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The Problem of Liquid Helium—Some Recent Aspects

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CONTENTS

PREFACE

TWENTY-SIX years have now passed since the explicit discovery of the two different modifications of liquid helium and since the introduction of the nomenclature "helium I" and "helium II" by W. H. Keesom. It seems appropriate, therefore, to attempt a review at this time of the more recent aspects of the liquid helium problem, the more particularly since it has received a great deal of detailed attention both experimentally and theoretically in recent years. In the somewhat incomplete review that follows attention has only been directed to work on liquid helium II; since, being the superfluid phase, helium II appears to be of greater interest. Only such properties of helium I have been reviewed as seem of importance in the interpretation of helium II. No data, either early or recent, on solid helium or on He^3 have been included. Moreover, the authors realize that there are many matters concerning liquid helium II which are not commented on herein, matters which have had to be omitted for the sake of brevity of presentation.

1. INTRODUCTION

1.1. The Liquid-Liquid Phase Transformation in Helium

The first hint of the fact that liquid helium undergoes a modification at the temperature of 2.18°K was found by Kamerlingh-Onnes (K.11)† in 1911 during measurements of the density of the liquid. He found (K.24) that the coefficient of expansion changed sign at this temperature and also that the density was a maximum at 2.18°K (see Fig. 1.1). The decisive experimental evidence for the two modifications of liquid helium was provided, on the other hand, largely by W. H. Keesom and his collaborators, and was furnished by measurements on such assorted physical properties as, for example, the dielectric constant

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† References in parentheses refer to the Bibliography at the end of the article.

(K.28a and K.28b), and the heat of vaporization (K.32c), which showed anomalies at the temperature of 2.18°K. Keesom concluded that two states of liquid helium exist which pass one into the other at the temperature mentioned. He designated the liquid at temperatures between 2.18°K and the boiling point (4.2°K), helium I; and the modification below 2.18°K, helium II (K.28a, K.27).

Keesom's measurements of the specific heat of liquid helium (K.32a, K.32b, K.33a, K.35) showed most convincingly the profound nature of the change. (A curve showing experimental values of the specific heat against temperature is given in Fig. 1.2.) Indeed it was on account of the shape of this curve that Ehrenfest (E.33) referred to the transition temperature as the λ point.

Further work by Keesom (K.33a) showed that there was no latent heat at the transition from helium I to helium II, and that the two modifications could not coexist in equilibrium. Reviews in greater detail on these matters have been given by Keesom (K.42), by Burton, Grayson-Smith, and Wilhelm (B.40), and by Squire (S.53b).

2. THE VISCOSITY AND THE "NORMAL" DENSITY

2.1. The Viscosity of Helium I

The first measurements of the viscosity of liquid helium I were made by Wilhelm and co-workers (W.35) by a method dependent on the observation of the torsional oscillations of a cylinder immersed in the liquid. The results, however, are open to question, as was pointed out by Kapitza (K.38a), owing to the probability of turbulent motion occurring in the measurements.

Later measurements, using the torsional oscillations of a disk immersed in the liquid, have been carried out by Keesom and MacWood (K.38b), Keesom and Keesom (K.41a), Smith (S.50a), and deTroyer and co-workers (T.51a) in Leiden. The most recent results, given by the latter authors, are probably the most reliable and these are shown graphically in Fig. 2.1.

Measurements of the viscosity of helium I have also been made by flow methods (Poiseuille flow), in which the rate of flow through capillary tubes was observed, by Johns and co-workers (J.39a) by Bowers and Mendelssohn (B.49a and B.50c). The results of these measurements are also shown in Fig. 2.1, from which it will be seen that both methods yield results in fair agreement with each other.

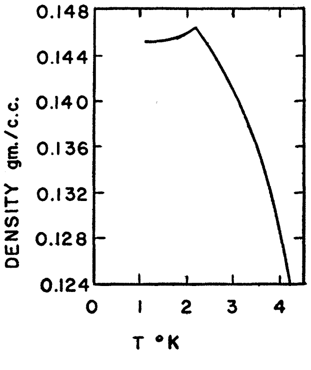

FIG. 1.1. Density of liquid helium in g/cm^3 *versus* temperature from Kamerlingh-Onnes and Boks (K.24).

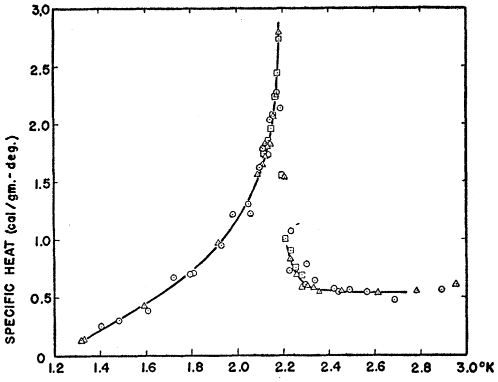

FIG. 1.2. Specific heat (cal/g-deg) of liquid helium. The circles give data of Keesom and Clusius (K.32a) and the squares and triangles data of Keesom and Miss Keesom (K.32b).

The main features of the results may be summarized as follows: (a) The viscosity η of helium I is practically independent of temperature in the temperature range 2.8°K to 4.2°K at a value of about 30 micropoise; (b) On lowering the temperature through the λ temperature (2.18°K), no discontinuity in η occurs.[1] The value of η, however, decreases markedly on passing through the λ temperature and appears to take on values, at say 1°K, about twenty-five times smaller than those at 4°K.

In Fig. 2.1 the measured (I.38) values of the viscosity of helium gas is also included for comparison with the liquid data. It is noteworthy that the viscosity of liquid helium I is not much greater than that of helium gas at the same temperature and that it does not rise in value with decreasing temperature, as does that of all normal liquids. This seems to suggest that even in the liquid form, helium has many gas-like properties, as might be expected for a liquid of such low density. It may be of interest to draw attention here to the recent measurements by Tjerkstra (T.53a) of the viscosity of liquid helium I under pressure. He employed pressure from 5 to 50 kg/cm^2 and found that as the pressure was raised the viscosity became like that of a normal liquid, i.e., viscosity increased with decreasing temperature. The highest viscosity occurring in this work was about 120×10^{-6} poise at a liquid density of 0.186 g/cm^3.

2.2. The Viscosity of Liquid Helium II

The results of the oscillating disk method for the viscosity in liquid helium II are typified by the curve given in Fig. 2.1. The results of attempts to measure

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[1] The earlier work of Keesom and co-workers (K.38b and K.41a) suggested a discontinuity at T_λ. This, however, has not been reproduced by the more recent work.

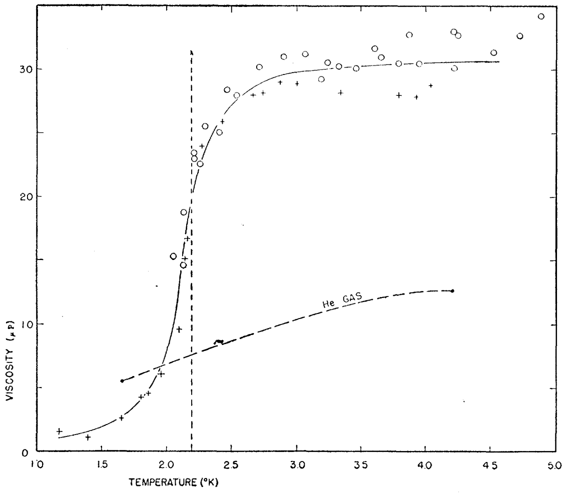

FIG. 2.1. Viscosity of liquid helium in μ poise *versus* temperature. The circles are data of Bowers and Mendelssohn (B.49a and B.50c), and the crosses of deTroyer *et al.* (T.51a). The broken curve is for helium gas (I.38).

the viscosity by flow methods in liquid helium II were, however, anomalous. Such experiments were first performed by Allen and Misener (A.38a) and by Kapitza (K.38a) and they showed that the flow of liquid helium II through narrow channels was quite unlike the flow of normal liquids and indicated that the flow was nonviscous.

The results showed that the flow of liquid helium II through the narrowest channels (of width of order of magnitude 10^{-4} to 10^{-5} cm) is almost independent of the pressure head causing the flow, and independent of the length of the path. Moreover, for the smallest channels, the volume flow appeared approximately proportional to the amount of surface present and not to the cross section. (See subsection 2.5.) Summaries of this early work have been given, for example, by Keesom (K.42), Jones (J.39b) and Darrow (D.40). The later work is discussed below in subsection 2.5.

The most recent estimate for the upper limit of the viscosity of helium II as measured by flow through narrow slits has been given by Kapitza (K.41b and K.41c) and indicates that the viscosity is certainly less than 10^{-11} poise. This value is to be compared with that for liquid helium I just above the λ temperature of 2×10^{-5} poise.

The results of the flow measurements provided a conception of flow without viscous losses, a process which is now referred to as "superfluidity," a term coined by Kapitza (K.38a), and which appears to be the basic feature of helium II.

The contradictory results which gave values for the viscosity of helium II of less than 10^{-11} poise in the flow measurements and of more than 10^{-6} poise in oscillating disk measurements were put into a coherent picture by Tisza (T.38a; T.38b; T.38c; T.40), who put forward the hypothesis that at finite temperatures the atoms which take part in the "superfluid" flow represent only a fraction of the total number of atoms, the remainder providing viscous drag for the oscillating disk experiments. To account for the experimental results, therefore, it is convenient to consider liquid helium II as made up of two constituents, which can be termed "superfluid" and "normal," so that the total density of the liquid may be written:

$$\rho=\rho_s+\rho_n \tag{2.1}$$

where ρ_s and ρ_n are the densities of the superfluid and normal constituents. Care, however, must be taken in consideration of this two-fluid model, since the two constituents do not appear to form distinct separate phases in configurational space, nor does helium II appear to show any strongly marked space-ordered structure different from that of helium I. The latter feature is supported by the x-ray analysis carried out by Keesom and Taconis (K.36a) and by Reekie. (R.40, R.47 and R.53a).

2.3. The Determination of the Normal Density

The hypothesis of the two-fluid model of liquid helium II, introduced by Tisza (T.38a; T.38b; T.38c; T.40) to account for the viscosity problem of liquid helium II, was implicit in the early theoretical work of F. London (L.38a; L.38b; L.39a) who suggested that the λ phenomenon in helium II was analogous to the

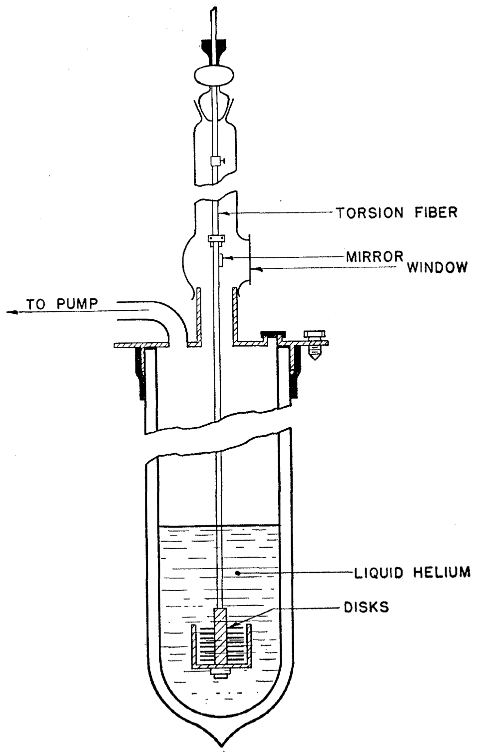

FIG. 2.2. Sketch of Andronikashvili's apparatus (A.46) for measuring ρ_n/ρ by oscillation of a pile of plates.

degeneracy occurring in a Bose-Einstein gas. The hypothesis posed the problem of determining the normal and superfluid densities experimentally. In 1946 Andronikashvili (A.46) reported measurements whereby the relative density of the normal constituent, ρ_n/ρ, could be measured directly. The experiment, based on a suggestion put forward by Landau (L.41a), consisted in measuring the moment of inertia of a pile of closely-spaced aluminum disks (diameter 3.45 cm, spacing between the disks 0.21 mm) hung in a bath of liquid helium II as shown in Fig. 2.2. The superfluid constituent has no effect on the rotation of the disks. The normal or viscous constituent, however, will in part be carried around with the movement of the disks when they are set in oscillatory motion. By measuring the variation in the period of oscillation with change in temperature, it is possible to calculate the variation of the total moment of inertia of the oscillating system and hence obtain the relative density of the normal constituent present at any given temperature. For details of this calculation see Hollis-Hallett (H.52a).

The experiment has subsequently been repeated by Andronikashvili (A.48a) and by Hollis-Hallett (H.50a and H.52a) and the results are shown in Fig. 2.3. It will be seen that within experimental error the results of the two workers are in good agreement.

In order to obtain an approximate interpolation formulation for the variation of ρ_n/ρ with temperature, the results can be expressed in the form:

$$\rho_n/\rho = (T/T_\lambda)^\sigma, \tag{2.2}$$

where T_λ is the λ temperature of 2.18°K. Andronikashvili's results show that a unique value of σ cannot hold over the entire range of temperature employed. It appears that for $1.3°\text{K} < T < 1.7°\text{K}$, $\sigma = 6.8$ and for $1.8°\text{K} < T < 2.18°\text{K}$, $\sigma = 5.3$. These results, therefore, might appear somewhat at variance with the statements made earlier by F. London (L.45 and L.46a) and Tisza (T.47) who suggested a single value of σ equal to 5.5. However at the lower temperature of measurement, the percentage of the normal constituent present is small and consequently the possible percentage error of measurement is greater than at the higher temperatures. The value of ρ_n/ρ, therefore, in the lower temperature range is more accurately assessed from second sound measurements.

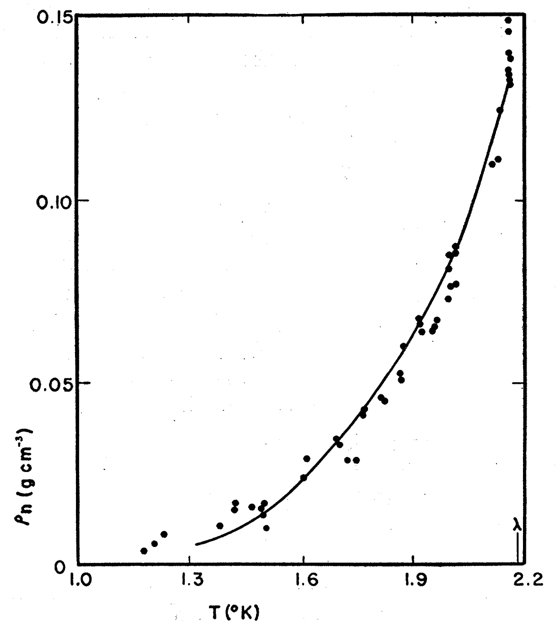

FIG. 2.3. The "normal" density ρ_n in g/cm.³ The full curve gives Andronikashvili's results (A.48a) and the points are due to Hollis-Hallett (H.52a).

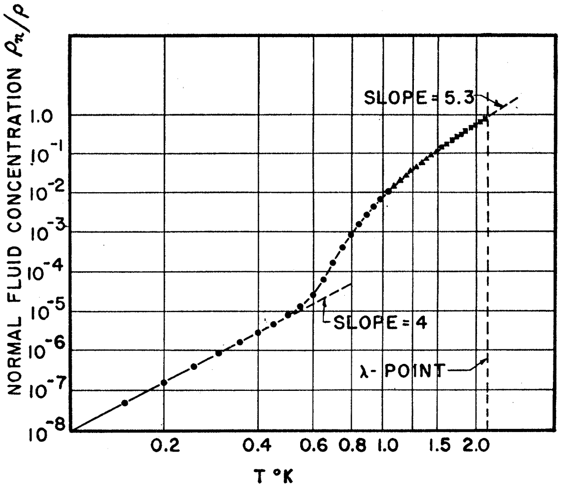

FIG. 2.4. Normal fluid concentration ρ_n/ρ *versus* T, taken from deKlerk, Hudson, and Pellam (K.53b). ● represents data of de Klerk, Hudson, and Pellam (K.53a). ▲ represents data from Peshkov's (P.46b, P.48c) and Maurer and Herlin's (M.49d) second-sound measurements. ■ represents data from Andronikashvili's (A.46) oscillating disks experiments. To avoid confusion, the overlap between the three sets of measurements is not shown.

The values of ρ_n/ρ can be calculated from measurements of the velocity of second sound, (see Sec. 9), as has been done for example by Peshkov (P.46b, P.48c), by Band and Meyer (B.48a) and most recently by deKlerk, Hudson and Pellam (K.53b). These evaluations are in fair agreement with the results of the direct methods quoted above.

An additional indirect method of assessing ρ_n comes from measurements of second sound with a Rayleigh disk, as reported by Pellam and co-workers (P.50a and P.52b). (See also Sec. 9.6.) These measurements evaluate the quantity ρ_n/S^2 at the lower temperatures (about 1.3°K to 1.6°K), where S is the entropy of liquid helium II per g. Using data on the entropy previously measured by Kapitza (K.41b), Pellam and Hanson find fair agreement between their evaluations of ρ_n and those of Andronikashvili, (A.48a).

Recent measurements of the velocity of second sound below 1°K by deKlerk, Hudson, and Pellam, (K.53a) allow evaluation of ρ_n/ρ in this temperature region The evaluations given by deKlerk, Hudson, and Pellam (K.53b), using the specific heat data of Kramers *et al.*, (K.52a), are graphically displayed in Fig. 2.4. If the interpolation formula (2.2) is retained, then it will be seen that, for $0.7<T<1.5$°K, σ starts at a high value of 13 and diminishes to about 6; whereas, for $0.3<T<0.6$°K, $\sigma=4$. It would seem therefore that, with the possible exception of the region below 0.6°K, a formula of the type 2.2 is entirely inadequate. (See Sec. 4.7 and 9.4.)

2.4. Oscillating Disks and Cylinders in Liquid Helium II

By use of the values of the normal density ρ_n as discussed previously in subsection 2.3, further deductions may be made with regard to the viscosity of the normal constituent of helium II as measured by oscillating disk methods. The results of such measurements provide only an evaluation of a product term of viscosity×density. Both Tisza (T.47) and Landau (L.41a) have pointed out that instead of evaluating the results as giving $\eta\rho$, where ρ is the total density of the liquid, as was done in order to obtain the viscosity values shown in Fig. 2.1, one should rather consider the experiments as determining the quantity $\eta_n\rho_n$, where η_n is the viscosity associated with the normal constituent.

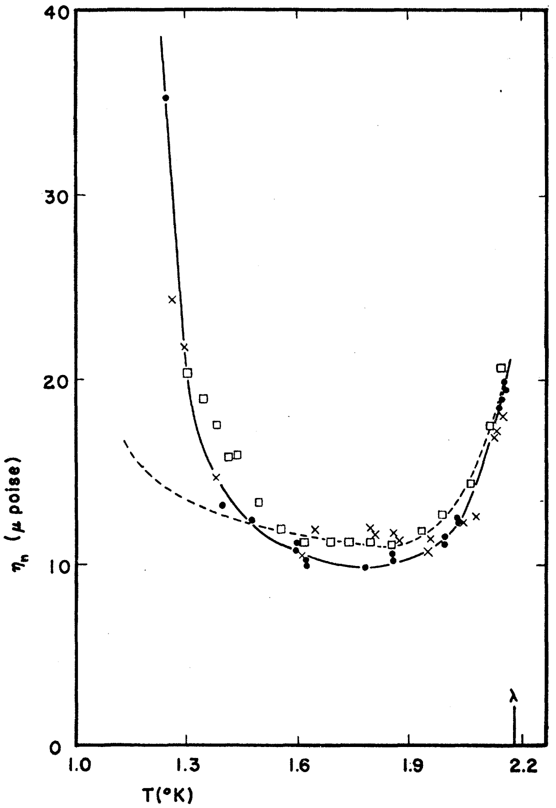

FIG. 2.5. The "normal" viscosity η_n (μ poise) in helium II. The circles are due to Hollis-Hallett (H.52a), the squares due to Andronikashvili (A.48a) and the crosses due to deTroyer *et al.* (T.51a). The broken curve gives Hollis-Hallett's (H.52b) results with the rotating cylinder viscometer, as given in a review by Atkins (A.52a).

The results for η_n, calculated in this way, from the most recent and reliable measurements on the oscillating disk viscometer in helium II, obtained by Andronikashvili (A.48b), deTroyer *et al.* (T.51a), and by Hollis-Hallett (H.52a) are shown in Fig. 2.5. In the evaluations made by the latter author, ρ_n was taken from the second sound measurements of Peshkov (P.46b). It will be seen that the agreement between the results of the various workers is good.

More recently Hollis-Hallett (H.52b), as reported in a recent review by Atkins (A.52a),‡ has made measurements of the "normal" viscosity η_n in helium II by a rotating cylinder arrangement, in which the torque on an inner cylinder was measured as a function of the speed of rotation of a concentric outer cylinder placed around it. In such an arrangement, the parameter measured is η_n directly, rather than $\eta_n\rho_n$ as obtained in the oscillating disk viscometer. Unfortunately the observed couple on the inner cylinder was not linearly proportional to the speed of rotation of the outer cylinder as one would expect for normal liquids. However by extrapolating the results to zero rotational speed, provisional evaluations of η_n could be made and the results are shown by the broken curve of Fig. 2.5. It will be seen that η_n derived in this way is in good agreement with η_n derived from the oscillating disk method for temperatures from T_λ down to about 1.6°K. Below 1.6°K the two sets of results diverge, those found by the rotating cylinder method lying lower. This discrepancy at the lower temperatures may reflect an unreliability in the assumed ρ_n values, taken from the second sound data and used in evaluating η_n from the oscillating disk experiments.

It is noteworthy that the viscosity η_n now shows relatively little temperature variation, as compared with the first type of computations shown in Fig. 2.1. As is shown in Fig. 2.5, η_n does not change by more than a factor of 2 between T_λ and 1.3°K. Furthermore, it appears from both methods of observation (i.e., both disk and cylinder methods) that at the lower temperatures η_n begins to show a negative temperature coefficient, similar to that occurring in normal liquids (see liquid hydrogen—K.38c), whereas in helium I $\partial\eta/\partial T$ is positive. This negative temperature coefficient of η_n below 1.7°K, is of opposite sign to that which one would expect for a gas type behavior of the "normal" constituent.

A theory explaining the rise in viscosity η_n with decreasing temperature below 1.7°K has been put forward by Landau and Khalatnikow (L.49a), based essentially on the assumption of a rapidly diminishing number of scattering centers (rotons) with diminution

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‡ *Note added in proof.*—For full report see Proc. Cambridge Phil. Soc. **49**, 717 (1953).

of temperature. A fuller discussion of this is given in Sec. 9.1.

2.5. Flow of Helium II Through Narrow Channels

The superfluid flow of liquid helium II through very narrow channels was observed first by Kapitza (K.38a) and by Allen and Misener (A.38a). Although the early work has been reviewed previously (K.42, J.39b, and D.40), a brief outline of the salient results is of significance to a full discussion of the flow phenomena and is therefore included herewith.

Allen and Misener investigated the properties of superfluid flow in considerable detail (A.38b and A.39a) and other workers have also reported early measurements on flow through narrow channels, notably Kapitza (K.41b), Giauque, Stout, and Barieau (G.38, G.39), and Johns, Wilhelm, and Grayson-Smith (J.39a).

The method by which Allen and Misener produced their narrowest capillary channels is of interest and has been extensively used in other experiments. (See Brown and Mendelssohn—B.47a.) They placed a bundle of fine stainless steel wires (about 1000 wires of about 6×10^{-2}-cm diameter) in a nickel-silver tube and drew the tube through a succession of steel dies. In this way channel widths of order of magnitude 10^{-5} cm were produced.

The general results obtained by Allen and Misener with the very fine capillaries can be summarized as follows:[2]

(a) The volume flow per second became independent of the pressure head at temperatures below 2°K.

(b) The velocity of flow was not proportional to the square of the radius of the channel, as would be expected for a normal viscous liquid; but instead the velocity increased with decreasing channel width, indicating that the flow was predominantly a surface effect.

(c) The volume flow was not inversely proportional to the length of the channel as would be expected from ordinary hydrodynamics; but tended to become independent of the length of the channel.

(d) The velocity of flow (as determined by measuring the volume flow per second and dividing by the channel cross section) was a function of temperature as shown in Fig. 2.6. This variation with temperature resembles very closely both in magnitude and shape that for flow through surface films (see Sec. 7.1).

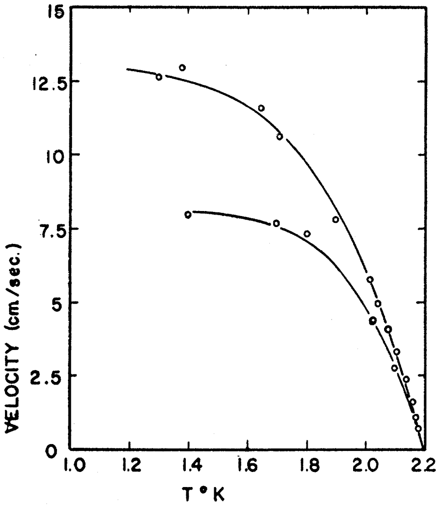

FIG. 2.6. Critical (or maximum) velocity of flow of helium II in cm/sec through narrow channels at a pressure head of 160 dynes/cm² as measured by Allen and Misener (A.39a). The upper curve is for channel diameter 1.2×10^{-5} cm; the lower for diameter 7.9×10^{-5} cm.

With such a dependence (or independence) on the various parameters concerned, it is clear that ordinary hydrodynamical equations do not apply and that no value for a viscosity, as normally defined, can be arrived at for helium II flowing through narrow channels. For the flow through intermediate size channels, the results, as might be expected, showed a behavior representing a complex mixture of viscous hydrodynamical flow and of the "superfluid" flow of the very fine channels. Their quantitative interpretation therefore is very difficult (see Sec. 8.1). Moreover, as is discussed in Sec. 4.1, it is now known that superfluid flow is strongly dependent on small temperature differences established at the ends of the channels. Consequently in the asbence of direct observation of the exact temperatures in the experiments of Allen and Misener, although presumably they were quasi-isothermal, a complete quantitative interpretation is not possible.

The marked similarity of the results obtained with the narrowest channels with those for flow through surface films (see Sec. 7.1), led Allen and Misener to conclude that there existed two flows: first a surface flow that was superfluid and which could be identified with the flow through surface films, and secondly a volume flow which perhaps might obey ordinary hydrodynamics. As the average channel width was reduced, therefore, the surface or superfluid flow would become predominant, and resulted in establishing a critical velocity of flow, v_{crit}, which itself was a marked function of temperature as shown in Fig. 2.6.

The question of the magnitude of this critical velocity for superfluid flow, v_{crit}, in narrow channels and its dependence on the channel width has received much experimental attention; but, as will be seen, the

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[2] Allen and Misener (A.39a) also made experiments on the flow of helium II through powder-packed tubes, using powder of particle size about 10^{-5}-cm diameter. They found a marked dependence of the volume flow on pressure head at all temperatures, and consequently it could be concluded that the features of flow through powders were not necessarily the same as for flow through capillary tubes. Such conclusions have been drawn by Bowers, Chandrasekhar, and Mendelssohn (B.50b and C.53a) from experiments with powders, and by Bowers and White (B.51b), and by White (W.51a) from experiments on flow through porous membranes.

TABLE 2.1. Critical velocities of superfluid flow through narrow channels.

Temp. °K	d cm $\times 10^5$	v_{crit} cm/sec	$v_{\text{crit}}\cdot d$ cm^2/sec $\times 10^4$	$v_{\text{crit}}\cdot d^{\frac{1}{2}}$ cgs units $\times 10^3$	Reference
1.2	1.2	13	1.6	45	A.39a
1.2	7.9	8	6.3	71	A.39a
1.54	3	40	12	220	K.41b
1.54	30	14	42	240	K.41b
1.58	2	23	4.6	105	M.47a
1.52	3	20	6	110	M.47a
1.66	10	15	15	150	M.47a
1.35	12	25	30	275	B.52
Surface Film					
1.5	0.2	40	0.8	57	see Sec. 7.4

results obtained are somewhat discordant and it is not possible at this time to give an accurate picture of the phenomenon. It is of interest to compare the values of v_{crit} obtained by various workers at the lowest temperatures of measurement where, as is evident for example from Fig. 2.6, v_{crit} tends to become temperature independent. Most of the known results are set out in Table 2.1, which gives the observed values of v_{crit} as measured by Allen and Misener (A.39a), Kapitza (K.41b), by Meyer and Mellink (M.47a), and by Bowers and Mendelssohn (B.52). It is evident from the Table 2.1 that the agreement between the various workers is one of order of magnitude only.

It has been suggested, as is detailed in Sec. 7.5, that for superfluid flow the following relationship between v_{crit} and the channel width d is valid;

$$mv_{\text{crit}}\cdot d \approx h, \tag{2.3}$$

where m is the mass of the helium atom. Numerically this gives

$$v_{\text{crit}}\cdot d \approx 10^{-4}\ \text{cm}^2/\text{sec}, \tag{2.4}$$

and in the case of helium surface films this is found experimentally to be approximately correct (see Sec. 7.5 and the values included in the last line of Table 2.1). For the known results for the superfluid flow through very narrow channels, however, as is indicated in Table 2.1, the observed values of the product $v_{\text{crit}}\cdot d$ is by no means independent of d, as is suggested by Eq. (2.3). Moreover, except for the observations of Allen and Misener using their narrowest channel ($d = 1.2\times 10^{-5}$ cm), this product is numerically much larger than that demanded by Eq. (2.4). Recent experiments on the flow of unsaturated surface films by Bowers, Brewer, and Mendelssohn (B.51a) also lead the authors to conclude that $v_{\text{crit}}\cdot d$ is not constant. In the absence of more accurate and definitive experiments on this question, therefore, considerable reserve should be exercised in the application of Eq. (2.3) to superfluid flow phenomena in the bulk liquid through narrow channels.

Mott (M.49b) has proposed an alternative mechanism to account for the establishment of a maximum average velocity of superfluid flow, which would make v_{crit} vary with $d^{-\frac{1}{2}}$, rather than d^{-1} as given by Eq. (2.3). Mott's model assumes a boundary between the superfluid in flow and the remaining superfluid at rest, and the surface tension at this boundary would counteract the tendency for the volume at rest to grow at the expense of the volume in motion due to the Bernoulli forces. The critical or maximum velocity of flow then would be given when the surface tensional forces could just maintain the boundary in equilibrium (see Sec. 2.6 for further discussion of this).

Detail of the method of measurement of v_{crit} by Kapitza (K.41b) and by Meyer and Mellink (M.47a) is left until Sec. 4.2.

A number of experiments on the flow of helium II through narrow channels have been carried out relatively recently by Mendelssohn and co-workers (Bowers and Mendelssohn—B.50a and B.52, Bowers, Chandrasekhar, and Mendelssohn—B.50b, Bowers and White—B.51b) with a view to investigating the pressure gradients existing within the channels themselves. Although the results obtained are complex, especially when the channels were composed of packed powder, the question of their interpretation offers interesting problems. The principle of the experimental method is illustrated diagrammatically in Fig. 2.7 which shows three reservoirs, A, B, and C, interconnected by narrow channels P and Q. A, B, and C are maintained in the same liquid helium bath and therefore supposedly at the same temperature. Superfluid flow is allowed to take place from A through B to C, initiated by having the level in A higher than that in C. By the choice of geometry of the system it can be arranged that in steady flow the critical velocity of flow, v_{crit}, is reached in channel P while the velocity, v_Q, in channel Q is less than v_{crit}, or vice versa. It was found that if, for example, $v_P < v_{\text{crit}}$ and $v_Q = v_{\text{crit}}$ then in the steady state the liquid in vessels A and B took up the *same* level and that level difference occurred only between B and C. This was interpreted as meaning that superfluid flow could take place from A to B, at a rate *less* than the maximum critical rate, without there being any pressure gradient across the interconnecting channel; and, moreover, that all the pressure discontinuity was concentrated at the channel showing the higher "resistance" to flow. The experiments showing these results were carried out using channels of approximately 10^{-4} cm width, formed either between

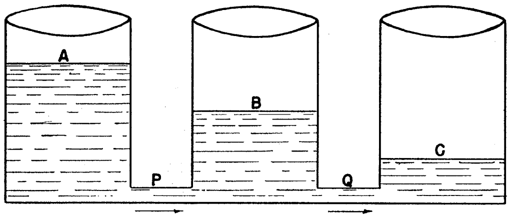

FIG. 2.7. Schematic diagram of flow experiments reported by Mendelssohn and co-workers (B.50a, B.50b, B.52, and C.53a).

ground-glass surfaces or in porous membranes. Other experiments (B.50b, C.53a) of a similar nature using powder-filled channels showed more complex behavior. These results indicate that considerable caution must be exercised in consideration of the gradient of pressure in a narrow channel and suggest that perhaps the pressure gradient exists only at the entrances or exits of such channels. A brief discussion of such possible effects has been given by Meyer and Band (M.48a). Moreover, since such superfluid flow must be accompanied by thermal flow (see Sec. 4.), it would appear that a full interpretation of the phenomena may depend on further experimentation in which detailed temperature as well as pressure measurements are made.

To conclude this discussion on the flow of helium II through narrow channels, some remarks on the question of the frictional forces seem appropriate as, for example, have been discussed previously by Atkins (A.50b). From the facts enumerated above that (a) for the narrowest channels the maximum or "critical" (average) velocity of flow is independent of the pressure head and that (b) for velocities of flow less than v_{crit} the pressure head is zero or at least very small, it may be concluded that the frictional force opposing the flow is zero or nearly zero for all velocities less than v_{crit} and that for velocities greater than v_{crit} the frictional force is extraordinarily large. Such a variation of the frictional force with velocity is illustrated in Fig. 2.8 by curve *A*. Nearly all the evidence on the flow properties of the surface films also (see Sec. 7) indicates that the frictional forces involved in surface film flow are also of the type illustrated by the curve *A*.

There is, however, still some doubt as to the form of the curve for velocities just below the critical velocity. For example it has been found by Mendelssohn and White (M.50a) and by Atkins (A.50b) that for surface film flow out of beakers the velocity of flow fell slightly but continuously below the critical velocity when the pressure head was reduced from values of about 3 mm of liquid helium to zero; whereas for pressure heads, Δp, greater than about 3 mm, the velocity of flow remained constant, independent of further increase of Δp (see Fig. 7.7). This observed smoothing out of the Δp *versus* velocity relationship for $\Delta p \rightarrow 0$, could be explained by a frictional force diagram given in Fig. 2.8 by the broken part of the curve *B*. Furthermore, because of the observed similarities in the behavior between the flow through very narrow channels and the flow through the surface films, it might be concluded that the modification *B* to the frictional curve *A* of Fig. 2.8 could be applicable also to flow of the bulk liquid through narrow channels.

An alternative explanation of the diminution of velocity with diminution of Δp might be found in the energy balance between potential and kinetic energy. For free fall, for example, from heights of 3 mm or less no velocity greater than 24 cm/sec can be expected, and the velocity *versus* Δp curve should be parabolic for any velocity less than v_{crit}, although this explanation appears to be not supported by the recent observations of Picus (P.53a), who has examined the film flow at small pressure heads, or by the experiments of Mendelssohn and co-workers (B.50a, B.52, B.50b, B.51b) described immediately above in this subsection; it seems, as has been emphasized above, that further experiments are necessary to make a decisive evaluation of these questions of superflow.

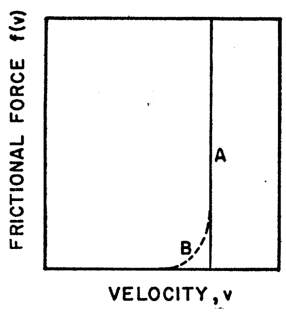

FIG. 2.8. Dependence of frictional force on velocity (after Atkins A.50b). Curve A. Simple critical velocity theory. Curve B. Modified critical velocity theory.

2.6. Flow of Helium II Through Wide Channels

The flow of liquid helium through narrow channels and through surface films is characterized by a critical velocity and by an almost complete independence of this velocity on pressure head. This can be interpreted as indicating that for velocities just equal to v_{crit} large frictional forces come into play, which make any further increase of velocity prohibitively difficult.

On the other hand, the flow of helium II through the wide channels ($d \geq 10^{-3}$ cm) does not reveal clearly any "critical" velocity (if a critical velocity exists it must be very small) and the frictional forces are not so large as to prohibit a continuous increase of the flow velocity with increasing pressure head. The flow through wide channels therefore is not characterized by the complete superfluidity, which is the outstanding feature of flow through narrow channels. Therefore, a study of the properties of flow through wide channels in consequence is of interest in evaluating the character of the frictional forces at play in the nonsuperfluid part of helium II flow, forces which are discussed in more detail in Sec. 8.1.

Detailed experimental studies of the flow of helium II through wide channels were made some time ago by Allen and Misener (A.39a) and by Johns, Wilhelm, and Grayson-Smith (J.39a). More recently the topic has been opened up again in the light of subsequent theoretical development by the experimental work of Atkins (A.51a) and of Hung, Hunt, and Winkel (H.52c). All but the latter workers used essentially the same experimental method, namely that of observing the isothermal flow of liquid helium II out of a cylindrical reservoir of known internal diameter through a capillary attached to the bottom of the reservoir. In such a technique corrections have to be applied for secondary effects, such as the flow out of

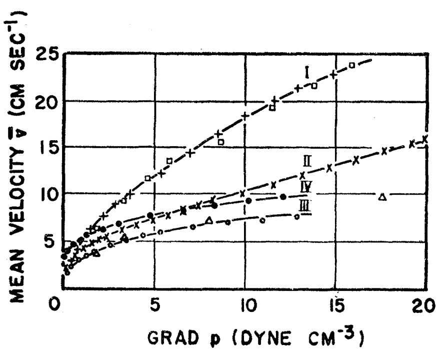

FIG. 2.9. Average flow velocity as function of pressure head, according to Atkins (A.51a). Temperature 1.22°K.

