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On the Convergence of Born Expansions*

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A study is made of the convergence of the Born expansions in collision problems involving a static central potential. The scattering of each partial wave, characterized by the phase shift η_l, is treated separately. Estimates of the radii of convergence of $\tan\eta_l$ and of $S_l \equiv e^{2i\eta_l}$ (as well as of the corresponding wave functions) are established, and the truncation errors due to breaking off the Born expansions are investigated. An appendix deals with the expansion of η_l itself.

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1. INTRODUCTION

ONE of the most widely used techniques of quantum mechanical collision theory is the development of the wave function as a power series in a parameter λ, which measures the strength of the interaction. This series, when broken off after the nth power of λ, is generally called the nth Born approximation.[1] Evidently there are two questions of importance:

(1) In a given physical problem, does the series converge?

(2) If the series converges but is broken off after a finite number of terms, how large an error is introduced?

Recently the general nature of Born expansions has been clarified by Jost and Pais[2] for the case of a non-relativistic particle scattered by a static potential. However, these authors did not investigate in any detail the actual magnitude of the radius of convergence, or the truncation error.

In the present paper we shall deal with the particularly simple case of static central potentials, $V(r)$, which are sectionally continuous, have at most an r^{-1} singularity at the origin, and are short range in the sense that for some positive ϵ

$$\lim_{r\to\infty} r^{2+\epsilon}V(r) = 0.^3 \tag{1.1}$$

For such central potentials the three-dimensional Schroedinger equation can be separated, leading to the radial equations

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2\right)\psi_l(r) = \lambda V(r)\psi_l(r) \tag{1.2}$$

corresponding to angular momentum l, and the boundary conditions

$$\psi_l(o) = 0. \tag{1.3}$$

In this very simple situation it is, of course, not necessary to resort to a power series expansion since the solution can be obtained for any value of λ by numerical integration. Nevertheless, it has seemed to us worth while to investigate the convergence of the Born expansions, in order to have at least some reliable results which may serve as guide for more complicated problems. Accordingly, we shall in the present paper derive estimates of the radii of convergence of the Born expansions of the solutions of (1.2), as well as of the error due to breaking off these expansions after a finite number of terms.

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[1] M. Born, Z. Physik **38**, 803 (1926).

[2] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).

[3] Some of our discussions will require further restrictions on $V(r)$. On the other hand, in several places, it will be permissible to include the limiting case of δ-function-like potentials.

Before proceeding further we must first complete the definition of ψ_l by specifying its behavior for large r. When dealing with a single angular momentum one customary choice is

$$r\to\infty: \psi_l(r)\to\sin\left(kr-\frac{l\pi}{2}\right)+\tan\eta_l\cos\left(kr-\frac{l\pi}{2}\right), \quad (1.4)$$

where η_l is the so-called phase shift entering the well-known expression for the scattering cross section.[4] The expansion of this function—and in particular of $\tan\eta_l$—is studied in Part I, Secs. 2–9. The estimates of the radius of convergence λ_c are collected in Sec. 8, and the truncation error is treated in Sec. 9.

Another behavior at infinity is also of interest. Frequently the total three-dimensional wave function, which satisfies the Schroedinger equation

$$(\nabla^2+k^2)\Psi(\mathbf{r})=\lambda V(r)\Psi(\mathbf{r}), \quad (1.5)$$

is expanded as a power series in λ. This is accomplished by iterating the integral equation

$$\Psi(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}+\frac{\lambda}{4\pi}\int\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}V(r')\Psi(\mathbf{r}')d\mathbf{r}', \quad (1.6)$$

where $\mathbf{k}$ represents the wave vector of the incident particle. It would be very desirable to obtain information about the Born expansion of this three-dimensional function. This will not be attempted in the present paper. We shall, however, in Secs. 10–12, discuss briefly the Born expansions of each partial wave of which Ψ is composed. This leads to a study of the radial functions $\varphi_l(r)$, which satisfy (1.2) and (1.3) but have the following behavior for large r:

$$r\to\infty: \varphi_l(r)\to\sin\left(kr-\frac{l\pi}{2}\right)+\frac{S_l-1}{2i}\exp\left[i\left(kr-\frac{l\pi}{2}\right)\right], \quad (1.7)$$

where $S_l(\equiv\exp[2i\eta_l])$ is the scattering matrix element corresponding to the angular momentum l. Estimater of the radius of convergence, λ_c', of φ_l—and in particules of S_l—are summarized at the end of Sec. 11, and the truncation error is discussed in Sec. 12.

The Appendix contains a brief discussion of the Born expansion of η_l itself, which gives the best results at high energies.

Some general comments on Born expansions will be found in Sec. 13.

PART I. BORN EXPANSION OF ψ_l AND OF TANη_l

2. General Properties

In most of the following considerations we shall deal with one value of l at a time and will therefore frequently omit the subscript l.

[4] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), second edition, p. 24.

Meromorphic Character of ψ and $\tan\eta$

Following reference 2 we show first that the function $\psi(r)$, defined by (1.2), (1.3), and (1.4), is a meromorphic function of λ, whose power series development is the Born expansion. Let $f_l(\lambda;k,r)$ be that solution of (1.2) which satisfies the initial condition

$$\lim_{r\to\infty} e^{i(kr-l\pi/2)}f_l(\lambda;k,r)=1. \quad (2.1)$$

This is an entire function of λ, because it is defined by initial conditions independent of λ (see reference 2, III). Calling

$$\left.\begin{aligned}(2l-1)!!&\equiv 1 & l&=0\\ &\equiv 1\cdot3\cdot5\cdots(2l-1), & l&>0\end{aligned}\right\}, \quad (2.2)$$

we now define

$$f_l(\lambda;k)=\lim_{r\to0}\frac{(kr)^l}{(2l-1)!!}f_l(\lambda;k,r), \quad (2.3)$$

which is a finite and, in general, nonvanishing complex number; for near $r=0$ there are two independent solutions of (1.2) behaving like r^{l+1} and r^{-l}, respectively. Clearly $\psi(r)$ is given by the following linear combination [see (1.3), (1.4), and (2.1)]:

$$\psi(r)=\frac{i}{f(\lambda;k)+f(\lambda;-k)}[f(\lambda;-k)f(\lambda;k,r)-f(\lambda;k)f(\lambda;-k,r)]. \quad (2.4)$$

Comparison with (1.3) shows that

$$i\tan\eta=\frac{f(\lambda;k)-f(\lambda;-k)}{f(\lambda;k)+f(\lambda;-k)}. \quad (2.5)$$

Equations (2.4) and (2.5) exhibit the meromorphic character of $\psi(r)$ and $\tan\eta$. Their common singularities are located at those points λ_s at which

$$f(\lambda_s;k)+f(\lambda_s;-k)=0. \quad (2.6)$$

The Born expansions of $\psi(r)$ and $\tan\eta$ converge for

$$|\lambda|<\lambda_c, \quad (2.7)$$

where λ_c is the magnitude of that singularity λ_0 which is located nearest the origin.

Integral Equation

The differential equation (1.2) and the boundary conditions (1.3), (1.4) can be combined into an integral equation.

We define two solutions of the "free" equation (1.2) (i.e., with $\lambda=0$) as follows:

$$u_l(r)=(kr)j_l(kr), \quad (2.8)$$

$$v_l(r)=(kr)n_l(kr), \quad (2.9)$$

where the so-called spherical Bessel functions j_l and n_l are

$$j_l(x)=\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}J_{l+\frac{1}{2}}(x), \tag{2.10}$$

$$n_l(x)=\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}Y_{l+\frac{1}{2}}(x). \tag{2.11}$$

u_l and v_l have the following properties:

$$r\rightarrow 0: \quad u_l(r)\rightarrow\frac{(kr)^{l+1}}{(2l+1)!!}; \quad v_l(r)\rightarrow-\frac{(2l-1)!!}{(kr)^l}. \tag{2.12}$$

$$r\rightarrow\infty: \quad u_l(r)\rightarrow\sin(kr-l\pi/2);$$

$$v_l(r)\rightarrow-\cos(kr-l\pi/2). \tag{2.13}$$

Next we define the Green's function,

$$G_l(r,r')=\begin{cases}\frac{1}{k}u_l(r)v_l(r'), & r\leq r',\\ \frac{1}{k}u_l(r')v_l(r), & r\geq r'.\end{cases} \tag{2.14}$$

Equations (1.2)–(1.4) are then clearly equivalent to the integral equation

$$\psi_l(r)=u_l(r)+\lambda\int_0^\infty G_l(r,r')V(r')\psi_l(r')dr'. \tag{2.15}$$

If in this equation we let $r\rightarrow\infty$ and compare with (1.4) we find

$$\tan\eta_l=-\frac{\lambda}{k}\int_0^\infty u_l(r)V(r)\psi_l(r)dr. \tag{2.16}$$

The Born expansion of $\psi_l(r)$ arises from iteration of (2.15), that of $\tan\eta_l$ by substituting this series in (2.16).

Position of the Singularities in the Complex λ Plane

At a λ_s at which $\tan\eta_l$ becomes infinite, the wave function

$$\chi_l(r)=\frac{k^l}{2}[f_l(\lambda_s;k,r)+f_l(\lambda_s;-k,r)] \tag{2.17}$$

satisfies (1.2) and (1.3) [see (2.1) and (2.3)] and has the asymptotic behavior

$$r\rightarrow\infty: \quad \chi_l(r)\rightarrow-k^lv_l(r). \tag{2.18}$$

It is, therefore, a solution of the following homogeneous equation:

$$\chi(r)=\lambda_s\int_0^\infty G(r,r')V(r')\chi(r')dr'. \tag{2.19}$$

Multiplying by $V(r)\chi^*(r)$ and integrating from 0 to ∞ one obtains

$$\lambda_s=\frac{\int_0^\infty|\chi(r)|^2V(r)dr}{\int_0^\infty dr\int_0^\infty dr'\chi^*(r)V(r)G(r,r')V(r')\chi(r')}. \tag{2.20}$$

If $V(r)$ does not change sign, the numerator is real and not zero, and since $G^*(r,r')=G(r',r)$, the denominator is real. Hence all singularities λ_s are real.

If however $V(r)$ does change sign, both integrals in (2.20) may vanish, leaving open the possibility of a complex λ_s. That this is, in fact, not unusual can be verified by using as an example

$$V(r)=A\delta(r-a)+B\delta(r-b). \tag{2.21}$$

In this case (2.19) leads to a quadratic secular equation for λ_s which can be explicitly solved. One finds that for certain ranges of A, B, a, b, and k the solutions are real, while for others they are complex.

At vanishing energy, however, the λ_s are real regardless of possible sign changes of $V(r)$. This is most simply seen by multiplying the appropriate Schroedinger equation

$$\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right)\chi(r)=\lambda_sV(r)\chi(r) \tag{2.22}$$

by $\chi^*(r)$ and integrating from zero to infinity. Since

$$\left.\begin{array}{l}\chi(o)=0,\\ r\rightarrow\infty: \quad \chi(r)\rightarrow\frac{(2l-1)!!}{r^l}\end{array}\right\}, \tag{2.23}$$

we obtain after an integration by parts

$$\lambda_s=-\frac{\int_0^\infty\left\{\left|\frac{d\chi}{dr}\right|^2+\frac{l(l+1)}{r^2}|\chi|^2\right\}dr}{\int_0^\infty|\chi|^2V(r)dr}. \tag{2.24}$$

Both numerator and denominator are real, and since the former is not zero, it follows that λ_s must be real.

