion production to within energetically feasible limits, but is a somewhat dubious, if necessary, approximation. The calculation proceeds as before, yielding, in process, the integral

$$4\pi\int_0^{\infty}d\lambda\cdot\lambda^2\cdot(\lambda^2-\alpha^2)^{-2n}\exp\{i\lambda^2sM/2\alpha\}.\quad(3.22)$$

TABLE I. A comparison of statistical weights for massless particles. The numbers in brackets show the weights relative to the case $N=2$.

Method	$N=2$	$N=3$	$N=4$	$N=5$
Fermi	$12.6W^2$	$5.26W^5$	$0.394W^8$	$0.0100W^{11}$
	(1)	$(0.42W^3)$	$(0.031W^6)$	$(0.00079W^9)$
Exact	$1.57W^2$	$0.287W^5$	$0.0127W^8$	$0.000218W^{11}$
	(1)	$(0.18W^3)$	$(0.0081W^6)$	$(0.00014W^9)$

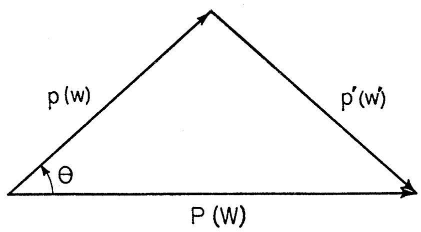

FIG. 1. Momentum configuration for two particles.

Although perfectly convergent, because of the infinitesimal negative imaginary part of α, this integral leads on computation to Fresnel integrals which are resistant to further analysis. Lepore and Stuart make the approximation that the denominator may be replaced by α^{4n}. Thus it follows that

$$\left(\frac{d\mathcal{V}_W}{dW}\right)_{\text{Approx.}}=\left[\frac{M^{3s/2}(2\pi)^{3(s-1)/2}}{(sM)^{3/2}}\right]\cdot\left[2^{3n}\pi^n\right]\cdot\frac{\left[W-sM-n\mu\right]^{3s/2+3n-5/2}}{\Gamma\left[3(s-1)/2+3n\right]}. \quad (3.23)$$

It is seen that this result is essentially the product of Eqs. (3.20) and (3.17), where energy conservation among the increased number of particles has been taken into account through the adoption of a common energy factor, and through a suitable change in the argument of the gamma-function. Essentially, this approximation assumes that momentum conservation holds only for the s nucleons, while the n pions are allowed to take on any energies consistent with over-all energy conservation. If there are a variety of classical particles they may again be accounted for by the substitution in Eq. (3.21).

From the above formula Lepore and Stuart have calculated the relative probabilities for various pion multiplicities, as obtained by substituting Eq. (3.23) in Eq. (2.7) for $S(n,s)$. Their results for several different *total* energies in the center-of-mass system are given in Table II,[3] where $s=2$ for all cases. In these calculations the interaction volume Ω was taken to be, following Fermi,[1]

$$\Omega=(2M/W)(4/3)\pi R^3,\quad R=\hbar/\mu c=1.4\times10^{-13}\text{ cm}. \quad (3.24)$$

This is simply the volume of the nucleonic pion cloud

TABLE II. The approximate weights for pion multiplicities in nucleon-nucleon collisions (reference 3) where $s=2=$number of nucleons emitted, $n=$number of pions produced, and $W=$total center-of-mass energy. No account is taken of charge states or of particle indistinguishability.

n	$W=$ 2.5 Bev	3.9 Bev	4.71 Bev
0	0.5846	0.0074	0.0008
1	0.4127	0.2302	0.0647
2	0.0027	0.3679	0.2957
3	0.0000	0.0919	0.2547
4		0.0050	0.0601
5		0.0001	0.0046

contracted by the appropriate Lorentz factor. The variation of multiplicity with energy is quite apparent in Table II. An accurate allowance for pion momentum conservation would probably somewhat reduce the above multiplicites. Calculations for the production of classical heavy mesons and of nucleon pairs are also to be found in the report of Lepore and Stuart.[3]

E. Step-by-Step Calculations

However satisfactory or unsatisfactory it may be in nucleon-nucleon collisions to require momentum conservation of the heavy particles alone, such an approximation clearly will not suffice for meson-nucleon collisions. This is particularly true for energies at which only one nucleon is expected to be present. Rather than attempting to solve the problem in general terms by an attack on the formula in Eq. (3.13), we may proceed in the manner now to be described.

Consider two particles, having total energies w, w' and momenta $\mathbf{p}$ and $\mathbf{p}'$, respectively. Suppose that these two particles are combined into a system having a fixed

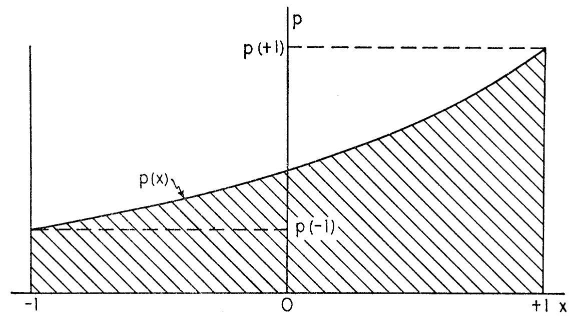

FIG. 2. Region of momentum integration.

total momentum $\mathbf{P}=\mathbf{p}+\mathbf{p}'$. We then ask, what is the total volume of momentum space available to one of these particles, such that the total energy of the pair, $w+w'$, is less than or equal to some fixed energy W. The momentum configuration may be drawn as shown in Fig. 1. Let the vector $\mathbf{p}$ be taken to represent the independent particle of the system. Then the momentum volume available to this particle will be given by

$$\mathcal{V}_{W,P}=2\pi\int_0^{p(W)}dp\cdot p^2\int_{w+w'\leq W}d(\cos\theta). \quad (3.25)$$

