Theory of Angular Correlation of Nuclear Radiations*

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CONTENTS

I. DEFINITION OF THE ANGULAR CORRELATION PROCESS

IT is a well-known fact that the coincidence observation of two radiations emitted by a nucleus yields a correlation in their relative propagation directions. This fact, which was first pointed out by Dunworth (22),[1] has been utilized very extensively as a method for measuring various properties of nuclear energy levels, and rather elaborate development of the theory is now available thanks to the efforts of many investigators (30, 57, 65). In the following we shall attempt to synthesize these various theoretical developments with particular emphasis on the interpretation which must be given to the various possible formulations of the theory (Sec. II). At the same time it is our purpose to discuss in detail the various special cases of angular correlation with which the experimentalist is apt to be confronted (Sec. III, IV, and V), and to provide a reasonably complete set of numerical results so that the present article, it is hoped, can be used as a "handbook" of angular correlation, so far as the theory is concerned. We shall not attempt a discussion of specific experimental results inasmuch as such discussion as is pertinent at present will be given in a companion article by Frauenfelder (34, see also 21 and 35). However, as will be seen from the following, it is necessary to recognize the relation of the theory, as it appears here, to the circumstances under which the observations may be carried out.

The simplest type of angular correlation process consists of the measurement of the coincidence rate of two successive nuclear radiations as a function of the angle between the propagation directions of these radiations. We shall refer to such a process as a direction-direction (double) cascade. The principal information obtainable from such a measurement is the angular momenta of the nuclear levels involved and that of the emitted radiations. In addition, in all cases except emission of electromagnetic radiation a determination of relative parity can be made. The case of γ correlations in which at least one of the transitions is not a pure multipole also provides a more sensitive means of obtaining the mixing ratio (ratio of intensities of magnetic to electric multipoles) than total intensity (for example, conversion coefficient) measurements do.[2] Possible generalizations and alternative processes in which one is interested are: (1) the polarization-direction correlation in which, for one of the radiations, the polarization state is also measured (thereby fixing relative parity for pure multipole γ-radiation) and (2) the case of nonsuccessive radiations where more than two radiations are involved in the cascade. The latter case, in which one may also observe the propagation vectors of two radiations only, is properly classified as a triple correlation, Sec. V. This list of alternative correlations processes is not meant to be complete. In particular, in the above we have omitted reference to the correlation process in a magnetic field (internal or external) which merits

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[1] Numbers in parentheses refer to references which are arranged alphabetically in bibliography at end of paper. Equation numbers are prefixed by the customary abbreviation "Eq." wherever ambiguity may arise. The bibliography should represent a fairly complete guide to the literature as of September, 1952.

[2] An additional parameter depending on nuclear structure resulting from the correlation observation is the relative phase of electric and magnetic multipole matrix elements, see further Sec. II and references (3) and (55).

separate consideration, Sec. IV. In the following references to the angular correlation without qualifying phrases or adjectives will generally be concerned with the direction-direction double cascade in which the radiations are emitted consecutively.

From a descriptive point of view a simple physical explanation of the existence of the angular correlation can be given as follows: If we consider only one transition, either the first or second, then clearly the angular distribution of the emitted radiation is isotropic. Proofs of this result too numerous to mention have been given in the literature; it is moreover obvious (or expected, to say the least) from the fact that no direction in space has been singled out. Each substate (magnetic sublevel of an isolated nucleus) emits anisotropically but on summing over these substates with equal populations and random (relative) phases, so that the sum is incoherent, the total intensity is independent of angle. However, in the angular correlation process the angular distribution of one radiation is observed when it is known that the other has a *fixed* direction. The prescription of a fixed direction, of course, singles out a direction in space; stated otherwise, the observation of the aforementioned propagation vector acts as a kind of projection operation in giving different weights to the various substates. In other words it gives information about the substates of the nuclear level formed after the emission of this radiation. In particular it asserts that these sublevels are not uniformly populated.[3] Consequently, the radiation subsequently emitted will not be isotropic [except in very rare cases where isotropy results as an accident (9)].

This rather elementary point is belabored in order to emphasize the fact that the quantitative description of the angular correlation process depends critically on the fact that we are considering an *isolated* nucleus. It is also clear that isolated here means that there is no appreciable interaction depending on the orientation of the nuclear spin in the intermediate state of the cascade. Even more specifically, the assumption is that the mean life for emission of the second radiation is short compared to the precession period of the nuclear spin, in the intermediate state, under the influence of possible spin couplings.

We may now examine the circumstances under which this assumption may realistically be regarded as valid. In general, it is clear that the only interactions of importance are those for which the energy splittings in the intermediate state are of the order of or larger than the total width of this intermediate state.

We first consider direct interactions. Then there are two well-known types of spin couplings to be considered: (1) the coupling of the intermediate state magnetic moment to a magnetic field and (2) the coupling of the quadrupole moment in the intermediate state to an electric field gradient. Since the intermediate state angular momentum j must fulfill $j \geqslant 1$ in order to have an anisotropic correlation neither of these couplings can be ruled out *a priori*. It is evident that the presence of such spin-couplings will attenuate the correlation in the sense that it will be more isotropic than that of a bare nucleus. Experimentally it is observed that the correlation can be wiped out entirely, although a detailed description of the physical circumstances giving rise to this effect cannot be given at present.

We may now examine the importance of the spin-couplings listed below.

(1) Magnetic Coupling

Here we are primarily concerned with hyperfine fields. The multiplet splitting caused by hyperfine interaction ranges from 10^{-3} to about 1 cm^{-1} corresponding to precession periods τ_{hfs} in the range 10^{-7} to 10^{-10} or perhaps 10^{-11} sec.[4] Therefore, for some cases, high energy and small angular momentum of the second radiation, no coupling effect would be important. However, a majority of cases, for which angular correlation would be a useful tool, fall into the range of lifetime corresponding to important hfs coupling perturbative effects. An indication of how these affect the angular correlation is given in Sec. IV. Of course, no difficulty arises if for the ground state the electronic shell has zero angular momentum, $J_e=0$. However, even if such were the case the electron shell cannot be expected to remain in the ground state, or indeed, to remain in any stationary state. The nuclear recoil in a heavy particle reaction and/or excitation and ionization resulting from β-emission or K capture which will initiate the cascade in almost every case will also initiate a chain of electronic transitions. Thus, following K capture (or internal conversion, as well) the emission of Auger electrons and x-rays transfers a K hole to the outer orbits and, in the case of the former, results in the formation of a multiply-charged ion (20). While these electronic transitions generally proceed very rapidly [$10^{-15}-10^{-14}$ sec—see (62)], it is to be expected that in most cases a non-negligible magnetic interaction will occur in the equilibrium state. In those cases in which $J_e=0$ for the ground state, one may attempt to eliminate the magnetic coupling by providing the radioactive nucleus with an environment such that the transition to the electronic ground state is very rapid. Presumably this is what is accomplished by the surface film technique [embedding in thin metallic layers formed by evaporation, electrodeposition, etc. (1, 33, 73) and by the use of dilute ionic

[3] As an example, in the emission of spinless particles the magnetic quantum numbers (defined with respect to the propagation direction) of the two states connected by the transition must be the same. Thus, in the simple case of a $j=0 \rightarrow j=1$ transition, only one of the substates ($m=0$) is populated.

[4] This, of course, is based on the very reasonable assumption that nuclear magnetic moments in excited states are of the same order of magnitude as ground state moments. It also involves the equally reasonable assumption that the electronic states involved in the correlation have roughly the same properties as those involved in optical transitions.

solutions (48, 73)]. However, the physical processes involved are not very well understood at present, and there is no *a priori* method whereby one may be sure that all spin-coupling has been eliminated.

(2) Quadrupole Coupling

The quadrupole interactions may be as large as 100 megacycles, or $\tau_Q \sim 10^{-8}$ sec. While not as large as the magnetic interaction, the quadrupole interaction cannot be ignored for a very appreciable number of cascades (1a). Little needs to be added to the above except to say that the formation of an environment which has the property of eliminating magnetic interactions must also eliminate field gradients at the nucleus.

It would seem that an alternative procedure, which avoids the complications of imperfectly understood solid-state properties of thin films and the properties of dilute solutions could be utilized wherein the correlation is observed in a strong external magnetic field. Then, under conditions of complete Paschen-Back effect the spins are decoupled even if quadrupole coupling is present. As discussed in Sec. IV, one of the radiations must then be parallel, or nearly parallel, to the field. However, the very preliminary and admittedly meager experimental evidence [correlation in Ni^{60}, (36)] indicates that little or no decoupling takes place even with fields of the order 10^4 gauss. There seems to be no reason to doubt the validity of the reasoning with regard to spin decoupling by such a field for an isolated atom. In the actual source the presence of indirect couplings between neighboring ions may explain the failure of the strong field method. The presence of such couplings would, of course, imply an effect of chemical environment on the angular correlation. Even if this is the case a combination of a strong field and suitably prepared (magnetically dilute) sources may produce the desired spin decoupling. Whether one uses these sources with or without a magnetic field, an additional complication arises in observing charged particles due to multiple scattering which is necessarily present with the requisite thicknesses. However, corrections can be made if the scattering is not excessive (32). Even with thin sources the presence of a strong field could be tolerated only for correlations not involving light charged particles.

In any case the difficulties discussed above are such that little or no help can be expected in the way of quantitative calculations. Certainly, the technique of preparing suitable sources which is closely associated with the question of understanding the concomitant physical problems, is largely an experimental one. Nevertheless, the starting point for the analysis of any experimental results must be the theory for the isolated, uncoupled nucleus. This must be so whether the angular correlation is regarded as a method of nuclear spectroscopy or as a method of investigating nonstationary state processes in the solid (or liquid) state.[5] For this reason, and because the purely nuclear aspect of the theory is so well developed, we restrict our attention chiefly to the angular correlation without spin-coupling. The only noteworthy exception is the case in which hfs and other internal couplings can be eliminated. Then the application of a magnetic field, for which the coupling is completely defined, permits a determination of the intermediate state gyromagnetic ratio and magnetic moment by comparison with the zero-field correlation, (2, 5). This will be discussed in Sec. IV. It goes almost without saying that, for purposes of nuclear spectroscopy, every effort to eliminate the essentially unknown spin-coupling should be made. To the extent that this can be done the results of this paper would apply to the observed correlation, and the correlation measurement becomes an extremely valuable tool for the investigation of nuclear properties. It should be emphasized, however, that in many cases the spin-coupling effects are either too small to be observed or are small enough so as not to render ambiguous spin (and parity) assignments. Recent experimental results on Co^{60} shows that when care is taken in the preparation of sources and in making appropriate corrections for geometry and statistics the agreement with theory is excellent.

II. GENERAL FORMULATION

A. Introduction

It is our purpose in this section to set up the problem of the angular correlation of successive nuclear radiations in the most general fashion possible in order to provide a common framework for the discussion of the various specialized problems of practical interest. The detailed applications appear in succeeding sections. The common features of all correlation problems is to be found in the fact that they involve an initial nuclear state of sharp angular momentum j_1 and parity that undergoes successive transformations either emitting or absorbing radiations through intermediate nuclear states of sharp angular momenta $j_a, j_b \cdots$ with sharp parity and terminating as a nucleus with sharp angular momentum j_2 and parity. The properties of the radiations, the observation or lack thereof of their directions of motion, polarization properties, the presence or absence of perturbation of the intermediate nuclear states by external fields (magnetic (hfs) and possibly quadrupole interactions) condition the specific correlation discussed. Our general assumption of sharp angular momenta and parity for the nuclear states is restrictive, however, and distinguishes the correlation problem from, say, the closely related problem of the angular distribution of nuclear reactions (15, 16). One addi-

[5] It is perhaps worthy of note that this interweaving of nuclear and atomic as well as solid-state physics arises from the fact that we are interested in the *angular* properties of the nuclear transitions and because the nuclear lifetimes are long enough to compare with the characteristic transition times for the atomic and solid state processes.

tional restriction will be made in the general formulation: namely, that the initial and final nuclear states are randomly oriented. This is a mild delimitation of the problem, excluding only such specialized processes as the radiations from oriented nuclei in certain crystalline lattices at very low temperatures (72). It turns out that in several instances such problems can nonetheless be treated as special cases of our formulation. In any case such processes are not our immediate concern.

The initial and final transitions thus occupy a preferred position in this formulation, and focusing our attention on the two transitions leading from and to these states we can write the general correlation function in the form

$$W(\mathbf{A}_1, \mathbf{A}_2; \mathbf{A}', \mathbf{A}'', \cdots) \sim \mathfrak{S}\sum \langle j_1m_1|H_1(\mathbf{A}_1)|j_am_a\rangle^* \times\langle j_1m_1|H_1(\mathbf{A}_1)|j_am_a'\rangle S(m_am_a'; m_nm_n') \times\langle j_nm_n|H_2(\mathbf{A}_2)|j_2m_2\rangle^*\langle j_nm_n'|H_2(\mathbf{A}_2)|j_2m_2\rangle. \quad (1)$$

Here the summation $\sum$ is over m_1, m_a, m_a', m_n, m_n', m_2, and the symbol $\mathfrak{S}$ refers to the averaging over all of the unobserved properties of the radiations. As indicated irrelevant constant factors have been omitted since we are interested only in the dependence of the correlation function on observed parameters $\mathbf{A}_1$ and $\mathbf{A}_2$ such as propagation directions, polarization. We omit scale factors in the sequel usually without comment. Further, in Eq. (1) $H_1(\mathbf{A}_1)$ is the interaction Hamiltonian for the emission of the initial radiation described by the set of vectors $\mathbf{A}_1$ and H_2 describes the emission of the second radiation in a similar way; $S(m_am_a'; m_nm_n')$ is a matrix that describes all the other properties of the correlation (the intermediate radiations and extraneous perturbations, see Secs. IV and V). The vectors $\mathbf{A}'$, $\mathbf{A}''\cdots$ describe intermediate observed radiations and are contained in the S coefficient. For the emission of two successive radiations this matrix represents the spin coupling of the nucleus in the intermediate state with extranuclear fields. In the following sections the form of this matrix will be derived in detail as it is needed for the cases of interest. For the simple case of two successive radiations (double cascade), i.e., where only the initial and final radiations occur and the intermediate state of spin $j_a=j_n=j$ is unperturbed, $S(m_am_a'; m_nm_n')=\delta(m_am_n)\delta(m_a'm_n')$. In such cases the superfluous subscripts a and n will be omitted. For the same case but with the intermediate state perturbed by hfs, or for the case where but a single unobserved radiation connects intermediate states j_a and j_n and the states j_a and j_n are unperturbed [see reference (42)] $S=\delta(m_a-m_a', m_n-m_n')f(m_a, m_a', m_n)$. This particular form may be deduced in general for those cases that do not introduce any new angular information. The case of an external magnetic field with incomplete Paschen-Back effect constitutes an example to the contrary.

We may rearrange Eq. (1) so as to bring it to the more useful form:

$$W(\mathbf{A}_1\mathbf{A}_2\mathbf{A}'\mathbf{A}''\cdots) \sim\sum E^{(1)}(m_am_a')S(m_am_a'; m_nm_n')E^{(2)}(m_n'm_n). \quad (2)$$

The summation is to be carried out over m_a, m_a', m_n and m_n', and we have defined:‡

$$E^{(1)}(m_am_a')\equiv\mathfrak{S}_1\sum_{m_1}\langle j_1m_1|H_1|j_am_a\rangle^* \times\langle j_1m_1|H_1|j_am_a'\rangle \quad (3)$$

with $\mathfrak{S}_1$ designating an average over the unobserved properties of radiation 1; E^2 is defined in a similar manner. These E matrices are Hermitian so that $E(mm')=E^*(m'm)$. The interpretation of the E matrices is given below in Sec. II-B.

The rearrangement leading to Eq. (2) is of more than calculational interest, since it allows one to break the angular correlation problem into much simpler parts: the initial and final transitions (which may now be considered independently and, moreover, involve the same kind of treatment), and the link between these radiations which depends on spin-coupling in the intermediate state. The E matrices can be greatly simplified by the methods of Racah (63, 64, 65), as originally shown by Gardner (38). To do this we first expand the interaction Hamiltonian H_1 for the emission of a particle along the quantization (z) axis in terms of tensor operators with definite angular momentum, parity, and time-reversal properties:

$$H_1(\mathbf{A}_1)=\sum_{L,M,\pi}\alpha(LM,\pi;\mathbf{A}_1)T(LM,\pi;\mathbf{X}_1). \quad (4)$$

Here the $\alpha(LM, \pi; \mathbf{A})$ are variables that characterize the particle emitted, and the $T(LM, \pi)$ are irreducible nuclear operators of degree L (see Eq. (5) below) with parity $\pi=\pm1$. The arguments $\mathbf{X}_1$ refer to nuclear configuration and spin coordinates. Note that L can be a half-integer as well as an integer. For a transition with a definite angular momentum and parity change, i.e., a pure multipole transition, only a single one of the $T(LM, \pi)$ enters. In the following sections, the $\alpha(LM, \pi)$ or their equivalent are given explicitly for the cases of interest. For the present purpose it suffices to say that the $\alpha(LM, \pi)$ are characteristic parameters of the particles; for example, for alpha particles the $\alpha(LM, \pi)$ vanish unless $\pi=(-)^L$ and $M=0$ since the projection

‡ *Note added in proof.*—We recognize that, strictly speaking, the H operators are not Hermitian but are the emission (or absorption) parts of Hermitian operators. Actually in Eq. (2) $E^{(2)}(m_n'm_n)$ should be replaced by a density matrix constructed in the same way except that H_2 is replaced by H_2^+. The distinction between these two density matrices is a phase $(-)^\nu$, [see Eq. (38), for example], which would appear in every correlation function. However, except in the impractical case in which the circular polarization of *two* γ-rays is measured, ν will be an even integer, and the distinction is academic.

of orbital angular momentum along the axis of motion (z axis in the present notation) must be zero and no spin angular momentum is involved. In this case the parity is related to the angular momentum L by $\pi=(-)^L$. For γ-rays $M=\pm 1$ only, because of the transverse character of the electromagnetic field, and now for given L (multipolarity) the π index essentially characterizes the transitions as electric or magnetic. Other cases in which we shall be interested, especially Dirac particles, are somewhat more complicated and will be examined in connection with the special cases treated in the sequel.

If we are interested in the radiation emitted in an arbitrary direction, we make use of the rotational properties of the T operators and transform them to a new set of axes. The operators $T(LM, \pi)$ transform as $(2L+1)$ dimensional irreducible representations of the 3-dimensional rotation group, that is

$$T(LM, \pi; \mathbf{X}_1)=\sum_{\mu} T(L\mu, \pi; \mathbf{X}_1')D(L, \mu M; \mathfrak{R}), \quad (5)$$

where the $\mathbf{X}_1$ and $\mathbf{X}_1'$ refer to the original and rotated coordinate system, respectively. The argument $\mathfrak{R}$ of the rotation matrix D is a rotation, with Euler angles $\alpha\beta\gamma$, such that the coordinate system describing the radiation (propagation vector and polarization direction) is carried over into the quantization coordinate system. The three Euler angles are necessary, so far as problems considered below are concerned, only in the case of emission of linearly polarized light. If the observation of the radiation does not include a measurement of linear polarization, the angle γ is ignorable and can be set equal to zero. In this case the azimuth angle α is meaningful only in the E matrix but, as will be seen and as is to be expected, for the correlation only a single angle β will be pertinent. However, until the two links in the correlation function are formed the rotation $\mathfrak{R}$ will define a direction (α, β) which will coincide with the propagation direction. With $\mathfrak{k}$ denoting the *unit* vector in this direction we shall write $\mathfrak{k}$ as argument of the rotation matrix in this case. (We follow the practice of using boldface German letters for unit vectors.)

Introducing Eq. (4) and Eq. (5) into Eq. (3) and dropping the subscript a we find

$$\begin{aligned}E^{(1)}(mm')=\mathfrak{S}_1\sum\alpha^*(LM, \pi)\alpha(L'M', \pi')\\ \times D^*(L, \mu M)D(L', \mu'M')\langle j_1m_1|T(L\mu, \pi)|jm\rangle^*\\ \times\langle j_1m_1|T(L'\mu', \pi')|jm'\rangle, \quad (6)\end{aligned}$$

and the summation is now over m_1, L, M, μ, π and L', M', μ', π'. Now the Wigner-Eckart theorem (77) gives

$$\begin{aligned}\langle j_1m_1|T(L\mu, \pi)|jm\rangle\\ =C(j_1Lj; m_1m-m_1)(j_1\|T(L\pi)\|j)\delta(\pi, \pi_1\pi_a), \quad (7)\end{aligned}$$

where π_1 and π_a are the parities of the nuclear states and $C(j_1Lj; m_1m-m_1)$ is the vector addition coefficient or Wigner coefficient[6] corresponding to the addition of j_1 and L with components m_1 and $m-m_1$ to give a resultant j with component m (19), while $(j_1\|T\|j)$ is a reduced matrix element. We use now the relation (77)

$$D^*(L, \mu M)=(-)^{M-\mu}D(L, -\mu, -M) \quad (8)$$

and the Clebsch-Gordan series (77)

$$\begin{aligned}D(L, -\mu, -M)D(L', \mu'M')\\ =\sum_{\nu} C(LL'\nu; -\mu\mu')C(LL'\nu; -MM')\\ \times D(\nu, \mu'-\mu, M'-M). \quad (9)\end{aligned}$$

Using Eqs. (7), (8), and (9) in Eq. (6) yields

$$\begin{aligned}E^{(1)}(mm')=\mathfrak{S}_1\sum(-)^{M-m+m_1}(j_1\|T(L\pi)\|j)^*\\ \times(j_1\|T(L'\pi)\|j)\alpha^*(LM, \pi)\\ \times\alpha(L'M'\pi)C(j_1Lj; m_1m-m_1)\\ \times C(j_1L'j; m_1\, m'-m_1)C(LL'\nu; -MM')\\ \times C(LL'\nu; m_1-m, m'-m_1)\\ \times D(\nu, m'-m, M'-M; \mathfrak{R}). \quad (10)\end{aligned}$$

The summation here is over m_1, L, L', ν, M and M'. We can perform the m_1 sum using Racah's techniques

$$\begin{aligned}\sum_{m_1} (-1)^{-m+M+m_1}C(j_1Lj; m_1m-m_1)\\ \times C(j_1L'j; m_1m'-m_1)C(LL'\nu; m_1-m, m'-m_1)\\ =(2j+1)(-)^{j_1-m+M}C(jj\nu; m, -m')\\ \times W(jjLL'; \nu j_1). \quad (11)\end{aligned}$$

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[6] Equation (7) is in the form given by Wigner (77). At first glance it appears that this result implies the convention

$$(a|H|b)\equiv\int\psi_b{}^*H\psi_a d\tau.$$

Although this is a perfectly legitimate convention, we prefer to use the more commonly used notation in which the indices a and b are interchanged in the left- (or right-) hand side of the above. It would then appear that the proper manner in which the Wigner-Eckart theorem should be written is

$$\begin{aligned}\langle j_1m_1|T(L\mu, \pi)|jm\rangle\\ =B(jj_1)(j_1\|T(L\pi)\|j)C(jLj_1; mm_1-m)\delta(\pi, \pi_1\pi_a), \quad (7a)\end{aligned}$$

where $B(jj_1)$ is a constant depending only on the indicated arguments and its specification merely fixes the normalization of the reduced matrix element $(j_1\|T(L\pi)\|j)$. For example, Racah [(63), his Eq. (28)] writes the Wigner-Eckart theorem in this form with $B(jj_1)=(2j_1+1)^{-\frac{1}{2}}$. Actually Eqs. (7) and (7a) give identical results with the understanding that different definitions of the reduced matrix elements are involved in each case. To see this in a simple way we consider that the tensor operator $T(LM, \pi)$ in Eq. (7) is actually the adjoint of $T(LM, \pi)$ in Eq. (7a) where the phase choice is $T^+(LM, \pi)=(-)^{L-M}T(L-M, \pi)$, say, (see Sec. II-C below). That is, we replace emission by absorption which doesn't have any effect on the results. Then, if we equate the corresponding sides of Eqs. (7) and (7a), we find that the reduced matrix elements differ by a phase $(-)^L$, aside from j and j_1 dependent factors which are nonessential and which can moreover be removed by suitable normalization. Thus, our result given in Eq. (38) below can be transcribed to Racah's form by insertion of a phase factor $(-)^{L_1+L_1'+L_2+L_2'}$ which is equivalent to a redefinition of the reduced matrix elements. Here and throughout we suppress the third magnetic quantum number in the Clebsch-Gordan (vector addition) coefficients since it is always the sum of the first two.

Following Racah we also define the set of parameters

$$c_{\nu\tau}(LL')\equiv\mathfrak{S}_1\sum_M \alpha^*(LM,\pi)\alpha(L'\tau+M,\pi) \times(-)^{L-M}C(LL'\nu;-M,\tau+M). \quad (12)$$

Then Eq. (10) reduces to

$$E^{(1)}(mm')=\sum(-)^{j_1-m+L}(2j+1)(j_1\|T(L\pi)\|j)^* \times(j_1\|T(L'\pi)\|j)c_{\nu\tau}(LL')C(jj\nu;m,-m') \times W(jjLL';\nu j_1)D(\nu,m'-m,\tau;\mathfrak{R}), \quad (13)$$

and the sum is over L, L', ν and τ.

The various terms in Eq. (13) have a clear-cut meaning. The factors $(j_1\|T(L\pi)\|j)$ are physical parameters (that is, reduced matrix elements) characteristic of the nuclear transition involved and, in particular, of the state emitting or absorbing radiation, and are thus independent of orientation and of the directions of motion. These are physical parameters in the sense that they are dependent on a physical model. The factors $c_{\nu\tau}(LL')$ are characteristic of the particles emitted, and upon the multipolarity and parity of the transitions considered through the parameters $\alpha(LM,\pi)$, but are, in addition, dependent on the information specified as defined by the physical experiment under discussion. This is clear by virtue of the averaging process indicated in Eq. (12). The remaining factors are to be interpreted as intrinsically geometrical in nature in contrast to the foregoing factors which are determined by the physics of the process.

Some general properties of the $c_{\nu\tau}$ can be obtained rather readily. If only the direction of motion of the particle is observed, then $c_{\nu 0}$ alone differs from zero. This can be seen simply from the fact that the angle γ refers to a rotation about the direction of motion, and the physical experiment by assumption is independent of γ. Thus $\tau=0$. Similarly if the experiment measures circular polarization, the experimental results are again independent of γ and $\tau=0$. Thus, for this case only $c_{\nu 0}\neq 0$. These conclusions can be verified in detail for each specific case from the definitions given above.

B. Statistical Interpretation of Fano (30)

A number of fruitful interpretations of the formalism of angular correlation are possible and have been given in the literature. At this point we consider an interesting interpretation of the meaning of Eq. (13) in terms of the density matrix as given by Fano. The density or statistical matrix was introduced into quantum mechanics to provide for the discussion of states about which less than maximal information is available. This is exactly the situation we have used earlier where we assumed the initial nuclear state was randomly oriented (an impure state) and only limited information provided on the state of the radiation (only its direction of motion specified, as is the usual case). The treatment of the problem was first to describe it as if we had maximal information, that is, the procedure is to introduce pure states and then average our final result to accord with our limited knowledge. The use of the Racah functions appeared as an algebraic convenience to handle the sums. As pointed out by Blatt (16) and especially by Fano (30), a direct procedure should be possible that never introduces the extraneous details of this method. In carrying out this program by means of the density matrix the geometrical significances of the Racah functions will become more apparent.

Consider a pure state ψ defined by an expansion in terms of an complete orthonormal set of wave functions $u_j{}^m$, i.e.,

$$\psi=\sum_m a_m u_j{}^m.$$

One constructs the matrix

$$P_{mm'}=a_{m'}{}^*a_m.$$

Then all properties of the state ψ, that is to say, the mean value of any arbitrary operator F can be found from the trace rule

$$\langle F\rangle=\sum_{mm'} a_{m'}{}^*a_m F_{m'm}=tr(FP).$$

The state ψ is therefore completely described by the density matrix P. The density matrix, however, can also describe impure states, which can be looked upon as the weighted sum of the density matrices P_n of pure states; thus $P=\sum_n p_n P_n$ where p_n are the weight factors.

The matrix $E(mm')$ introduced above is seen to be the density matrix of the intermediate state j. The information on the intermediate state j, as represented by the matrix $E(mm')$, has its origin in the coupling of the density matrices for the initial state, about which we know only its spin j_1, and the radiation, about which we also have limited information. Equations (4) and (5) can be considered as the expansion of the radiation into pure states, $T(L\mu,\pi)$, with coefficients $\sum_M D(L,\mu M)\alpha(LM,\pi;\mathbf{A}_1)$, while the initial state is expanded on the set $\psi_{j_1}{}^{m_1}$. Equation (6) is then the expression of the density matrix for the state j in terms of the density matrices of the initial and radiation states.

In more detail one has for the initial state

$$\psi_{\text{in}}=\sum_{m_1} A(m_1)\psi_{j_1}{}^{m_1},$$

for the radiation

$$\psi_{\text{rad}}=\sum_{LM} B(LM)\psi_L{}^M, \quad (14)$$

and for the resultant state

$$\psi_{\text{res}}=\sum_m C(jm)\psi_j{}^m,$$

where

$$C(jm)=\sum_{m_1LM} A(m_1)B(LM)\langle jm|LM|j_1m_1\rangle.$$

Forming the density matrices for the resultant state, in which the transition ends, one obtains

$$P_{mm'}(\text{res})=\sum P_{m_1m_1'}(\text{in})P_{LML'M'}(\text{rad})\times\langle jm|LM|j_1m_1\rangle\langle jm'|L'M'|j_1m_1\rangle^*. \quad (15)$$

The summation is over the indices L, L', M, M', m_1 and m_1'. Equation (15) is the desired relation for the density matrices, but it is in a very unwieldy form. Fano introduced at this stage the concept of "statistical tensors" $R(kq)$, defined by the relation

$$R(kq,jj')=\sum_{mm'}(-)^{j-m}C(jj'k;m,-m')\times P(jj'mm')\delta(m',m-q), \quad (16)$$

where $P(jj'mm')$ is the density matrix for states not sharp in j.

This step is motivated by the observation that the density matrix is dependent upon the choice of the quantization axes, and under rotation of coordinates transforms as the Kronecker product of the two representations $\mathbf{j}'$ and $\mathbf{j}$. This product, $(\mathbf{j}')^*\times\mathbf{j}$ is, of course, completely reducible into a sum of terms with rank k, $(|j'-j|\leqslant k\leqslant j'+j)$, and, in the usual fashion, the coefficient of each representation is the vector addition coefficient, which allows us to separate each term by means of Eq. (16).