3. Lower and Upper Bounds for the Radius of Convergence in the Limit of Vanishing Energy

We adopt the following notation:

$$\left.\begin{array}{l}\lambda_0\text{: singularity of smallest absolute value,}\\ \lambda_c=|\lambda_0|\text{: radius of convergence,}\\ \lambda_{uc}\geq\lambda_c\text{: upper bound of radius of convergence,}\\ \lambda_{lc}\leq\lambda_c\text{: lower bound of radius of convergence.}\end{array}\right\} \tag{3.1}$$

We consider first the case of vanishing energy. If we call

$$\psi^{(0)}(r)=\lim_{k\to 0}\frac{(2l+1)!!}{k^{l+1}}\psi(r), \tag{3.2}$$

and

$$G^{(0)}(r,r')=\left.\begin{array}{ll}-\dfrac{1}{2l+1}\dfrac{r^{l+1}}{r'^{l}}, & r<r' \\ =-\dfrac{1}{2l+1}\dfrac{r'^{l+1}}{r}, & r>r'\end{array}\right\}, \tag{3.3}$$

the integral Eq. (2.15) becomes

$$\psi^{(0)}(r)=r^{l+1}+\lambda\int_0^\infty G^{(0)}(r,r')V(r')\psi^{(0)}(r')dr'. \tag{3.4}$$

Lower Bound

Clearly, for all r and r'

$$|G^{(0)}(r,r')|\leq\frac{1}{2l+1}\frac{r^{l+1}}{r'^{l}}. \tag{3.5}$$

Hence the series expansion of (3.4),

$$\psi^{(0)}(r)=r^{l+1}+\lambda\int_0^\infty G^{(0)}(r,r')V(r')(r')^{l+1}dr' +\lambda^2\int_0^\infty G^{(0)}(r,r')V(r')dr' \times\int_0^\infty G^{(0)}(r',r'')V(r'')(r'')^{l+1}dr''+\cdots, \tag{3.6}$$

is dominated by the series

$$S(r)\equiv r^{l+1}\left\{1+\frac{|\lambda|}{2l+1}\int_0^\infty r|V(r)|dr +\left(\frac{|\lambda|}{2l+1}\int_0^\infty r|V(r)|dr\right)^2+\cdots\right\}, \tag{3.7}$$

which converges, provided that

$$|\lambda|\int_0^\infty r|V(r)|dr<2l+1. \tag{3.8}$$

Hence

$$\lambda_{lc}=(2l+1)\bigg/\int_0^\infty r|V(r)|dr. \tag{3.9}$$

TABLE I. Square well, various l. $V(r)=-1,\ r\leq 1=0,\ r>1$.

l	0	1	2	3	Large
$\lambda_{lc}=4l+2$[a]	2.00	6.00	10.00	14.00	$\sim 4l$
λ_c[a]	2.46	9.87	20.16	33.22	$\sim l^2$
$\lambda_{uc}=\frac{1}{2}(2l+1)(2l+5)$[a]	2.50	10.50	22.50	38.50	$\sim 2l^2$

[a] See (3.1) for definitions of λ_{lc}, λ_c, and λ_{uc}.

TABLE II. Various potentials, $l=0$.

	Square well $V(r)=-1,\ r<1$; $=0,\ r\geq 1$	Yukawa $V(r)=e^{-r}/r$	Exponential $V(r)=e^{-r}$
λ_{lc}	2.00	1.00	1.00
λ_c	2.46	1.68	1.45
λ_{uc}	2.50	2.00	1.60

The result (3.9) which was derived by means of the seemingly very crude inequality (3.5) is nevertheless the greatest lower bound of the form $A/\int_0^\infty r|V(r)|dr$ which is valid for all potentials. For with the particular potential

$$V(r)=\delta(r-a) \tag{3.10}$$

(3.4) leads to the series

$$\psi^{(0)}(a)=a^{l+1}+\frac{\lambda a}{2l+1}a^{l+1}+\left(\frac{\lambda a}{2l+1}\right)^2a^{l+1}+\cdots, \tag{3.11}$$

whose radius of convergence coincides with the lower bound given by (3.9).

In Tables I and II, λ_{lc} is compared to λ for several l values in the case of a square-well potential and for $l=0$ for three common potential shapes. It will be noted that for large l, λ_{lc} is of a lower order of magnitude than λ_c and hence not a very useful estimate. A more practical estimate for high l is derived in Sec. 7.

Upper Bounds

When the potential does not change sign, upper bounds for λ_c are readily obtained. In this case the singularities are the real eigenvalues of the homogeneous equation

$$\psi^{(0)}(r)=\lambda\int_0^\infty G^{(0)}(r,r')V(r')\psi^{(0)}(r')dr'. \tag{3.12}$$

It is then well known that the expression

$$\lambda_{uc}=\frac{\left|\int_0^\infty \psi^2(r)V(r)dr\right|}{\int_0^\infty\int_0^\infty \psi(r)V(r)G^{(0)}(r,r')V(r')\psi(r')drdr'} \tag{3.13}$$

represents an upper bound of the magnitude of the numerically smallest eigenvalue, i.e., of the radius of convergence, for an arbitrary function $\psi(r)$.[5]

A very simple, though crude, choice of $\psi(r)$ is the "free" solution r^{l+1}, which gives

$$\lambda_{uc}=\frac{2l+1}{2}\frac{\left|\int_0^\infty dr\, r^{2l+2}V(r)\right|}{\int_0^\infty dr\, r^{2l+2}V(r)\int_0^\infty dr' r' V(r')}. \tag{3.14}$$

[5] This may be verified by expanding ψ in a series of eigenfunctions of (3.12).

As Table I shows, this expression gives good estimates of λ_c for reasonably small l. Table II contains a comparison of λ_{uc} and λ_c for three common potential shapes, all for $l=0$.

4. Behavior of the Radius of Convergence at Low Energies

It is of interest to know how λ_c changes when the energy changes from zero to small positive values.

We consider first the simplest case of a cut-off potential

$$V(r)=0, \quad r\geq a. \tag{4.1}$$

We denote by $\lambda_0(E)$ the singularity of smallest absolute value corresponding to energy $E=k^2$ and begin by calculating $(d\lambda_0/dE)_{E=0}$. Let $\psi(r;E)$ be that solution of the Schroedinger equation

$$\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+E\right)\psi(r;E)=\lambda_0(E)V(r)\psi(r,E) \tag{4.2}$$

which vanishes at $r=0$ and for $r\geq a$ has the form

$$\psi(r,E)=-\frac{k^l}{(2l-1)!!}(kr)n_l(kr), \quad r\geq a, \tag{4.3}$$

where $(2l-1)!!$ is defined in (2.2). The factor in front is chosen so that, for small E,

$$\psi(r,E)=\frac{1}{r^l}+\frac{E}{4l-2}\frac{1}{r^{l-2}}+\cdots, \quad r\geq a. \tag{4.4}$$

For $E=0$, (4.2) becomes

$$\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right)\psi(r;0)=\lambda_0(0)V(r)\psi(r;0). \tag{4.5}$$

We now multiply (4.2) by $\psi(r;0)$, (4.5) by $\psi(r;E)$, subtract, and integrate from 0 to b, where $b\geq a$. This gives

$$\left[\psi(r;0)\frac{d}{dr}\psi(r;E)-\psi(r;E)\frac{d}{dr}\psi(r;0)\right]_0^b +E\int_0^b \psi(r;0)\psi(r;E)dr=[\lambda_0(E)-\lambda_0(0)] \times\int_0^b V(r)\psi(r;0)\psi(r;E)dr. \tag{4.6}$$

When this equation is divided by E and the limit $E\rightarrow 0$ is taken, one obtains, by (4.4),

$$\frac{1}{(2l+1)}\frac{1}{b^{2l-1}}+\int_0^b \psi^2(r;0)dr =\left(\frac{d\lambda_0(E)}{dE}\right)_{E=0}\int_0^b V(r)\psi^2(r;0)dr. \tag{4.7}$$

This equation holds identically in b. We now let $b\rightarrow\infty$ and write $\psi(r;0)\equiv\psi(r)$. This gives for $l=0$

$$l=0:\quad \left(\frac{d\lambda_0(E)}{dE}\right)_{E=0}=-\frac{\lim_{b\rightarrow\infty}\left[b-\int_0^b \psi^2(r)dr\right]}{\int_0^\infty V(r)\psi^2(r)dr} =-\frac{\int_0^\infty\{\psi^2(\infty)-\psi^2(r)\}dr}{\int_0^\infty V(r)\psi^2(r)dr}, \tag{4.8}$$

where we have used the fact that, by (4.4), $\psi(\infty)=1$. For $l>0$, one finds, as $b\rightarrow\infty$,

$$l>0:\quad \left(\frac{d\lambda_0(E)}{dE}\right)_{E=0}=\frac{\int_0^\infty \psi^2(r)dr}{\int_0^\infty V(r)\psi^2(r)dr}. \tag{4.9}$$

Equations (4.8) and (4.9) hold not only for cut-off potentials but also under the weaker assumption that

$$\int_0^\infty r^3|V(r)|dr<\infty. \tag{4.10}$$

The proof will now be outlined. We call

$$F(\lambda;E)\equiv f(\lambda;k)+f(\lambda;-k), \tag{4.11}$$

so that $\lambda_0(E)$ satisfies the equation

$$F(\lambda_0(E);E)=0. \tag{4.12}$$

Under the assumption (4.10), $\partial F/\partial E$ exists for all E including $E=0$, as may be verified from the expansion of F in powers of λ (cf. (5.4) below, for $l=0$). Further, as F is an entire function of λ (cf. reference 2), $\partial F/\partial\lambda$ exists everywhere. Hence by (4.12)

$$\left(\frac{\partial\lambda_0(E)}{\partial E}\right)_{E=0}=-\left(\frac{\partial F/\partial E}{\partial F/\partial\lambda}\right)_{E=0,\lambda=\lambda_0(0)}. \tag{4.13}$$

One now considers the family of cut-off potentials

$$\begin{aligned} V^{(a)}(r)&=V(r), \quad r\leq a\\ &=0 \qquad\ \ r>a, \end{aligned} \tag{4.14}$$

to each of which (4.8) and (4.9) apply and can show without difficulty that as $a\rightarrow\infty$, $\psi^{(a)}/\psi\rightarrow 1$, $\partial F^{(a)}/\partial E\rightarrow\partial F/\partial E$, and $\partial F^{(a)}/\partial\lambda\rightarrow\partial F/\partial\lambda$, so that (4.8) and (4.10) hold also for a $V(r)$ satisfying (4.10).