Here the upper limit to the p integration has been indicated as $p(W)$, or the momentum that the one particle would have were it given the entire energy of the system. In general, momentum conservation forces the upper limit to be somewhat less than this, and the angular integral will vanish except for certain ranges of p. By definition, the integral in Eq. (3.25) is taken over values of p and θ such that $w'\leqslant(W-w)$, or such that, equivalently, $p'\leqslant$[some known function of $(W-w)$]. Now w is a known function of p, and $p'^2=p^2$

$+P^2-2pP\cos\theta$. Hence, for fixed W and P, the boundary of the region of integration will be specified by a definite equation relating p and $x=\cos\theta$, from which we may calculate the functions $p(x)$ or $x(p)$. The region of integration is shown in Fig. 2. In practice, calculations are easier if one inverts the order of integration, to obtain

$$\mathcal{V}_{W,P}=2\pi\int_{-1}^{1}dx\int_{0}^{p(x)}dp\cdot p^2. \tag{3.26}$$

To add a third particle to the system one repeats the above process by taking the dependent variables p' and w' to represent the energy and momentum of the combined system formed by the first two particles. These two variables may be calculated in terms of p and w which are now the variables of the independent third particle. The total momentum space available to the *two* independent particles of this system is then given by

$$\mathcal{V}_{W,P}=2\pi\int_{-1}^{1}dx\int_{0}^{p(x)}dp\cdot p^2\cdot\mathcal{V}'_{w'(p,x),\,p'(p,x)}, \tag{3.27}$$

where $\mathcal{V}'$ is the momentum-space volume previously calculated for the first two particles. Here again $p(x)$ enforces the conservation of energy and momentum upon the system. In general, $p(x)$ may be calculated by taking the first root of the integrand $\mathcal{V}'_{w',p'}$. Then $\mathcal{V}'$ will vanish when p' is too large in relation to w'. Understood, of course, in these formulas are W and P, the total energy and net momentum of the entire system. The dynamic complexities of the particles involved are entirely contained in the computation of the limit $p(x)$. Aside from calculational difficulties, one could iterate the above process to include in the system as many particles as desired. To deal with a center-of-mass system one takes $P=0$ in the final result, and to establish the density of states at a given energy W one differentiates $\mathcal{V}_{W,0}$ with respect to W.

F. Special Cases—Extreme Relativistic Particles

We shall now calculate a number of special cases, in order of increasing complexity. First, let us reconsider the case of extreme relativistic particles in Sec (III.B). One may denote the total phase-space volume for n such particles by $\mathcal{V}(n)$. Then it would clearly follow that

$$\mathcal{V}(1)=0, \tag{3.28}$$

since only one particle is involved. To calculate $\mathcal{V}(2)$ we observe that the integration limits are determined by

$$p^2+P^2-2pPx=(p')^2=(w')^2=(W-p)^2,$$

or

$$p(x)=\frac{W^2-P^2}{2[W-Px]}, \tag{3.29}$$

which yields

$$\mathcal{V}(2)=2\pi\int_{-1}^{1}dx\cdot\tfrac{1}{3}[p(x)]^3=(\pi/6)W(W^2-P^2). \tag{3.30}$$

The first root of this is at $P=W$. Hence, to find $\mathcal{V}(3)$, we may use the same limit $p(x)$ as found in Eq. (3.29) for $\mathcal{V}(2)$. Thus,

$$\mathcal{V}(3)=\frac{\pi^2}{2\cdot 6!}[W^2-P^2]^2[7W^2-P^2]. \tag{3.31}$$

Similarly,

$$\mathcal{V}(4)=\frac{3\pi^3}{2\cdot 9!}W[W^2-P^2]^3[11W^2-3P^2]. \tag{3.32}$$

The corresponding momentum-space weight factors are then found by taking $P=0$ and differentiating with respect to W. One obtains

$$\begin{aligned}\frac{d\mathcal{V}(1)}{dW}&=0,\\ \frac{d\mathcal{V}(2)}{dW}&=(\pi/2)W^2,\\ \frac{d\mathcal{V}(3)}{dW}&=\frac{7\pi^2}{2\cdot 5!}\cdot W^5,\\ \text{and}\quad \frac{d\mathcal{V}(4)}{dW}&=\frac{3\cdot 11\pi^3}{2\cdot 8!}W^8.\end{aligned} \tag{3.33}$$

These results are the same as those given by the general formula in Eq. (3.16).

G. One Mass-N Particle with n Extreme-Relativistic Particles

With but little extra difficulty it is possible to include one mass-N particle in a system with n massless particles, a calculation of particular relevance to pion-nucleon events. One calculates $\mathcal{V}(n)$ as for the n massless particles, and then adds the mass-N particle and the simultaneous requirement that the total momentum $P=0$. In all cases the momentum limit is determined from the equation

$$p^2=(p')^2=(w')^2=(W-w)^2=[W-(N^2+p^2)^{1/2}]^2, \tag{3.34}$$

which yields

$$p=(W^2-N^2)/2W, \tag{3.35}$$

a quantity independent of the variable x. Let the momentum volume be denoted by $\mathcal{V}(1N,n0)$. Then for $n=1$ we have

$$\mathcal{V}(1N,10)=2\pi\int_{-1}^{1}dx\int_{0}^{p(x)}p^2dp=\frac{\pi}{6}\left[\frac{W^2-N^2}{W}\right]^3. \tag{3.36}$$

Other results follow similarly. Since the formulas are complicated, however, we shall quote only the energy derivatives, which are of paramount interest. These include

$$\frac{d\mathcal{V}(1N,10)}{dW}=\frac{\pi}{2}W^2[1-\nu^2][1-\nu^4], \tag{3.37}$$

$$\frac{d\mathcal{V}(1N,20)}{dW}=\frac{\pi^2W^5}{2\cdot 5!}\left\{(1-\nu^2)(7-43\nu^2-23\nu^4-3\nu^6+2\nu^8)+120\nu^4\ln\frac{1}{\nu}\right\}, \tag{3.38}$$

and

$$\frac{d\mathcal{V}(1N,30)}{dW}=\frac{3\pi^3W^8}{2\cdot 8!}\left\{11-196\nu^2-\frac{4697}{3}\nu^4+\frac{3430}{3}\nu^6+\frac{1855}{3}\nu^8-\frac{28}{3}\nu^{10}-\frac{7}{3}\nu^{12}+\frac{2}{3}\nu^{14}+280\nu^4[7+10\nu^2+\nu^4]\ln\frac{1}{\nu}\right\}. \tag{3.39}$$