For a completely random state but with sharp j we know that the density matrix $P(jj'mm')=\delta(jj')\delta(mm')$ and using Eq. (16) we find that only the scalar $R(00,jj)$ can be defined; i.e., a random state looks spherically symmetric, as is to be expected. In general a state of spin j can define tensors of rank less than or equal to $2j$, as Eq. (16) shows. Using the unitary property of the vector addition coefficients, we can invert Eq. (16) to give the density matrix in terms of the tensor parameters. Introducing the tensor parameters in Eq. (15) gives the result

$$P_{mm'}(\text{res})=\sum_{kq}(-1)^{j-m}C(jjk;m,-m')\times R(kq)\delta(q,m-m') \quad (17\text{a})$$

with

$$R(kq)=\sum(j_1\|L\|j)^*(j_1\|L'\|j)R(k_1q_1)\times R(k_rq_r)C(k_rk_1k;q_r,-q_1)\delta(q_1,q_r+q)\times\{\sum_\alpha(2\alpha+1)(2j+1)[(2k_1+1)(2k_r+1)]^{\frac{1}{2}}\times(-1)^{L'+q_r+k_1+\alpha}W(j_1\alpha j_1L;jk_1)\times W(j\alpha jL';j_1k)W(k_1\alpha k_rL';Lk)\}. \quad (17\text{b})$$

The first summation is over L, L', k_1, q_1, k_r, q_r. The bracketed term in Eq. (17b) is essentially a recoupling coefficient that allows one to relate the tensor parameters of the initial and radiation states to the tensor parameters of the resultant state. The tensor parameters of the initial state represent the coupling $\mathbf{j}_1+\mathbf{j}_1=\mathbf{k}_1$, while the radiation tensor parameters represent the coupling $\mathbf{L}+\mathbf{L}'=\mathbf{k}_r$.[7] The tensor parameters of the resultant state, however, are expressed in the system $\mathbf{j}+\mathbf{j}=\mathbf{k}$ with $\mathbf{j}=\mathbf{L}+\mathbf{j}_1=\mathbf{L}'+\mathbf{j}_1$. The bracketed coefficient in Eq. (17b) is just the change of coupling coefficient to go from the coupling of resultant tensor parameters,

$$(\mathbf{j}_1+\mathbf{L})+(\mathbf{j}_1+\mathbf{L}')=\mathbf{k},$$

to the coupling

$$(\mathbf{j}_1+\mathbf{j}_1=\mathbf{k}_1)+(\mathbf{L}+\mathbf{L}'=\mathbf{k}_r)=\mathbf{k}. \quad (18)$$

For the present problem where the initial state is random, only the scalar tensor parameter is defined, i.e., $k_1=q_1=0$. Then

$$R(kq)=\sum_{LL'}(j_1\|L\|j)^*(j_1\|L'\|j)R(k,-q;LL';\text{rad})\times(2j+1)(-1)^{L'+q+j-j_1}W(jjLL';kj_1). \quad (19)$$

Using Eq. (19) in Eq. (17a) we find

$$P_{mm'}(\text{res})=\sum_{kq}\sum_{LL'}(-)^{m+L'-j_1+q}(2j+1)\times C(jjk;m,-m')\delta(q,m'-m)\times R(kq;LL';\text{rad})(j_1\|L\|j)^*\times(j_1\|L'\|j)W(jjLL';kj_1). \quad (20)$$

Comparing Eqs. (20) and (13) gives the desired interpretation of Fano. Namely, (a) the $(-1)^{j-m}C(jj\nu;m,-m')$ is the coefficient that relates the tensor parameters to the density matrix, (b) $W(jjLL';\nu j_1)$ is the coefficient that changes the coupling schemes, (c) the $(j_1\|L\|j)$ are physical coefficients that weight the various multipoles (d) finally, $(-)^{L+L'+q}\sum_\tau c_{\nu\tau}(LL')\times D(\nu,q\tau;\mathcal{R})$, which appear in Eq. (13), [see also Eqs. (42a) and (71) below] are the tensor parameters of the radiation [$R(kq)$ in Eq. (20)] which express directly the information available on the radiation by virtue of the physical measurements performed. For example, if the radiation were unobserved only a scalar parameter can be defined, and Eq. (20) shows that the intermediate nuclear state is also random.

C. The Tensor Parameters of the Radiation

This formulation shows that the key point of the correlation problem is the determination of the tensor parameters of the radiations. Let us limit ourselves for the moment to a single emergent nonrelativistic[8] particle. Then the tensor parameters of this radiation are obtained from the measurements:

(1) the type of particle (this includes its spin magnitude),

(2) the detection of its direction of motion,

(3) a spin polarization measurement (carried out, we shall assume, with respect to a set of coordinates defined by the direction of motion).

[7] The order in which the vectors $\mathbf{L}$ and $\mathbf{L}'$ are added is important for the phases, (19).

[8] By this we mean that the spin is treated in Pauli approximation.

The measurement of a direction of motion is, of course, the basic correlation measurement. Interpreted quantum-mechanically it means that the particle is in a plane wave state, $\exp(i\mathbf{K}_\sigma\cdot\mathbf{r}_\sigma)$ (leaving aside the irrelevant center of mass motion). The subscript σ here is a particle label. The spin measurements determine the spin wave function to be $\chi_s{}^m$, assuming for the moment that the observation corresponds to pure m states, rather than to a more general linear combination of m states. This information must now be used to deduce the density matrix for the radiation in terms of eigenstates of the total angular momentum.

We employ the Rayleigh plane wave expansion

$$e^{iKz}=2\pi^{\frac{1}{2}}\sum_l j_l(Kr)(2l+1)^{\frac{1}{2}}(i^lY_l{}^0).$$

The $j_l(Kr)$ are, of course, the spherical Bessel functions. We shall use the set of functions $(i^lY_l{}^0)$ as our angle functions since they have the time-reversal property we desire, namely,

$$K_0(i^lY_l{}^m)=(-)^{l-m}Y_l{}^{-m},$$

where K_0 is here the complex conjugation operation. Similarly we choose a representation such that

$$K_t\chi_s{}^m=(-)^{s-m}\chi_s{}^{-m}.$$

For example, the time reversal operator K_t for a spin $\frac{1}{2}$ particle is $K_t=i\sigma_yK_0$. In the coordinate system determined by the direction of motion we have therefore

$$\begin{aligned}e^{iKz}\chi_s{}^M&=\sum_l j_l(Kr)[4\pi(2l+1)]^{\frac{1}{2}}(i^lY_l{}^0)\chi_s{}^M\\&=\sum_{J,l}[4\pi(2l+1)]^{\frac{1}{2}}j_l(Kr)\\&\qquad\times C(lsJ;0M)\Phi_{J,l}{}^M,\end{aligned}\quad(21)$$

where

$$\Phi_{J,l}{}^M\equiv\sum_\mu C(lsJ;\mu,M-\mu)(i^lY_l{}^\mu)\chi_s{}^{M-\mu}\quad(21\text{a})$$

and

$$K_t\Phi_{J,l}{}^M=(-)^{J-M}\Phi_{J,l}{}^{-M}.\quad(21\text{b})$$

The wave functions $\Phi_{J,l}{}^m$ combine with the initial state wave functions, $\psi_{j_1}{}^{m_1}$, to form the compound state $\psi_j{}^m$. Since the Hamiltonian that effects this combination is invariant under rotations and commutes with the time reversal operator K_t, we find that the reduced matrix elements, the "physical parameters," appearing in Eq. (20) for example, are purely real. To see this we note that

$$(j_1m_1;JlM|H|jm)=(j_1Jl\|H\|j)C(j_1Jj;m_1M)$$

and

$$\begin{aligned}(j_1Jl\|H\|j)^*C(j_1Jj;m_1M)\\=(K_tj_1m_1;K_tJlM|H|K_tjm)\\=(-1)^{j_1-m_1+J-M-j+m}(j_1Jl\|H\|j)C(j_1Jj;-m_1,-M)\end{aligned}$$

or

$$(j_1Jl\|H\|j)^*=(j_1Jl\|H\|j).\quad(22)$$

This result is the general form of a theorem enunciated for special cases by Lloyd (55), by Longmire and Messiah (59), and by Fuchs (37). These authors have stated the theorem, for electromagnetic radiation and β particles, in the form of the reality of the product of interfering matrix elements. This, of course, follows trivially from Eq. (22). The foregoing is not intended as a rigorous demonstration of the phase theorem. However, a rigorous proof of this theorem, which moreover demonstrates the fact that it is generally valid and does not depend on the use of perturbation theory, is given below in Sec. II-D (see Eq. (30) and following text).

For an arbitrary direction of motion, we use the property of the $\Phi_{J,l}{}^M$ that this set of wave functions transforms with a $2J+1$ dimensional representation of the rotation group. Hence

$$(e^{iKz}\chi_s{}^m)'=2\pi^{\frac{1}{2}}\sum_{J,M}D(J,Mm;\mathfrak{k})\sum_l(2l+1)^{\frac{1}{2}}j_l(Kr)\times C(lsJ;0m)\Phi_{J,l}{}^M,$$

where the prime refers to the rotated coordinate system.

The parity of the states $\Phi_{J,l}{}^M$ is $\pi_\sigma(-)^l$ where π_σ is the intrinsic parity of the particle σ. Hence we note that only states with l's differing by an *even* integer can mix. The density matrix for the radiation is

$$\begin{aligned}E(JMl;J'M'l')=4\pi[(2l+1)(2l'+1)]^{\frac{1}{2}}\\\times C(l'sJ';0m)C(lsJ;0m)\\\times D^*(J',M'm)D(J,Mm).\end{aligned}\quad(23)$$

If, instead of $\chi_s{}^m$ being measured, the measurement fixes the spin function to be the linear combination $\chi_s=\sum_m a_m\chi_s{}^m$ then the density matrix is

$$\begin{aligned}E(J'M'l';JMl)=4\pi[(2l+1)(2l'+1)]^{\frac{1}{2}}\sum_{mm'}a_{m'}{}^*a_m\\\times C(l'sJ';0m')C(lsJ;0m)D^*(J',M'm')D(J,Mm).\end{aligned}$$

Introducing the concept of tensor parameters leads to the results

$$\begin{aligned}R(k_\sigma q_\sigma;\text{spin})=\sum_{mm'}(-)^{s-m}a_{m'}{}^*a_m\\\times C(ssk_\sigma;m,-m')\delta(q_\sigma,m-m'),\end{aligned}\quad(24\text{a})$$

and

$$\begin{aligned}R(kq,JJ'll';\text{rad})=\sum_{k_\sigma q_\sigma\lambda}(-)^{J-J'+l+q-q_\sigma-\lambda+k}\\\times[(2k+1)(2k_\sigma+1)(2J+1)(2J'+1)\\\times(2l+1)(2l'+1)]^{\frac{1}{2}}R(k_\sigma q_\sigma)C(l'k_\sigma\lambda;0q_\sigma)\\\times C(lk\lambda;0q_\sigma)D(k,qq_\sigma)W(l'J'k_\sigma s;s\lambda)\\\times W(lskJ';J\lambda).\end{aligned}\quad(24\text{b})$$

If no measurement is made on the spin polarization then $k_\sigma=q_\sigma=0$. The result for the tensor parameters of the radiation is then

$$\begin{aligned}R(kq,JJ'll')=(-)^{J-s+q}\\\times[(2J+1)(2J'+1)(2l+1)(2l'+1)]^{\frac{1}{2}}\\\times C(ll'k;00)W(lJl'J';sk')D(k,q0;\mathfrak{k}).\end{aligned}\quad(25)$$

This example shows the general feature of all correlation problems: that the tensor parameters of the radiation are basically no more and no less than the *information contained in the transformation from linear momentum to angular momentum representations*, suitably coupled with the information obtained from a spin polarization measurement, if such is present. The situation for relativistic particles, and photons, needs to be treated more carefully, but the essentials are completely analogous to the nonrelativistic case above. We give the results in detail in Sec. III.

D. Relation to Nuclear Reactions

It is instructive to consider now the relstion of the angular correlation problem to the very closely related problem of the angular distribution of nuclear reactions, and scattering (78). In so doing one can eliminate any reference to perturbation theory, or to the specific interactions responsible for the transitions, and concentrate solely on the properties of angular momenta which account for the correlation phenomena (23). As this implies, the angular correlation formalism is very general and is based on fundamental quantum-mechanical concepts.

Consider a (compound) state of an excited nucleus with angular momentum J and parity π that can decay into pairs of particles of spins j_1 and j_2, respectively. The specific pair will be denoted by α (channel label); the vector sum of $\mathbf{j}_1$ and $\mathbf{j}_2$ will be denoted by $\mathbf{s}$ (channel spin).

In the region where the particle pair α are well separated, that is, the "external region" where the only interparticle interaction is, at most, the Coulomb interaction, the two particles will be in relative motion with angular momentum l, and we can write the eigenstates with sharp energy in the form

$$\psi(JM\pi)=\psi(JM;\alpha ls;\text{in}) + \sum_{\alpha'l's'} S(J\pi;\alpha ls;\alpha'l's')\psi(JM;\alpha'l's';\text{out}), \quad (26)$$

where

$$\psi(JM;\alpha ls)\equiv\pm(ir_sv_s^{\frac{1}{2}})^{-1}\exp\pm i\left(K_sr_s-\frac{l\pi}{2}\right)\sum_{m\nu} \times C(j_1j_2s;m_1,\nu-m_1)C(slJ;\nu,M-\nu) \times\chi_{j_1}{}^{m_1}\chi_{j_2}{}^{m_2}Y_l{}^{M-\nu}(\vartheta,\varphi). \quad (26a)$$

(The $\chi_{j_i}{}^{m_i}$ are the wave functions of the particles i, while v_s is their relative velocity. For an in-going spherical wave we use the minus signs in Eq. (26a); for an outgoing wave, the plus signs are used.) $S(J\pi;\alpha ls;\alpha'l's')$ is the scattering matrix, which must be unitary and symmetric in order to satisfy the two requirements of (1) conservation of the total number of systems and (2) reciprocity. The wave functions given above are continuable into the internal region, and are therefore suitable eigenfunctions to use as a basis for describing the compound state. The compound state is assumed to be a pure state, decaying into pairs of particles which are detected by the apparatus envisaged. According to the preceding discussion, the detection of a direction of motion of a particle corresponds to information which determines the density matrix of the compound state. Such information, when combined with the information that leads to this state, yields an angular correlation.

The states given by Eq. (26) are, however, not pure states, since the in- and out-going parts correspond to different mixtures of the various channels. What we desire is an eigenstate of the scattering matrix, that is, a state (we label it by n) composed of all channels, pairs of particles and orbital angular momenta, with the weights $X(n,\alpha ls)$, such that the in- and out-going parts belong to the same mixture. Since this is an eigenstate of the scattering matrix, we have by definition

$$\sum_{\alpha'l's'} S(J\pi;\alpha ls;\alpha'l's')X(n,\alpha'l's')=e^{2i\delta_n}X(n,\alpha ls) \quad (27)$$

with real δ_n. The desired state is therefore

$$\psi(JM\pi,n)=\sum_{\alpha ls} X(n,\alpha ls) \times\{\psi(JM;\alpha ls;\text{in})+e^{2i\delta_n}\psi(JM;\alpha ls;\text{out})\}. \quad (28)$$

Strictly speaking, since the compound state is decaying, the energy is unsharp and one should take a summation of the states above over an energy range. We shall not put this in evidence since it does not affect the arguments that follow.

The experiment envisaged measures now the type of particles (α) and their spin magnitudes (j_1 and j_2) as well as their (back-to-back) direction of motion. Hence we must expand the wave function above into eigenstates of the linear momentum, exactly the situation considered in Sec. II-C above. When this is done, one finds for the density matrix of the compound state

$$E(JJMM')=\tfrac{1}{2}\sum(-)^{s-M}i^{l'-l}(2J+1) \times[(2l+1)(2l'+1)]^{\frac{1}{2}}C(ll'\nu;00) \times C(JJ\nu;M,-M')X(n,\alpha ls) \times X^*(n,\alpha l's)W(JJll';\nu s) \times D(\nu,M-M',0;\mathfrak{k}), \quad (29)$$

and the summation is to be carried out over s, l, l' and ν.

This result has just the form given in Eq. (20); since (a) $[(2l+1)(2l'+1)]^{\frac{1}{2}}C(ll'\nu;00)D(\nu,M'-M,0;\mathfrak{k})$ are the tensor parameters of the radiation, inasmuch as no spin polarization is considered, (b) $C(JJ\nu;M,-M')\times(-)^{s-M}$ are the coefficients that connect the tensor parameters with the density matrix, and (c) the $W(JJll';\nu s)$ are the change of coupling or Racah coefficients. The $X(n,\alpha ls)$, $X^*(n,\alpha l's)$ are the "physical parameters" that give the weights with which various subsystems mix in the nth eigenstate.

It remains only to observe that the $X(n,\alpha ls)$ are (or can be chosen to be) *real*. Since the $X(n,\alpha ls)$ correspond in a perturbation calculation to the reduced matrix elements, one sees in particular that the relative phase of any mixture of reduced matrix elements is always 0 or π.

The proof for the reality of the $X(n, \alpha ls)$ is immediate. Since $S^{+}=S^{-1}$ (unitary property) and $\tilde{S}=S$ (symmetric property), we see that S can be always written in the form $S=U^{-1}e^{i\Delta}U$ where U is real and orthogonal and Δ is real and diagonal.

Since

$$SX(n)=e^{2i\delta_n}X(n) = U^{-1}e^{i\Delta}UX(n),$$

we have then

$$e^{i\Delta}UX(n)=e^{2i\delta_n}UX(n),$$

and we can choose $UX(n)$ to have the form of a column vector with unity in the nth place, zero elsewhere. Designating this column vector by w_n, that is,

$$\text{trans}w_n=(\cdots 0\,0\,0\,1\,0\cdots),$$

we have

$$X(n)=U^{-1}w_n. \tag{30}$$

Then $X(n, \alpha ls)$ is real since U is real. This is the proof of the reality of the matrix elements, and *a fortiori*, of the theorem on relative phases of interfering matrix elements, referred to in subsection C above.

Previous proofs of the reality of the reduced matrix elements have involved the commutability of the interactions with time reversal operators. The above proof also uses time reversal but in a concealed fashion since the properties of the S matrix were deduced from demanding reciprocity. The form of the proof above shows that this result on the relative phase of the reduced matrix elements is more general than the strict applicability of the equations would indicate. For photon emission or absorption, the use of wave functions as in the above equations is dubious at best. However (78), the form of the results should nonetheless hold, and, indeed, the result on the relative phase of the matrix elements has been verified for photon emission directly, (55). Since the generalization to relativistic wave functions can be readily carried out (42a), the result can be seen to be valid in this case also.

In the double cascade correlation experiment, the initial compound state is randomly oriented and the observation of the first radiation determines the density matrix of the residual compound state. This introduces only formal changes as compared to the case considered above where the observation of the radiation determined the density matrix of the *radiating* state. As a result, however, one sees that the cascade correlation involves the properties (i.e., the "physical parameters" $X(n, \alpha ls)$) of *two* compound states, unlike the nuclear reaction problem where the same compound state occurs in both transitions. An additional distinction, as mentioned earlier, is the fact that the nuclear reaction cannot in general be considered as occurring via a pure state; rather one must add, with equal weights, the contributions of all spins (J), parities (π) and eigenstates (n) of the compound system.

The density matrix for a compound state where J, π, n are not sharp has the form

$$\begin{aligned}E(JJ'MM')=\tfrac{1}{2}\sum(-)^{s-M}i^{l'-l}\\ &\times[(2J+1)(2J'+1)(2l+1)(2l'+1)]^{\frac{1}{2}}\\ &\times C(ll'\nu;00)C(JJ'\nu;M,-M')\\ &\times W(lJl'J';s\nu)D(\nu,M'-M,0;\mathbf{k})\\ &\times X(n,\alpha ls)X(n',\alpha'l's)\\ &\times\begin{Bmatrix}\exp[2i(\delta_n-\delta_{n'})] & \text{out}\\ 1 & \text{in}\end{Bmatrix},\end{aligned} \tag{31}$$

and the sum is over s, l, l', ν; the "out" and "in" refer to the compound state decaying or being formed, respectively. For a nuclear reaction, we have the *same* compound state formed by an incident radiation (1) and decaying into an outgoing radiation (2). The angular distribution has the form (see Eqs. (35) and (38) below),

$$W(\mathbf{k}_1,\mathbf{k}_2)=\sum E(JJ'MM';1)E^*(JJ'MM';2), \tag{32}$$

with the summation over M, M', J, J', π, π', n and n'. In Eq. (32) $\mathbf{k}_1$ and $\mathbf{k}_2$ are the unit propagation vectors of the radiations (1) and (2). The sum over J, J', π, π', n, n' is necessary since the reaction proceeds via all compound states. The sum over M' and M is readily performed giving the result

$$W(\mathbf{k}_1,\mathbf{k}_2)=\tfrac{1}{4}\sum_L B_L P_L(\mathbf{k}_1\cdot\mathbf{k}_2),$$

where

$$\begin{aligned}B_L=\sum X(n,\alpha l_1 s)X(n',\alpha l_1's)X(n,\alpha'l_2s')\\ &\times X(n',\alpha'l_2's')(-)^{s-s'}\exp[2i(\delta_{n'}-\delta_n)]\\ &\times[i^{l_1'-l_1+L}\{(2J+1)(2J'+1)(2l_1+1)(2l_1'+1)\}^{\frac{1}{2}}\\ &\times C(l_1l_1'L;00)W(l_1Jl_1'J';sL)]\\ &\times[i^{l_2'-l_2+L}\{(2J+1)(2J'+1)(2l_2+1)(2l_2'+1)\}^{\frac{1}{2}}\\ &\qquad\times C(l_2l_2'L;00)W(l_2Jl_2'J';s'L)].\end{aligned} \tag{33}$$

The latter sum is over J, J', l_1, l_1', l_2, l_2', s, s', n, n'. The quantities in square brackets are just the $Z(l_1Jl_1'J';s\nu)$ coefficients arising in the angular distribution of nuclear reactions and defined in reference (15).

The sum over n, n' can be carried out since we note that this is just the "spectral form" of the scattering matrix. That is,

$$\sum_n X(n,\alpha l_1 s)X(n,\alpha'l_2s')e^{-2i\delta_n} = S^{+}(J\pi;\alpha l_1 s;\alpha'l_2s') \tag{34a}$$

$$\sum_{n'} X(n',\alpha l_1's)X(n',\alpha'l_2's')e^{2i\delta_{n'}} = S(J'\pi';\alpha l_1's;\alpha'l_2's'). \tag{34b}$$

Finally if we wish to find the scattering angular distribution as well as the reaction distribution, we must cancel the cross section given above for no reaction, i.e., $S=1$. In other words, we replace S by $S-1$ in the above formula for the reaction cross section, and it becomes valid for scattering as well. Putting these steps together we find for the differential cross section

$$W(\mathbf{f}_1, \mathbf{f}_2) \sim d\sigma_{\alpha\alpha'} = \frac{1}{4}\frac{\lambda\!\!\!^{-}{}_\alpha^2}{(2j+1)(2j_0+1)}\sum_L B_L P_L(\mathbf{f}_1\cdot\mathbf{f}_2),$$

where j and j_0 are the spins of incident particle and target nucleus and $\lambda\!\!\!^{-}{}_\alpha$ is the reduced wavelength in the incident channel α;

$$B_L = \sum (S(J\pi)-1)^+(S(J'\pi')-1)(-)^{s-s'} \times Z(l_1Jl_1'J';sL)Z(l_2Jl_2'J';s'L). \quad (34)$$

The sum in the expression for B_L is over J, J', π, π', l_1, l_1', l_2, l_2', s, s'. This is exactly the result for nuclear scattering and reactions given in reference (15).

We emphasize again that this simple reduction of the distribution to its dependence on the scattering matrix was possible only because (a) we summed over all possible compound states, and (b) the two radiations involved the *same* compound state. The angular correlation problem is distinguished from the above by just these features, and we therefore must give our results in the less compact spectral form.

E. The Double Correlation

We consider here the correlation of two successive radiations for the case of an unperturbed intermediate state. In (1) below the general form of the correlation function is obtained. In (2) the direction-direction correlation is considered. The polarization-direction correlation is of practical interest only in the case of an observation of the linear polarization of γ rays and for this reason we defer the discussion of this correlation to Sec. III.

1. The General Double Correlation Function

Here, and in the following, we make explicit use of our choice of phases for which the reduced matrix elements are *real*. They are not Hermitian (i.e., symmetric). Instead

$$(2j+1)^{\frac{1}{2}}(j_1\|L\|j) = (-)^{j_1+L-j}(2j_1+1)^{\frac{1}{2}}(j\|L\|j_1).$$

Having set up the density matrices $E(mm')$, the problem now is to relate this to the correlation function of a simple two radiation correlation. Let us suppose that the intermediate nucleus is unperturbed, i.e., that $S(m_am_a';m_nm_n') = \delta(m_am_n)\delta(m_a'm_n')$. Then using Eqs. (2) and (13) we have

$$\begin{aligned} W &= \sum_{mm'} E^{(1)}(mm')E^{(2)*}(mm') \\ &= (2j+1)^2 \sum c_{\nu_1\tau_1}(L_1L_1')c_{\nu_2\tau_2}{}^*(L_2L_2') \\ &\quad \times(-1)^{j_1-j_2+L_1-L_2}(j_1\|L_1\|j)(j_1\|L_1'\|j) \\ &\quad \times(j_2\|L_2\|j)(j_2\|L_2'\|j)C(jj\nu_1;m,-m') \\ &\quad \times C(jj\nu_2;m,-m')W(jjL_1L_1';\nu_1j_1) \\ &\quad \times W(jjL_2L_2';\nu_2j_2)D(\nu_1,m'-m,\tau_1;\mathcal{R}_1) \\ &\quad \times D^*(\nu_2,m'-m,\tau_2;\mathcal{R}_2). \quad (35) \end{aligned}$$

The summation in Eq. (35) is over $\pi_1\pi_2L_1L_1'L_2L_2'$ $\nu_1\nu_2\tau_1\tau_2mm'$. $\mathcal{R}_1$ and $\mathcal{R}_2$ are the rotations corresponding to radiations 1 and 2 as explained below Eq. (5). We can perform the sum over m holding $\mu\equiv m-m'$ fixed. This involves

$$\sum_m C(jj\nu_1;m\mu-m)C(jj\nu_2;m\mu-m) = \delta(\nu_1\nu_2). \quad (36)$$

That is, $\nu_1=\nu_2=\nu$. Finally we can perform the sum over μ

$$\begin{aligned} \sum_\mu D(\nu,\mu\tau_1;\mathcal{R}_1)D^*(\nu,\mu\tau_2;\mathcal{R}_2) &= \sum_\mu D(\nu,\mu\tau_1;\mathcal{R}_1)D(\nu,\tau_2\mu;\mathcal{R}_2^{-1}) \\ &= D(\nu,\tau_2\tau_1;\mathcal{R}_2^{-1}\mathcal{R}_1) \quad (37) \end{aligned}$$

and $\mathcal{R}_2^{-1}$ is the inverse rotation to $\mathcal{R}_2$. The rotation $\mathcal{R}_2^{-1}\mathcal{R}_1$ is that rotation which carries the coordinate system of the first radiation into the coordinate system of the second radiation. We designate the Euler angles of the $\mathcal{R}_2^{-1}\mathcal{R}_1$ rotation by $\alpha\beta\gamma$. The result is

$$\begin{aligned} W(\alpha\beta\gamma) \sim \sum &(-1)^{L_1+L_2}c_{\nu\tau_1}(L_1L_1')c_{\nu\tau_2}{}^*(L_2L_2') \\ &\times(j_1\|L_1\|j)(j_1\|L_1'\|j)(j_2\|L_2\|j)(j_2\|L_2'\|j) \\ &\times W(jjL_1L_1';\nu j_1)W(jjL_2L_2';\nu j_2) \\ &\times D(\nu,\tau_2\tau_1;\alpha\beta\gamma). \quad (38) \end{aligned}$$

The sum is over L_1, L_1', L_2, L_2', τ_1, τ_2, and ν. This is the form of the general double correlation as given by Racah (65) and equivalently by Lloyd (57).

It is of interest to note that the sum performed in Eq. (37) furnishes a direct proof of the conjecture of Falkoff and Uhlenbeck (26) since it shows in particular thst only the angles of the reference axes attached to $\mathbf{f}_1$ measured with respect to $\mathbf{f}_2$ are significant in the correlation.[9] In other words, one may arbitrarily pick either of the radiations to define the coordinate axes without loss of generality—the principle of spectroscopic stability in another terminology. This fact has been extensively discussed in the literature from many points of view, see especially (44) and (26, 53, 51, 71, 75). It should be noted further that the density matrix may or may not be diagonal for this choice of axes, the diagonality being irrelevant (75).

[9] In fact, if only propagation vectors are measured, the only nonignorable angle is β which is the angle between $\mathbf{f}_1$ and $\mathbf{f}_2$, the unit vectors in the propagation directions. See paragraph (2) below.

For the important cases where the intermediate state is not unperturbed (hfs, for example), the correlation calculation is more difficult but the methods are the same. We shall discuss these cases in Sec. IV.

2. Direction-Direction Correlation

Of greatest interest is the angular correlation in which one observes only the propagation vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ of the two radiations and, as discussed in subsection II-A, for this case $\tau_1=\tau_2=0$ in Eq. (38) and α and γ are ignorable. Then, using

$$D(\nu,\lambda 0;\alpha\beta 0)=\left(\frac{4\pi}{2\nu+1}\right)^{\frac{1}{2}}Y_\nu^{\lambda *}(\beta,\alpha), \quad (39a)$$

and thus

$$D(\nu,00;\alpha\beta 0)=P_\nu(\cos\beta)=P_\nu(\mathbf{f}_1\cdot\mathbf{f}_2), \quad (39b)$$

where P_ν is the Legendre polynomial, Eq. (38) becomes

$$\begin{aligned}W(\beta)=\sum(-)^{L_1+L_2}c_{\nu 0}(L_1L_1')c_{\nu 0}(L_2L_2')\\ \times(j_1\|L_1\|j)(j_1\|L_1'\|j)(j_2\|L_2\|j)\\ \times(j_2\|L_2'\|j)W(jjL_1L_1';\nu j_1)\\ \times W(jjL_2L_2';\nu j_2)P_\nu(\cos\beta). \quad (40)\end{aligned}$$

The sum is over ν, L_1, L_1', L_2, and L_2'. As discussed in subsection G below, ν is an even integer in the range 0 to the smallest even integer of the set $2(L_1)_{\max}$, $2(L_2)_{\max}$ or $2j$. Here $(L_1)_{\max}$ is the largest value assumed by L_1 and/or L_1' and similarly for $(L_2)_{\max}$.