Capillary	Diameter	Length	Length
I	4.40 ×10^{-2} cm	8.04 cm (+)	1.75 cm (□)
II	2.03 ×10^{-2} cm	7.90 cm (×)	
III	0.815 ×10^{-2} cm	8.03 cm (○)	0.83 cm (△)
IV	0.262 ×10^{-2} cm	7.76 cm (●)	

the reservoir upwards through the surface film and for the existence of possible temperature differences between the inside and outside of the reservoir. For details of these corrections, the original papers should be consulted.

As an illustration of the marked difference in the flow behavior between that in wide channels and that in narrow channels, the results obtained by Atkins (A.51a) at 1.22°K for flow through circular capillaries of various diameters, as shown in Figs. 2.9 and 2.10, are typical. In these figures the observed average velocity of flow is plotted as a function of the pressure head Δp. First, it will be seen that it cannot be concluded unambiguously that all the curves have a finite intercept on the velocity axis for $\Delta p=0$, such as would be required if there existed a "critical" velocity for frictionless flow, although informed guesses at v_{crit} can be made. Secondly, by increasing the pressure head the velocity continually increases, showing no sign of "saturation" as in the case of flow through narrow channels. The slope of the curves does not correspond in general, however, to what one would expect for normal hydrodynamic flow, indicating that the frictional forces involved are complex in character. It appears that some combination of a term involving critical velocities together with other frictional terms is necessary to describe the results over the whole temperature range. This conclusion was arrived at by H. Jones (J.39b) in his detailed analysis of Allen and Misener's (A.39a) results.

Atkins (A.51a) attempted to interpret these results by means of the equation

$$v=v_1(d)+v_2\ (\mathrm{grad}p)+v_3(d,\ \mathrm{grad}p), \tag{2.5}$$

and found that

$$v_3=\alpha(d)\ \mathrm{grad}p. \tag{2.6}$$

Unfortunately this forces the conclusion that $\alpha(d)$ has a minimum for $d=8\times10^{-3}$ which is surprising and seems somewhat arbitrary when it is recalled that only four different sizes have been measured. This conclusion can only be avoided by rejecting the data for the smallest diameter tube used ($d=0.262\times10^{-2}$ cm).

It is customary to attempt to describe these complex frictional forces by means of the two fluid theory. Then, for the case of steady *isothermal* flow (zero acceleration of either the superfluid or the normal fluid), one may write quite generally (see Sec. 8)[3]

$$\begin{aligned}-(\rho_s/\rho)\ \mathrm{grad}p&=F_r+F_s,\\ -(\rho_n/\rho)\ \mathrm{grad}p&=-F_r+F_n-\eta_n\nabla^2\mathbf{v}_n,\end{aligned} \tag{2.7}$$

where F_r is a mutual frictional force due to relative motion of the superfluid and normal fluid ($F_r=f(\mathbf{v}_s-\mathbf{v}_n)$), F_s and F_n are undetermined frictional forces characteristic only of the superfluid velocity v_s and of the normal velocity v_n, respectively, and where the third term on the rhs of Eq. (2.7) describes the usual effect of viscosity.

Various proposals have been offered to describe the detailed character of the forces F_r, F_s, and F_n, some of which are outlined below in Sec. 8. The most extensively investigated proposal is that of Gorter and Mellink (G.49a) who proposed, in interpreting data on heat conductivity in helium II (see Sec. 5), that the dominating term was F_r the mutual frictional force, and that F_r was proportional to the third power of the relative velocity ($\mathbf{v}_n-\mathbf{v}_s$).

According to Gorter and Mellink then, for isothermal steady flow under a pressure gradient:

$$\begin{aligned}(\rho_s/\rho)\ \mathrm{grad}p&=A\rho_s\rho_n(\mathbf{v}_n-\mathbf{v}_s)^3,\\ (\rho_n/\rho)\ \mathrm{grad}p&=-A\rho_s\rho_n(\mathbf{v}_n-\mathbf{v}_s)^3-\eta_n\nabla^2\mathbf{v}_n,\end{aligned} \tag{2.8}$$

where A is a constant characteristic of the mutual friction. As has been shown by Atkins, (A.51a) the above equations lead to the following evaluation of

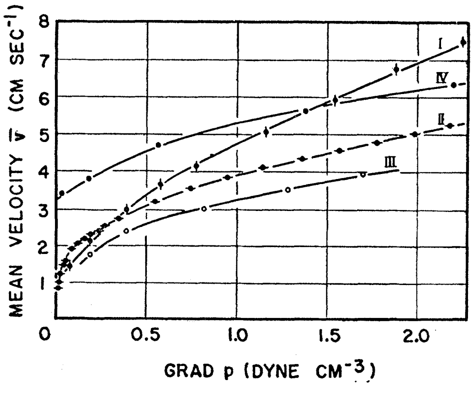

FIG. 2.10. Average flow velocity as function of pressure head, according to Atkins (A.51a). Temperature 1.22°K.

Capillary	Diameter	Length	Symbol
I	4.40 ×10^{-2} cm	48.6 cm	⦶
II	2.03 ×10^{-2} cm	46.6 cm	-●-
III	0.815 ×10^{-2} cm	8.03 cm	○
IV	0.262 ×10^{-2} cm	7.76 cm	●

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[3] For a recent review of this question see also Atkins (A.52a).

the average velocity of flow $\bar{v}$ through a cylindrical capillary of radius r:

$$\bar{v}=\frac{\rho_s}{\rho}\left(\frac{-\operatorname{grad}p}{A\rho\rho_n}\right)^{\frac{1}{3}}+\frac{d^2(-\operatorname{grad}p)}{32\eta_n}. \tag{2.9}$$

In comparing his experimental results for flow of helium II through wide capillaries with this theory, Atkins (A.51a) concluded that the observed variation of $\bar{v}$ with grad$p$ was in agreement with that of Eq. (2.9), indicating that the mutual frictional force would depend on the third power of $(\mathbf{v}_s-\mathbf{v}_n)$. By making the assumption that in capillary III (see legend to Fig. 2.9) v_3 [see Eq. (2.5)] was negligible, Atkins evaluated the Gorter-Mellink constant A and found approximate agreement with Gorter and Mellink's values. Moreover A was constant to an order of magnitude. However, the observed dependence of $\bar{v}$ on d was not in agreement with the theory. The more recent measurements of Hung, Hunt, and Winkel (H.52c), which are described briefly in Sec. 5, also for flow through wide capillaries, also indicated that at sufficiently high relative velocities $F_r \propto (\mathbf{v}_s-\mathbf{v}_n)^3$. However, these new measurements showed that the constant A (see Eq. 2.8) was not a constant, but depended strongly on channel width, especially at the lower temperatures ($T\sim 1°$K).

Atkins suggested that turbulence should be considered in flow through wide channels, but cautioned that if the fundamental postulate of Landau and London that curl $v_s=0$ (see Sec. 3) is accepted, turbulence does not seem possible, since turbulence would not exist in the superfluid and could hardly exist in the normal fluid for the Reynold's numbers observed (for the normal fluid $R=\frac{1}{2}\bar{v}d\rho_n/\eta_n\leq 750$).

However, several years ago Jones (J.39b) indicated how, in a qualitative way, the results of Allen and Misener for the flow in large capillaries could be interpreted by assuming that turbulent flow occurred in some of the capillaries. It is our belief, also, that among presently postulated or known mechanisms, turbulence is necessary to explain the behavior of helium II in wide capillaries. For it seems very significant that in their widest capillary Allen and Misener found that the velocity of flow *fell* as the temperature was lowered from the λ point, while for streamline flow the second term of Eq. (2.9) predicts an increase in velocity. Moreover, no choice of the temperature dependence of the Gorter-Mellink constant A will compensate for the behavior of the streamline flow term in Eq. (2.9).

In order to show the possibility of turbulence we will compare Allen and Misener's and Atkins' data at 1.2°K with the semiempirical formula

$$v=\frac{a}{d^{\frac{1}{2}}}+bd^{\frac{1}{2}}(\operatorname{grad}p)^{\frac{1}{2}}, \tag{2.10}$$

TABLE 2.2.

Capillary	Diameter d, cm	v_{crit} at $\nabla p=0$ cm/sec	$v_{\mathrm{crit}}\cdot d^{\frac{1}{2}}$ cgs units	$v_{\mathrm{crit}}\cdot d$ cm²/sec	Reference
A	4×10^{-4}	4	0.08	16×10^{-4}	
B	8×10^{-5}	8	0.07		
C	1.2×10^{-5}	13	0.04	1.6×10^{-4}	A. and M. (A.39a)
I	4.4×10^{-2}	0.5	0.10	220×10^{-4}	
II	2.03×10^{-2}	0.7	0.10		
III	0.815×10^{-2}	1.0	0.08		Atkins (A.51a)
IV	0.262×10^{-2}	3.1	0.15	81×10^{-4}	

in which a and b are constants. The first term in this equation corresponds to Mott's (M.49b) expression for the critical velocity (see Sec. 2.5); the second gives approximately the relationship between velocity, diameter and pressure for classical turbulent flow.

The constant a of Eq. (2.10) has been determined from (*i*) Allen and Misener's data on three specimens in which flow took place between close-packed wires, (it was assumed that in such fine interstices the flow was pressure independent at 1.2°K), and (*ii*) from Atkins' results shown in Figs. 2.9 and 2.10, after reasonable guesses have been made as to the critical velocities. The data used are set out in Table 2.2 and yield the value $a=0.07$ cgs units. From Table 2.2 also it is clear that $v_{\mathrm{crit}}\cdot d^{\frac{1}{2}}$ is more nearly constant than is $v_{\mathrm{crit}}\cdot d$, a feature that was not so marked in Table 2.1.

To check the form of Eq. (2.10) we have plotted log $(v-v_{\mathrm{crit}})$ *versus* log (grad$p$) from Atkins' data (see Figs. 2.9 and 2.10) and found that the slopes of these curves lie between 0.43 and 0.59, so that we may take the exponent of grad p in Eq. (2.10) equal to 0.5 with sufficient accuracy.

In evaluating b from Atkins' figures, we have found for all curves, and for $5<\operatorname{grad}p<15$ that $20\leq b\leq 40$. If one assumes $b=30$, Eq. (2.10) can be written:

$$v=\frac{0.07}{d^{\frac{1}{2}}}+30d^{\frac{1}{2}}(\operatorname{grad}p)^{\frac{1}{2}}. \tag{2.11}$$

When we compare this equation with Allen and Misener's results (A.39a) we find semiquantitative agreement with their curves for velocity *versus* diameter, the position and value of the minima being predicted with little error. A comparison of the experimental results with Eq. (2.11) is given in Fig. 2.11.

2.7. Problem of the Viscous Forces

The question of the nature of the frictional forces, occuring, for example, in the motion of helium II through wide channels, has been investigated experimentally in some detail by Hollis-Hallett (H.50a, H.52a). These experiments, together with more recent ones by the same author (H.52b) using a rotating cylinder viscometer, have been quantitatively discussed at some length in a recent review by Atkins

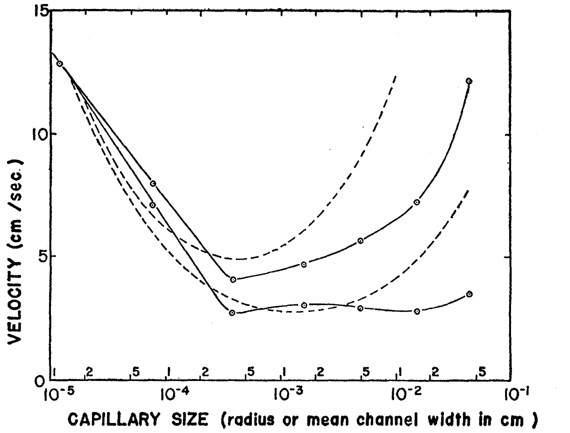

FIG. 2.11. The variation of velocity with capillary size for two different pressure gradients at 1.2°K. The solid curves are taken from Allen and Misener's experimental results (A.39a) (their Fig. 6). The broken curves are computed from our Eq. (2.11) in the text. Lower pair of curves are for grad$p$=0.75 dynes cm^{-3}. Upper pair of curves are for grad$p$=8.0 dynes cm^{-3}.

(A.52a), and consequently we will confine ourselves here to a qualitative outline of the main results.

In his measurements of the viscosity of helium II with an oscillating disk viscometer (see Sec. 2.4), Hollis-Hallett extended his observations to larger amplitudes of oscillation than had previously been carried out. If θ_n is the amplitude of the nth swing of the disk, then the logarithmic decrement δ given by

$$\delta=-\frac{1}{2\pi}\frac{d\log\theta_n}{dn}, \tag{2.12}$$

provides a measure of the damping forces acting on the disk. For an ordinary viscous liquid δ is independent of amplitude, as Hollis-Hallett found for helium I. In helium II, however, he found that the logarithmic decrement δ was independent of amplitude for $\theta<0.1$ radian; but that for $\theta>0.1$ radian, δ increased markedly with increasing amplitude, especially at the lower temperatures of measurement.

In order to explain this variation in the damping term δ with amplitude it is necessary to introduce frictional forces which are more complex than that associated with normal viscosity. The mutual frictional forces due to the relative motion of the superfluid and normal constituents of the liquid, as introduced by Gorter and Mellink (G.49a) and given by (see Secs. 2.6 and 5.5)

$$F_r=A\rho_s\rho_n(\mathbf{v}_s-\mathbf{v}_n)^3, \tag{2.13}$$

presents a possibility of explaining Hollis-Hallett's results. The solution of the hydrodynamic equations based on the Gorter-Mellink mutual friction theory for an oscillating disk in the two-fluid liquid has been given by Zwanikken (Z.50b) and the results of these computations are given by the full curve of Fig. 2.12. In this figure the excess decrement $(\delta-\delta_0)$ is plotted against temperature for observations at an amplitude of 0.215 radian and of period 3.7 sec. (The excess decrement is the excess value over the decrement δ_0 at zero amplitude.) Also in Fig. 2.12 are plotted Hollis-Hallett's observed results and the broken curve plots ρ_s scaled to fit the results. It will be seen that the Gorter-Mellink frictional force, represented by the full curve, is inadequate to account fully for the variation of δ with temperature. It is of interest, moreover, to note that the excess decrement $(\delta-\delta_0)$ appears to be proportional to the superfluid density ρ_s, a feature which may indicate that the significant frictional force involved is a function of v_s only.

These interesting results obtained with the oscillating disk viscometer at large amplitudes, i.e., for large peripheral velocities, have been complemented by further work by Hollis-Hallett (H.52a) on the oscillations of a pile of plates in liquid helium II. This is essentially a repetition of Andronikashvili's experiment (A.46), (see Sec. 2.3), in which, however, large amplitudes of oscillation are used. Hollis-Hallett found for the oscillations of the pile of plates that not only did the logarithmic decrement δ increase with increasing amplitude θ as for the single disk, but also that the period of oscillation increased with increasing θ. From the increase in period it is concluded that the superfluid constituent is dragged more and more with the pile of plates at higher velocities, as would be expected if there were either a force of mutual friction F_r or a force of friction F_s, between the plates and the superfluid. The detailed results indicate that, whereas a mutual frictional force F_r, at least similar to that postulated by the Gorter-Mellink theory, is undoubtedly present, there must exist other frictional forces of unspecified character.

A further experiment indicating the possibility of dragging the superfluid constituent into rotational motion has been carried out by Osborne (O.50). In this experiment a cylindrical beaker (inside diameter 1.4 cm) was rotated about its vertical axis with angular speeds of from eight to sixteen rps and the shape of the meniscus was observed. It was found that over a range of temperature, the meniscus was parabolic and corresponded to that which would be expected if all the liquid were rotating with the speed of the beaker. Osborne concluded that the peripheral velocities involved (35 to 70 cm/sec) were higher than the

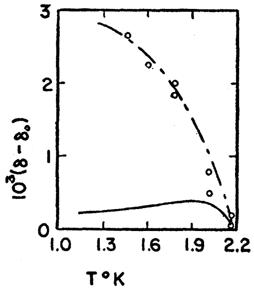

FIG. 2.12. Comparison with the Gorter-Mellink theory of the values of the excess decrement $(\delta-\delta_0)$ found with a single disk at an amplitude of 0.215 radian and a period of 3.78 sec. The full curve is the theory, the circles the observed results, and the broken curve ρ_s scaled to fit the points. (Hollis-Hallett, H.52a.)

"critical" velocities for superfluid motion and, hence, the superfluid constituent was dragged around with the normal fluid.

This conclusion is in agreement with the conclusions drawn from the experiments of Hollis-Hallett mentioned previously. Very little, however, can be deduced from the experiment concerning small peripheral velocities of value less than the "critical" velocity; and no conclusions, therefore, can be drawn regarding the postulate of Landau (L.41a) and London (L.46a) that curl v_s is zero for subcritical velocities.

Experiments also on liquid helium in rotation have recently been carried out by Andronikashvili (A.52c), being in general outline similar to a suggested experiment put forward by H. London (L.46b). A system of a pile of circular horizontal plates, fitting inside a copper cylinder similar to that used also by Andronikashvili for his determination of ρ_n (A.46), was suspended from a torsion fiber in liquid helium II and set into steady rotation of 0.5 rps at 2.17°K. It was then cooled to 1.5°K and stopped by means of an electromagnetic damping arrangement. The damping was then removed and the system heated to 1.65°K in 15 sec, so that ρ_n increased from 12 percent to 22 percent in the heating process. If the superfluid was in rotation at the lower temperature, then, on heating, the fraction of the liquid transformed into normal fluid should give up its momentum to the vessel. No experimental evidence of the vessel acquiring such momentum was observed.

3. THEORETICAL BASES OF THE TWO-FLUID THEORY

3.1. General Remarks

It has been shown that the explanation of the peculiar viscosity of helium II is facilitated by the introduction of a two-fluid model of the liquid in which

$$\rho=\rho_s+\rho_n. \tag{3.1}$$

The macroscopic implications of the two-fluid model, characterized by this and a few other equations which we shall presently display, were first extensively discussed by Tisza (T.38a, T.38b, T.38c, T.40). These implications we shall describe more fully in Secs. 4.7 and 8.1; here we wish to discuss briefly the microscopic theories which on the one hand justify this model and on the other hand elucidate other properties of helium II. Our treatment will be qualitative and suggestive rather than exhaustive for neither is there room in a general review of this nature for great detail nor is there need for it since the publication of Dingle's review of the theories of helium II (D.52a). Four basic theories will first receive consideration: that associated with the name of F. London (L.38a, L.38b), that of Landau (L.41a, L.41b, L. 44a), that of Green (G.48a, G.48b), and that of Prigogine and Philippot (P.52a, P.53b, P.53c).§ The first of these chronologically was that of F. London. It predated and inspired the phenomenological theory of Tisza.

3.2. London's Theory

London (L.38a, L.38b) pointed out the similarity between the transformation of helium I into helium II and the degeneration which can occur at sufficiently low temperatures in a perfect Bose-Einstein gas. In the perfect Bose-Einstein gas the degeneration or "condensation" consists in a finite fraction of all the particles dropping into the lowest energy state where they are characterized by a uniform distribution in configuration space but by only a single momentum; i.e., they are ordered or "condensed" in momentum space. This condensation corresponds to a third-order phase change (there is a discontinuity in the slope of the specific heat). Below the degeneracy temperature, given by the temperature of the peak appearing in the specific heat curve, the particles drop into the lowest energy level in increasing numbers and the entropy of the gas drops rapidly toward zero.

Since non-interacting helium atoms would obey Bose-Einstein statistics, London sought to explain the properties of helium II by this gas model. The uniform distribution of the gas in configuration space he considered to be mirrored by the liquid helium II in that the latter shows no regularities of structure not already present in helium I (K.36a, R.40, R.47) and the ordering in momentum space to be mirrored in the superflow of helium II. Of course liquid helium is by no means an ideal gas, so the model is far from perfect; however, the degeneracy temperature for the model was found to be 3.13°K in fair agreement with the experimental value of 2.18°K for the λ temperature of helium II.

In other respects the perfect gas model differs quantitatively from helium. The specific heat anomaly is much less strong for the gas and the variation of ρ_n and of the entropy with temperature is slower for the model than for the experimental substance. [σ in Eq. (2.2) is only $\frac{3}{2}$ for the gas.] These quantitative differences prompted London (L.39a) to modify the theory by changing the density of energy states of the particles from that appropriate for free particles to that approximately appropriate for particles moving in the average potential field of their neighbors. In this way, by an *ad hoc* choice of the density-of-states function, London was able to make the theory follow the experimental specific heat and to increase the exponent σ to 5. However, a difficulty arose in this treatment, in that it was necessary to postulate that only a fraction (0.136) of the particles "condensed"—even at the absolute zero—in order to obtain the desired specific heat curve. Such a postulate is in-

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§ *Note added in proof.*—Feynman's theory was published unfortunately after this paper was completed. See Appendix for references to it and see footnote ||.

consistent with the observed entropy (see Sec. 4), and has since been abandoned by London (L.45).

The conclusion reached originally by F. London, that modifications of the perfect Bose-Einstein theory to take account of interatomic forces were necessary, has been reaffirmed by work of Goldstein (G.41), Lamb and Nordsieck (L.41c), Becker (B.50b), Leibfried (L.50b), and Halpern (H.52e and H.52f) who have considered the effects due to external force fields.

Following London's first paper on the subject, a great deal of work has been done on the theory of the Bose-Einstein gas and its applicability and application to the helium problem. We shall refrain from discussing this work in any detail here. We will, however, mention a few refinements: Bijl, deBoer, and Michels (B.41) have discussed a smoothed potential model of a Bose-Einstein liquid, in which they justify intuitively the inclusion of an energy gap between the ground state of the molecules and all excited states. In one modification of their theory the value of the energy gap is taken to be a linear function of the occupation of the ground state. The result of these measures is to bring the theoretical specific heat curve into closer agreement with experiment near T_λ. Toda (T.51d) has introduced the idea of molecules of several helium atoms in equilibrium with the ground state in order to justify an energy gap. His results are similar to Bijl, deBoer, and Michels. Goldstein (G.46) has considered a Bose-Einstein liquid in which deviations from the ideal gas character due to interatomic forces are treated by averaging over the volume of the system. The resultant effects on the order of the transition are discussed.

More rigorous developments of the gas model have been made by Feynman‖ (F.53a) who has shown that interactions between helium atoms do not obliterate the λ transition, and by Friedman and Butler (F.53b) who have approximately treated the partition function of a Bose-Einstein gas with interactions. Considering only repulsive forces between particles, Friedman and Butler obtain 2.4°K for T_λ, and a profile of the specific heat more closely resembling the experimental curve than does the specific heat of the ideal gas.

It should be stated that none of the above theories gives a specific heat similar to that observed at temperatures below 0.6°K. The observed specific heat varies as T^3 in this region (see Sec. 3.3 and Sec. 4) and is apparently that due to phonons in the liquid.

3.3. Landau's Theory

Landau (L.41a, L.44a) rejected the idea that the type of statistics obeyed by the helium atoms has anything to do with the superfluid properties of helium II. To him, the essential property of helium was its remaining liquid down to absolute zero. At absolute zero the liquid would be in its ground state, which state was assumed to be free of vorticity. Landau attempted to prove this assumption by demonstrating that vortex motions, which would be quantized, would require the addition of a finite energy increment to the system, just as in atomic systems the quantization of angular momentum leads to finite energy increments. The energy increment Δ was assumed to be positive; i.e., the system with one unit or more of vortex excitation would have an energy greater than the ground state. Although this assumption does not *prove* that the ground state is vortex free, it is natural to suppose that vorticity would increase the entropy of the system, and hence excitation of vorticity would represent departure from the zero temperature state.

Departures from the ground state could also arise, as Landau pointed out, from the excitation of one or more units of sound-wave energy, or "phonons." Landau proposed analogously that the unit of vortex excitation be called a "roton." States near the ground state, therefore, were characterized by the numbers and energies of the phonons and rotons superimposed on the ground state.

As long as this superposition is valid, the specific heat and entropy of the system are identical with the specific heat and entropy of the excitations, which quantities may be calculated when their energy spectra are known. For the phonons, as is generally assumed $E=pu_1$, where E is the energy, p is the magnitude of the momentum of the phonon, and u_1 is the velocity of sound. For the rotons, Landau assumed the spectrum

$$E=\Delta+p^2/2\mu, \tag{3.2}$$

where Δ is the energy increment or gap mentioned before, p is the linear momentum of the roton, and μ is the effective mass of the roton. By treating Δ and μ as adjustable parameters, Landau (L.41a) was able to fit the specific heat curve of helium below the λ point. To fit the experimental curve in the temperature range 1°K to 2.18°K the values of the parameters chosen were $\Delta/k=8$ to 9°K, $\mu\approx 7$ to 8 He^4 masses.[4]

Landau arrived at the conclusion that the ground state and the excitations, respectively, play the role of superfluid and normal fluid. The excitations are "normal" because they may be scattered and reflected, and hence show viscosity. If, for some reason such as the presence of a temperature gradient, the excitations acquire a drift velocity, their drift momentum per unit volume may be calculated as a function of that drift velocity. Landau said that the drift momentum density divided by the drift velocity is the normal

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‖ *Note added in proof.*—The full presentation of Feynman's theory of liquid helium occurred after this paper was completed in September, 1953. The reader is referred to the original papers [Phys. Rev. **91**, 1291, 1301 (1953); **94**, 262 (1954)], since any attempt to review this work in a footnote would only do it an injustice.

[4] For the phonon specific heat Landau (L.41a) calculated the value $C_{ph}=4.4\times10^{-3}T^3$ cal/g-deg from the observed compressibility data of Keesom (K.36b). This is to be compared with $C_{ph}=5.6\times10^{-3}T^3$ cal/g-deg as recently experimentally observed by Kramers, Wasscher, and Gorter (K.52a). It is to be noted that for $1.0°\text{K}<T<2.18°\text{K}$, C_{ph} would be negligibly small compared with C_{roton}, using the chosen values of Δ and μ given above.

density ρ_n. If ρ_n is less than ρ, the difference is the density of the ground state ρ_s; i.e., $\rho_s = \rho - \rho_n$. So defined,[5] ρ_n is an increasing function of temperature and the temperature for which ρ_n first equals ρ is the λ temperature. The λ temperature calculated in this way—by treating the excitations as an ideal gas—was found to be 2.3°K, close to the observed λ temperature.

To show that the fluid associated with the ground state was superfluid, Landau considered the flow of helium II in a tube and deduced the result that the helium could not absorb a phonon from the walls of the tube unless it was flowing with a velocity relative to the tube greater than the velocity of sound. By a similar argument he found that rotons could not be absorbed at relative velocities less than a "critical velocity" equal to $(2\Delta/\mu)^{\frac{1}{2}}$. Below the lesser of these two velocities (they are of the same order of magnitude) the flowing helium would not interact with the walls and, hence, would be superfluid—unless, as Landau said, some other mechanism yet undetermined limited the flow.

By such arguments Landau justified a two-fluid model. It is a model in which Eq. (3.1) and other equations of Tisza's macroscopic model have significance. However, it is important to understand that the expression "two fluid" is not to be taken literally in this treatment. There is but one fluid in which, because of low temperature, all the modes of motion are not excited. The excitations present can transport momentum independently from the fluid as a whole and *act* like a normal fluid. The unexcited residue *acts* like a superfluid.

In a later paper Landau (L.47a) modified the roton energy spectrum, and suggested that:

$$E = \Delta + (p - p_0)^2/2\mu. \qquad (3.3)$$

This modification with its three adjustable parameters was introduced in order to bring Landau's evaluation for the velocity of second sound (see Sec. 9) into better agreement with experiment. He proposed that $\Delta/k = 9.6$°K, $\mu = 0.77$ He^4 masses, and $p_0/\hbar = 1.95 \times 10^8$ cm^{-1}. Unfortunately, these values of the constants cannot be checked independently. At best it can be stated that with these values of Δ, μ, and p_0 Eq. (3.3) as a whole is not inconsistent with the specific heat measurements.

We should make some comments on this theory. First, we note that the theory has little to say about the λ-point transition. It can be made to reflect the shape of the observed specific heat curve below the λ point but does not predict a λ discontinuity in the specific heat. However, the question of the λ discontinuity and the associated question of the order of the phase change has been discussed by Goldstein (G.53) who has emphasized the importance of the role of the phonons even at the transition temperature. Secondly, it is seen that although the fitting of the entropy and the calculation of ρ_n depend critically on the sign and magnitude of the energy gap Δ this quantity cannot be approached from first principles. There is no way of telling, for example, whether Δ for liquid He^3 should be of such sign and magnitude as to lead also to superfluidity. Lastly, the exact nature of the "roton" is not clear. About all that may be said is that the roton is an excitation with an energy given by Eq. (3.3).

Considerable progress has been made toward rendering Landau's theory more rigorous. Kronig and Thellung (K.52b) have given a quantum theory of the irrotational hydrodynamical field in which the concept of phonons is made more precise. More recently Thellung (T.53b) has succeeded in quantizing a hydrodynamical field containing vortices. His Hamiltonian contains a phonon term, a roton term, and an interaction term. Unfortunately, he has not been able to determine the energy levels of the rotons.

3.4. Green's Theory

Green's theory (G.48a, G.48b) is based on the quantal formulation of the kinetic theory of liquids of Born and Green (B.47b, B.47c). The essential property of the quantum treatment, according to Green, is the distinction between intensive properties when (*i*) defined thermodynamically, and (*ii*) defined by kinetic theory. Especially important is the difference between the thermodynamic pressure p, given by the thermodynamic formula $p \cdot dV = dW$ (the work done in moving a boundary), and p_1, the kinetic pressure whose gradient determines the mean acceleration of the molecules. The thermodynamic pressure is given by a series, $p = p_1 + p_3 + p_5 + \cdots$ in which the p_i vary in relative importance with temperature. When $p_3 + p_5 + \cdots$ become comparable with p_1, the liquid behaves in a manner not predicted by classical thermodynamical formulae; in particular, the first law of thermodynamics takes different forms depending on whether the liquid is at rest, undergoing steady motion, or undergoing periodic motion.

On this difference between classical and quantal liquids Green based the explanation of the behavior of He II. A specific calculation showed that $\pi = p_1 - p$ first becomes significant when energies of the order of $2\hbar^2/mr_1^2$ are excited (r_1 is the interatomic distance for which the potential energy is a minimum and which is about 3 a.u. for helium) due to contributions from the states of angular momentum $l = 2$, 4, etc. This corresponds to a characteristic temperature T_0 given by

$$T_0 = 2\hbar^2/mkr_1^2 \approx 2.3\text{°K}, \qquad (3.4)$$

which Green identified with the λ temperature. Below this temperature he showed that thermal waves arise:

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[5] Temperley (T.51b) has suggested that at extremely low temperatures this "linear" definition of ρ_n may differ from the "rotational" one, as measured by the oscillations of piles of plates (see Sec. 2.3). This difference would arise if in the rotational case some excited states corresponded to zero angular momentum. Dingle (D.52a) has expressed a contrary opinion.

these waves propagate heat and mass in opposite directions. The anomalous specific heat and density of helium II were attributed to the presence of these waves and the mechanocaloric effects (see Sec. 4) were explained by the same mechanism.

Green, moreover, pointed out that at the characteristic temperature, T_0 the radial distribution function in the liquid would undergo considerable modification, such that for $T<T_0$ the maxima in the radial distribution function characteristic of a normal liquid disappear, as shown in Fig. 3.1. Since it is known that the effect of the higher angular momentum states, which are excited for $T>T_0$, is to contribute to the normal maxima in the radial distribution function, Green concluded that the unusual features of liquid helium would disappear above T_0.

Green considered that the analysis of the density according to Eq. (3.1) is artificial. He stated that fluctuations in the pressure can be *interpreted* as density fluctuations by the formula $\pi_0=\pi+\Delta\pi=\pi_0(\rho_s/\rho)+\pi_0(\rho_n/\rho)$, but warned that the hypostatization of these densities may lead to unreal and absurd predictions. In his theory the two-fluid model is all but rejected.

A more complete discussion of this theory is beyond the scope of this review for the mathematical complexities are great.

Speaking now of its criticism, it must be noted that several authors have differed with Green at the very basis of the theory. They have found that $p_1=p$ identically. For bibliography on this question the reader is referred to the review by Dingle (D.52a), where he will find a further discussion. Another question concerns the very marked change in the radial distribution function which Green predicts should take place at the λ transition. (See Fig. 3.1.) It is surprising that the x-ray investigations of Keesom and Taconis (K.36a) and of Reekie (R.40, R.47, R.53a) have not shown the existence of such a change.

3.5. Prigogine and Philippot's Theory

The recent theory of Prigogine and Philippot (P.52a, P.53b, P.53c) gives a model of the thermal properties of helium II. No attempt is made to discuss the dynamical properties of the liquid. The model is a generalization of the cell model of an ordinary liquid developed by Lennard-Jones and Devonshire in which the total partition function for the liquid is obtained from the partition function at one typical molecule which moves in a cell determined by the molecule's neighbors. Prigogine and Philippot generalize this model by considering the fluctuations in the number of particles occupying a cell. In ordinary liquids, far from their critical points, the volume of a cell is only great enough to accommodate one molecule, but in helium even at low temperatures, the large zero-point energy of each atom insures that the volume of a cell ($=V_{\mathrm{mol}}/N_{\mathrm{mol}}$) is much greater (i.e., about 3 times greater) than the "hard sphere" volume of the molecule. Therefore, only in helium may multiple occupation of a cell be expected. The authors first develop the partition function for the case that 0, 1, or 2 particles may occupy a cell and find a specific heat in excess of that of an ordinary liquid, which excess specific heat for a reasonable choice of energy constants reveals a hump of about 1 cal/mole degree high at 1.5°K. A second model in which cells of volume $=2V/N$ are occupied by 1, 2, or 3 particles yields essentially the same results for *B-E* particles, but for Fermi particles the specific heat excess is smeared out with a maximum at a much higher temperature (actually a temperature near the critical temperature). In summary, this simple model shows roughly how a λ type of specific heat anomaly may arise in He^4, but not be found in He^3, that is, the model shows the analog of a Bose-Einstein condensation.

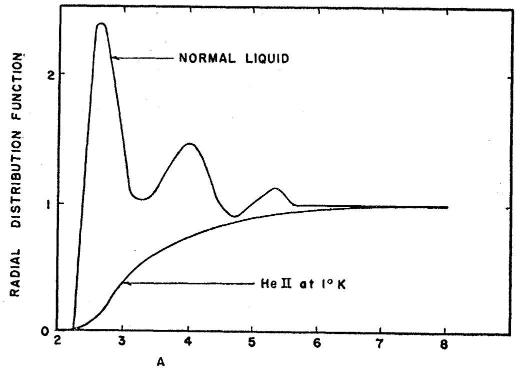

FIG. 3.1. The radial distribution function. Upper curve for a normal liquid; lower curve according to Green (G.48b) for liquid helium II at 1°K. (Curves due to Mendelssohn, M.49a.)

In their two subsequent papers, Prigogine and Philippot show, first, how their model leads to a negative coefficient of expansion of the liquid below the "λ point," i.e., the maximum of the specific heat anomaly; and, second, how a cooperative effect among the fluctuations leads to an infinity in the specific heat at T_λ rather than a smooth hump. This cooperative effect arises, they suggest, because the "excitation energy" for two molecules in a cell decreases with increasing number of fluctuations, i.e., as the temperature is increased.

3.6. Temperley's Fusion of the Landau and London Models

London's theory of the Bose-Einstein liquid has had considerable success in explaining the properties of liquid helium in the neighborhood of T_λ. Besides the explanation of the specific heat anomaly, it furnishes also a logical basis for an explanation of the viscosity of liquid helium, which in this temperature region is gas-like rather than liquid-like (see Sec. 2.1). Furthermore, evidence to be discussed in Sec. 3.7 below indicates that the statistics of the particles plays an important role in the behavior of the liquid since liquid He^3 is in many ways different from liquid He^4.

On the other hand, Landau's model is in closer accord with experiment below $T=0.6°K$. As has already been pointed out, the low temperature specific heat is apparently that of longitudinal phonons, while the experiments on second sound (Sec. 9.4) also indicate that the normal density ρ_n includes the effective mass of the phonons as Landau predicted. In fact, in order for a theory based on the Bose-Einstein condensation to be successful in the region below about 1°K, it has been necessary to add the phonon specific heat and mass to those quantities derived for the gas.

In view of these results, Temperley (T.52a) has concluded that the Landau approach is the proper one at lowest temperatures, but loses its validity above say, 0.6°K, while the London model is valid above, say, 1.5°K. Between these temperatures there is a transition region in which neither a solid nor an almost perfect gas furnishes a good approximate model. In attempting to reconcile the two models named above Temperley (T.52b) has recently outlined a theory in which large clusters in coordinate space (showing the properties of a Debye lattice) are in statistical equilibrium with small clusters in momentum space which play the role of the superfluid component.[6] The λ transition announces the appearance of the clusters in momentum space. The possibility of the formation of the two kinds of clusters in equilibrium with each other Temperley ascribes to the known fact that the binding forces in helium are just about strong enough to form a stable molecule. Because of the different symmetry properties of He^3 and He^4, Temperly does not expect He^3 capable of forming clusters in momentum space.

3.7. Experimental Evidence Concerning the Statistical Basis for the λ Transition

In attempts to test the merits of these theories great interest has centered on the question of the importance of the Bose-Einstein statistics and in consequence experimental attention has been focused on the rare helium isotope of mass three. Since the He^3 atoms obey Fermi-Dirac statistics, liquid He^3 should not show superfluidity if, as would seem reasonable from the large molar volume of He^3, liquid He^3 is as well approximated by an ideal Fermi-Dirac gas as London has postulated He^4 to be by an ideal Bose-Einstein gas. This criterion was suggested independently by Pollard and Davidson (P.42), by Franck (F.46), and by Onsager (O.48a).