Here $\nu=N/W$ represents the relative effect of the mass-N particle over what it would have as a mass-zero particle. It is to be noted that the above weights tend to their extreme-relativistic values as $N\rightarrow 0$, and to zero as $N\rightarrow W$. The complexity of these formulas, and particularly the presence of the logarithm, presumably reflect the presence of the branch point in the Hankel functions in Eq. (3.13). In addition one should observe that the particle of mass N has been treated with all due relativistic rigor. Numerical calculations with the above formulas must be made with considerable care since many of the large individual terms cancel one another, leaving but a small result.

These formulas permit one to calculate the relative probabilities for the production of various numbers of mesons in pion-nucleon collisions. We have carried out such a calculation for a center-of-mass energy of 1.92 Bev (corresponding to 1.37 Bev pions in the laboratory system) and for a nucleon mass of 0.938 Bev. In this case $\nu=0.487$. Neglecting the effects of factors arising from the consideration of charge states and the identity of particles, the relative statistical weights may be calculated from Eq. (2.7). Denoting the relative probability for the emission of n pions and one nucleon by $S(n,1)$, one finds that

$$S(1,1)=\left(\frac{\Omega}{8\pi^3\hbar^3}\right)\cdot\frac{\pi W^2}{2}\cdot(0.720),$$

$$S(2,1)=\left(\frac{\Omega}{8\pi^3\hbar^3}\right)^2\cdot\frac{\pi^2W^5}{2\cdot 5!}(1.547), \tag{3.40}$$

and

$$S(3,1)=\left(\frac{\Omega}{8\pi^3\hbar^3}\right)^3\cdot\frac{3\pi^3W^8}{2\cdot 8!}(0.485).$$

The total energy appearing in Eq. (3.40) is actually expressed in momentum units, and the factor in W becomes

$$\frac{W}{2\pi\hbar}=\frac{W}{Nc}\cdot\frac{Nc}{2\pi\hbar}=\frac{1}{\nu\lambda_N}. \tag{3.41}$$

Here $\lambda_N=(2\pi\hbar/Nc)=1.2\times 10^{-13}$ cm is approximately the extent of the meson cloud around a nucleon. Defining further the nucleon volume, $\Omega_0=(4\pi/3)\lambda_N^3$, one may express the results in terms of the ratio of the interaction volume Ω to the nucleon volume Ω_0. Thus

$$\frac{S(2,1)}{S(1,1)}=2.04\left(\frac{\Omega}{\Omega_0}\right),\quad \frac{S(3,1)}{S(2,1)}=0.318\left(\frac{\Omega}{\Omega_0}\right),$$

and

$$\frac{S(3,1)S(1,1)}{[S(2,1)]^2}=0.156. \tag{3.42}$$

It is quite clear that the interaction volume plays a predominant role in the determination of the meson multiplicity.

H. Several Particles of Finite Mass

To consider correctly the case of several particles of nonvanishing mass involves one in calculations which, while elementary from the mathematical standpoint, still offer great practical difficulties. Initially let us discuss the case of two particles having masses M and N. For zero total momentum the limit function $p(x)$ is independent of x and is given by the equation

$$p^2=p'^2=w'^2-N^2=(W-w)^2-N^2=[W-(M^2+p^2)^{1/2}]^2-N^2,$$

or

$$p=1/2W[(W^2-M^2-N^2)^2-4M^2N^2]^{1/2}. \tag{3.43}$$

From this we find that

$$\mathcal{V}(1M,1N)=\frac{\pi}{6}\left[\frac{(W^2-M^2-N^2)^2-4M^2N^2}{W^2}\right]^{3/2}. \tag{3.44}$$

It is to be noticed that this reduces to the correct forms, Eqs. (3.36) and (3.33), as M and N tend to zero. Also, $\mathcal{V}=0$ when $W=M+N$, as is to be expected from energy conservation.

Since we would like to apply our results to the multiple production of pions, let us next consider the case where two particles (the pions) have a mass M, and one particle (the nucleon) has a mass N. We first must derive a formula analogous to Eq. (3.30) for the two particles of mass M. The function $p(x)$ is determined from

$$p^2+P^2-2Ppx=p'^2=w'^2-M^2=[W-(M^2+p^2)^{1/2}]^2-M^2. \tag{3.45}$$

If we define the dimensionless quantities

$$y=\frac{p}{W},\quad \beta=\frac{P}{W},\quad \mu=\frac{M}{W},\quad \nu=\frac{N}{W},\tag{3.46}$$

we find that

$$\frac{p(x)}{W}=y(x)$$
$$=\left[\beta x+\left(1-\frac{1-\beta^2x^2}{1-\beta^2}\cdot\frac{4\mu^2}{1-\beta^2}\right)^{1/2}\right]\frac{[1-\beta^2]}{2[1-\beta^2x^2]}.\tag{3.47}$$

From this one may calculate, by elementary methods, that

$$\mathcal{V}(2M)=\frac{\pi W^3}{6}(1-\beta^2)\left(1-\frac{4\mu^2}{1-\beta^2}\right)^{3/2}.\tag{3.48}$$

When $P=0$ this volume vanishes for $M=(1/2)W$, as is to be expected. To include the third particle, of mass N, the new limit function is determined from the first root of $\mathcal{V}(2M)$ above. Expressed in terms of the variables w' and p', this integrand is

$$\mathcal{V}'(2M)=\frac{\pi(w')^3}{6}\left[1-\frac{p'^2}{w'^2}\right]\left[1-\frac{4M^2}{(w')^2\left(1-\frac{p'^2}{w'^2}\right)}\right]^{3/2}.\tag{3.49}$$