Equation (40) refers to the correlation in which the emitted radiation is a superposition of different angular momentum states. This case may occur not infrequently in the emission of α particles (Sec. III-B). Also, in the case of unfavorable parity change in γ-ray emission, a superposition of magnetic dipole and electric quadrupole has been observed to occur in several instances. Perhaps the more interesting case is the one in which only one of the transitions is not pure and consists in the superposition of only two angular momenta L_1 and L_1', say. Then the correlation function becomes

$$W(\beta)=W_I(\beta)+W_{II}(\beta)+W_{III}(\beta), \quad (41)$$

where

$$\begin{aligned}W_I(\beta)=(-)^{L_1+L_2}(j_2\|L_2\|j)^2(j_1\|L_1\|j)^2\\ \times\sum_\nu c_{\nu 0}(L_1L_1)c_{\nu 0}(L_2L_2)W(jjL_1L_1;\nu j_1)\\ \times W(jjL_2L_2;\nu j_2)P_\nu(\cos\beta), \quad (41a)\end{aligned}$$

$$\begin{aligned}W_{II}(\beta)=(-)^{L_1'+L_2}(j_2\|L_2\|j)^2(j_1\|L_1'\|j_1)^2\\ \times\sum_\nu c_{\nu 0}(L_1'L_1')c_{\nu 0}(L_2L_2)W(jjL_1'L_1';\nu j_1)\\ \times W(jjL_2L_2;\nu j_2)P_\nu(\cos\beta), \quad (41b)\end{aligned}$$

$$\begin{aligned}W_{III}(\beta)=(-)^{L_2}(j_2\|L_2\|j)^2(j_1\|L_1\|j)(j_1\|L_1'\|j)\\ \times\sum_\nu[(-)^{L_1}c_{\nu 0}(L_1L_1')+(-)^{L_1'}c_{\nu 0}(L_1'L_1)]\\ \times c_{\nu 0}(L_2L_2)W(jjL_1L_1';\nu j_1)\\ \times W(jjL_2L_2;\nu j_2)P_\nu(\cos\beta). \quad (41c)\end{aligned}$$

W_I is the correlation function which applies when the first transition is pure with radiation angular momentum L_1. Similarly, W_{II} applies when the first radiation angular momentum is L_1' only. W_{III} is an interference term which (see subsection II-F) does not contribute to the total intensity. In this term the minimum $\nu=2$. The maximum value of ν is the smallest even integer of the sets $2(L_1L_2j)$, $2(L_1'L_2j)$, $(L_1+L_1', 2L_2, 2j)$ in W_I, W_{II}, and W_{III}, respectively, (see subsection II-F). As written the correlation function W is not normalized so as to be necessarily positive definite. However, the insertion of a factor $(-)^{j_1-j_2}$ will insure this.

It will be observed that for both transitions pure the reduced matrix elements are merely scale factors so that in this case the correlation function can be written

$$\begin{aligned}W(\beta)=(-)^{L_1+L_2}\sum_\nu c_{\nu 0}(L_1L_1)c_{\nu 0}(L_2L_2)\\ \times W(jjL_1L_1;\nu j_1)W(jjL_2L_2;\nu j_2)P_\nu(\cos\beta). \quad (41d)\end{aligned}$$

The result for the double correlation given in Eq. (40) shows that a useful representation of the correlation functions for different radiations is obtained as follows: (a) we adopt a standard cascade, *viz*, the $\gamma-\gamma$ cascade. For a transition in which a γ ray is emitted the $c_{\nu 0}(LL')$ will be denoted by $c_\nu(LL';\gamma)$. (b) For any other type of particle (x) emitted, the $c_{\nu 0}(LL')$ will be denoted by $c_\nu(LL';x)$. (c) We define the particle parameters $b_\nu(LL';x)$ by[10]

$$b_\nu(LL';x)=c_\nu(LL';x)/c_\nu(LL';\gamma). \quad (42)$$

From the discussion following Eq. (20) we have in this case ($\tau=0$) for the tensor parameters of radiation of type x

$$R(\nu q,LL';x)=(-)^{q+L'+L}D(\nu,q0;\mathbf{f})c_{\nu 0}(LL';x). \quad (42a)$$

Therefore an alternative definition of the particle parameters is

$$b_\nu(LL';x)=R(\nu q,LL';x)/R(\nu q,LL';\gamma), \quad (42b)$$

and it is to be noted that the ratio is independent of q. (d) The correlation function as written in Eq. (41 a, b, c) may be expressed in the form

$$W=\sum_i W_i, \quad (43)$$

where $i=I, II, III$, and each of these may be written in the form

$$W_i=\sum_\nu A_\nu^{(i)}P_\nu(\cos\beta). \quad (43a)$$

In Sec. III-A results are given whereby the coefficients $A_\nu^{(i)}$ for the standard $\gamma-\gamma$ correlation may be immediately obtained. We denote these coefficients by

[10] In subsequent applications involving pure radiations it is sometimes convenient to adopt a normalization in which $b_0(LL;x)=1$. This, of course, can always be accomplished by redefining b_ν as $c_\nu(LL;x)c_0(LL;\gamma)/c_\nu(LL;\gamma)c_0(LL;x)$.

$A_\nu^{(i)}(\gamma\gamma)$. Then, if a radiation ($x$) replaces the pure γ, the correlation function is

$$W(\beta)=\sum_i \sum_\nu b_\nu(L_2L_2;x)A_\nu^{(i)}(\gamma\gamma)P_\nu(\cos\beta). \quad (43b)$$

When the mixed γ-radiation is replaced by the radiation x, the coefficients $A_\nu^{(I)}$ and $A_\nu^{(II)}$ are merely multiplied by $b_\nu(L_1L_1;x)$ and $b_\nu(L_1'L_1';x)$, respectively. For the interference term we first note that

$$(-)^L c_\nu(LL';\gamma)=(-)^{L'}c_\nu(L'L;\gamma) \quad (44)$$

as is shown in Sec. III-A. Therefore the factor by which $A_\nu^{(III)}$ is to be multiplied is

$$\tfrac{1}{2}[b_\nu(LL';x)+b_\nu(L'L;x)].$$

(e) For the correlation of radiations x and y the correlation function is obtained by performing the steps given in (d) for each particle. Thus, for the $x-y$ correlation with both radiations pure the correlation function is[11]

$$W_{x-y}(\beta)=\sum_\nu b_\nu(L_1L_1;x)b_\nu(L_2L_2;y) \times A_\nu^{(I)}(\gamma\gamma)P_\nu(\cos\beta). \quad (45)$$

Of course, the choice of the $\gamma-\gamma$ correlation as the standard one is arbitrary. We could have chosen the $\alpha-\alpha$ correlation which, from the point of view of analytical simplicity and availability of numerical results, would have been almost equally suitable. However, the fact that the parity restriction precludes the mixing of even and odd L militates against this choice. For γ-rays the parity for given L depends on the character of the radiation (electric or magnetic) and mixtures of all L, L' pairs (consistent with angular momentum conservation) are permissible in principle. Of course, in the case of γ rays $L\geqslant 1$ and the omission of the $L=0$ value is a slight disadvantage when one wishes to consider the emission of radiations with a mixture of $L=0$ and a nonvanishing L (in the emission of α or β particles, for example). However, in this infrequently occurring case direct use of Eq. (40) can be made.

F. Symmetry Properties of the Double Correlation

The double-correlation problem shows a number of symmetries, which are the consequence of (a) the "Hermitian" character of the matrix elements, (b) the invariance of the interactions [Eq. (4)] under reversal of coordinates and/or time reversal and (c) the special symmetries characteristic of particular observations, (for example, observing only a direction of propagation). Consider first the Hermitian symmetry. From Eq. (1), particularized to simple double correlation, ($S(m_am_a';m_nm_n')$ a unit matrix), we find that the Hermitian property requires the correlation function for $j_1(L_1)j(L_2)j_2$ to be identical with that corresponding to $j_2(L_2)j(L_1)j_1$. That is, initial and final states as well as first and second radiations may be reversed without any change in the correlation. This "degeneracy" is present even when the intermediate state is perturbed (Secs. IV and V-B). In the tabulation of the coefficients for the double correlation, given in Sec. III, use is made of this symmetry.

This symmetry property was apparent from Eq. (2). This property, it can be seen, holds even for non-Hermitian H operators. For example, in the case of no spin-coupling considered here, the correlation function is given by the trace of the product of the two density matrices; i.e., $W=tr(E^1E^2)=tr(E^2E^1)$ and the density matrices E^1 and E^2 are defined in precisely corresponding ways. The symmetry is not entirely obvious in Eq. (38) but can be readily checked by using

$$c_{\nu\tau}^*(LL')=(-)^{L+L'-\tau}c_{\nu,-\tau}(L'L),$$

which is directly derivable from the definition given in Eq. (12), and the unitary character of the rotation group matrices D^ν. It must be emphasized, however, that this symmetry was built into the theory, so to speak by our definition of the reduced matrix elements. Thus, Lloyd [(56) and especially (52)] writes the reduced matrix elements so that they always appear in the form $(j_f\|L\|j_i)$ where j_f and j_i, respectively, refer to final and initial states in each transition of the cascade. This differs from our convention only for the second transition of the cascade. Using the fact that Hermitian conjugation of the reduced matrix elements introduces a factor $(-)^{j_f-j_i+L}$, one finds that for the second transition

$$(j\|L_2\|j_2)(j\|L_2'\|j_2)=(-)^{L_2-L_2'}(j_2\|L_2\|j)(j_2\|L_2'\|j).$$

Consequently, with Lloyd's definition of the reduced matrix elements one must introduce a sign change in the interference term w_{III} when the second transition is a mixture with $|L_2-L_2'|$ an odd integer—the practical case (and the first pure), but no sign change if the first transition is a similar mixture (the second pure). This point is emphasized because the sign of the reduced matrix element product $(j_i\|L_i\|j)(j_i\|L_i'\|j)$ (with $|L_i'-L_i|=1$) can be measured and any attempt to compare a calculation of this phase on the basis of some nuclear model with the measured result must take into account the definition of the matrix elements used in the analysis of the experimental data.

The invariance of the interaction to time inversion allows us to conclude that the relative phase of the reduced matrix elements appearing in Eq. (38) is 0 or π. This point has already been discussed in subsections II-C and -D above.

[11] For the K conversion-conversion correlation the validity of Eq. (45) is based on the fact that the lifetime of the K shell hole, formed after the first conversion, is much shorter than the lifetime of the intermediate nuclear state, (62). Therefore, when the second conversion transition occurs the atomic state is initially the same as for the first conversion transition. If this were not the case, there would be appreciable interference contributions between the two alternative ways of reaching the final state corresponding to the two possible orders of emission of the two K electrons (50).

In order to derive the properties of $W(\mathfrak{k}_1 \cdot \mathfrak{k}_2)$ under coordinate reversal ($\beta \rightarrow \pi - \beta$), we shall assume (a) that the parity is a good quantum number and (b) that our measurements do not distinguish between right and left-handed coordinate systems. The only possible cases where the latter would not apply involve measurements of circular polarization of γ rays and/or polarization of spin $\frac{1}{2}$ particles and these cases are not particularly amenable to experiment. In the case where only a direction of motion is observed, the tensor parameters of the radiation involve $D(\nu, q0; \mathfrak{k})$ which under reflection is multiplied by $(-)^\nu$. Since this must be identical with the tensor parameters before reflection we deduce that ν is *even*. The correlation is therefore symmetric around $\beta = \frac{1}{2}\pi$. If the second condition is violated (imagine for example measuring the circular polarization of gamma-rays), then the argument is invalid, and in fact odd values of ν enter. In contrast to some arguments hitherto advanced, the good quantum number property of the parity is insufficient to guarantee that only even ν enter the double correlation. For linear polarized gammas, the measurements are unchanged by coordinate reflection and thus ν is indeed even in this case.

The *complexity* of the double correlation, that is the number of Legendre polynomials with $\nu > 0$ that enter Eq. (40), has been the subject of much discussion in the literature, (24, 79, 80). In the formulation above the allowed values for ν are determined by the properties of the Racah coefficients, so that this question is solved in detail. The Racah coefficient $W(abcd; ef)$ vanishes unless we satisfy the triangle conditions (abe), (cde), (acf), and (bdf). A triangle condition such as (abe) implies that a, b, and e can form a triangle (including the case of vanishing area). As a result, we see from Eq. (40) that ν is restricted by

(1) $0 \leqslant \nu \leqslant 2j$,

(2) $0 \leqslant \nu \leqslant 2(L_1)_{\max}$,

(3) $0 \leqslant \nu \leqslant 2(L_2)_{\max}$.

In the light of the density matrix formulation these rules are fairly obvious. For a radiation of multipolarity 2^L, we can define tensor parameters of rank at most $2L$. Since these tensor parameters determine the tensor parameters of the intermediate state, it is clear that the rank of the tensor parameters of the intermediate state is less than or equal to the minimum of $(2L, 2j)$. Finally, the tensor parameters of the intermediate state determine the tensor parameters of the second radiation, which must therefore be the minimum of $2(L_1, L_2, j)$. If j is a half-integer then j in the above is replaced by $j - \frac{1}{2}$. Of course, for $j = \frac{1}{2}$ one always has $\nu = 0$ only and therefore an isotropic correlation function.

Of course, an experimental determination of the maximum ν is difficult if the coefficient of the Legendre polynomial with maximum ν is small. However, the coefficient of P_2 is almost always sufficiently large to give an observable anisotropy. Thus, if isotropy is observed one can conclude (a) in the case of γ emission (or conversion electrons) $j = 0$ for even-mass nuclei and $j = \frac{1}{2}$ for odd-mass nuclei; (b) for particles which can have $L = 0$ (α particles, β decay, nucleons) either one of the L's has a zero value or $j = 0$ or $\frac{1}{2}$ depending on the mass number of the nucleus in the intermediate state.

G. Relation to the Falkoff-Uhlenbeck Method (26)

It is useful at this point to show the relation of the present methodology to the procedures of Falkoff-Uhlenbeck. Falkoff and Uhlenbeck consider the angular correlation problem in two steps, wherein each transition is represented in terms of the emission of a particle current multiplied by suitable weight factors (the relative probability of transitions between various sublevels of the nuclear states involved, see Eq. (47) below). In the case of gamma emission, this current is the familiar Poynting flux from an array of multipoles all with the same magnetic quantum number. Since the Falkoff-Uhlenbeck procedure considers the radiation with a well-defined magnetic quantum number, emerging from a completely defined initial state, it is clear that the intermediate state similarly has a fixed magnetic quantum number. Consequently, their procedure is more restrictive than the procedure given in the foregoing, in that it does not make provision for coherent processes in general. Lacking the flexibility to handle coherent processes, the Falkoff-Uhlenbeck procedure forces one to link the two successive emissions via an incoherent intermediate state, which is simply enough accomplished by letting one or the other of the two radiations define the quantization axis for the problem, subject only to the restriction (necessary in the Falkoff-Uhlenbeck method) that the choice must diagonalize the density matrix (see discussion at the end of II-E). It is for this reason that the "choice of axes" held such a central position in their work.

To show the relation of the two procedures in more detail let us consider the double correlation problem for the unperturbed intermediate state. The correlation is then

$$W(\mathfrak{k}_1 \cdot \mathfrak{k}_2) = \mathfrak{S} \sum_{m_1 m_2 m} |\langle j_1 m_1 | H_1(\mathfrak{k}_1) | jm \rangle|^2 \times |\langle jm | H_2(\mathfrak{k}_2) | j_2 m_2 \rangle|^2. \quad (46)$$

Here we have made the sums incoherent by taking $\mathfrak{k}_1$ (for example) to define the quantization axis. Falkoff and Uhlenbeck now introduce the function

$$\mathfrak{S} |\langle j_1 m_1 | H(\mathfrak{k}) | jm \rangle|^2 \equiv [C(j_1 L j; m_1, m - m_1)]^2 F_L{}^{m-m_1}(\beta) \quad (47)$$

for a pure multipole transition. Hence

$$W(\mathfrak{k}_1 \cdot \mathfrak{k}_2) \sim \sum_{m_1 m_2 m} [C(j L_1 j; m_1, m - m_1)]^2 \times F_{L_1}{}^{m-m_1}(0) [C(j L_2 j_2; m, m_2 - m)]^2 F_{L_2}{}^{m_2 - m}(\beta), \quad (48)$$

again for pure multipoles and unpolarized radiations. The result for polarized radiation can be obtained from Eq. (50) below. In this form the angular correlation problem has a very simple physical interpretation. Equation (48) states that the populations of the sublevels m of the intermediate state are determined by the current associated with the first radiation, $F_{L_1}{}^{m-m_1}(0)$, multiplied by the Zeeman intensity coefficients $[C(j_1L_1j;m_1, m-m_1)]^2$. The subsequent radiation is the product then of the flux of the second radiation, $F_{L_2}(\beta)$, the Zeeman intensity coefficients and the population of the intermediate sublevels obtained from the first transition. Of course, as pointed out earlier, it is equally valid to choose $\mathbf{f}_2$ as the axis of quantization.

The connection with the density matrix formulation is easily seen to be

$$E_{mm}{}^{(1)}\sim\sum_{m_1}[C(j_1L_1j, m_1, m-m_1)]^2F_{L_1}{}^{m-m_1}(\beta). \quad (49)$$

It is more convenient to consider Eq. (47) (generalized for mixed multipoles), however, to define the mixed flux $F_{LL'}{}^M$ (49). Introducing Eqs. (4), (5), (7), (9), yields

$$\begin{aligned}F_{LL'}{}^M(\beta,\alpha)&=\sum_\nu(j_1\|L\|j)(j_1\|L'\|j)\\&\quad\times C(LL'\nu;M,-M)(-)^M\\&\quad\times\mathfrak{S}\sum_{\mu\mu'}\alpha^*(L\mu,\pi)\alpha(L'\mu',\pi)(-)^\mu\\&\quad\times C(LL'\nu;-\mu,\mu')D(\nu,0\mu'-\mu;\mathbf{f}_1)\\&=\sum_{\nu\tau}c_{\nu\tau}(LL')(j_1\|L'\|j)(j_1\|L\|j)\\&\quad\times\left(\frac{4\pi}{2\nu+1}\right)^{\frac{1}{2}}(-)^{L'-\nu}Y_\nu{}^\tau(\beta\alpha)\\&\quad\times(-1)^MC(LL'\nu;M,-M). \quad (50)\end{aligned}$$

The $F_{LL'}{}^M$ of reference (49) is normalized differently and is, in fact, the above without the reduced matrix elements.

This form for the generalized $F_{LL'}{}^M$ shows a number of interesting features. (a) In order that the use of the Falkoff-Uhlenbeck procedure be correct, it is necessary that the choice of axes diagonalize the density matrix. This will not be the case unless $\tau=0$. (An example when $\tau\neq0$ is an experiment detecting linearly polarized γ-rays.) We must thus choose the other radiation to define the axis, if it has $\tau=0$, or if not, apply the more general techniques discussed earlier. (b) The integrated intensity from the $F_{LL'}{}^M(\beta\alpha)$ consists of only the term with $\nu=\tau=0$. Since $C(LL'0;M,-M)$ vanishes unless $L=L'$ we see that only the $F_{LL}{}^M(\beta)$, i.e., the "self terms," contribute to the integrated intensity. The interference terms affect the angular distribution but not the total intensity. This was shown first by Casimir (18). As shown earlier, this result follows from the general correlation function given in Eq. (38). (c) One of the central problems for angular correlations is the determination of the particle parameters of the radiation, depending on the $c_{\nu\tau}(LL')$. From Eq. (50) it is clear that the problem of determining $F_{LL'}{}^M(\beta\alpha)$ is completely equivalent to that of determining the $c_{\nu\tau}(LL')$. In practice, the formulation of a radiation problem in terms of the emitted flux is generally simpler conceptually than the equivalent formulation in terms of the tensor parameters for the radiation. Consequently, the relation given in Eq. (50) is very often useful.

It remains only to observe that introducing Eq. (50) into Eq. (49), and using Eq. (11) twice, reduces the correlation function $W(\mathbf{f}_1,\mathbf{f}_2)$ to the form of Eq. (38). As noted before the radiation that is chosen to define the axes must have $\tau=0$ so that the Falkoff-Uhlenbeck procedure is not completely general even for the double correlation. However, the use of the $F_{LL'}{}^M(\beta\alpha)$ to define the particle parameters is nonetheless completely general.

III. SPECIAL CASES OF ANGULAR CORRELATION

In this section we take up the correlation in the case of specific radiations. We consider in turn the tensor parameters for (a) gamma radiation (b) alpha particles, (c) conversion electrons, and (d) β particles. It follows from Sec. II-E that from these results one can immediately obtain the correlation function for the double cascade in which any pair of these radiations (e.g., $\gamma-\gamma$, $\beta-\gamma$, etc.) are emitted. The numerical results are presented in the form of tables given below.

A. Gamma Radiation

(1) General Properties

We shall describe the photon field by means of the vector potential, which we shall treat as the "wave function" for the light quantum, (42). The tensor parameters of the quantum, according to our general recipe, are then found from the transcription of the measured plane wave states, taking proper account of polarization, into the spherical eigenfunction states natural to the description of the nuclear process. The intrinsic (unit) spin of the photon will be described by the wave function $\chi_1{}^P$, the sharp momentum of the γ-ray by $e^{i\mathbf{K}\cdot\mathbf{r}}$. The transverse nature of light waves allows P to be ±1 only, measured in a coordinate system attached to the direction of motion. Thus P is a polarization index ($P=-1$ corresponding to right circular polarization). Because of this restriction, a polarization measurement determines the spin state to be a linear combination of these two basic states, with complex coefficients, that is,

$$\chi_s=a\chi_1{}^1+b\chi_1{}^{-1};\ |a|^2+|b|^2=1. \quad (51)$$

A convenient description of an arbitrary polarization is obtained by introducing the rotation matrices for spin $\frac{1}{2}$,

$$\chi_s=\sum_\mu D(\tfrac{1}{2},\tfrac{1}{2}\mu;0ba)\chi_1^{2\mu}. \quad (52)$$

Of course, $\mu=\pm\frac{1}{2}$. The Euler angles b, a constitute a full description of a completely polarized quantum, and are, of course, closely related to the Stokes parameters, as shown in detail later. Setting the first Euler angle in Eq. (52) equal to zero merely fixes a nonessential constant phase. Consider now the wave function (vector potential) for a circularly polarized plane wave quantum travelling along the z direction[12]

$$\chi_1^P e^{iKz}=\sum_l (2l+1)^{\frac{1}{2}}(4\pi)^{\frac{1}{2}}j_l(Kr)i^l Y_l^0(\mathbf{r})\chi_1^P. \quad (52a)$$

Using the vector addition model we write

$$i^l Y_l^0\chi_1^P=\sum_{L=l\pm1,l} C(l1L;0P)\Phi_{L,l}^P, \quad (53)$$

where

$$\Phi_{Ll}^P=i^l\sum_\tau C(l1L;P-\tau,\tau)Y_l^{P-\tau}\chi_1^\tau. \quad (53a)$$

Thus

$$\chi_1^P e^{iKz}=(2\pi)^{\frac{1}{2}}\sum_L[(L+1)^{\frac{1}{2}}j_{L-1}(Kr)\Phi^P{}_{L,L-1}+L^{\frac{1}{2}}j_{L+1}(Kr)\Phi^P{}_{L,L+1}-P(2L+1)^{\frac{1}{2}}j_L(Kr)\Phi^P{}_{L,L}]. \quad (54)$$

In Eq. (54) the first two terms correspond to an electric multipole vector potential and the last term to a magnetic multipole potential.

If we rotate the coordinate system, so that the wave propagates in an arbitrary direction $\mathfrak{k}$, we note that the spin direction rotates (rigidly) with the propagation direction so that we rotate the $\Phi_{L,l}^P$ as a single entity.

$$(\chi_1^P e^{iKz})'=\pi\sum_{L,M}(2L+1)^{\frac{1}{2}}D(L,MP;\mathfrak{R})\times(A^e(LM)+(-P)A^m(LM)). \quad (55)$$

The $A^e(LM)$ and $A^m(LM)$ are the usual (standing wave) vector potentials, (42), normalized to $(\pi^2\hbar K)^{-1}$ quanta-sec, redefined only to the extent of a phase. For magnetic multipoles (m) this phase is i^L, for electric multipoles (e) it is i^{L+1}, since we require that $K_tA^{e,m}(LM)=(-1)^{L-M}A^{e,m}(L-M)$ in order to make the reduced matrix elements explicitly real.

At present, it is experimentally feasible to measure only unpolarized and linearly polarized gamma-rays (17). Nonetheless we shall not specialize our treatment at the outset to these important cases alone but rather shall handle a general partially polarized gamma-quantum and specialize at the end to the relevant cases. Let us represent the result of the polarization measurement by the vector $\mathbf{P}$; that is to say, a vector with the Euler angles a, b as azimuth and polar angles, respectively, in a coordinate frame attached to the direction of motion. Note that this polarization vector is *not* along the direction of the electric field vector, $\mathbf{E}$. Then our measured state is represented by

$$(\chi_s e^{iKz})'=\pi\sum_{LM\mu}(2L+1)^{\frac{1}{2}}D(\tfrac{1}{2},\tfrac{1}{2}\mu;\mathfrak{P})\times D(L,M2\mu;\mathfrak{R})[A^e(LM)-2\mu A^m(LM)]. \quad (56)$$

We write the phase (-2μ), that occurs for magnetic multipoles only, in the form $[\pi(-)^L]^{\frac{1}{2}+\mu}$ when π is the parity of the radiation. Here $\pi=(-)^{L+1}$ for magnetic multipoles and $\pi=(-)^L$ for electric multipoles. Then we can treat all the multipoles symmetrically

$$(\chi_s e^{iKz})'=\pi\sum(2L+1)^{\frac{1}{2}}D(\tfrac{1}{2},\tfrac{1}{2}\mu;\mathfrak{P})\times D(L,M2\mu;\mathfrak{R})[\pi(-)^L]^{\frac{1}{2}+\mu}A^\pi(L,M), \quad (57)$$

where $A^\pi(L,M)=A^e(L,M)$ and $A^m(L,M)$ for $\pi=(-)^L$ and $(-)^{L+1}$, respectively. The sum is over L, M, μ, and π.[13]

The tensor parameters of the radiation, with parity a good quantum number, are obtained as described in Sec. II-B and one finds[14]

$$R(\nu q,LL'\pi)=[(2L+1)(2L'+1)]^{\frac{1}{2}}\times\sum(-)^{(\frac{1}{2}+\mu)L+(\frac{1}{2}+\mu')L'}(-)^{L-M}\pi^{\mu-\mu'}\times C(LL'\nu;M,-M')\delta(q,M-M')\times D^*(\tfrac{1}{2},\tfrac{1}{2}\mu';\mathfrak{P})D(\tfrac{1}{2},\tfrac{1}{2}\mu;\mathfrak{P})\times D^*(L',M'2\mu;\mathfrak{R})D(L,M2\mu;\mathfrak{R}). \quad (58)$$

The sum is over M, M', μ, μ'. Now

$$D^*(\tfrac{1}{2},\tfrac{1}{2}\mu';\mathfrak{P})D(\tfrac{1}{2},\tfrac{1}{2}\mu;\mathfrak{P})=\tfrac{1}{2}\delta(\mu\mu')+\tfrac{1}{2}|\mathbf{P}|(-)^{\frac{1}{2}-\mu}D(1,00;\mathfrak{P})\delta(\mu\mu')+2^{-\frac{1}{2}}|\mathbf{P}|(-)^{\frac{1}{2}+\mu}D(1,02\mu;\mathfrak{P})\delta(\mu,-\mu'). \quad (59)$$

To account for partial polarization, we have assigned a magnitude to the polarization vector, $0\leqslant|\mathbf{P}|\leqslant1$, such that $|\mathbf{P}|=0$ corresponds to unpolarized γ rays and $|\mathbf{P}|=1$ corresponds to completely polarized γ rays.

Using Eq. (9) we find that the tensor parameter consist of three terms

$$R(\nu q,LL'\pi)=R_I+R_{II}+R_{III}. \quad (60)$$

The first term is

$$R_I\equiv(-)^{q+L'+1}[(2L+1)(2L'+1)]^{\frac{1}{2}}\times C(LL'\nu;1-1)D(\nu,q0;\mathfrak{k}) \quad (60a)$$

---

[12] To clarify the notation we observe that

$$\chi_1^1=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad\chi_1^{-1}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

and denoting the vector potential by $\mathbf{A}$, Eq. (52a) gives $-2^{-\frac{1}{2}}(A_x+iA_y)$ for $P=1$ and $2^{-\frac{1}{2}}(A_x-iA_y)$ for $P=-1$.

[13] The distinction between the numeric π (occurring in Eq. (57) and elsewhere) and the parity symbol should be fairly obvious. Wherever π occurs in the tensor parameters $R(\nu q)$, it refers to parity.

[14] Since the tensor parameters refer here to γ rays, we have suppressed the radiation index γ and to emphasize the parity dependence the π symbol is put in evidence in the $R(\nu q)$.

for ν an even integer. This is the polarization independent term.

$$R_{II}\equiv(-)^{q+L'+1}|\mathbf{P}|[(2L+1)(2L'+1)]^{\frac{1}{2}}\times C(LL'\nu;1-1)D(1,00;\mathfrak{P})D(\nu,q0;\mathfrak{k}) \quad (60b)$$

for ν an *odd* integer. This is the "circular" polarization contribution.

$$R_{III}\equiv(-)^{q}\pi 2^{-\frac{1}{2}}|\mathbf{P}|[(2L+1)(2L'+1)]^{\frac{1}{2}}\times C(LL'\nu;11)[D(1,01;\mathfrak{P})D(\nu,q2;\mathfrak{R})-(-)^{\nu}D(1,0-1;\mathfrak{P})D(\nu,q-2;\mathfrak{R})]. \quad (60c)$$

This is the linear polarization contribution. There is no restriction on ν at this stage of the calculation, but, since the reduced matrix elements are explicitly real, the $R(\nu q)$ are restricted to be "real" in the sense that $K_0R(\nu q)=(-)^{-q}R(\nu-q)$ (see II-C). This condition implies that ν is an *even* integer in R_{III}.

The tensor parameters for the most general partially polarized γ quanta have thus been obtained. Having introduced a magnitude for $\mathbf{P}$ to describe the degree of polarization, it is now clear that this description in terms of $\mathbf{P}$ corresponds to the Stokes parameter description of a quantum of unit intensity with polarization vector $\mathbf{P}$ in the Poincaré sphere (29).