The first experimental indications of the probable importance of the statistics were given by Daunt and co-workers (D.47a, D.47b) in measurement of the superflow of dilute mixtures of He^3 in He^4. They found that the He^3 atoms did not partake in superflow either in films or in the bulk liquid, a result confirmed by Lane and co-workers (L.48a). Subsequently, more decisive measurements were made on the flow of pure liquid He^3 by Osborne, Weinstock, and Abraham (O.49a) who found no superfluidity at any temperature between 1.05°K and 3.2°K, the normal boiling point. Later work by Daunt and Heer (D.50a) on the λ temperatures of solutions of He^3 in He^4, with measurements extending to solutions containing 89 percent He^3 (which last solution showed a λ temperature equal to 0.38°K), indicated that pure He^3 could not be expected to show superfluidity above 0.25°K.

There is, however, one piece of experimental evidence which complicates the picture presented in the last paragraph and which has yet to receive satisfactory explanation. Recent measurements of Hammel and Schuch (H.52d) indicated an anomalous film flow of He^3 out of solutions of He^3 and He^4. The He^3, comprising about four percent of the solution, was observed to flow through *unsaturated* surface films in the same manner as does He^4, i.e., apparently with zero pressure gradient, although at a much slower rate. However, these investigators have not detected any superflow in pure He^3, so the previous experiments of Osborne, Weinstock, and Abraham are not contradicted. Since the presence of the He^4 appears to be necessary to produce this anomalous flow in the He^3 through the unsaturated film, and since no anomalous He^3 flow has yet been observed through saturated films, it is not to be concluded from these experiments that He^3 is necessarily the same kind of liquid as He^4.

Other indirect evidence exists that liquid He^3 is not completely unlike liquid He^4. Many of the detailed properties of liquid He^3 were beautifully predicted by deBoer and Lunbeck (B.48b), using deBoer's "Quantum Law of Corresponding States" (B.48c) without consideration of the statistics involved. In particular, the vapor pressure equation was predicted quite accurately. Of course this is not very strong evidence against the "Bose-Einstein hypothesis" of F. London; it is rather evidence that at least some of the properties of liquids at very low temperatures may be insensitive to the nature of the particles—or to put the matter another way, that many properties of He^4 (and He^3) may be "explained" without a fundamental knowledge of the system involved. For a parallel example of a virtual stalemate, the reader is referred to the discussion of the variation of the λ temperature in He^3-He^4 mixtures presented by Daunt (D.52b) in a review of the properties of He^3 at low temperatures. There it is shown that a broad range of models, some involving consideration of statistics and some not, lead to pre-

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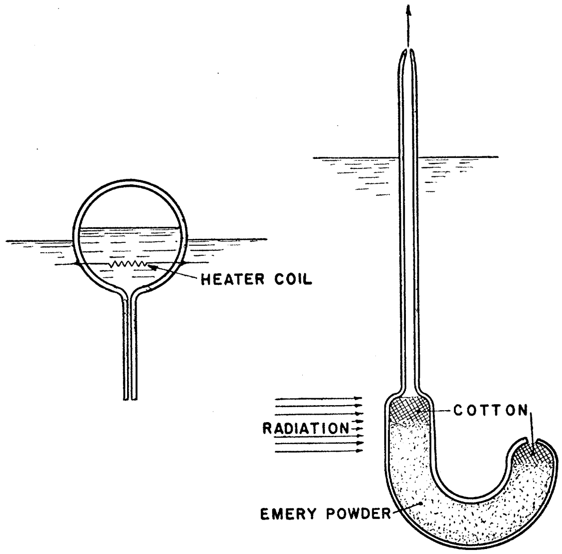
[6] The thermodynamics resulting from the assumptions of the occurrence of molecular association below T_λ have been discussed by Rice (R.49 and R.50). Although it is difficult to decide unambiguously from the existing experimental evidence whether such molecular complexes are in fact formed in configurational space, measurements of the electric polarizability (K.28a, K.28b, and G.50c), of the refractivity (J.38), and of the light scattering (J.43) have failed to reveal any significant difference in these quantities either in liquid helium I or liquid helium II.



Figs. 4.1 and 4.2. Apparatus used for observing the "fountain effect" Allen and Jones (and Misener) (A.38c).

dictions that can scarcely be distinguished experimentally.

From our discussion of the experimental evidence it should appear that it is not necessary to assume that liquid He^3 possesses the peculiar properties of helium II. The experimental evidence seems to point the other way at present. However specific heat measurements or vapor pressure measurements at still lower temperatures for liquid He^3 appear necessary to make the conclusions more decisive.¶

4. THERMOMECHANICAL EFFECTS AND THE SUPERFLUID AND NORMAL ENTROPIES

4.1. The Thermomechanical Effect

Some idea of the thermal state of the superfluid and normal constituents of helium II can be experimentally obtained from measurements on the thermomechanical effects in the liquid. Although no experiments on these effects have been performed more recently than about ten years ago [except for some very recent measurements below 1°K briefly noted by Bots and Gorter (B.53a) and by Rogers and Herlin (R.53b)], it seems appropriate to review here the earlier results, since in recent years a large number of theoretical papers have been concerned with the interpretation of the experiments.

The first experimental observations of the thermomechanical effects which can be produced were made by Allen, Peierls, and Uddin (A.37) who in attempts at measuring the heat conduction of liquid helium II noticed that their observations were disturbed by a peculiar flow of liquid occurring in the opposite direction to the flow of heat. The observations, which were first formally reported by Allen and Jones (A.38c), were made with an apparatus, illustrated in Fig. 4.1, consisting of a closed reservoir containing liquid helium II and connected to the main helium bath through a narrow capillary channel. Under isothermal conditions, as would be expected, the liquid levels inside and outside the reservoir would remain at the same height. When heat was introduced electrically inside the reservoir, it was found that liquid flowed into the reservoir and, as a result, the inside level rose above that of the liquid in the bath. This unusual behavior was strikingly emphasized by the following experiment of Allen, and Jones,(and Misener) (A.38c), employing an apparatus illustrated in Fig. 4.2, in which the reservoir was in the form of a narrow vertical glass tube partially immersed in a bath of liquid helium II. The lower end of the tube was constricted by a multiplicity of fine channels formed by the interstices of a tightly-packed plug of emery powder. Heat could be supplied by illuminating the powder

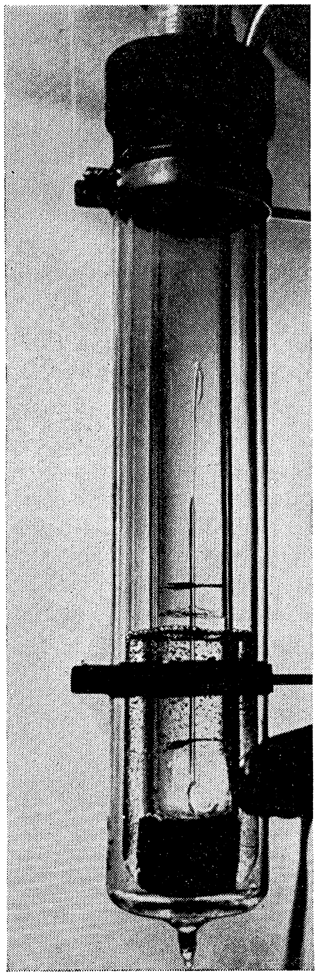


Fig. 4.3. Photo of helium II fountain. (Courtesy of the Research Laboratory of Electronics, Massachusetts Institute of Technology and A. D. Little, Inc.)

¶ *Note added in proof.*—Measurements of the specific heat of liquid He^3 down to about 0.5°K have just been reported by de Vries and Daunt [Phys. Rev. **93**, 631 (1954)], Roberts and Sydoriak [Phys. Rev. **93**, 1418 (1954)] and by Osborne, Abraham, and Weinstock [Phys. Rev. **94**, 202 (1954)]. No λ anomaly was observed.

with a lamp external to the cryostat, and it was found that on illumination liquid helium flowed so quickly through the powder into the tubular reservoir that it was ejected from the top of the tube in the form of a fountain. Evidently the superfluid constituent of the liquid, which alone could flow easily through the powder, is constrained to move in a direction towards the source of heat.

A photograph of such a helium "fountain" is shown in Fig. 4.3. Not unnaturally this thermomechanical effect in liquid helium II has subsequently acquired the title of "The Fountain Effect."

These exploratory qualitative measurements of Allen and co-workers were followed by quantitative observations by Allen and Reekie (A.39b) who used powder-filled tubes to form the narrow channel between the helium reservoir and the bath, by Kapitza (K.41b) who employed optically polished glass disks to form narrow channels, and by a group of workers at Leyden University, namely Duyckaerts (D.43), Keesom and Duyckaerts (K.47a), Mellink (M.47b), and Meyer and Mellink (M.47a) all of whom used apparatus essentially similar to that of Kapitza.

It was shown, moreover, by Daunt and Mendelssohn (D.39a) that the fountain effect could also be obtained with the helium reservoir linked to the bath of liquid helium II by means of a surface film only. This result was confirmed by Strelkov (S.40) and more recently by many others (see Sec. 7), and has been given a detailed interpretation by Daunt and Mendelssohn (D.50b). Finally, the fountain effect has also been shown by Long and Meyer (L.50a and L.52a) to occur between two volumes connected together only by an adsorbed unsaturated helium film below the λ temperature. (See Sec. 6.4.)

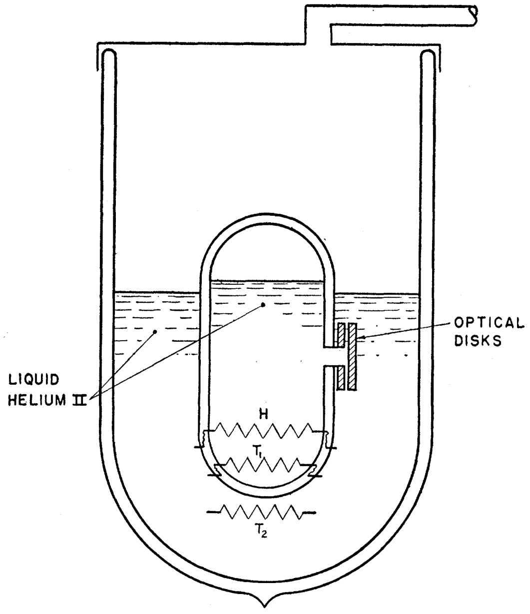


FIG. 4.4. Diagram of apparatus used for observing thermomechanical effects, etc. (After Kapitza, K.41b) H is a heater coil; T_1 and T_2 resistance thermometers.

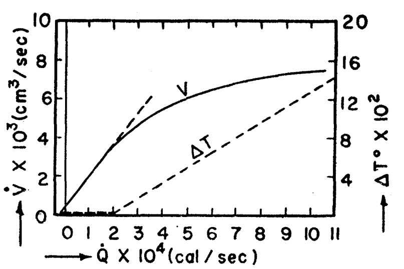


FIG. 4.5. Plot of the volume flow $\dot{V}$ in cm³/sec of helium II and of the temperature difference ΔT on either side of the narrow channel against heat input to the hotter volume as observed by Kapitza (K.41b).

4.2. The Thermomechanical Effect and Entropy Determination

Kapitza (K.41b), and Meyer and Mellink (M.47a) used the thermomechanical effect in helium II to make measurements of the difference between the entropy of the liquid partaking in the superfluid flow and that of the bulk liquid. Kapitza's apparatus is diagrammatically sketched in Fig. 4.4. It consisted of a Dewar vessel having a narrow channel between it and the surrounding helium bath, the channel being formed between two optically polished glass plates. The mean channel width could be adjusted between 3×10^{-5} and 30×10^{-5} cm, and this range of channel width was sufficiently narrow to ensure that preponderantly only the superfluid constituent of the liquid could pass through during the periods of experimentation. Resistance thermometers were placed inside and outside the Dewar vessel and an electrical heater was mounted inside. On supplying heat to the liquid inside the Dewar vessel the superfluid part of the helium II flowed from the bath through the narrow channel into the Dewar, and the rate of flow was determined by direct observation of the rate of rise of the inner level. Figure 4.5 gives a typical plot of the simultaneously observed rate of flow $\dot{V}$ in cm³/sec and the temperature difference ΔT between the inside and outside as a function of the rate of heat input $\dot{Q}$. For small heat inputs the volume flow $\dot{V}$ was linearly proportional to $\dot{Q}$ and this result can be expressed by:

$$\dot{Q}=\rho\dot{V}\cdot T\cdot\Delta S, \tag{4.1}$$

where ρ is the liquid density and where ΔS is the entropy difference per gm between the superfluid and the bulk liquid. $T\cdot\Delta S$, therefore, is the heat absorbed per gm of liquid on passing from the superfluid state into the bulk liquid. The values of ΔS obtained in this way by Kapitza fit on a smooth curve obtained by drawing the ΔS values of Table 4.1[7] as a function of temperature.

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[7] The ΔS values given in Table 4.1 are also experimentally observed results due to Kapitza (K.41b) but were obtained by the method described in Sec. 4.5.

TABLE 4.1. Entropy evaluations from the mechano-caloric effect by Kapitza (K.41b).

T °K	ΔS joules/g-deg
2.105	1.33
2.04	1.10
1.965	0.91
1.880	0.735
1.746	0.49
1.580	0.284
1.470	0.18
1.345	0.11
1.33	0.10

Meyer and Mellink (M.47a) reported similar results.

In order for Eq. (4.1) to be valid, it is necessary for the following two conditions to hold:[8] first none of the heat input Q must be dissipated in raising the temperature of the liquid which is in the reservoir. As can be seen from Fig. 4.5 Kapitza observed no change in the temperature inside the Dewar ($\Delta T=0$) for all flow rates below some critical flow rate $\dot{V}_{\mathrm{crit}}$ indicating that this condition was in fact met for small heat inputs. Secondly, none of the heat $\dot{Q}$ must be dissipated by conduction through the narrow channel. As is shown in Sec. 5, the true heat conductivity of the liquid is small, heat transport being due to circulation of the superfluid and normal constituents in opposite directions. In the absence of any significant flow of normal fluid through the narrow channel, therefore, as was the case in these experiments, the heat dissipated by conduction through the narrow channel was small. Confirmation of this comes from the observation that the results for ΔS were independent of the width of the channel over the range 3×10^{-5} cm to 30×10^{-5} cm.

It is to be noted in Fig. 4.5 that Kapitza's work showed that at a relatively well defined critical velocity of flow the linear dependence of $\dot{V}$ on $\dot{Q}$ broke down and at the same time temperature differences ΔT began to be set up between the inside and outside of the Dewar. Presumably at this critical flow rate $\dot{V}_{\mathrm{crit}}$ the superfluid character of the flow through the channel breaks down and excess heat is required to provide for the viscous irreversible processes then introduced. In this way, therefore, critical velocities of flow can be computed, and the results of the observations of Kapitza and of Meyer and Mellink of these critical velocities have already been presented in Sec. 2.5.

Meyer and Mellink (M.47a) reported that for velocities through the narrow channels greater than the critical velocity the heat absorbed per g of liquid as it flows from the channel into the bulk liquid, or vice versa, differed considerably from that found for flow velocities less than the critical velocity. The exact nature of the irreversible processes involved for flow velocities greater than critical and the exact location of the temperature differences then set up are not readily defined. A possible mechanism, however, involving temperature gradients just at the entrance or exit of the channels has been briefly discussed by Meyer and Band (M.48a).

4.3. Specific Heat Measurements and the Superfluid and Normal Entropies

The measurements reported in the previous section on the thermomechanical effect allow an evaluation to be made of ΔS, the difference in entropy per g between that of the superfluid constituent of liquid helium II and that of the bulk liquid. In order to evaluate either independently, comparison must be made with the total entropy of liquid helium II in bulk, as determined by direct calorimetric observations. These observations have been made often during the past 20 years.

Early measurements of the specific heat of liquid helium II between 1°K and 3°K were made by Keesom and Clusius (K.32a), Keesom and Miss Keesom (K.32b, K.35) and are illustrated in Fig. 1.2. Measurements between 0.6°K and 1.5°K with paramagnetic salts as the cooling agents have been made by Keesom and Westmijze (K.41d) and by Hull, Wilkinson and Wilks (H.50b), and the results indicated that the specific heat was approximately a sixth power function of the absolute temperature in this temperature range. The most recent measurements are those of Kramers, Wasscher, and Gorter (K.52a) which extend down to 0.25°K and which indicate that: for $0.25°\mathrm{K}<T<0.6°\mathrm{K}$, $C_{\mathrm{sat}}=0.0223T^3$ cal/mole-deg, for $0.7°\mathrm{K}<T<1.4°\mathrm{K}$, $C_{\mathrm{sat}}=0.106T^{6.7}$ cal/mole-deg, and which also indicate that above the T^3 temperature-dependent region the specific heat cannot be very accurately described by a single power of T but rather appears more like an exponential function.

Evidently the T^3 region of the temperature dependence of the specific heat indicates that longitudinal compressional waves exist in the liquid (i.e., phonon excitation), as is evident by comparison of the theoretical value with the experimental value of the specific heat. The absolute value of the phonon specific heat is given by: [as has been further discussed recently by Kronig and Thellung (K.52b)]

$$C_{ph}=\frac{16}{15}\pi^5\frac{k^4}{h^3}\cdot\frac{1}{\rho}\frac{T^3}{u_1{}^3},\tag{4.2}$$

where u_1 is the longitudinal velocity of sound. Using the value of $u_1=237$ m/sec given by extrapolating to low temperatures the recent measurements by Atkins and Chase (A.51b), one gets

$$C_{ph}=0.020T^3 \text{ cal/mole-deg},$$

which is in excellent agreement with the measured C_{sat} below 0.6°K.

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[8] A correction of course must be applied, as was pointed out by Meyer and Mellink (M.47a), for the heat liberated by the vapor in the Dewar when it condenses to a smaller volume to accommodate the inflowing liquid.

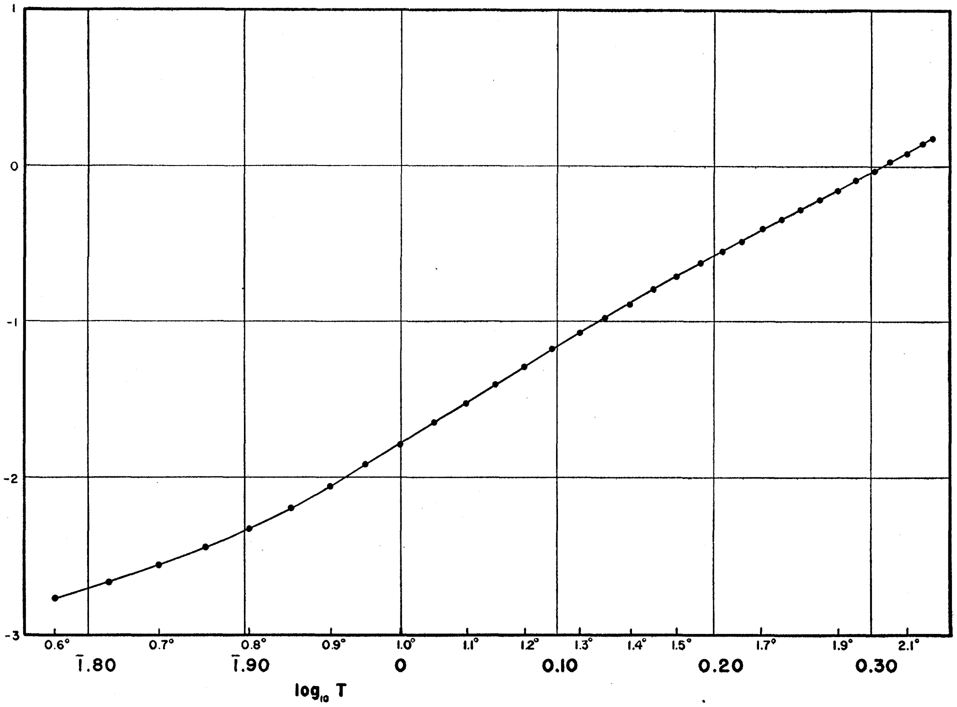

FIG. 4.6. The entropy of helium II from data of Kramers *et al.* (K.52a). The ordinate plots $\log_{10} S$, where S is in joules/g-deg.

Evidently the much higher values of C_{sat} obtained at temperatures above 0.7°K than would be expected from phonon excitation indicate the establishment of other excitations, the possible nature of which have been briefly discussed in Sec. 3 above.

Using the results of the earlier of these specific heat measurements, computations of the total entropy S have been made by Gorter, Kasteleijn, and Mellink (G.50a), and by Band and Meyer (B.48a). As these computations were made before the recent observations of Kramers *et al.* (K.52a), they were based on an erroneous extrapolation for the specific heat of liquid helium as a T^6 function from 1°K down to 0°K. The computations implicitly assumed the well established conclusion that helium remains liquid down to 0°K.[9] These computations have now been superseded by the more recent ones of Kramers *et al.* (K.52a), using a T^3 extrapolation for the specific heat from 0.6°K to 0°K, and these new evaluations are given in Table 4.2 and are shown in Fig. 4.6. This figure is plotted logarithmically and it will be seen that even above 0.9°K the curve is not linear. This means that if it is desired to express S by the relation, as has been suggested by Tisza (T.47) and by London (L.45),

$$S/S_\lambda = (T/T_\lambda)^\xi, \tag{4.3}$$

then a unique value of ξ cannot accurately describe the results even over the temperature range 1°K to 2.1°K. From Fig. 4.6 it is apparent that very approximately ξ equals 6.7 for the temperature range $0.9°\text{K} < T < 1.45°\text{K}$, and ξ equals 5.4 for $1.45°\text{K} < T < 2.1°\text{K}$.

If now the values of ΔS, of Table 4.1, computed from the experimental results of the thermomechanical effect of Kapitza (K.41b), be compared with the calorimetric determinations of the total entropy S given in Table 4.2, then it will be found that ΔS and S are essentially the same within the probable experimental error. Since ΔS is the difference in entropy between that of the bulk liquid and that of the liquid taking part in superfluid flow, one must conclude from these results, as was pointed out by Kapitza (K.41b), that *the liquid that partakes in superfluid flow has zero entropy*. This very fundamental result has also been arrived at by other experimental lines of approach, as is discussed as follows in Sec. 4.5, and it can be regarded as well established. The result can also be expressed by saying that the "normal" constituent of liquid helium II carries all the entropy of the liquid.

It should be remarked parenthetically here that the possible existence of a T^3 region of the entropy of the liquid below about 0.7°K was suggested in 1938 in preliminary statements of results of measurement of the specific heat of liquid helium II by Pickard and Simon (unpublished, but see Bleaney and Simon, B.39a); yet it was not until 1952 that its existence was definitely proven experimentally by Kramers, Wasscher, and Gorter (K.52a). Such a T^3 variation of the specific heat at the lowest temperatures was, however, interpreted by the earlier group of workers as indicating that the "anomalous" region of liquid helium II had been

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[9] Some recent work by Cwilong (C.52a) on the measurement of the melting curve appeared to throw doubt on this conclusion, but it was subsequently withdrawn (C.52d).

TABLE 4.2. Smoothed values of the specific heat, C_{sat}, and the total entropy S of helium II according to Kramers, Wasscher, and Gorter (K.52a).

T °K	C_{sat} joule/g-deg	S joule/g-deg
0.60	0.0051	0.00169
0.65	0.0068	0.00215
0.70	0.0098	0.00276
0.75	0.0146	0.00358
0.80	0.0332	0.00475
0.85	0.0343	0.00644
0.90	0.0510	0.00885
0.95	0.0743	0.0122
1.00	0.1042	0.0168
1.05	0.142	0.0227
1.10	0.191	0.0304
1.15	0.250	0.0402
1.20	0.322	0.0523
1.25	0.410	0.0672
1.30	0.516	0.0853
1.35	0.634	0.1069
1.40	0.780	0.132
1.45	0.944	0.162
1.50	1.127	0.197
1.55	1.330	0.238
1.60	1.572	0.284
1.65	1.83	0.336
1.70	2.11	0.395
1.75	2.46	0.461
1.80	2.80	0.535
1.85	3.19	0.617
1.90	3.63	0.709
1.95	4.27	0.812
2.00	4.95	0.929
2.05	5.82	1.061
2.10	6.92	1.215
2.15	8.61	1.40
2.18	11.6	1.53
2.186	14.3	1.57

passed in reducing the temperature below 0.7°K and that one was dealing with a remanent "lattice" type entropy. Such a view, essentially postulating a lower "transition" temperature at about 0.7°K, below which thermomechanical effects, etc., should disappear, had also been taken by Tisza (T.47 and T.49a) and by Kurti and Simon (K.38d), the latter authors using the hypothesis to explain their observation of very small heat conductivities of liquid helium in the temperature range 0.2°K to 0.5°K. Recent measurements of the velocity of second sound (see Sec. 9.4), however, as well as the results reported in this section and in section 4.5, clearly show that the "phonon" entropy, evidenced by the T^3 specific heat region, belongs only to the normal constituent of helium II and that, as observed above, *the superfluid constituent of helium II has zero entropy at all temperatures from 0°K to* T_λ.

One may recall here that in a superconductor, which provides another example of superfluid flow, Daunt and Mendelssohn (D.38a, D.46a) have shown experimentally by measurement of the Thomson heat that the electrons engaged in superflow are also energetically at absolute zero. In both cases, for helium II and for superconductors, as has been re-emphasized by Mendelssohn (M.45), the superfluid particles remain in the same energy states that they have at the absolute zero, although the rest of the material may be at finite temperatures. The evident similarity between the phenomena of superflow in helium II and in superconductors, for which the superfluid constituent in each has zero entropy, has been a subject for much speculation and discussion by many authors, for example by Landau (L.41a), Daunt and Mendelssohn (D.42 and D.46b), Mendelssohn (M.45), F. London (L.45).

4.4. The Mechano-Caloric Effect

The primary feature of the thermomechanical effect, or "Fountain" effect, was that on application of heat superfluid helium II flowed towards the source of heat. If the effect were reversible, it would be expected, as was first suggested by Tisza (T.38a, T.38c), that superfluid flow should be accompanied by a development of heat (or cold). This reverse effect, the mechanocaloric effect, was first experimentally observed by Daunt and Mendelssohn (D.39a), using an arrangement as shown in Fig. 4.7. Liquid helium II was allowed to flow out of a closed Dewar vessel through a plug P filled with fine powder, through which the flow was substantially superfluid owing to the narrowness of the channels. Measurement of the temperature of the liquid remaining in the Dewar vessel by means of the resistance thermometer T indicated a momentary temperature rise, showing that the liquid removed by the superfluid flow was in a lower thermal state than that of the liquid as a whole. Moreover, on lowering the vessel, so that the liquid now flowed from the bath into the vessel, the liquid in the vessel was found to cool. The results corresponded approximately to an entropy of the superfluid constituent equal to zero.

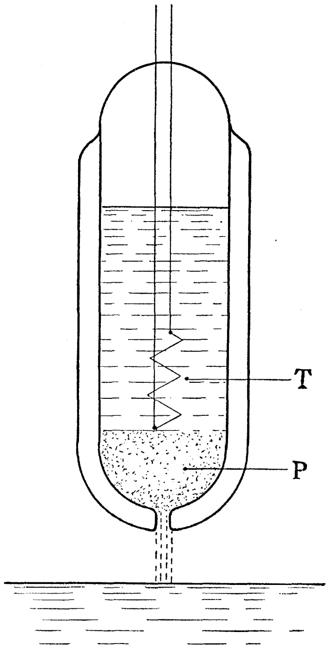

FIG. 4.7. Apparatus used by Daunt and Mendelssohn (D.39a) for observing the mechano-caloric effect. P is packed powder plug; T is resistance thermometer.

Similar experiments were subsequently carried out by Kapitza (K.41b) who observed the flow of liquid helium II into a thermally isolated bulb through a narrow slit formed between two optically polished glass disks, the experimental arrangement being the same as that described in Sec. 4.2 and illustrated in Fig. 4.4. He also found that if the liquid level in the bulb was set higher than that in the main helium bath, so that superfluid flow took place from the bulb into the bath, the temperature inside rose. The thermal isolation of the bulb was sufficiently good to allow the level differences and the temperature differences to be maintained for long periods of time and consequently estimates could be made of the equilibrium dependence of the temperature difference ΔT between the temperature of the liquid helium II in the bulb and that in the bath, on the pressure difference Δp between them. The results obtained are given in Fig. 4.8, in which it will be seen that at least for small temperature differences there is a linear relationship between Δp and ΔT at each average temperature of measurement.

Similar accurate measurements of the same kind have been made for flow through narrow slits ($d \leq 10^{-4}$ cm) by Meyer and Mellink (M.47a) and their results for $\Delta p/\Delta T$ as a function of temperature T are given in Fig. 4.9. These results are in good agreement with those of Kapitza and represent very satisfactorily the relationship between the so-called "fountain pressure," Δp and the corresponding temperature differences that are set up between two volumes of helium II connected by a very narrow channel.

When the channel is made wider, as has been investigated for channel widths from 1 micron to 100 microns by Keesom and Duyckaerts (K.47a) and by Mellink (M.47b), the relationship between the fountain pressure Δp and the temperature difference ΔT becomes more complicated than is found for the very narrow channels. This is due to the fact that for the wider channels some flow of the normal or viscous constituent of the liquid helium II can take place through the channel, and this flow takes place naturally from the volume at higher pressure to that at the lower pressure, i.e., in a direction opposite to that of the superfluid flow. This circulation of matter results in a large heat transfer from one volume to the other through the channel and, as a result, the fountain pressure cannot be maintained without a continuous supply of heat being injected into the hotter volume. Such a process is more properly discussed in connection with other observations on the heat conductivity of helium II and is postponed to Sec. 5.

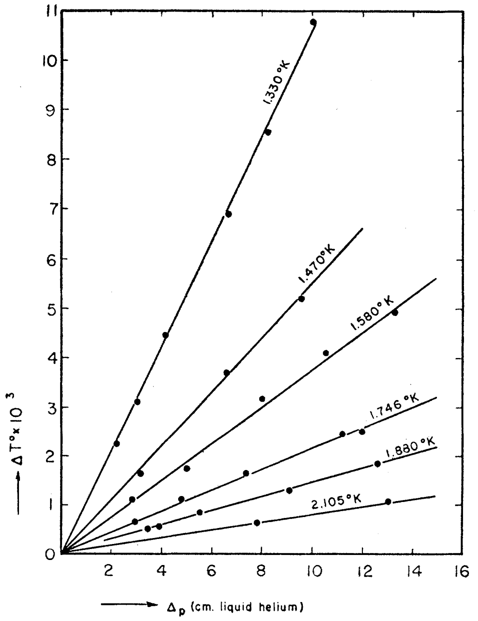

FIG. 4.8. Observed relation between ΔT and Δp for fountain effect. From Kapitza (K.41b). ΔT is plotted in millidegrees against Δp in cm of liquid helium.

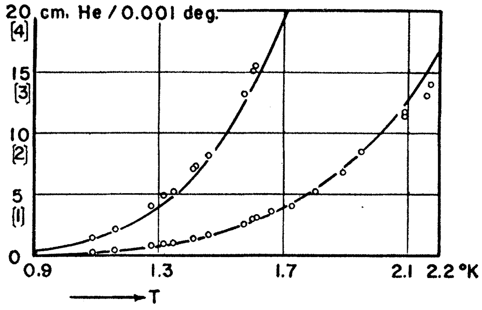

FIG. 4.9. Observed fountain pressure in cm liquid helium per 0.001°K temperature difference as a function of temperature. Points are data of Meyer and Mellink (M.47a). The full curves give the theoretical values from Eq. 4.4. The upper curve is 5 times greater scale than lower one.

4.5. Further Evaluation of the Superfluid and Normal Entropies

As is discussed in Sec. 4.7, the following formula, first proposed by H. London (L.38c and L.39b), is applicable to these measurements of the fountain pressures and their accompanying temperature differences for flow of liquid through very narrow channels:

$$(dp/dT)_{\Delta T \to 0} = \rho \cdot \Delta S, \qquad (4.4)$$

where ρ is the liquid density and ΔS is the entropy difference between the entropy S per g of total fluid in bulk and the entropy S_s per g of superfluid that takes part in superfluid flow.[10] If now the values of ΔS obtained from this formula 4.4, using observed values of dp/dT (Table 4.1 sets out the values obtained in this way by Kapitza (K.41b)), be compared with the calorimetric determination of the total entropy S, given in Table 4.2, then it will be found that ΔS and S are essentially the same within the experimental error. It is concluded, therefore, as from the experiments

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[10] A more sophisticated significance for this quantity has been given by Gorter (G.49b) which is discussed in more detail in Sec. 4.7.

reported in Sec. 4.2, that *the liquid that partakes in superfluid flow has zero entropy*. This conclusion has been re-emphasized in recent briefly reported experimental work by Rogers and Herlin (R.53b).[11]

It might be remarked here that the evaluation of ΔS from Meyer and Mellink's results on the thermomechanical effect in the temperature range 1.0°K to 1.4°K appears about 10 percent greater than the total entropy, as computed by Gorter, Kastelijn, and Mellink (G.50a). The latest evaluations of S by Kramers *et al.* (K.52a), however, have largely removed this discrepancy.

A brief report of recent observations of the fountain effect below 1°K by Bots and Gorter (B.53a) suggests that deviations from Eq. (4.4) occur below 1°K. Comment, however, is withheld pending further detailed information.

4.6. The Mechano-Caloric Effect as a Cooling Agent

The fact that the mechano-caloric effect results in heatings and coolings being created through the agency of superfluid flow has suggested this process as a possible cooling method. Kapitza (K.41b and K.41c) has reported the results of preliminary experiments to this end in which he states that temperature drops of 0.4°K had been easily attained. No detail of his experimental arrangements, however, was given.

The method has been discussed and criticized in detail by Simon (S.50b). His criticism was not of the experimental feasibility of using the mechano-caloric effect as a cooling agent, but rather concerned the lack of usefulness of helium as a working substance on the grounds of its small and monotonously decreasing entropy as 0°K is approached.

4.7. Theories of the Thermomechanical Effect

The qualitative explanation of the experiments discussed above was given by Tisza (T.38a, *et seq.*). Seizing on London's suggestion that helium II may be a partially condensed Bose-Einstein liquid, Tisza gave a qualitative explanation of the fountain effect and predicted the inverse mechano-caloric effect. As has been noted, Tisza split the liquid into two fluids. The superfluid, whose density he took to be a function of T only, possessed no viscosity and exerted no pressure; the normal fluid exhibited both. When helium II passed through a fine capillary, the normal fluid was left behind because of viscous drag, whereupon the increase in superfluid density induced a temperature gradient (and a pressure gradient). In a capillary so fine that the normal fluid would be completely immobilized the flow could be reversible, and Tisza recognized that the reverse effect is the fountain effect: a temperature gradient gives rise to a gradient of the superfluid density and hence to a pressure gradient. Tisza later (T.38c) extended these ideas by noting that if the superfluid has no entropy, as the Bose-Einstein theory suggests, liquid flowing through a fine capillary leaves its entropy behind and produces a heating effect in the vessel being emptied and a cooling effect in the vessel being filled. This is the mechano-caloric effect.

A quasi-thermodynamic theory of these effects was given by H. London (L.38c; L.39b). London employed a method analogous to Kelvin's method for deriving the thermoelectric equations in which it is assumed that reversible effects can be treated separately from irreversible effects. He considered a helium circuit consisting of two reservoirs joined by two flow paths, one path through a fine capillary, the other through an engine. The two reservoirs were at different temperatures and pressures. The analog of the thermal emf was (dp/dT) for the helium in the reservoir, the analog of the Peltier heat was the heat left behind by the superfluid helium entering the fine capillary, and the analog of the Thomson heat of the helium flowing in the capillary up or down a temperature gradient was introduced, also. When London introduced Tisza's assumption that the superfluid particles are particles in the ground state of the system and therefore flow reversibly, i.e., without interaction, through the capillary so that the Thomson heat vanished, he arrived at Eq. (4.4) for (dp/dT), the "thermal emf," and at (4.1) for the mechano-caloric coefficient, the "Peltier heat."

Before proceeding further in this discussion of theories of these effects, we give the following derivation of Eq. (4.4), which may be of interest because of its simplicity and which is essentially the argument presented by Tisza (T.47). Since the experiments discussed in the early parts of Sec. 4 have shown that in sufficiently small capillaries and for sufficiently small velocities, flow of superfluid helium is reversible, while the theories outlined in Sec. 3 also support this point of view, purely thermodynamic arguments may be considered to apply. Imagine two thermally isolated vessels connected by a capillary so fine that only superfluid can pass (or connected by the superfluid film) and let one be characterized by the coordinates (p,T) and the other by the coordinates $(p+\Delta p, T+\Delta T)$. For equilibrium to obtain in the system the Gibbs function per g of superfluid must then be the same in both vessels. This implies (since the Gibbs function per gm of normal particles must equal the Gibbs function per gm of superfluid particles) that

$$G(p,T)=G(p+\Delta p, T+\Delta T),$$

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[11] From a detailed numerical analysis of the results of measurement of second sound (see Sec. 9.4) Koide, Matsudaira, and Usui (K.52d), using the thermodynamic theory of Usui (U.51a), have proposed giving the superfluid entropy S_s small finite values in order to account for discrepancies between theory and experiment. These discrepancies, however, are not larger than about 3 percent, and it appears to us possible that the experimental errors may account for them.

or in the limit of small ΔT

$$\mathrm{grad}G=0=\frac{1}{\rho}\,\mathrm{grad}p-S\,\mathrm{grad}T,$$

where the thermodynamic quantities G and S are per g of *total* fluid. This is, of course, H. London's equation

$$\mathrm{grad}p/\mathrm{grad}T=\rho S. \tag{4.5}$$

The unusual thing about this application of the equilibrium condition is that different temperatures obtain in the two vessels. Reversible flow between two reservoirs at different temperatures is possible, of course, only because the fluid taking part in the flow (and linking the reservoirs) has zero entropy. If the substance which flows possesses entropy then there is thermal conduction and hence production of entropy in the process, so that the rigorous derivation of the fountain effect should be made according to the discipline of irreversible thermodynamics.