Hence the condition for $p(x)$ is, with $(P=0)$,

$$4M^2=w'^2-p'^2=[W-(N^2+p^2)^{1/2}]^2-p^2,\tag{3.50}$$

giving

$$y(x)=\left[\left(\frac{1+\nu^2}{2}-2\mu^2\right)^2-\nu^2\right]^{1/2}.\tag{3.51}$$

From this we may write the desired three-particle momentum volume for zero net momentum, and total energy W. It is

$$\mathcal{V}(2M,1N)=\frac{2\pi^2W^6}{3}\int_0^{y(x)}dy\cdot y^2[1-(\nu^2+y^2)^{1/2}]$$
$$\cdot\{[1-(\nu^2+y^2)^{\frac{1}{2}}]^2-y^2\}$$
$$\times\left\{1-\frac{4\mu^2}{[1-(\nu^2+y^2)^{\frac{1}{2}}]^2-y^2}\right\}^{3/2}.\tag{3.52}$$

This quantity resists ready calculation in closed form. Although one is tempted to assume that in practical cases the relative pion mass, μ, will be small with respect to unity, and to expand the integral in Eq. (3.52) to the first order in μ^2, such a procedure turns out to be quite inaccurate. Indeed, this approximation will overestimate the effect of the pion mass by a factor of nearly two. To see the actual effect of the pion mass, the above quantity, or rather its more interesting energy derivative, may be calculated numerically from the equivalent formula

$$\frac{d\mathcal{V}(2M,1N)}{dW}=(4/3)\pi^2W^5\mu^4\int_1^{(1-\nu)^2/4\mu^2}du\cdot(u-1)^{3/2}u^{-1/2}$$
$$\cdot[(1+\nu^2-4\mu^2u)^2-4\nu^2]^{-1/2}$$
$$\cdot\{(1-\nu^2+\nu^4-2\nu^6+\nu^8)$$
$$-4\mu^2u(1+2\nu^2-2\nu^4+4\nu^6)$$
$$+16\mu^4u^2(1+2\nu^2+6\nu^4)$$
$$-64\mu^6u^3(2+4\nu^2)+256\mu^8u^4\}.\tag{3.53}$$

The calculations were performed for a total energy of 1.920 Bev (laboratory pions of 1.37 Bev), with a nucleon mass of 0.938 Bev ($\nu=0.487$), and a pion mass of 0.136 Bev ($\mu=0.071$). Using Simpson's method, except at the upper limit in Eq. (3.53) where the irrational singularity is calculated in closed form, we find the result to be

$$\frac{d\mathcal{V}(2M,1N)}{dW}=\frac{\pi^2W^5}{2\cdot 5!}(1.162).\tag{3.54}$$

The exact one-pion, one-nucleon weight may also be readily found from Eq. (3.44) and is

$$\frac{d\mathcal{V}(1M,1N)}{dW}=\frac{\pi W^2}{2}(0.714).\tag{3.55}$$

The inclusion of the pion mass has thus reduced the one-pion weight in Eq. (3.37) by a factor of 0.99, and the two-pion weight in Eq. (3.38) by a factor of 0.75. Hence the ratio of the relative production probabilities becomes

$$\left.\frac{S(2,1)}{S(1,1)}\right|_{\text{Exact}}=1.54\left(\frac{\Omega}{\Omega_0}\right).\tag{3.56}$$

The exact calculation of the relative probability for the production of three or more pions would be very complex. It is of interest, therefore, to compare the results of the exact calculations above with those for zero pion mass as seen in Eqs. (3.37), (3.38), and (3.39), under the simple assumption that the total energy available is merely reduced by the total mass of the produced pions. Thus, the total energy of 1.920 Bev is reduced to 1.784 Bev for one pion, to 1.648 Bev for two pions, and to 1.512 Bev for three pions. These give $\nu=0.525$ for one pion, $\nu=0.569$ for two pions, and $\nu=0.619$ for three pions. Upon direct calculation from Eqs. (3.37), (3.38), and (3.39) the relative probabilities become

$$\left.\frac{S(2,1)}{S(1,1)}\right|_{\text{Approx.}}=0.550\left(\frac{\Omega}{\Omega_0}\right),$$
and
$$\left.\frac{S(3,1)}{S(2,1)}\right|_{\text{Approx.}}=0.028\left(\frac{\Omega}{\Omega_0}\right).\tag{3.57}$$

It is evident from the disparity between Eq. (3.56) and Eq. (3.57) that the simpler approximations do not suffice to give results which do justice to the theory. Since just such an approximation was made in deriving Eq. (3.23) for the nucleon-nucleon case, one may perhaps infer that more exact calculations are also necessary here. It has been reported that Yang and Christian at Brookhaven have made such calculations by a numerical integration of Eq. (3.1).[5]

I. Elaborations of the Fermi Theory

Clearly, the preceding calculations do not represent the ultimate in deductions from the Fermi theory. First of all, angular momentum conservation was grossly neglected in deriving Eq. (2.7) from Eq. (2.6). It would evidently be desirable to have this procedure checked by a rigorous calculation, based, perhaps, upon the insertion of a suitable additional delta function in Eq. (3.1). An attempt of the author's in this direction has met with little success in the face of mathematical complications.