What we are really interested in is the rate of change of $\lambda_c=|\lambda_0|$ with energy. We consider first the simpler case of $l>0$. By (2.24), the expression (4.9) can be rewritten as

$$\left(\frac{d\lambda_0(E)}{dE}\right)_{E=0} = -\lambda_0(0)\frac{\int_0^\infty \psi^2(r)dr}{\int_0^\infty \left\{\left(\frac{d\psi}{dr}\right)^2 + \frac{l(l+1)}{r^2}\psi^2\right\}dr}; \tag{4.15}$$

we have here used the fact that since λ_0 is real, $\psi(r)$, which is real for $r\rightarrow\infty$, must be real everywhere. Thus we see that the absolute value of λ_0, i.e. λ_c, decreases with increasing energy, so that we may write

$$l>0:\quad \left(\frac{d\lambda_c(E)}{dE}\right)_{E=0} = -\frac{\int_0^\infty \psi^2(r)dr}{\left|\int_0^\infty V(r)\psi^2(r)dr\right|}. \tag{4.16}$$

Illustrations of this behavior are shown in Figs. 1 and 2.

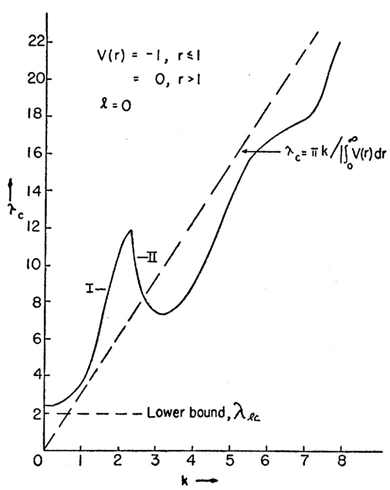

FIG. 1. Radius of convergence, λ_c, for the square well, $l=1$. Note the initial decrease of λ_c.

The fact that, for $l>0$, λ_c decreases as the energy increases from zero is perhaps at first sight surprising. It may, however, be qualitatively understood as follows: At very low energies the centrifugal barrier keeps the particle away from the scattering potential, making it thus effectively weak. As the energy is raised, the particle penetrates further into the region of the potential, which thus becomes effectively stronger, so that λ_c decreases. This decrease will continue until interference effects resulting from oscillations of ψ inside the potential become important, i.e., until

$$ka\sim l+\tfrac{1}{2}, \tag{4.17}$$

where a is some "range" of the potential. This picture is confirmed by Figs. 1 and 2.

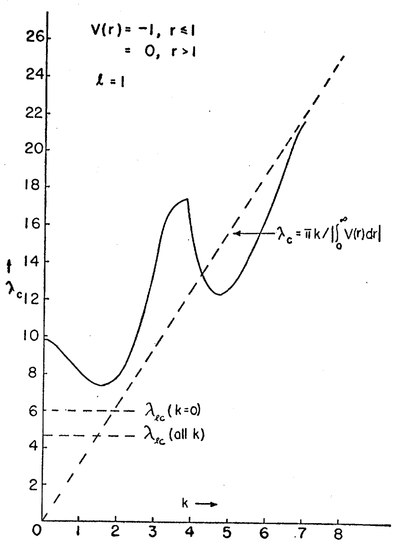

FIG. 2. Radius of convergence, λ_c, for the square well, $l=2$.

For $l=0$, we find, by using (2.24) as before,

$$l=0:\quad \left(\frac{d\lambda_c(E)}{dE}\right)_{E=0} = \frac{\int_0^\infty \{\psi^2(\infty)-\psi^2(r)\}dr}{\left|\int_0^\infty V(r)\psi^2(r)dr\right|}. \tag{4.18}$$

If $V(r)$ does not change sign, then by (2.24),

$$\lambda_0(0)V(r)\leq 0,$$

so that by (2.22) (with $l=0$), $|\psi(r)|$ is a non-decreasing function of r and $\psi^2(\infty)\geq\psi^2(r)$. Under these

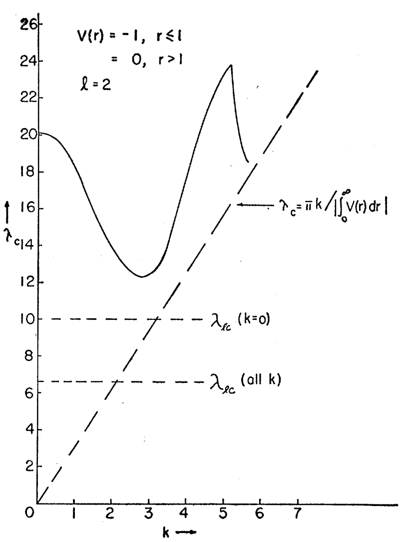

FIG. 3. Radius of convergence, λ_c, for the square well, $l=0$. For $k<2.3$ the singularity of smallest absolute value is positive (attractive potential), for $k>2.3$, negative. Thus the portions I and II represent the absolute values of two different singularities.

circumstances $(d\lambda_c/dE)_{E=0}>0$, which is the intuitively expected result. However, if $V(r)$ changes sign, the numerator of (4.18) may well become negative, so that $(d\lambda_c/dE)_{E=0}<0$. This may be verified by using a suitable δ-function potential of the type (2.21).

5. Asymptotic Behavior of the Radius of Convergence at High Energies

In the limit of high energies regular potentials and those with an r^{-1} singularity near the origin require separate treatment.

Regular Potentials

We shall discuss in detail only the S scattering by a "regular" potential, satisfying the additional condition

$$\int_0^1 |V(r)|dr<\infty. \tag{5.1}$$

We exclude from the start those exceptional cases for which $\int_0^\infty V(r)dr$ vanishes.

By (2.6) and (3.1), $\lambda_0(k)$ is defined as the root of

$$f(\lambda;k)+f(\lambda;-k)=0 \tag{5.2}$$

which has the smallest absolute value. We are therefore led to a study of $f(\lambda;k)$ for large k.

It can be directly verified that the function $f(\lambda;k,r)$ [see (2.1)] is defined by the integral equation

$$f(\lambda;k,r)=e^{-ikr}+\left(\frac{\lambda}{k}\right)\int_r^\infty dr' \sin k(r'-r) \times V(r')f(\lambda;k,r'). \tag{5.3}$$

When this is iterated, one obtains the series

$$f(\lambda;k,r)=\sum_0^\infty \left(\frac{\lambda}{k}\right)^n z_n(k,r), \tag{5.4}$$

where

$$z_0(k,r)=e^{-ikr}$$

$$z_n(k,r)=\int_r^\infty dr' \sin k(r-r')V(r')\int_{r'}^\infty dr''\cdots \times\int_{r^{(n-1)}}^\infty dr^{(n)} \sin k(r^{(n)}-r^{(n-1)}) \times V(r^{(n)})e^{-ikr^{(n)}}, \quad n>0. \tag{5.5}$$

Clearly,

$$|z_n(k,0)|\le\int_0^\infty dr'|V(r')|\int_{r'}^\infty dr''|V(r'')|\cdots \times\int_{r^{(n-1)}}^\infty dr^{(n)}|V(r^{(n)})|=\frac{1}{n!}\left[\int_0^\infty dr'|V(r')|\right]^n. \tag{5.6}$$

Thus the series (5.4) for $f(\lambda;k)=f(\lambda;k,0)$ converges uniformly in the domain

$$|\lambda|\le M|k|, \tag{5.7}$$

where M is an arbitrary positive number.

We note further that, by (5.5),

$$\lim_{k\to\infty} z_n(k,0)=\frac{1}{n!}\left[\frac{W}{2i}\right]^n \tag{5.8}$$

where

$$W\equiv\int_0^\infty V(r)dr. \tag{5.9}$$

Let us now assume that when k exceeds a sufficiently large k_1, λ_0 satisfies the inequality (5.7) for some M. It then follows from (5.8) and the uniform convergence of (5.4) that

$$\lim_{k\to\infty}[f(\lambda_0(k),k)-e^{\lambda_0(k)W/2ik}]=0. \tag{5.10}$$

Adding to this its complex conjugate and recalling that $\lambda_0(k)$ satisfies (5.2), we obtain

$$\lim_{k\to\infty}\cos\frac{\lambda_0(k)W}{2k}=0. \tag{5.11}$$

Hence,

$$k\to\infty:\quad \lambda_c(k)=\pi k\bigg/\left|\int_0^\infty V(r)dr\right|+o(k), \tag{5.12}$$

where

$$\lim_{k\to\infty} o(k)/k=0.$$

It will be noted that (5.11) is consistent with the assumption following (5.9).

A rigorous proof of this assumption may be given as follows: Consider only real positive λ such that

$$|\lambda|\le\frac{2\pi}{\left|\int_0^\infty V(r)dr\right|}\cdot k. \tag{5.13}$$

It can then be shown directly, by virtue of (5.6) and (5.8), that when k exceeds a sufficiently large k_1,

$$f(\lambda;k)+f(\lambda;-k)=\cos\frac{\lambda W}{2k}+R(\lambda;k), \tag{5.14}$$

where R is real and $|R|<\frac{1}{2}$. It follows that for every $k>k_1$, (5.2) has a real positive solution λ_1 satisfying (5.13). *A fortiori*, since by its definition $|\lambda_0|\le|\lambda_1|$, $|\lambda_0|$ also satisfies (5.13), which is of the required form (5.7).

The discussion of $l>0$ is analogous and leads to the same result (5.12).

Illustrations of (5.12) can be seen in Figs. 1–3.

Singular Potentials

We shall now consider potential shapes for which

$$\lim_{r\to 0} rV(r)=1. \tag{5.15}$$

For such potentials it can be shown that, for $l=0$,

$$\lim_{k\to\infty} z_n(k,0) \Big/ \frac{1}{n!}\left(\frac{\log ka}{2i}\right)^n = 1, \quad (5.16)$$

where a is some "range" of the potential.[6] This corresponds to (5.8), when one bears in mind that the range $0\leq r \lesssim k^{-1}$ does not contribute significantly to the expression (5.5) for z_n.

One can then show, exactly as in the case of regular potentials, that

$$k\to\infty: \quad \lambda_c(k) = \frac{\pi k}{\log ka} + o\left(\frac{k}{\log ka}\right). \quad (5.17)$$

Two features of this result are worth noting: λ_c increases less rapidly with energy than for regular potentials [see (5.8)], a fact which is intuitively plausible. Further, we see that at high energies λ_c is governed by the singularity at the origin and independent of the other details of the potential shape, since, for $k\to\infty$, (5.17) becomes independent of the range a.[6]

The results (5.16) and (5.17) also hold for $l>0$.

Instead of giving in detail the rather tedious proof of (5.16), we shall discuss two examples.