It should be pointed out that a derivation of Eq. (4.5), essentially equivalent to the one given here, has been given by Temperley (T.52a) in a paper in which he discusses the theoretical "no-man's-land" between the London and the Landau theories. His derivation, however, comes after some rather special arguments. We have taken the point of view here that, thanks to the observed properties of the superfluid, the simplest thermodynamic arguments apply.

A possible interpretation of Eq. (4.5) is that even in the absence of a hydrostatic pressure gradient a "fountain pressure" acts on the superfluid helium in the presence of a temperature gradient. One is then led to postulate that the usual term in grad$p$ in the ordinary hydrodynamical equations must be supplemented in the case of helium II with a term involving grad$T$ to give the proper equations of motion. The same result may be obtained by generalizing intuitively the argument leading to Eq. (4.5), i.e., by saying that in the absence of equilibrium the superfluid connecting the two reservoirs will be accelerated in the direction of decreasing Gibbs function. Therefore, since grad$G$ is a force per unit mass,

$$dv_s/dt=-\mathrm{grad}G,$$

or in linear approximation

$$\rho_s\left(\frac{\partial v_s}{\partial t}\right)=-\left(\frac{\rho_s}{\rho}\right)\mathrm{grad}p+\rho_s S\,\mathrm{grad}T. \tag{4.6}$$

This equation is the same as the linear approximation to Landau's (L.41a) hydrodynamical equation for the superfluid and in fact the "derivation" given here is essentially the same as his.

Equation (4.6) which is a generalization of (4.5) may also be thought of profitably as a generalization of ordinary hydrodynamics in which the acceleration is the negative gradient of the Gibbs function but in which the effects of temperature are wholly neglected. (These matters of the hydrodynamical equations are discussed more fully in Sec. 8.1.)

Equation (4.6), while it is adequate to describe the behavior of the superfluid in oscillatory processes such as occur in second sound (see Sec. 9), does not give more than a qualitative insight into the flow of superfluid through a capillary or a surface film from a higher to a lower level. In the first place such flows are characterized by a critical velocity which implies additional terms in (4.6). In the second place in, say, steady-state flow through a capillary continuity requires that grad$G$ vanish inside the capillary so that at least part of the gradients of p and T must occur in a complicated way in the reservoir *before* the superfluid enters the capillary. Furthermore, every flow from one level to a lower or from one temperature to a higher involves a dissipation of free energy: Eq. (4.6) says nothing about this, that is, how the superfluid loses its momentum and gets mixed with the bulk liquid in the "lower" reservoir.

To return to H. London's equation (Eq. 4.4) and its history: first, it is clear that such a fountain pressure equation can always be derived from the hydrodynamical equations as a special result for static conditions (compare Sec. 8.1) (unless, of course, the fountain pressure equation has already been used in the derivation of the hydrodynamic equations.) Such conclusions drawn from the hydrodynamics are subject to the limitation that most hydrodynamical discussions leave out or drop by the way all irreversible effects. As an example, we obtain from the hydrodynamical equations derived by Dingle (D.49), in which the superfluid as well as the normal fluid is considered to carry entropy,

$$\mathrm{grad}p/\mathrm{grad}T=\rho(S-S_s), \tag{4.7}$$

where S_s is the entropy per gm of superfluid.

Gorter (G.49b) has offered a revision of H. London's equation based on reasonings applied to a quasi-thermodynamic model for reversible flow of both constituents in a wide capillary. He obtains for the analog of Eq. (4.5)

$$\mathrm{grad}p/\mathrm{grad}T=\rho x\left(\frac{\partial S}{\partial x}\right);\quad x=\frac{\rho_n}{\rho}. \tag{4.8}$$

This reduces to London's equation only if

$$S=x(\partial S/\partial x)_{p,T}. \tag{4.9}$$

Equation (4.9) leads to the result that

$$S=xS_0(p,T), \tag{4.10}$$

which is a generalization of an assumption made by Tisza (T.47) that $S=xf(p)$ only. Note that Eq. (4.10) shows that S and x need not vary in the same way with temperature (although over much of the superfluid temperature range they do approximately), since

$$\left(\frac{\partial S}{\partial T}\right)_p = x\left(\frac{\partial S_0}{\partial T}\right)_p + S_0(p,T)\cdot\left(\frac{\partial x}{\partial T}\right)_p.$$

Usui (U.51a) has also derived Eq. (4.8, 4.9, 4.10) and has pointed to the distinction to be made between $(\partial S/\partial x)_{p,T}$ and $(\partial S/\partial x)_p$.

Dingle (D.51) has criticized Gorter's work severely. He neither agrees with the result nor justifies the treatment. According to Dingle, Gorter has implicitly equated the concentration x of the normal fluid to the effective mass, whereas if these quantities be defined and derived in the manner in which Landau does, they are not equal. It should be pointed out in this respect however, as has been shown by F. London,[12] that if the entropy be derived from the Gibbs function $G(p,T,x)$, in the usual manner, then Eq. (4.9) follows immediately, if one assumes S_s (the superfluid entropy) equals zero; and this without any appeal to a particular microscopic model or to a Gedanken experiment.

Some authors have objected to the use of thermodynamics or quasi-thermodynamics in the derivation of equations describing the fountain and mechanocaloric effects. For example, Meixner (M.38) in an interesting paper in which the analogy between the fountain effect in helium and thermoelectric effects in conductors is emphasized, questions the validity of H. London's use of the second law of thermodynamics in his derivation. Meixner calculates the production of entropy in a helium circuit consisting of two reservoirs at different temperatures connected by two capillaries. He suggests that no experimental arrangement can make the irreversible effects (those due to thermal conductivity and viscosity) small compared to "Peltier" heat, so that the methods of thermodynamics need not be expected to yield correct results.

In any ordinary system these objections are certainly valid: rigorous treatments of problems of this kind must employ the methods of irreversible thermodynamics. This has been done in the case of helium II by de Groot and his co-workers (G.47a; G.50b; G.51a; G.51b). (See especially, G.50b and G.51b.) De Groot *el al.* obtain expressions for the fountain pressure and the mechano-caloric effect for the general case of two fluids which can undergo a chemical reaction: $1 \rightleftharpoons 2$. They then introduce the special assumptions usually made for helium II, (*i*) that the two fluids react infinitely rapidly, (*ii*) that only the superfluid flows through the capillary, and (*iii*) that the superfluid does not interact with the walls of the capillary, i.e., that it is transferred in bulk. In this case their equations reduce to Gorter's equations, i.e., Eq. (4.8) and the appropriate analog of Eq. (4.1), namely,

$$\dot{Q} = \rho\dot{V}Tx\left(\frac{\partial S}{\partial x}\right). \tag{4.11}$$

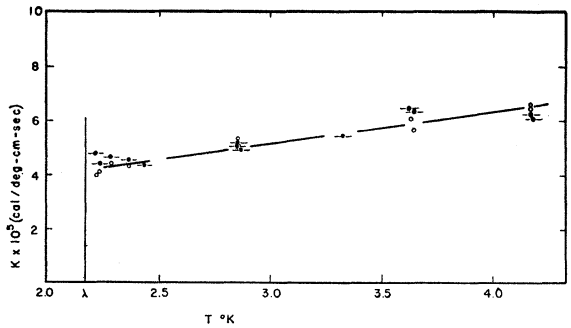

FIG. 5.1. Observed heat conductivity of liquid helium I due to Grenier (G.51c).

There is, of course, no conflict between the thermodynamic and the irreversible-thermodynamic approach to the problem of helium II. (E.g., in helium II the irreversible effects discussed by Meixner apparently do not arise in small capillaries, so that H. London's equations are directly applicable.) On the one hand the authors last quoted have preferred to proceed in as general a manner as possible, applying their results to helium II as a special case. One might say that they admit nothing special about helium II until they have to. The authors referred to earlier, on the other hand, have preferred to emphasize the peculiar properties of helium II—the vanishing entropy and viscosity of the superfluid—and as a consequence have felt justified in taking the simple thermodynamic point of view.

5. HEAT TRANSPORT

5.1. Heat Conductivity in Liquid Helium I

The heat conductivity of liquid helium I has been measured between T_λ and 4.2°K by Grenier (G.51c) and by Bowers and Mendelssohn (B.51c) and a solitary observation at 3.3°K has been reported by Keesom and Miss Keesom (K.36c). The results of all these authors are in good agreement with each other, and they indicate that helium I behaves normally with a relatively small value of the heat conductivity. The results obtained by Grenier, using a modified Lee's disk method are shown in Fig. 5.1, in which it will be seen that the heat conductivity increases approximately linearly from a value of 4.5×10^{-5} cal/deg-cm-sec at 2.24°K to 6.3×10^{-5} cal/deg-cm-sec at 4.2°K.

It is interesting to note that the heat conductivity shows no indication of anomalously large values just above T_λ. This lack of any apparent preparation for the λ transition as the temperature is reduced towards T_λ in the macroscopic property of the heat conductivity is in marked contrast to definite signs of such preparation revealed in the viscosity measurements (see Sec. 2.1), in the attenuation of first sound (see Sec. 9.1) and in the observation of the specific heat (see Fig. 1.2).

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[12] We are indebted to the late Professor F. London for kindly pointing out to us this conclusion presented in the manuscript of his book, "Superfluids, Vol. II."

It is to be noted also that if the simple gas kinetic formulation is used for theoretical computation of the heat conductivity, namely,

$$K = 2.5\eta C_v, \tag{5.1}$$

one obtains a value of the same order of magnitude as the observed value. At 4.2°K the gas kinetic value is 4.4×10^{-5} cal/deg-cm-sec and the variation with temperature is essentially the same as for the observed heat conductivity, indicating once again that helium I can in first approximation be treated effectively as a gas.

5.2. Heat Transport in Liquid Helium II

The heat conductivity of liquid helium II was noticed at an early date by Keesom (K.35) and by Kurti, Rollin, and Simon (K.36d) to be appreciably much larger than that of helium I and the first direct observations of the apparent heat conductivity were reported by Keesom and Miss Keesom (K.36c), who found that through capillaries of 0.6 mm diameter the heat conductivity of helium II reached values of about 200 times that of copper at room temperatures or about 5×10^6 times greater than that of helium I.

Early measurements of the heat conductivity of helium II, all using wide capillaries (of diameter from 0.2 mm to 1.6 mm), were reported by Allen, Peierls, and Uddin (A.37), Keesom, Keesom, and Saris, (K.38e), Allen and Ganz (A.39c), Kapitza (K.41f), Keesom and Saris (K.40a) and by Keesom, Saris, and Meyer (K.40b), some review of which has already been given by Keesom (K.42) and by Darrow (D.40). All these measurements showed that it was *not* possible to describe the heat transport in capillaries containing helium II by the usual equation for heat conduction, namely,

$$\dot{q} = \dot{Q}/A = K \cdot \Delta T/l, \tag{5.2}$$

where $\dot{q}$ is the heat current density, $\dot{Q}$ the total heat current, A the cross section of the capillary, K the heat conductivity, ΔT the temperature difference between the ends of the capillary and l the capillary length. Instead the following general conclusions were arrived at:

(a) the heat current $\dot{Q}$ was proportional to the total cross section of the capillary, as is expected from Eq. (5.2).

(b) the heat conductivity K was independent of the length of the capillary, as is expected in Eq. (5.2).

(c) the heat current density $\dot{q}$ is proportional to the third root of the temperature gradient, i.e., $\dot{q} \propto (dT/dl)^{\frac{1}{3}}$, a result totally unexpected from Eq. (5.2) and a result which renders it meaningless to quote numerical values for a "heat conductivity," as normally understood. For the temperature range 1°K to 1.6°K,

$$\dot{q} = 0.623 T^5 (dT/dl)^{\frac{1}{3}} \text{ watts-cm}^{-2}. \tag{5.3}$$

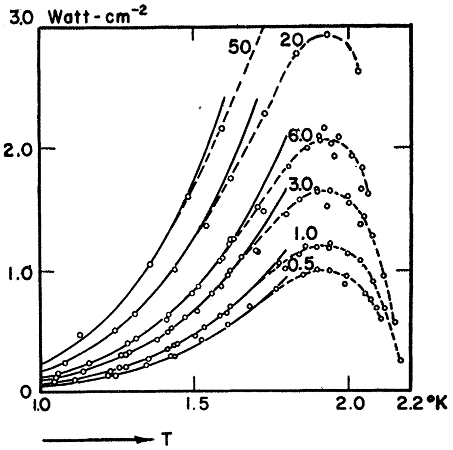

Fig. 5.2. Observed heat current density in watts/cm² in liquid helium II through wide capillaries against temperature as reported by Keesom, Saris, and Meyer (K.40b). The numbers attached to the curves give the temperature gradient $\times 10^3$ in deg/cm. The full drawn curves were calculated from Eq. (5.3). The broken curves give the experimental results.

Some data are shown in Fig. 5.2.

The results of these early measurements clearly showed that heat transport in helium II was an unusual process, which could not be explained by the formalism of heat conduction. Instead the two fluid theory, developed by Tisza (T.38a), provided an insight into the mechanisms involved in the heat transport process. In considering two volumes of helium II connected by a capillary, such as was used for the measurement of the heat transport, Tisza proposed that there was a circulation of mass between them, such that the superfluid constituent of the liquid would move without viscous losses towards the source of heat (as in the fountain effect) and such that the "normal" constituent would move in the opposite direction at a rate determined by the frictional forces that may be present. Such a circulation of mass would provide a mechanism for transportation of very large heat currents, as was first clearly emphasized by H. London (L.38c), in a manner analogous to the Peltier effect, as follows: the superfluid on arriving at the volume at the higher temperature would be "excited" and would thereby absorb heat in amount given by Eq. (4.1), since an amount of heat $T \cdot \Delta S$ per g would be required to convert the superfluid into bulk liquid; similarly at the entrance to the capillary at the colder volume a corresponding amount of heat would be released. The total heat transport $\dot{Q}$, therefore, would be in this process,

$$\dot{Q} = \dot{M} \cdot T\Delta S, \tag{5.4}$$

where $\dot{M}$ is the mass flow of either constituent. There would in addition be an extra amount of heat carried by the normal fluid given by $\dot{M} \cdot C_v \cdot \Delta T$, but this clearly is negligibly small compared with that of

Eq. (5.4).[13] This qualitative explanation has been further amplified and put on a firmer quantitative basis by many authors since that time (see Secs. 5.4 and 5.6) and it is now accepted as being correct.

It is of interest to outline in passing some of the early experiments of Kapitza (K.41f), in which he demonstrated elegantly the existence of the circulating mass currents that accompany heat transportation. In one series of experiments he connected a closed bulb containing a heater to the liquid helium II bath by means of a horizontal capillary of diameter 0.55 mm, and suspended on a torsion balance a flat vertical disk just in front of the exit of the capillary, as shown in Fig. 5.3. When the liquid inside the bulb was heated, the disk was found to be forced away from the capillary exit, as if a jet of liquid was issuing therefrom. By relative movement of the parts of the apparatus, he studied the topology of the jet and found it to be of diameter the same as that of the capillary. Moreover, he found from measurement of the reaction force that the jet velocity was of order of magnitude 5 cm/sec. It is interesting to note that he found the disk and torsion balance a more sensitive method of detecting heat inputs to the bulb than by direct measurement of the temperature changes within the bulb with a phosphor-bronze resistance thermometer.

In another series of experiments he attached the bulb and capillary to the torsion balance and measured the reaction forces directly. He found, not unexpectedly, that helium jet issuing from the capillary had the same properties as obtained previously. From these experiments it was concluded that the observable jet was due to the motion of the normal constituent down the capillary away from the source of heat. The superfluid motion through the capillary towards the source of heat is unobservable.

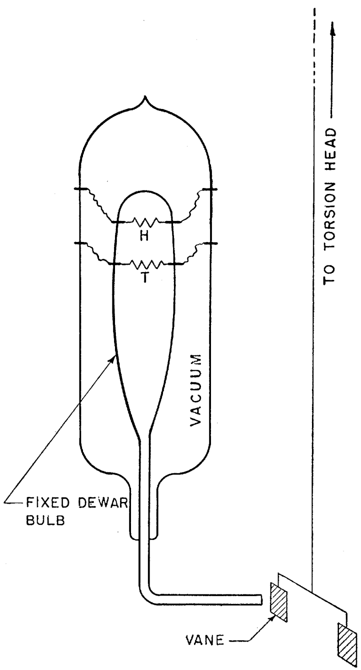

FIG. 5.3. Diagram of apparatus used by Kapitza (K.41f) for observing the helium jet. The bulb and vane system were immersed in liquid helium II. H is a heater, T a thermometer.

TABLE 5.1. Temperatures above which the linear relationship between the heat flow $\dot{Q}$ and the temperature difference ΔT is invalid. (From Meyer and Mellink, M.47a.)

Channel width (microns)	Temperature (°K)
10	1.3
5	1.6
2	1.9
1	2.1
0.3	2.16

5.3. Heat Transport Through Narrow Channels (Experimental)

In considering the process of heat transport in liquid helium II in more detail, it is necessary, as in the problem of mass flow through tubes, to distinguish between the effects occurring (a) in heat transport through narrow channels and (b) heat transport through wide channels. The "internal convection" process outlined above in Sec. 5.2 demands that there exist a "normal" mass flow away from the source of heat equal in magnitude to the superfluid mass flow towards the source of heat. For extremely narrow channels, therefore, it would be expected that the flow of the normal constituent would be negligibly small, owing to its finite viscosity, and hence the heat transport also by the internal convection process would be expected to be negligibly small. As the width of the channel is increased, more and more normal constituent could flow through, and hence it would be expected that this flow together with the associated heat transport would be limited by the boundary conditions imposed on the normal flow. These boundary conditions, for so called narrow channels (width 0.1 to 10 microns) seem with few exceptions to be those associated with Poiseuille laminar flow of the normal constituent through the channel, i.e., conditions determined largely by surface viscous forces. In case (b), for the so-called wide channels (with 10 microns to 2 mm), the boundary conditions are not so clearly defined. It appears however, as is described below in Sec. 5.6, that the main forces limiting the normal flow are due to mutual friction between the superfluid flow and the normal flow throughout the whole cross section of the channel. In practice the outstanding observable difference between case (a) and case (b) is that in the former case the heat flow is accurately proportional to the temperature difference between the ends of the channel, whereas for case (b) the heat flow is approximately

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[13] Heat current given by the amount $\dot{M}\cdot C_v\cdot\Delta T$ is, of course, that which is normally associated with convection. Without postulating fantastically high velocities of flow (10^4 to 10^5 cm/sec) through the capillaries, it is impossible to account for the observed heat currents by this convection process alone. The process associated with Eq. (5.4), i.e., a kind of Peltier process, must in fact be operative, and in order to distinguish it from ordinary convection, we term it here, following London, "internal convection."

proportional to the third root of the temperature gradient, a feature mentioned in Sec. 5.2.

The first observations on the heat flow through narrow channels were made by Allen and Reekie (A.39b), using channels formed between the interstices of tightly packed fine powder. They noted that at the lowest temperature of measurement (1.24°K) the heat flow $\dot{Q}$ was proportional to the temperature difference ΔT between the ends of the channels. Later measurements made with narrow channels, by Keesom and Duyckaerts (K.47a), Mellink (M.47b), Meyer and Mellink (M.47a) and by Hung, Hunt, and Winkel (H.52c), using apparatus essentially the same as that reported by Kapitza (K.41b) and as illustrated in Fig. 4.4, have all shown that for sufficiently low temperature and sufficiently narrow channel $\dot{Q}$ varies directly with ΔT. They have shown, moreover, as would be expected, that the change over from the region where $\dot{Q}\propto\Delta T$ [case (a)] to the region where $\dot{Q}\propto(\Delta T)^{\frac{1}{3}}$ [case (b)] is also a function of temperature. Table 5.1, taken from Meyer and Mellink, tabulates the temperatures for various channel widths above which the linear relationship between $\dot{Q}$ and ΔT no longer is observed. An illustration of the change over from one region to another, taken from the work of Hung, Hunt, and Winkel (H.52c), is given in Fig. 5.4.

The work referred to above on observation of heat flow through narrow channels has led to the following conclusions:

(1) at sufficiently low temperatures (see Table 5.1) the heat current density $\dot{q}$ is proportional to the temperature difference ΔT.

(2) insufficient evidence exists for formulating the variation of the heat flow as a function of the length of the channel.

(3) the heat current density is not independent of the channel width. Instead $\dot{q}$ appears to vary with some power of the channel width d between 1 and 2. The data are unfortunately not too accurately known for the channel widths themselves, and it appears that the variation of $\dot{q}$ with d is also a function of temperature.

(4) the heat current density is a strong function of temperature for constant temperature difference ΔT. For a 5 micron channel Mellink (M.47b) reports that $\dot{q}\propto T^{13}$; for a 1 micron channel Meyer and Mellink (M.47a) report $\dot{q}\propto T^{9}$ and for diminishing channel width they report diminishing variation with T.

(5) numerical values for the observed "heat conductivity" as computed by use of Eq. (5.2) are tabulated in Table 5.2. [This table is taken from the work of London and Zilsel (L.48b).] Table 5.2 shows that even in the narrowest channels the apparent heat conductivity is many orders of magnitude greater than that of helium I.

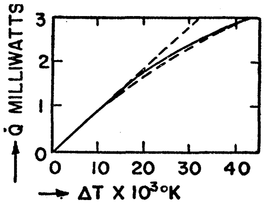

FIG. 5.4. Observed heat flow, $\dot{Q}$ (full curve) as a function of temperature difference ΔT across a 5 micron channel as measured by Hung, Hunt, and Winkel (H.52c). The upper broken curve plots the linear relation between $\dot{Q}$ and ΔT and the lower broken curve the cube root relation between $\dot{Q}$ and ΔT. ($\dot{Q}\propto(\Delta T)^{\frac{1}{3}}$.)

TABLE 5.2. The experimental and theoretical values K_{exp} and K_{theor} of the apparent heat conductivity of helium II through narrow channels. Channel width is d. Channel length is l. Taken from London and Zilsel (L.48b).

d microns	l cm	T °K	K_{exp}	K_{theor} cal/deg-cm-sec	$K_{\text{exp}}/K_{\text{theor}}$	Authors
1.75	0.275	1.960	17.2	10.2	1.69	K.47a
		1.705	3.5	1.86	1.88	
		1.476	0.61	0.32	1.92	
		1.223	0.066	0.032	2.05	
1.15	0.275	2.170	24	15.5	1.55	K.47a
		1.989	17.1	5.28	3.25	
		1.799	5.35	1.55	3.45	
1.0	0.1	2.159	31	10.8	2.9	M.47a
		1.948	21.4	3.06	7.0	
		1.802	12.4	1.19	10.4	
		1.411	1.05	0.060	17.5	
0.75	0.275	2.097	30.2	4.3	7.0	K.47a
		1.600	1.85	0.156	11.8	
		1.403	0.39	0.031	12.5	
0.5	0.1	1.659	2.48	0.108	23	M.47a
		1.315	0.354	0.0064	55	
		1.274	0.277	0.0043	64	
		1.086	0.124	0.0006	202	
0.3	0.1	1.652	1.92	0.0365	54	M.47a
		1.558	1.35	0.018	75	
		1.358	0.48	0.0034	140	
		1.226	0.25	0.00097	260	

5.4. Heat Transport Through Narrow Channels (Theoretical)

The theory of the heat transport in helium II through narrow channels has been discussed in detail by London and Zilsel (L.48b) on the basis (a) of heat transport being by the "internal convection" process and (b) of the limitation of the mass flow in the circulation being given by laminar (Poiseuille) flow of the normal constituent through the channel. The matter has been discussed further by Andronikashvili (A.49), by Gorter and Mellink (G.49a) and by Atkins (A.52a).

A formulation for the heat flow can be obtained as follows: first, if the only limitation to the flow is due to the viscosity η_n of the normal constituent flowing laminarly through the channel, then the pressure gradient in the channel must be given by

$$\text{grad}p=\eta_n\nabla^2\mathbf{v}_n, \tag{5.5}$$

where v_n is the velocity of the normal constituent. Next one notes that this pressure gradient is the "fountain" pressure gradient set up because of the temperature gradient across the channel. The connection between the fountain pressure and the temperature gradient has been discussed earlier in Sec. 4.5 and is given by Eq. (4.4). Using Eq. (5.4) for the total heat current, therefore, and combining with Eqs. (4.4) and (5.5), one obtains for a parallel-sided channel of width d and unit breadth:

$$\dot{Q}=-K_{\text{theor}}\cdot d\,\text{grad}T=-\frac{\rho^2S^2Td^3}{12\eta_n}\,\text{grad}T,\qquad(5.6)$$

where S, the entropy per g of liquid, has been identified with ΔS, and where K_{theor} is the apparent heat conductivity.

Values of K_{theor} from Eq. (5.6) have been computed by London and Zilsel (L.48b), using experimentally determined values of S and η_n. They also used for comparison the values of d and grad$T$ occurring in the experimental measurements of Keesom and Duyckaerts (K.47a), and of Meyer and Mellink (M.47a); the results of their computations are set out in Table 5.2. It will be seen that K_{theor} is of the same order of magnitude as K_{exp} for channel widths d greater than about 1 micron. For channel widths less than 1 micron, however, the theoretical evaluations are far too small. For one of the smallest channels used by Meyer and Mellink, namely 0.3 microns, $K_{\text{exp}}/K_{\text{theor}}$ reaches values as high as 260 at 1.23K.

Whereas the approximate agreement between K_{theor} and K_{exp} for channel widths of from 1 to 10 microns is satisfactory, the marked discrepancies for the smaller channels is disturbing. Gorter and Mellink (G.49a), who also noted these discrepancies, consider them to be due to the fact that for the smallest channels the width d becomes comparable with the mean free path which determines the viscosity, and that the "normal" flow is then analogous to Knudsen flow in low pressure gases. No satisfactory quantitative explanation, however, has yet been given for the anomalously high heat flow through channels of width less than 1 micron, and one must conclude that much work remains to be done in the elucidation of this problem.

Since the dependence of S on temperature in the liquid helium temperature region is very marked (see Sec. 4.3) and since η_n does not vary greatly in this temperature region (see Fig. 2.5) it is clear from Eq. (5.6) that the heat conductivity K_{theor} could be approximately considered as being proportional to T^n, where n is a number between 10 and 15, in approximate agreement with the experimental observations. London and Zilsel (L.48b) have suggested $n=12.2$, but in view of more recent measurements of S and η_n since the time they wrote their paper, this value should be revised. The diminution in the observed value of n, as noted by Meyer and Mellink (M.47a), with decreasing channel width must presumably be associated with the anomalously high heat conductivities in the narrowest channels, and remains unexplained.

5.5. Heat Transport Through Wide Channels (Experimental)

The early measurements and the results of these measurements of the heat transport through wide channels (diameter from 0.2 mm to 1.6 mm) have been noted already in Sec. 5.2. More recent work on heat transport, in the region where the heat flow $\dot{Q}$ is *not* linearly proportional to the temperature difference ΔT has been carried out by Keesom and Duyckaerts (K.47a), by Mellink (M.47b), and by Hung, Hunt, and Winkel (H.52c). The interest in this work lies in the information it yields concerning the resistive forces acting throughout the bulk of the liquid, i.e., across the whole cross section of the channel.

The apparatus used in this work was the same as that used by the authors in their observations on narrow channel heat transport, as illustrated in principle in Fig. 4.4. The channel widths employed varied between 1 and 15 microns, for which by operating at a sufficiently high temperature (see Table 5.1) the heat flow $\dot{Q}$ was observed to be nonlinear in dependence on ΔT.

The results obtained were the same in character as those for the very much wider channels (see Sec. 5.2), namely, (a) the total heat current $\dot{Q}$ was approximately proportional to the cross section A of the channel and (b) the heat current density $\dot{q}$ ($\dot{q}=\dot{Q}/A$), was found to vary with $(\Delta T)^{\frac{1}{3}}$. Measurements were not made extensively enough to determine the influence of the length of the channel on the results.

The most recent results of Hung, Hunt, and Winkel (H.52c) indicate that the nonlinear region, i.e., the region in which $\dot{Q}\propto(\Delta T)^{\frac{1}{3}}$, may not extend right down to zero heat input. It appears that for channels from 1 to 5 microns width as employed by them, linearity between $\dot{Q}$ and ΔT exists for sufficiently small heat input, even at temperatures above those critical values tabulated in Tables 5.1.

Such a change-over from the $(\Delta T)^{\frac{1}{3}}$ region to the linear region has already been illustrated in Fig. 5.4.

In many of the measurements on heat transport reported by the above authors, as well as the early measurements of Allen and Reekie (A.39b), the pressure difference across the channel connecting the two volumes of liquid helium II was measured as well as the other pertinent parameters $\dot{Q}$ and ΔT. This pressure difference is the so-called fountain pressure, caused by the application of heat to the hotter volume of helium II and fundamentally due, on the two-fluid theory, to the motion of the superfluid constituent towards the source of heat. When wide channels are used, however, the fountain pressure cannot be maintained without a continuous supply of heat being provided in the hotter volume, due to the fact that "normal" flow takes place along the fountain pressure gradient and heat is transported to the colder volume by the "internal convection" process.

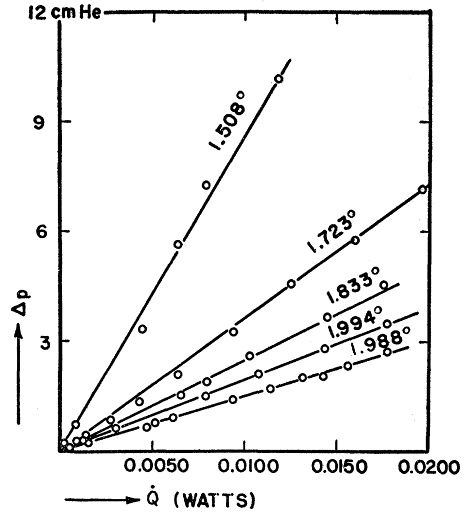

FIG. 5.5. Plot of observed fountain pressure Δp_{obs} in cm of liquid helium against heat flow $\dot{Q}$ in watts through a channel 10.5 microns wide, at various constant temperatures as reported by Mellink (M.48b).

In the case of extremely small channels, it is possible, as has been discussed in Sec. 4.4 to maintain fountain pressures once initiated for considerable periods of time without further application of heat, and ideally one supposes that these periods could be made indefinitely long by considering channels impervious to the flow of the normal constituent.

The main results of these measurements of the heat flow and the fountain effect across wide channels indicate:

(a) at constant temperature there is a strict proportionality between the heat current $\dot{Q}$ and the observed fountain pressure Δp_{obs} regardless of whether $\dot{Q}$ is linearly proportional to ΔT or not. This fact was first noted by Allen and Reekie (A.39b), and data collected by Mellink (M.48b) clearly illustrating this result are given in Fig. 5.5.

(b) the ratio of the observed fountain pressure to the temperature difference $\Delta p_{\text{obs}}/\Delta T$ is not in general equal to $\rho\Delta S$, as found for the narrow channels (see Sec. 4.5). The ratio $\Delta p_{\text{obs}}/\Delta T$ is found to be a complex function of the temperature, channel width, and ΔT, and a typical curve due to Mellink (M.48b) showing the variation of Δp_{obs} with T for various constant values of ΔT at a channel width of 5 microns is shown in Fig. 5.6. In this figure the computed values of $\rho\Delta S\cdot\Delta T$ or rather $\rho S\cdot\Delta T$ are shown by the broken curves.

5.6. Heat Transport Through Wide Channels (Theoretical)

A possible explanation of the nonlinear variation of the heat flow $\dot{Q}$ of helium II through wide channels with the temperature difference ΔT, has been put forward by Gorter (G.48c) and has been discussed in detail by Gorter and Mellink (G.49a) and by Gorter, Kasteleijn, and Mellink (G.50a). The explanation is based on the assumption that there exists a force of mutual friction between the superfluid and normal constituents when in relative motion given by,

$$F_r = A\rho_s\rho_n(\mathbf{v}_s-\mathbf{v}_n)^3, \tag{5.7}$$

where F_r is the force per unit volume and where A is a term which may be temperature dependent. The influence of the mutual frictional force proportional to the cube of the relative velocity on the isothermal flow of helium II, etc., has already been noted in Secs. 2.6 and 2.7.

Under the influence of such a frictional force, acting on the superfluid as well as the normal fluid, the simple expression for the fountain pressure given in Eq. (4.4) is no longer valid, since the Eq. (4.4) was deduced on the assumption of no irreversible forces such as F_r. For flow of superfluid in this case the theoretical fountain pressure gradient $\rho S\ \text{grad}T$ is larger than the actual observed pressure gradient $\text{grad}p_{\text{obs}}$ because of the mutual friction. We have, therefore, in the steady state,

$$\rho S\ \text{grad}T = \text{grad}p_{\text{obs}} + A\rho\rho_n(v_s-v_n)^3. \tag{5.8}$$

By measuring, as was done for example by Hung, Hunt, and Winkel (H.52c), grad T and grad p_{obs} for stationary flow or for flow with negligible acceleration through channels, it is possible from Eq. (5.8) to compute the mutual frictional term F_r.

To determine the heat flow by the "internal convection" process when the mutual friction is present it is necessary to combine Eq. (5.8) with some equation relating the heat current density $\dot{q}$ with grad p. To do this, anticipating Sec. 8.1 somewhat, one combines

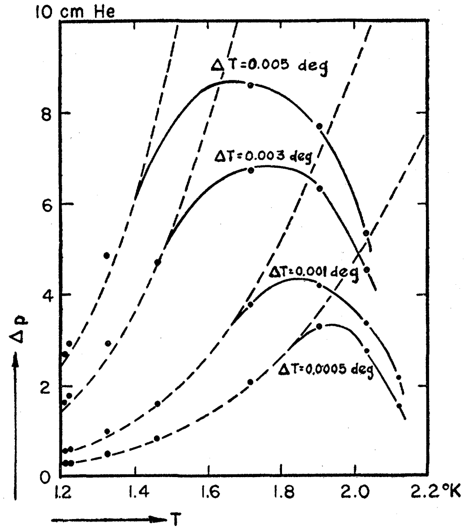

FIG. 5.6. Observed fountain pressure in cm of liquid helium as a function of temperature for various constant values of ΔT due to Mellink (M.48b). Measurements made for channel width of 5 microns. The broken lines give the theoretical maximum fountain pressure as presented in Eq. (4.4).

the two hydrodynamical equations of motion for the superfluid and the normal fluid as proposed by Gorter and Mellink, and one obtains,

$$\mathrm{grad}\,p_{\mathrm{obs}}=\eta_n\nabla^2\mathbf{v}_n, \tag{5.9}$$

which together with the basic equation for the internal convection process (Eq. 5.4) gives

$$\dot{q}=-\frac{\rho STd^2}{12\eta_n}\,\mathrm{grad}\,p_{\mathrm{obs}} \tag{5.10}$$

for linear flow, for example, in a parallel-sided channel of width d. This Eq. (5.10) is interesting in that it indicates that the normal fluid motion is entirely determined by the boundary conditions imposed by the walls of the channel even when mutual friction is present. It shows, moreover, that an exact linear proportionality should exist between the heat current density $\dot{q}$ and the *observed* fountain pressure gradient $\mathrm{grad}\,p_{\mathrm{obs}}$ across the channel regardless of whether $\mathrm{grad}\,p_{\mathrm{obs}}$ satisfies the simple relation (4.4) or the more complex one Eq. (5.8). This latter result is satisfactory as it serves to explain the experimental results first noted by Allen and Reekie (A.39b), which indicate that $\dot{q}\propto\Delta p_{\mathrm{obs}}$. (See Fig. 5.5.) It should be noted here, however, that this linear relationship between $\dot{q}$ and Δp_{obs} as exemplified in Eq. (5.10) would be lost theoretically, if irreversible forces other than those of *mutual* friction were assumed to act on the superfluid motion.

By combining Eq. (5.8) and (5.10) it can readily be shown that

$$\mathrm{grad}\,T=-\frac{12\eta_n}{\rho^2S^2Td^2}\cdot\dot{q}-\frac{A\rho_n}{S(\rho_sST)^3}\cdot\dot{q}^3. \tag{5.11}$$

For narrow channels the first term on the right-hand side of Eq. (5.11) is the dominant term determining the heat flow, and it will be seen to be the same as the linear equation (5.6). As d the channel width becomes larger, the first term on the right-hand side diminishes and eventually one would expect that $\mathrm{grad}\,T\propto\dot{q}^3$. This variation of $\dot{q}$ with the cube root of the temperature gradient is what is observed in practice. Indeed it was because of this observed variation that Gorter and Mellink assumed the mutual frictional term F_r would vary with the relative velocity cubed, i.e., with $(\mathbf{v}_s-\mathbf{v}_n)^3$.