For the cases of interest we have first unpolarized γ-rays; $|\mathbf{P}|=0$. Then

$$R(\nu q,LL'\pi)=(-)^{q+L'+1}[(2L+1)(2L'+1)]^{\frac{1}{2}}\times C(LL'\nu;1-1)D(\nu,q0;\mathfrak{k}), \quad (61)$$

where ν is an even integer, and

$$c_{\nu 0}(LL')=(-)^{L+1}C(LL'\nu;1-1)\times[(2L+1)(2L'+1)]^{\frac{1}{2}}. \quad (62)$$

If linear polarization is measured, the Euler angles of $\mathbf{P}$ are a, $b=\pi/2$, so that $D(1,00;\mathfrak{P})=0$ and $D(1,0\pm1;\mathfrak{P})=\mp2^{-\frac{1}{2}}e^{\pm ia}$. Then one finds

$$R(\nu q,LL'\pi)=[(2L+1)(2L'+1)]^{\frac{1}{2}}[(-)^{q+L'+1}\times C(LL'\nu;1-1)D(\nu,q0;\mathfrak{k})+\tfrac{1}{2}(-)^{q+1}\pi C(LL'\nu;11)(e^{ia}D(\nu,q2;\mathfrak{R})+e^{-ia}D(\nu,q-2;\mathfrak{R}))]. \quad (63)$$

The first term is simply R_I which is polarization independent. The second term is the contribution of the linear polarization, (R_{III}). It should be noted that the tensor parameters for linear polarization, in contrast to the unpolarized case, depend explicitly upon the parity of the radiation. It is this feature which motivates one to perform the more difficult polarization measurements, and shows, incidentally, that a polarization-polarization correlation (in which one measures the polarization of both quanta) has essentially nothing new to offer over the simpler direction-polarization correlations (60). The origin of this dependence on parity is not far to seek, since it is simply an expression of the duality theorem for multipole radiation. Changing the parity of the radiation but keeping L, L' fixed takes $e\leftrightarrow m$ multipoles, which by duality, takes $\mathbf{E}\rightarrow\mathbf{H}$, $\mathbf{H}\rightarrow-\mathbf{E}$, thus changing a by 180°; this reverses the sign of the linear polarization term, R_{III}. The usual linear polarization angle (defined by the $\mathbf{E}$ vector) is here $\frac{1}{2}a$. Since $D(L,q\pm2;\alpha\beta\gamma)=e^{iq\alpha}e^{\pm i2\gamma}\,d(L,q\pm2;\beta)$ (where $d(L,\mu\mu';\beta)$ is the Jacobi polynomial), one sees that the polarization angle a can be absorbed into γ, i.e., $\gamma\rightarrow\gamma+\frac{1}{2}a$. This introduces the usual polarization angle into the formalism above, see Eq. (73) below.

(2) *Correlation Function for Unpolarized Radiation*

Only the first term in Eq. (63) for the γ-tensor parameter enters and using Eqs. (40) and (42a) the $\gamma-\gamma$ correlation function is

$$W(\beta)=\sum(j_1\|L_1\|j)(j_1\|L_1'\|j)(j_2\|L_2\|j)(j_2\|L_2'\|j)\times C(L_1L_1'\nu;1-1)C(L_2L_2'\nu;1-1)\times W(jjL_1L_1';\nu j_1)W(jjL_2L_2';\nu j_2)\times[(2L_1+1)(2L_1'+1)(2L_2+1)(2L_2'+1)]^{\frac{1}{2}}\times P_\nu(\cos\beta). \quad (64)$$

The sum is over L_1L_1', L_2L_2' and ν.

For the cascade emission of a mixed 2^{L_1} plus $2^{L_1'}$ pole and a pure 2^{L_2} pole the correlation is, explicitly,

$$W(\beta)=(j_1\|L_1\|j)^2w_I+(j_1\|L_1'\|j)^2w_{II}+2(j_1\|L_1\|j)(j_1\|L_1'\|j)w_{III}, \quad (65)$$

where

$$w_I=(2L_1+1)(2L_2+1)\times\sum_{\nu\geqslant0}C(L_1L_1\nu;1-1)C(L_2L_2\nu;1-1)\times W(jjL_1L_1;\nu j_1)W(jjL_2L_2;\nu j_2)P_\nu(\cos\beta) \quad (65a)$$

is the correlation function for a pure 2^{L_1}—pure 2^{L_2} pole cascade, w_{II} is the same as w_I but with L_1' replacing L_1 and

$$w_{III}=[(2L_1+1)(2L_1'+1)]^{\frac{1}{2}}(2L_2+1)\times\sum_{\nu\geqslant2}C(L_1L_1'\nu;1-1)C(L_2L_2\nu;1-1)\times W(jjL_1L_1';\nu j_1)W(jjL_2L_2;\nu j_2)P_\nu(\cos\beta) \quad (65b)$$

is the interference term. See also Eqs. (70, 70a, 70b, 70c). The correlation function for two mixed radiations can be readily obtained from Eq. (64). In practice the γ-ray mixture will be a 2^L and 2^{L+1} pole mixture. Then if $L_1=L$, the first term of Eq. (64) refers to the 2^L pole radiation and the second to the 2^{L+1} pole. From (Appendix A11)

$$C(LL0;1-1)W(jjLL;0j_1)=(-)^{j_1-j-1}(2L+1)^{-1}(2j+1)^{-\frac{1}{2}} \quad (66)$$

it follows that the 2^{L_1+1} pole to 2^{L_1} pole intensity ratio is

Table I(a). $F_2(1j_1j)$ for integer spins.

$j \backslash j_1$	0	1	2	3	4
1	0.7071	−0.3536	0.0707	0	0
2	0	0.4183	−0.4183	0.1195	0
3	0	0	0.3464	−0.4330	0.1443
4	0	0	0	0.3134	−0.4387
5	0	0	0	0	0.2944

Table Ib. $F_2(2j_1j)$ for integer spins.

$j \backslash j_1$	0	1	2	3	4
1	0	−0.3535	0.3535	−0.1010	0
2	−0.5976	−0.2988	0.1281	0.3415	−0.1707
3	0	−0.4949	−0.1237	0.2268	0.3093
4	0	0	−0.4477	−0.0448	0.2645
5	0	0	0	−0.4206	0

Table I(c). $F_4(2j_1j)$ for integer spins.

$j \backslash j_1$	0	1	2	3	4
2	−1.069	0.7127	−0.3054	0.0764	−0.0085
3	0	−0.4467	0.6700	−0.4467	0.1489
4	0	0	−0.3044	0.6087	−0.4981
5	0	0	0	−0.2428	0.5665

Table I(d). $F_2(3j_1j)$ for integer spins.

$j \backslash j_1$	0	1	2	3	4
1	0	0	−0.4243	0.5303	−0.1768
2	0	−0.7171	−0.1793	0.3287	0.4482
3	−0.8660	−0.6495	−0.2742	0.1443	0.4330
4	0	−0.7835	−0.4701	−0.0855	0.2678
5	0	0	−0.7360	−0.3680	0.0170

Table I(e). $F_4(3j_1j)$ for integer spins.

$j \backslash j_1$	0	1	2	3	4
2	0	0.0891	−0.1336	0.0891	−0.0297
3	0.2132	0.0355	−0.1066	−0.0355	0.1044
4	0	0.1453	−0.0484	−0.1012	0.0132
5	0	0	0.1159	−0.0773	−0.0852

Table I(f). $F_6(3j_1j)$ for integer spins.

$j \backslash j_1$	0	1	2	3	4
3	1.3056	−0.9792	0.5440	−0.2176	0.0593
4	0	0.4214	−0.7585	0.6895	−0.3831
5	0	0	0.2420	−0.6049	0.6979

Table I(g). $F_2(4j_1j)$ for integer spins.

$j \backslash j_1$	0	1	2	3	4
1	0	0	0	−0.4293	0.6010
2	0	0	−0.7257	−0.0726	0.4288
3	0	−0.8763	−0.5258	−0.0956	0.2995
4	−0.9687	−0.8234	−0.5554	−0.2101	0.1447
5	0	−0.9099	−0.6825	−0.3787	−0.0437

Table I(h). $F_4(4j_1j)$ for integer spins.

$j \backslash j_1$	0	1	2	3	4
2	0	0	0.1718	−0.3436	0.2811
3	0	0.4112	−0.1371	−0.2866	0.0374
4	0.6034	0.3017	−0.0901	−0.2860	−0.1408
5	0	0.4814	0.0802	−0.2239	−0.2592

Table I(i). $F_6(4j_1j)$ for integer spins.

$j \backslash j_1$	0	1	2	3	4
3	0	0.0218	−0.0392	0.0356	−0.0198
4	0.0674	−0.0034	−0.0346	0.0104	0.0243
5	0	0.0387	−0.0290	−0.0190	0.0242

Table I(j). $F_8(4j_1j)$ for integer spins.

$j \backslash j_1$	0	1	2	3	4
4	−1.4809	1.1847	−0.7539	0.3770	−0.1450
5	0	−0.3893	0.7785	−0.8384	0.5989

related to the ratio of reduced matrix elements by

$$\delta^2 \equiv \frac{I_{L_1+1}}{I_{L_1}} = \frac{(j_1\|L_1+1\|j)^2}{(j_1\|L_1\|j)^2}. \tag{67}$$

Very often $I_{L_1+1}/I_{L_1} \ll 1$ and only linear terms in δ need be retained, (56). Of course, while the correlation function for pure multipoles is parity independent, the case of mixed multipoles gives a parity determination only if it is assumed that a M_L, E_{L+1} mixture is much more likely than an E_L, M_{L+1} mixture (76a).

It should also be emphasized that the presence of interference in the correlation may change the correlation markedly and the correlation measurement is a sensitive method for the determination of mixture ratios.

Equations (64) and (65) define the standard $\gamma-\gamma$ correlation. In particular Eq. (65a) is the standard $\gamma-\gamma$ correlation for pure multipoles. We write the latter in the form

$$w(\beta) = \sum_\nu A_\nu P_\nu(\cos\beta) \tag{68}$$

and renormalize so that $A_0 = 1$ corresponding to unit value for the average of the correlation function w. Then

$$A_\nu = F_\nu(L_1j_1j)F_\nu(L_2j_2j), \tag{69a}$$

where

$$F_\nu(Lj_1j) = (-)^{j_1-j-1}(2j+1)^{\frac{1}{2}}(2L+1) \times C(LL\nu;1-1)W(jjLL;\nu j_1) \tag{69b}$$

so that also $F_0 = 1$.

TABLE I(k). $F_2(1j_1j)$ for half-integer spins.

$j \backslash j_1$	1/2	3/2	5/2	7/2	9/2	11/2	13/2
3/2	0.5000	−0.4000	0.1000	0	0	0	0
5/2	0	0.3742	−0.4276	0.1336	0	0	0
7/2	0	0	0.3273	−0.4364	0.1528	0	0
9/2	0	0	0	0.3028	−0.4404	0.1651	0
11/2	0	0	0	0	0.2876	−0.4425	0.1738
13/2	0	0	0	0	0	0.2774	−0.4437

TABLE I(l). $F_2(2j_1j)$ for half-integer spins.

$j \backslash j_1$	1/2	3/2	5/2	7/2	9/2	11/2	13/2
3/2	−0.5000	0	0.3571	−0.1429	0	0	0
5/2	−0.5345	−0.1909	0.1909	0.3245	−0.1909	0	0
7/2	0	−0.4676	−0.0779	0.2494	0.2962	−0.2182	0
9/2	0	0	−0.4325	−0.0197	0.2752	0.2752	−0.2359
11/2	0	0	0	−0.4109	0.0158	0.2890	0.2596
13/2	0	0	0	0	−0.3962	0.0396	0.2972

TABLE I(m). $F_4(2j_1j)$ for half-integer spins.

$j \backslash j_1$	1/2	3/2	5/2	7/2	9/2	11/2	13/2
5/2	−0.6172	0.7054	−0.3968	0.1176	−0.0147	0	0
7/2	0	−0.3581	0.6367	−0.4775	0.1736	−0.0253	0
9/2	0	0	−0.2684	0.5857	−0.5124	0.2102	−0.0338
11/2	0	0	0	−0.2237	0.5505	−0.5309	0.2359
13/2	0	0	0	0	−0.1970	0.5254	−0.5418

TABLE I(n). $F_2(3j_1j)$ for half-integer spins.

$j \backslash j_1$	1/2	3/2	5/2	7/2	9/2	11/2	13/2
3/2	0	−0.6000	0.1500	0.5000	−0.2500	0	0
5/2	−0.8018	−0.4410	0.0267	0.4009	0.4009	−0.3341	0
7/2	−0.8183	−0.5455	−0.1637	0.2182	0.4474	0.3273	−0.3819
9/2	0	−0.7569	−0.4129	−0.0275	0.3028	0.4541	0.2753
11/2	0	0	−0.7191	−0.3319	0.0522	0.3476	0.4504
13/2	0	0	0	−0.6934	−0.2774	0.1040	0.3744

TABLE I(o). $F_4(3j_1j)$ for half-integer spins.

$j \backslash j_1$	1/2	3/2	5/2	7/2	9/2	11/2	13/2
5/2	0.1543	−0.0772	−0.0772	0.1029	−0.0444	0.0070	0
7/2	0.1709	−0.0190	−0.1076	−0.0069	0.1019	−0.0627	0.0121
9/2	0	0.1281	−0.0660	−0.0932	0.0278	0.0935	−0.0735
11/2	0	0	0.1067	−0.0849	−0.0778	0.0469	0.0850
13/2	0	0	0	0.0940	−0.0940	−0.0653	0.0585

It is clear that for the numerical tabulation of the pure $\gamma-\gamma$ correlation it is most economical to tabulate only the F_ν and this gives a complete numerical description of the correlation. In Tables I(a) through I(j) we give numerical values for the F_ν for $L=1, 2, 3, 4$ and all the necessary ν values and for $j_1=0, 1, 2, 3, 4$ and $j\leqslant 5$ (integer values). In Tables I(k) through I(t) we give F_ν for the same range of L and ν and for all half-integer values of j_1 in the range $\frac{1}{2}$ to 13/2 inclusive and for all pertinent $j\leqslant 13/2$.

In using the tables one selects an F value (for each ν) corresponding to each transition and then the correlation function is completely defined by Eqs. (68) and (69a).

For the correlation in which one γ-ray is not pure we can use the normalization given above if the intensity ratio δ^2 is introduced in Eq. (65). The correlation function is simply

$$W=w_I+\delta^2 w_{II}+2\delta w_{III}, \tag{70}$$

where δ^2 is the ratio of intensities of the $2^{L_1'}$ pole to that

TABLE I(p). $F_6(3j_1j)$ for half-integer spins.

$j \backslash j_1$	1/2	3/2	5/2	7/2	9/2	11/2	13/2
7/2	0.6528	−0.8704	0.6528	−0.3165	0.0989	−0.0183	0.0015
9/2	0	0.3077	−0.6714	0.6994	−0.4304	0.1614	−0.0344
11/2	0	0	0.1998	−0.5533	0.6916	−0.4918	0.2075
13/2	0	0	0	0.1498	−0.4793	0.6741	−0.5287

TABLE I(q). $F_2(4j_1j)$ for half-integer spins.

$j \backslash j_1$	1/2	3/2	5/2	7/2	9/2	11/2	13/2
3/2	0	0	−0.6071	0.2429	0.5409	−0.3091	0
5/2	0	−0.8113	−0.3245	0.1711	0.4898	0.3924	−0.4130
7/2	−0.9274	−0.6889	−0.3397	0.0482	0.3734	0.5059	0.2867
9/2	−0.9358	−0.7444	−0.4558	−0.1155	0.2127	0.4466	0.4861
11/2	0	−0.8890	−0.6326	−0.3175	0.0122	0.3001	0.4770
13/2	0	0	−0.8573	−0.5572	−0.2266	0.0934	0.3521

TABLE I(r). $F_4(4j_1j)$ for half-integer spins.

$j \backslash j_1$	1/2	3/2	5/2	7/2	9/2	11/2	13/2
5/2	0	0.2976	−0.2976	−0.1082	0.2976	−0.1602	0.0291
7/2	0.5128	0.0733	−0.2597	−0.2131	0.1235	0.2474	−0.2188
9/2	0.5322	0.1693	−0.1762	−0.2786	−0.0800	0.2065	0.1893
11/2	0	0.4434	0.0171	−0.2509	−0.2363	0.0087	0.2372
13/2	0	0	0.3906	−0.0651	−0.2744	−0.1916	0.0668

TABLE I(s). $F_6(4j_1j)$ for half-integer spins.

$j \backslash j_1$	1/2	3/2	5/2	7/2	9/2	11/2	13/2
7/2	0.0435	−0.0348	−0.0071	0.0317	−0.0259	0.0110	−0.0025
9/2	0.0492	−0.0213	−0.0269	0.0195	0.0172	−0.0292	0.0167
11/2	0	0.0320	−0.0326	−0.0123	0.0264	0.0061	−0.0284
13/2	0	0	0.0240	−0.0348	−0.0024	0.0275	−0.0014

TABLE I(t). $F_8(4j_1j)$ for half-integer spins.

$j \backslash j_1$	1/2	3/2	5/2	7/2	9/2	11/2	13/2
9/2	−0.6623	0.9633	−0.8429	0.5187	−0.2316	0.0741	−0.0162
11/2	0	−0.2624	0.6460	−0.8075	0.6460	−0.3533	0.1330
13/2	0	0	−0.1497	0.4791	−0.7336	0.6904	−0.4315

of the 2^{L_1} pole and

$$w_I = \sum_\nu A_\nu(L_1L_2)P_\nu(\cos\beta), \tag{70a}$$

$$w_{II} = \sum_\nu A_\nu(L_1'L_2)P_\nu(\cos\beta), \tag{70b}$$

where the A_ν are given in Eq. (69a) and

$$w_{III} = (-)^{j-j_1-1}[(2j+1)(2L_1+1)(2L_1'+1)]^{\frac{1}{2}} \times\sum_\nu G_\nu(L_1L_1'j_1j)F_\nu(L_2j_2j)P_\nu(\cos\beta), \tag{70c}$$

where F_ν is given in Eq. (69b) and

$$G_\nu(L_1L_1'j_1j) = C(L_1L_1'\nu;1-1)W(jjL_1L_1';\nu j_1) \tag{70d}$$

symmetric in L_1 and L_1'.

In Tables II(a) through II(l) numerical values of the coefficients $G_\nu(LL'j_1j)$ are presented for $\nu=2$, 4, 6, $L=1$, 2, 3 with $L'=L+1$ and the range of j_1 and j is the same as Table I. In the table headings the first argument in G_ν is the value of L. Special cases of the mixed correlation were considered by Ling and Falkoff (49).

(3) Correlation Function for Linear Polarization-Direction Correlation (45)

The correlation function which applies here was given in Eq. (38). In order to make direct use of the results of this section we evaluate the $c_{\nu\tau}$ from the tensor param-

TABLE II(a). $G_2(1j_1j)$ for integer spins.

$j \backslash j_1$	1	2	3	4
1	0.1581	0.0707	0	0
2	−0.1080	0.0707	0.0756	0
3	0	−0.0926	0.0423	0.0704
4	0	0	−0.0809	0.0289
5	0	0	0	−0.0725

TABLE II(b). $G_2(2j_1j)$ for integer spins.

$j \backslash j_1$	1	2	3	4
1	0	−0.0617	−0.0369	0
2	−0.0404	0.0432	−0.0132	−0.0382
3	0.0296	−0.0378	0.0279	0
4	0	0.0298	−0.0322	0.0195
5	0	0	0.0284	−0.0279

TABLE II(c). $G_4(2j_1j)$ for integer spins.

$j \backslash j_1$	1	2	3	4
2	0.0753	0.0604	0.0246	0.0047
3	−0.0667	0	0.0472	0.0332
4	0	−0.0507	−0.0171	0.0346
5	0	0	−0.0409	−0.0224

TABLE II(d). $G_2(3j_1j)$ for integer spins.

$j \backslash j_1$	1	2	3	4
1	0	0	0.0326	0.0223
2	0	0.0246	−0.0213	0.0027
3	0.0145	−0.0203	0.0193	−0.0105
4	−0.0114	0.0168	−0.0180	0.0141
5	0	−0.0125	0.0161	−0.0155

TABLE II(e). $G_4(3j_1j)$ for integer spins.

$j \backslash j_1$	1	2	3	4
2	0	−0.0331	−0.0382	−0.0199
3	−0.0386	0.0180	0.0194	−0.0190
4	0.0317	−0.0242	−0.0033	0.0221
5	0	0.0295	−0.0138	−0.0106

TABLE II(f). $G_6(3j_1j)$ for integer spins.

$j \backslash j_1$	1	2	3	4
3	0.0429	0.0429	0.0259	0.0101
4	−0.0429	−0.0153	0.0206	0.0286
5	0	−0.0288	−0.0236	0.0071

eters. By making use of (see Eq. (20) *et seq.*)

$$R(\nu q, LL'\pi) = (-)^{q+L'+L} \sum_{\tau} D(\nu, q\tau; \mathfrak{R}) c_{\nu\tau}(LL'\pi), \quad (71)$$

and we have used the fact that we are concerned only with linear polarization (therefore $\mathfrak{R}$ appears instead of $\mathfrak{k}$). Comparing Eqs. (71) and (63) we see that

$$c_{\nu\tau}(LL'\pi) = (-)^{L+1} e^{\frac{1}{2}i\tau a} [(2L+1)(2L'+1)]^{\frac{1}{2}} \times \left\{ \begin{array}{cc} & C(LL'\nu; 1-1) \\ \frac{1}{2}(-)^{\sigma'} & C(LL'\nu; 11) \end{array} \right\}, \quad (72)$$

where the uppper line in the bracket refers to $\tau=0$, and the lower refers to $\tau=\pm 2$. Also $\sigma'=1$ for magnetic and 0 for electric radiation and the prime in σ' denotes the $2^{L'}$ multipole.

For pure multipoles in which the polarization of the first (2^{L_1} pole) quantum is observed and correlated with the direction of a 2^{L_2} pole quantum, we obtain the correlation function

$$W_1(\beta, \varphi) = \sum_{\nu} A_\nu \left\{ P_\nu + (-)^{\sigma_1} \frac{C(L_1 L_1 \nu; 11)}{C(L_1 L_1 \nu; 1-1)} \times \left[\frac{(\nu-2)!}{(\nu+2)!} \right]^{\frac{1}{2}} \cos 2\varphi P_\nu^2(\cos\beta) \right\}, \quad (73)$$

where A_ν is defined in Eq. (69a and b) and σ_1 refers to radiation 1. In Eq. (73) φ is the angle between the polarization vector and the normal to the $\mathfrak{k}_1 - \mathfrak{k}_2$ plane and P_ν^2 is the associated Legendre polynomial; in the second term of Eq. (73) $\nu \geqslant 2$. The explicit dependence on the character of the radiation will be noted. Equivalent results have been obtained by Zinnes (81) and Lloyd (58).

The ratio of vector addition coefficients can be expressed in the form

$$\frac{C(LL\nu; 11)}{C(LL\nu; 1-1)} = \frac{2\nu(\nu+1)L(L+1)}{\nu(\nu+1)-2L(L+1)} \left[\frac{(\nu-2)!}{(\nu+2)!} \right]^{\frac{1}{2}} \quad (74)$$

with ν even, so that with Table I numerical results for the correlation function for linear polarization can be readily obtained. Special cases have been considered by Falkoff (25).

If the linear polarization of the second quantum is measured, the correlation function is obtained by interchanging L_1 and L_2 which affects only the ratio of vector addition coefficients given in Eq. (74) and the phase $(-)^\sigma$ if the radiations are of different character. Designating this correlation function by W_2, what will be observed is the average correlation function

$$W = \eta_{12} W_1 + \eta_{21} W_2,$$

where η_{12} is the over-all efficiency for detection of photon 1 in the polarization sensitive detector and photon 2 in the polarization insensitive detector, while η_{21} is the over-all efficiency for the photons interchanged.

The importance of the relative over-all detector efficiencies η_{12} and η_{21} can be seen as follows. If the efficiencies are the same (so that $W \sim W_1 + W_2$) the polarization sensitive term in W will be absent if

$$\frac{\nu(\nu+1)-2L_2(L_2+1)}{\nu(\nu+1)-2L_1(L_1+1)} = \pm \frac{L_2(L_2+1)}{L_1(L_1+1)}$$

TABLE II(g). $G_2(1j_1j)$ for half-integer spins.

j \ j_1	1/2	3/2	5/2	7/2	9/2	11/2	13/2
3/2	−0.1118	0.1000	0.0764	0	0	0	0
5/2	0	−0.1000	0.0535	0.0732	0	0	0
7/2	0	0	−0.0863	0.0345	0.0677	0	0
9/2	0	0	0	−0.0764	0.0246	0.0628	0
11/2	0	0	0	0	−0.0691	0.0187	0.0586
13/2	0	0	0	0	0	−0.0635	0.0148

TABLE II(h). $G_2(2j_1j)$ for half-integer spins.

j \ j_1	1/2	3/2	5/2	7/2	9/2	11/2	13/2
3/2	0	0.0535	−0.0286	−0.0391	0	0	0
5/2	0.0261	−0.0405	0.0344	−0.0049	−0.0366	0	0
7/2	0	0.0302	−0.0349	0.0231	0.0031	−0.0332	0
9/2	0	0	0.0292	−0.0299	0.0168	0.0066	−0.0305
11/2	0	0	0	0.0275	−0.0262	0.0129	0.0082
13/2	0	0	0	0	0.0259	−0.0234	0.0102

TABLE II(i). $G_4(2j_1j)$ for half-integer spins.

j \ j_1	1/2	3/2	5/2	7/2	9/2	11/2	13/2
5/2	−0.0753	0.0225	0.0551	0.0308	0.0070	0	0
7/2	0	−0.0578	−0.0111	0.0402	0.0339	0.0096	0
9/2	0	0	−0.0452	−0.0205	0.0300	0.0334	0.0109
11/2	0	0	0	−0.0375	−0.0235	0.0233	0.0320
13/2	0	0	0	0	−0.0322	−0.0243	0.0187

TABLE II(j). $G_2(3j_1j)$ for half-integer spins.

j \ j_1	1/2	3/2	5/2	7/2	9/2	11/2	13/2
3/2	0	0	−0.0298	0.0119	0.0234	0	0
5/2	0	−0.0195	0.0225	−0.0149	−0.0020	0.0215	0
7/2	−0.0094	0.0165	−0.0193	0.0164	−0.0074	−0.0063	0.0193
9/2	0	−0.0122	0.0166	−0.0167	0.0123	−0.0035	−0.0079
11/2	0	0	−0.0126	0.0156	−0.0145	0.0095	−0.0013
13/2	0	0	0	−0.0124	0.0145	−0.0127	0.0076

TABLE II(k). $G_4(3j_1j)$ for half-integer spins.

j \ j_1	1/2	3/2	5/2	7/2	9/2	11/2	13/2
5/2	0	0.0405	0.0078	−0.0282	−0.0238	−0.0068	0
7/2	0.0295	−0.0317	0.0045	0.0222	−0.0122	−0.0241	−0.0092
9/2	0	0.0310	−0.0183	−0.0078	0.0208	−0.0037	−0.0223
11/2	0	0	0.0280	−0.0104	−0.0122	0.0178	0.0009
13/2	0	0	0	0.0251	−0.0057	−0.0137	0.0150

TABLE II(l). $G_6(3j_1j)$ for half-integer spins.

j \ j_1	1/2	3/2	5/2	7/2	9/2	11/2	13/2
7/2	−0.0526	0	0.0312	0.0292	0.0142	0.0039	0.0005
9/2	0	−0.0347	−0.0213	0.0128	0.0268	0.0178	0.0060
11/2	0	0	−0.0244	−0.0242	0.0030	0.0227	0.0188
13/2	0	0	0	−0.0187	−0.0236	−0.0023	0.0190

for all possible ν; the $(+, -)$ sign refers to the case of radiations of (different, same) character. This actually occurs for

(a) radiations of opposite character and $L_1=L_2$,

(b) radiations of the same character, and L_1, $L_2=1$, 2 or 2, 1 and for no other cases.[15] In both cases there is an *over-all* parity change in the cascade. Therefore, if there is no polarization effect, with equal efficiencies, there must have been an over-all parity change (61), but the converse does not follow. That is, it is not true (as has been stated (61)) that the cascades in which a polarization effect is observed ($\eta_{12}=\eta_{21}$) correspond to no over-all parity charge. It should be recognized that when the detector efficiencies are unequal there will also be cases in which the polarization can cancel, or nearly cancel, accidentally. No conclusions should be drawn in such a case and the relative detector efficiencies should be varied.

As mentioned above, when one of the radiations is a mixture, the correlation function depends on the parity change in the transition in which the mixed radiation is emitted. To determine the parity change in the transition in which the pure radiation is emitted one may make a polarization-direction correlation, observing the polarization of the pure radiation. Of course, unless one can devise detectors for which the efficiency is zero for the mixed radiation going to the polarization sensitive detector, it is also necessary to consider the case in which the polarization of the mixed radiation is observed. We therefore consider these two cases. We consider a M_{L_1}, E_{L_1+1} (or alternatively, E_{L_1}, M_{L_1+1}) mixture. The pure radiation is again a 2^{L_2} pole.

(*a*) *Polarization-direction correlation with polarization of pure radiation measured.*—From Eqs. (38) and (72) we find for the correlation function

$$\bar{W}_1=\varpi_I+\delta^2\varpi_{II}+2\delta\varpi_{III}, \tag{73a}$$

where δ^2 is the intensity ratio $I(L_1+1)/I(L_1)$ and

$$\varpi_I=\sum_\nu A_\nu(L_1L_2)\mathcal{P}_\nu(L_2L_2;\beta\varphi). \tag{73b}$$

This is the polarization-direction correlation for the pure 2^{L_1}-pure 2^{L_2} pole. Here we define

$$\mathcal{P}_\nu(LL';\beta\varphi)=P_\nu(\cos\beta)+(-)^{\sigma(L')}\times\left[\frac{(\nu-2)!}{(\nu+2)!}\right]^{\frac{1}{2}}\frac{C(LL'\nu;11)}{C(LL'\nu;1-1)}\cos2\varphi P_\nu^2(\cos\beta). \tag{73c}$$

The notation is the same as in Eq. (73) and the parity index $\sigma(L')$ is determined by the character of the $2^{L'}$ pole. Thus, in Eq. (73b) the phase $(-)^{\sigma(L_2)}$ is -1 for M_{L_2} radiation, 1 for E_{L_2} radiation. In a similar way ϖ_{II} is the correlation function for a pure 2^{L_1+1} pole-2^{L_2} pole and is obtained by replacing L_1 by L_1+1 in Eq. (73b)

$$\varpi_{II}=\sum_\nu A_\nu(L_1+1,L_2)\mathcal{P}_\nu(L_2L_2;\beta\varphi) \tag{73d}$$

and ϖ_{III} is the interference term

$$\varpi_{III}=(-)^{j-j_1-1}[(2j+1)(2L_1+1)(2L_1+3)]^{\frac{1}{2}}\times\sum_\nu G_\nu(L_1L_1+1j_1j)F_\nu(L_2j_2j)\mathcal{P}_\nu(L_2L_2;\beta\varphi). \tag{73e}$$

In Eq. (73e) $\nu\geqslant 2$.