We consider first the Schroedinger equation corresponding to a cut-off Coulomb potential,

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2\right)\psi_l(r) = \lambda\frac{1}{r}\psi_l(r), \quad r\leq a$$
$$= 0, \quad r>a. \quad (5.18)$$

At the (real) $\lambda_s(k)$ where $\tan\eta_l=\infty$, the solution of (5.18) has the form

$$\psi_l(r) = r^{l+1}e^{ikr}F\left(i\frac{\lambda_s(k)}{k}+l+1,\, 2l+2,\, kr\right), \quad r\leq a$$
$$= \text{const}\cdot v_l(r). \quad r>a.^7 \quad (5.19)$$

Now, provided that $\lambda_s(k)/k$ remains bounded and r remains fixed, then as $k\to\infty$,

$$\psi_l(r)\to \text{const}\cdot\sin\left(kr - \frac{l\pi}{2} + \gamma_l - \frac{\lambda_s}{2k}\log 2kr\right), \quad r\leq a$$
$$\to \text{const}\cdot\cos\left(kr - \frac{l\pi}{2}\right), \quad r>a, \quad (5.20)$$

where

$$\gamma_l = \arg\Gamma\left(l+1+i\frac{\lambda_s}{2k}\right). \quad (5.21)$$

The condition that the two expressions (5.18) join smoothly at $r=a$ is evidently

$$\lim_{k\to\infty}\left(\gamma_l - \frac{\lambda_s}{2k}\log 2ka + \frac{\pi}{2} + n_s\pi\right) = 0, \quad (5.22)$$

where n_s is some integer depending on s, provided only that

$$\lim_{k\to\infty} \lambda_s/2k = 0.$$

We see that the smallest λ_s corresponds to either $n_s=0$ or $n_s=-1$, so that

$$\lim_{k\to\infty}\left(\frac{\lambda_c}{2k}\log 2ka - \frac{\pi}{2}\right) = 0 \quad (5.23)$$

in agreement with (5.17).

Another example which can easily be worked out is the Hulthén potential

$$V(r) = -\frac{1}{a}\frac{e^{-r/a}}{1-e^{-r/a}}, \quad (5.24)$$

which has an r^{-1} singularity at the origin. For this potential and $l=0$, $f(\lambda;k)$ is explicitly known,[2]

$$f(\lambda;k) = \prod_1^\infty\left(1+\frac{\lambda a}{n(n+2ika)}\right), \quad (5.25)$$

and, by (2.5), λ_0 is the numerically smallest value for which

$$\arg f(\lambda;k) = (2n+1)\frac{\pi}{2}. \quad (5.26)$$

Elementary considerations of the logarithm of (5.25) lead again to the result (5.17).

The behavior of λ_c at high energies for the Hulthen potential is shown in Fig. 4.

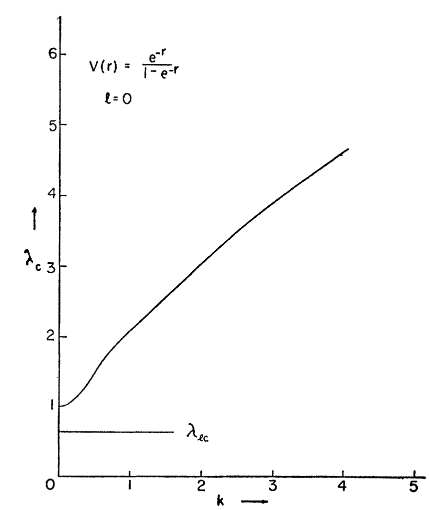

FIG. 4. Radius of convergence, λ_c, for the Hulthen potential, $l=0$.

[6] Since $\lim_{k\to\infty} \log ka_1/\log ka_2 = 1$, for arbitrary a_1 and a_2, Eq. (5.16) is independent of the exact choice of a.

[7] See reference 3, p. 53.

6. Lower and Upper Bounds of the Radius of Convergence Valid at All Energies

Lower Bound

In the preceding three sections we have derived estimates of λ_c in the limit of low and high energies. We shall now establish, for each l, a lower bound of λ_c valid at all energies.

The discussion is very similar to that of Sec. 3. We iterate the integral equation (1.5), which gives the series

$$\begin{aligned}\psi_l(r)&=u_l(r)+\lambda\int_0^\infty dr' G_l(r,r')V(r')u_l(r')\\&+\lambda^2\int_0^\infty dr' G_l(r,r')V(r')\\&\times\int_0^\infty dr'' G_l(r',r'')V(r'')u_l(r'')+\cdots\\&=(kr)^{\frac{1}{2}}\left[\frac{u_l(r)}{(kr)^{\frac{1}{2}}}+\lambda\int_0^\infty dr'\frac{G_l(r,r')}{(rr')^{\frac{1}{2}}}r'V(r')\frac{u_l(r')}{(kr')^{\frac{1}{2}}}\right.\\&+\lambda^2\int_0^\infty dr'\frac{G_l(r,r')}{(rr')^{\frac{1}{2}}}r'V(r')\int_0^\infty dr''\\&\left.\times\frac{G_l(r',r'')}{(r'r'')^{\frac{1}{2}}}r''V(r'')\frac{u_l(r'')}{(kr'')^{\frac{1}{2}}}+\cdots\right].\end{aligned}\tag{6.1}$$

Now for every l, $|u_l(r)/(kr)^{\frac{1}{2}}|$ and $|G_l(r,r')/(rr')^{\frac{1}{2}}|$ are bounded for all r, r', and k [see Eqs. (2.8)–(2.11) and (2.14)]. Let us call

$$m_l\equiv|u_l(r)/(kr)^{\frac{1}{2}}|_{\max}\tag{6.2}$$

$$\frac{1}{t_l}\equiv|G_l(r,r')/(rr')^{\frac{1}{2}}|_{\max}.\tag{6.3}$$

Then clearly the series (6.1) is dominated by

$$(kr)^{\frac{1}{2}}m_l\left[1+\frac{|\lambda|\int_0^\infty r|V(r)|dr}{t_l}+\left[\frac{\int_0^\infty r|V(r)|dr}{t_l}\right]^2+\cdots\right]\tag{6.4}$$

so that the following is a lower bound of λ_c:

$$\lambda_{lc}=t_l\Big/\int_0^\infty r|V(r)|dr.\tag{6.5}$$

The constants t_l can be found from tables of spherical Bessel functions[8] and are listed in Table III for the first few values of l. To obtain the last entry in this table the asymptotic form of the Bessel functions was used. This lower bound is indicated in Figs. 1–4.

TABLE III. Lower bound of λ_c valid at all energies. $\lambda_{lc}=t_l/\int_0^\infty r|V(r)|dr$.

l	0	1	2	3	Large
t_l	1.000	2.344	3.339	4.198	$1.036(2l+1)^{\frac{2}{3}}$

(For $0\leq l\leq 9$, $|G_l(r,r')/(rr')^{\frac{1}{2}}|$ attains its largest value for $r=r'$. In that case it can be shown as in Sec. 3 [see Eqs. (3.10) and (3.11)] that the estimates of Table III are the best estimates of the form $A/\int_0^\infty r|V|dr$. However, for $l\geq 10$, $|G_l(r,r')/(rr')^{\frac{1}{2}}|$ attains its largest value for $r\neq r'$. In this case the lower bounds of Table III are no longer attained with the δ-function potential (3.10) and we do not know whether or not they are attained with any potential. However, (6.5) evidently cannot be improved above the λ_c of the δ-function potential (3.10) and the latter lies never more than 11.2 percent above (6.5). [For (3.10), as the energy varies from 0 to ∞, the smallest λ_c has the following asymptotic behavior: $\lim_{l\to\infty}\lambda_c/1.157(2l+1)^{\frac{2}{3}}=1$.]

Upper Bounds

For potentials of fixed sign, an upper bound can be obtained as at the end of Sec. 3:

$$\begin{aligned}\lambda_{uc}&=\left|\frac{\int_0^\infty\psi^2(r)V(r)dr}{\int_0^\infty\int_0^\infty\psi(r)V(r)G(r,r')V(r')\psi(r')drdr'}\right|\\&=\frac{k}{2}\left|\frac{\int_0^\infty\psi^2(r)V(r)dr}{\int_0^\infty dr\psi(r)u_l(r)V(r)\int_r^\infty dr'\psi(r')v_l(r')V(r')}\right|,\end{aligned}\tag{6.6}$$

where $\psi(r)$ is an arbitrary function. Unfortunately this is a rather complicated expression and hence not very useful.

7. The Radius of Convergence at Arbitrary Fixed Energy in the Limit of Large l

So far we have derived various estimates for the radius of convergence for any given l. However, most of these [(3.9), (3.14), (5.12), (5.17); see also Table I], if applied to scattering at a given energy, become increasingly poor as l becomes large. It is therefore fortunate that the foregoing estimates can be supple-

[8] *Tables of Spherical Bessel Functions* (Columbia University Press, New York, 1947).

mented by an asymptotic expression for λ_c which holds for any given E in the limit as $l\to\infty$.

We restrict ourselves in the present section to $l\geq 1$.

First we consider the case of $E=0$, in which λ_0 is real (see Sec. 2). The wave function corresponding to λ_0 has the following behavior at 0 and ∞:

$$r\to 0:\quad \psi(r)\sim r^{l+1}, \tag{7.1}$$

$$r\to\infty:\quad \psi(r)\sim r^{-l}, \tag{7.2}$$

Thus λ_0 is the numerically smallest value of λ for which a square integrable wave function with binding energy zero exists.

It is convenient to write the wave equation as

$$\left.\begin{aligned} \frac{d^2\psi}{dr^2}&=U(r)\psi(r),\\ U(r)&=\frac{l(l+1)}{r^2}+\lambda V(r). \end{aligned}\right\} \tag{7.3}$$

Clearly, since the logarithmic derivative of $\psi(r)$ changes sign between $r=0$ and $r=\infty$, $U(r)$ must be negative for some r. Now by (1.1) the quantity $|r^2V(r)|$ has a finite maximum value, and hence λ_c must satisfy the inequality

$$\lambda_c>\frac{l(l+1)}{|r^2V(r)|_{\max}}. \tag{7.4}$$

This is rigorously correct for all l.

To avoid uninteresting complications, let us now assume that $|r^2V(r)|$ reaches its maximum value at only one point, say r_0, and let us choose the sign of $V(r)$ such that $V(r_0)<0$ [the sign of $V(r)$ does not, of course, affect λ_c]. If we call

$$\Lambda\equiv l(l+1)/|r^2V(r)|_{\max}, \tag{7.5}$$

we shall next show that

$$\lim_{l\to\infty}\frac{\lambda_c}{\Lambda}=1. \tag{7.6}$$

Consider (7.3) with

$$\left.\begin{aligned} &\lambda=\Lambda(1+\epsilon),\quad \epsilon>0,\\ &U(r)=l(l+1)\left\{\frac{1}{r^2}\left(1-\frac{r^2V(r)}{r_0^2V(r_0)}\right)-\epsilon\frac{V(r)}{r_0^2V(r_0)}\right\}. \end{aligned}\right\} \tag{7.7}$$

Now keep ϵ fixed and let $l\to\infty$. The shape of $U(r)$ does not change, but its magnitude increases. In the neighborhood of r_0, U is negative [see (7.7)] and as $l\to\infty$, every solution of (6.3) acquires an increasing number of nodes in this neighborhood. Since, however, the solution of (7.3) corresponding to λ_0 has no nodes, it follows that

$$\lim_{l\to\infty}\frac{\lambda_0}{\Lambda(1+\epsilon)}\leq 1,\quad \text{all}\quad \epsilon>0. \tag{7.8}$$

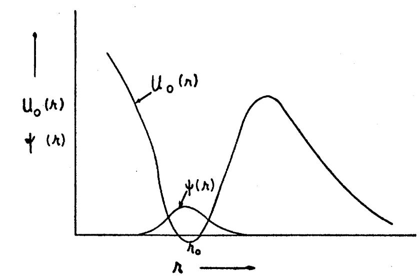

FIG. 5. Schematic diagram of $U_0(r)$ and $\psi(r)$ (Eqs. (7.9), (7.1), (7.2)) for a large value of l. Near r_0, $U_0(r)$ resembles a parabola and $\psi(r)$ an oscillator ground-state wave function.