As is explained in some detail in Secs. 2.6 and 2.7, it is not clear that the mutual frictional term of Gorter and Mellink is sufficient to explain all the observations on the normal viscosity, heat conductivity, etc, in spite of its initial success, a success emphasized also by the extensive computations of Nakajima, Tomita, and Usui (N.50a) which were made with Gorter and Mellink's mutual frictional term. Moreover it may be, as the results obtained by Hung, Hunt, and Winkel (H.52c), (some of which are illustrated in Fig. 5.4) clearly show that the mutual frictional term is essentially absent until some critical velocity of flow is reached. This conclusion, that flows below critical are essentially frictionless, had previously been arrived at by Chandrasekhar and Mendelssohn (C.51) and by Atkins (A.51a) from observations of flow through surface films and through wide channels, respectively. The suggestion has been put forward that

$$F_r=A\rho_s\rho_n[(\mathbf{v}_s-\mathbf{v}_n)-v_{\mathrm{crit}}]^3. \tag{5.12}$$

Recently Kasuya (K.53e) presented in a short note an interpretation of the results of Hung, Hunt, and Winkel (H.52c) using a mutual frictional force of the type given in (5.12) together with an *ad hoc* frictional force on the superfluid motion proportional to v_s^3. He calculated the critical velocities resulting from this phenomenological approach. A more detailed discussion, however, must await further information.

In general it seems clear, as is shown also in Sec. 8.1, that there are many possible choices for the irreversible terms in the hydrodynamic equations of motion, and their intensive study, either theoretically or experimentally, is now only beginning.

5.7. Boundary Layers

Apart from the problem of heat transport within liquid helium II itself, interesting studies have been made of the heat transport across the solid-liquid boundary of a surface immersed in liquid helium II. The first studies of this were made by Kapitza (K.41f). His apparatus consisted of metal parallelepipeds which contained internal resistance thermometers and heaters and which were suspended freely by fine wires in a constant temperature bath of liquid helium II. Dissipation of heat in any one parallelepiped resulted in its temperature increasing above that of the helium bath due to the finite thermal boundary resistance across the solid-liquid surface. Kapitza found that this thermal boundary resistance was in first approximation independent of the nature of the solid surface, and that the temperature drop across the surface was located in a thin liquid layer near the surface less than 10^{-3} cm thickness. He measured, moreover, the thermal transfer coefficient n, i.e., the number of watts that can be dissipated per cm^2 per degree rise in temperature given by

$$n=W/\Delta T, \tag{5.13}$$

and found that for small temperature differences ($\Delta T<10^{-2}$°K) n was approximately proportional to T^3. His results are shown in Fig. 5.7.

Similar measurements using, however, a more indirect technique have subsequently been made by White, Gonzales, and Johnston (W.53a), and their results confirm approximately in magnitude and temperature dependence the results of Kapitza (K.41f). Using observations on second sound, Osborne (O.51) has come also to the same conclusions.

Gorter, Taconis, and Beenakker (G.51d) have given

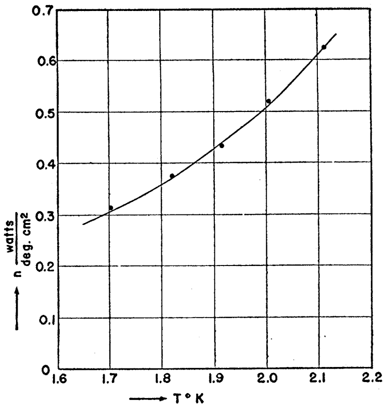

FIG. 5.7. The value of the thermal transfer coefficient n across a solid-liquid surface in liquid helium II (in watts/cm^2-deg) as a function of temperature [Kapitza (K.41f)].

a formal interpretation of the thermal transfer coefficient across solid boundaries to liquid helium II. They consider the motion of the superfluid and normal fluid perpendicular (in the z direction) to a solid surface and use the conclusion that, if in this case the temperature is assumed continuous at the boundary, normal fluid must be transformed into superfluid (or vice versa) near this boundary. This transformation requires a different temperature (T) near the boundary than far out in the liquid (T_0) if the transformation proceeds at a finite conversion rate. Using their notation by putting x equal to the fraction of normal fluid present ($x=\rho_n/\rho$), then they assume that the increment in x is linearly proportional to $(T-T_0)$. Then continuity of x requires

$$\operatorname{div} x v_n + a(T-T_0)=0, \tag{5.13}$$

and the total conduction of heat W from unit surface will be given by;[14]

$$W=v_n\rho TS^* - \lambda \operatorname{grad} T = \text{constant}, \tag{5.14}$$

where λ is the normal classical type coefficient of thermal conductivity of the liquid. The first term here describes the internal convection process of heat transport.

Equations (5.13) and (5.14) yield an exponential solution for T given by

$$T-T_0=\Delta T\exp(-z/\delta), \tag{5.15}$$

and, hence,

$$n=\frac{W}{\Delta T}=\frac{\lambda}{\delta}=[\lambda\rho TaS^*/x]^{\frac{1}{2}} \tag{5.16}$$

and from which the effective thickness δ of the liquid layer supporting the temperature drop is given by

$$\delta^2=x\lambda/\rho TaS^*. \tag{5.17}$$

It is to be noted that a must be associated with a relaxation time for superfluid to normal conversion, which appears also in theories by Gorter *et al.* (G.50a) of the attenuation of second sound (see Sec. 9.7). By using this connection between a and the results of measurement on second sound, Gorter and co-workers (G.51d) concluded that formula (5.16) is not in disagreement with experiment.

A further development of this problem has been made by Kronig and co-workers. In considering the theory of the propagation and attenuation of first and second sound (see Sec. 9.1) in liquid helium II, Kronig and Thellung (K.50) concluded that a third type of wave motion is possible, namely, one which has an almost purely imaginary wave number and, hence, one which is aperiodic in space with exponential damping. They associate this with surface effects near solid-liquid boundaries and concluded that the liquid thickness over which such effects are appreciable are less than or equal to 10^{-5} cm. Kronig, Thellung, and Woldringh (K.52h) in extending this theory, write for the effective thickness δ,

$$\delta=\delta'+\delta'', \tag{5.18}$$

where δ' is the result of the third type of wave motion introduced earlier by them (for the limit $\omega=0$), and where δ'' is the contribution resulting from relaxation as discussed by Gorter, Taconis, and Beenakker. They give, for δ',

$$\delta'^2=\eta\lambda/\rho^2S^{*2}T. \tag{5.19}$$

(Note here that S^* has the significance recorded above.)

Recently Khalatnikov (K.52c) has suggested that the thermal transfer at the boundary is preponderantly due to radiation of sound from the vibrations of the solid. At the time of writing insufficient detail on this paper is available.

6. THE SURFACE FILMS (SATURATED AND UNSATURATED)

6.1. The Film Thickness and Specific Heat

It was first pointed out by Rollin (R.36), and by Rollin and Simon (R.39) that films are formed on all solid surfaces which dip into liquid helium. These films have been the subject of numerous investigations (perhaps chiefly of their thickness and rate of flow) since the films are one of the most characteristic manifestations of helium II.

The thickness of the film was first measured by Daunt and Mendelssohn (D.38b and D.39b) and by Kikoin and Lazarew (K.38f). The first pair mentioned employed a large surface of copper (of area 103 cm^2) held by a string near the surface of a bath of liquid helium. By means of a small wire attached to the copper and dipping into the bath, a film could move onto the copper at the temperature of the bath. Subsequently the copper was removed to room temperature and the quantity of helium evaporated from it in this

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[14] Here S^* represents $-x(\partial^2 G/\partial x\partial T)$ or $x(\partial S/\partial x)$ which Gorter (G.49b) has introduced instead of the simpler picture presented elsewhere in this article.

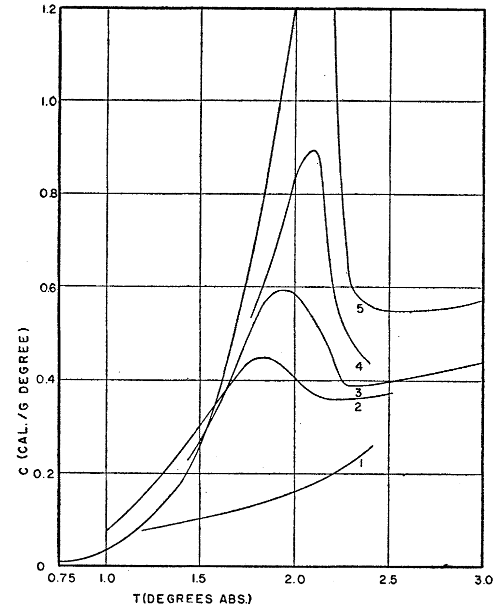

FIG. 6.1. Specific heats in cal/g-deg of adsorbed helium (after Frederikse, F.49a): curve (1) 3–4 layers, curve (2) 5–6 layers, curve (3) 7–9 layers, curve (4) 9–12 layers, curve (5) liquid helium in bulk. The number of layers indicated here are as given by Frederikse. For a re-interpretation see Sec. 6.3.

process was observed by noting the increase in height of the level of the liquid in the bath. By this method they could detect films of thickness equal to or greater than 10^{-7} cm. The results gave an average thickness of the helium surface film of 3.5×10^{-6} cm, essentially independent of temperature for temperatures between 1.59°K and 2.14°K. Above the λ point (2.18°K) the thickness was found to be less than 10^{-7} cm.

Using a different experimental arrangement which also depended on the evaporation of the film from a large surface, Kikoin and Lazarew (K.38f) found an average film thickness, measured at an unstated temperature, of from 2 to 3×10^{-6} cm. These authors also imply the qualitative result that the film forms very rapidly; they speak of a "jump" in the bath level as the helium rushes onto their large sample.

We should digress a moment from our discussion of film thickness to mention a difficulty in the interpretation of the data on films and the resolution of that difficulty. The thickness measurements just discussed are really volume measurements—volumes of the *bulk* liquid. In order to proceed to a thickness it must be assumed that the density of the helium film is substantially the same as that of the bulk liquid. That this is justified by more than expediency may be claimed perhaps from Frederikse's measurements of the specific heat of adsorbed helium films (F.49a). The specific heats were measured in a calorimeter in which known quantities of helium gas were allowed to be adsorbed on a surface of jeweller's rouge (Fe_2O_3) of surface area about 4000 m². The results appear in Fig. 6.1. It was found that, for the thinnest films of helium measured, about 4 layers thick according to Frederikse, the specific heat differs profoundly from the bulk value, showing no λ point, but that after about 20 layers the average specific heat is almost identical with that for the bulk helium. The experiments show very prettily a smearing out of the λ transition and its disappearance as the film is made thinner. From these results it is reasonable to conclude that the average properties of the usual film >100 layers thick are very close to those of the bulk liquid.

The experiments quoted above yield only average thicknesses and say nothing about the variation of film thickness with height above the bath. But intuition, and perhaps the image in the mind of water wetting a glass, suggest that the film thickness may vary with height. The experiments of Jackson and Burge (J.49a and B.51d) were designed to measure this variation and to test the theories of the film which had already been put forward.

Jackson and Burge have performed an elegant series of measurements of the thickness of the surface film *in situ*. Their method, based on the method used by Rothen (R.45) for the measurement of the thickness of barium stearate layers, allows the thickness of the helium film adhering to a stainless steel mirror, the lower end of which dips into a bath of liquid helium II, to be assessed from measurement of the state of polarization of light reflected from the mirror. Incident plane polarized light when reflected from the film-covered mirror is elliptically polarized, the eccentricity and orientation of the ellipse giving a measure of the film thickness. (Of course, one must know the index of refraction of the film. As in the similar situation mentioned above, the bulk property is used for want of better data.) The main advantage of this method is that the film thickness can be measured at different heights above the bath. Measurements can be made on stationary or on flowing films equally well. Some of Burge and Jackson's results (B.51d) for stationary films are shown in Figs. 6.2 and 6.3. In these figures, H is the height above the bath level and ΔN is a

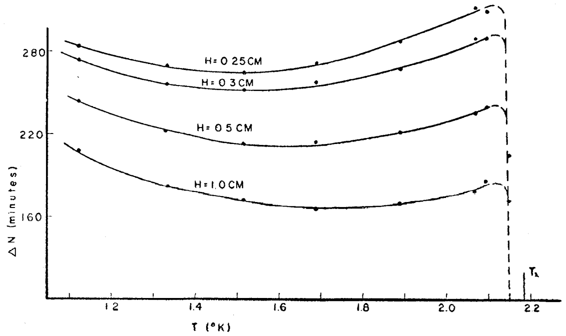

FIG. 6.2. Results of Burge and Jackson (B.51d) for film thickness as a function of temperature for various heights H. The Nicol rotational angle ΔN is proportional to the film thickness.

number proportional to the film thickness. It will be seen

(a) that at a given height H the film thickness varies little with temperature between the extremes of temperature used,[15] namely, 1.1°K to just below T_λ, a result which is in agreement with that previously obtained by Daunt and Mendelssohn (D.39b).

(b) that the film thickness falls to an immeasurably small value at the λ point (2.18°K) and above, again as previously observed (D.39b). The behavior near the λ point could be reproduced when the bath temperature was either slowly rising or slowly falling. These results near T_λ are of some theoretical significance, as we shall discuss later. Subsequent observations by Jackson and Henshaw (J.53) show, however, that above T_λ the helium film is about 10 layers thick. This observation requires that a constant thickness of about 10 layers must be added to the results obtained from Figs. 6.2 and 6.3. This surface film is easily removed by radiation falling on it.

Burge and Jackson attempted to express the variation of film thickness with H at a given temperature by means of the formula

$$d = d_0/H^{1/z}, \tag{6.1}$$

where d_0 is the film thickness at a height of one cm. This form was used because it had already appeared in some of the theorizing on the film thickness. However, in the experiments z was found to vary both with temperature and with the height H. That is, even at a fixed temperature Eq. 6.1 is only an approximation. For example, in the measurements on the stationary film $z_{\text{ave}} = 2.5$ at 2.1°K and increased to $z_{\text{ave}} = 3.5$ at 1.1°K, but (at an unspecified temperature) z varied from 2.9 to 3.1 as H varied from 0.25 cm to 1.2 cm.

Unfortunately the results for z quoted in the preceeding paragraph must be corrected by the addition of the increment reported by Jackson and Henshaw (J.53). Since Jackson and Henshaw have not yet given definitive values of the corrected thickness, we shall not recompute z for their experiments. Bowers, in two recent papers (B.53b and B.53c), has added some important data to this discussion. Bowers has investigated the thickness of static helium films above and below T_λ for both saturated and unsaturated films. His method consisted in weighing the film formed on an aluminum foil by means of a microbalance built into his cryostat. In his experiments with saturated films below T_λ, Bowers used a foil 7.6 cm high which dipped into liquid helium II by means of a fine wire tail about 3 cm long so the thicknesses he measured were average values with H_{ave} (Eq. 6.1) about 5 cm. By integrating Eq. 6.1 over his foil, Bowers found his results represented by

$$n = 295/H^{1/z}, \tag{6.2}$$

where n is the number of layers and $z = 2.0 \pm 0.3$. Bowers warns that 295 does not necessarily lead to the film thickness at 1 cm above the fluid since his foil was, on the average, much higher than this. [For his method of relating n with the film thickness see Bowers (B.53d).] These results for z do not agree with the results already quoted of Jackson and co-workers or with Atkins (see following). By averaging Eq. (6.2) over the heights involved, Bowers obtains agreement with the much earlier results on the film thickness of Daunt and Mendelssohn (D.39b). Above T_λ Bowers found a film about 10 layers thick in agreement with Jackson and Henshaw (J.53). Bowers also notes that the film thickness is extremely sensitive to radiation. With regard to the variation of film thickness with temperature, Bowers and Burge and Jackson are in very satisfactory agreement.

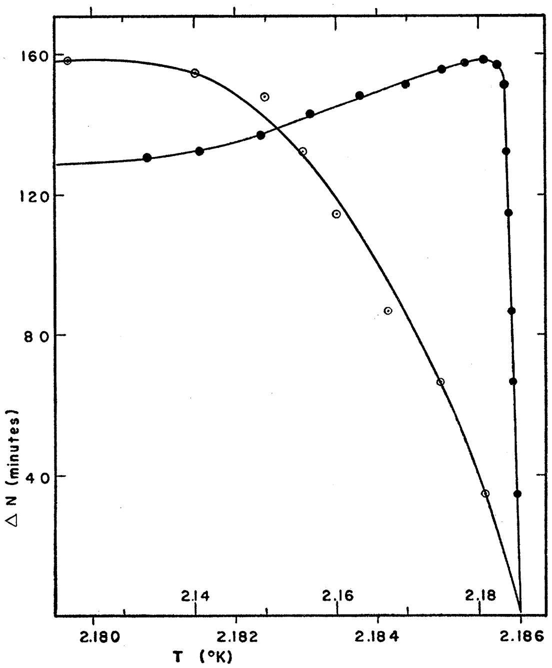

FIG. 6.3. Results of Burge and Jackson (B.51d) for film thickness as a function of temperature near T_λ. The Nicol rotational angle ΔN is proportional to the film thickness. The full circles refer to the upper temperature scale; the open circles to the lower.

The results quoted in the previous paragraphs are for stationary films. By placing a heat source on the top of their steel mirror, Burge and Jackson were also able to observe films moving across their mirror. They found that in the flowing film the thickness at a given height was as much as 20 percent greater than in the stationary film. Also they found that the variation of thickness with height was different in the two cases. In fitting the data from these experiments on flowing films to Eq. (6.1) the authors found that now z varied

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[15] Henshaw and Jackson (H.51) using the same technique, however, have published a curve showing a small monotonic variation of the thickness with temperature. It is not clear to us which results should be taken as definitive.

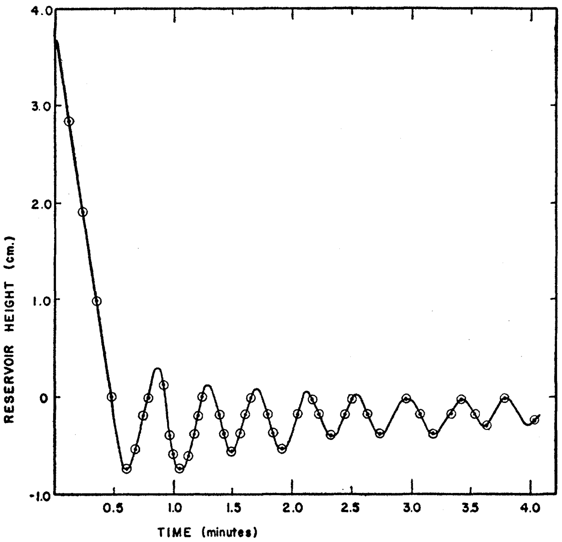

FIG. 6.4. Graph showing oscillatory character of superfluid flow out of a reservoir around zero pressure head. Allen and Misener (A.39a).

from 2.5 to 9 as H varied from 0.25 to 1.8 cm. These data were offered only tentatively by the authors (B.51d) because at that time they had not satisfied themselves as to the thermal state of the flowing film.

As yet nothing has been said about Burge and Jackson's value of d_0 of Eq. (6.1). In their preliminary reports (J.49a and J.49b) Jackson and Burge quoted $d_0 = 1.9\times10^{-6}$ cm at 1.5°K as a provisional value. In later work on the flowing film, Jackson and Henshaw (J.50) gave values of d_0 which are consistent with the number quoted earlier so that it may be regarded as established. These values are also in approximate agreement with the values found by Atkins, which will be given later. (The provisional nature of the value first quoted arose from the difficulty in making an absolute as compared to a relative measurement of thickness.)

Atkins' (A.50a and A.50b) experiments on the film thickness were also made on flowing films. His method was a quantitative investigation of an effect first noted for bulk liquid flow by Allen and Misener (A.39a) who observed that the level of a reservoir of liquid helium II, out of which liquid was flowing through a very narrow channel, did not immediately come to its equilibrium value (at the same level as that of the helium bath in which the reservoir was immersed) but performed oscillations[16] (of about 1 mm amplitude) about the equilibrium level. A typical observation of these oscillations is shown in Fig. 6.4.

Atkins observed similar oscillations in the level of liquid helium II in a reservoir emptying through film flow only, when the level approached that of the bath. In interpreting these oscillations formally, he made certain assumptions regarding the motion (such as, for example, that the velocity of motion was constant across the film). He arrived at the following expression for the period of oscillation τ,

$$\tau = 2\pi\left\{\frac{\rho}{\rho_s}\frac{r}{2g}\left(1+\frac{r}{R}\right)\int_0^l \frac{dH}{d}\right\}^{\frac{1}{2}}. \tag{6.3}$$

The experimental situation covered by this equation is that in which the film moves over the inner and outer surfaces of a tube of radii r and R, respectively, extending vertically a distance l above the surface of a helium bath. If one assumes that Eq. (6.1) gives the variation of d with H, one can, by measuring τ as a function of l and by using Eq. (6.3,) obtain values for both d_0 and z of the Eq. (6.1). In this way Atkins found that $d_0 = 1.5\times10^{-6}$ cm at 1.1°K, and $d_0 = 2.4\times10^{-6}$ cm at 2.0°K, and that $z \approx 7$. These results for both d_0 and z are in fair agreement with Burge and Jackson's; but it must be remembered that they cannot be compared directly since Atkins obtained an average z in a region in which Burge and Jackson showed it to be changing markedly. (For analogous reasons, it is not clear whether an exact comparison can be made between the results of Jackson and co-workers and those of Bowers.)

Atkins' results have been criticized by Kaganov and Eselson (K.51a) who, by altering Eq. (6.3), have obtained a dependence of d_0 on temperature which they feel is more consistent with the results of Burge and Jackson. However, R. B. Dingle in a private communication has derived Atkins' equation in a particularly straightforward manner, which leaves no doubt that it is the correct result for the assumptions made.

6.2. Theories of the Formation of the Film

Theories on the formation of thick films on vertical surfaces dipping into a helium II bath have been advanced by Frenkel (F.40), Schiff (S.41), by Bijl, deBoer, and Michels (B.41), and by Temperley (T.49b). Frenkel and Schiff supposed that the film is formed by the van der Waals forces between the atoms of the wall and the film atoms. Such forces, when integrated over all atoms in the essentially semi-infinite wall, yield an inverse cube potential. When such a potential is added to the gravitational potential it is seen that the equation for the free surface is just Eq. (6.1) with $z = 3$. So far this result is in fair agreement with experiment, its trouble being, however, that it explains too much. It does not indicate any difference between helium I and helium II. Of course, strictly it is only a theory of the absolute zero since internal energy

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[16] These oscillations were isothermal or quasi-isothermal, since the reservoir was connected *via* the vapor phase to the helium bath. If, however, two reservoirs connected by a narrow channel are well isolated from one another then the oscillations that may occur when their levels approach each other will be enhanced by the fountain effect. An analysis of such adiabatic oscillations has been made by Robinson (R.51) who concludes that the experimental observations of such oscillations would lead to an effective way of measuring the temperature dependence of the entropy of "normal" helium II.

rather than free energy is considered; but if the observed approximate constancy of film thickness between 1°K and 2°K is adduced to justify the theory in this temperature range then there is no reason for the film's not persisting in the helium I region. Schiff suggested that the observed difference in the film thickness for helium I and helium II could be accounted for by the flow properties in the two phases: in the superfluid region the superflow of matter forms the film when the temperature is lowered and balances any evaporation from the film as the temperature is raised, whereas in the helium I region viscosity prevents the formation of the film and permits the evaporation of any film already formed below the λ point when the temperature is raised through the λ point. Burge and Jackson have discussed this point at some length, however. Jackson and co-workers have shown experimentally that the film falls abruptly to a very small value (~10 layers) at the λ point and that the temperature of its abrupt fall is independent of the rate at which heat is supplied to the film. This appears to preclude the possibility that the extinction of the film at T_λ occurs merely because of the flow properties of the helium II film.

Bijl, deBoer, and Michels (B.41) have approached the problem of the film from a different point of view. They consider all helium atoms in the film to be particles in an ideal Bose-Einstein gas which have condensed into the lowest energy level. This lowest energy is determined by the thickness d of the film. The part of the wave function that depends on motion perpendicular to the wall is sinusoidal, the energy for each particle being

$$\epsilon = h^2/8md^2, \tag{6.4}$$

where h is Planck's constant and m is the mass of the helium atom. If the gravitational potential energy is added to this zero-point energy, and the number of particles per unit area of the film is taken to be proportional to the thickness: $n=Kd$, then the energy per unit area of film at the constant height H is

$$E=Kd(mgH+h^2/8md^2). \tag{6.5}$$

The thickness d at constant H is found from the fact that $dE/dd=K(mgH-h^2/8md^2)=0$. Thus, at equilibrium,

$$d=(h^2/8m^2g)^{\frac{1}{2}}H^{-\frac{1}{2}}. \tag{6.6}$$

This variation of d with H is in agreement with the result of Bowers and is perhaps not too far from the experimental findings of Burge and Jackson. The constant factor in Eq. 6.6, which contains no arbitrary terms, is within a factor of 5 or so of the observed value of d_0, certainly a striking result. To conclude, on this theory the change in the film thickness on passing through T_λ is explained qualitatively by the initial assumption that film particles are Bose-Einstein particles in their lowest state. As the temperature is raised through the λ point, the film vanishes because the number of particles available to form the film falls to zero.

However, it should be noticed that Eq. (6.5) contains only positive terms. At equilibrium $E=2KdmgH$, so that there is no apparent reason for the film's forming at all. To overcome this difficulty it is necessary that an interaction with the wall must be introduced—something powerful enough to lift particles from the bath level where the energy given by Eq. (6.4) would be practically zero. Any reasonable wall potential however may be expected to alter the wave functions, and hence the dependence of the energy on the thickness. Another serious objection has been offered by Mott (M.49b), who pointed out that the sine wave functions required by Eq. (6.4) imply a great variation of density across the film, with all the particles concentrated at the center of the film. For any interaction among the particles this could not correspond to the state of lowest energy. A more reasonable wave function would have constant amplitude over most of the film and, therefore, would not result in an energy varying as d^{-2}.

A more sophisticated theory of the film was given by Temperley (T.49b). He started with a one-dimensional wave equation which included the van de Waals energy of the helium atom in the presence of the wall and in the presence of the other helium atoms in the film; that is, the other helium atoms furnish a self-consistent field. His wave equation is

$$\frac{\partial^2\psi}{\partial x^2}+\left(\frac{8\pi^2 mE}{h^2}+\frac{k^2}{(s+d+x)^3}\right.$$
$$\left.+\frac{l^2}{(s+x)^3}-\frac{l^2}{(s+d+x)^3}\right)\psi=0. \tag{6.7}$$

In this equation s is the atomic diameter, d is the film thickness, k^2 is a measure of the energy of the atom due to the presence of the wall, and l^2 is a measure of the energy of the atom in the field of the other film atoms. By means of a variational treatment, Temperly found the lowest energy level of this problem to be

$$E=-\frac{h^2}{32\pi^2 m}\left(a+\frac{b}{d^2}\right)^2, \tag{6.8}$$

where a is a constant depending on s and l, and b is a constant depending on k and l. The gravitational energy is added to Eq. (6.8) and the equipotential is taken as the equation of the film surface: i.e.,

$$mgH-(h^2/32\pi^2 m)\{a+b/d^2\}^2=\text{constant}. \tag{6.9}$$

The experimental data hardly admit an accurate evaluation of the constants but Temperley finds an order of magnitude agreement between the slope of the film surface as determined from Eq. (6.9), using his values of a and b on the one hand and the experimental results of Jackson and Burge on the other.

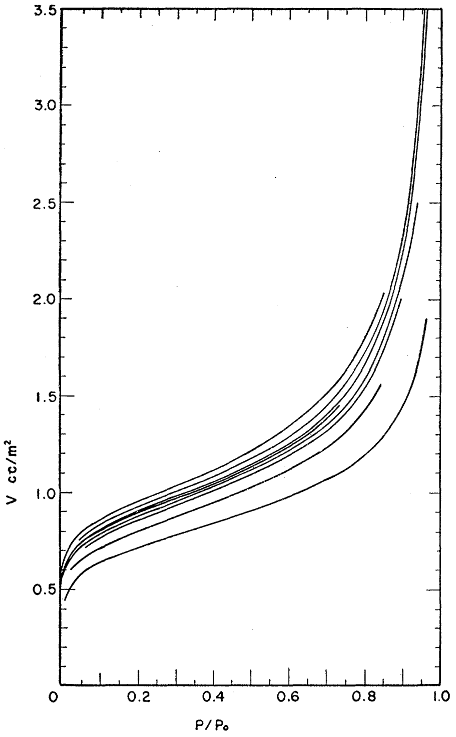

Fig. 6.5. Adsorption isotherms of helium on jeweller's rouge in cc (S.T.P.)/m^2 as a function of saturation at 4.21°K—lowest curve: 3.02°K, 2.42°K, 2.25°K, 2.14°K, 2.01°K, 1.80°K, and 1.59°K—top curve. Due to Strauss [see Long and Meyer (L.53a)].

Incidentally, the form of Eq. (6.9) emphasizes the fact that an Eq. like (6.1) may be entirely inadequate to fit the experimental results. Indeed, we have already mentioned the wide variations of z which arise when the experimental results are inserted into Eq. 6.1.

Temperley has also given a qualitative prediction of the variation of film thickness with temperature. He expected the thickness to have a flat maximum (when plotted against temperature) at about 1.8°K and to fall to zero at the λ point. This prediction is not in agreement with the experimental results shown in Fig. 6.2.

6.3. The Unsaturated Helium Film—Static Properties

The unsaturated helium film is the film which is in equilibrium with its vapor at less than the vapor pressure. Its properties have been measured by several investigators with—it must be admitted at once—most divergent results. The results and the concomitant problems involved have recently been reviewed by Long and Meyer (L.53a), to whom the reader is referred for further detail; but findings announced since their review are not in keeping with all the results surveyed by them.

The chief subjects for study are the adsorption isotherms. Such isotherms have been obtained by Keesom and Schmidt (K.33b) on glass; Keesom and Schweers (K.41e) on glass; by Kistemaker (K.47b) on glass; by Long and Meyer (L.49b) on jeweller's rouge; by Sheaffer, Smith, and Wendell (S.49) on carbon; by Frederikse and Gorter (F.50) on jeweller's rouge and steel; by Mastrangelo and Aston (M.51a) on TiO_2; by Strauss [see (L.53a)] on jeweller's rouge; by Brewer and Mendelssohn (B.53e) on glass; and by Bowers (B.53d) on aluminum foil.

The general pattern obtained is indicated by Fig. 6.5, which shows Strauss' isotherms on Fe_2O_3; however there are deviations from the pattern. Frederikse and Gorter's results are similar to Strauss', whereas Long and Meyer found that below T_λ the isotherms could be superimposed on one another. More startling, however, is the result of Brewer and Mendelssohn shown in Fig. 6.6, which implies that for $T > T_\lambda$ the isotherms fall lower on a plot such as shown in Fig. 6.5.

Besides the discrepancies noted above, there is considerable uncertainty as to the maximum thickness of the adsorbed film below the λ temperature for p, the observed vapor pressure, just less than p_0, the saturated vapor pressure. Kistemaker's data, when re-evaluated by Frederikse and Kistemaker [see long and Meyer (L.53a), p. 11] led to a thickness just below saturation of 150 layers. Long and Meyer (L.49b) reported a comparable figure. Bowers, however, (B.53d) observed not more than 20 layers up to 99.92 percent saturation. Another divergent result has recently been reported by Brewer and Mendelssohn (B.53f) [see also Brewer and Mendelssohn (B.53g)] from experiments in which a helium II film was deposited on a stack of glass plates spaced about 10^{-4} cm apart and the "run-off" subsequently collected and measured after the temperature of the plates had been raised above T_λ. The amount of helium exceeded by a factor of ten or so the amount to be expected for the usual helium II film.[17] The excess

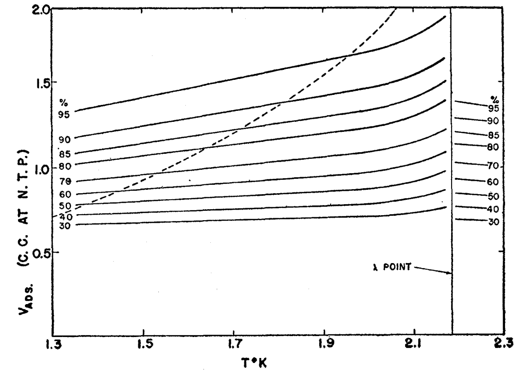

Fig. 6.6. Adsorption isotherms of helium on 0.5 m^2 of glass for various percentages of saturation due to Brewer and Mendelssohn (B.53e). The broken line indicates the temperature of the onset of superfluidity as a function of percentage saturation.

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[17] Similar unpublished experiments by W. B. Gager and one of us (J.G.D.) carried out in 1952 using horizontal piles of brass, glass, and Al plates separated by wire spacers 0.005 cm diameter,

they attribute to the formation of clusters due to an anomalous surface tension.

Despite the inconsistencies already mentioned for nearly saturated or saturated films, the isotherms may be discussed conveniently in terms of the B.E.T. [Brunauer-Emmett-Teller (B.38)] theory of multilayer adsorption for small values of (p/p_0). The theory predicts the shape of the isotherms and also permits the evaluation of certain physical constants of the adsorbate. In particular, one may obtain the volume of the first layer of the absorbate from the isotherm data. Helium deviates from the theoretical behavior in that the volume of the first adsorbed layer as calculated from the isotherms is about four times as large as the volume of a unilayer having the properties of helium in bulk. This indicates that the first adsorbed layer has a higher density than bulk liquid helium. The factor four is just the ratio of the volume of atoms having liquid spacing to the volume of close-packed helium atoms whose diameter is that of the hard-sphere approximation. The exceptionally large volume of the bulk liquid is attributed to the high zero-point energy [see Simon (S.34), and Benewitz and Simon (B.23)]: in the adsorbed film the large attractive forces of the wall overcome the zero-point repulsion and bring about close packing, so that the first layer is essentially a two-dimensional solid rather than a liquid layer. This result leads to a re-interpretation of Frederikse's data on the specific heat of the adsorbed film. (See Fig. 6.1.) Frederikse's first four layers are really one highly compressed layer which has the properties of a two-dimensional solid—as the specific heat shows.

The modification of the B.E.T. theory necessitated by the anomalous packing of the first layer mentioned above has been undertaken by Band (B.49b) and by Aston and Mastrangelo (A.51c). Both attempts lead to improved agreement between theory and experiment for values of (p/p_0) sufficiently small. (Deviations from the B.E.T. isotherms are quite generally observed for values of (p/p_0) greater than about 0.4.) Band (B.51e) has also given a possible explanation of the adsorption isotherms in terms of the condensation of an ideal Bose-Einstein gas into layers of a two-dimensional gas bound to a surface by an energy $W(n)$, where n is the number of the layer counting out from the solid surface. The shape of the resulting isotherm depends on the relationship between W and n. Unfortunately, the best fit to the data does not occur for the most reasonable choice of this relationship.

Note added in proof.—Recently Tjerkstra, Hooftman, and von der Meydenberg [Physica. **19**, 935 (1953)] have used an ingenious differential McLeod gauge to measure the adsorption isotherms of He II on glass. Within the limits of accuracy of their measurements the isotherms taken at temperatures from 1.5°K to T_λ cannot be distinguished from one another.

6.4. Superfluid Properties of Unsaturated Films

Long and Meyer have reported on experiments undertaken to detect superfluidity in adsorbed films (L.50a and L.52a). In one set of experiments (their method I) they found that as they lowered the temperature of their system superfluidity began at a definite temperature below 2.18°K, which temperature decreased as the film thickness decreased. However, in another set of experiments (their method II)—in which presumably their system was nearer to mechanical equilibrium—superfluidity appeared at T_λ (bulk) for films of all thicknesses. The latter would appear to favor the conclusion that the λ point is not a function of the film thickness. However, experiments by Bowers, Brewer, and Mendelssohn (B.51a) on heat transfer in unsaturated films support the opposite conclusion. These investigators found that for a given ratio (p/p_{sat}) the heat current, itself presumably linked to superfluidity, decreased as the temperature rose. For a thin film the heat current fell to zero at a low temperature; whereas for a thick film the extinction of the heat current occurred at a higher temperature. Recently Long and Meyer (L.52c) have repeated the experiments of Bowers, Brewer, and Mendelssohn and find results in agreement with these investigators. The temperatures at which heat flow just begins in these experiments, moreover, agree for given film thicknesses with the temperatures found in method I of reference L.52a, indicating a variation of T_λ with film thickness.

Further data on this point has been furnished by the work of Brewer and Mendelssohn (B.53e) mentioned in Sec. 6.3 and shown in Fig. 6.6. The onset of superfluidity came at increasingly lower temperature as the thickness of the film was decreased, but did not appear to correspond to any anomaly in the observed isotherm. Brewer and Mendelssohn maintained therefore that by examination of the isotherms no indication can be seen of the change from superfluidity to nonsuperfluidity in the adsorbed film.

6.5. Theories of the Unsaturated Film

Besides comparing the observed isotherms with theories of adsorption isotherms several authors have attempted more far-reaching thermodynamic interpretations of the data. These efforts include calculations to correlate the adsorption isotherm data (Sec. 6.3) with the data on superfluidity (Sec. 6.4) and inspections of the isotherm data for evidence of a λ transition. (The onset of superfluidity as the temperature is lowered does not necessarily coincide with the appearance of thermodynamic anomalies.) Deductions concerning the specific heat, entropy, enthalpy, etc. have been made from the adsorption data, for example, by Mastrangelo and Aston (M.51a and A.51c); by

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also yielded anomalous values for the apparent saturated film thickness ($\sim 50\times 10^{-6}$) for the brass and Al. The results for the glass were not anomalous.