(*b*) *Polarization-direction correlation with polarization of mixed radiation measured.*—An entirely similar procedure gives for the correlation function

$$\bar{W}_2=\varpi_I'+\delta^2\varpi_{II}'+2\delta\varpi_{III}'. \tag{73f}$$

Here, δ is as in Eq. (73a) and

$$\varpi_I'=\sum_\nu A_\nu(L_1L_2)\mathcal{P}_\nu(L_1L_1;\beta\varphi) \tag{73g}$$

$$\varpi_{II}'=\sum_\nu A_\nu(L_1+1,L_2)\mathcal{P}_\nu(L_1+1,L_1+1;\beta\varphi), \tag{73h}$$

which are the polarization-direction correlations for pure multipoles and

$$\varpi_{III}'=(-)^{j-j_1-1}[(2j+1)(2L_1+1)(2L_1+3)]^{\frac{1}{2}}\times\sum_\nu G_\nu(L_1L_1+1j_1)F_\nu(L_2j_2j)\times\mathcal{P}_\nu(L_1L_1+1;\beta\varphi) \tag{73i}$$

which is the interference term. Again in Eq. (73i) $\nu\geqslant 2$. In Eq. (73i) one needs (52)

$$\frac{C(LL'\nu;11)}{C(LL'\nu;1-1)}=\left[\frac{(\nu-2)!}{(\nu+2)!}\right]^{\frac{1}{2}}(L'-L)(L'+L+1), \tag{73j}$$

where $L+L'+\nu$ is an odd integer. Equivalent results have been obtained by Zinnes (81).

In the actual observation the quantity measured is, of course, the weighted average of $\bar{W}_1$ and $\bar{W}_2$ with weight factors given by the over-all detector efficiencies for the pure radiation and the mixed radiation going to the polarization sensitive detector, respectively. The results given above have a common normalization and can be averaged directly.

B. Spinless Particles

The only spinless particle in which we are interested is the alpha particle and the specific results given here apply to this case. However, for other nuclear particles (neutrons, protons, deuterons) for which only directions of motions are observed, the results given here could apply equally well in the manner indicated below.

The determination of the tensor parameters for alpha particles is a very simple problem, as one might expect, since the only possible measurement is the determination of a direction of motion, uncomplicated by any spin polarization questions. Let us first omit consideration of the charge of the emitted particles. Then the physical measurement determines that the particle is in the plane wave state, $e^{i\mathbf{K}\cdot\mathbf{r}}$, and in accord with the discussion of Sec. II-C, the tensor parameters of the radiation are determined from the transformation to

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[15] Aside from the trivial cases ($j=0$ or $\frac{1}{2}$) there is one other case which leads to *complete* isotropy; *viz.*, a cascade in which $j_1=\frac{3}{2}$, $j=\frac{3}{2}$ with pure quadrupole emission (9).

angular momentum eigenfunctions. Using the well-known Rayleigh formula we have

$$e^{i\mathbf{K}\cdot\mathbf{r}}=4\pi\sum_{lm} j_l i^l Y_l^m(\mathbf{r})Y_l^{m*}(\mathbf{f})$$
$$=\sum_{lm}[(2l+1)4\pi]^{\frac{1}{2}}j_l(Kr)D(l,m0;\mathbf{f})i^lY_l^m(\mathbf{r}). \quad (75)$$

The tensor parameters, $R(\nu q)$, of this radiation are

$$R(\nu q,ll'\pi)=4\pi(2l+1)^{\frac{1}{2}}(2l'+1)^{\frac{1}{2}}\sum_m(-)^{l-m}$$
$$\times D^*(l',q-m,0;\mathbf{f})D(l,m0;\mathbf{f})$$
$$\times C(ll'\nu;mq-m). \quad (76)$$

By use of Eq. (9), we find that

$$R(\nu q,ll'\pi)=4\pi[(2l+1)(2l'+1)]^{\frac{1}{2}}(-)^{l-q}$$
$$\times C(ll'\nu;00)D(\nu,q0;\mathbf{f}). \quad (77)$$

In order that parity be a good quantum number for the radiation (as indicated by the variable $\pi_l=(-)^l=(-)^{l'}$ in the notation for $R(\nu q)$), we must require that l and l' differ by an even integer. Since the vector addition coefficient, $C(ll'\nu;00)$, vanishes for $l+l'+\nu\neq$even integer we see that ν is even.

In terms of the $c_{\nu\tau}$ of Eq. (12) we find

$$c_{\nu\tau}=\delta(\tau 0)(-)^l[(2l+1)(2l'+1)]^{\frac{1}{2}}C(ll'\nu;00). \quad (78)$$

If we now consider charged particles the changes that occur are of two kinds. Firstly the discrimination against the emission of high angular momentum particles will become less marked, because of the Coulomb barrier, so that the interference of different angular momenta will be more pronounced. This effect[16] is, of course, contained in the physical parameters (the relative size of the reduced matrix elements) and does not influence the tensor parameters above. The second effect is to introduce phase shifts, $e^{i\delta_l(Z)}$, multiplying the lth spherical wave of our basis.[17]

The result for charged, spinless particles (alpha particles) is found to be

$$c_{\nu\tau}(\alpha)=4\pi\delta(\tau 0)(-)^l e^{i(\delta_l-\delta_{l'})}[(2l+1)(2l'+1)]^{\frac{1}{2}}$$
$$\times C(ll'\nu;00). \quad (78a)$$

For a single angular momentum,[16] the phase shift difference drops out. However, for those cases in which more than one angular momentum is important the phase shifts will remain and may change the correlation functions appreciably. We consider explicitly only the former case. Since we have chosen the gamma-ray correlation to be the standard directional correlation, the most useful parameters are the b_ν, defined in Eq. (42). Using the results of the preceding section we find for pure multipoles (unique orbital angular momentum l) that (normalized to $b_0=1$)

$$b_\nu(ll;\alpha)=-C(ll\nu;00)/C(ll\nu;1-1)$$
$$=\frac{2l(l+1)}{2l(l+1)-\nu(\nu+1)}. \quad (79)$$

The $\alpha-\gamma$ correlation, for example, is then

$$W_{\alpha-\gamma}(\beta)=\sum_\nu A_\nu\frac{2l(l+1)}{2l(l+1)-\nu(\nu+1)}P_\nu(\cos\beta), \quad (80)$$

with A_ν from Eq. (69a, b) with l replacing L_1 or L_2 according as to whether the α particle is emitted in the $j_1\rightarrow j$ or $j\rightarrow j_2$ transition.

The results can be extended to the emission or absorption of a nuclear particle with a pure angular momentum in the following way. If l is the particle angular momentum, i its intrinsic spin and I the target nucleus then the channel spin s is any of the values consistent with

$$\mathbf{s}=\mathbf{i}+\mathbf{I}.$$

Assume for the moment that only a single value of s need be considered. The correlation is then the same as for α particles if j_1, the initial state angular momentum is replaced by s. For the more realistic case wherein more than one channel spin is relevant, the correlation is an incoherent sum over s of such contributions with weight factors corresponding to the relative probability for the occurrence of the channel spin s in the assumed pure compound state. An example of this procedure is given in reference (11).

C. Conversion Electrons

For heavy elements in which conversion electrons are more prevalent than γ rays, it may be advantageous to measure the correlation with such electrons. The problem of correlations involving conversion electrons, in either or both transitions, has been considered by a number of authors. A preliminary discussion was given by Fierz (31) and nonrelativistic approximations applicable to electric conversion of s electrons were considered by Berestetzky (10) and Gardner (38, 39). An approximate but incomplete consideration of magnetic conversion was also given by Gardner (40), see also Ling (50). The relativistic treatment of the conversion of an s electron was given by Rose, Biedenharn, and Arfken (68) and numerical results were given for the K shell when the conversion transition is pure. For mixed conversion transitions the results appeared in a later publication by the same authors (70). At present all numerical results in a relativistic calculation are limited to K conversion correlation (69, 68).

The internal conversion process is viewed as the ejection of an extranuclear electron, originally in a stationary state (say the K shell), via the electromag-

[16] This question is discussed at length by H. A. Bethe, Revs. Modern Phys. **9**, 69 (1937), see Sec. 72.

[17] We may consider the Coulomb potential to be screened, so as to eliminate the logarithmic terms. In the final result the limit for zero screening can be taken. Also from Eq. (78a) below it is seen that the logarithmic term would drop out in any case since only phase differences enter.

magnetic field which is excited by a nuclear transition. The tensor parameters of the radiation are nonetheless the tensor parameters of the nuclear gamma radiation for we have but substituted, in place of a direct observation of the gamma radiation, an indirect observation of its effect on the surrounding electrons. The tensor parameters of the gamma radiation are thus determined by the coupling of (a) the tensor parameters of the ejected electron-fixed by our observation of a plane wave state with a definite direction of motion but unobserved spin orientation, and (b) the tensor parameters of the initial state of the electron, which are quite simple since this state is random.

The first problem is to determine the tensor parameters of the final state of the ejected electrons. For the moment, let us neglect the effect of the Coulomb field on the motion of the electrons.

The observations imply that the ejected electrons are detected in plane wave states, $e^{iKz}D_\tau$, where we have taken the direction of motion to define the z axis and the D_τ are Dirac plane wave spinors (with $\tau=+$ taken to be spin "up," i.e., along the z axis and $\tau=-$, spin "down"). In relativistic units and with $K=p$

$$D_+=\begin{bmatrix} -p/E+1 \\ 0 \\ 1 \\ 0 \end{bmatrix};\quad D_-=\begin{bmatrix} 0 \\ p/E+1 \\ 0 \\ 1 \end{bmatrix}. \tag{81}$$

Since the spin is not observed, we must average over τ. These plane wave states can be expanded into a sum of Dirac spherical eigenfunctions, $\phi_\kappa{}^\mu$, (references 68, 67 and Eq. (83) below) and the result is

$$e^{iKz}D_\tau=-\left(\frac{\pi}{p(E+1)}\right)^{\frac{1}{2}}\sum_\kappa (4\pi|\kappa|)^{\frac{1}{2}}i^{-l(\kappa)} \times[-S(\kappa)]^{\tau-\frac{1}{2}}\phi_\kappa{}^\tau. \tag{82}$$

$\kappa=\pm(j+\frac{1}{2})$ for $j=l\mp\frac{1}{2}$ and l determines the parity (67). The sum is over all positive and negative integers, except zero. The notation is

$$\begin{aligned} S(\kappa)&\equiv\text{sign of }\kappa=\pm1 \\ l(\kappa)&=|\kappa|+\tfrac{1}{2}[S(\kappa)-1] \\ j&=|\kappa|-\tfrac{1}{2} \end{aligned} \tag{82a}$$

$$\phi_\kappa{}^\mu=\left(\frac{p}{\pi}\right)^{\frac{1}{2}}\begin{pmatrix} i(E-1)^{\frac{1}{2}}j_{l(-\kappa)}(pr)\chi_{-\kappa}{}^\mu \\ -S(\kappa)(E+1)^{\frac{1}{2}}j_{l(\kappa)}(pr)\chi_\kappa{}^\mu \end{pmatrix} \tag{83}$$

$$\chi_\kappa{}^\mu=\sum_\tau C(l(\kappa)\tfrac{1}{2}j;\mu-\tau,\tau)\chi_{\frac{1}{2}}{}^\tau Y_{l(\kappa)}{}^{\mu-\tau}. \tag{83a}$$

$\chi_{\frac{1}{2}}{}^\tau$ in Eq. (83a) is a Pauli spinor and μ is the eigenvalue of the z component of the total angular momentum j. The total energy $E=(p^2+1)^{\frac{1}{2}}$.

For convenience we give the following properties of the central field spinors which are used below

$$(\boldsymbol{\sigma}\cdot\mathbf{L}+1)\chi_\kappa{}^\mu=-\kappa\chi_\kappa{}^\mu \tag{83b}$$

$$\sigma_r\chi_\kappa{}^\mu=-\chi_{-\kappa}{}^\mu, \tag{83c}$$

and $\boldsymbol{\sigma}$ is the usual Pauli spin operator, $\mathbf{L}$ the orbital angular momentum operator.

If now our measuring axes are taken to be arbitrarily oriented with respect to the direction of motion of the plane wave (but the spin still quantized along this direction of motion), we must consider the above equation in a rotated system of coordinates. Since the $\phi_\kappa{}^\mu$ transform like wave functions with total angular momentum $j=|\kappa|-\frac{1}{2}$, we find

$$(e^{iKz}D_\tau)'=\frac{2\pi}{[p(E+1)]^{\frac{1}{2}}}\sum_{\kappa\mu}|\kappa|^{\frac{1}{2}}i^{-l(\kappa)} \times[-S(\kappa)]^{\tau-\frac{1}{2}}D(j,\mu\tau)\phi_\kappa{}^\mu. \tag{84}$$

The prime again denotes rotated.

The tensor parameters for the electron in the continuum are now readily seen to be (discarding a factor $2\pi/[p(E+1)]^{\frac{1}{2}}$)

$$\begin{aligned} R(\nu q,\kappa\kappa')&=\sum_{\mu\mu'\tau}(-)^{j-\mu}i^{l(\kappa')-l(\kappa)}C(jj'\nu;\mu,-\mu',q) \\ &\quad\times[S(\kappa)S(\kappa')]^{\tau-\frac{1}{2}}|\kappa\kappa'|^{\frac{1}{2}}D^*(j',\mu'\tau)D(j,\mu\tau) \\ &=i^{l(\kappa')-l(\kappa)}(-)^{\kappa+q}|\kappa\kappa'|^{\frac{1}{2}}D(\nu,q0;\mathfrak{k}) \\ &\quad\times\sum_\tau[-S(\kappa)S(\kappa')]^{\tau-\frac{1}{2}}C(jj'\nu;\tau,-\tau). \end{aligned} \tag{85}$$

$\mathfrak{k}$ is the unit propagation vector for the electron and $j'=|\kappa'|-\frac{1}{2}$. Using now the identity $(2l+1)^{\frac{1}{2}}C(l\frac{1}{2}j;0\tau)=|\kappa|^{\frac{1}{2}}[-S(\kappa)]^{\tau+\frac{1}{2}}$, one readily works out the sum over τ from

$$\begin{aligned} \sum_\tau[-S(\kappa)S(\kappa')]^{\tau+\frac{1}{2}}C(jj'\nu;\tau,-\tau) &=[(2l+1)(2l'+1)]^{\frac{1}{2}}(-)^{j+j'-\nu} \\ &\quad\times C(ll'\nu;00)W(ljl'j';\tfrac{1}{2}\nu). \end{aligned} \tag{86}$$

Hence the tensor parameters of the ejected electron can be written in the simpler form

$$\begin{aligned} R(\nu q,\kappa\kappa')&=i^{l(\kappa')-l(\kappa)}(-)^{\nu+q+j'-\frac{1}{2}} \\ &\quad\times[(2l+1)(2l'+1)|\kappa\kappa'|]^{\frac{1}{2}}S(\kappa)S(\kappa') \\ &\quad\times C(ll'\nu;00)W(ljl'j';\tfrac{1}{2}\nu)D(\nu,q0;\mathfrak{k}). \end{aligned} \tag{87}$$

So far we have calculated the tensor parameters for the field-free case. If there is a central field, decreasing more rapidly at infinity than a Coulomb field, then the only effects are to modify the radial part of the base wave function $\phi_\kappa{}^\mu$ and to multiply these functions by a phase shift $e^{i\Delta_\kappa}$. The tensor parameters are then

$$\begin{aligned} R(\nu q,\kappa\kappa')&=[(2l+1)(2l'+1)|\kappa\kappa'|]^{\frac{1}{2}} \\ &\quad\times(-)^{\nu+q+j'-\frac{1}{2}}i^{l(\kappa')-l(\kappa)}\exp[i(\Delta_\kappa-\Delta_{\kappa'})] \\ &\quad\times S(\kappa)S(\kappa')C(ll'\nu;00)W(ljl'j';\tfrac{1}{2}\nu) \\ &\quad\times D(\nu,q0;\mathfrak{k}). \end{aligned} \tag{88}$$

For the Coulomb field the well-known logarithmic term in the phase shift enters, but this clearly drops out in the above expression since only differences of the phase shifts enter and the logarithmic terms are independent of κ and the phase shift difference is therefore independent of r. Hence the above expression applies to the Coulomb case as well. The phase shift Δ_κ is equal to $\delta_\kappa(Z)-\delta_\kappa(0)$ in the notation of reference (67) and the arguments here refer to the atomic number; thus Δ_κ vanishes in the case of no field. The phase shift $\delta_\kappa(0)=\frac{1}{2}\pi[\frac{1}{2}(1+S(\kappa))-|\kappa|]$. The absolute *sign* of the phase shifts (the relative signs are unique) in Eq. (88) above is determined from the requirement that the Green's function for the emission of the electrons corresponds to outgoing waves at infinity. Strictly speaking, this method of deriving the tensor parameters from the field-free tensor parameters is ambiguous in regard to the absolute sign of the phase shifts. A more thorough derivation which treats this point in detail is given in (68).

Since the tensor parameters of the initial state ($\kappa=-1$) are $R(\nu q;\text{in})=\delta_{\nu 0}\delta_{q0}$ it is now a simple matter to determine the radiation parameters, using the procedure discussed in connection with Eq. (17b). The result is

$$R(\nu q, LL'\pi)=\sum_{\kappa\kappa'}|\kappa\kappa'|^{\frac{1}{2}}W(\tfrac{1}{2}jL'\nu;Lj')Q(\kappa L\sigma)\times Q^*(\kappa'L'\sigma')R(\nu q,\kappa\kappa'). \quad (89)$$

The $Q(\kappa L\sigma)$ are the reduced matrix elements for the electronic transition from $\kappa_i=-1$ to κ by an outgoing 2^L pole electromagnetic field and parity $(-)^{L+\sigma}$. These are defined explicitly in references (68) and (69). The same sign and normalization conventions are used here. The σ parameter (Sec. III-A) is used in the Q's in lieu of the parity symbol π. Since for pure multipoles we have two possible κ values for the final state ($\kappa=L$, $-L-1$ for electric multipoles and $\kappa=-L$, $L+1$ for magnetic multipoles), there will always be coherent mixing of the final states, and thus the Coulomb phase shifts, as well as the relative size of the electronic matrix elements, will become important. In general, exact calculation of these matrix elements using Dirac wave functions for the Coulomb field is consequently essential [see further, (68)].

The radiation tensor parameters for a pure 2^L multipole ($\sigma=\sigma'$) are then

$$\begin{aligned}R(\nu q, LL\sigma)=\sum_{\kappa\kappa'}&[(2l+1)(2l'+1)]^{\frac{1}{2}}|\kappa\kappa'|Q(\kappa L\sigma)\\&\times Q^*(\kappa'L\sigma)(-)^{L+q}i^{l(\kappa')-l(\kappa)}\\&\times\exp[i(\Delta_\kappa-\Delta_{\kappa'})]C(ll'\nu;00)\\&\times S(\kappa)S(\kappa')D(\nu,q0;\mathfrak{k})W(ljl'j';\tfrac{1}{2}\nu)\\&\times W(LjLj';\tfrac{1}{2}\nu)\end{aligned} \quad (90)$$

and $l\equiv l(\kappa)$, $l'\equiv l(\kappa')$.

Using the fact that for 2^L-poles the allowed κ values are L, $-L-1$ one finds in both cases, $l(\kappa)=L$. The above equation then takes the simpler form

$$\begin{aligned}R(\nu q,&LLe)\\&=(2L+1)(-)^{L+q}C(LL\nu;00)D(\nu,q0;\mathfrak{k})\\&\times\{L^2|Q(LLe)|^2W^2(LLL-\tfrac{1}{2}L-\tfrac{1}{2};\nu\tfrac{1}{2})\\&+(L+1)^2|Q(-L-1Le)|^2W^2(LLL+\tfrac{1}{2}L+\tfrac{1}{2};\nu\tfrac{1}{2})\\&-2L(L+1)Re[(e^{i\Delta_L}Q(LLe))^*e^{i\Delta_{-L-1}}\\&\quad\times Q(-L-1,Le)]W^2(LLL-\tfrac{1}{2}L+\tfrac{1}{2};\nu\tfrac{1}{2})\}.\end{aligned} \quad (91)$$

The occurrence of $C(LL\nu;00)$ shows that ν is an even integer. If we use the relation

$$C(LL\nu;1-1)=C(LL\nu;00)\frac{\nu(\nu+1)-2L(L+1)}{2L(L+1)}$$

for ν even and the explicit formulas for the Racah coefficients (see Appendix, A8, A9, A10) then $R(\nu q,LLe)$ takes the form

$$\begin{aligned}R(\nu q,LLe)=&\frac{2L(L+1)(2L+1)}{\nu(\nu+1)-2L(L+1)}\\&\times(-)^{L+q}C(LL\nu;1-1)D(\nu,q0;\mathfrak{k})\\&\times\Bigg\{|Q(LLe)|^2\frac{(2L-\nu)(2L+\nu+1)}{4(2L+1)^2}\\&+|Q(-L-1,L,e)|^2\frac{(2L+1-\nu)(2L+2+\nu)}{4(2L+1)^2}\\&-\frac{\nu(\nu+1)}{2(2L+1)^2}Re[(e^{i\Delta_L}Q(LLe))^*e^{i\Delta_{-L-1}}\\&\qquad\times Q(-L-1,L,e)]\Bigg\}\end{aligned} \quad (92)$$

or

$$\begin{aligned}R(\nu q,LLe)=&\{(-)^{L+q+1}(2L+1)\\&\times C(LL\nu;1-1)D(\nu,q0;\mathfrak{k})\}\\&\times\frac{L(L+1)}{2(2L+1)^2}[2L(L+1)-\nu(\nu+1)]^{-1}\\&\times\Bigg[\frac{|T_e|^2}{L}2(2L+1)+2(L+1)(2L+1)\\&\qquad-\nu(\nu+1)\left(\frac{|T_e|^2}{L^2}-\frac{2}{L}\text{Re}(T_e)+1\right)\Bigg],\end{aligned} \quad (93)$$

where we have used the definition

$$T_e\equiv L\frac{e^{i\delta_L}Q(LLe)}{e^{i\delta_{-L-1}}Q(-L-1,L,e)};$$

$$\Delta_\kappa=\delta_\kappa-\frac{\pi}{2}\left[\frac{1+S(\kappa)}{2}-|\kappa|\right]. \quad (94)$$

The term in curly brackets in Eq. (93) will be recognized as just the tensor parameters for pure 2^L pole γ rays. We can therefore write, after a little algebraic manipulation, that the particle parameter b_ν' for conversion electrons [see Eq. (42b)] has the form

$$b_\nu'(LLe)=\frac{|Q(-L-1,L,e)|^2}{2L(2L+1)}(|T_e|^2+L(L+1))\times\left[1+\frac{\nu(\nu+1)}{2L(L+1)-\nu(\nu+1)}\frac{L}{2L+1}\times\frac{|L+1+T_e|^2}{L(L+1)+|T_e|^2}\right]$$
$$=c_e(L)\left[1+\frac{\nu(\nu+1)}{2L(L+1)-\nu(\nu+1)}\frac{L}{2L+1}\times\frac{|L+1+T_e|^2}{L(L+1)+|T_e|^2}\right]\equiv c_e(L)b_\nu(LLe). \quad (95)$$

In this last equation the conversion coefficient, $c_e(L)$ (aside from a factor-reference (69)), has been inserted, so that the $b_\nu(LLe)$ may be normalized to $b_0=1$. Thus

$$b_\nu(LLe)=1+\frac{\nu(\nu+1)}{2L(L+1)-\nu(\nu+1)}\frac{L}{2L+1}\times\frac{|L+1+T_e|^2}{L(L+1)+|T_e|^2}. \quad (95a)$$

The tensor parameters for pure magnetic multipoles may also be put in this explicit form by similar manipulation, involving explicit formulas for the Racah coefficients given in the Appendix (A6, A8, A9, A10). The result is

$$R(\nu q,LLm)=\{(-)^{L+1+q}(2L+1)\times C(LL\nu;1-1)D(\nu,q0;\mathfrak{k})\}\times\frac{|Q(-LLm)|^2L}{2(L+1)(2L+1)}(L+1+L|T_m|^2)\times\left[1+\frac{\nu(\nu+1)}{2L(L+1)-\nu(\nu+1)}\times\frac{L(L+1)}{2L+1}\frac{|1-T_m|^2}{L+1+L|T_m|^2}\right]$$
$$=R(\nu q,LL;\gamma)c_m(L)b_\nu(LLm). \quad (96)$$

The notation is $c_m(L)\equiv$ the 2^L multipole conversion coefficient (69), that is,

$$c_m(L)=\frac{L}{2(L+1)(2L+1)}|Q(-L,L,m)|^2\times(L+1+L|T_m|^2) \quad (96a)$$

and

$$T_m\equiv-\frac{L+1}{L}\frac{e^{i\delta_{L+1}}Q(L+1,L,m)}{e^{i\delta_{-L}}Q(-L,L,m)}. \quad (96b)$$

Therefore,

$$b_\nu(LLm)=1+\frac{\nu(\nu+1)}{2L(L+1)-\nu(\nu+1)}\times\frac{L(L+1)}{2L+1}\frac{|1-T_m|^2}{L+1+L|T_m|^2}. \quad (97)$$

From Eqs. (95a) and (97) it is clear that for both electric and magnetic transitions the normalized coefficients $b_\nu(LL;e/m)$ for $\nu>2$ are readily obtained from the corresponding coefficients $b_2(LL;e/m)$ by the recurrence formula

$$b_\nu(LL;e/m)=1+\frac{\nu(\nu+1)[L(L+1)-3]}{3[2L(L+1)-\nu(\nu+1)]}\times[b_2(LL;e/m)-1]. \quad (98)$$

As a result, one need tabulate only the coefficients b_2. Tables III(a)–III(l) give $b_2(LL;e)$ and $b_2(LL;m)$ for the K shell for 12 values of Z in the range 10–96, 5 or 6 values of the transition energy ($\equiv kmc^2$) and $L=1$ through 5.

The experimental problem of observing correlations with conversion electrons is somewhat complicated by the multiple scattering that is necessarily present with thick sources. If thick sources are necessary, the requisite corrections can be made as described by Frankel, (32).

For purposes of extrapolation the following limiting cases may be of interest. For small k values, in the case of $b_2(LLe)$, extrapolation toward threshold can be made with the aid of the nonrelativistic limit

$$b_2(LLe)=\frac{L(L+1)}{L(L+1)-3}\quad(\alpha Z\ll1;k\ll1).$$

For the magnetic transitions, the "nonrelativistic" limit for $b_2(LLm)$ is Z, k dependent.

The high-energy limit can be obtained by using the Casimir approximation [asymptotic forms of the radial Dirac functions are used (67)]. One finds immediately that

$$Q(\kappa,L,m)\to Ne^{-i\delta\kappa}(\kappa-1)\frac{1}{[L(L+1)]^{\frac{1}{2}}}(\delta_{\kappa,-L}+\delta_{\kappa,L+1})$$
$$Q(\kappa,L,e)\to Ne^{-i\delta\kappa}(\kappa+1)\frac{1}{[L(L+1)]^{\frac{1}{2}}}(\delta_{\kappa,L}+\delta_{\kappa,-L-1})$$

where N is an irrelevant normalization constant. Thus $T_e\to-(L+1)$ and $T_m\to1$. As a result one finds that both $b_\nu(LLe)$ and $b_\nu(LLm)$ approach the value unity (68). This implies that at high energies the conversion electrons for pure multipoles give the same correlation function as the corresponding cascade with a photon replacing the conversion electron.

For $Z=0$ arbitrary k one finds $b_\nu(LLm)=1$ (for all ν) and

$$b_2(LLe)=1+\frac{12L}{[L(L+1)-3][(L+1)k^2+4L]}.$$

TABLE III(a). b_2 Coefficients for $Z=10$.[a]

	$k=0.3$	0.5	1.0	1.8	3.0	5.0
E_1	−1.8624	−1.6544	−1.0000	−0.1757	0.4157	0.7506
E_2	1.9611	1.9015	1.7023	1.4281	1.2176	1.0942
E_3	1.3209	1.3020	1.2376	1.1458	1.0737	1.0316
E_4	1.1699	1.1600	1.1264	1.0777	1.0389	1.0163
E_5	1.1069	1.1007	1.0796	1.0489	1.0243	1.0100
M_1	0.9288	0.9551	0.9769	0.9875	0.9930	0.9963
M_2	1.0291	1.0186	1.0100	1.0058	1.0035	1.0020
M_3	1.0115	1.0073	1.0040	1.0024	1.0015	1.0010
M_4	1.0070	1.0045	1.0024	1.0015	1.0009	1.0006
M_5	1.0050	1.0032	1.0017	1.0010	1.0007	1.0004

[a] E_L, M_L designate electric, magnetic 2^L-pole transitions, respectively.

TABLE III(b). b_2 Coefficients for $Z=20$.

	$k=0.3$	0.5	1.0	1.8	3.0	5.0
E_1	−1.8539	−1.6419	−1.0000	−0.2077	0.3730	0.7188
E_2	1.9527	1.8850	1.6717	1.4004	1.2044	1.0913
E_3	1.3167	1.2940	1.2219	1.1290	1.0632	1.0270
E_4	1.1670	1.1546	1.1159	1.0661	1.0311	1.0126
E_5	1.1047	1.0966	1.0718	1.0403	1.0184	1.0071
M_1	0.7685	0.8430	0.9119	0.9488	0.9696	0.9830
M_2	1.0931	1.0637	1.0369	1.0226	1.0144	1.0087
M_3	1.0361	1.0248	1.0145	1.0091	1.0061	1.0038
M_4	1.0214	1.0149	1.0087	1.0055	1.0036	1.0024
M_5	1.0149	1.0104	1.0061	1.0038	1.0026	1.0017

TABLE III(c). b_2 Coefficients for $Z=30$.

	$k=0.3$	0.5	1.0	1.8	3.0	5.0
E_1	−1.8452	−1.6292	−1.0000	−0.2411	0.3260	0.6815
E_2	1.9411	1.8631	1.6338	1.3679	1.1877	1.0874
E_3	1.3105	1.2827	1.2018	1.1099	1.0518	1.0225
E_4	1.1625	1.1466	1.1022	1.0531	1.0232	1.0090
E_5	1.1009	1.0903	1.0615	1.0308	1.0126	1.0045
M_1	0.5810	0.6951	0.8130	0.8830	0.9261	0.9560
M_2	1.1646	1.1206	1.0755	1.0491	1.0328	1.0210
M_3	1.0623	1.0460	1.0292	1.0193	1.0132	1.0091
M_4	1.0362	1.0271	1.0172	1.0114	1.0079	1.0054
M_5	1.0246	1.0186	1.0119	1.0079	1.0055	1.0038

TABLE III(d). b_2 Coefficients for $Z=40$.