When this is combined with (7.4), one obtains the desired result (7.6).

We can now easily obtain a refinement of (7.6). Let

$$U_0(r)\equiv\frac{l(l+1)}{r^2}+\lambda_0(l)V(r), \tag{7.9}$$

where $\lambda_0(l)$ is the smallest singularity corresponding to angular momentum l. Then by (7.6)

$$\lim_{l\to\infty}\frac{U_0(r)}{l(l+1)}=\frac{1}{r^2}\left(1-\frac{r^2V(r)}{r_0^2V(r_0)}\right), \tag{7.10}$$

while for any finite l

$$\frac{U_0(r_0)}{l(l+1)}<0 \tag{7.11}$$

(see Fig. 5). Thus as $l\to\infty$, the neighborhood of r_0 in which $U_0(r)<0$, say $r_1(l)<r<r_2(l)$, shrinks to zero. Since $\psi(r)$ increases with r for $r<r_1(l)$ and decreases for $r>r_2(l)$, it is now clear that λ_0 is the smallest λ which can "trap" the particle with zero binding energy near r_0. We therefore write (7.3), in the vicinity of r_0, as

$$\frac{d^2\psi}{dr^2}=\{U_0(r_0)+U_0'(r_0)(r-r_0)+\tfrac{1}{2}U_0''(r_0)(r-r_0)^2\}\psi, \tag{7.12}$$

where $U_0(r)$ is given by (7.9), and determine λ_0 from the condition that (7.12) has a solution, without nodes, which vanishes as $r-r_0\to\pm\infty$. [In (7.12) the continuity of $U''(r)$ near r_0 is presupposed; in cases where U cannot be so expanded—e.g. the square well—a separate but analogous discussion is necessary; see (7.15) below.] This solution is, of course, a harmonic oscillator ground-state wave function, and an elementary calculation gives

$$\lambda_c=\frac{1}{|r_0^2V(r_0)|}\left\{l(l+1)+\left(3-\frac{r_0^2V''(r_0)}{2V(r_0)}\right)^{\frac{1}{2}}[l(l+1)]^{\frac{1}{2}}\right\}+o[l(l+1)]^{\frac{1}{2}}. \tag{7.13}$$

TABLE IV. λ_c calculated from the asymptotic expressions (7.13) and (7.14).

l	Square well $V(r)=1,\ r\leq 1$; $=0,\ r>1$ Exact	Approx	Gaussian $V(r)=\exp(-r^2)$ Exact	Approx	Exponential $V(r)=e^{-r}$ Exact	Approx	Yukawa $V(r)=e^{-r}/r$ Exact	Approx
0	2.5		2.7		1.5		1.7	
1	9.9	7.9	12.3	10.9	7.2	6.3	9.4	9.3
2	20.1	18.3		25.7	14.6	15.6		23.0

A rigorous justification of this procedure presents no difficulty. We remark here only that as $l\rightarrow\infty$, the spread of the oscillator function behaves like $l^{-\frac{1}{2}}$, so that the omitted terms in the expansion in (7.12) become negligible.

In Table IV we compare, for a number of common potentials, the results obtained from the asymptotic expression (7.13) with the exact results. (We have also included the exact results for $l=0$.) In the case of the square well where $V'(r_0)$ and $V''(r_0)$ do not exist, we have used the separately derived expression

$$\lambda_c=\frac{1}{a^2|V_0|}\{l(l+1)+3.71151[l(l+1)]^{\frac{2}{3}}\}+o[l(l+1)]^{\frac{2}{3}},\quad (7.14)$$

where a is the range and V_0 is the depth. It will be seen that even for $l=1$, the asymptotic expression (7.13) gives remarkably good results.

When the energy E is positive, an entirely analogous discussion shows that the expression (7.13) still holds, provided that $V(r)$ does not change sign[9] and (4.10) is satisfied. It can, in fact, be shown that in this case

$$\lim_{l\rightarrow\infty}(\lambda_c(E)-\lambda_c(o))=-\frac{E}{|V(r_0)|},\qquad (7.15)$$

which is consistent with (4.16), when one recalls that as $l\rightarrow\infty$, $\psi(r)$ approaches an increasingly narrow oscillator function centered at $r=r_0$.[10] Thus the energy-dependent term is $o[l(l+1)]^{\frac{1}{2}}$.

8. Summary of Estimates of the Radius of Convergence λ_c

Most of the results of Secs. 3–7 are here collected for convenient reference. The following notation will be used:

- λ_c denotes the radius of convergence of the power series expansion of the solution of (1.2) with boundary conditions (1.3) and (1.4), and in particular of $\tan\eta_l$.
- λ_{lc} and λ_{uc} denote a lower and upper bound of λ_c, respectively.

[9] This assumption enables one to use the fact that λ_c is real. It is conjectured, however, that (7.13) and (7.15) hold independently of this assumption, but no proof has been given.

[10] (7.15) holds even when $V'(r)$ and $V''(r)$ are not continuous near r_0, e.g., for the square well.

$E\equiv k^2$ [see Eq. (1.2)].

$o(x)$ denotes some $f(x)$ with the property

$$\lim_{x\rightarrow\infty}f(x)/x=0.$$

Numbers in [ ] indicate sections where details are given.

Vanishing Energy

$$\lambda_{lc}=(2l+1)\Big/\int_0^\infty r|V(r)|dr,\qquad [3]$$

$$\lambda_{lc}^{(\prime)}=l(l+1)/|r^2V(r)|_{\max}.\qquad [7]$$

$$\lambda_{uc}=\frac{2l+1}{2}\left|\int_0^\infty dr\, r^{2l+2}V(r)\Big/\int_0^\infty dr\, r^{2l+2}V(r)\times\int_r^\infty dr'r'V(r')\right|,\qquad [3]$$

for $V(r)$ of fixed sign.

Behavior at Low Energies

$l=0$: $(d\lambda_c(E)/dE)_{E=0}$

$$=\int_0^\infty\{\psi^2(\infty)-\psi^2(r)\}dr\Big/\left|\int_0^\infty V(r)\psi^2(r)dr\right|,\qquad [4]$$

$l\geqslant 1$: $(d\lambda_c(E)/dE)_{E=0}$

$$=-\left|\int_0^\infty\psi^2(r)dr\Big/\int_0^\infty V(r)\psi^2(r)dr\right|.\qquad [4]$$

In these expressions $\psi(r)$ is the wave function corresponding to $E=0$ and the numerically smallest λ for which $\psi(0)=0$ and as $r\rightarrow\infty$, $\psi(r)\sim r^{-l}$.

High Energies

"Regular" potential

$$\left(\int_0^1|V(r)|dr<\infty\right),\quad \text{all } l:$$

$$\lambda_c(k)=\pi k\Big/\left|\int_0^\infty V(r)dr\right|+o(k).\qquad [5]$$

"Singular" potential

$$(\lim_{r\rightarrow 0}rV(r)=1),\quad \text{all } l:$$

$$\lambda_c(k)=\pi k/\log ka+o(k/\log ka),\qquad [5]$$

where a is some "range."

Arbitrary Energy

$$\lambda_{lc}=t_l\Big/\int_0^\infty r|V(r)|dr.\qquad [6]$$

See Table III for values of t_l.

High Angular Momenta

Vanishing energy:

$$\lambda_c = \frac{l(l+1)}{|r^2V(r)|_{\max}} + o[l(l+1)], \qquad [7]$$

where $o[l(l+1)]$ is positive.

$$\lambda_c = \frac{1}{r_0^2|V(r_0)|}\left\{ l(l+1) + \left(3 - \frac{r_0^2V''(r_0)}{2V(r_0)}\right)[l(l+1)]^{\frac{1}{2}}\right\} + o([l(l+1)]^{\frac{1}{2}}), \qquad [7]$$

provided that $V(r)$ is "smooth" at the point $r=r_0$ where $r^2|V(r)|$ has its maximum value.

Arbitrary fixed energy:

If $V(r)$ does not change sign and satisfies (4.10), the last two estimates hold for arbitrary E. Further,

$$\lim_{l\to\infty}(\lambda_c(E) - \lambda_c(o)) = -E/V(r_0). \qquad [7]$$

If $V(r)$ changes sign it is believed (but not proved) that the same estimates still hold.

9. Truncation Error

From a practical standpoint it is important to know, besides λ_c, an estimate for the error in $\tan\eta_l$, due to breaking off the Born expansion after the nth term. On the basis of the preceding sections some estimates of this truncation error can be directly obtained.

Arbitrary Energy

Using the expansion (6.1) with the definition (2.14) for G, we have for large r

$$\psi_l(r) = u_l(r) - \tan\eta_l v_l(r), \qquad (9.1)$$

where

$$\tan\eta_l = -\frac{\lambda}{k}\int_0^\infty dr' u_l^2(r')V(r') - \frac{\lambda^2}{k}\int_0^\infty dr' u_l(r')V(r') \times \int_0^\infty dr'' G_l(r',r'')V(r'')u_l(r'') - \cdots. \qquad (9.2)$$

This expansion, when broken off after the nth term, will be denoted by $(\tan\eta_l)^{(n)}$. To estimate the truncation error, we introduce m_l and t_l defined in (6.2) and (6.3) and obtain

$$|\tan\eta_l - (\tan\eta_l)^{(n)}| \le \frac{|\lambda|^{n+1}m_l^2}{t_l^n}\int_0^\infty dr'r'|V(r')|\cdots \times\int_0^\infty dr^{(n+1)}r^{(n+1)}|V(r^{(n+1)})|dr^{(n+1)} + \cdots$$

$$= \frac{|\lambda|^{n+1}\left[\int_0^\infty r|V(r)|dr\right]^{n+1} m_l^2}{t_l^n} \times \frac{1}{1-|\lambda|\int_0^\infty r|V(r)|dr/t_l}, \qquad (9.3)$$

provided the last denominator is positive. Values of t_l and m_l^2 are listed in Tables III and V.

TABLE V. $m_l = |x^{\frac{1}{2}}j_l(x)|_{\max}$.

l	0	1	2	3	Large
m_l^2	0.725	0.434	0.329	0.271	$1.136(2l+1)^{-\frac{2}{3}}$

Vanishing Energy

For simplicity we restrict ourselves here to potentials such that

$$\int_0^\infty r^{2l+2}|V(r)|dr < \infty. \qquad (9.4)$$

From (9.2) we have in the limit $k\to 0$

$$\lim_{k\to 0}\frac{\tan\eta_l}{k^{2l+1}} = \frac{1}{[(2l+1)!!]^2}\left[-\lambda\int_0^\infty dr'(r')^{2l+2}V(r') - \lambda^2\int_0^\infty dr'(r')^{l+1}V(r') \times\int_0^\infty dr'' G_l^{(0)}(r',r'')V(r'')(r'')^{l+1} - \cdots\right], \qquad (9.5)$$

where $G_l^{(0)}$ is defined in (3.3). Hence, using the estimate (3.5) for $G_l^{(0)}$ and defining

$$T_l = \lim_{k\to 0}\frac{\tan\eta_l}{k^{2l+1}}, \qquad (9.6)$$

we obtain directly

$$|T_l - (T_l)^{(n)}| \le \frac{1}{[(2l+1)!!]^2} \times \frac{|\lambda|^{n+1}\left[\int_0^\infty r|V(r)|dr\right]^n \int_0^\infty r^{2l+2}|V(r)|dr}{(2l+1)^n} \times\left[1 - \frac{1}{1-|\lambda|\int_0^\infty r|V(r)|dr/(2l+1)}\right], \qquad (9.7)$$

provided the last denominator is positive.