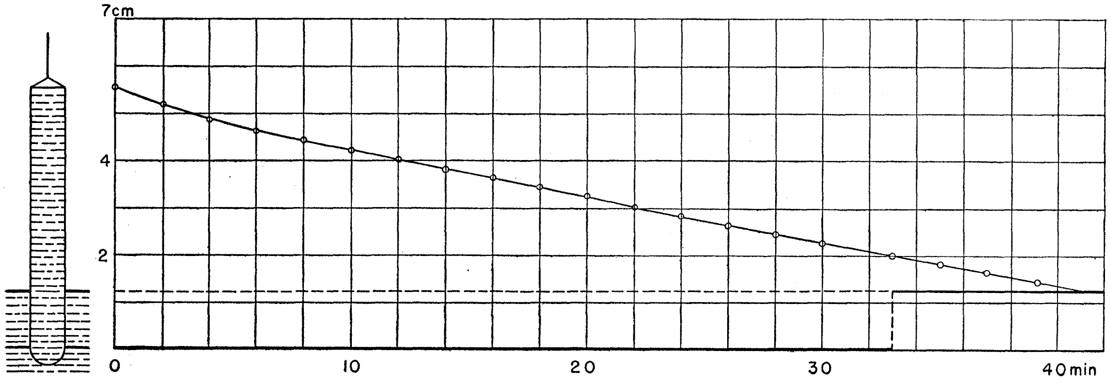

FIG. 7.1. Film transfer out of a glass vessel as a function of time. Daunt and Mendelssohn (D.39c). The curve plots the liquid level in the vessel as a function of time.

Meyer and Long (M.52); by Long and Meyer (L.53a and L.53b), and by Rice and Widom (R.53c). In view, however, of the divergences among the data discussed in Sec. 6.3 the writers of the present article feel that arguments based on the derived thermodynamic quantities are scarcely to be relied on.

Of a somewhat different nature are the discussions of the helium film in terms of the properties of an ideal Bose-Einstein assembly. Osborne (O.49b) has shown that a two-dimensional Bose-Einstein gas does not show a condensation comparable to that found in a three-dimensional gas, but rather a gradual "accumulation" into the lowest quantum state. This accumulation occurs at a temperature very much smaller than the condensation in the bulk gas. Ziman (Z.53a) has investigated the properties of slab-shaped assemblies. If L_i is the finite length of the edge of such an assembly, and $L_x \leq L_y = L_z$, the λ temperature is less than that of the cubical assembly $(L_z)^3$. By assuming that helium liquid both in bulk and in films can be divided into "domains" of the order 10^{-5} cm on an edge, Ziman was able to find a change of λ temperature with thickness of the ideal Bose-Einstein film which represents quite well the behavior of the onset of superfluidity in the experiments of Bowers, Brewer, and Mendelssohn (B.51a) already discussed.

7. FILM FLOW RATES

7.1. Early Experimental Results

The remarkable properties of flow of the surface films formed from helium II, observed and investigated by Rollin and Simon (R.36; R.39) were the subject of extensive study by Daunt and Mendelssohn (D.38b, D.38c, D.39b, D.39c). The general properties of the flow of the surface films found by them have been reviewed frequently (K.42, B.40, D.40, K.48, M.49c, S.53b), but in view of the relevance of the early results to some recent investigations, it is considered that a brief summary of them would be of value. For detail of the experimental procedures, however, the previous reviews should be consulted.

The first investigations of Daunt and Mendelssohn were confined to isothermal conditions in which the flow of helium II took place from one reservoir to another through the interconnecting surface film. The run of a typical experiment is graphically displayed in Fig. 7.1, which shows a plot of the height of the liquid in a glass beaker of helium II measured as a function of time, the transfer from the beaker to the bath taking place through the film on the surface of the beaker. It will be noticed that, except for an anomalous region where the inner level is near the rim of the beaker, the rate of drop of the inner level is constant and that it remained constant even when, as at minute 33, an abrupt change was made in the difference in height between the two levels. From an extended series of measurements Daunt and Mendelssohn concluded that for temperatures below the λ temperature:

1. Under isothermal conditions liquid helium II always collected at the lowest available level.
2. This transfer of helium from one reservoir to another took place in a film formed on the solid surface between the reservoirs.
3. The rate of transfer was practically independent of the difference in height between the levels.[18]
4. The rate of transfer was independent of the material of the substrate.
5. The rate of transfer was proportional to the width of the connecting surface.
6. The transfer from one reservoir to another was restricted by the narrowest part of the connecting surface above the height of the upper level. At places below the upper level liquid could collect in drops from the film.[19]

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[18] Daunt and Mendelssohn observed a small variation of the rate of transfer with height when observations were made over height differences as large as 6 cm. The variation in the rate was approximately 2 percent per cm change in height.

[19] Elegant visual confirmations of this have been provided recently by Jackson and Henshaw (J.53), and by Ham and Jackson (H.53) using an optical technique (see also Sec. 7.4).

7. The rate of transfer was within experimental error independent of the length of the connecting film.

8. The rate of transfer measured in cc of liquid transferred over glass surfaces per cm width of connecting film per sec was a function of temperature, the numerical values being given in Fig. 7.2.

Other experiments of Daunt and Mendelssohn (D.39a, D.39b), as well as those of Rollin and Simon (R.39), showed that under nonisothermal conditions the helium flow in the film took place towards the region sustaining the maximum heat influx at a rate identical with that obtained under isothermal conditions.

7.2. Recent Measurements Over Glass Surfaces

The established picture, codified above, was questioned in 1948 and early 1949 by the experiments of Atkins (A.48c) and of de Haas and van den Berg (H.49), which yielded results strongly at variance with those quoted above. As a consequence, interest was renewed in helium film flow phenomena, and much work was done on this topic between 1949 and 1952, all of which has led to the unequivocal re-establishment of the early results of Daunt and Mendelssohn.

In view of the theoretical significance of the transfer rates in films, a significance which is discussed later, some comment on the work of Atkins and of de Haas and van den Berg may be of value. The experiments of Atkins concerned the isothermal flow of liquid helium II out of glass beakers through the surface film and he found that his results were not in agreement with the findings of Daunt and Mendelssohn summarized by items 3, 5, 7, and 8 in Sec. 7.1. Atkins (A.48c) observed transfer rates more than 5 times greater than those given in Fig. 7.2 for the same temperature, and he found that marked variations with the length of the connecting film existed.

Atkins, moreover, reported a dependence of the transfer rate on the difference in levels and for constant level difference the transfer rate depended in a complicated way on the width of the connecting film and on the geometry of the beaker arrangement. de Haas and van den Berg reported briefly experiments also on the isothermal flow of helium II out of glass beakers. They reported transfer rates as high as 140×10^{-5} cm^3/cm width-sec at 1.4°K (see the value of 7.5×10^{-5} cm^3/cm width-sec given in Fig. 7.2) with a strong dependence on the level difference. de Haas and van den Berg attributed their result to the fact that in the experimental arrangement employed by them no radiation from external sources fell on the surface film.

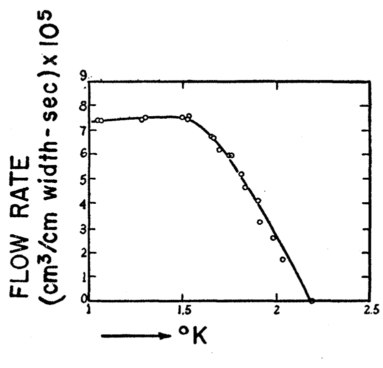

Fig. 7.2. The film transfer rate over glass surfaces in cm^3/cm width-sec as a function of temperature. Daunt and Mendelssohn (D.39b).

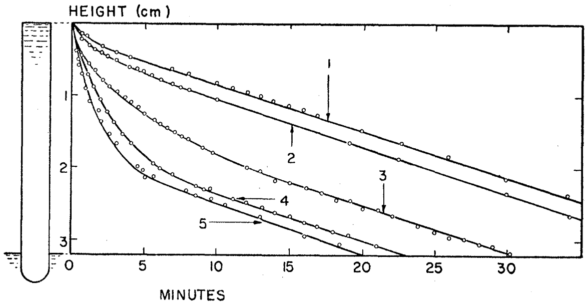

Fig. 7.3. Film transfer out of glass vessels plotted against time for different conditions of the glass surface (see text). Bowers and Mendelssohn (B.50e).

Subsequent experimenters observing the transfer through the film, also out of glass beakers, and using a variety of techniques, however, have all found transfer rates closely similar to the early results of Daunt and Mendelssohn. This confirmatory evidence for the accuracy of the early work was obtained, for example, by Lane and co-workers (W.49, F.49b), Eselson and Lazarew (E.51), Boorse and Dash (B.50d), and also later by Atkins (A.50b), and de Haas and van den Berg (B.51f) themselves.

However, it has been largely through the recent work of Mendelssohn and co-workers that the physical phenomena underlying the abnormal results of Atkins (A.48c) and de Haas and van den Berg (H.49) have been understood.

In a series of elegant experiments Bowers and Mendelssohn (B.49c, B.50e) have shown that the abnormal effects observed by Atkins and by de Haas and van den Berg are of a secondary character and do not reflect the properties of pure film transfer. From a comparison of the techniques employed by the different workers, Bowers and Mendelssohn concluded that the marked variations apparent in the abnormal results might be due to variations in the condition of the substrate of the film and in particular due to contamination of the substrate by condensed gases. They therefore carried out experiments on the isothermal flow of helium II out of glass beakers through the surface film, introducing controlled contamination of the substrate. The result of a typical series of experiments is given in Fig. 7.3, showing the height of the liquid helium II inside a glass beaker as a function of time. Curve 1 gives the result for a very pure surface of the glass beaker. The flow, again except for a short region near the rim of the beaker, is independent of the difference in levels (see Fig. 7.1) and

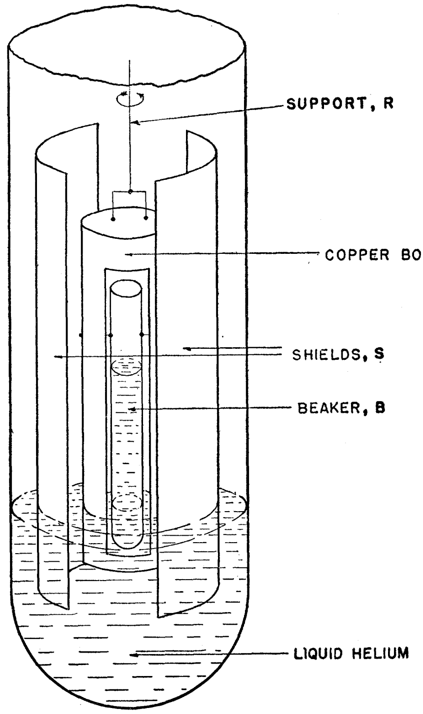

FIG. 7.4. Experimental arrangement used by Bowers and Mendelssohn (B.50e) to investigate the possible effect of radiation on film transfer rates.

corresponds to a transfer rate equal to that previously reported by Daunt and Mendelssohn (D.39c). Curve 2 shows the results after a thin layer of solid air had been deposited on the surface of the beaker. This layer was obtained by allowing helium gas with a small air impurity to condense down into the helium vessel. Here the initial transfer rate has been increased a little, but the main features of the flow remained the same. Curves 3, 4, and 5 represent the results after three further successive condensations of air were allowed to solidify on the beaker at about 2°K. It will be seen that the character of the transfer was completely altered. The initial transfer rate was increased many fold and its variation with level difference assumed the character reported by Atkins (A.48c) and by de Haas and van den Berg (H.49). Curve 5 represents a "saturation" curve, in that further deposits of air gave no appreciable change in the results. It is to be noted that these deposits of condensed gases were quite transparent and colorless and that similar results were obtained using H_2 and Ne as the impurity condensate. Moreover deposition of the solid impurity occurred immediately the impure gas was introduced into the system, regardless of whether the beaker was out of or submerged in the liquid helium II. After raising the temperature of the contaminated beaker to room temperature and cleaning its surface by continued pumping, the results of curve 1 could be reproduced.

It was concluded, therefore, that the high transfer rates, the level difference dependence of the transfer rate, and other abnormal results were due to contamination of the substrate, the random and probably granular character of which could amply explain the complicated nature of the results observed by Atkins and by de Haas and van den Berg. It is probable, moreover, in view of many of the results obtained with the transfer over metal surfaces, that the increase in the transfer rate over contaminated surfaces is due to the increased perimeter due to the deposit.

The possibility of there being an effect on the transfer rate due to incidence of external radiation of the film surface, as had been suggested by de Haas and van den Berg, was also investigated by Bowers and Mendelssohn (B.50e). In their experiments, they suspended a glass beaker *B* (see Fig. 7.4) in liquid helium II, the beaker being enclosed in a copper vessel *C* in which two narrow slits had been cut so that the level of the liquid in the beaker could be observed. This arrangement was situated inside another copper shield *S* and could be rotated by the support *R* so that the beaker either was completely shielded from outside radiation or could be viewed for level measurement. The transfer of helium II out of the beaker was observed alternately with and without radiation, and the results are shown in Fig. 7.5. As will be seen from the curve of Fig. 7.5, all the points lie on the same straight line, showing that no change in the transfer rate occurred while the radiation was excluded. The absolute value of the transfer rate, moreover, was in agreement with the early work (D.39c) as would have been expected for clean surfaces. This result on the absence of any effect of radiation was subsequently confirmed by Atkins (A.50b).

Further work by Brown and Mendelssohn (B.50f) on the flow out of clean glass beakers of different geometrical shapes also confirmed the early work of Daunt and Mendelssohn and indicated that the anomalous results of Atkins were of a secondary character induced by surface contamination.

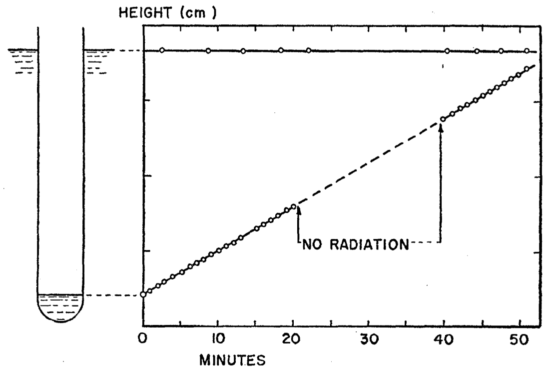

FIG. 7.5. Typical results for film transfer obtained by Bowers and Mendelssohn (B.50e) using the arrangement of Fig. 7.4. The curve plots the liquid level inside the beaker as a function of time when filling.

Mendelssohn and White (M.50a) made detailed studies of the film transfer rates over uncontaminated glass surfaces as a function of temperature and their results together with the early results of Daunt and Mendelssohn (D.39c) and the later results of Webber, Fairbank, and Lane (W.49) are shown in Fig. 7.6. It will be seen that the agreement of all these measurements amongst themselves is satisfactory. Mendelssohn and White suggested an expression of the following type to describe their results:

$$R = A[1-(T/T_\lambda)^\alpha] \tag{7.1}$$

where R is the transfer rate in cm³/cm width-sec, and where α lies between 6 and 8. The experimental uncertainty unfortunately does not allow a more exact determination of α. The constant A lies between 7.3×10^{-5} and 7.65×10^{-5} cm³/cm width-sec for glass.

Mendelssohn and White, moreover, confirmed the early results of Daunt and Mendelssohn (a) that the transfer rate was sensibly independent of the height of the film and (b) that when the level is very close to the rim of the beaker the transfer rate is higher. They also reported that when the difference between the levels is less than about 3 mm, the transfer rate appears to be lower. A similar effect was reported also by Atkins (A.50b), using clean glass beakers, and this is shown in Fig. 7.7. No satisfactory explanation of these end effects has been given, although an interesting observation has been made by Jackson and co-workers on them, to be described below. The question of the film flow at very small level differences has again been taken up experimentally by Picus. In a preliminary communication (P.53a) he reported full transfer rates, (corresponding to those observed at large level differences) even for level differences as small as 1×10^{-2} cm. Further comment, however, is withheld pending further information.

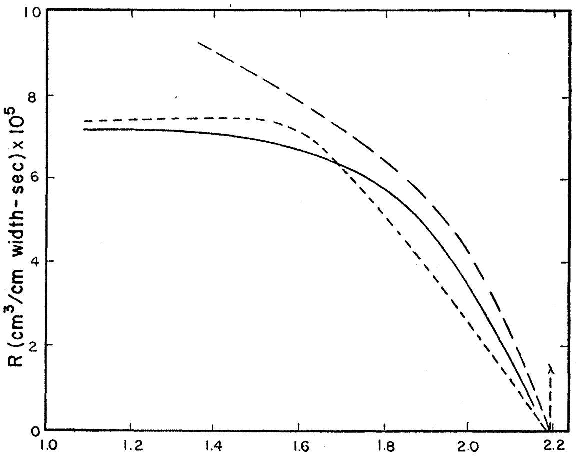

Fig. 7.6. Film transfer rates over glass surfaces in cm³/cm width-sec as a function of temperature. The full curve is due to Mendelssohn and White (M.50a), the curve marked — — — — due to Webber *et al.* (W.49), and that marked - - - - - - - - due to Daunt and Mendelssohn (D.39c).

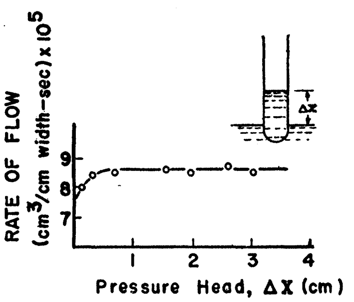

Fig. 7.7. The film transfer rate over glass as a function of level difference (pressure head). Atkins (A.50b).

With regard to the temperature dependence of the transfer rate Ambler and Kurti (A.52b) have recently reported measurements below 1°K. They observed the transfer out of a glass beaker that had been cooled to temperatures as low as 0.15°K by direct contact with a paramagnetic salt. Their preliminary results are shown in Fig. 7.8. There was a scatter in the numerical values of the transfer rate from one experiment to another, and all of them were higher than the accepted ones (M.50a) for glass in the temperature range 1°K to 2.18°K. However in Fig. 7.8 the results have been "normalized" by putting all the observed values equal at 1.2°K. The interesting part of the results is the apparent steady rise in the transfer rate at temperatures below about 0.5°K. Lesensky and Boorse (L.52b) have made flow measurements over copper surfaces down to 0.75°K and their results tend to show this transfer rate rise below 1°K also.

7.3. Measurements of Flow over Metal and Plastic Surfaces

In their early work Daunt and Mendelssohn (see Sec. 7.1) reported that the transfer rate of helium II through surface films was the same over polished copper as over glass. They noted however that over drawn copper wires the transfer rate was somewhat higher. This question of the influence of the substrate (uncontaminated) on the transfer rate has been experimentally investigated a great deal in the past few years, and it appears that the results are somewhat complex.

Observations of the transfer rate of helium II over copper, stainless steel, lucite, Fe, and Pb in various states of preparation have been made by Boorse and Dash (B.50d, B.50g, B.51g), on Pt and Ni by Mendelssohn and White (M.50a, M.50b), on stainless steel by Jackson and Henshaw (J.50), on Lucite and Perspex by Chandrasekhar (C.52b) and on stainless steel by Chandrasekhar and Mendelssohn (C.52c). The general shape of the curves for the variation of the transfer rate with temperature for the flow over metal surfaces was found to be the same as that over glass, although in general the absolute values were much higher. For example Boorse and Dash (B.51g) report a transfer

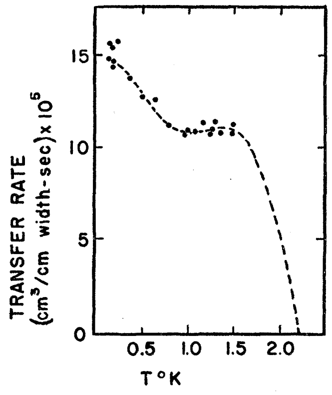

FIG. 7.8. Film transfer rates over glass below 1°K. Ambler and Kurti (A.52b).

rate over etched copper as large as 50×10^{-5} cm^3/cm width-sec at 1.5°K.

In connection with the measurements of flow out of opaque beakers, which must be performed to obtain the transfer rates over metallic surfaces, the method of Boorse and Dash (B.50d, B.50g, B.51g) is of interest in that it differed from the optical methods employed by all other workers. In their method the dielectric property of liquid helium was employed to indicate the total amount of liquid between the walls of a cylindrical capacitor. Variations in height of the liquid helium contained in the annular space of the capacitor were observed as changes of electric capacity, by observing changes in the frequency of a resonant circuit containing the capacitor. The capacitor served in this way as a depth gauge, and was located within the beaker of the material over whose surface the transport rate was to be measured.

The surfaces of metals and plastics as obtained from an average machine shop are in general rougher and more prone to surface defects than those of annealed glass. It is to be expected, therefore, that such surface irregularities associated with metals and plastics would cause higher transfer rates than those over smooth uncontaminated glass surfaces, in the same way that the artificial introduction of surface irregularities on glass by the solidification thereon of gaseous impurities (see Sec. 7.2) increases the transfer rates. That this conclusion is correct is evidenced (a) by the work of Boorse and Dash (B.50g, B.51g) who showed that, whereas the transfer rate, R, at 1.5°K over machined copper was 14.8×10^{-5} cm^3/cm width-sec, the same vessel when etched showed a value of R equal to 49×10^{-5} cm^3/cm width-sec at the same temperature and who found that the surface roughness of the etched surface was easily visible with a microscope, (b) by the work of Chandrasekhar (C.52b) who found that the transfer rate over Perspex at 1.5°K changed from 16.0×10^{-5} to 10.7×10^{-5} cm^3/cm width-sec when the commercial polish was further smoothed by rouge polishing, and (c) by Chandrasekhar and Mendelssohn (C.52c) who found that the transfer rate over stainless steel of exceptionally high finish was identical with that over clean glass ($R=8.0\times10^{-5}$ cm^3/cm width-sec), whereas the same surface after it had been brought to red heat, thus destroying the smooth finish, and then lightly polished gave a value of $R=12.8\times10^{-5}$ cm^3/cm width-sec.

It would appear, therefore, especially from the recent work of Chandrasekhar and Mendelssohn mentioned above, that the transfer rate over substances other than glass does not differ from that over uncontaminated glass, *if the other substance can be obtained in an equally smooth state*, a result which was shown to be so for copper in the original work of Daunt and Mendelssohn. This conclusion has been essentially reconfirmed by recent work of Smith and Boorse (S.53a).

It presumably must also be concluded that the increase in transfer rate over rough surfaces is due to their increased periphery available for flow, as is the case in the contaminated glass surfaces. This is further discussed in subsection 7.5.

Finally an item may be mentioned which reflects the complexity of film transfer over rough surfaces, namely, the observed maximum (B.50g, B.51g, C.52b) at about 1.5°K in the curve plotting the transfer rate as a function of temperature for flow over Lucite and Perspex. Chandrasekhar (C.52b) suggested that this may indicate the existence of two competing mechanisms of transport, one a pure superfluid transfer as over glass, and the other a syphoning process through the semicapillary nature of the rough surfaces. There does not appear to be sufficient evidence on the processes, however, to conclude in detail the mechanisms involved, particularly since Smith and Boorse (S.53a) do not observe such a maximum. It would appear that further work would benefit by exact specifications of the microstructure of the various surfaces employed.

7.4. Determination of the Velocity of Flow

In order to determine the velocity of flow in the film the measurements on the transfer rates, which measure the volume flow of liquid per cm width of film per sec, must be supplemented by data on the film thickness. To this end one elegant series of measurements, in which the film thickness and the transfer rates were measured simultaneously, have been carried out by Jackson and Henshaw (J.50).

TABLE 7.1.[a]

T °K	R cm^3/cm width-sec	d cm	$\bar{v}_{\mathrm{crit}}$ cm/sec	$\bar{v}_s$ (critical) cm/sec
1.1	16.9×10^{-5}	1.63×10^{-6}	104	105
1.3	16.9	1.63	104	110
1.5	16.8	1.66	101	114
1.7	16.1	1.82	88.5	118
1.9	12.8	1.94	66	118

[a] The values given in columns 1, 2, and 3 are taken from D. G. Henshaw and L. C. Jackson, "Symposium on low temperature physics," Proc. Natl. Bur. Standards 185 (1952), and from private communications from Dr. L. C. Jackson.

Their apparatus is illustrated diagrammatically in Fig. 7.9. It consisted of a cylindrical stainless steel beaker *A* which had a 2 mm wide flat strip ground on its outside parallel to the axis and polished to form a plane mirror. A barium stearate film was laid down on this mirror and the thickness of the helium film moving over it could be determined by the optical method developed by Jackson and described above in Sec. 6.1. A glass capillary tube *B* was attached to the bottom of the beaker by means of a Kovar-glass seal *K* and this allowed observations of the liquid level in the beaker. It was possible thus to observe the volume rate of emptying or filling of the beaker and to measure the moving film thickness simultaneously. The results obtained are collected in Table 7.1 following in which the film thickness *d* was observed at a height of 1 cm above the outer liquid bath level.

Column 4 of Table 7.1 gives the critical velocity of film flow on the inside surface of the beaker at 1 cm height above the helium level, obtained by dividing the transfer rate *R* (column 2) by the film thickness *d* at 1 cm height.[20] This computation implicitly assumes that (a) the average density is the same as the bulk liquid density, (b) the film on the inside of the beaker varies with height in the same way as the observed film on the outside, and (c) that the whole of the liquid in the film is moving with the same average velocity.

Assumptions (a) and (b) above are probably valid. Assumption (c), however, is open to question. If, instead of assumption (c) above, one assumes that only the superfluid constituent in the film moves and that this superfluid constituent is given by the bulk superfluid density, ρ_s, (see Sec. 2), the average critical velocity $\bar{v}_s$ for superfluid flow in the film is given by Column 5 of Table 7.1 in which the values of Column 4 are multiplied by ρ/ρ_s.

The following conclusions may be drawn from these results:

(1) The value of the average critical velocity, $\bar{v}_s$ for superfluid flow in the film is sensibly independent of temperature. The theoretical interpretation of this is discussed below in Sec. 7.5.

(2) The numerical values of $\bar{v}_s$ are about 110 cm/sec, which are greater by a factor of about two than the values calculated according to Eq. (7.2). Whether this discrepancy really indicates the lack of validity of Eq. (7.2) is open to doubt, since (a) the transfer rates given in Table 7.2 are about twice those obtained for helium film transfer over glass and over highly polished stainless steel (see Sec. 7.2 above), and (b) the film thickness in these experiments was measured for films on a barium stearate subsurface, whereas the critical transfer rates were for flow over stainless steel.

TABLE 7.2.

R cm³/cm width-sec	d cm	v_{crit} cm/sec
7.4×10^{-5}	1.9×10^{-6}	39
7.8	2.1	37
8.15	2.3	35

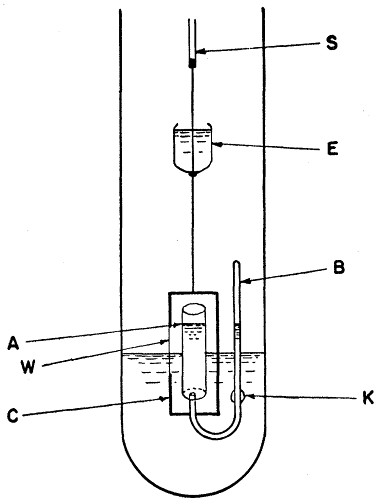

FIG. 7.9. Apparatus used by Jackson and Henshaw (J.50) for simultaneous measurement of film flow and thickness. *A* is steel beaker with glass tube *B* attached to it by Kovar seal *K*. *C* is copper shield with viewing window, *W*. *E* is liquid helium thermal shield and *S* is suspension.

A further observation reported by Jackson and Henshaw concerns the condition of the film during transfer out of a beaker when the inner level is within a few millimeters of the rim of the beaker. As was initially reported by Daunt and Mendelssohn (see Sec. 7.1) the transfer rate is anomalously high for the first few mm of level fall starting with a full beaker. After the inner level has fallen below this anomalous region, the transfer rate takes on its characteristic pressure independent flow as described in detail above in Sec. 7.2. By using their optical method of observing the film thickness, Jackson and Henshaw noted that in the anomalous flow region bright specks due to the formation of drops of liquid could be seen in the field of view, and that these drops were about $\frac{1}{10}$ to $\frac{1}{3}$ mm long and moved downward comparatively sluggishly with a velocity of about 1 cm/sec. When the transfer had settled down to its characteristic pressure independent flow, the drops disappeared.

Simultaneous measurements of the film thickness and transfer rates for flow over glass surfaces have been carried out by Atkins (A.50b). The transfer rates were observed by direct observation of the rate of emptying or filling of the glass beakers and the film thickness was computed, as described above in Sec. 6.1, from observation of the oscillations of the liquid level in the beaker about the equilibrium level. The results given by Atkins are tabulated in Table 7.2, which are smoothed values taken from his curves showing the filling of a beaker at 1.47°K. In this table the different values of *d* all at the same temperature of 1.47°K

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[20] These values are numerically different from those presented by Henshaw and Jackson, since the latter authors calculated the film velocity on the *outside* of the beaker. The velocities on the outside are not the *critical* velocities, since the transfer rate is limited by the narrowest film width above the upper liquid surface.

correspond to different heights above the liquid bath level of from 0.5 to 3.5 cm. These results indicate first, that the film thickness on glass is not greatly different from that quoted by Jackson and Henshaw (Table 7.2) for films on barium stearate and secondly, that the critical velocity v_{crit} has a value of about 37 cm/sec. This critical velocity v_{crit} obtained by dividing Column 1 of Table 7.2 by Column 2, will also be essentially the same as the average critical velocity of superfluid flow $\bar{v}_s$ since at a temperature of 1.47°K the superfluid density is almost the same as the total density. The value of 37 cm/sec for $\bar{v}_s$ is in good agreement with the prediction of Eq. (7.2) discussed below.

Using a technique, as yet only very briefly described, of observation of film flow along copper spirals, Knudsen and Dillinger (K.53d) also report a velocity of 37 cm/sec at 1.3°K, and a film thickness of about 2×10^{-6} cm, which results are in very remarkable agreement with those of Atkins above.

7.5. The Critical Velocity in Films and Its Interpretation

The characteristic velocities of flow in the film of about 50 cm/sec, as reported in the previous subsection, are relatively small and correspond to the velocity that would be attained, for example, by frictionless free fall under gravity through distances of only about 3 mm. All measurements of the isothermal flow out of beakers, as for example the one shown in Fig. 7.1, involve differences in level or pressure heads considerably greater than 3 mm. Consequently, it must be concluded that there is some mechanism which limits the film flow velocity to these relatively low "critical" velocities, regardless of the pressure head causing the motion, as is indicated by the observed independence of the film flow rate on pressure head.

The facts that these critical velocities of film flow are unaffected by the length of the path traversed and that the film can perform relatively undamped oscillations (see Sec. 6.1) indicate that at these critical velocities, and at smaller velocities, there is essentially no frictional resistance to the flow.[21] The film flow therefore can be considered as superfluid flow. It can be shown, furthermore, that for velocities greater than the critical velocities, the resistance to flow increases sharply to unmeasurably large values. This question has been investigated by observing nonisothermal film flow, as described below.

As was shown initially by Rollin and Simon (R.39), film flow can be caused by thermal gradients as well as gravitational gradients, and the velocities reached in such flow are identical with the critical velocities observed in isothermal flow under gravity. This film flow under a thermal gradient, the film moving towards the source of heat, has subsequently been investigated in some detail by Daunt and Mendelssohn (D.39b, D.39c) particularly in their initial experiments on the thermomechanical effects in films (D.39a, D.50b) and by Daunt *et al.* (D.47a), by Atkins (A.48c), by Brown and Mendelssohn (B.50f), and by Chandrasekhar and Mendelssohn (C.51).[22] In the most recent work, which is by the latter authors, helium II was made to flow through the film into a completely enclosed Dewar vessel through a narrow slit above the liquid surfaces, by introducing a source of heat electrically inside the Dewar as is shown in Fig. 7.10. Also in Fig. 7.10, the measured rate of flow is shown as a function of the heat input at 2.09°K. It will be seen that the flow increased linearly with heat input until a critical value (2.4×10^{-4} cm^3/sec) was reached and thereafter further increase in the heat input produced no measurable increase in the flow rate. The sharp break in the curve can be interpreted as indicating that for velocities below the critical value the flow is frictionless, and that large frictional effects must prohibit velocities greater than the critical.

The observed fact that the flow through films, whether produced by gravitational or thermal gradients, is superfluid until it reaches a limiting critical value, which is a unique function of temperature, seems of primary importance in the phenomena of liquid helium II. The same effect also appears in the flow through very narrow channels in the bulk liquid (see Sec. 2.5); although for flow in the bulk liquid, even in the narrowest channels, the effects are not always quite so clear cut as for flow in films. The effect has been likened by Daunt and Mendelssohn (D.42) to the limiting superfluid current density permissible on the surface of a superconductor above which critical current, as is well known, superconductivity is de-

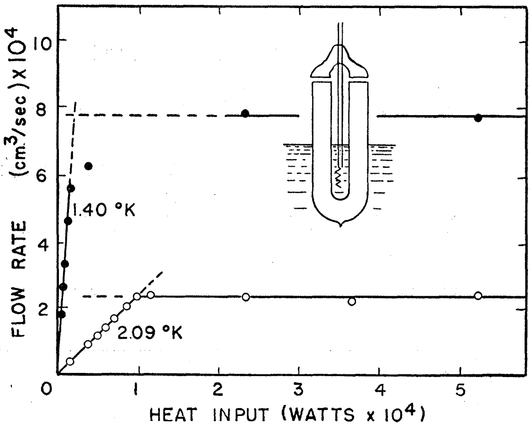

FIG. 7.10. Apparatus and results of Chandrasekhar and Mendelssohn (C.51) in observations of film flow, showing flow rate as a function of heat input inside the reservoir.

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[21] Kasuya (K.53f) has tentatively explained the damping of the oscillations as being due to slow temperature equalization via the common copper boundary and has also concluded that the motion of the superfluid is perfectly frictionless.

[22] The significance of these effects to the design of cryostats has been recently studied experimentally by Ambler and Kurti (A.52d).

stroyed. (i.e., the "threshold current.") These authors further suggested the hypothesis that the variation of this limiting volume flow with temperature can be associated with the variation of the density of the superfluid constituent that can partake in superfluid flow. Such a relationship would imply a temperature independent maximum velocity for superfluid flow. The results obtained for film flow in helium given in the preceding subsection, are not in contradiction with this hypothesis. By use of this hypothesis, the value of the fractional "normal" density ρ_n/ρ, i. e. the ratio of the density of the normal constituent to the total density, can be evaluated from the curves for the rate of volume flow. Unfortunately, the rate of flow curves (see Fig. 7.6) are not known with sufficient accuracy to permit exact computation. However, the formula 7.1 given for the flow rate as a function of temperature with α equal to 6, is the same as formula 2.2 for $[1-(\rho_n/\rho)]$. Realizing the limitations in accuracy, therefore, the values of ρ_n/ρ that can be obtained in this way are in general agreement with those obtained directly by Andronikashvili and others (see Sec. 2.3) in the same temperature range and do not contradict the hypothesis of a temperature independent maximum speed for superfluid flow.

In connection with flow velocities in films a further interesting observation has been made by Bijl, de Boer, and Michels (B.41), independently by F. London (L.45), and, moreover, implicitly adopted sometime previously by Gogate and Rai (G.44a): namely, the product of the average momentum of the superfluid atoms moving at their critical velocity multiplied by the film thickness is of the order of Planck's constant, i.e.,

$$mv\cdot d\approx\hbar. \tag{7.2}$$

The statement by London (L.45) is that the rate of transfer in grams transferred per second through one cm width of surface divided by the number of superfluid particles per cc has the dimensions of angular momentum and appears to be numerically equal to $\hbar$. London showed, moreover, that in the cases where sufficient data are available, this quantum relationship holds not only for helium II films but also for the superfluid flow of electrons in superconductors, thereby reinforcing the analogy between superconductivity and liquid helium II proposed by Daunt and Mendelssohn (D.42).

The relationship of Eq. (7.2) indicates that regardless of the value of film thickness d the transfer rate R in cm^3/cm width-sec is constant, since R is proportional to vd. Thus although the film thickness varies with height above the liquid surface, R should be independent of this height; and this is experimentally so to a high degree of approximation. The experimental evidence for this height invariance of R seems to be well established for heights up to about 6 cm. (See Sec. 7.1 and for example Fig. 7.7.) Recently, however, Eselson and Lazarew (E.52) have reported peculiar variations with height. A description of the exact nature of these effects is unfortunately not yet available to us. Kasuya (K.53e) has suggested that the mutual friction term due to Gorter determines the film transfer rate at large heights. He concluded that R should vary as $(\Delta p)^{\frac{1}{3}}$ at large heights. No experimental evidence in favor of this is yet available.

A further conclusion that can be drawn from Eq. (7.2) is that, although the film thickness may vary from one substrate to another, again the total transfer rate R should be independent of the nature of the substrate. This is in agreement with the latest experimental data on film flow over stainless steel (C.52c), which give identical R values as for flow over glass. Measurements of the film thicknesses on different substrate materials, however, would be of interest.

Various attempts have been made to account theoretically for the existance of critical velocities of superfluid flow in liquid helium II and for their numerical values. Landau (L.41a) considered the initiation of viscous resistance to flow just above the critical velocity by studying the interaction of the liquid at rest with a wall moving through it. He postulated that the motion of the liquid must begin with the excitation of either phonons or rotons. For the former process the relative velocity must be larger than the velocity of sound, u_1 (i.e., $v_{\text{crit}}\geq u_1$), and for the second process v_{crit} must be greater than $(2\Delta/\mu)^{\frac{1}{2}}$, where Δ and μ have the significance described in subsection 3.3.[23] However, the absolute values given by these inequalities make $v_{\text{crit}}\approx 10^4$ cm/sec, which is well beyond the observed limit. A thermodynamic argument put forward later by Ginsburg (G.44b) also resulted in a v_{crit} many orders of magnitude too great.

In 1949 Ginsburg (G.49c) reiterated in some detail an argument previously put forward by Daunt and Mendelssohn (D.46b) that, by taking account of the zero-point energy, v_{crit} should be associated with the zero-point velocity that is conferred by the uncertainty principle on a particle confined to a space of linear dimension equal to the film thickness, d. This immediately results in the Eq. (7.2). This viewpoint has been further developed by Mendelssohn (M.45) who has supposed that both for superfluid flow in liquid helium II as well as for superconductivity the transport of mass is carried out by diffusion which is due to zero points motion. Such a process would give a temperature independent average velocity of flow, and the relationship given by Eq. (7.2) would be automatically satisfied.