	$k=0.3$	0.5	1.0	1.8	3.0	5.0
E_1	−1.8361	−1.6160	−1.0000	−0.2759	0.2743	0.6375
E_2	1.9246	1.8333	1.5863	1.3298	1.1703	1.0824
E_3	1.3008	1.2662	1.1764	1.0885	1.0400	1.0174
E_4	1.1549	1.1346	1.0849	1.0392	1.0156	1.0058
E_5	1.0944	1.0804	1.0485	1.0209	1.0075	1.0024
M_1	0.4009	0.5359	0.6898	0.7908	0.8583	0.9095
M_2	1.2298	1.1787	1.1205	1.0831	1.0586	1.0398
M_3	1.0850	1.0668	1.0454	1.0317	1.0228	1.0160
M_4	1.0486	1.0387	1.0266	1.0186	1.0136	1.0096
M_5	1.0325	1.0262	1.0182	1.0128	1.0093	1.0067

TABLE II(c). b_2 Coefficients for $Z=54$.

	$k=0.3$	0.5	1.0	1.8	3.0	5.0
E_1	−1.8231	−1.5966	−1.0000	−0.3273	0.1936	0.5624
E_2	1.8868	1.7698	1.4983	1.2660	1.1402	1.0737
E_3	1.2764	1.2293	1.1303	1.0564	1.0241	1.0113
E_4	1.1347	1.1068	1.0546	1.0200	1.0067	1.0027
E_5	1.0765	1.0579	1.0269	1.0085	1.0022	1.0007
M_1	0.1917	0.3244	0.4961	0.6231	0.7174	0.7974
M_2	1.3024	1.2518	1.1863	1.1390	1.1054	1.0778
M_3	1.1094	1.0919	1.0686	1.0514	1.0393	1.0295
M_4	1.0613	1.0521	1.0393	1.0297	1.0228	1.0172
M_5	1.0403	1.0346	1.0264	1.0200	1.0154	1.0118

TABLE III(f). b_2 Coefficients for $Z=64$.

	$k=0.3$	0.5	1.0	1.8	3.0	5.0
E_1	−1.8129	−1.5819	−1.0000	−0.3661	0.1293	0.4972
E_2	1.8400	1.6984	1.4152	1.2131	1.1162	1.0668
E_3	1.2444	1.1876	1.0905	1.0344	1.0149	1.0085
E_4	1.1082	1.0768	1.0311	1.0088	1.0027	1.0019
E_5	1.0540	1.0354	1.0122	1.0025	1.0006	1.0010
M_1	0.0800	0.1954	0.3566	0.4842	0.5837	0.6736
M_2	1.3410	1.2952	1.2312	1.1817	1.1446	1.1131
M_3	1.1220	1.1064	1.0840	1.0660	1.0527	1.0415
M_4	1.0677	1.0596	1.0474	1.0375	1.0299	1.0236
M_5	1.0441	1.0392	1.0316	1.0251	1.0201	1.0159

For a *mixed* conversion transition the tensor parameters are found from Eq. (89). We specialize now to a magnetic 2^L and electric 2^{L+1} mixture. If we choose the unprimed variables to refer to the magnetic multipole, κ is restricted to $-L$, $L+1$ (and $\sigma=1$) while κ' is restricted to $L+1$, $-L-2$ (and $\sigma'=0$). $R(\nu q)$ then consists of four terms which, by using Eq. (A7) of the Appendix, can be reduced to

$$R(\nu q)[L, L+1, \pi=(-)^L] = (-)^{L+q} D(\nu, q0; \mathfrak{f}) C(LL+1\nu; 1-1)\Omega \quad (99)$$

and

$$\Omega = \tfrac{1}{2}(L+1)\left[\frac{L(L+2)}{(2L+1)(2L+3)}\right]^{\frac{1}{2}} \{Q(-L, L, m) \times Q^*(L+1, L+1, e)e^{i(\Delta_{-L}-\Delta_{L+1})} + Q(-L, L, m) \times Q^*(-L-2, L+1, e)e^{i(\Delta_{-L}-\Delta_{-L-2})} - Q(L+1, L, m)Q^*(L+1, L+1, e) - Q(L+1, L, m)Q^*(-L-2, L+1, e) \times e^{i(\Delta_{L+1}-\Delta_{-L-2})}\}$$

$$= \tfrac{1}{2}(L+1)\left[\frac{L(L+2)}{(2L+1)(2L+3)}\right]^{\frac{1}{2}} \times \{Q(-L, L, m)e^{i\Delta_{-L}} - Q(L+1, L, m)e^{i\Delta_{L+1}}\} \times \{Q(L+1, L+1, e)e^{i\Delta_{L+1}} + Q(-L-2, L+1, e)e^{i\Delta_{-L-2}}\}^*. \quad (99a)$$

Comparing these results with the tensor parameters for mixed multipole γ rays, [Eq. (60a)], and noting again that the γ-ray reduced matrix elements are real, we find

$$b_\nu'[L, L+1, \pi=(-)^L] = \tfrac{1}{2}(L+1)\frac{[L(L+2)]^{\frac{1}{2}}}{(2L+1)(2L+3)} \times \mathrm{Re}\{[Q(-L, L, m)e^{i\Delta_{-L}} - Q(L+1, L, m)e^{i\Delta_{L+1}}] \times [Q(L+1, L+1, e)e^{i\Delta_{L+1}} + Q(-L-2, L+1, e)e^{i\Delta_{-L-2}}]^*\}. \quad (100)$$

TABLE III(g). b_2 Coefficients for $Z=72$.

	$k=0.3$	0.5	1.0	1.8	3.0	5.0
E_1	−1.8041	−1.5693	−1.0000	−0.3986	0.0735	0.4368
E_2	1.7801	1.6155	1.3348	1.1678	1.0968	1.0615
E_3	1.2039	1.1423	1.0572	1.0194	1.0096	1.0074
E_4	1.0772	1.0480	1.0147	1.0031	1.0017	1.0027
E_5	1.0311	1.0171	1.0038	1.0004	1.0009	1.0021
M_1	0.0152	0.1104	0.2520	0.3687	0.4615	0.5460
M_2	1.3647	1.3242	1.2645	1.2158	1.1781	1.1455
M_3	1.1296	1.1160	1.0950	1.0773	1.0637	1.0517
M_4	1.0716	1.0645	1.0533	1.0436	1.0358	1.0292
M_5	1.0464	1.0422	1.0352	1.0289	1.0238	1.0195

TABLE III(h). b_2 Coefficients for $Z=78$.

	$k=0.3$	0.5	1.0	1.8	3.0	5.0
E_1	−1.7968	−1.5592	−1.0000	−0.4240	0.0288	0.3861
E_2	1.7133	1.5325	1.2672	1.1339	1.0832	1.0584
E_3	1.1621	1.1023	1.0344	1.0111	1.0075	1.0082
E_4	1.0499	1.0269	1.0059	1.0013	1.0024	1.0042
E_5	1.0149	1.0065	1.0007	1.0007	1.0022	1.0036
M_1	−0.0189	0.0589	0.1808	0.2838	0.3652	0.4365
M_2	1.3789	1.3427	1.2874	1.2408	1.2039	1.1717
M_3	1.1343	1.1220	1.1025	1.0855	1.0719	1.0601
M_4	1.0739	1.0676	1.0573	1.0479	1.0403	1.0336
M_5	1.0478	1.0440	1.0376	1.0316	1.0266	1.0222

TABLE III(i). b_2 Coefficients for $Z=83$.

	$k=0.5$	1.0	1.8	3.0	5.0
E_1	−1.5499	−1.0000	−0.4460	−0.0103	0.3399
E_2	1.4467	1.2083	1.1073	1.0734	1.0569
E_3	1.0680	1.0188	1.0071	1.0078	1.0100
E_4	1.0125	1.0016	1.0016	1.0041	1.0062
E_5	1.0014	1.0003	1.0020	1.0040	1.0053
M_1	0.0247	0.1279	0.2166	0.2853	0.3407
M_2	1.3562	1.3051	1.2609	1.2255	1.1944
M_3	1.1265	1.1083	1.0922	1.0789	1.0673
M_4	1.0698	1.0603	1.0514	1.0439	1.0373
M_5	1.0453	1.0394	1.0338	1.0289	1.0245

Here one uses the result that interchanging L and $L+1$ in Ω changes this factor into its negative complex conjugate.

This result can be greatly simplified by the introduction of the T_e and T_m given earlier for the pure multipole case, as well as the explicit introduction of the conversion coefficients $c_e(L+1)$ and $c_m(L)$:

$$b_\nu'[L, L+1, \pi=(-)^L]$$

$$=[c_e(L+1)c_m(L)]^{\frac{1}{2}}\left[\frac{L(L+2)}{(2L+1)(2L+3)}\right]^{\frac{1}{2}}$$

$$\times\frac{\mathrm{Re}\left\{e^{i(\theta_e-\theta_m)}\left(1+\frac{L+1}{LT_m}\right)^*\left(1-\frac{L+1}{T_e}\right)\right\}}{\left[\left(1+\frac{L+1}{L|T_m|^2}\right)\left(1+\frac{(L+2)(L+1)}{|T_e|^2}\right)\right]^{\frac{1}{2}}}, \quad (101)$$

TABLE III(j). b_2 Coefficients for $Z=88$.

	$k=0.5$	1.0	1.8	3.0	5.0
E_1	−1.5404	−1.0000	−0.4687	−0.0514	0.2899
E_2	1.3461	1.1503	1.0841	1.0661	1.0573
E_3	1.0367	1.0081	1.0059	1.0097	1.0129
E_4	1.0033	1.0007	1.0036	1.0068	1.0088
E_5	1.0001	1.0018	1.0044	1.0064	1.0075
M_1	−0.0012	0.0820	0.1548	0.2086	0.2457
M_2	1.3678	1.3214	1.2803	1.2468	1.2176
M_3	1.1302	1.1137	1.0985	1.0858	1.0745
M_4	1.0718	1.0630	1.0547	1.0475	1.0410
M_5	1.0465	1.0411	1.0358	1.0312	1.0269

TABLE III(k). b_2 Coefficients for $Z=92$.

	$k=0.5$	1.0	1.8	3.0	5.0
E_1	−1.5321	−1.0000	−0.4875	−0.0856	0.2470
E_2	1.2581	1.1081	1.0696	1.0631	1.0596
E_3	1.0172	1.0044	1.0081	1.0128	1.0166
E_4	1.0004	1.0022	1.0064	1.0097	1.0114
E_5	1.0015	1.0043	1.0069	1.0086	1.0095
M_1	−0.0160	0.0511	0.1105	0.1522	0.1743
M_2	1.3758	1.3335	1.2951	1.2636	1.2362
M_3	1.1330	1.1179	1.1036	1.0912	1.0805
M_4	1.0731	1.0651	1.0572	1.0503	1.0440
M_5	1.0473	1.0424	1.0374	1.0329	1.0287

TABLE III(l). b_2 Coefficients for $Z=96$.

	$k=0.5$	1.0	1.8	3.0	5.0
E_1	−1.5231	−1.0000	−0.5069	−0.1213	0.2013
E_2	1.1701	1.0734	1.0603	1.0634	1.0643
E_3	1.0050	1.0042	1.0115	1.0173	1.0207
E_4	1.0013	1.0057	1.0103	1.0132	1.0146
E_5	1.0046	1.0075	1.0098	1.0112	1.0117
M_1	−0.0253	0.0258	0.0719	0.1019	0.1106
M_2	1.3828	1.3446	1.3093	1.2801	1.2548
M_3	1.1354	1.1214	1.1081	1.0965	1.0862
M_4	1.0744	1.0670	1.0596	1.0531	1.0471
M_5	1.0480	1.0435	1.0389	1.0346	1.0306

where we have defined

$$\frac{Q(L+1, L, m)}{|Q(L+1, L, m)|}=e^{i\theta_m}, \quad (101a)$$

$$\frac{Q(L+1, L+1, e)}{|Q(L+1, L+1, e)|}=e^{i\theta_e}. \quad (101b)$$

This is just the result given in reference (70). The b_ν' here defined is that coefficient by which one multiplies the cross term in a mixed 2^L, 2^{L+1} multipole γ-ray correlation to convert this term into the corresponding term of a mixed conversion electron correlation. It should be noted that this coefficient is independent of ν.

Just as for the pure multipole cases, we normalize by dividing out the conversion coefficient terms, in this case removing $[c_e(L+1)c_m(L)]^{\frac{1}{2}}$. Let

$$b[L, L+1, \pi=(-)^L]=b_\nu'(L, L+1, \pi)[c_e(L+1)c_m(L)]^{\frac{1}{2}}.$$

Then the correlation function for a cascade involving a mixed conversion electron (E_{L+1}, M_L mixture) and a

TABLE IV(a). Mixed conversion particle parameter b for $k=0.3$.

Z	$L=1$	2	3	4
10	−0.1928	−0.1899	−0.1898	−0.1909
20	−0.1989	−0.2083	−0.2166	−0.2243
30	−0.1979	−0 2287	−0.2491	−0.2669
40	−0.1894	−0.2536	−0.2905	−0.3225
54	−0.1483	−0.2916	−0.3692	−0.4311
64	−0.08891	−0.3130	−0.4353	−0.5235
72	−0.006616	−0.3200	−0.4825	−0.5857
78	0.08917	−0.2972	−0.4953	−0.6059

TABLE IV(b). Mixed conversion particle parameter b for $k=0.5$.

Z	$L=1$	2	3	4
10	−0.3094	−0.3032	−0.3016	−0.3020
20	−0.3203	−0.3288	−0.3365	−0.3447
30	−0.3244	−0.3567	−0.3772	−0.3958
40	−0.3198	−0.3877	−0.4251	−0.4576
54	−0.2924	−0.4383	−0.5106	−0.5658
64	−0.2490	−0.4759	−0.5785	−0.6460
72	−0.1908	−0.5003	−0.6242	−0.6947
78	−0.1266	−0.5126	−0.6486	−0.7094
83	−0.05448	−0.5077	−0.6527	−0.7118
88	0.03913	−0.4913	−0.6447	−0.6951
92	0.1322	−0.4610	−0.6231	−0.6728
96	0.2419	−0.4144	−0.5895	−0.6410

TABLE IV(c). Mixed conversion particle parameter b for $k=1.0$.

Z	$L=1$	2	3	4
10	−0.5417	−0.5324	−0.5291	−0.5287
20	−0.5557	−0.5643	−0.5715	−0.5792
30	−0.5623	−0.5948	−0.6150	−0.6324
40	−0.5593	−0.6267	−0.6610	−0.6869
54	−0.5324	−0.6665	−0.7209	−0.7563
64	−0.4902	−0.6867	−0.7552	−0.7903
72	−0.4370	−0.6976	−0.7697	−0.7986
78	−0.3822	−0.6966	−0.7694	−0.7907
83	−0.3246	−0.6882	−0.7620	−0.7792
88	−0.2547	−0.6717	−0.7461	−0.7596
92	−0.1892	−0.6490	−0.7241	−0.7404
96	−0.1148	−0.6272	−0.7056	−0.7181

pure γ multipole is

$$W(\beta)=I_m c_m(L)W_m(\beta)+I_e c_e(L+1)W_e(\beta)$$
$$\pm 2(I_e I_m)^{\frac{1}{2}}[c_m(L)c_e(L+1)]^{\frac{1}{2}} bW_{em}(\beta). \quad (102)$$

Here the conversion coefficients can be taken from reference (69) since only a common factor has been dropped. In Eq. (102) I_m and I_e are proportional to the intensities of the magnetic and electric radiations in the mixed transition (Eq. 67) and the sign in the third (interference) term depends on the nuclear structure and must be regarded as an adjustable parameter. This sign is the same as in the mixed γ-pure γ-correlation. In Eq. (102) W_m is the correlation function for a pure magnetic 2^L pole (M_L) conversion electron replacing the mixed radiation

$$W_m(\beta)=\sum_\nu b_\nu(LLm)A_\nu P_\nu(\cos\beta), \quad (102a)$$

where A_ν is obtained from Eq. (69a) by setting $L_1=L$.

TABLE IV(d). Mixed conversion particle parameter b for $k=1.8$.

Z	$L=1$	2	3	4
10	−0.7533	−0.7473	−0.7451	−0.7449
20	−0.7608	−0.7709	−0.7785	−0.7850
30	−0.7609	−0.7906	−0.8075	−0.8200
40	−0.7504	−0.8063	−0.8314	−0.8472
54	−0.7104	−0.8159	−0.8506	−0.8675
64	−0.6574	−0.8100	−0.8505	−0.8651
72	−0.5958	−0.7979	−0.8400	−0.8501
78	−0.5363	−0.7819	−0.8248	−0.8317
83	−0.4768	−0.7634	−0.8075	−0.8129
88	−0.4078	−0.7411	−0.7868	−0.7901
92	−0.3453	−0.7170	−0.7640	−0.7703
96	−0.2763	−0.6955	−0.7455	−0.7489

TABLE IV(e). Mixed conversion particle parameter b for $k=3.0$.

Z	$L=1$	2	3	4
10	−0.8826	−0.8805	−0.8805	−0.8813
20	−0.8829	−0.8909	−0.8977	−0.9033
30	−0.8770	−0.8981	−0.9101	−0.9181
40	−0.8614	−0.8991	−0.9148	−0.9236
54	−0.8149	−0.8878	−0.9093	−0.9170
64	−0.7564	−0.8674	−0.8926	−0.8990
72	−0.6895	−0.8443	−0.8717	−0.8761
78	−0.6255	−0.8230	−0.8525	−0.8544
83	−0.5623	−0.8009	−0.8325	−0.8339
88	−0.4897	−0.7758	−0.8101	−0.8108
92	−0.4248	−0.7535	−0.7903	−0.7912
96	−0.3540	−0.7296	−0.7696	−0.7705

TABLE IV(f). Mixed conversion particle parameter b for $k=5.0$.

Z	$L=1$	2	3	4
10	−0.9507	−0.9501	−0.9509	−0.9521
20	−0.9481	−0.9531	−0.9572	−0.9604
30	−0.9410	−0.9511	−0.9579	−0.9632
40	−0.9263	−0.9463	−0.9549	−0.9590
54	−0.8835	−0.9266	−0.9382	−0.9410
64	−0.8271	−0.9014	−0.9162	−0.9184
72	−0.7591	−0.8759	−0.8940	−0.8941
78	−0.6915	−0.8508	−0.8721	−0.8722
83	−0.6229	−0.8266	−0.8514	−0.8523
88	−0.5430	−0.8004	−0.8292	−0.8296
92	−0.4712	−0.7763	−0.8085	−0.8110
96	−0.3931	−0.7527	−0.7888	−0.7909

Similarly W_e is the correlation function for a pure electric 2^{L+1} pole (E_{L+1}) conversion electron replacing the mixed radiation

$$W_e(\beta)=\sum_\nu b_\nu(L+1,\, L+1,\, e)A_\nu P_\nu(\cos\beta), \quad (102b)$$

and here A_ν is obtained from Eq. (69a) by replacing L_1 by $L+1$. The interference term W_{em} is given in Eq. (70c) if one sets $L_1=L$, $L_1'=L+1$.

Of course, as mentioned above, the interference term (which is absent in the total conversion coefficient) constitutes a sensitive indicator of mixtures. Numerical results for this mixed conversion correlation coefficient $b[L, L+1, \pi=(-)^L]$ are given in Table IV(a)–IV(f).

Corresponding to the discussion of limiting cases for the pure conversion transitions one may examine the coefficient b for the mixed conversion transition for these limits. It is of interest to inquire whether in the high-energy limit the conversion electrons in a mixed transition behave like photons as was the case for the pure transitions. To examine this question in a general way we consider the high-energy limit of $R(\nu q)$. By use of Eq. (89) and the notation $\epsilon=(-)^{\sigma}$, one sees that

$$R(\nu q, LL'\pi)\rightarrow|N|^2(-)^{L+q}[L(L+1)L'(L'+1)]^{-\frac{1}{2}}D(\nu, q0;\mathbf{f})$$
$$\times\sum_{\kappa\kappa'}[(2l+1)(2l'+1)]^{\frac{1}{2}}|\kappa\kappa'|(\kappa+\epsilon)(\kappa'+\epsilon')$$
$$\times(\delta_{\kappa,\epsilon L}+\delta_{\kappa,-\epsilon(L+1)})(\delta_{\kappa',\epsilon'L'}+\delta_{\kappa',-\epsilon'(L'+1)})$$
$$\times C(ll'\nu;00)W(ljl'j';\tfrac{1}{2}\nu)W(LjL'j';\tfrac{1}{2}\nu). \quad (89a)$$

There are three cases to consider: (a) electric 2^L-electric $2^{L'}$ poles, (b) magnetic 2^L-magnetic $2^{L'}$ poles (for both of these $L-L'$ is an even integer), and (c) electric 2^L-magnetic $2^{L'}$ poles (here $L-L'$ is an odd integer). We note that for all cases we have $|\kappa|(\kappa+\epsilon)=S(\kappa)L(L+1)$.

Consider the $E-E$ case first. Then $l=L$, $l'=L'$ and we have

$$R(\nu q, LL';EE)\rightarrow|N|^2(-)^{L+q}$$
$$\times[L(L+1)(2L+1)L'(L'+1)(2L'+1)]^{\frac{1}{2}}C(LL'\nu;00)$$
$$\times D(\nu, q0;\mathbf{f})\sum_{\kappa\kappa'}S(\kappa)S(\kappa')W^2(LjL'j';\tfrac{1}{2}\nu)$$
$$=\tfrac{1}{2}|N|^2(-)^{L+q+1}[(2L+1)(2L'+1)]^{\frac{1}{2}}C(LL'\nu;1-1)D(\nu, q0;\mathbf{f}),$$

where we have used the results

$$\sum_{\kappa\kappa'}S(\kappa)S(\kappa')W^2(LjL'j';\tfrac{1}{2}\nu)=\frac{L(L+1)+L'(L'+1)-\nu(\nu+1)}{4L(L+1)L'(L'+1)}$$

and

$$C(LL'\nu;1-1)=C(LL'\nu;00)\frac{\nu(\nu+1)-L(L+1)-L'(L'+1)}{2[L(L+1)L'(L'+1)]^{\frac{1}{2}}}$$

for ν an even integer.

From Eq. (61) we find therefore

$$R(\nu q, LL';EE)\rightarrow\tfrac{1}{2}|N|^2R(\nu q, LL';\gamma)$$

using $L-L'$ = even integer.

Exactly the same results holds for the magnetic-magnetic case if we first employ the identity given in (A6) of the Appendix.

The $E-M$ case presents no additional difficulties if we use the identity given in (A7) of the Appendix. Using this one gets

$$R(\nu q, LL';E-M)\rightarrow\tfrac{1}{2}|N|^2(-)^{L+q+1}$$
$$\times[(2L+1)(2L'+1)]^{\frac{1}{2}}C(LL'\nu;1-1)D(\nu, q0;\mathbf{f})$$
$$=-\tfrac{1}{2}|N|^2R(\nu q, LL';\gamma).$$

(The use of the identity (A7) shows, moreover, that the $b_\nu(LL';E-M)$ are always independent of ν. We found this to be true earlier for the special case of $E_{L+1}-M_L$ mixtures.)

Collecting our results, we can now assert that the high-energy conversion electron correlation is exactly the same as the correlation with a photon replacing the conversion electron, except that all cross-terms involving electric and magnetic multipoles have the *opposite* sign.

The $Z=0$ limit for the $E_{L+1}-M_L$ mixture is found to be

$$b=\left[1+\frac{4}{k^2}\frac{L+1}{L+2}\right]^{-\frac{1}{2}}.$$

The nonrelativistic limit is Z, k dependent.

D. Beta Radiation

1. Description of the Problem

The problem of determining the particle parameters b_ν for β radiation is analogous to the determination of the b_ν for internal conversion electron emission, as treated in III-C above. In the latter case, one considers a bound electron, a K shell electron to be explicit, making a transition to an outgoing wave state. In the β-decay case the initial state is considered to be a Dirac particle without charge or rest mass, a neutrino, which undergoes a transition from a plane wave state into an outgoing wave electron state.[18] The fact that the initial state is characterized by zero charge and rest mass does not change the formalism of III-C in any essential way. The physical measurement in both the β-decay and internal conversion cases is the same, namely, one observes the direction of motion $\mathbf{f}$ of the emitted electron.

The direction of motion of the neutrino and its spin polarization are, of course, unobserved. Consequently, the initial state is random, exactly as was the case for the conversion problem. In contrast, however, we can no longer confine our attention to an initial state with $\kappa=-1$.

The simplicity which arises in the conversion of an s electron ($\kappa=-1$) is twofold. First, the fact that κ is fixed removes a summation and second, this particular value of κ (for which $j=\frac{1}{2}$) allows considerable algebraic simplification. Equivalent remarks would apply for $p_{\frac{1}{2}}$ electrons ($\kappa=1$).

From the point of view of precedure, however, there is a one-to-one correspondence between the treatment of the two problems, although the greater complexity of detail for the β-decay problem tends to obscure this parallelism. The origin of this complexity is to be found in the difference in the interactions. Whereas in the conversion problem one has a field interaction (electromagnetic) with known coupling, in the β-decay problem one deals with a point interaction (between leptons and nucleons) with essentially unknown coupling. This coupling must be some combination of the five relativistic invariants that can be formed with the four Dirac spinors, corresponding to the four spin $\frac{1}{2}$ particles involved, i.e., contraction of the covariants scalar (S), vector (V), tensor (T), axial vector (A) and pseudoscalar (P), (47). We shall consider only pure invariants in the following, although the treatment of mixtures would add no difficulty in principle.

As a model for the β-radiation problem, let us consider the essential steps involved in the conversion problem. By making a multipole expansion of the electromagnetic field, and with the justifiable neglect of the finite size of the nucleus, one separates the interaction into a sum of terms, each of which is factorable into a product of nuclear and electronic matrix elements (76). The separate terms in the expansion are characterized by their transformation properties, indicated by L and π. The analogous step for the β-radiation problem

[18] We follow the customary formulation of the theory in which the emission of *two* particles is equivalent to the emission of a particle and the absorption of an antiparticle. Thus, for negative electron emission the initial state is actually an antineutrino (charge-conjugate neutrino) state. To discuss positron emission one applies the charge conjugation operator to the entire system and, as is well known, this is equivalent to reversing the sign of the external field, that is, Z is changed to $-Z$ in the result for the negative electron case.

For the present problem there is no distinction between Dirac and Majorana neutrinos.

is effected by an expansion of the point interaction (a δ function) into spherical harmonics, Eq. (104), with appropriate modifications for each of the five invariants, $SVTAP$, as required (see Eq. (103b) and the discussion following). However, L and π do not constitute a sufficient description of these "multipoles" and an additional index α is used to distinguish between them. Having separated the nuclear and electronic matrix elements by this procedure, the tensor parameters, and hence the particle parameters $b_\nu(\beta)$, follow exactly as in III-C.

The procedure whereby the β-decay process is described as an analog of the conversion process clearly necessitates the use of the angular momentum representation, in contrast to methods previously used (27, 37, 47). This is, understandably, the natural representation for angular correlation problems. Analogous to the ordering of the contributions of the various possible multipoles in the electromagnetic (or conversion) problems by the retardation expansion, is the corresponding ordering which here leads to the usual classification in terms of orders of forbiddenness of the β-transitions. The results obtained below appear most most directly in terms of a notation different than that already familiar in β-decay theory and therefore tables of transcriptions to the customary notation are given.

The $\beta-x$ correlation has been treated by Falkoff and Uhlenbeck (27) who considered free electrons. The formalism for the case of electrons in a Coulomb field was given by Fuchs (37) and that paper also gives explicit results for the Z dependence valid for light nuclei. For heavy emitters results of sufficient precision require numerical evaluation[19] of certain combinations of electronic radial functions, see Table VII and Table 10 of reference 37.

2. Explicit Calculation

The Hamiltonian density H for the β-decay interaction, corresponding to the emission of negative electrons, is taken to be a scalar formed by the contraction of covariants made up of the lepton and nucleon wave functions, evaluated at the same point in space. The most general form for H is

$$H=\sum_s C_s(\Psi_{N_f}{}^+|O_s|\Psi_{N_i}):(\psi_e{}^+|O_s|\psi_\nu). \qquad (103)$$

In Eq. (103) the sign: means contraction. The $^+$ symbol is the Pauli adjoint, $\psi^+=i\psi^*\gamma_4$. The interactions O_s are

$$\begin{array}{ccccc} S & V & T & A & P \\ 1 & \gamma_\mu & \gamma_\mu\gamma_\nu & \gamma_\mu\gamma_\nu\gamma_\tau & \gamma_\mu\gamma_\nu\gamma_\tau\gamma_\sigma. \end{array} \qquad (103a)$$

Of course, in Eq. (103) above the O_s operators appearing are in the separate nucleon and lepton spaces, respectively.

The C_s can be taken to be real constants. Since we shall treat only pure interactions, we can take $C_s=g\delta_{ss'}$ where g is the coupling constant. The γ_μ are Dirac operators; the Greek indices in Eq. (103a) run from 1 to 4. A product such as $\gamma_\mu\gamma_\nu\gamma_\tau$ implies that all combinations of μ, ν, τ are to be used, subject to $\mu>\nu>\tau$.

It is our aim now to classify the operators in the nucleon space by their rotational properties (L) and parity (π). One can accomplish this most easily by the following two steps: (a) Formally separate the nucleon and lepton spaces by introducing the delta function, $\delta(\mathbf{r}_N-\mathbf{r}_L)$ (integrating over the lepton space), and (b) by classifying the interaction O_s by their 3-space rotational properties. This latter step is easily accomplished by introducing the Dirac direct product notation for the γ's, i.e., $\gamma_4=\rho_3$, $\gamma_k=\rho_2\sigma_k$. Under 3-space rotations the ρ transform as scalars, the $\boldsymbol{\sigma}$ as a vector. Under reflections ($\pi_{Op}=\rho_3P_s$ where P_s mean space inversion), ρ_3, $\boldsymbol{\sigma}$ are unchanged, while ρ_1 and ρ_2 change sign. The interactions O_s become in this representation

$$\begin{array}{cccccc} & S & V & T & A & P \\ \text{Scalar} & 1 & \rho_3 & & i\rho_2 & \rho_1 \\ \text{Vector} & & \rho_2\boldsymbol{\sigma} & i\rho_1\boldsymbol{\sigma} & i\rho_3\boldsymbol{\sigma} & . \\ & & & i\boldsymbol{\sigma} & & \end{array} \qquad (103b)$$

In order to avoid possible confusion we repeat that the "scalar" and "vector" here refer to the transformation properties of the constituents of the O_s operators under 3-space rotations and not under Lorentz transformations. The phase i is fixed by direct substitution of the ρ and $\boldsymbol{\sigma}$ operators in (103a). A real phase (±1) has been ignored since it disappears under contraction.[20] Next one introduces the expansion for the delta function

$$\delta(\mathbf{r}_N-\mathbf{r}_L)=\frac{1}{r^2}\delta(r_N-r_L)\sum_{L,M}Y_L{}^{M*}(\mathbf{r}_N)Y_L{}^M(\mathbf{r}_L). \qquad (104)$$

For the "scalar" part of the interactions, i.e., those with the operators 1, ρ_k, this provides the desired multipole expansion since we may write for the nucleon operators

$$\left\{\begin{array}{c}1\\ \rho_k\end{array}\right\}Y_L{}^{M*}(\mathbf{r}_N)$$
$$\times\left[\int d\Omega\left(\psi_e{}^+\left|\left\{\begin{array}{c}1\\ \rho_k\end{array}\right\}Y_L{}^M(\mathbf{r}_L)\right|\psi_\nu\right)\right]_{r_L=r_N}. \qquad (104a)$$

Here the integration is over the common lepton angular variables. This operator is a function of r_N, it will be observed, but it is customary to evaluate it at $r_N=R$, the nuclear radius. In the sequel we shall refer to the first part (the coefficient of the term in square brackets) as the nuclear operator and the term in square brackets will be referred to as the lepton matrix element.