High Energies

We begin with a consideration of "regular" potentials [see (5.1)]. Let $f^{(n)}(\lambda;k)$ be the term of order λ^n in the expansion of $f(\lambda;k)$. We have seen in Sec. 5 that for large k it has the following asymptotic behavior [see (5.4), (5.8)]:

$$k\to\infty:\ f^{(n)}(\lambda;k)=\frac{\lambda^n}{n!}\left[\left(\frac{\int_0^\infty V(r)dr}{2ik}\right)^n+o(k^{-n})\right].\quad(9.8)$$

This expression holds for all values of l. If we now denote by $\tau_l^{(n)}$ the nth term of the Born expansion (9.2) of $\tan\eta_l$, we find, on combining (9.8) and (2.5), that

$$k\to\infty:\ \tau_l^{(n)}=\lambda^n\left[-c^{(n)}\left(\frac{\int_0^\infty V(r)dr}{2k}\right)^n+o(k^{-n})\right],\quad(9.9)$$

where $c^{(n)}$ is the nth coefficient in the expansion of $\tan x$:

$$c^{(1)}=1,\quad c^{(3)}=\tfrac{1}{3},\ \cdots;\ c^{(2n)}=0.\quad(9.10)$$

We see that, for large k, the odd terms of the Born expansion of $\tan\eta_l$ behave like k^{-n}, while the even ones decrease more rapidly. This is illustrated by the Born expansion corresponding to the exponential potential λe^{-r}, whose first two terms are, for $l=0$,

$$\tan\eta_0=-\left(\frac{\lambda}{2k}\right)\left[\frac{4k^2}{4k^2+1}\right]-\left(\frac{\lambda}{2k}\right)^2\left[\frac{4k^5-5k^3}{(4k^2+1)^2(k^2+1)}\right]-\cdots.\quad(9.11)$$

Clearly, the truncation error of $(\tan\eta_l)^{(n)}$ is just $\tau_l^{(n+1)}$, to within terms of $o(k^{-(n+1)})$. A somewhat more useful expression than (9.9) for high l is

$$\tau_l^{(n)}=\lambda^n\left[-c^{(n)}\left(\frac{\int_{l/k}^\infty V(r)dr}{2k}\right)^n+o(k^{-n})\right],\quad(9.12)$$

which takes into account the fact that the particle "sees" only that part of the potential for which $r\gtrsim l/k$, or, mathematically speaking, that $u_l(r)\approx 0$ for $r\lesssim l/k$. Of course (9.9) and (9.12) are quite equivalent, since the difference between their leading terms is clearly $o(k^{-n})$.

In the case of singular potentials [Eq. (5.15)], entirely similar considerations, based on (5.16), give the following result:

$$k\to\infty:\ \tau_l^{(n)}=\lambda^n\left\{-c^{(n)}\left(\frac{\log ka}{2k}\right)^n+o\left[\left(\frac{\log ka}{2k}\right)^n\right]\right\}.\quad(9.13)$$

Equation (9.13) is analogous to Eq. (9.12), so that we can write down the following expression valid for regular as well as singular potentials:

$$k\to\infty:\ \tan\eta_l-(\tan\eta_l)^{(n)}=\lambda^{n+1}\left\{-c^{n+1}\left(\frac{\int_{\bar{l}/k}^\infty V(r)dr}{2k}\right)^{n+1}+o\left[\left(\frac{\int_{\bar{l}/k}^\infty V(r)dr}{2k}\right)^{n+1}\right]\right\},\quad(9.14)$$

where $\bar{l}=l$, except when $l=0$ and the potential is singular, in which case $\bar{l}=1$.

For odd n the leading term in (9.14) vanishes, since $c^{(n+1)}=0$, so that (9.14) indicates only the general behavior of the truncation error with increasing k. In particular, (9.14) does not include an asymptotic expression for the truncation error of the so-called first Born approximation. However, for potentials which are bounded and have a bounded first derivative such an expression can easily be obtained. From (9.2) and (2.14) we have

$$\tau_l^{(2)}=-\frac{\lambda^2}{k^2}\left[\int_0^\infty dr u_l(r)v_l(r)V(r)\times\int_0^r dr' u_l^2(r)V(r')+\int_0^\infty dr u_l^2(r)V(r)\times\int_r^\infty dr' u_l(r')v_l(r')V(r')\right].\quad(9.15)$$

By suitable integrations by parts the leading terms, for $k\to\infty$, can be isolated and give

$$\tau_l^{(2)}=-\left(\frac{\lambda}{2k}\right)^2\left[\frac{1}{2k}\int_{l/k}^\infty V^2(r)dr+o(k^{-1})\right].\quad(9.16)$$

[See the remarks following (9.12) regarding the lower limit of integration.]

Combining (9.14) and (9.16) we have

$$\tan\eta_l-(\tan\eta_l)^{(1)}=\frac{1}{8k^3}\left[-\lambda^2\int_{l/k}^\infty V^2(r)dr-\frac{\lambda^3}{3}\left(\int_{l/k}^\infty V(r)dr\right)^3\right]+o(k^{-3})\quad(9.17)$$

for bounded potentials with bounded first derivative. This result is illustrated for the exponential potential and $l=0$ in Table VI.

PART II. BORN EXPANSION OF φ_l AND OF $S_l \equiv e^{2i\eta_l}$

10. General Relationships

When the scattering potential has spherical symmetry, the three-dimensional solution $\Psi(\mathbf{r})$ of (1.6) can be broken up into partial waves,

$$\Psi(\mathbf{r})=\sum_{l=0}^{\infty}(2l+1)i^{-l}\frac{1}{kr}\varphi_l(r)P_l(\cos\theta), \qquad (10.1)$$

where θ is the angle between $\mathbf{r}$ and the direction of incidence and P_l is the Legendre polynomial of order l. $\varphi_l(r)$ satisfies the same wave equation (1.2) and boundary condition (1.3) as $\psi_l(r)$, but for large r,

$$r\to\infty:\quad \varphi_l(r)\to\frac{i}{2}\left(\exp\left[-i\left(kr-\frac{l\pi}{2}\right)\right]-S_l\exp\left[i\left(kr-\frac{l\pi}{2}\right)\right]\right)=\sin\left(kr-\frac{l\pi}{2}\right)+\frac{S_l-1}{2i}\exp\left[i\left(kr-\frac{l\pi}{2}\right)\right]. \qquad (10.2)$$

The first form of (10.2) shows that S_l is the amplitude of the outgoing wave corresponding to unit incoming wave and this, by definition, is the (diagonal) element of the scattering matrix corresponding to angular momentum l.

Since φ_l and ψ_l both satisfy (1.2) and (1.3), their ratio must be independent of r. Comparing them for large r [Eqs. (1.4) and (10.2)] we find that for each l

$$S=e^{2i\eta}, \qquad (10.3)$$

$$\varphi(r)=\cos\eta e^{i\eta}\psi(r). \qquad (10.4)$$

The connection with $f(\lambda;k,r)$ of Eq. (2.1) also follows from the asymptotic forms:

$$S=f(\lambda;k)/f(\lambda;-k), \qquad (10.5)$$

$$\varphi(r)=\frac{i}{2f(\lambda;-k)}[f(\lambda;-k)f(\lambda;k,r)-f(\lambda;k)f(\lambda;-k,r)]. \qquad (10.6)$$

Since $f_l(\lambda;k,r)$ is an entire function of λ (see reference 2), $\varphi_l(r)$ and S_l are meromorphic functions of λ, whose poles are determined by the equation

$$f_l(\lambda;-k)=0. \qquad (10.7)$$

It is shown in reference 2 that for $k\neq 0$, these poles have a nonvanishing imaginary part.

The differential equation (1.1) and boundary conditions (1.2) and (10.2) can be combined in the following integral equation for φ_l:

$$\varphi_l(r)=u_l(r)+\lambda\int_0^\infty K_l(r,r')V(r')\varphi_l(r')dr', \qquad (10.8)$$

where

$$K_l(r,r')=\left.\begin{aligned}&\frac{1}{k}u_l(r)w_l(r'),\quad r\leq r'\\&=\frac{1}{k}w_l(r)u_l(r'),\quad r\geq r'\end{aligned}\right\}, \qquad (10.9)$$

and

$$u_l(r)=(kr)j_l(kr), \qquad (10.10)$$

$$w_l(r)=(kr)[n_l(kr)-ij_l(kr)]. \qquad (10.11)$$

[See (2.10), (2.11) for definitions of j_l and n_l.] On letting $r\to\infty$ in (10.8) and comparing with the second form of (10.2), one finds

$$S_l=1+\frac{2\lambda}{ik}\int_0^\infty u_l(r)V(r)\varphi_l(r)dr. \qquad (10.12)$$

The Born expansion of φ_l is obtained by iterating (10.8), and that of S_l by substituting the series of φ_l into (10.12). Both series converge for $|\lambda|<\lambda_c'$ where λ_c' is the magnitude of that solution of (10.7) which has the smallest absolute value.

11. Estimates of the Radius of Convergence, λ_c'

We consider now the radius of convergence, λ_c', of the Born expansion of φ_l, Eq. (10.1), and in particular of the scattering matrix element S_l, Eqs. (10.2), (10.3), (10.5). Much of the discussion is very similar to that of Secs. 2–7 and will therefore be only briefly sketched. Our results for λ_c', particularly in the limit of high energies, are less complete than those for λ_c.

Vanishing Energy

In the limit of vanishing energy we have, by (2.3) and (10.6),

$$\lim_{k\to 0}\frac{\varphi(r)}{\psi(r)}=\lim_{k\to 0}\frac{f(\lambda;k)+f(\lambda;-k)}{2f(\lambda;-k)}=1, \qquad (11.1)$$

so that the power series expansions of φ and ψ have the same radius of convergence. Thus for each l,

$$k=0:\quad \lambda_c'=\lambda_c \qquad (11.2)$$

[see (3.9), (7.4), and (3.14)].

TABLE VI. Truncation error of the first Born approximation.

$\lambda V(r)=-2e^{-r};\ l=0.$

k	$\tan\eta_0$	$(\tan\eta_0)^{(1)}$	$\tan\eta_0-(\tan\eta_0)^{(1)}$	$\frac{1}{12k^3}$ [a]
2	0.489122	0.470588	0.018534	0.010417
4	0.247844	0.246154	0.001690	0.001303
6	0.165960	0.165517	0.000443	0.000386

[a] This is the leading term of the right-hand side of (9.17).