Another sort of refinement has been suggested by Lepore, Stuart, and Neuman,[4,14] and is based upon the requirement of the Lorentz covariance for the interacting system. Each particle of the system may be described by a space-time four-vector

$$x_i=(ix_1,ix_2,ix_3,ct),$$

and by an energy-momentum four-vector

$$u_i=(ip_1,ip_2,ip_3,W).$$

From these one may form a skew-symmetric tensor of rank two

$$L_{ij}=x_iu_j-x_ju_i=-L_{ji}, \tag{3.58}$$

which is the general angular momentum tensor of the particle. In the absence of an external interaction the sum of these tensors taken for all the particles of the system will be conserved. Therefore,

$$\textstyle\sum_s L_{ij}^{(s)}=\text{const}, \tag{3.59}$$

where the index s represents the particle taken in the sum. This equation, for i and $j=1, 2, 3$, represents the law of conservation of angular momentum as discussed above. The ($i4$) component written in vector form is

$$\textstyle\sum_s (\mathbf{x}^{(s)}W^{(s)}-\mathbf{p}^{(s)}t^{(s)})=\text{const}. \tag{3.60}$$

Using the fact that in a particular coordinate system the time factor for all particles will be the same, and that the total momentum will be conserved, one deduces a law for the conservation of the center of energy

$$\langle\mathbf{x}\rangle=(\textstyle\sum_s \mathbf{x}^{(s)}W^{(s)})/(\sum_s W^{(s)}). \tag{3.61}$$

This reduces to the usual center-of-mass conservation law in the nonrelativistic limit.

In addition, Lepore, Stuart, and Neuman modify the matrix element in Eq. (2.3) to allow for the fact that the higher the energy of a virtual particle created momentarily within the interaction volume, the more closely confined that particle will be to the point of origin. This property is included numerically in the form of a Gaussian weighting factor

$$\exp[-W_i^2x_i^2/\hbar^2c^2]. \tag{3.62}$$

The following modified formula is then used for the statistical weight:

$$S(N)=(2\pi\hbar)^{-3(N-1)}\times\prod_{i=1}^{N}\left\{\int d^3\mathbf{x}_i\int d^3\mathbf{p}_i\exp[-W_i^2x_i^2/\hbar^2c^2]\right\}\times\delta(\textstyle\sum\mathbf{p}_i)\delta(W-\sum W_i)\delta(\sum\mathbf{x}_iW_i/W). \tag{3.63}$$

Upon calculation of the Gaussian spatial integral one derives

$$S(N)=\frac{2^{N-3}\pi^{3(N-1)/2}}{(2\pi\hbar)^{3(N-1)}N^{3/2}}\cdot\left(\frac{W}{\hbar}\right)^3\prod_{i=1}^{N}\left[\int\frac{d^3\mathbf{p}_i\hbar^3}{[p_i^2+M_i^2]^{3/2}}\right]\times\delta(\textstyle\sum\mathbf{p}_i)\delta(W-\sum W_i). \tag{3.64}$$

The most striking effect of these new requirements is the factor

$$\hbar^3[p_i^2+M_i^2]^{-3/2}, \tag{3.65}$$

included in the momentum integral for each particle. This represents a reduction in the phase space available to high-momentum particles, and corresponds to the reduced overlap between the higher energy bound virtual particles and the corresponding free particles. One would expect these factors taken together to favor the production of more low-energy particles and fewer high-energy particles, that is, to increase the multiplicity. The originators of this modification to the Fermi theory have not yet published any numerical results.[14] Since their approach would seem to yield an increase in the expected multiplicities, while previous refinements of the momentum-space integrations have tended to decrease these multiplicities, their results will be of considerable interest. We shall further discuss these various calculations in Sec. V.

IV. CONSERVATION OF CHARGE AND ISOTOPIC SPIN

It has been suggested by Fermi[2] that, besides charge conservation, one might also impose isotopic spin conservation upon the interacting pion-nucleon system. The background relating to this restriction has been thoroughly discussed in the literature.[15] We shall here describe only its application to multiple meson production. For simplicity we shall describe the theory as applied to pion-nucleon collisions. Following this we

[14] Lepore, Neuman, and Stuart, Phys. Rev. **94**, 788 (1954).

[15] Henley, Ruderman, and Steinberger, "Reactions of pi-mesons with nucleons," in *Annual Review of Nuclear Science* (Stanford, California, 1953), Vol. 3, pp. 1–38.

shall append a tabulation of Fermi's earlier results for nucleon-nucleon collisions.[16]

The behavior of the isotopic spin of a system of particles is conveniently described in the language of a fully equivalent system of angular momenta. Thus a nucleon, with respect to its charge state, is considered to behave like a particle of total "charge" angular momentum, or isospin, one-half. The two possible projections of this charge vector along a given axis in charge space represent the two possible charge states of a nucleon, that is, a proton or a neutron. Similarly a pion, which can have positive, zero, or negative charge states, is represented by a total isospin of unity. One may summarize this correspondence as

Nucleons: Isospin $T^N=1/2$;
State of $T_z^N=+1/2$ is a proton (p)
$T_z^N=-1/2$ is a neutron (n)

Pions: Isospin $T^\pi=1$; State of $T_z^\pi=+1$ is a π^+
$T_z^\pi=\ \ 0$ is a π^0
$T_z^\pi=-1$ is a π^-.

If a number of pions and nucleons are combined to form a joint system, the total isospin vector is formed in the same way that one forms the total angular momentum of a number of particles. The resultant isospin is thus the vector sum of the isospins of the component particles, and may be written

$$\mathbf{T}=\sum_i \mathbf{T}^{(i)}.$$

The total charge of the composite system is related to the z component of $\mathbf{T}$ by

$$Q=+e[T_z+(1/2)s],$$

where s is the number of nucleons present. Charge conservation is equivalent to the requirement that T_z be a good quantum number during the interaction. Isotopic spin conservation similarly is equivalent to the requirement that the total isospin T be a good quantum number.

We shall illustrate this with the case of a system formed from a negative pion and either a neutron or a proton. Only two resultant total isospins are possible in this case, $T=1/2$ and $T=3/2$. Because of charge conservation we are only interested in states having $T_z=-3/2$, for interactions with neutrons, and $T_z=-1/2$, for interactions with protons. Henceforth we shall adopt the notation (T,M) for states having total isospin T and charge component $T_z=M$; and also, for example, the notation $(p++-0)$ for a state containing one proton, two positive, one neutral, and one negative pion. The ordering of the pions is of no significance here. In order to relate the description in terms of isospin with that in terms of the charges of the two particles

TABLE III. Clebsch-Gordon coefficients for a pion-nucleon system (all other entries are zero).