For the "vector" part of the interactions, one uses instead of the delta function the unit dyadic delta

[19] This work has been undertaken at this laboratory. In addition, it is necessary to change the sign of the phase δ which appears in the formal results of reference (37).

[20] An alternative way to fix the phase i is to check the requirement of commutability with the time reversal operator $K_t=i\sigma_yK_0$.

function:

$$\mathbf{I}\delta(\mathbf{r}_N-\mathbf{r}_L)=\frac{1}{r^2}\delta(r_N-r_L)\sum_{J,M,L}\times\mathbf{\Phi}^{M+}{}_{J,L}(\mathbf{r}_N)\mathbf{\Phi}^M{}_{J,L}(\mathbf{r}_L). \quad (105)$$

The orthonormal vector functions $\mathbf{\Phi}^M{}_{J,L}$ have the operational definitions

$$\mathbf{\Phi}^M{}_{J=L,L-1}=[L(2L+1)]^{-\frac{1}{2}}(r\nabla+L\mathbf{r})Y_L{}^M \quad (105a)$$

$$\mathbf{\Phi}^M{}_{J=L,L}=[L(L+1)]^{-\frac{1}{2}}\mathbf{L}Y_L{}^M \quad (105b)$$

$$\mathbf{\Phi}^M{}_{J=L,L+1}=[(L+1)(2L+1)]^{-\frac{1}{2}}\times(r\nabla-(L+1)\mathbf{r})Y_L{}^M. \quad (105c)$$

An equivalent definition of these vector eigenfunctions was given in Eq. (53a) except that, for convenience, the phase i^l appearing in that definition has been omitted here. (The magnetic quantum number M which replaces P in that definition is not restricted to the values ± 1.) Here $\mathbf{L}$ is again the rotation operator, $-i\mathbf{r}\times\nabla$.

Using this formalism the operators corresponding to (104a) for the "vector" parts become

$$\begin{Bmatrix}1\\ \rho_k\end{Bmatrix}\boldsymbol{\sigma}\cdot\mathbf{\Phi}^{M+}{}_{J,L}\times\left[\int d\Omega\left(\psi_e{}^+\left|\begin{Bmatrix}1\\ \rho_k\end{Bmatrix}\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{J,L}\right|\psi_\nu\right)\right]_{r_L=r_N}. \quad (105d)$$

The terminology, nuclear operator and lepton matrix element, already introduced applies here also.

For low values of J, $L\leqslant 3$ the operators $\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{J,L}$ are the familiar tensor operators which appear explicitly in Konopinski-Uhlenbeck (47). The correspondences are given in Table V. The great advantage of using the functions $\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{J,L}$ is that by means of the operational definitions, Eqs. (105a, b, c), one can readily evaluate the lepton matrix elements, $\int d\Omega(\psi_e{}^+|(1 \text{ or } \rho_k)\boldsymbol{\sigma}\cdot\mathbf{\Phi}|\psi_\nu)$.

For clarity, the various interaction operators are tabulated below in Table VI. These operators are characterized by their rotational properties (L) and their parity (π). Since this is insufficient to distinguish them completely, a third index α is introduced as shown explicitly.

The lepton wave functions are plane waves at infinity (using once more the technique of a screened Coulomb field so that the concept of a plane wave is meaningful) and we again go to the spherical eigenfunction representation. The coefficients of the expansion, in the now familiar way, lead to the tensor parameters of the electron and neutrino [Eq. (88)], the latter being trivial since the neutrino state is random. We denote the electron quantum numbers by κ, μ; the neutrino by κ_ν, μ_ν. We must now evaluate the lepton matrix elements

$$M(\kappa_\nu;L\pi\alpha)=\left[\int d\Omega(\phi_\kappa{}^\mu(e)^+|O_\alpha|\phi_{\kappa_\nu}{}^{\mu_\nu}(\nu))\right]_{r_L=R}. \quad (106)$$

The operators O_α are explicitly given in Table VI.

Making use of the operational form of the $\mathbf{\Phi}^M{}_{J,L}$ one readily evaluates the (lepton) matrix elements, $M(\kappa\kappa_\nu;L\pi\alpha)$. It is convenient to split off the magnetic quantum number dependence, i.e., to define reduced

TABLE V. Correspondence of operators.[a]

Interaction	Operator	$K-U$	L
S	$Y^M{}_L$	β	0
		$\beta\mathbf{r}$	1
V	$\rho_3 Y^M{}_L$	$R^\beta{}_{\mu\nu}/r^2$	2
		1	0
		$\mathbf{r}$	1
	$\rho_3\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L}$	$R_{\mu\nu}/r^2$	2
		$\boldsymbol{\alpha}\times\mathbf{r}$	1
	$\rho_2\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L-1}$	$\boldsymbol{\alpha}$	1
		$A_{\mu\nu}/r$	2
T	$\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L}$	$\beta\boldsymbol{\sigma}\times\mathbf{r}$	1
		$T^\beta{}_{ij}/r^2$	2
	$\rho_1\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L+1}$	$\beta\boldsymbol{\alpha}\cdot\mathbf{r}$	0
	$\rho_1\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L-1}$	$\beta\boldsymbol{\alpha}$	1
		$A^\beta{}_{ij}/r$	2
	$\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L-1}$	$\beta\boldsymbol{\sigma}$	1
		$B^\beta{}_{\mu\nu}/r$	2
		$S^\beta{}_{ijk}/r^2$	3
	$\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L+1}$	$\beta\boldsymbol{\sigma}\cdot\mathbf{r}$	0
A	$\rho_3\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L}$	$\boldsymbol{\sigma}\times\mathbf{r}$	1
		T_{ij}/r^2	2
	$i\rho_2 Y^M{}_L$	γ_5	0
	$\rho_3\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L+1}$	$\boldsymbol{\sigma}\cdot\mathbf{r}$	0
	$\rho_3\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L-1}$	$\boldsymbol{\sigma}$	1
		$B_{\mu\nu}/r$	2
		S_{ijk}/r^2	3
P	$\rho_1 Y^M{}_L$	$\beta\gamma_5$	0
		$\beta\gamma_5\mathbf{r}$	1
		$R^{\beta\gamma_5}{}_{\mu\nu}/r^2$	2

[a] Whereas the operator given here is associated with the use of the Pauli adjoint (in forming matrix elements) the $K-U$ equivalent is associated with the ordinary Hermitian conjugate nuclear wave function.

TABLE VI. Lepton operators.[a]

$\pi=(-1)^L$		$\pi=(-1)^{L+1}$	
α	Operator	α	Operator
S	$Y^M{}_L$	P	$\rho_1 Y^M{}_L$
T_0	$i\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L}$	T_0'	$i\rho_1\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L}$
V	$\rho_3 Y^M{}_L$	A	$i\rho_2 Y^M{}_L$
A_0	$i\rho_3\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L}$	V_0	$\rho_2\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L}$
T_1	$i\rho_1\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L+1}$	T_1'	$i\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L+1}$
T_{-1}	$i\rho_1\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L-1}$	T'_{-1}	$i\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L-1}$
V_1	$\rho_2\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L+1}$	A_1	$i\rho_3\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L+1}$
V_{-1}	$\rho_2\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L-1}$	A_{-1}	$i\rho_3\boldsymbol{\sigma}\cdot\mathbf{\Phi}^M{}_{L,L-1}$

[a] The indexes in the first and third columns, on the T, V, and A symbols refer to the three $\mathbf{\Phi}$ functions introduced in Eqs. (105a, b, c). The prime distinguishes the two operators of different parity for given L.

TABLE VII. Reduced matrix elements.[a]

α	$Q(\kappa\kappa_\nu; L\pi\alpha)$
(a) $\pi=(-1)^L$	
S	$(f_\kappa g_{\kappa_\nu}-g_\kappa f_{\kappa_\nu})(\kappa\|L\|-\kappa_\nu)$
V	$-(f_\kappa g_{\kappa_\nu}+g_\kappa f_{\kappa_\nu})(\kappa\|L\|-\kappa_\nu)$
T_0	$i(\kappa+\kappa_\nu)(f_\kappa g_{\kappa_\nu}+g_\kappa f_{\kappa_\nu})(\kappa\|L\|-\kappa_\nu)N_0$
A_0	$-i(\kappa+\kappa_\nu)(f_\kappa g_{\kappa_\nu}-g_\kappa f_{\kappa_\nu})(\kappa\|L\|-\kappa_\nu)N_0$
T_1	$-i[f_\kappa f_{\kappa_\nu}(L+1-\kappa-\kappa_\nu)+g_\kappa g_{\kappa_\nu}(L+1+\kappa+\kappa_\nu)](\kappa\|L\|-\kappa_\nu)N_1$
T_{-1}	$-i[f_\kappa f_{\kappa_\nu}(-L-\kappa-\kappa_\nu)+g_\kappa g_{\kappa_\nu}(-L+\kappa+\kappa_\nu)](\kappa\|L\|-\kappa_\nu)N_{-1}$
V_1	$[f_\kappa f_{\kappa_\nu}(L+1-\kappa-\kappa_\nu)-g_\kappa g_{\kappa_\nu}(L+1+\kappa+\kappa_\nu)](\kappa\|L\|-\kappa_\nu)N_1$
V_{-1}	$[f_\kappa f_{\kappa_\nu}(-L-\kappa-\kappa_\nu)-g_\kappa g_{\kappa_\nu}(-L+\kappa+\kappa_\nu)](\kappa\|L\|-\kappa_\nu)N_{-1}$
(b) $\pi=(-)^{L+1}$	
P	$i(f_\kappa f_{\kappa_\nu}+g_\kappa g_{\kappa_\nu})(\kappa\|L\|\kappa_\nu)$
A	$i(f_\kappa f_{\kappa_\nu}-g_\kappa g_{\kappa_\nu})(\kappa\|L\|\kappa_\nu)$
T_0'	$i(\kappa_\nu-\kappa)(f_\kappa f_{\kappa_\nu}-g_\kappa g_{\kappa_\nu})(\kappa\|L\|\kappa_\nu)N_0$
V_0	$-(\kappa_\nu-\kappa)(f_\kappa f_{\kappa_\nu}+g_\kappa g_{\kappa_\nu})(\kappa\|L\|\kappa_\nu)N_0$
T_1'	$-i[f_\kappa g_{\kappa_\nu}(\kappa-\kappa_\nu-L-1)+g_\kappa f_{\kappa_\nu}(\kappa-\kappa_\nu+L+1)](\kappa\|L\|\kappa_\nu)N_1$
T_{-1}'	$i[f_\kappa g_{\kappa_\nu}(-\kappa+\kappa_\nu-L)-f_{\kappa_\nu}g_\kappa(\kappa-\kappa_\nu-L)](\kappa\|L\|\kappa_\nu)N_{-1}$
A_1'	$-i[f_\kappa g_{\kappa_\nu}(-\kappa+\kappa_\nu+L+1)+g_\kappa f_{\kappa_\nu}(\kappa-\kappa_\nu+L+1)](\kappa\|L\|\kappa_\nu)N_1$
A_{-1}'	$i[f_\kappa g_{\kappa_\nu}(\kappa-\kappa_\nu+L)-f_{\kappa_\nu}g_\kappa(\kappa-\kappa_\nu-L)](\kappa\|L\|\kappa_\nu)N_{-1}$

[a] $N_\mu=[1-\delta(\mu,1)+(2-\mu^2)L]^{\frac{1}{2}}[L(L+1)(2L+1)]^{-\frac{1}{2}}$.

matrix elements $Q(\kappa\kappa_\nu; L\pi\alpha)$ (the analog of the $Q(\kappa L\pi)$ of III-C).

$$M(\kappa\kappa_\nu; L\pi\alpha)\equiv Q(\kappa\kappa_\nu; L\pi\alpha)C(Lj_\nu j; M\mu_\nu). \quad (107)$$

Table VII gives the explicit forms for these reduced matrix elements. In Table VII the f_κ and g_κ are radial wave functions for the electron which are given explicitly in reference (67). The notation differs from that used in (67) only in the use of κ as an index. Note also that the normalization is to one particle in a sphere of unit radius. Moreover, in Table VII, f and g are evaluated at R. The radial functions with κ_ν as index refer to the neutrino. The identification of the radial functions can be made by the definition

$$\phi_\kappa{}^\mu=\begin{pmatrix} -if_\kappa\chi_{-\kappa}{}^\mu \\ g_\kappa\chi_\kappa{}^\mu \end{pmatrix},$$

which may be compared with Eq. (83), noting that for the neutrino $E=p$, $(E-1$ and $E+1\rightarrow E)$.

In Table VII the reduced angular matrix elements are given by

$$(\kappa|L|\kappa_\nu)=(-)^{j_\nu-\frac{1}{2}}\left[\frac{(2j_\nu+1)(2l+1)(2l_\nu+1)}{4\pi}\right]^{\frac{1}{2}} \times C(ll_\nu L, 00)W(jlj_\nu l_\nu; \tfrac{1}{2}L), \quad (108)$$

where again $l\equiv l(\kappa)$, $l_\nu\equiv l(\kappa_\nu)$ and similarly for j and j_ν. It is useful to note that $(-\kappa|L|-\kappa_\nu)=(\kappa|L|\kappa_\nu)$. The occurrence of the matrix element with $-\kappa_\nu$ replacing κ_ν in the case $\pi=(-)^L$ should be noted. The two cases, $\pi=(-)^L$ and $\pi=(-)^{L+1}$, can be considered together by introducing the notation $\delta=\pi(-)^{L+1}$. Then the reduced matrix elements in Eq. (108) have the common form $(\kappa|L|\delta\kappa_\nu)$.

The formal work of determining the tensor parameters of the nuclear transition is complete, upon using Eq. (89), which couples the electron and neutrino tensor parameters, noting that now an additional summation, over κ_ν, is required

$$R(\nu q, L\pi\alpha; L'\pi\alpha')=\sum_{\kappa\kappa'\kappa_\nu} Q(\kappa\kappa_\nu; L\pi\alpha) \times Q^*(\kappa'\kappa_\nu; L'\pi\alpha')|\kappa\kappa'|^{\frac{1}{2}}W(jj_\nu\nu L'; Lj') \times R(\nu q, \kappa\kappa'; el), \quad (109)$$

where the tensor parameter for the β-particle $R(\nu q, \kappa\kappa'; el)$ is given explicitly in Eq. (87). The summation is over all κ_ν, which thus sums all states of the neutrino with equal weight but incoherently. The electron states, on the other hand, are seen to interfere coherently, as is to be expected.

The reflection symmetry (that is, symmetry about 90° in the angle between the two radiations) follows from Eq. (109). From Eq. (108) one sees that

$$l(\kappa)+l(\delta\kappa_\nu)+L=\text{even integer}$$

with $\delta=\pi(-)^{L+1}$ and

$$l(\kappa')+l(\delta'\kappa_\nu)+L'=\text{even integer}$$

with $\delta'=\pi(-)^{L'+1}$. From the results of III-C, the tensor parameter for the electron vanishes unless

$$l(\kappa)+l(\kappa')+\nu=\text{even integer}.$$

Moreover, $L-L'$ and $\delta-\delta'$ have the same character (are both even or both odd). Therefore it follows that ν is always an even integer which guarantees the reflection symmetry (since ν is always the degree of the Legendre polynomial in the correlation function). This result is a direct consequence of the fact that the β-transition is characterized by a definite parity change π (so that no interference between π and $-\pi$ is possible) and that only a propagation direction $\mathbf{f}$ is observed.

At this point we may verify from Eq. (109) that isotropy is obtained in the following two cases: (a) the limit of zero electron momentum (since this implies that Q vanishes unless $\kappa=\pm 1$; but then $j(\kappa)=\frac{1}{2}$ and the Racah coefficient $W(jj_\nu\nu L';Lj')$ with $j=j'=\frac{1}{2}$ vanishes unless $\nu=0$, which means isotropy). (b) If the energy spectrum has the allowed shape (this implies that the energy correction factor (47) is independent of the electron momentum, which in turn implies that $\kappa=\pm 1$. Hence $\nu=0$ as in (a).

One can reduce the large number of nuclear operators that are seen to occur in the β-decay problem to manageable proportions, by making the customary "order of forbiddenness" expansion. The physical basis of this is twofold: firstly, the lepton matrix elements, $Q(\kappa\kappa_\nu;L\pi\alpha)$, are functions of the small parameters (pR) and (qR), i.e., the ratio of the nuclear radius to the electron and neutrino de Broglie wavelengths, respectively.[20] At least in the small Z limit, one can classify the Q in terms of the lowest order of (pR) and (qR) occurring. Secondly, some of the nuclear operators involve the off-diagonal operators (in the Dirac space), ρ_1 and ρ_2; hence since small and large components of the nuclear wave function are thereby coupled, the nuclear matrix elements are of the order v/c (where v is the nucleon velocity). This is a small quantity of the same order as (pR) or (qR). We shall evaluate the "order of forbiddenness" in the $Z=0$ limit and assume that this same ordering holds for $Z\neq 0$. Table VIII lists the results.

TABLE VIII. Order of forbiddenness classification.[a]

$\pi=(-1)^L$		$\pi=(-1)^{L+1}$	
α	Order of forbiddenness	α	Order of forbiddenness
S	L	P	$L+1$
T_0	L	T_0'	$L+1$
V	L	A	$L+1$
A_0	L	V_0	$L+1$
T_1	$L+2$	T_1'	$L+1$
T_{-1}	L	T'_{-1}	$L-1$
V_1	$L+2$	A_1	$L+1$
V_{-1}	L	A_{-1}	$L-1$

[a] The operator in each case is as in Table VI.

It will be observed from Table VIII that the well-known correlation between order of forbiddenness and parity is obtained. That is, successive orders of forbiddenness are characterized by opposite parity change and the allowed transitions correspond to parity change "no" ($\pi=1$).

TABLE IX. Interfering operators for the Lth forbidden transitions ($\pi=(-)^L$).[a]

Interaction	Operators
S	$Y^M{}_L$
V	$i\rho_3\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^M{}_{L-1,L-1}$; $\rho_3 Y^M{}_L$; $\rho_2\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^M{}_{L,L-1}$; $\rho_2\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^{M(\mathrm{x})}{}_{L-2;L-1}$
T	$i\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^M{}_{L,L}$; $i\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^{M(\mathrm{x})}{}_{L-2,L-1}$; $i\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^M{}_{L,L-1}$; $i\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^{M(\mathrm{x})}{}_{L-1,L-1}$; $i\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^{M(\mathrm{x})}{}_{L-1,L}$; $i\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^M{}_{L+1,L}$
A	$i\rho_3\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^M{}_{L,L}$; $i\rho_2 Y^{M(\mathrm{x})}{}_{L-1}$; $i\rho_3\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^{M(\mathrm{x})}{}_{L-1,L}$; $i\rho_3\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^M{}_{L+1,L}$
P	$\rho_1 Y^M{}_{L-1}$

[a] The operators marked with a cross (x) are generally omitted as small corrections to operators with the same parity and L but two orders of forbiddenness lower, provided the L value is large enough for this latter operator to exist.

Specification of the order of forbiddenness and the nature of the interaction ($SVTAP$) does not uniquely select a nuclear operator, as is also well known. In general, interference between various operator matrix elements occur, and this is illustrated in Table IX. In this table interferences occur for a given pure interaction wherever there is more than one entry (V, T, A). These interferences are of two types. (a) Interference between operators of different rotational properties (indicated by the first subscript on $\boldsymbol{\Phi}^M{}_{JL}$). This type of interference is the analog of the E_L, $M_{L'}$ multipole mixtures in the case of electromagnetic radiation. (b) Interference between different operators of the same rotational properties. Interferences of type (b) occur in the total intensity of β-emission whereas those of type (a) are present only in angular distributions (37).

For the T and A interactions, for given L, there is a single operator $i\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}^M{}_{L+1,L}$ which does not interfere with any others. This is the unique matrix element case in the Gamow-Teller interactions.

Having determined the $R(\nu q, L\pi\alpha; L'\pi\alpha')$ by Eq. (109), it is now a straightforward matter to define the particle parameters b_ν for β-radiation. The first step is to weight $R(\nu q, L\pi\alpha; L'\pi\alpha')$ by the reduced nuclear matrix elements

$$R(\nu q, LL';\beta)=\sum_{\alpha\alpha'} M_N(L\pi\alpha)M_N^*(L'\pi\alpha') \times R(\nu q, L\pi\alpha; L'\pi\alpha'). \quad (110)$$

The nuclear reduced matrix elements, $M_N(L\pi\alpha)$, are defined as usual to be

$$(\Psi_J{}^\mu|O_\alpha{}^{+(L\pi)}|\Psi_{J'}{}^{\mu'}) \equiv M_N(L\pi\alpha)C(J'LJ;\mu'M)\delta(\mu, M+\mu'). \quad (111)$$

Here the primed Ψ functions refer to initial state and the unprimed Ψ's to the final state. The relative phases of these reduced matrix elements are real, i.e., $M_N(L\pi\alpha)\times M_N^*(L'\pi\alpha')$ is a real number, see Sec. II. The b_ν parameters are then obtained from Eq. (42b).

[20] That is, p and q are the electron and neutrino momenta, respectively.

TABLE X. Partial particle parameters for β radiation.[a]

Matrix element product	b_ν
First forbidden	
S: $\left\|\int \beta \mathbf{r}\right\|^2_{(L=1)}$	$b_0=\frac{1}{4}\left[\frac{1}{3}(q^2+p^2)-\frac{2}{9}\frac{qp^2}{E}+\frac{2\xi}{3}\left(\frac{p^2}{E}-q\right)+\xi^2\right]$ $b_2=\frac{p^2}{6}\left(\frac{2q}{3E}-1-\frac{2\xi}{E}\right)$
V: $\left\|\int \mathbf{r}\right\|^2_{(L=1)}$	$q\rightarrow -q$ in $\left\|\int \beta\mathbf{r}\right\|^2$
$\left\|\int \boldsymbol{\alpha}\right\|^2_{(L=1)}$	$b_0=\frac{1}{4}$
$\pm\left\|\int \boldsymbol{\alpha}\right\|\left\|\int \mathbf{r}\right\|_{(L=1)}$	$b_0=\frac{1}{2}\left(\frac{q}{3}+\frac{p^2}{3E}+\xi\right)$ $b_2=-p^2/3E$
T: $\left\|\int \beta\boldsymbol{\alpha}\right\|^2_{(L=1)}$	Same as $\left\|\int \boldsymbol{\alpha}\right\|^2$
$\left\|\int \beta\boldsymbol{\sigma}\times\mathbf{r}\right\|^2_{(L=1)}$	$b_0=\frac{1}{4}\left[\frac{1}{6}(q^2+p^2)+\frac{2}{9}\frac{qp^2}{E}+\frac{2}{3}\left(\frac{p^2}{E}+q\right)\xi+\xi^2\right]$ $b_2=\frac{p^2}{6}\left(\frac{q}{3E}+\frac{1}{4}+\frac{\xi}{E}\right)$
$\pm\left\|\int \beta\boldsymbol{\sigma}\times\mathbf{r}\right\|\left\|\int \beta\boldsymbol{\alpha}\right\|_{(L=1)}$	$b_0=\frac{1}{2}\left[\frac{1}{3}\left(q+\frac{p^2}{E}\right)+\xi\right]$ $b_2=p^2/6E$
First forbidden—*continued*	
$\sum_{ij}\|B_{ij}^\beta\|^2_{(L=2)}$	$b_0=\frac{1}{48}(q^2+p^2)$ $b_2=\frac{7}{240}p^2$
A: $\left\|\int \boldsymbol{\sigma}\times\mathbf{r}\right\|^2_{(L=1)}$	$q\rightarrow -q$ in $\left\|\int \beta\boldsymbol{\sigma}\times\mathbf{r}\right\|^2$
$\sum_{ij}\|B_{ij}\|^2_{(L=2)}$	Same as $\sum_{ij}\|B_{ij}^\beta\|^2$
P: $\left\|\int \beta\gamma_5\mathbf{r}\right\|^2_{(L=1)}$	Same as $\left\|\int \beta\mathbf{r}\right\|^2$
Second forbidden	
S: $\sum_{ij}\|R_{ij}^\beta\|^2_{(L=2)}$	$b_0=\frac{q^4+p^4}{120}+\frac{1}{36}q^2p^2-\frac{qp^2}{90E}(q^2+p^2)+\xi\left[\frac{1}{60}\left(\frac{p^4}{E}-2q^2\right)-\frac{1}{36}qp^2\left(1-\frac{2q}{E}\right)\right]+\frac{\xi^2}{12}\left(q^2+\frac{1}{4}p^2\right)$ $b_2=\frac{p^2}{30}\left\{\frac{p^2}{2}+\frac{7}{6}q^2-\frac{1}{15}\frac{q}{E}(7q^2+10p^2)+\xi\left[\frac{3p^2+7q^2}{3E}-\frac{7q}{6}\right]+\frac{7}{8}\xi^2\right\}$ $b_4=-\frac{p^4}{60}\left(\frac{3}{4}-\frac{q}{E}+\frac{3}{2}\frac{\xi}{E}\right)$

[a] The cross terms which occur in Table X are to be taken only once since a factor 2 has been inserted in the corresponding b_ν.

The determination of the $b_\nu(\beta)$ is complicated by the coherent mixing of different operators, for different α, having the same L and π, and the same order of forbiddenness. Table IX shows that this occurs for the V, T, A interactions (for example, T_0 and T_{-1} mix as do V and V_{-1}).

In order to give explicit results for the parameters b_ν it would be necessary to evaluate the nuclear matrix elements $M_N(L\pi\alpha)$ in the cases where interference occurs. Then the b_ν are obtained at once from Eqs. (42b), (109) and (110). In the absence of information concerning the nuclear matrix elements one is forced to resort to the usual artifice of treating these as adjustable parameters. In the cases where there is no interference between matrix elements with different L values, the matrix elements which occur in the angular correlation are the same as those which occur in the total intensity. Thus, the pertinent matrix element could be empirically determined from an analysis of the β spectrum. In any case it is important to define partial parameters $b_\nu(LL';\beta\alpha\alpha')$ as follows:

$$b_\nu(LL';\beta\alpha\alpha')=\frac{R(\nu q,\, L\pi\alpha,\, L'\pi\alpha')}{R(\nu q,\, LL';\gamma)}. \tag{112}$$

When there are no interferences (for example S, P interactions and the favorable parity change transitions for Gamow-Teller interactions) these parameters, defined by Eq. (112), constitute the factors by which the $\gamma-x$ correlation is modified. When interference is present, the required parameters are weighted sums of the partial parameters of Eq. (112) with weight factors given by the product of matrix elements appearing in Eq. (110).

TABLE X.—*Continued.*

Matrix element product	b_ν
Second forbidden—*continued*	
V: $\left\|\int \boldsymbol{\alpha}\times\mathbf{r}\right\|^2_{(L=1)}$	Same as $\left\|\int \boldsymbol{\sigma}\times\mathbf{r}\right\|^2$
$\sum_{ij} \|R_{ij}\|^2_{(L=2)}$	$q\rightarrow -q$ in $\sum_{ij} \|R_{ij}{}^\beta\|^2$
$\sum_{ij} \|A_{ij}\|^2_{(L=2)}$	Same as $\sum_{ij} \|B_{ij}\|^2$
$\pm\|\sum_{ij} R_{ij}A_{ij}{}^*\|_{(L=2)}$	$b_0=\frac{1}{12}\left[\frac{1}{5}\left(q^3+\frac{p^4}{E}\right)+\frac{1}{3}qp^2\left(1+\frac{q}{E}\right)+\xi\left(q^2+\frac{1}{2}p^2\right)\right]$
	$b_2=\frac{p^2}{30}\left[\frac{p^2}{E}+\frac{7}{6}q\left(1+\frac{q}{E}\right)+\frac{7}{4}\xi\right]$
	$b_4=-\frac{1}{40}\frac{p^4}{E}$
T: $\sum_{ij} \|A_{ij}{}^\beta\|^2_{(L=2)}$	Same as $\sum_{ij} \|B_{ij}\|^2$
$\sum_{ij} \|T_{ij}{}^\beta\|^2_{(L=2)}$	$b_0=\frac{1}{360}\left[\frac{1}{2}(q^2+p^2)+\frac{5}{3}q^2p^2+\frac{qp^2}{E}(p^2+q^2)\right]+\frac{\xi}{120}\left[q^3+\frac{5}{6}qp^2+\frac{1}{2}\frac{p^4}{E}+\frac{5}{3}\frac{q^2p^2}{E}\right]+\frac{5\xi^2}{240}\left(q^2+\frac{1}{4}p^2\right)$
	$b_2=\frac{p^2}{60}\left[\frac{1}{6}\left(\frac{p^2}{2}+\frac{7}{6}q^2+\frac{qp^2}{E}\right)+\frac{\xi}{4}\left(\frac{7}{3}q+\frac{p^2}{E}\right)+\frac{7}{16}\xi^2\right]$
	$b_4=\frac{p^4}{360}\left(\frac{q}{E}+\frac{3\xi}{2E}+\frac{1}{2}\right)$
$\pm\|\sum_{ij} T_{ij}{}^\beta A_{ij}{}^{\beta*}\|_{(L=2)}$	$b_0=\frac{1}{120}\left\{q^3+\frac{p^4}{E}+\frac{5}{3}qp^2\left(1+\frac{q}{E}\right)+5\xi\left(q^2+\frac{1}{2}p^2\right)\right\}$
	$b_2=\frac{p^2}{120}\left(\frac{7}{3}q+\frac{p^2}{E}+\frac{7}{2}\xi\right)$
	$b_4=\frac{1}{120}\frac{p^4}{E}$
$\sum_{ijk} \|S_{ijk}{}^\beta\|^2_{(L=3)}$	$b_0=\frac{1}{864}\left[\frac{1}{5}(p^2+q^2)+\frac{2}{3}q^2p^2\right]$
	$b_2=\frac{p^2}{3780}\left(p^2+\frac{7}{3}q^2\right)$
	$b_4=\frac{11p^4}{15120}$
A: $\sum_{ij} \|T_{ij}\|^2_{(L=2)}$	$q\rightarrow -q$ in $\sum_{ij} \|T_{ij}{}^\beta\|^2$
$\sum_{ijk} \|S_{ijk}\|^2_{(L=3)}$	Same as $\sum_{ijk} \|S_{ijk}{}^\beta\|^2$
P: $\sum_{ij} \|R_{ij}{}^{\beta\gamma_5}\|^2_{(L=2)}$	Same as $\sum_{ij} \|R_{ij}{}^\beta\|^2$

Pending numerical work in progress we give the analytical results of Fuchs (37), valid for light elements, for the $b_\nu(LL';\beta\alpha\alpha')$ in Table X below. In this table the notation in common use has been employed. E is the total energy of the electron ($E=(p^2+1)^{\frac{1}{2}}$ and $q=E_0-E$, where E_0 is the end point energy). The Coulomb field effect is contained in the terms with $\xi\equiv\alpha Z/2R$, where α is the fine structure constant. When no entry is given the corresponding parameter (b_ν) is zero. In the table the pseudoscalar interaction, $\beta\gamma_5\mathbf{r}$, has been grouped with the first forbidden case although in actuality the pseudoscalar interaction can only enter in mixtures and from this point of view it might more appropriately be classed with second forbidden (in harmony with the discussion above). A similar remark applies to $\sum_{ij}|R_{ij}{}^{\beta\gamma_5}|^2$.