Low Energies

For $l=0$ no simple general result has been obtained for the behavior of the radius of convergence λ_c' at low energies. The following examples, in which the behavior of λ_c and λ_c' are compared, may be of interest:

Square well:

$$V(r)=-1, \quad r\leq a \qquad = 0, \quad r>a, \tag{11.3}$$

$$\left.\begin{aligned}\lambda_c&=\frac{\pi^2}{4a^2}+k^2+\cdots\\ \lambda_c'&=\frac{\pi^2}{4a^2}+0.216k^2+\cdots\end{aligned}\right\}. \tag{11.4}$$

Hulthén potential [Eq. (5.24)]:

$$\left.\begin{aligned}\lambda_c&=a\left(\frac{1}{a^2}+3k^2+\cdots\right)\\ \lambda_c'&=a\left(\frac{1}{a^2}+2k^2+\cdots\right)\end{aligned}\right\}. \tag{11.5}$$

For $l\geq 1$, considerations similar to those of Sec. 4 show that (4.16) holds also for λ_c'.

Furthermore, it can be shown from the expansion of $f(\lambda;k)$ [see (5.4)] that for all l

$$ka\ll l+\tfrac{1}{2}:\quad \lambda_c'=\lambda_c+0[(ka)^{4l+2}], \tag{11.6}$$

where a is a "range," provided that the potential satisfies (9.4). This is illustrated by the δ-function potential (3.10) for which

$$\lambda_c=\frac{ka}{|u(ka)v(ka)|},$$

$$\lambda_c'=\frac{ka}{|u(ka)w(ka)|}\approx\frac{ka}{|u(ka)v(ka)|}\times\left(1-\frac{(ka)^{4l+2}}{2[(2l+1)!!(2l-1)!!]^2}\right),(ka)\ll l+\tfrac{1}{2}. \tag{11.7}$$

See also Fig. 6, showing the close agreement of λ_c and λ_c' for $(ka)\ll l+\frac{1}{2}$.

High Energies

Regular potentials [see (5.1)]:

In this case we shall establish the result that for all l

$$\lim_{k\to\infty}\lambda_c'/k=\infty, \tag{11.8}$$

showing that, for large k, λ_c' increases more rapidly than λ_c [see Eq. (5.12)].

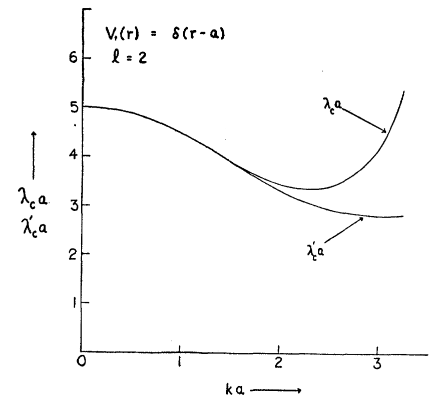

FIG. 6. The radii of convergence for $\tan\eta(\lambda_c)$ and for $e^{2i\eta}(\lambda_c')$, for the potential $\delta(r-a)$ and $l=2$. Note the close agreement for $ka\ll 2+\frac{1}{2}$.

We shall treat, for simplicity, only the case of $l=0$. Let M be an arbitrary positive number. Then by virtue of (5.10) we can choose a k_1 such that for all $k\geq k_1$, and for all λ satisfying the inequality

$$|\lambda|\leq Mk, \tag{11.9}$$

$$\left|f(\lambda;-k)-\exp\left[-\lambda\int_0^\infty V(r)dr/2ik\right]\right| <\tfrac{1}{2}\exp\left[-M\int_0^\infty V(r)dr/2\right]. \tag{11.10}$$

If now, for $k\geq k_1$, there exists a root of (10.7) satisfying (11.9), the left-hand side of (11.10) would exceed the right-hand side by at least a factor 2, leading to a contradiction. Thus for all $k\geq k_1$, $\lambda_c'>Mk$, which proves Eq. (11.8).

A similar proof can be given for higher l.

A special study of S scattering by the square well (11.2) gave the result

$$\lambda_c'=(k/a)\log ka+o\left(\frac{k}{a}\log ka\right). \tag{11.11}$$

Singular potentials [see (5.15)]: Here we have the result, for all l,

$$\lim_{k\to\infty}\lambda_c'\Big/\left(\frac{k}{\log ka}\right)=\infty, \tag{11.12}$$

which can be established as above by using (5.16) in place of (5.8).

The Hulthén potential, (5.24), gives the rigorous expression,[2]

$$\lambda_c'=2k\left(1+\frac{1}{4(ka)^2}\right)^{\frac{1}{2}}, \tag{11.13}$$

as is evident from (5.25) and (10.7).

Arbitrary Energy

Proceeding exactly as in Sec. 6, we find the following lower bound:

$$\lambda_{lc}' = s_l \Big/ \int_0^\infty r|V(r)|dr, \tag{11.14}$$

where

$$\frac{1}{s_l} = |K_l(r,r')/(rr')^{\frac{1}{2}}|_{\max}. \tag{11.15}$$

Numerical values of s_l are given in Table VII.

The lower bound (11.14) is attained, for a certain k, with the δ-function potential (3.10), for all values of l. This follows from the fact that for all l, $|K_l(r,r')/(rr')^{\frac{1}{2}}|$ attains its maximum value for $r=r'$ (see the remarks at the end of Sec. 6).

Large l at Fixed Energy

At zero energy $\lambda_c'=\lambda_c$, so that (7.13) applies. For potentials of fixed sign which obey (4.10) we also have (7.15). [See (11.6) and Fig. 6].

Summary of Estimates of λ_c'

For convenience we summarize below the results of this section (see Sec. 8). In the following λ_c' denotes the radius of convergence of the Born expansion of the solution of (1.2) with the boundary conditions (1.3) and (10.2). The other symbols are explained at the beginning of Sec. 8.

Vanishing Energy

$$\lambda_c'=\lambda_c.$$

See Sec. 8 for estimates.

Behavior at Low Energies

$$l\geq 1: \quad \left(\frac{d\lambda_c'}{dE}\right)_{E=0} = \left(\frac{d\lambda_c}{dE}\right)_{E=0}.$$

See Sec. 8. Furthermore, for all l, and $ka \ll l+\frac{1}{2}$,

$$\lambda_c'=\lambda_c+0[(ka)^{4l+2}],$$

where a is some "range," provided $\int|V(r)|r^{2l+2}dr<\infty$.

High Energies

Regular potentials, all l:

$$\lim_{k\to\infty} \lambda_c'/k=\infty.$$

TABLE VII. Lower bound of λ_c' valid at all energies. $\lambda_{lc}'=s_l/\int_0^\infty r|V(r)|dr$.

l	0	1	2	3	Large
s_l	1.000	2.047	2.783	3.416	$0.850(2l+1)^{\frac{1}{3}}$

Singular potentials, all l:

$$\lim_{k\to\infty} \lambda_c' \Big/ \left(\frac{k}{\log k}\right) = \infty.$$

Arbitrary Energy

$$\lambda_{lc}' = s_l \Big/ \int_0^\infty r|V(r)|dr.$$

See Table VII for values of s_l.

High Angular Momenta

Same results as for λ_c, Sec. 8.

12. Truncation Error

The Born expansion of $S_l \equiv e^{2i\eta_l}$ is obtained by substituting into (10.12) the series for φ_l obtained by iterating the integral equation (A.2) (see Appendix):

$$S_l = 1+\frac{2\lambda}{ik}\int_0^\infty dr\, u_l^2(r)V(r) + \frac{2\lambda^2}{ik}\int_0^\infty dr\, u_l(r)V(r) \times \int_0^\infty dr' K_l(r,r')V(r')u_l(r')+\cdots. \tag{12.1}$$

We inquire into the error of the series $S_l^{(n)}$ obtained by breaking off (12.1) after the term proportional to λ^n.

Arbitrary Energy

Proceeding exactly as in Sec. 9, we find

$$|S_l-S_l^{(n)}| \leq \frac{|\lambda|^{n+1}\left[\int_0^\infty r|V(r)|dr\right]^{n+1} m_l^2}{s_l^n} \times \frac{1}{1-|\lambda|\int_0^\infty r|V(r)|dr/s_l}, \tag{12.2}$$

where s_l and m_l^2 are defined in (10.19) and (6.2) and listed in Tables VI and V, respectively.

Vanishing Energy

Here the Born expansions of $(S_l-1)/2i$ and $\tan\eta_l$ are identical, so that if the potential satisfies (9.4),

$$\lim_{k\to 0} \frac{S_l-S_l^{(n)}}{k^{2l+1}} = 2i(T_l-T_l^{(n)}) \tag{12.3}$$

and an estimate for the latter is given in (9.7).

High Energies

We denote by $\sigma_l^{(n)}$ the term of order λ^n in the expansion (11.1). Using the connection

$$S_l = f_l(\lambda;k)/f_l(\lambda;-k) \tag{12.4}$$

which follows at once from a comparison of (10.6) and (10.10), and the asymptotic behavior (5.8) and (5.16) of the terms in the series expansion of $f(\lambda;k)$, we obtain the following results:

Regular potentials [see (5.11)]:

$$k\rightarrow\infty:\quad \sigma_l^{(n)} = \frac{\lambda^n}{n!}\left\{\left(-\frac{i\lambda}{k}\int_0^\infty V(r)dr\right)^n + o(k^{-n})\right\}. \tag{12.5}$$

Singular potentials [see (5.15)]:

$$k\rightarrow\infty:\quad \sigma_l^{(n)} = -\frac{\lambda^n}{n!}\left\{\left(-\frac{i\log ka}{2k}\right)^n + o\left[\left(\frac{\log ka}{k}\right)^n\right]\right\}. \tag{12.6}$$

As in Sec. 9 we can therefore summarize the high-energy results in the expression,

$$k\rightarrow\infty:\quad S_l - S_l^{(n)} = -\frac{1}{n!}\left[-\frac{i\lambda}{k}\int_{\bar{l}/k}^\infty V(r)dr\right]^n + o\left[\left(\frac{1}{k}\int_{\bar{l}/k}^\infty V(r)dr\right)^n\right] \tag{12.7}$$

which is valid for both regular and singular potentials. The term $\bar{l}$ is defined following Eq. (9.14).

13. Conclusion

In this paper the convergence of Born expansions for a specially simple class of scattering problems has been studied and estimates for the radius of convergence and truncation error have been established. In the past a good many rules of thumb have been used to form an idea of the validity of Born expansions but as the present paper shows, several of these are quite unreliable. It may be useful to comment on some of them.

Unreliable Rules of Thumb

A. $\eta_l \ll \pi/2$.—Since $\tan\eta_l = \infty$, for $|\eta_l| = \pi/2$, the above inequality is sometimes used as a criterion for the rapidity of convergence of the Born expansion of $\tan\eta_l$. This is an extremely treacherous criterion.

(a) Considering first only potentials of fixed sign, we must recall that the Born expansion breaks down as soon as the phase shift corresponding to either $+$ or $-|\lambda|V(r)$ becomes $\pi/2$.