	(M_1,M_2)	(T,M):	$(3/2,-3/2)$	$(3/2,-1/2)$	$(1/2,-1/2)$
$(p+)$	$1/2$	1			
$(p0)$	$1/2$	0			
$(p-)$	$1/2$	-1		$\sqrt{(1/3)}$	$-\sqrt{(2/3)}$
$(n+)$	$-1/2$	1			
$(n0)$	$-1/2$	0		$\sqrt{(2/3)}$	$\sqrt{(1/3)}$
$(n-)$	$-1/2$	-1	1		

one must know the transformation coefficients connecting the two representations. These coefficients are just the Clebsch-Gordon, or vector-addition, coefficients, and may be computed from formulas arising in the theory of atomic spectra.[17] Using the notation $(T_1,T_2,M_1,M_2|T_1,T_2,T,M)$, the coefficients of interest in the present case are those with $T_1=1/2$ and $T_2=1$. Here M_1 represents the charge component of the nucleon, and M_2 that of the pion. These coefficients are given in Table III. From this table one may see immediately that a system composed of a negative pion and a neutron will surely be in a state $T=3/2$, $M=-3/2$, or $(3/2,-3/2)$, for short. A negative pion and a proton, on the other hand, will, with a probability of 1/3, be found in a combined state $(3/2,-1/2)$ and with a probability of 2/3 in $(1/2,-1/2)$. Assuming charge and isospin conservation, only these three states will be needed for the following computations. Conversely, one may also deduce from this table that a system composed of one nucleon and one meson, known to be in a state $(3/2,-1/2)$, will, with a probability of 1/3 be observed as a $(p-)$, and with a probability of 2/3 as a $(n0)$. If one assumes that the $(\pi^-$-$p)$ interaction cross section is the same for both possible states of isospin, then final states of $T=3/2$ and $T=1/2$ will be formed with probabilities 1/3 and 2/3, respectively. The probability of observing $(p-)$ will then be: $(1/3\cdot 1/3)+(2/3\cdot 2/3)=5/9$, and of observing $(n0)$: $(1/3\cdot 2/3)+(2/3\cdot 1/3)=4/9$. The two possible one-pion final states thus will be observed in the ratio $(p-):(n0)=5:4$. Actually the assumption of equal cross sections is a rather dubious one in the light of evidence pertaining to medium energy pion scattering, which suggests the existence of a resonance or near-resonance in the $T=3/2$ state.[15,18] For negative pions colliding on neutrons no charge exchange scattering is possible, and the $(n-)$ final state will be produced with unit probability, assuming no other pions are produced.

When one admits the possibility of producing more than one pion, the situation becomes a little more complicated. With two pions and one nucleon, for example, the state $(3/2,-3/2)$ can be formed in one way when the total isospin of the two pions is equal to two $(T_{2\pi}=2)$, and another when $T_{2\pi}=1$. Following

[16] A thorough description of the isotopic spin calculations for nucleon-nucleon collisions is contained in a set of hectographed notes transcribed by G. Yodh from a course in Meson Physics given by Professor H. Anderson at the University of Chicago during the fall of 1953.

[17] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Cambridge, England, 1935), pp. 76 ff.

[18] Anderson, Fermi, Martin, and Nagle, Phys. Rev. **91**, 155 (1953).

TABLE IV. The statistical weights for the products of (π^--nucleon) collisions. For (π^+-nucleon) collisions exchange (+) for (−) and (p) for (n) everywhere. The numbers in brackets represent the total statistical weight of the state under which they are entered.

No. Pions	Final	Initial: (3/2, −3/2)	(3/2, −1/2)	(1/2, −1/2)	(π^--n)	(π^--p)
1	(p−)		1/3	2/3		5/9
	(n−)	1			1	
	(n0)		2/3	1/3		4/9
	Total	(1)	(1)	(1)	(1)	(1)
2	(p−−)	4/5			4/5	
	(p0−)		14/15	2/3		34/45
	(n0−)	6/5			6/5	
	(n00)		4/15	1/3		14/45
	(n+−)		12/15	3/3		42/45
	Total	(2)	(2)	(2)	(2)	(2)
3	(p0−−)	8/5			8/5	
	(p+−−)		6/5	6/5		18/15
	(p00−)		5/5	4/5		13/15
	(n+−−)	10/5			10/5	
	(n00−)	7/5			7/5	
	(n+0−)		12/5	9/5		30/15
	(n000)		2/5	1/5		4/15
	Total	(5)	(5)	(4)	(5)	(13/3)
4	(p+−−−)	80/35			80/35	
	(p00−−)	88/35			88/35	
	(p+0−−)		152/35	192/60		752/210
	(p000−)		44/35	48/60		200/210
	(n+0−−)	200/35			200/35	
	(n000−)	52/35			52/35	
	(n++−−)		87/35	123/60		461/210
	(n+00−)		129/35	165/60		643/210
	(n0000)		8/35	12/60		44/210
	Total	(12)	(12)	(9)	(12)	(10)

the spirit of Fermi's statistical theory we assume that these two possibilities are equally excited, and that this degeneracy provides the state (3/2, −3/2) with the over-all weight 2. To proceed with the calculations one must first relate the various two-pion states according to their total isospin $T_{2\pi}$, and total charge component $M_{2\pi}$. This relation is provided by the set of coefficients, $(1, 1; M_1, M_2|1, 1; T_{2\pi}, M_{2\pi})$, evaluated for $T_{2\pi}=2$, 1 0. Only the intensities are of interest. They are, writing $(T_{2\pi},M_{2\pi})$, given compactly by

$$(2,2)=(++) \qquad (2,1)=(+0)$$
$$(2,0)=1/3(+-)+2/3(00) \qquad (2,-1)=(0-)$$
$$(2,-2)=(--) \qquad (1,1)=(+0)$$
$$(1,0)=(+-) \qquad (1,-1)=(0-)$$
$$(0,0)=2/3(+-)+1/3(00).$$

Once in possession of these two-pion states one may add the nucleon as before by utilizing the coefficients $(T_{2\pi},1/2; M_{2\pi},M_N|T_{2\pi},1/2; T,M)$. Thus one finds directly, for example, that a system in the state (3/2, −3/2) will form two-pion, one-nucleon states with the relative weights for (p−−): 4/5 and for (n0−): 6/5. The sum of these two weights is just 2, representing the over-all degeneracy of the (3/2, −3/2) state.