Mott (M.49b) has proposed an alternative mechanism to account for the establishment of a maximum average velocity of superfluid flow, which would make v_{crit} vary with $d^{-\frac{1}{2}}$, rather than d^{-1} as given by Eq. (7.2.) Mott's model has already been briefly outlined in Sec. 2.5. The experimental evidence to date, however,

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[23] See also R. B. Dingle (D.52a) for further discussion of v_{crit}.

does not allow a completely clear choice between this model and that summarized by Eq. (7.2.)

A possible and tempting explanation for the existence of the critical velocity v_{crit} has been put forward by one of us, (R.S.S., unpublished), in which the film is regarded as being a collection of idealized chains of lattice constant a_1 and all of length d, all parallel to the z axis, i.e., perpendicular to the film substrate surface. Now if the lattice of the substrate (of lattice constant $a_0 \neq a_1$) is moved along the x axis with velocity v relative to the film, the lower ends of each chain will be subject to periodic forces, the fundamental frequency of which will be $\nu_0 = v/a_0$. Waves will be generated in the chains as a result; but, for given d and v sufficiently small, the reflected waves will be of such phase that the chain will absorb no energy from the moving crystal. As v is increased, however, the chain will finally resonate, the lowest energy phonon will be created in the chain and energy will be absorbed from the crystal. This velocity will be the critical velocity v_{crit} at which viscosity is first displayed and it will clearly be given when

$$v_{\text{crit}} = a_0 \cdot \nu_0 = a_0(u_1/\lambda_{\max}) = a_0 u_1/4d, \tag{7.3}$$

where u_1 is the velocity of sound in the chain.

It is of interest to note that the formula 7.3 developed above indicates that $v_{\text{crit}} \cdot d$ is constant. Moreover, if one uses the typical numerical values of a_0 and the known value of u_1 in liquid helium, one obtains $v_{\text{crit}} \cdot d \approx 10^{-4}$ cm^2/sec which is in agreement with the experimental findings.

It is to be noted that this somewhat idealized picture assumes that the film possesses no microscopic rigidity, as has been explicitly pointed out previously in the theory of Tisza (T.47). Otherwise the shear forces set up by the moving crystal substrate would lead to wave propagation in directions other than that perpendicular to the substrate with fundamental frequencies much lower than $(u_1/4d)$ and hence interaction would occur at vanishingly small relative velocities.

8. TWO-FLUID HYDRODYNAMICS

8.1. The Thermo-Hydrodynamical Equations

The usual hydrodynamical equations of a single fluid must be augmented in the two-fluid theory in order to account for the extra effects that can arise. In the usual one-fluid case the motion of the fluid in the absence of exterior forces is determined by the Euler equations, the equation of continuity, and the boundary conditions. The equations are

$$\begin{aligned}\rho\frac{Dv}{Dt} &= \rho\frac{\partial v}{\partial t} + \rho\mathbf{v}\cdot\operatorname{grad}\mathbf{v}\\ &= -\operatorname{grad}p + \eta\left(\frac{4}{3}\operatorname{grad}\operatorname{div}\mathbf{v} - \operatorname{curl}\operatorname{curl}\mathbf{v}\right),\end{aligned} \tag{8.1}$$

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\rho\mathbf{v} = 0, \tag{8.2}$$

where the volume viscosity coefficient has been set equal to zero in accord with the Stokes' approximation.[24] In this section we shall devote ourselves to the question of what equations replace (8.1) and (8.2) in two-fluid hydrodynamics.

We have already (in the section on viscosity) discussed how the density of helium II is split. In the same way it is natural to split the mass current density (the momentum density) into two terms. Thus,

$$\text{Density:} \qquad \rho = \rho_s + \rho_n, \tag{8.3}$$

$$\text{Momentum density:} \quad \mathbf{j} = \rho\mathbf{v} = \rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n. \tag{8.4}$$

Consistent with the ideas expressed in Secs. 2, 4, and 5, $\mathbf{v}_s$ and $\mathbf{v}_n$ in (8.4) may be quite different depending on the experimental conditions, in particular any one of $\mathbf{v}$, $\mathbf{v}_s$, and $\mathbf{v}_n$ may be zero while the other two are not. Also consistent with results outlined in Secs. 2, 4, and 5 is the concept that a different viscosity affects the flow of the normal fluid from that acting on the superfluid. From these results we are led to take $\eta_s = 0$, $\eta_n \neq 0$.

Also from the above mentioned results (or as an assumption consistent with the theories discussed in Sec. 3), we may write an equation for the conservation of entropy (when irreversible processes are absent or negligible) of the form

$$\frac{\partial(\rho S)}{\partial t} + \operatorname{div}(\rho S\mathbf{v}_n) = 0, \tag{8.5}$$

where S is the entropy per gm of the total liquid. This means that the normal fluid only carries entropy. Some authors prefer to assume only that the entropy of the superfluid is as yet unmeasurably small; we will assume that it is zero, i.e.,

$$\rho S = \rho_n S_n, \tag{8.6}$$

(where S_n is the entropy per g of the normal constituent) rather than

$$\rho S = \rho_s S_s + \rho_n S_n.$$

If Eqs. (8.3) and (8.4) are put into the equation of continuity (8.2) we get two equations whose sum is zero, thus:

$$\begin{aligned}&\frac{\partial\rho_n}{\partial t} + \operatorname{div}(\rho_n\mathbf{v}_n) = \Gamma,\\ &\frac{\partial\rho_s}{\partial t} + \operatorname{div}(\rho_s\mathbf{v}_s) = -\Gamma.\end{aligned} \tag{8.7}$$

In (8.7) Γ is the rate of production of normal fluid per unit volume: in most discussions of two-fluid hydrodynamics the approximation has been that Γ is equal to zero.

We have still not written the two fluid analogs of (8.1). At the present time the exact equation (one for

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[24] See Lord Rayleigh, *The Theory of Sound*, Vol. II, p. 345.

$D\mathbf{v}_n/Dt$ and one for $D\mathbf{v}_s/Dt$) are not known with certainty. Equations which are linear in the velocities and which are valid in the limit of small velocities are known, but terms in the squares of the velocities have been given differently by different authors. The first-order equations, neglecting irreversible effects, on which all are agreed are

$$\begin{aligned}\rho_n\frac{\partial\mathbf{v}_n}{\partial t}&=-\frac{\rho_n}{\rho}\,\mathrm{grad}p-\rho_sS\,\mathrm{grad}T,\\ \rho_s\frac{\partial\mathbf{v}_s}{\partial t}&=-\frac{\rho_s}{\rho}\,\mathrm{grad}p+\rho_sS\,\mathrm{grad}T.\end{aligned}\tag{8.8}$$

The two equations are first given (in a somewhat different form) by Tisza (T.40) in 1940, although he had already written one of them earlier ((T.38b) and (T.38c)) whereby the other was implicit. Tisza wrote the terms which are here proportional to grad$T$, as terms proportional to the gradient of an osmotic pressure p_n; however, when H. London's equation (see Sec. 4.7) for the fountain pressure is used for this osmotic pressure, Eqs. (8.8) follow. In arriving at (8.8) Tisza assumed that Γ (Eq. 8.7) is zero; this assumption is always made when deducing Eq. (8.8). [A clear and complete derivation of the hydrodynamical equations in first-order approximation is given by Dingle (D.49).]

The same equations follow from the treatment of Landau (L.41a), although from a different starting point: Landau assumed that the Gibbs function of helium II with excitations present (i.e., with normal fluid in motion) equals the Gibbs function of the pure superfluid plus the kinetic energy of the excitations relative to the superfluid. If the conservation of momentum is also invoked, Eqs. (8.8) follow as the linear approximation.

In a later paper, in which he derived the equations for second sound (see Sec. 9.2), Tisza (T.47) employed a method which also leads to Eqs. (8.8). Dingle (D.49) has used this method to arrive at (8.8). The equations may also be obtained from Hamilton's principle applied in a linear approximation.

When a better approximation than (8.8) is sought, one finds conflicting results. Equations with terms quadratic in the velocities have been derived (or postulated) by several authors: Landau (L.41a), Gorter and Mellink (G.49a), Nakajima, Tomita, and Usui (N.50a), Zilsel (Z.50a), Temperley (T.51c), Prigogine and Mazur (P.51b), Usui (U.51a), and Dingle (D.52a). The interest in these equations stems chiefly from the fact that the superfluid shows something like a critical velocity (see Sec. 7); a complete set of hydrodynamical equations would elucidate the critical velocity (or critical region of velocity) whereas Eqs. (8.8) have obviously nothing to say on this matter.

We have not space to display and discuss the theories of all these authors in any detail, nor is it completely necessary for Dingle in his review has given them considerable attention. However, we will list the equation of motion of the superfluid component given by each author (using a standard notation as far as possible) and then comment briefly on the terms which appear.

Landau:

$$\rho_s\frac{D\mathbf{v}_s}{Dt}=-\frac{\rho_s}{\rho}\,\mathrm{grad}p+\rho_sS\,\mathrm{grad}T+\rho_s\,\mathrm{grad}\left[\frac{\rho_n}{2\rho}(v_n-v_s)^2\right].$$

Gorter and Mellink:

$$\rho_s\frac{D\mathbf{v}_s}{Dt}=-\frac{\rho_s}{\rho}\,\mathrm{grad}p+\rho_sx\left(\frac{\partial S}{\partial x}\right)\mathrm{grad}T-A\rho_s\rho_n(\mathbf{v}_s-\mathbf{v}_n)^3.$$

Nakajima, Tomita, and Usui:

$$\rho_s\frac{D\mathbf{v}_s}{Dt}=-\frac{\rho_s}{\rho}\,\mathrm{grad}p+\rho_sS\,\mathrm{grad}T-\frac{\Gamma}{2}(\mathbf{v}_n-\mathbf{v}_s)+\mathbf{f}.$$

Zilsel:

$$\rho_s\frac{D\mathbf{v}_s}{Dt}=-\frac{\rho_s}{\rho}\,\mathrm{grad}p+\rho_sS\,\mathrm{grad}T+\frac{\rho_n\rho_s}{2\rho}\,\mathrm{grad}(\mathbf{v}_n-\mathbf{v}_s)^2$$

$$\rho_n\frac{D\mathbf{v}_n}{Dt}=-\frac{\rho_n}{\rho}\,\mathrm{grad}p-\rho_sS\,\mathrm{grad}T-\frac{\rho_n\rho_s}{2\rho}\,\mathrm{grad}(\mathbf{v}_n-\mathbf{v}_s)^2-\Gamma(\mathbf{v}_n-\mathbf{v}_s).$$

Temperley:

$$\rho_s\frac{D\mathbf{v}_s}{Dt}=-\frac{\rho_s}{\rho}\,\mathrm{grad}p+\rho_sS\,\mathrm{grad}T-\frac{\Gamma}{2}(\mathbf{v}_n-\mathbf{v}_s).$$

Prigogine and Mazur:

$$\begin{aligned}\rho_s\frac{D\mathbf{v}_s}{Dt}=&-\frac{\rho_s}{\rho}\,\mathrm{grad}p+\rho_sx\left(\frac{\partial S}{\partial x}\right)\mathrm{grad}T\\ &+\frac{\rho_n\rho_s}{\rho}\,\mathrm{grad}(\mu_n-\mu_s)-\frac{\rho_s}{\rho}\Gamma(\mathbf{v}_n-\mathbf{v}_s)\\ &+\zeta(\mathbf{v}_n-\mathbf{v}_s)^2(\mathbf{v}_n-\mathbf{v}_s).\end{aligned}$$

Dingle:

$$\rho_s\frac{D\mathbf{v}_s}{Dt}=-\frac{\rho_s}{\rho}\,\mathrm{grad}p+\rho_sS\,\mathrm{grad}T+\rho_s\,\mathrm{grad}\left[\frac{1}{2\rho}\int_0^{v=v_n-v_s}\rho_n d(v^2)\right].$$

As has been mentioned before, Landau arrived at his equation by postulating the form of the Gibbs function in a "two fluid." To obtain a complete set of equations, he also used the conservation laws (mass, entropy, momentum, and energy must be conserved in reversible processes).

Gorter and Mellink obtained their equations by generalizing Eq. (8.8). The last term on the right in their equation is the force due to *mutual friction* (already discussed in Secs. 2 and 5) which acts with equal magnitude but opposite sense on each fluid: this they postulated in order to explain the heat flow experiments.[25] The second term on the right of their equation contains the combination $x(\partial S/\partial x)$, $(x=\rho_n/\rho)$, which follows from a generalization of Tisza's entropy postulate (see Sec. 4.7) made by Gorter (G.49b). If Eq. (8.6) is correct, this combination is just equal to S of (8.6).

Nakajima, Tomita, and Usui derived their equations from the conservation laws. The vector **f** is a splitting factor introduced because the conservation laws alone give only the behavior of the sum $(\rho_s(D\mathbf{v}_s/Dt) + \rho_n(D\mathbf{v}_n/Dt))$. The third term in their equation marks the appearance of Γ (Eq. 8.7) in the equations. These authors have a term of similar form in their expression for $(D\mathbf{v}_n/Dt)$. In discussing their equations in the light of experiment, Nakajima *et al.* set Γ equal to zero, and take **f** to be Gorter and Mellink's mutual friction force.

Zilsel obtained his equations by using Eckart's (E.38) variational principle. This principle bears a formal resemblance to Hamilton's principle; however, it is not a formulation of classical mechanics because the motion of particles is not followed and velocities rather than displacements are varied. We have written both of Zilsel's equations in order to show that they are not symmetrical in Γ as are those of Nakajima *et al.* and Temperley. Zilsel shows that the asymmetry is consistent with the concept that the superfluid particles are "condensed" in the sense of the Bose-Einstein condensation. It is a peculiarity of Eckart's principle that curl $v_s=0$ appears automatically—recall that this condition was postulated by Landau. Notice, also, that a term appears in Zilsel's equations which is very similar to the last term in Landau's equation.

Temperley's equations derive from the conservation laws. They are also symmetrical in Γ. (We have changed Temperley's notation somewhat by using the relationship between Γ and (DS_n/Dt) implied by (8.5), (8.6), and (8.7).)

Prigogine and Mazur derived their equations by using the methods of irreversible thermodynamics. Beside using the conservation laws these authors introduce the concept of two partial pressures each of which acts on one of the fluids. The equation we have written here is a combination of Eqs. (3.41) and (5.2) of their paper. The μ's appearing in their equation are the chemical potentials of the two fluids which need not be equal if $x=\rho_n/\rho$ has not its equilibrium value. It can be shown that Prigogine and Mazur's term in grad $(\mu_n-\mu_s)$ is equivalent to Zilsel's term in grad $\frac{1}{2}(\mathbf{v}_n-\mathbf{v}_s)^2$. The last term in Prigogine and Mazur's equation is equivalent to the Gorter-Mellink mutual friction, which is postulated by these authors in order to obtain the Gorter-Mellink equations.

We have omitted Usui's equation since it differs from that of Nakajima, Tomita, and Usui only in that $x(\partial S/\partial x)$ replaces S.

A general comment is in order on all equations containing a term proportional to $\Gamma(\mathbf{v}_n-\mathbf{v}_s)$. In all derivations made from the conservation laws some arbitrariness exists as to the way this term is to be divided between the two fluids, and the divisions displayed represent additional assumptions. On the other hand, Zilsel's asymmetrical splitting follows from the assumption that Eckart's principle may be applied to the problem.

Dingle's equation is a generalization of Landau's equation. Whereas Landau explicitly limited his considerations to terms involving $(\mathbf{v}_n-\mathbf{v}_s)^2$, Dingle has extended the calculation to include all powers of the relative velocity.

A complete set of thermohydrodynamical equations (for a nonviscous, nonheat-conducting, fluid) which includes the definitions (8.3), (8.4), and (8.6), the conservation laws (8.5) and (8.7), and Euler equations such as (8.8) have usually been derived, in whole or in part, by thermodynamic arguments. Quite recently, however, Kronig (K.53g) has used a statistical mechanical approach to derive the Eqs. (8.5) and (8.8) in linear approximation. Kronig's derivation is based on the notions of Landau (L.41a) and applies rigorously only to that region in which it may be assumed that the only excitation present in the fluid are phonons which are in local equilibrium, i.e., below about 0.6°K. Kronig's work is an extension and modification of similar results pertaining chiefly to the second-sound velocity derived by Dingle (D.52c), and Ward and Wilks (W.51b) and (W.52). These other results will be cited in Sec. 9.3 on second sound. H. A. Kramers (K.52e) undertook an analysis along the same lines as that of Kronig just discussed. Kramers considered excitations of a general character of which phonons and rotons are special cases. Kramer's work, however, leads to an expression for the second-sound velocity (see Sec. 9) which is correct only in the limit $T\rightarrow 0$°K.

Work in the same spirit as Kronig's, but starting from the Bose-Einstein gas model has been undertaken by Bogolubov (B.47d) and Zilsel (Z.53b). Bogolubov considers a nonperfect gas and by using second quantization shows that in the case of small interactions, the excited states of the gas can be described as a perfect Bose-Einstein gas of "quasi-particles." The energy spectrum of the quasi-gas is not known precisely, but it is shown that for special forms of $E(p)$ (essentially that $E(p)$ be not convex toward the p axis as $p\rightarrow 0$) the gas divides into excited and condensed fractions

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[25] Kasuya (K.53c) has suggested supplementing the Gorter-Mellink mutual friction with a superfluid friction force of the form $f_s=-B\rho_s(v_s)^3$ with $B=0$ for $v_s<v_{\text{crit}}$. This *ad hoc* addition would enable one to describe experimental results somewhat better than the mutual friction term alone. It is obvious how the hydrodynamical equations are to be modified in this case.

which may have differing average velocities if the relative velocity be small enough.

Zilsel has similarly discussed the possibility of two velocity fields in a Bose-Einstein gas, finding the same sort of restriction on the energy spectrum. In particular he considers the spectrum of Bijl, de Boer, and Michels (B.41), and discusses how a critical velocity can arise.

8.2. Experimental Investigations on the Nonlinear Equations of Motion

Few experiments have been performed to test or differentiate between the nonlinear equations just displayed. The first-order equations (8.8) have been verified, chiefly by the experiments on second sound (Sec. 9.). It has been mentioned that the complete equations should display the effect of a critical velocity. However, it is not obvious how such a velocity arises. Fried and Zilsel (F.52) have used Zilsel's equations to obtain a criterion for the validity of the linear equations. They find that the linear approximation loses validity at $v_s \cdot d \approx 5 \times 10^{-4}$ cm^2/sec in rough agreement with Eq. 7.2.

The equations of Gorter and Mellink also lead to a rather smeared out critical velocity since the mutual friction term increases very rapidly with relative velocity. Gorter and Mellink's equations are the only equations which have been compared with observations in a thoroughgoing way. The original formulation was made in order to explain the experimental findings on heat conduction. These experiments, together with other experiments designed to test these equations, have been reviewed above in Secs. 2 and 5, and have also been discussed at some length in a recent review by Atkins (A.52a).

From these experiments we must conclude that the equations of Gorter and Mellink do not describe completely the behavior of helium II. Unfortunately the other equations proposed and listed above have not been subjected to experimental test.

9. FIRST AND SECOND SOUND

9.1. The Velocity and Attenuation of First Sound

The two separate velocity fields in helium II give rise to the possibility of two types of wave motion in the fluid. This fact was first perceived and investigated by Tisza (T. 38b *et seq.*). In the absence of any temperature gradient the Eq. (8.8) may be summed to give the one fluid hydrodynamical equation, thus ordinary (first) sound is possible in which waves of density or pressure may be propagated at constant entropy with the velocity

$$u_1 = [(\partial p/\partial \rho)_s]^{\frac{1}{2}}. \tag{9.1}$$

The first measurements of the velocity u_1 were made by Findlay, Pitt, Grayson-Smith, and Wilhelm (F.38), using an ultrasonic standing wave technique and generating the sound waves at 1.338 Mc/sec by a quartz crystal oscillator immersed in the liquid helium. Subsequent measurements have been made by Pellam and Squire (P.47) using pulse techniques at 15 Mc/sec and by Atkins and Chase (A.51b), also using pulse techniques at 14 Mc/sec. All the results are in good agreement with each other and a graph showing the value of the velocity of first sound u_1 as a function of temperature under saturated vapor pressure due to Atkins and Chase is given in Fig. 9.1. This graph shows an anomalous minimum at the λ temperature of 2.18°K due to the transformation from liquid helium I to liquid helium II. The maximum at about 2.5°K, corresponding to a minimum in the adiabatic compressibility, is also of interest. The agreement between all workers using different operating frequencies shows that, at least above 1.6°K, the dispersion is small (<1 percent).

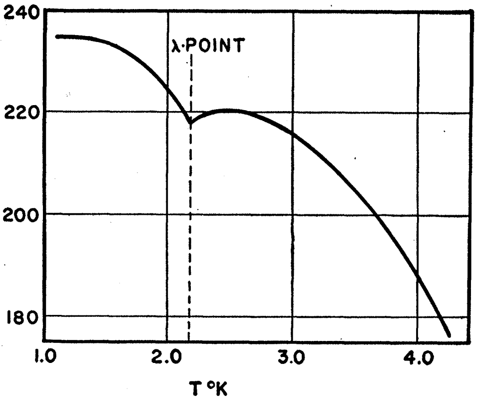

FIG. 9.1. Velocity of first sound u_1 (m/sec) in liquid helium, due to Atkins and Chase (A.51b).

One of the striking features of the curve is the sharp dip at T_λ. Atkins and Chase studied this region in detail and concluded that it was impossible to know if there was a discontinuity here (as would be expected from Ehrenfest's relations, if the transition were of second order). They supposed, on the other hand, that both branches of the u_1 *versus* T curves, above and below T_λ, may drop together to much lower values than have yet been observed.

From Fig. 9.1, noting that the measurements extend to 1.2°K and that the curve appears to be flattening out, it is possible to make an informed extrapolation of the value of u_1 to absolute zero, particularly since on theoretical grounds $(\partial p/\partial \rho)_s$ is expected to be practically constant from this temperature down. Atkins and Chase's extrapolation gives $u_1 = 237 \pm 2$ m/sec as $T \rightarrow 0$°K.

The pulse technique of measurement, first used by Pellam and Squire (P.47) allowed them to observe the attenuation of first sound and their results for the attenuation coefficient α are shown in Fig. 9.2 by the full curve. Subsequent measurements of α at 14 Mc/sec have been made by Atkins and Chase [unpublished but see Atkins (A.52a) and (C.53b)] in substantial agree-

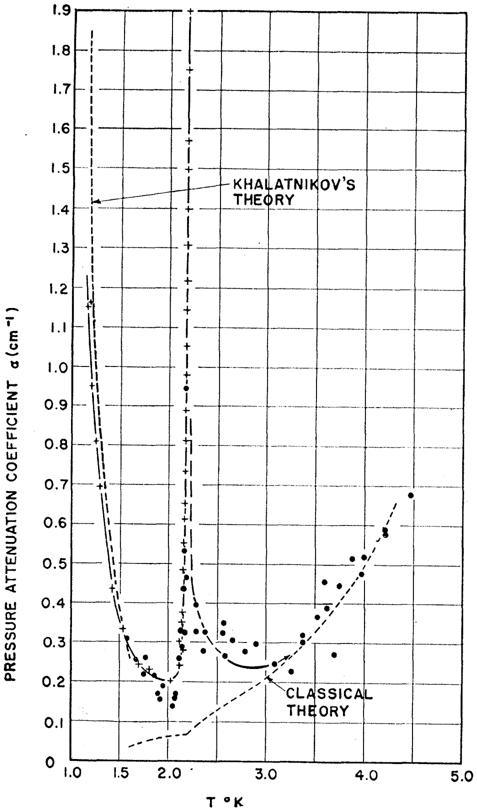

FIG. 9.2. Ultrasonic pressure attenuation coefficient α *versus* temperature. Pellam and Squire (P.47). For the range 1.6°K to 1.2°K the curve is due to more recent work of Atkins and Chase (see A.52a). The broken curve above 1.6°K gives the theoretical evaluation of α made by Pellam and Squire (P.47) using classical methods. The broken curve below 1.6°K gives Khalatnikov's (K.50b) calculation of α, obtained by fitting his results to experiment at the high temperature end. Curve due to Atkins (A.52a). ● Measurements of Pellam and Squire (P.47). + Measurements of Atkins and Chase (see A.52a).

ment with the earlier work of Pellam and Squire. The more recent workers extended the measurements to temperatures below 1.6°K, where Pellam and Squire left off, and these results down to 1.2°K are also included in Fig. 9.2. The very high attenuation in liquid helium I as T_λ is approached is noteworthy, as is also the steady rise in α in liquid helium II as the temperature is reduced.

The theoretical value of the attenuation coefficient α shown by the broken curve of Fig. 9.2, above 1.6°K was calculated by Pellam and Squire by assessing separately the contribution to absorption due both to viscous losses and to the thermal conduction by strictly classical methods. [See, for example, Bergman (B.39b).] It will be seen that down to 3°K the agreement between theory and experiment is good, due probably, as pointed out by Pellam and Squire, to the monatomic character of the helium which, therefore, disposes of relaxation phenomena connected with inner degrees of freedom in molecules or with association. On the other hand, the marked increase in the experimental value of α below 3°K is outstanding, apparently leading to infinity at T_λ. The same authors have suggested that this is due to forced transitions from liquid helium I to liquid helium II taking place locally at temperatures around T_λ caused by the pressure variations in the sound waves. This explanation is in line with that put forward by Keesom (K.42) in explanation of the specific heat curve just above T_λ. Pippard (P.51a) has considered in detail this question of local fluctuational transitions from helium I to helium II and vice versa. He derived expressions for the velocity and attenuation of first sound in an inhomogeneous fluid containing spherical inclusions of compressibility different from that of the matrix in which they are embedded. In applying these results to liquid helium he noted that compression of helium II inclusions would cause them to cool, whereas compression of the helium I matrix causes it to be warmed. The subsequent trend to equilibrium must be characterized as a relaxation process, which would provide anomalous absorption of pressure waves. By making plausible assumptions in order to apply the theory to liquid helium, he has accounted for the observed u_1 *versus* T curve around T_λ and has found that anomalous attenuation in liquid helium I near T_λ can be explained by considering the inclusions in it of liquid helium II to be of size of about 850 atoms.

The rise in attenuation in liquid helium II with falling temperature was discussed by Pellam and Squire. They noted that normal viscous and conductive losses were entirely inadequate to account for the observed rise and they suggested that it might be due to dissipative effects again of a relaxation type. In assessing the magnitude of the contribution to α by ordinary viscous losses, the evaluations of η_n given in Sec. 2.4 (see Fig. 2.5) could now be used, with results in agreement with Pellam and Squire. Moreover, a theoretical computation of η_n is now available, due to Landau and Khalatnikov (L.49a). In view of the use made subsequently of their computation by Khalatnikov (K.50b) in his theory of the attenuation of first sound in liquid helium II, some comment on it might be appropriate here.

Landau and Khalatnikov's (L.49a) theory of the viscosity of liquid helium II has as its basis the detailed picture of the liquid due to Landau, such that the normal constituent is a "gas" of excitations—phonons and rotons having the energy spectra given in Sec. 3. The theory is built in terms of the momentum scattered from one layer of the gas of excitations into an adjacent layer, when the two layers have slightly different macroscopic velocities. In order to do this the cross sections of scattering of phonons by phonons, phonons by rotons, and rotons by rotons were first calculated, and from them the collision integrals which give the rate of change of the distribution function. It turned out that the rotons behave like heavy particles and the phonons like light ones. Rotons are scattered only by rotons at all temperatures, and such processes contribute a temperature independent term to the viscosity. Above 0.9°K the effect of phonon-phonon collisions is small compared to phonon-roton collisions, whereas below this temperature phonon-phonon collisions may not be neglected and contribute the major effect below 0.7°K. The problem is further complicated because the rate of creation and destruction of phonons is not negligible compared to the rate of scattering processes, in fact, below 0.9°K the first rate is the greater, while above 0.9°K the second is. This entails two separate calculations of the viscosity, one valid below about 0.8°K and the other valid above 1.0°K.

The details of these heroic calculations cannot be reproduced here. It is found that the phonon contribution to the viscosity is strongly temperature dependent. When this contribution is subtracted from Andronikashvili's experimental data (A.48a), the constant roton viscosity is found. The final expressions for the viscosity coefficient[26] (in poise) are

$$\eta\times 10^{-5}=1+8.7\times 10^{-4}T^{\frac{1}{2}}\exp(\Delta/kT),\quad T>1.0°\mathrm{K}, \tag{9.2}$$

$$\eta\times 10^{5}=7.8\times 10^{-5}[T^{\frac{1}{2}}\exp(-\Delta/kT)+4.8\times 10^{-6}T^{5}]^{-1},\quad T<0.8°\mathrm{K}.$$

Viscosities in the interval between 0.8 and 1.0°K are found by interpolation. It is apparent that at fairly low temperatures $\eta\propto T^{-5}$; this is due to the phonon-phonon scattering.

The authors claim only a limited accuracy for their work owing to the uncertainty in the value of certain parameters such as $\partial^2\Delta/\partial\rho^2$ and $\partial p_0/\partial\rho$. By making plausible assumptions as to the magnitude of these terms, however, agreement with experiment is fair. More importantly, on the other hand, they believe the temperature dependence has been obtained with sufficient accuracy.

Good use of the above theory has been made by Khalatnikov (K.50b) in describing the attenuation of first sound. He pointed out that in the pressure wave the local nonequilibrium phonon and roton densities can revert back to the equilibrium values only in finite times and this provides a relaxation mechanism which causes attenuation. He considered two relaxation times, characterizing the two most important collision processes by which equilibrium is established, namely: phonon-phonon collisions and phonon-roton collisions. From this he was able to assess the temperature variation of the attenuation coefficient due to relaxation effects and he found that α should increase rapidly as the temperature is reduced. By fixing the absolute values by fitting on Pellam and Squire's experimental results above 1.6°K, Khalatnikov arrived at the curve shown in Fig. 9.2 by the broken curve below 1.6°K.

More recently Khalatnikow (K.52f and K.52g) has considered these effects in more detail and has shown that three coefficients of second viscosity should be considered together with a coefficient analogous to the coefficient of heat conductivity in ordinary liquids. He asserts, however, that these newer considerations do not change the conclusions regarding the attenuation of first sound reported earlier (K.50b), but that for second sound they should lead to appreciable absorption at frequencies of 10^4 cps.

Kronig, Thellung and Woldringh (K.52h) have also developed a relaxation theory to account for the attenuation of first sound. Using the two fluid model previously investigated by Kronig and Thellung (K.50a) they have considered the relaxation of one fluid to the other and concluded from a comparison of experimental values of α that near T_λ the relaxation time must be either less than about 10^{-12} sec or greater than about 10^{-4} sec.

9.2. Tisza's and Landau's Equations for Second Sound

Besides the fluid motion corresponding to first sound, in which the density oscillates at any point in the fluid, Tisza (T.38b) first recognized that another wave motion, in which the density and the pressure (to first approximation) remain constant but in which the entropy (or the temperature) fluctuates, is also possible. (In such a motion the normal and superfluid move 180° out of phase.) In his 1947 paper Tisza discussed the two wave motions more thoroughly, showed that the coupling between density waves and entropy waves is indeed very small, and arrived at the equation for the velocity of the temperature wave, u_2, (the so-called "second sound"),

$$u_2=\left[-\left(\frac{\partial T}{\partial(1/S_n')}\right)\cdot\frac{\rho_s}{\rho_n}\right]^{\frac{1}{2}}, \tag{9.3}$$

where S_n' is the entropy of the *normal* constituent per gm of *total* liquid.

In order to reduce Eq. 9.3 for the velocity of second sound to numerical values, Tisza adopted a number of assumptions partly based on experimental evidence. These were:

(a) The entropy, S_n' of Eq. (9.3) was taken to be the observed total entropy of the liquid in the temperature range 1°K to the λ point. This was based on the experimental evidence (outlined in Sec. 4) which showed that the entropy of the superfluid constituent was negligible in this temperature range.

(b) The superfluid as well as the normal constituent was taken to be capable of sustaining compressional (Debye) thermal excitations. As a result it was supposed that at a sufficiently low temperature ($0.4°\mathrm{K}<T$

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[26] Unfortunately, it is not immediately clear whether these results of Eq. (9.2) really give η or whether they give η_n. In arriving at the numerical evaluations, however, Landau and Khalatnikov compared their results with the experimental values of η_n.

$<1.0°\mathrm{K}$) mechano-caloric effects would disappear. This hypothesis, however, was neglected in the calculations of u_2 between 1°K and T_λ. Its effect would be that at temperatures below the second transition temperature second sound would be completely damped out.

(c) By a quasi-thermodynamic argument it was suggested that

$$\frac{\rho_n}{\rho}=\frac{S}{S_\lambda}=\left(\frac{T}{T_\lambda}\right)^{5.5} \tag{9.4}$$

in the temperature range 1°K to T_λ, where the Debye phonon excitations were neglected. This relationship, as shown in Sec. 4, is a fair approximation to experimental results in the temperature range 1.6°K to T_λ.

By use of items (a) and (c) Tisza arrived at the following type of expression for u_2:

$$u_2=26\left\{\frac{T}{T_\lambda}\left[1-\left(\frac{T}{T_\lambda}\right)^{5.5}\right]\right\}^{\frac{1}{2}}\ \mathrm{m/sec.} \tag{9.5}$$

The numerical constant was first given by F. London (L.46a) and the exponent 5.5 represented the best fit of Eq. (9.4) to the experimental results [see Eq. (2.2) and (4.3) of Secs. 2.3 and 4.3]. Equation (9.5) was found to represent with some accuracy the experimental values of u_2 (see Sec. 9.4) between 1.2°K and T_λ. It is to be noted, however, that for lower temperatures Eq. (9.5) expresses u_2 as a monotonously decreasing function as T decreases, such that for $T\rightarrow 0$, $u_2\rightarrow 0$. This is markedly at variance with the experimental findings.

Landau, too (L.41a), derived from his hydrodynamical equations two velocities for propagation of waves —which he called waves of first and second sound. Landau's nomenclature persists—perhaps because the idea of "second sound" is vaguely exciting—although the second motion is almost purely a temperature wave and cannot be excited by ordinary mechanical means (such as a microphone) since in the first order there are no density or pressure variations. Starting from the assumption that the entropy of the superfluid constituent is zero, Landau obtained the equation for the second sound velocity

$$u_2=\left[\frac{\rho_s}{\rho_n}\cdot\frac{S^2T}{C}\right]^{\frac{1}{2}}=\left[-\frac{\rho_s}{\rho_n}\left\{\frac{\partial T}{\partial(1/S_n')}\right\}\right]^{\frac{1}{2}}. \tag{9.6}$$

Here C is the specific heat.

Although the formulas for the velocity of second sound given by Tisza and Landau are almost identical, the two authors interpreted the quantities which appear therein differently. The basic ideas proposed by Landau were:

(a^1) The entropy S_n' of Eq. (9.6) was taken to be the observed total entropy S of the liquid for *all* temperatures below T_λ.

(b^1) Only the normal constituent was taken to be capable of sustaining phonon excitation. Hence mechano-caloric effects and second sound should be observable at infinitely low temperatures.

(c^1) The value of ρ_n/ρ was assessed in two different manners:[27] (1) by appeal to experiments such as those of Andronikashvili (A.46 and A.48a) (see Sec. 2.3); (2) from Landau's microscopic theory of liquid helium II (L.41a and L.47a) which gives an explicit expression for ρ_n in terms of the parameters Δ, μ, and p_0 (see Sec. 3.).

The first explicit computation of u_2 as a function of temperature using Landau's theory was by Lifshitz (L.44b), who used Landau's earlier (L.41a) evaluations of the microscopic parameters Δ and μ to provide a knowledge of ρ_n. The result was in qualitative agreement with experiment (see Sec. 9.4) between 1.2°K and T_λ, but more importantly it showed that u_2 should rise in value markedly at about 1°K and eventually flatten out, as was emphasized first by Landau, to a value given by $u_1/\sqrt{3}$ as $T\rightarrow 0°\mathrm{K}$. This remarkable rise at the lower temperatures, due to the significance of the phonon field in this region, has been amply confirmed by subsequent experiment. Lifshitz's evaluation of u_2, which was found to be only in qualitative agreement with experiment, has been revised by Landau (L.47a) who introduced new numerical data for Δ, μ, and p_0 from which ρ_n could be computed. The revised scheme led to a dependence of u_2 on T which agrees roughly with Tisza's equation above about 1.5°K, possesses a minimum at about 1.1°K, rises rather steeply below this, and then levels off to the value $u_1/\sqrt{3}$ as T approaches zero.

To anticipate the results of experimental findings (see Secs. 9.4 and 9.6) somewhat, it appears, first, that in the temperature range 1.2°K to T_λ both theories are qualitatively in agreement. Secondly, in the temperature range below 1.2°K Tisza's interpretation fails, partly perhaps due to the inflexibility of the relationship (c), but more significantly because the assumption (b) prevents the rise of u_2 at lower temperatures. Thirdly, the evaluation of u_2 by Landau indicates the importance of the phonon contribution to the normal entropy at lower temperatures and enables a numerical estimate to be made for it (see Sec. 4.3). This contribution indicates that Landau's hypothesis (b^1), namely, the existence of mechano-caloric effects down to absolute zero, is correct.

In writing Eqs. (9.1) and (9.6) we have neglected the coupling between the density waves and the temperature waves. This coupling, which is proportional to the difference between the specific heats C_p-C_v divided by C_p, is indeed negligible in helium II. This point has been discussed by Lifshitz (L.44b), Peshkov (P.48a), and Dingle (D.50c).