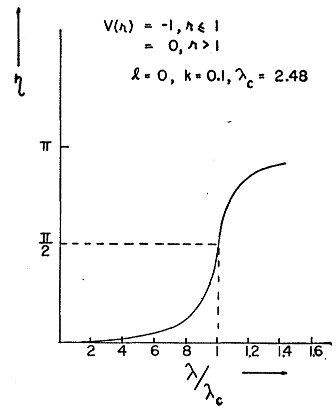

FIG. 7. Behavior of the phase shift as function of λ for low energies. Note that even where $\eta = 0.3 \ll \pi/2$, $\lambda/\lambda_c \approx 0.8$, so that the Born expansion will converge only slowly.

(b) If the potential changes sign, the numerically smallest singularity is sometimes complex, so that the Born expansion may diverge for a certain real λ_1, even though for all real λ of absolute value $<|\lambda_1|$ the phase shift is in absolute value less than $\pi/2$.

(c) Finally it must be noted that at low energies [$ka \ll l+\frac{1}{2}$, where a is some "range" of $V(r)$] the phase shift remains very small, even though $|\lambda|/\lambda_c$ may be very close to 1, and hence the convergence of the Born series may be very slow. This is illustrated in Fig. 7.

As far as the convergence of the S-matrix element S_l is concerned, the value $|\eta_l| = \pi/2$ has no special significance (see Sec. 10). Take, for example, the case of a regular potential. As shown in the Appendix, Eq. (A.9), when the energy tends to infinity, the λ for which $\eta_l = \pm\pi/2$ has the following asymptotic behavior:

$$\lim_{k\rightarrow\infty}\frac{\lambda}{k} = \mp\frac{\pi}{\int_0^\infty V(r)dr}.$$

On the other hand, we have seen in Sec. 11 that asymptotically the radius of convergence has the following behavior:

$$\lim_{k\rightarrow\infty}(\lambda_c'/k) = \infty.$$

Thus at sufficiently high energies the Born expansion for S_l converges, even though $\eta_l \gg \pi/2$.

B. $|\lambda V'|/E \ll 1$.—It is sometimes believed that at high energies the Born expansion converges, provided that a typical depth of the potential, $\lambda V'$, is much less than the total (or kinetic) energy. We have seen in Secs. 5 and 11 that this criterion is not generally true. Thus λ_c, the radius of convergence for the expansion of $\tan\eta_l$, behaves like $E^{\frac{1}{2}}$ for regular potentials and like $E^{\frac{1}{2}}/\log E$ for singular ones; while in the two examples mentioned in Sec. 11 the radius of convergence for S_l, λ_c' behaved like $E^{\frac{1}{2}}\log E$ (square well) and like $E^{\frac{1}{2}}$ (Hulthén potential), respectively. Thus in all of these cases the radius of convergence increases less rapidly than with the first power of E.

C. The convergence improves with increasing energy.—According to Secs. 5 and 11, this is correct for sufficiently high energies. However, for low energies the radii of convergence for both $\tan\eta_l$ and S_l are decreasing functions of the energy, whenever $l\geq 1$, and sometimes even for $l=0$ (Secs. 4 and 11).

D. No bound states.—It is sometimes stated that if the potential is too weak to have a bound state, the Born expansions for $\tan\eta_l$ and S_l will converge. We shall first show that this is not generally correct by a consideration of potentials of fixed sign, and $l\geq 1$. Let λ_1 be that strength of potential (say positive, with $V(r)\leq 0$) which just gives binding with energy zero. This λ_1 is the radius of convergence for $\tan\eta_l$ and S_l in the limit of vanishing energy. (See Secs. 7 and 11). Now consider $(\lambda_1-\epsilon)V(r)$ where ϵ is a small positive number. This potential has no bound states, and—it is true—the Born expansion at zero energy just converges. But at a somewhat higher energy the radii of convergence λ_c and λ_c' are smaller than λ_1 (see Secs. 4 and 11), so that if ϵ is sufficiently small, the Born expansion no longer converges. Thus we have the following situation:

$$\left.\begin{array}{ll} 0<E<E_1: & \text{convergence} \\ E_1<E<E_2: & \text{divergence} \\ E\to\infty: & \text{convergence}\end{array}\right\}.$$

The same situation occurs sometimes even for $l=0$, if the potential changes sign. This may be verified with a suitable δ-function potential of the type (2.21).

It is suggested that, when possible, these rules of thumb be replaced by the convergence criteria summarized in Secs. 8 and 11, and by the estimates of the truncation error given in Secs. 9 and 12.

In the body of this paper we have dealt with the Born expansions of the quantities $\tan\eta_l$ and S_l, which arise most naturally in collision problems. However, if in a practical problem one wishes to achieve a certain accuracy in η_l with the smallest number of terms, the best quantity to expand, particularly at high energies, is the phase shift η_l itself. This is discussed and illustrated in the Appendix, where it is shown that, provided the energy is sufficiently high, η_l—no matter how large—is accurately represented by the first Born approximation.

We have dealt exclusively, in this study, with the radii of convergence for each partial wave, $\lambda_c(l)$ and $\lambda_c'(l)$. If the potential is central and falls off, at infinity, like (1.1), then the radius of convergence Λ_c of the three-dimensional problem (1.6) is the smallest of all the $\lambda_c'(l)$. Unfortunately, the only general result we can derive from this is that at all energies

$$\Lambda_c\geq 1\Big/\int_0^\infty r|V(r)|dr,$$

as already proved in reference 2; in our treatment this result is a consequence of (11.3) and Table VII. About the interesting problem of the energy dependence of Λ_c we cannot draw any detailed conclusions.

A few remarks about nuclear scattering may be of interest. Jost and Pais[2] have emphasized the failure of the Born approximation for *n-p* triplet scattering. We should like to add that this failure arises entirely from the S wave. With the usual potential shapes (square well, exponential, Yukawa) an analysis of the scattering data[11] shows that the actual strength of the potential, λ, exceeds the radius of convergence for $l=0$ at zero energy by a factor of about 1.4. On the other hand, for $l=1$, λ is about equal to the lower bounds of the radius of convergence valid at all energies, as given in Secs. 6 and 11. Specifically, for the square well and $l=1$, we have for all energies $\lambda/\lambda_c\leq 0.48$, indicating a rather comfortable convergence of the Born expansion. The situation is similar for the other potentials.

For sufficiently high energies, the Born expansions for *n-p* triplet scattering converge of course also for $l=0$. In the case of the square well, the critical energies are 20 Mev for the expansion of $\tan\eta$ and 100 Mev for that of $S\equiv e^{2i\eta}$.

14. ACKNOWLEDGMENTS

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APPENDIX: THE BORN EXPANSION OF η_l

The Radius of Convergence

Let us first consider real λ. By (10.3) and (10.5) we have

$$e^{2i\eta}=\frac{f(\lambda;k)}{f(\lambda;-k)}=\frac{f(\lambda;k)}{f^*(\lambda;k)}, \tag{A.1}$$

so that

$$\eta=Im[\log f(\lambda;k)], \tag{A.2}$$

and

$$\eta=0 \quad \text{when} \quad \lambda=0, \tag{A.3}$$

in conformity with the usual convention.

We now extend the definition (A.2) to complex λ. Since $f(\lambda;k)$ is an entire function of λ, $\log f(\lambda;k)$ has

[11] E. E. Salpeter, Phys. Rev. **82**, 60 (1951); J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).

TABLE VIII. Various first Born approximations.[a] $V(r)=-e^{-r}$, $l=0$.

k	λ	ηexact	$\eta^{(1)}$	$\frac{1}{2}\arg[(e^{2i\eta})^{(1)}]$	$\tan^{-1}[(\tan\eta)^{(1)}]$
4	16.1	1.8	2.0	0.66	1.1
6	23.3	1.8	1.9	0.66	1.1

[a] $X^{(1)}$ denotes the first Born approximation of the quantity X.

branch points at those λ where

$$f(\lambda; k)=0. \quad \text{(A4)}$$

If one recalls that $f(\lambda; k)=f^*(\lambda^*; -k)$, comparison of (A.4) with (10.7) shows that the singularities of η are the complex conjugates of the singularities of S. Hence the radius of convergence of η is identical with that of S (see Sec. 11).

The First Born Approximation of η

In spite of the fact just proved, we shall now show that at sufficiently high energies the first Born approximation of η is much superior to that of S or $\tan\eta$. At low energies the first Born approximations of η, S, and $\tan\eta$ are in general about equally good.

Let η be a fixed phase shift and consider the Born expansions of η, S, and $\tan\eta$ as both k and λ (real) increase. Let us take for example a regular potential. Then, provided that $|\lambda/k|$ remains bounded, we have, by (5.10),

$$\lim_{k\to\infty}\left(f(\lambda; k)-\exp\left[\lambda\int_0^\infty V(r)dr/2ik\right]\right)=0, \quad \text{(A.5)}$$

and hence by (A.2) and (A.3)

$$\lim_{k\to\infty}\left(\eta+\frac{\lambda}{2k}\int_0^\infty V(r)dr\right)=0. \quad \text{(A.6)}$$

This equation shows first of all that for fixed η, $|\lambda/k|$ remains bounded, thus justifying (A.5), and secondly that no matter how large η is, it is correctly represented by the first Born approximation as $k\to\infty$.

In other words, if we denote the first Born approximation by the superscript$^{(1)}$ we have, for constant η

$$\lim_{k\to\infty}\eta^{(1)}=\eta; \quad \text{(A.7)}$$

on the other hand, by (A.5) and (A.6),

$$\lim_{k\to\infty}(\tan\eta)^{(1)}=\eta \quad \text{and} \quad \lim_{k\to\infty}(e^{2i\eta})^{(1)}=1+2i\eta, \quad \text{(A.8)}$$

showing that unless $\eta\ll 1$, the first Born approximations of $\tan\eta$ and $e^{2i\eta}$ remain substantially in error, even as $k\to\infty$. Table VIII compares the phase shifts calculated from the first Born approximation of η, S, and $\tan\eta$, for the case of an exponential well, $l=0$, and the exact $\eta=1.8$. For large k, the results obtained by expanding η are evidently the best.

In the case of singular potentials the situation is entirely analogous.

A practical remark: The power series for η is most easily obtained by calculating that for $\tan\eta$, Eq. (9.2),

$$\tan\eta=a_1\lambda+a_2\lambda^2+a_3\lambda^3+\cdots, \quad \text{(A.9)}$$

from which we find

$$\eta=a_1\lambda+a_2\lambda^2+\left(a_3-\frac{a_1^3}{3}\right)\lambda^3+\cdots. \quad \text{(A.10)}$$

Notes added in proof: (1) T. Kikuta [Progr. Theoret. Phys. **10**, 653 (1953)] has recently made an interesting study of various approximation methods, including the Born approximation. Further papers on this subject by the same author are to be published. (2) P. Urban and K. Wildermuth [Z. Naturforsch. **8a**, 594 (1953)] state erroneously that for a cut-off Coulomb potential the radius of convergence λ_c' of $S\equiv\exp(2i\eta)$ increases monotonically with energy for all l. In fact, for $l\geq 1$, $d\lambda_c'/dE$ is negative at $E=0$ as can be explicitly verified by an argument analogous to that leading to (4.16) (see also our Sec. 11). Urban and Wildermuth arrive at their incorrect conclusion (page 596) by applying their Eq. (3), derived in the limit of infinitely high energy, to a discussion of the behavior of λ_c' at vanishing energy.