One may proceed in this way to build up the relative weights for as many pions and nucleons as desired. Results are given in Table IV for (π^--nucleon) collisions which produce one nucleon and up to four pions. The weights for (π^--p) events are calculated, assuming equal cross sections for the two possible charge states. Weights for one charge state or the other may be found directly from the appropriate column. One should also note that as a result of charge symmetry the table will also give the weights for (π^+-nucleon) collisions. One need only exchange everywhere in Table IV + for −, and p for n.

These weights include the over-all contribution resulting from the degeneracy of the charge states. The total charge space weights are written in brackets. In practice these factors will be nullified by the reduction in the over-all weight arising from the indistinguishability of the pions.

Fermi[2] has made similar calculations for the products of nucleon-nucleon collisions. These are given in Table V. Two protons are necessarily in the isospin state 1, 1, two neutrons in the state 1, −1, and a neutron and a proton in either 1, 0 or 0, 0. In the last case, provided the interaction cross sections are equal for $T=1$ and $T=0$, one may assume that the interacting n-p system is equally divided between the 1, 0 and 0, 0 states. The results in the table are for n-p and p-p collisions. Again, one may obtain n-n results from the p-p results by everywhere exchanging + for − and p for n.

V. DISCUSSION OF RESULTS

A. Application of the Fermi Theory to π^--p Events at 1.37 Bev

We shall illustrate, finally, the application of the Fermi theory to a particular case which corresponds to the conditions of some recent experiments at the Brookhaven Cosmotron. Negative pions are incident on protons with a laboratory energy of 1.37 Bev, corresponding to a center-of-mass energy of 1.92 Bev. We have already calculated the phase-space contributions to the statistical weights at this energy. For zero-mass pions we found, in Eq. (3.42), that

$$S(2,1)/S(1,1)=2.04(\Omega/\Omega_0),$$

and

$$S(3,1)/S(2,1)=0.32(\Omega/\Omega_0). \quad (3.42)$$

An exact calculation taking into account the pion mass gave, however,

$$[S(2,1)/S(1,1)]_{\text{Exact}}=1.54(\Omega/\Omega_0). \quad (3.56)$$

In the absence of similar calculations for the three meson case we may estimate approximately that

$$[S(3,1)/S(2,1)]_{\text{Exact}}\doteq 0.2(\Omega/\Omega_0). \quad (5.1)$$

These weights must be adjusted for the degeneracy of states in charge space (Table IV), and for the fact that the pions are identical particles as seen in Eq. (2.3). For one pion produced this factor is just $1/1!=1$. For two pions it is $2/2!=1$, and for three pions $(13/3)/3!=0.72$. We finally obtain the corrected weights

$$S_c(2,1)/S_c(1,1)=1.54(\Omega/\Omega_0),$$

and

$$S_c(3,1)/S_c(2,1)=0.14(\Omega/\Omega_0). \quad (5.2)$$

TABLE V. Statistical weights for the products of (nucleon-nucleon) collisions (after Fermi[2]). For *n-n* collisions use *p-p* results exchanging + for − and *p* for *n* everywhere.

No. Pions	Final	Initial: (1,1)	(1,0)	(0,0)	(*p-p*)	(*n-p*)
0	(*pp*)	1			1	
	(*np*)		1	1		1
	Total	(1)	(1)	(1)	(1)	(1)
1	(*pp*0)	1/2			1/2	
	(*pn*+)	3/2			3/2	
	(*pp*−)		1/2	1/3		5/12
	(*pn*0)		2/2	1/3		8/12
	(*nn*+)		1/2	1/3		5/12
	Total	(2)	(2)	(1)	(2)	(3/2)
2	(*pp*+−)	6/5			6/5	
	(*pp*00)	2/5			2/5	
	(*pn*+0)	9/5			9/5	
	(*nn*++)	3/5			3/5	
	(*pp*0−)		4/5	1/3		17/30
	(*pn*00)		3/5	1/3		14/30
	(*pn*+−)		9/5	3/3		42/30
	(*nn*+0)		4/5	1/3		17/30
	Total	(4)	(4)	(2)	(4)	(3)
3	(*pp*+0−)	154/60			154/60	
	(*pp*000)	18/60			18/60	
	(*pn*++−)	175/60			175/60	
	(*pn*+00)	121/60			121/60	
	(*nn*++0)	72/60			72/60	
	(*pp*00−)		9/10	2/5		13/10
	(*pp*+−−)		12/10	3/5		18/10
	(*pn*000)		6/10	1/5		8/10
	(*pn*+0−)		42/10	9/5		60/10
	(*nn*+00)		9/10	2/5		13/10
	(*nn*++−)		12/10	3/5		18/10
	Total	(9)	(9)	(4)	(9)	(13/2)

TABLE VI. Relative production probabilities for 1, 2, and 3 pions in π^--p collisions at 1.37 Bev.

Case *A* ($\Omega=\Omega_0$)	Case *B* ($\Omega=0.78\Omega_0$)	Case *C* ($\Omega=0.5\Omega_0$)
$S(1,1)=0.36$	$S(1,1)=0.42$	$S(1,1)=0.55$
$S(2,1)=0.56$	$S(2,1)=0.51$	$S(2,1)=0.42$
$S(3,1)=0.08$	$S(3,1)=0.07$	$S(3,1)=0.03$

TABLE VII. Distribution of secondary products in π^--p collisions at 1.37 Bev.