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[27] Currently the validity of Eq. (9.6) is undisputed and, instead of inserting evaluations of ρ_n into it to obtain predictions regarding u_2, it is customary to use Eq. (9.6) together with the observed values of u_2 to compute ρ_n.

9.3. Other Equations for Second Sound

Several derivations and discussions of the equations of second sound besides those of Tisza and Landau have been presented by various authors. We shall mention a few but not all of these. For a more complete summary the reader is referred to the review of Dingle (D.52a), previously mentioned.

Gogate and Pathak (G.47b) have given an elementary derivation of the equation for the velocity of second sound, which, while not completely rigorous, is very suggestive. The equation is the same as Landau's. Gorter, Kasteleijn, and Mellink (G.50a), and Usui (U.51a), on the other hand, have obtained a somewhat different equation starting from Gorter's (G.49b) equation for the entropy, $S^*=x(\partial S/\partial x)$, where $x=\rho_n/\rho$, mentioned previously in Sec. 4.7. They obtain Gorter, Kasteleijn, and Mellink:

$$u_2^2=x(1-x)G_{xx} \text{ (for high frequencies)}. \qquad (9.7)$$

Usui: $u_2^2=x(1-x)S_xT_x$. (9.8)

(Throughout this section and in Sec. 9.7 subscripts on thermodynamic functions indicate partial differentiations.) Gorter *et al.* have another equation for the velocity for low frequencies which is discussed briefly in Sec. 9.7. To the approximation that (9.7) and (9.8) are correct, they are also equivalent. (In one of them G_{px} has been neglected, in the other, G_{pT}. These quantities are probably negligible.)

Nakajima and Shimizu (N.50b; N.51) have used Usui's equation to evaluate numerically the second-sound velocity, using two different equations for the entropy of helium II *versus* temperature—one based on Landau's model, the other based on a generalization of Tisza's model in which the phonon entropy is included in the normal entropy. When they use Landau's model, they find that the low-temperature limit of u_2 is just three-fourths of the value predicted by Landau, which result is, as is discussed in section 9.4, if anything, too low. With regard to these equations, Dingle (D.51) has questioned the validity of Gorter's entropy equation.

Some ingenious derivations of the velocity of second sound have been given in which no recourse is made to any two-fluid theory. In all discussions of this sort the excitations of the fluid are treated like a gas of particles [following Landau (L.41a)]. From this viewpoint second-sound waves are density waves in the gas of excitations or "particles." In order that such waves propagate, the "particles" must satisfy certain conditions (discussed more fully in the references cited following) among which are (i) only elastic collisions occur between particles, (ii) the mean free path of the excitations must be short compared with the wavelength of the second sound, and (iii) dissipative processes must not attenuate the second-sound motion too severely for detection.

First among these discussions are those of Ward and Wilks (W.51b; W.52). They developed an analogy between phonons and photons in order to arrive at Landau's value for the second-sound velocity in the limit $T\rightarrow 0$; i.e., $u_2=u_1/\sqrt{3}$. They briefly discussed the possibility of second sound in crystals, the investigation of which had first been proposed by Peshkov (P.46a). (The analogy used by Ward and Wilks suggests the possibility of second-sound waves in a photon gas also, but, as was pointed out by F. London (L.51), photons do not satisfy condition (ii) given above.)

Later, Ward, and Wilks obtained the same result by applying Boltzmann's equation to the excitations.

Dingle has also considered this problem (D.52a; D.52c), proceeding along lines similar to those of Ward and Wilks. Dingle displayed the thermodynamic functions of the excitation gas and derived the result that

$$u_2^2=TS^2\rho/C_v\cdot\rho_n, \qquad (9.9)$$

which differs from Eq. (9.6) by a factor (ρ/ρ_s) so that agreement of the two arises only as $T\rightarrow 0$. Dingle has suggested (D.52a) that the difference between (9.6) and (9.9) is due to the neglect of the ground state. The derivation of the two-fluid equations by Kronig (K.53g) already mentioned in Sec. 8 carries this approach to the problem of second sound in helium to its conclusion inasmuch as Kronig found it possible to take the momentum of the ground state into account.

As we have just indicated, there has been considerable speculation as to the possibility of second-sound propagation in media other than helium II. Dingle (D.52.a) has recently calculated the velocity u_2 to be expected in several different media. Of course, the restrictions mentioned above must be met. In particular, if one is interested in the possibility of second sound in liquid He^3, it might be inferred from the fact that He^3 apparently never becomes superfluid [see Sec. (3.7)] that dissipative processes in He^3 might prevent the propagation of second-sound waves in this fluid. Nevertheless, investigations on second sound in pure liquid He^3 would be of extreme interest and significance.

9.4. Experiments on Second Sound

The first experiments on second sound in liquid helium II were performed by Peshkov. In a series of beautiful experiments reported from 1944 to 1949 (P.44, P.46a, P.46b, P.48b, P.48c, P.49a), Peshkov explored the temperature region from the λ point down to slightly above 1.0°K.

Peshkov used standing wave techniques: waves of temperature or entropy were generated at one end of a tube filled with liquid helium II by means of a heater consisting of a flat coil of wire through which a sinusoidal current was passed. Another flat-coiled phosphor-bronze resistance thermometer normal to the axis of the tube served as a detector and it could be moved up and down the tube. The relationship between

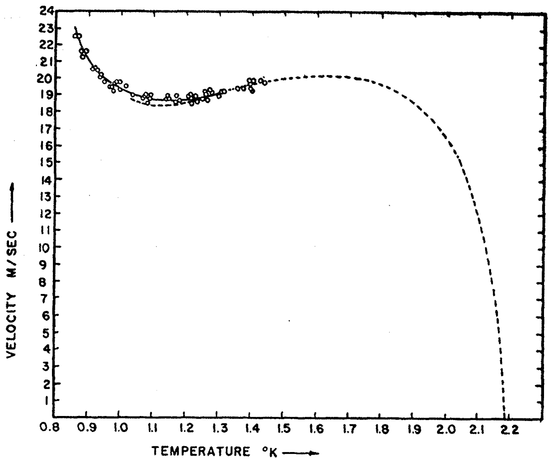

FIG. 9.3. The velocity of second sound u_2 as a function of temperature. The broken curve shows the results of Peshkov (P.44, P.46b, P.48b, P.48c, P.49a). Above about 1.25°K this broken curve also represents the results of Lane and co-workers (L.46c, L.47b), of Pellam (P.48d, P.49b), and of Osborne (O.48b), all of which are in good mutual agreement. The full curve, which is the best fit to the observed results shown by the circles, is due to Maurer and Herlin (M.49d).

wavelength and frequency could be studied either by producing resonance of the temperature waves in the tube and measuring the resonance response curve, or by moving the detector nearer or farther from the heater and observing the spacing of nodes and antinodes of temperature variation. Peshkov used frequencies from 10 to 10^4 cps and detected no dispersion in the waves. His results are shown in Fig. 9.3 by the broken curve. It will be noted that the value of second-sound velocity u_2 rises rapidly from zero at T_λ and reaches a maximum of 20.3 m/sec at about 1.65°K, which value is approximately maintained over a wide range of temperature. Before finally increasing to much higher values below 1°K, however, the value of u_2 passes through a minimum of about 18.4 m/sec just above 1°K.

Peshkov (P.48a; P.48b) also studied certain subsidiary effects which we have not here space to discuss completely. One interesting experiment consisted in producing second sound by placing a porous filter in front of an ordinary acoustic transducer. Between the transducer and the filter normal and superfluid moved together producing density, or first sound, waves; in the filter the normal fluid was slowed by viscous drag while the superfluid was not, giving rise to a relative motion of the two fluids and thus second sound beyond the filter.

In reaching to lower temperatures, it was Peshkov (P.48c) who first found that contrary to Tisza's Eq. (9.5) the second-sound velocity passes through a minimum just above 1.0°K, thereby furnishing experimental evidence for Landau's assumption that the density and entropy of phonons contribute to the normal fluid.

Other measurements of the second-sound velocity have been carried out by Lane, Fairbank, Schultz, and Fairbank (L.46c, L.47b) using a somewhat different method. In their arrangement a cylindrical Lucite cavity containing a heater at the bottom as a source of periodic temperature waves was partially filled with liquid helium II. At the top of the cavity a magnetic microphone was mounted, which served as a detector. According to a suggestion which had been put forward by Onsager, the "second sound" induced in the liquid was converted into normal sound in the vapor at the liquid surface with a high percentage transmission of energy. The pressure variations of the normal sound in the vapor, therefore, could activate the microphone. As the level of the liquid in the cavity fell slowly, it was found that resonances could be observed, and knowing the length of the liquid column at each resonance, it was possible to determine the velocity u_2 of the temperature waves to within ±0.5 percent. The results obtained were in agreement with the work of Peshkov, as given in Fig. 9.3.

More recent work on the second sound velocity has been done using pulse techniques developed by Pellam (P.48d, P.49b) and Osborne (O.48b). A short heat pulse (of about 10^{-4} sec duration) is delivered to the helium and picked up by a receiver. Both input and received pulse are displayed on an oscilloscope screen. For details of the experimental arrangements, the original papers must be consulted. The velocity is given by the ratio of the path length to the time increment between delivered and received pulses, the time increment being directly observed with the oscilloscope. This method has the advantage of convenience (resonance conditions do not have to be achieved) and of a low average power transmitted to the helium. This second advantage becomes more and more important as the temperature is lowered, and the entropy of helium II diminishes.

The results of these experiments provided ample confirmation of the accuracy of the measurements previously made by Peshkov, and by Lane and coworkers. They provided also the result that no measurable dispersion exists for second sound over a wide range of frequencies, as was found also by Peshkov.

In order to provide further theoretical insight into the mechanism of second sound, measurements below a temperature of about 1.1°K, the lower limit of the range used in the work quoted above, were of interest. Results of such measurements have been reported during the past few years, first by Peshkov down to 1.03°K (P.48c), then by Maurer and Herlin (M.49d) and by Peshkov (P.52c) down to 0.86°K and then by Pellam and Scott (P.49c), Atkins and Osborne (A.50c), and by de Klerk, Hudson, and Pellam (K.53a) to temperatures obtained by paramagnetic cooling. The results of Maurer and Herlin's and of Peshkov's work are given in Fig. 9.3 in which the older data is also included. These results, as already mentioned, indicated for the first time the existence of a minimum in the

u_2 *versus* T curve at about 1°K and the beginning of a rapid rise in u_2 below 1°K. These results, therefore, were the first to allow an unambiguous choice between the basic assumptions of Tisza on the one hand and of Landau on the other hand (see Sec. 9.2) made in their interpretation of the second-sound equation for liquid helium II.

The results for u_2 in the paramagnetic cooling region of temperature obtained by Atkins and Osborne (A.50c) and by de Klerk, Hudson, and Pellam (K.53a) both using pulse techniques are shown in Fig. 9.4. Leaving aside for the moment the question of the accuracy of the temperature measurement in Atkins and Osborne's data (later work has shown the temperature data to be in doubt), we note their results as a milestone in the measurements of the velocity of second sound. They observed that u_2 increases very markedly at about 0.4°K and then appears to "level off." Extrapolating their results to 0°K they obtained $u_2(0) \approx 152$ m/sec—a value which is about 10 percent higher than that expected from Landau's result: $u_2(0) = u_1(0)/\sqrt{3}$.

Very recently de Klerk *et al.*, working in the same temperature interval, found evidence of somewhat different behavior. First, they observed the steepest rise in temperature to come at about 0.6°K rather than at 0.4°K. This they explained by noticing that the apparatus of Atkins and Osborne had a very small warm-up time so that the helium involved probably never came to the low temperature of the paramagnetic salt used to cool it. Atkins and Osborne had reported the salt temperature rather than the helium temperatures. H. A. Kramers (K.52e) pointed out some time before the publication of the results of de Klerk *et al.* that the steep rise in u_2 as a function of T should occur at about the same temperature at which the T^3 dependency of the specific heat first became apparent, i.e., at the initiation of the pure phonon region. The measurements of Kramers *et al.* (K.52a) on the specific heat indicated that this temperature was about 0.6°K. It was therefore very reassuring when this was shown to be so by the experiments of de Klerk *et al.*

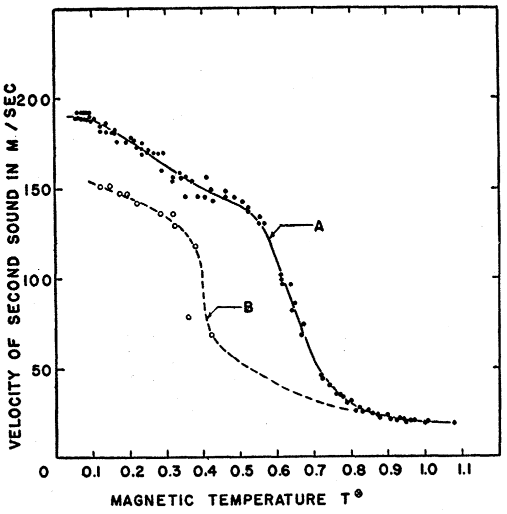

FIG. 9.4. The velocity of second sound *versus* temperature at low temperatures [from de Klerk, Hudson, and Pellam (K.53a)]. Curve *A* and associated data points due to de Klerk *et al.* Broken curve *B* and associated data points due to Atkins and Osborn (O.50c).

Furthermore, de Klerk *et al.* found that, for the lowest temperatures they have obtained, down to a nominal 0.01°K, the velocity of second sound is considerably above the Landau value and apparently is still increasing with diminishing T. (They report velocities as high as 192 m/sec.) These results are shown in Fig. 9.4.

De Klerk, Hudson, and Pellam have tentatively listed three possible explanations of their results, as follows:

(1) It is possible that the high velocity found is a real property of second-sound propagation; that is, the prediction of Landau concerning the low-temperature limiting velocity is in error.

(2) The effect may be a shock-wave phenomenon of the type observed by Osborne (O.51).

(3) The effect may be the result of a long mean free path of the phonons.

As to the first of these explanations little, of course, can be said. However, Dingle (D.52d) has recently made some calculations on the behavior of rectangular or δ function pulses in viscous or conducting media. The effect of the damping is to spread the pulse and to destroy its symmetry, so that the beginning of the received pulse arrives before the time $t = x/u_{20}$ (where u_{20} is the velocity computed from Eq. (9.6)). If one evaluates the velocity u_{start} with respect to the "start" of the pulse it will be too high. Dingle obtains at 0.2°K

$$u_{20} \approx u_{\text{start}}(1 - 60\sqrt{\eta}). \tag{9.10}$$

Evaluating[28] this equation he finds the observations of Atkins and Osborne (who measured to the start of the received pulse) in agreement with Landau's prediction. This form of the correction to the observed velocity fails at the lowest temperature used by de Klerk *et al.* On the other hand, if other points on the received pulse are used in the measurements, velocities several times smaller than those reported may be obtained. For this reason Pellam has suggested using pulsed carrier waves for the determinations so that the velocity at a particular frequency may be obtained. However, although CW pulses have been generated by de Klerk *et al.*, the contribution of the CW frequency has not been discriminated in the received signals.

Considering next the possibility of shock waves, it is clear that they will become serious as the temperature

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[28] In order to use (9.10) Dingle had to abandon Landau and Khalatnikov's value of η ($\eta \approx 1$ poise at 0.2°K) and use the approximation of Tisza (T.47) that $\eta \propto T^{\frac{1}{2}}$. Even with this assumption the approximation formula (9.10) fails at 0.06°K.

is lowered since any nonzero heat input into the system will cause an appreciable temperature rise in a medium of vanishing entropy. If such shock waves cannot be avoided they will lead in the limit, according to de Klerk, Hudson, and Pellam (K.53a), to an apparent velocity of second sound equal to the velocity of first sound. Shock waves have been discussed by Temperley (T.51c) and Khalatnikov (K.51b). Khalatnikov has criticized Temperley's work as being based on erroneous hydrodynamical equations. Khalatnikov starts from Landau's equations which he solves to second order. He obtains

$$u_{\text{start}}=u_{20}+\tfrac{1}{2}\alpha_2(v_{n1}+v_{n2}), \qquad (9.11)$$

$$\alpha_2=\frac{ST}{C}\frac{\partial}{\partial T}\left[\ln\left(u_{20}{}^3\cdot\frac{C}{T}\right)\right], \qquad (9.12)$$

and v_{n1} and v_{n2} are the normal fluid velocities just ahead and just behind the shock front. As the temperature changes α_2 changes signs, so that in some regions of temperature a front-edge shock and in others a back-edge shock wave develops. As $T\rightarrow 0$, if the normal velocity is zero ahead of the shock front,

$$u_{\text{start}}\rightarrow\frac{u_1}{\sqrt{3}}+\tfrac{1}{3}v_n. \qquad (9.13)$$

Since v_n becomes very large but cannot exceed u_1,

$$u_{\text{start}}\rightarrow u_1\left(\frac{1+\sqrt{3}}{3}\right). \qquad (9.14)$$

Shock waves have received considerable attention in the recent reviews of Dingle (D.52a) and Atkins (A.52a).

The third possible explanation of the observed velocity lies in a long mean free path of the phonons, as has been suggested by Ward (W.53b), Gorter (G.52), and Atkins (A.53). This may be explained as follows: That second sound arises at all requires a "local temperature" to be established in the liquid. Such a local temperature, implying local equilibrium, can only occur if the mean free path of phonons in the liquid is very much shorter than the wavelength of second sound. If the mean free path of the phonons becomes macroscopic at low temperatures, as has been suggested by the theory of viscosity of Landau and Khalatnikov (L.49a) (see Sec. 9.1), equilibrium may not be established in the pulse transit distance. In this case, the heat pulse will be transmitted with the velocity of the phonons, i.e., with the velocity of first sound.

In a recent investigation to test the mean-free-path effect Kramers, van den Berg, and Gorter (K.53h) have compared the propagation of pulses in two cavities, one 3 cm and the other 6 cm long, between 0.1°K and 1.0°K.

At the lowest temperatures the sharp start of the pulse had a velocity equal to 230 m/sec. As the temperature was raised the velocity decreased and the start of the received pulse became unsharp. This occurred at a lower temperature in the 6 cm cavity than in the 3 cm cavity, and up to 0.8°K the measured velocity was smaller in the longer cavity than in the shorter. At 0.8°K the velocity found was in agreement with that measured by other workers. The results therefore appear to confirm the postulated mean-free-path effect.

9.5. The Effect of Pressure on the Second Sound Velocity

Measurements of the effect of pressure on the second sound velocity have been made by Peshkov and Zinoveva (P.48e) in the temperature range 1.3°K to T_λ using pressures from the saturation vapor pressure to the freezing pressure, and by Maurer and Herlin (M.51b, M.51c), and by Mayper and Herlin (M.53) who extended the temperature range to well below 1°K using pressures up to 25 atmos. Above 1°K the results, as taken from the publication of Herlin *et al.*, are shown by the family of isobars given in Fig. 9.5. It will be seen that the maximum in the u_2 *versus* T curve shifts to lower temperatures as the pressure is increased. As Peshkov and Zinoveva pointed out, a plot of the line of the maxima (on a p *versus* $T_{\max}$ plot) has $dT/dp<0$ and is parallel to the λ line.

At very low temperatures (about 0.4°K, not shown in Fig. 9.5) the second-sound velocity rises sharply in the same manner as reported by de Klerk *et al.* (K.53a); however, as is suggested by Fig. 9.5, the greater the pressure the lower is the temperature at which the increase occurs. After the sharp increase in velocity there is a leveling off, as is to be expected, but here the curves cross and the ultimate velocity as the absolute zero is approached is higher, the higher the pressure.

These results are of some theoretical significance. They support the assumption first made by Landau that only the phonons comprise the normal fluid at lowest temperatures. The sharp rise in the velocity

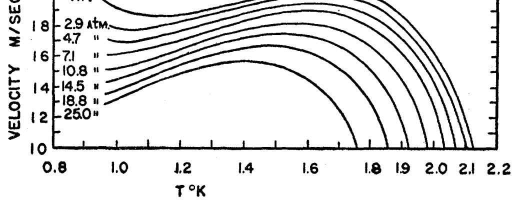

FIG. 9.5. The velocity of second sound as a function of temperature and pressure from Maurer and Herlin (M.51c). The experimental points, well spaced and about 250 in number, are not shown. The maximum deviation of points from the curve is ±1 percent. Pressures on the curves are given in atmospheres.

of second sound occurs at about the temperature at which the phonon entropy becomes an appreciable fraction of the total entropy, and the limiting velocity at 0°K is proportional to the velocity of first sound. Because the first sound velocity increases with increasing pressure, and because the phonon entropy varies inversely as the cube of the first sound velocity, increasing the pressure of the helium II should lower the temperature at which the velocity increase occurs and should also increase its limiting value. Both of these effects are observed. Like de Klerk, Hudson, and Pellam (K.53a), however, Mayper and Herlin observe that the limiting velocities obtained are greater than those to be expected from Landau's value: $u_1/\sqrt{3}$.

Kondoh, Nakajima, and Shimizu (K.51c) have compared the results of Maurer and Herlin in the range above 1°K with calculations based on Usui's equation for the second-sound velocity, Eq. (9.8). They obtain qualitative, not quantitative agreement.

9.6. The Rayleigh Disk and the Pitot Tube in Second-Sound Fields

Besides the measurements in which the second-sound velocity was of primary interest, there have been elegant experiments by Pellam and co-workers designed to test the two-fluid hypothesis. They have carried out an extensive series of experiments with a thermal pitot tube and with a thermal Rayleigh disk in second-sound fields.

In the experiments on the thermal pitot tube by Pellam (P.50b), standing waves of second sound were set up in liquid helium II in a cavity one-half wavelength long. The apparatus (see Fig. 9.6) was ingeniously constructed so that the pressure of the helium II could be measured at the midpoint of the cavity (the antinode of the wave) and at the end of the cavity (the node of the wave) by using the liquid helium itself as the manometric fluid. The fluid pressure was found to be higher at the node. Pellam explained this by an intuitive generalization of the Bernoulli equation in which a term accounting for the relative motion of the superfluid and normal fluid is included, namely:

$$\tfrac{1}{2}\rho v^2+\rho gh+p+\tfrac{1}{2}\rho(\rho_n/\rho_s)\left(\frac{\dot{H}}{\rho ST}\right)^2=\text{constant}, \qquad (9.15)$$

where $\dot{H}=\rho STv_n$ is the heat current. The inclusion of this fourth term in Eq. (9.15) shows that where the kinetic energy of relative motion of the two fluids is high, the pressure is low, and vice versa—a conclusion in agreement with the observations. Notice that in this experiment a pressure variation proportional to the square of the particle velocity was detected; as mentioned before, there is no pressure fluctuation proportional to the first power of the velocity. Only qualitative results of the experiments, however, have so far been reported.

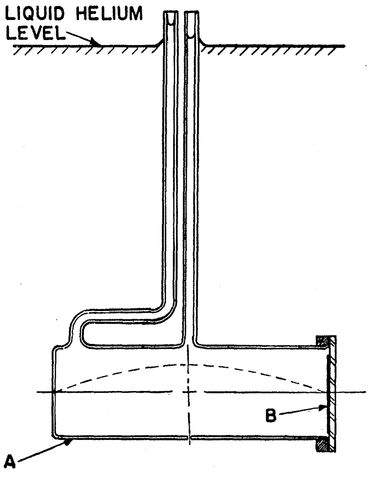

FIG. 9.6. Diagram of Pellam's (P.50b) thermal pitot tube. Dotted line within horizontal cavity A represents distribution of heat flow density (B is a plane electric heater surface) for condition of resonance. The resulting forced difference in the levels within the vertical tubes (as well as the net elevation due to capillarity) is exaggerated here.

The use of a thermal Rayleigh disk for the exploration of second-sound fields was first suggested by Pellam and Morse (P.50a); complete experimental results have recently been reported by Pellam and Hanson (P.52b). The fundamental idea in these experiments is that each fluid in liquid helium II will affect a Rayleigh disk as if the other were not present, because the disk responds to the square of the particle velocity rather than to the first power. It is well known that the maximum torque which can be exerted on a small disk suspended in a sound field is given by the expression[29]

$$\tau=(4/3)a^3\rho v^2, \qquad (9.16)$$

in terms of the radius of the disk, a; the density of the fluid ρ; and the fluid velocity v. We can write such an equation for both the superfluid and the normal fluid in helium II:

$$\begin{aligned}\tau_n&=(4/3)a^3\rho_n v_n^2,\\ \tau_s&=(4/3)a^3\rho_s v_s^2.\end{aligned} \qquad (9.17)$$

In a second-sound field the total torque is just the sum of the two terms (9.17). The two velocities are linked by the condition that the mass current density Eq. (8.4) is zero, and they can be replaced by the heat current $\dot{H}$ by the Eq. $\dot{H}=\rho STv_n$. Thus, as was shown in detail by Pellam (P.52b),

$$\langle\tau\rangle_{\text{Av}}=\langle\tau_n\rangle_{\text{Av}}+\langle\tau_s\rangle_{\text{Av}}=\frac{4}{3}a^3\rho\frac{\rho_n}{\rho_s}v_n^2=\frac{4}{3}a^3\rho\frac{\rho_n}{\rho_s}\left(\frac{\dot{H}}{\rho ST}\right)^2. \qquad (9.18)$$

The experimental arrangements used for the observation of the Rayleigh disk in liquid helium II by Pellam and co-workers is diagrammatically shown in Fig. 9.7. The disk D a $\frac{1}{2}$-inch galvanometer mirror, is suspended from a fine copper ribbon H in the center of a cylindrical horizontal cavity E. A light beam reflected from the disk to a conventional scale indicates small deflections of the mirror produced by the second-sound field.

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[29] See reference 24, p. 44.

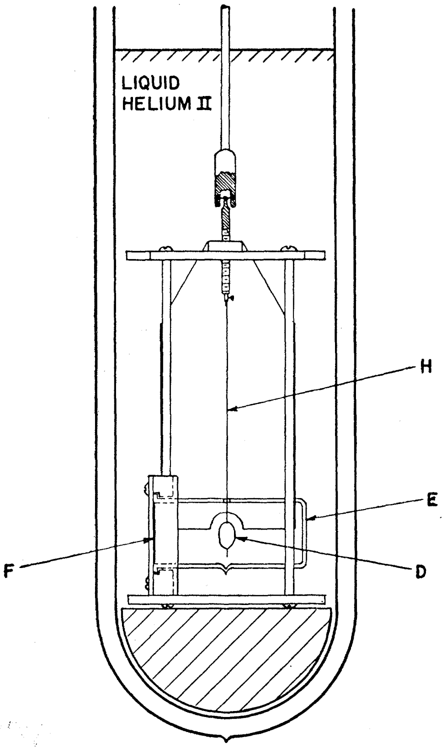

FIG. 9.7. Diagram of apparatus used by Pellam and co-workers (P.52b) for observations with a Rayleigh disk in liquid helium II. (See text for explanation of symbols.)

The second sound is produced by means of a flat electrical heater element *F* forming one vertical end of the cavity *E* which heater is energized sinusoidally. Standing waves of second sound are produced in the cavity by tuning the frequency of the heater current approximately to the fundamental resonance. At a given temperature, the torque on the disk is measured as a function of the frequency of the periodic heating, and from the shape of the resonance curve the effective heat current at the center of the cavity can be computed. From the frequency at maximum response, i.e., maximum torque, the second-sound velocity can be reckoned; it is found to give agreement with the other previously quoted results for this quantity. The torque itself is more interesting. It follows from Eq. (9.17) and the condition of no net mass flow that the fluid constituent which has the lesser density contributes the greater torque, and indeed as either density vanishes the torque becomes infinite. Thus near the λ point the torque is almost entirely due to the superfluid, while at and below about 1.2°K the torque is due almost entirely to the normal fluid. Theory is beautifully supported by experiment, each fluid *does* make its expected contribution to the torque as is shown in Fig. 9.8, taken from Pellam and Hanson's paper, thus giving elegant support to the two-fluid theory.

The theoretical curve in Fig. 9.8 follows from Eq. (9.18) if ρ_n and S are known. Andronikashvili's data for ρ_n (A.48a) and Kapitza's entropy data (K.41b) have been used. The agreement of experiment and theory provides a consistent check on all quantities. This has already been pointed out in Sec. (2.3). A little juggling of Eqs. (9.6) and (9.18) leads to an expression for the specific heat in terms of torque, heat current, and u_2. Thus the Rayleigh disk can be used as a calorimeter. Pellam and Hanson evaluate C_v in this way and obtain good agreement with other experimenters.

Pellam's approach to the equations of the thermal pitot tube and the thermal Rayleigh disk was intuitional rather than rigorous. Compare, on this point, Pellam's discussion of mechanical energy flow, radiation pressure, and transmission of heat current at a "classical" boundary for thermal pulses (P.49d). It is, therefore, very satisfactory that Usui (U.51b) has derived the equations appropriate to each experiment from the equations of motion of Nakajima, Tomita, and Usui. In particular, Usui has verified that the total torque acting on the Rayleigh disk is just the sum of the two Eqs. (9.17). As Usui pointed out, these experiments on the pitot tube and the Rayleigh disk mark the first appearance of the terms (v·grad) v which are present in the equations of motion.

9.7. Attenuation of the Second Sound

Equations expressing the effect on the propagation of second-sound waves of irreversible processes (due to viscosity, thermal conductivity, and a finite relaxation time governing the conversion of superfluid helium II into normal fluid at places where one of them is above its equilibrium concentration for the local temperature) have been discussed, in whole or in part, by Dingle (D.48, D.50c); Usui (U.51a); Kronig and Thellung (K.50a); Gorter, Kasteleijn, and

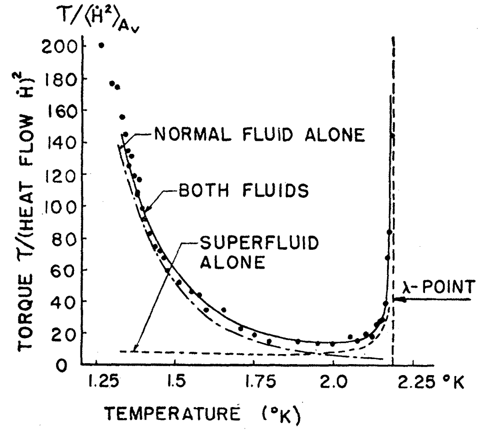

FIG. 9.8. Torque ratios $\tau/(\dot{H}^2)_{\text{Av}}$ as functions of temperature, T: experimental and theoretical values. The theoretical predictions indicated by curves: — - — - — represents $\tau_n/(\dot{H}^2)_{\text{Av}}$, the expected torque for the normal fluid; - - - - - - - - - represents $\tau_s/(\dot{H}^2)_{\text{Av}}$, the expected torque for the superfluid; and ———— represents $\tau/(\dot{H}^2)_{\text{Av}}$, that predicted for both. The circles are the observed values.

Mellink (G.50a); Kronig, Thellung, and Woldringh (K.52h); Khalatnikov (K.50b, K.52f, K.52g); and Band and Meyer (B.48a, B.48d, M.48d, B.49d).

Dingle derived expressions for the attenuation coefficient α of second-sound waves propagated in a large vessel due either to viscosity alone, or to thermal conduction alone. (α is the reciprocal of the distance in which the amplitude of the disturbance is reduced by the factor $1/e$.) These coefficients are:

$$\text{Viscosity alone:} \qquad \alpha_\eta = (\rho_s/\rho_n)(2\eta\omega^2/3\rho u_2^3), \tag{9.19}$$

$$\text{Thermal Cond. alone:} \ \alpha_K = K\omega^2/2\rho C_v u_2^3. \tag{9.20}$$

In these equations η is the viscosity coefficient for shearing, K is the coefficient of thermal conductivity, and C_v is the specific heat. Usui also discussed the effects of viscosity and arrived at an equation equivalent to (9.19). Kronig and Thellung carried out a similar analysis starting from more general equations which include coefficients of viscosity, thermal conductivity, and cubic expansion. They arrived at a cubic equation for the square of the wave vector, k ($k_i = \omega/u_i$). The three roots of this equation correspond to three types of wave propagation: first sound, second sound, and, newly named by the investigators, "third sound." Third sound is a skin or surface effect (see Sec. 5.7), since the wave vector k_3 is almost a purely imaginary quantity: the motion, caused by viscous drag, dies out in about 10^{-5} cm. (These waves are *not* the usual viscosity waves found in all viscous liquids. The usual viscosity waves, which are also found in helium II, are characterized by the condition that the *square* of the wave vector is a purely imaginary quantity.) Of more interest here is the fact that Kronig and Thellung's waves of second sound are attenuated with a coefficient which is just the sum of the two terms already given by Eqs. (9.19) and (9.20), if the Stokes relationship between the shear and the volume viscosity coefficients is assumed.

Gorter, Kasteleijn, and Mellink also considered the effect of relaxation on the second-sound velocity. They assumed that a characteristic parameter β related to a relaxation time, exists such that for nonequilibrium $(dT/dt) = \beta G_x$. Introducing this into the equations of motion yielded the result that u_2 is frequency dependent: at high frequencies ($\omega \gg \beta G_{xT}$) Eq. (9.7) holds; at low frequencies (the opposite inequality) Eq. (9.7) must be divided by the factor $(1 - G_{xx}G_{TT}/G^2{}_{xT})$. Kronig, Thellung, and Woldringh have discussed the effects of relaxation somewhat more thoroughly by expanding the equations of Kronig and Thellung (K.50a). They state that for second sound their results differ from those just quoted to an extent which lies beyond the precision of the present measurements. They do not calculate the dependence of the attenuation coefficient on the relaxation time.

Khalatnikov has discussed the relaxation process in terms of the emission and absorption of phonons and rotons which bring about equilibrium. This is an extension of the work of Landau and Khalatnikov (L.49a) on the viscosity mentioned above (Sec. 9.1). His results require the introduction of certain experimental data. With such data he finds that relaxation phenomena lead to an additional attenuation which (at 2°K for example) is given by $\alpha \approx 2 \times 10^{-14}\omega^2$. At the same temperature Eqs. (9.19) and (9.20) yield $\alpha_\eta \approx 7 \times 10^{-18}\omega^2$ and $\alpha_K \approx 4 \times 10^{-14}\omega^2$. Unfortunately, since the theories of Dingle and of Kronig and Thellung are macroscopic, while Khalatnikov's is microscopic, it is not clear whether all the attenuation coefficients are additive.

Band and Meyer also have considered the effects of relaxation. To do this they have introduced two relaxation times. The first relaxation time τ_1 characterizes momentum exchange between normal and superfluid and introduces a damping term into the wave equation. Thus, if z is proportional to the displacement of the normal constituent from the center of mass of a fluid element, they write:

$$(\partial^2 z/\partial t^2) + \frac{1}{\tau_1}(\partial z/\partial t) = v_s^2 \text{ grad div} z. \tag{9.21}$$

The second relaxation time τ_2 characterizes the rate of thermal adjustment between the constituents. τ_2 is apparently related to the characteristic time of Gorter and co-workers and Kronig and co-workers. τ_1 is introduced purely *ad hoc*. Band and Meyer's assumptions lead to the result that at very low frequencies the second-sound wave equation goes over into a thermal diffusion equation.

For the frequencies employed in second-sound measurements (*ca* 10^4 cps) the attenuations given by (9.19) and (9.20) are exceedingly small in the range of temperature from 1°K to 2°K. However, near the λ point (α_η) and (α_K) become very large, since here ρ_s and u_2 approach zero. At low temperature also, α_K becomes large as the specific heat vanishes.

The experimental data concerning attenuation are very slight and somewhat contradictory. Pellam (P.48d, P.49b) has found α to increase with temperature varying from 10^{-2} cm^{-1} at 1.65°K to 0.25 cm^{-1} at 2.1°K while above this last temperature it increases very rapidly in qualitative agreement with the predictions of the last paragraph. Osborne (O.51), however, has found much smaller attenuations which are practically temperature independent up to about 2.17°K. The difference in the magnitude of the effect found by the two experimenters is explained at least in part by differences in important frequencies: Pellam was using rectangular pulses of 150 μsec duration, Osborne 600 μsec pulses, which difference introduces (roughly) a factor of 16 in the expected attenuation coefficients.

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Appendix (*Added in proof*)

Since September 1953, when this paper was completed, many articles on liquid helium II have been published. The following list, which makes no claim to completeness, has been compiled to indicate the rate of publication on liquid helium (*excluding* articles on solid helium, He^3 or He^6) up to May, 1954.

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Worley, Zemansky, and Boorse, Phys. Rev. **93**, 45 (1954).

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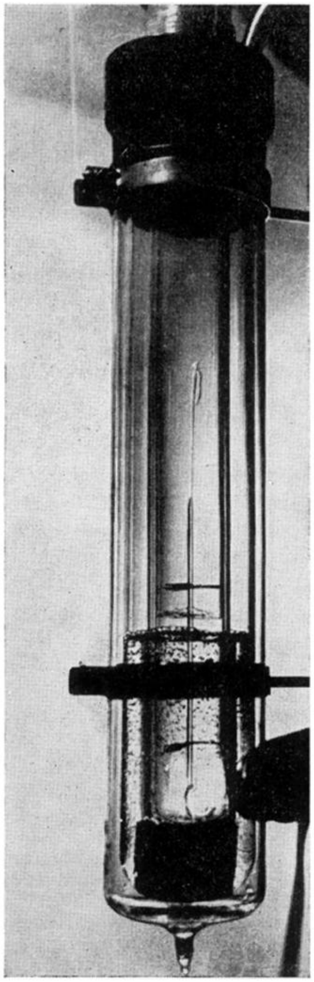

FIG. 4.3. Photo of helium II fountain. (Courtesy of the Research Laboratory of Electronics, Massachusetts Institute of Technology and A. D. Little, Inc.)