Secondary products	Case *A* ($\Omega=\Omega_0$)	Case *B* ($\Omega=0.78\Omega_0$)	Case *C* ($\Omega=0.5\Omega_0$)
(*p*−)	0.20	0.24	0.31
(*n*0)	0.16	0.18	0.24
(*p*0−)	0.21	0.19	0.16
(*n*00)	0.09	0.08	0.07
(*n*+−)	0.26	0.24	0.19
(*p*+−−)	0.02	0.02	0.01
(*p*00−)	0.02	0.02	0.01
(*n*+0−)	0.04	0.03	0.01
(*n*000)	0.00	0.00	0.00

TABLE VIII. Distribution of charged secondary products in π^--p collisions at 1.37 Bev.

Charged secondaries	Case *A* ($\Omega=\Omega_0$)	Case *B* ($\Omega=0.78\Omega_0$)	Case *C* ($\Omega=0.5\Omega_0$)	Experiment[19]
$(\pi^-+p)_{\text{elas}}$	0.27	0.32	0.45	0.11
$(\pi^-+p)_{\text{inel}}$	0.30	0.29	0.25	0.35
$(\pi^++\pi^-)$	0.40	0.36	0.29	0.50
$(\pi^++2\pi^-+p)$	0.03	0.03	0.01	0.04

Assuming that no more than three pions are produced in significant quantities, and making three possible choices for the interaction volume Ω, we find the normalized relative probabilities listed in Table VI. Case B, $\Omega=0.78\Omega_0$, assumes the interaction volume is that of the proton, Lorentz-contracted to allow for its motion in the center-of-mass system.

In terms of the above, and the charge distribution given in Table IV, we may predict the over-all distribution of secondary particles shown in Table VII.

Of somewhat greater experimental interest is a table in which only the observable charged secondaries are entered. Elastic scatterings are frequently distinguishable experimentally and hence are separated in Table VIII. Included are some tentative experimental results from photographs of 147 events in a diffusion cloud chamber filled with hydrogen at high pressure.[19]

The fit of theory to experiment is not very good for the theory predicts many more incoherent elastic

[19] Eisberg, Fowler, Lea, Shepard, Shutt, Thorndike and Whittemore, Phys. Rev. in press. Of the observed events 31 were identifiable only as inelastic collisions with two emerging charged prongs. The assignment of these events to the categories in Table VIII according to the theoretical weights quoted there for Case *B* was suggested by J. C. Street. The separation of the 147 events into 95 inelastic and 52 elastic cases, and the division of the elastic events into 40 coherent (diffraction) and 12 incoherent scatterings, are as quoted by Eisberg *et al.* The Fermi theory does not include diffraction scattering.

(π^--p) events than appear to occur. The experimental uncertainties in the data are considerable, however, and the fractioning off of 80 percent of the elastic scatterings as coherent may turn out to have been too arbitrary. Of course the theory was also oversimplified by neglecting angular momentum and other refinements in the calculation of the phase space factor. A somewhat different fit can be obtained if only $(3/2, -1/2)$ states are permitted to occur.

B. Nucleon-Nucleon Collisions

Unfortunately the fit of theory with experiment obtained in the last section deteriorates further when nucleon-nucleon collisions are considered. In a recent experiment to determine the meson production in (n-p) and (p-p) collisions at Cosmotron energies (1.0–2.2 Bev),[20] it was found that double-pion production occurred much more frequently than could be accounted for by a simple statistical theory. For example, neutrons of a median energy 1.7 Bev (total center-of-mass energy 2.6 Bev) produced two pions about 2.2 times as often as one pion. A Fermi theory calculation by Yang and Christian[5] predicted this ratio to be about 1/11 (cf. Table II). Even admitting many refinements it would seem to be difficult to reconcile these two figures. To explain this sort of result Peaslee has advanced a theory in which the two colliding nucleons become excited, separate, and finally, when some distance apart, decay to produce one, and perhaps more pions.[21] The intermediate, excited-nucleon states were considered to be states of angular momentum and isospin each equal to 3/2, and excitation energy about 160 Mev, to correspond to the apparent π^+-p resonance peak observed in scattering experiments. Peaslee found that such a hypothesis would suffice to give the anomalously large two-pion production observed experimentally. His theory also suggests that when the produced particles are ($pn+-$) there should be a correlation between the decay planes of the π^- and the n, arising from the excited neutron of charge state $(3/2, -3/2)$, and a similar correlation for the π^+ and the p, of charge state $(3/2, 3/2)$. Such a correlation was indeed indicated, if not assured, by the experimental data. A preliminary analysis of p-p collisions led the same authors[20] to conclude that the Fermi theory is again inadequate to predict the observed interactions, while the Peaslee model offers a much closer fit.

[20] Fowler, Shutt, Thorndike, and Whittemore, preprint, Brookhaven National Laboratory, Upton, Long Island; Phys. Rev. **95**, 1026 (1954).
[21] D. Peaslee, Phys. Rev. **94**, 1085 (1954).

VI. CONCLUSION

Pion-nucleon and nucleon-nucleon experiments conducted in the energy range of 5–10 Bev would shed great light upon the problem of multiple meson production. For the pion-nucleon case at such energies the Fermi theory predicts multiplicities of sufficient magnitude to give the one parameter fit, inconclusive at 1.4 Bev, a severe test, and may enable one to determine whether any particular charge states are formed preferentially. For the nucleon-nucleon case one would perhaps expect, following Peaslee's approach, that excited nucleons would be created which are capable of decaying into two or more pions. If so, then one would hope to be able to compare these excited nucleons with similar excited states formed in pion-nucleon collisions. Conclusions must await the acquisition of experimental data of suitable quality and quantity.