The matrix elements

$$\int \boldsymbol{\alpha},\quad \int \beta\boldsymbol{\sigma}\cdot\mathbf{r},\quad \int \beta\boldsymbol{\alpha},\quad \int \beta\boldsymbol{\alpha}\cdot\mathbf{r},\quad \int \boldsymbol{\sigma}\cdot\mathbf{r},\quad \int \beta\gamma_5$$

by themselves give isotropy as expected. Since $\xi\gg 1$, especially for heavy elements, the matrix elements

$$\int \mathbf{r},\quad \int \beta\mathbf{r},\quad \int \boldsymbol{\sigma}\times\mathbf{r},\quad \int \beta\boldsymbol{\sigma}\times\mathbf{r},\quad \int \beta\gamma_5\mathbf{r},\quad \int \boldsymbol{\alpha}\times\mathbf{r}$$

by themselves give a very weak correlation ($b_0\gg b_2$). The matrix elements

$$B_{ij},\ B_{ij}{}^\beta,\ A_{ij},\ A_{ij}{}^\beta,\ S_{ijk},\ S_{ijk}{}^\beta$$

by themselves give a correlation which is virtually Z independent and large. The remaining matrix elements are characterized by a correlation intermediate in magnitude.

In Table X above interference terms of type (b) are explicitly given. For interference terms of type (a) we distinguish two cases; namely, (1) those involving interference with a rotationally symmetric operator ($L=0$) like $\beta\boldsymbol{\sigma}\cdot\mathbf{r}$, or γ^5 and (2) those involving interferences between operators for both of which $L\neq 0$. This procedure is necessary because in the standard correlation $(\gamma-\gamma)$ there is no counterpart for the first case.

Interferences involving one operator with $L=0$ are given by Fuchs (37) for the following cases: $B_{ij}{}^{\beta}$ with $\int\beta\boldsymbol{\sigma}\cdot\mathbf{r}$, B_{ij} and $\int\boldsymbol{\sigma}\cdot\mathbf{r}$, B_{ij} and $\int\gamma_5$. In these cases we give the results in the form of the $\beta-\gamma$ correlation function. Noting that for the B_{ij} operator $L=2$, we find

$$W=B^2\sum_{\nu} A_\nu b_\nu(B_{ij})P_\nu(\cos\beta)+5M_0{}^2C_0 +2BM_0DF_2(Ljj)P_2(\cos\beta),$$

where F_2 is given in Eq. (69b), (note that $j_1=j$) and and $B^2=\sum_{ij}|B_{ij}|^2$, M_0 is the matrix element of $L=0$ operator and $b_\nu(B_{ij})$ is obtained from Table X. The constants C_0 and D for each case are as follows:

$$T:\quad M_0=\int\beta\boldsymbol{\sigma}\cdot\mathbf{r}$$

$$C_0=\frac{1}{9}(q^2+p^2)-\frac{2}{9}\frac{qp^2}{E}+\frac{2\xi}{3}\left(\frac{p^2}{E}-q\right)+\xi^2$$

$$D=(54)^{-\frac{1}{2}}p^2\left(1-\frac{q}{E}+\frac{3\xi}{E}\right).$$

For the axial vector interaction with $M_0=\int\boldsymbol{\sigma}\cdot\mathbf{r}$ the constants are obtained by changing q to $-q$ in the above. The remaining operator gives

$$A:\quad M_0=\int\gamma_5$$

$$C_0=1$$

$$D=6^{-\frac{1}{2}}(p^2/E).$$

For the other interfering terms we can express the results most simply by giving the values of the partial particle parameters b_ν as before. The cases considered were interferences between B_{ij} (or $B_{ij}{}^{\beta}$) with $L=1$ operators for which we may write the matrix element as M_1. Then we give only the particle parameter b_2' for the cross term (since only $\nu=2$ enters in those cases). The correlation function for β-(pure) 2^L pole γ ray is then

$$W=\tfrac{1}{5}B^2\sum_{\nu} A_\nu(2,L)b_\nu(B_{ij})P_\nu +\tfrac{1}{3}M_1{}^2\sum_{\nu} A_\nu(1,L)b_\nu(M_1)P_\nu-2M_1Bb_2'A_2'P_2,$$

where

$$A_2'=(2j+1)^{\frac{1}{2}}G_2(12j_1j)F_2(Lj_2j)$$

is obtained from the mixed-pure $\gamma-\gamma$ correlation. The coefficients b_2' are

$$T:\quad B^2=\sum_{ij}|B_{ij}{}^{\beta}|^2$$

$$M_1=\int\beta\boldsymbol{\sigma}\times\mathbf{r},\quad b_2'=\frac{p^2}{4(30)^{\frac{1}{2}}}\left(\frac{1}{2}+\frac{q}{3E}+\frac{\xi}{E}\right)$$

$$M_1=\int\beta\boldsymbol{\alpha},\quad b_2'=\frac{1}{4(30)^{\frac{1}{2}}}\frac{p^2}{E},$$

$$A:\quad B^2=\sum_{ij}|B_{ij}|^2$$

$$M_1=\int\boldsymbol{\sigma}\times\mathbf{r},\quad b_2'=\frac{p^2}{4(30)^{\frac{1}{2}}}\left(\frac{1}{2}-\frac{q}{3E}+\frac{\xi}{E}\right).$$

As was pointed out by Fuchs (37) the existence of these interference terms can give rise to a situation in which the energy spectrum has very nearly an allowed shape and a strongly anisotropic correlation. An example would be one in which $\int\beta\boldsymbol{\alpha}$ dominates. Alternatively, it is possible to have an almost isotropic correlation together with a strongly energy dependent correction factor for the β spectrum.

In all cases the normalization is such that integration of the angular correlation function gives the energy spectrum for the β transition, that is, b_0 is, to within an energy independent constant, equal to the usual energy-correction factor. Therefore, the integrated angular correlation is obtained by integrating the product of the angular correlation function and the allowed energy spectrum over the appropriate energy range.

IV. MAGNETIC FIELD EFFECTS IN ANGULAR CORRELATION

In this section we wish to consider three questions: (a) the effect on the angular correlation of the magnetic interaction due to the electron shell, (b) the role of an applied magnetic field in the angular correlation when no hyperfine interaction is present, and (c) the spin decoupling effect of an external magnetic field. In connection with (a) the fundamental assumption is made that the nucleus is coupled to a system which remains in a stationary state throughout the nuclear cascade. Our concern here is to indicate at least qualitatively, that the hyperfine interaction does affect the correlation function in an important way. The primary point of interest in connection with (b) is the possibility of measuring the gyromagnetic ratio and the magnetic moment (sign and magnitude) in the short-lived intermediate state. As would be expected, the elimination of hyperfine interaction in this case must be accomplished by the use of suitable prepared sources since the Paschen-Back effect discussed in (c) imposes geometrical requirements which make the measurement considered in (b) impossible.

A. The Correlation Function with Spin-Coupling

The starting point for all the effects here considered is the formalism developed by Goertzel (42). One considers the nucleus as part of a larger system (atom or ion) from which the radiations are emitted. The energy levels in the cascade are designated by the labels A, B, and C for initial, intermediate and final levels, respectively. Due to the spin-coupling these energy levels will now be nondegenerate; the substates of A, B, and C are designated by α, β, and γ, respectively.

While it is not essential to specify the nature of the radiation emitted, it is convenient, for concreteness, to discuss the emission of γ rays. Then in the first transition the radiation emitted is described by the index ρ, the second by σ—these indices giving the propagation vector and polarization of the emitted photons. With only slight changes we follow Goertzel's notation (but with units such that $\hbar=1$, and with a different sign convention on the time dependence); then the equations of "motion" for the probability amplitudes are

$$i\dot{a}_\alpha=\epsilon_\alpha a_\alpha+\sum_{\beta\rho}(\alpha|H_\rho|\beta)b_{\beta\rho}, \tag{113a}$$

$$i\dot{b}_{\beta\rho}=\epsilon_{\beta\rho}b_{\beta\rho}+\sum_{\alpha}(\alpha|H_\rho|\beta)^*a_\alpha+\sum_{\gamma\sigma}(\beta|H_\sigma|\gamma)c_{\gamma\rho\sigma}, \tag{113b}$$

$$i\dot{c}_{\gamma\rho\sigma}=\epsilon_{\gamma\rho\sigma}c_{\gamma\rho\sigma}+\sum_{\beta}(\beta|H_\sigma|\gamma)^*b_{\beta\rho}, \tag{113c}$$

where a, b, and c refer to initial, intermediate, and final state, respectively. Also

$$\epsilon_{\beta\rho}=\epsilon_\beta+\omega_\rho,\quad \epsilon_{\gamma\rho\sigma}=\epsilon_\gamma+\omega_\rho+\omega_\sigma, \tag{113d}$$

where ω_ρ and ω_σ are the quantum energies radiated, while ϵ_α, ϵ_β and ϵ_γ are eigenvalues of the zero-order Hamiltonian where the latter contains everything but the radiation operators H_ρ, H_σ.

With the initial condition

$$a_\alpha=e^{i\phi_\alpha}\quad (t=0),$$

where ϕ_α is a random phase, the above equations can be solved to give $c(t=\infty)$. This is conveniently done, for example, by the Laplace transform method (76). Thus, introducing

$$A_\alpha(s)=\int_0^\infty e^{-st}a_\alpha(t)dt$$

and similar definitions for $B_{\rho\sigma}(s)$ and $C_{\gamma\rho\sigma}(s)$ in terms of $b_{\beta\rho}$ and $c_{\gamma\rho\sigma}$, respectively, Eqs. (113) become

$$w_1A_\alpha=ie^{i\phi_\alpha}+\sum_{\beta\rho}(\alpha|H_\rho|\beta)B_{\beta\rho}, \tag{114a}$$

$$w_2B_{\beta\rho}=\sum_{\alpha}(\alpha|H_\rho|\beta)^*A_\alpha+\sum_{\gamma\sigma}(\beta|H_\sigma|\gamma)C_{\gamma\rho\sigma}, \tag{114b}$$

$$w_3C_{\gamma\rho\sigma}=\sum_{\beta}(\beta|H_\sigma|\gamma)^*B_{\beta\rho}. \tag{114c}$$

Here

$$w_1=is-\epsilon_\alpha,\quad w_2=is-\epsilon_{\beta\rho},\quad w_3=is-\epsilon_{\gamma\rho\sigma}. \tag{114d}$$

Neglecting all but the first terms on the right-hand side of Eqs. (114) gives A to zero order, B to first order, and C to second order in the radiation coupling. This result, which gives the correlation function of Eq. (114) with $S(m_am_a';m_nm_n')=\delta(m_am_n)\delta(m_a'm_n')$, contains no radiation damping and hence no spin-coupling effects. Using these results in Eq. (114) to obtain A to first order, thence B to third order and finally C to fourth order, one obtains the desired final result for $C_{\gamma\rho\sigma}$. Carrying out the inverse Laplace transform we find

$$\lim_{t\to\infty}c_{\gamma\rho\sigma}=\sum_{\alpha\beta}\frac{e^{-i\phi_\alpha}(\alpha|H_\rho|\beta)^*(\beta|H_\sigma|\gamma)^*e^{i(\omega_\rho+\omega_\sigma+\epsilon_\gamma)t}}{(\omega_\rho+\omega_\sigma-\epsilon_{\alpha,\gamma}-i\gamma_A)(\omega_\sigma-\epsilon_{\beta,\gamma}-i\gamma_B)}, \tag{115}$$

where $\epsilon_{\alpha,\gamma}=\epsilon_\alpha-\epsilon_\gamma$, etc., and[21]

$$\gamma_A=i\sum_{\beta\rho}\frac{|(\alpha|H_\rho|\beta)|^2}{w_2}, \tag{116a}$$

$$\gamma_B=i\sum_{\gamma\sigma}\frac{|(\beta|H_\sigma|\gamma)|^2}{w_3}. \tag{116b}$$

These quantities are real and independent of s and of the sublevel labels α, β, respectively (76). The total transition probabilities from levels A and B are in fact $2\gamma_A$ and $2\gamma_B$, respectively.

Averaging over the phases ϕ_α with

$$\langle e^{i(\phi_\alpha-\phi_{\alpha'})}\rangle_{\text{Av}}=\delta_{\alpha\alpha'}$$

and summing $|c_{\gamma\rho\sigma}(\infty)|^2$ from (115) over all polarization states of the emitted radiations gives finally the correlation function for propagation directions defined by the unit vectors $\mathfrak{k}_\rho$, $\mathfrak{k}_\sigma$:

$$W(\mathfrak{k}_\rho,\mathfrak{k}_\sigma)=\mathfrak{S}_{\rho\sigma}\int_0^\infty d\omega_\rho\int_0^\infty d\omega_\sigma\sum_\gamma\langle|\lim_{t\to\infty}c_{\gamma\rho\sigma}|^2\rangle_{\text{Av}}$$
$$=\sum_{\alpha\beta\beta'\gamma}\mathfrak{S}_{\rho\sigma}\frac{(\alpha|H_\rho|\beta)^*(\beta|H_\sigma|\gamma)^*(\alpha|H_\rho|\beta')(\beta'|H_\sigma|\gamma)}{1+i\epsilon_{\beta\beta'}\tau}. \tag{117}$$

Here, $\mathfrak{S}_{\rho\sigma}$ denotes the polarization average for the two radiations. Irrelevant constant factors have been discarded in obtaining Eq. (117). We have introduced the mean life of the intermediate state

$$\tau=1/2\gamma_B. \tag{118}$$

As expected on physical grounds only the intermediate state lifetime enters. The scale of the coupling

[21] Goertzel's Eqs. (4a) and (4b) and also his Eq. (5) contain misprints.

effects is clearly defined by the comparison of the "multiplet" splittings $\epsilon_{\beta\beta'}$ and $1/\tau$ in corresponding units. Comparing with the no spin-coupling case of Sec. II we see that the effect of this coupling is to introduce the energy denominator $(1+i\epsilon_{\beta\beta'}\tau)^{-1}$. It will be noted that even if the coupling is extremely large, $\epsilon_{\beta\beta'}\rightarrow\infty\,(\beta\neq\beta')$, the result expressed by Eq. (117) would imply that the correlation is not completely eliminated. In this case the terms $\beta=\beta'$ would yield a "minimum" correlation which would, in general, be less anisotropic than the no spin-coupling case, but would not be isotropic. This is not surprising in view of the assumptions made. The spin-coupling is expected to attenuate the correlation due to the transitions taking place between substates of different nuclear spin orientation in the intermediate nuclear state. However, these transitions (or, speaking classically, precession) of the nuclear spin vector does not represent a complete loss of information. In some actual experimental cases it is observed that the anisotropy actually disappears

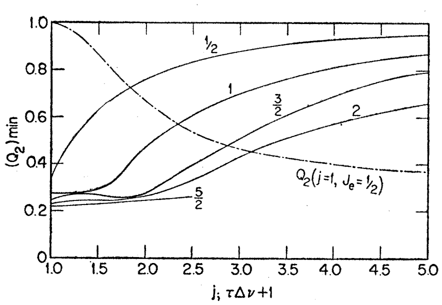

FIG. 1. The solid curves give the minimum attenuation factor $(Q_2)_{\min}$ *versus* the intermediate state spin j, see Eq. (124a). The numbers affixed to the curves give the value of J_e. $(Q_2)_{\min}$ gives a lower limit for the attenuation of the correlation for dipole transitions or for $j<2$. The dashed curve gives the full attenuation factor Q_2 as a function of $\tau\Delta\nu$ for $J_e=\frac{1}{2}$ and $j=1$ ($\tau\equiv$intermediate state lifetime, $\Delta\nu$=hfs doublet splitting). For $J_e=\frac{1}{2}$, $Q_\nu=1-[1-(Q_\nu)_{\min}](\tau\Delta\nu)^2[1+(\tau\Delta\nu)^2]^{-1}$ for any j.

(within the accuracy of the measurements), and it may be concluded that the above assumptions are invalid.

B. Magnetic Interaction

Here we consider an isolated atom (or ion) and represent the coupling by the usual hyperfine interaction $a\mathbf{j}\cdot\mathbf{J}_e$ where J_e is the angular momentum of the electron shell, assumed constant. The labels β, β' correspond to the zero-field quantum numbers F, m_F, where in the customary notation $\mathbf{F}=\mathbf{j}+\mathbf{J}_e$ and m_F is the eigenvalue of F_z.

We introduce the density matrix as in Sec. II

$$E^{(1)}(\beta\beta')=\mathfrak{S}_\rho\sum_\alpha(\alpha|H_\rho|\beta)^*(\alpha|H_\rho|\beta'),$$
$$E^{(2)}(\beta\beta')=\mathfrak{S}_\sigma\sum_\gamma(\gamma|H_\sigma|\beta)^*(\gamma|H_\sigma|\beta'),\qquad(119)$$

and write Eq. (117) in the form

$$W=\sum_{\beta\beta'}\frac{E^{(1)}(\beta\beta')E^{(2)}(\beta'\beta)}{1+i\epsilon_{\beta\beta'}\tau}.\qquad(120)$$

The reality of the result in Eq. (120) follows from the Hermitian property of $E^{(i)}$ (as does that of Eq. (117) from the Hermitian property of the matrix element occurring therein). Since the matrix elements of H_ρ and H_σ are diagonal in all non-nuclear quantum numbers, the density matrix is very easily written in terms of the decoupled (strong field) representation. In this case we introduce the projection quantum numbers m and μ for the nucleus and electrons, respectively. Then, for example,

$$E^{(1)}(mm')=(-)^{j_1-m}(2j+1)\sum_\nu C(jj\nu;m,-m')\times W(jjLL;\nu j_1)D(\nu,m'-m,0;\mathfrak{k}_1)\qquad(121)$$

for a pure 2^L pole. In Eq. (121) ν is even and, as usual, the transition is represented by $j_1\rightarrow j$. We shall write $\mathfrak{k}_1$ and $\mathfrak{k}_2$ for $\mathfrak{k}_\rho$ and $\mathfrak{k}_\sigma$, respectively. The density matrix in the weak field representation is obtained by the usual unitary transformation defined by the vector addition coefficients $C(jJ_eF;m\mu)$ (and $m_F=m+\mu$). A straightforward calculation gives

$$\begin{aligned}E(Fm_F;F'm_F')&=(-)^{J_e+m_F-J_1}[(2F+1)(2F'+1)]^{\frac{1}{2}}(2j+1)\sum_\nu\\&\times C(LL\nu;1-1)C(FF'\nu;-m_Fm_F')\\&\times W(jjFF';\nu J_e)W(jjLL;\nu j_1)\\&\times D(\nu,-\Delta m_F0;\mathfrak{k}),\end{aligned}\qquad(122)$$

where

$$\Delta m_F=m_F-m_F'.$$

Inserting Eq. (122) into Eq. (120), summing over m_F (keeping Δm_F fixed) and using

$$\sum_{m_F}C(FF'\nu;m_F,-m_F')C(FF'\bar{\nu};m_F,-m_{F'})=\delta(\nu\bar{\nu}),$$

we find

$$\begin{aligned}W=\sum_{FF'}\sum_{\Delta m_F}&(2F+1)(2F'+1)\frac{W^2(jjFF';\nu J_e)}{1+i\epsilon_{FF'}\tau}\\&\times C(L_1L_1\nu;1-1)C(L_2L_2\nu;1-1)\\&\times W(jjL_1L_1;\nu j_1)W(jjL_2L_2;\nu j_2)\\&\times D^*(\nu,-\Delta m_F0;\mathfrak{k}_1)D(\nu,-\Delta m_F0;\mathfrak{k}_2).\end{aligned}\qquad(122a)$$

Carrying out the sum over Δm_F as before and comparing with Eqs. (65a) and (68), we recognize that the correlation function can be written in the form

$$W=\sum_\nu A_\nu Q_\nu P_\nu(\cos\beta),\qquad(123)$$

which differs from the ordinary $\gamma-\gamma$ correlation by the additional attenuation factors Q_ν. These are

$$Q_\nu=\frac{1}{2J_e+1}\sum_{FF'}(2F+1)(2F'+1)\frac{W^2(jjFF';\nu J_e)}{1+(\epsilon_{FF'}\tau)^2}. \quad (124)$$

In Eq. (124) a normalization factor $(2J_e+1)^{-1}$ has been inserted so that $Q_0=1$ and $Q_\nu(\tau=0)=1$. The result given in Eq. (124) was originally given by Alder (4).[22] It is clear that this result also applies to correlations with mixed radiations.

In order to obtain an idea as to the order of magnitude of the hyperfine coupling effect one may insert

$$\epsilon_{FF'}=\frac{a}{2}[F(F+1)-F'(F'+1)].$$

Then, for particular values of j or J_e and of ν one can evaluate the sums in Eq. (124). For numerical results, however, it is easier to use the Racah Tables (13) and perform the sums arithmetically. A few cases are shown in Fig. 1 [see also (4)]. Here the "minimum" correlation factors $(Q_\nu)_{\min}$ are shown:

$$(Q_\nu)_{\min}=\frac{1}{2J_e+1}\sum_F(2F+1)^2W^2(jjFF;\nu J_e). \quad (124a)$$

From Fig. 1 we may draw the qualitative conclusion that, even if the assumption of a stationary state for the electronic shell is valid, the magnetic interaction produces a small alteration of the correlation only for $J_e=\frac{1}{2}$ and $j\gtrsim 5$ with the additional restriction that at least one of the radiations be a dipole. Hence we can conclude that, in general, the effect is not a minor one.

C. External Magnetic Fields

Here we shall assume that $J_e=0$. That this situation can be realized in practice is made highly plausible by the results of Aeppli *et al.* (1, 2). We are interested in the possibility of using an external field to measure the nuclear gyromagnetic ratio and for this purpose two methods have been suggested[23]:

(1) The delayed correlation (54).

(2) The average correlation in a field. Of these two methods (2) has been successfully applied experimentally (2).

(1) The Delayed Correlation

One measures the coincidence rate per dt (steradians^{-2} sec^{-1}) in the case that a time delay between t and $t+dt$ is introduced between the successive radiations. The spin precession in the presence of a magnetic field clearly alters the angular correlation only if a nonzero time delay exists. Conversely, without a field the time delay only affects the total intensity but not the angular correlation of the radiations.

The perturbation theory formalism leading to the correlation function in this case is very similar to that discussed in subsection A of this Section but with the following alterations. We observe the first radiation at time t_1, say. Then t_1 is taken as the time origin for the emission of the second radiation. If we designate the interval between t_1 and the time at which the second radiation is observed by t we have as initial conditions

$$b_{\beta\rho}=b_{\beta\rho}^{(0)}, \quad c_{\gamma\rho\sigma}=0: \quad t=0$$

to use with the equations of motion

$$w_2B_{\beta\rho}=ib_{\beta\rho}^{(0)}+\sum_{\gamma\sigma}(\beta|H_\sigma|\gamma)C_{\gamma\rho\sigma}$$

$$w_3C_{\gamma\rho\sigma}=\sum_\beta(\beta|H_\sigma|\gamma)^*B_{\beta\rho},$$

and $b_{\beta\rho}^{(0)}$ is obtained by a solution of the equations for A_α and $B_{\beta\rho}$ in which only the matrix elements of H_ρ are involved. The desired probability is then

$$\frac{d}{dt}\int|c_{\gamma\rho\sigma}(t)|^2d\omega\sigma.$$

The result for the probability that radiations 1 and 2 be emitted in the direction $\vartheta_1\varphi_1$ and $\vartheta_2\varphi_2$, respectively, with a time delay t to $t+dt$ is, per dt,

$$W_d(\vartheta_1\varphi_1;\vartheta_2\varphi_2;t)$$

$$=\frac{1}{\tau}e^{-t/\tau}\mathfrak{S}\sum_{\alpha\beta\beta'\gamma}(\alpha|H_1|\beta)^*(\alpha|H_1|\beta')(\beta|H_2|\gamma)^*\times(\beta'|H_2|\gamma)e^{i\epsilon_{\beta\beta'}t}. \quad (125)$$

In the external field H

$$\epsilon_{\beta\beta'}=\mu_Ng(H/\hbar)(m-m')=\omega_0(m-m'), \quad (125a)$$

where ω_0 is the Larmor frequency, μ_N the nuclear magneton, g the nuclear gyromagnetic ratio in the intermediate state, and m is the projection of the nuclear spin on the quantization axis ($\equiv$direction of magnetic field). Here α, β, β', γ all have the significance of nuclear magnetic quantum numbers for the appropriate states. In particular, β, $\beta'=m$, m'.

[22] The same result can also be obtained in an essentially equivalent manner by transforming the radiation matrix elements with the unitary (vector addition coefficients) transformation to obtain [see (42) and Eq. (2)]

$$S(m_am_a';m_nm_n')=\delta(m_a-m_a',m_n-m_n')S(m_am_nm_a')$$
$$S(mm'm'')=(2J_e+1)^{-1}\sum(1+\epsilon_{FF'}\tau)^{-2}\delta(\mu+m,\mu'+m')$$
$$\times C(jJ_eF;m\mu)C(jJ_eF;m'\mu')C(jJ_eF';m''\mu)C(jJ_eF;m'+m''-m,\mu')$$

and the sum is over F, F', μ, μ'. Application of the algebraic relations of the Racah coefficients [Appendix Eq. (A5)] yields the result given in Eq. (124). It is also possible to generalize the foregoing to other radiations x, y. It is easily seen that the only change is the insertion of the parameters $b_\nu(x)$ in the density matrices Eq. (121) and Eq. (122) and $b_\nu(x)b_\nu(y)$ in Eq. (122a). Thus, Q_ν is independent of the nature of the radiation.

[23] A third method, which would utilize an rf field so that at resonance one would observe a maximum attenuation in anisotropy, is actually impractical. The intermediate state lifetime and the available rf power are such that only a very small and probably unobservable effect would take place.

Since a rotation around the quantization axis through an angle α multiplies the product of the elements of the two density matrices in (125) by $e^{i\alpha(m-m')}$ it follows that

$$W_d(\vartheta_1\varphi_1;\vartheta_2\varphi_2;t)=\frac{1}{\tau}e^{-t/\tau}W_0(\vartheta_1\varphi_1;\vartheta_2,\ \varphi_2+\omega_0 t) \quad (126)$$

which is Lloyd's result (54). The delayed coincidence correlation may be regarded as the result of imparting to the instantaneous (or zero field) correlation W_0 a precession around the external field with the Larmor frequency. In this form the result seems an almost self-evident application of Larmor's theorem.

Expressing the W_0 correlation function in the usual form we find[24]

$$W_d=\frac{4\pi}{\tau}e^{-t/\tau}\sum_\nu \frac{A_\nu}{2\nu+1}\sum_{M=-\nu}^{\nu} \times Y_\nu^M(\mathfrak{k}_1)Y_\nu^{M*}(\mathfrak{k}_2)e^{-iM\omega_0 t}. \quad (127)$$

As one may expect, the delayed correlation contrasts with the average correlation (see paragraph (2) below) in that it provides a possible means of measuring g without measuring the mean life τ. However, this is only a formal difference and does not correspond to any essential difference in experimental techniques involved. We note in passing that interchanging the propagation directions $\mathfrak{k}_1$ and $\mathfrak{k}_2$ is equivalent to changing the sign of ω_0 or to changing the direction of the field. Hence, the sign of the g factor is determinable only if the over-all detector efficiencies changes when $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are interchanged. In addition, if the radiations and the field are coplanar, then reversing the directions of $\mathfrak{k}_1$ and $\mathfrak{k}_2$ is equivalent to changing the direction of $\mathbf{H}$. Since reversing the directions of $\mathfrak{k}_1$ and/or $\mathfrak{k}_2$ does not change W_d, it follows that a determination of the sign of g requires non-coplanarity of $\mathfrak{k}_1$, $\mathfrak{k}_2$ and $\mathbf{H}$. A particular case of coplanarity is coincidence between $\mathbf{H}$ and either $\mathfrak{k}_1$ or $\mathfrak{k}_2$. In this case the effect of the magnetic field disappears. This result is of interest in connection with the question discussed in IV-D below.

(2) Average Correlation

If an external field is applied and the correlation measured without regard to delay time (that is, all coincidences are accepted), the correlation function is

$$\bar{W}=\int_0^\infty W_d dt=4\pi\sum_\nu \frac{A_\nu}{2\nu+1}\sum_M \frac{Y_\nu^M(\mathfrak{k}_1)Y_\nu^{M*}(\mathfrak{k}_2)}{1+iM\omega_0\tau}. \quad (128)$$

Aside from the irrelevant factor 4π, Eq. (128) is equivalent to the result given by Alder (5).[25] A convenient way in which to carry out the measurement is the arrangement in which the magnetic field is perpendicular to the plane of the radiations. In this case

$$\bar{W}=\sum_{n=0}^{n_m}\frac{B_n}{1+(2n\omega_0\tau)^2}[\cos 2n\phi+2n\omega_0\tau\sin 2n\phi], \quad (129)$$

where $\phi=\varphi_1-\varphi_2$ is the angle between $\mathfrak{k}_1$ and $\mathfrak{k}_2$ and the coefficients B_n are given by

$$B_n=(2-\delta_{n0})\sum_{k=n}^{k_{\max}}\frac{A_{2k}}{2^{4k}}\frac{(2k+2n)!(2k-2n)!}{[(k+n)!(k-n)!]^2}. \quad (129a)$$

Here $2k=\nu$ and $2n_m=2k_{\max}=\nu_{\max}$. These are, of course, just the coefficients of the zero-field correlation when expanded in $\cos 2n\phi$. The same remarks relative to the sign of g made under (1) apply here. Thus, for equal over-all efficiencies

$$\bar{W}=\sum_{n=0}^{n_m}\frac{B_n}{1+(2n\omega_0\tau)^2}\cos 2n\phi, \quad (129b)$$

while with unequal efficiencies the sign of g is determined. Thus, if the efficiency for the arrangement $\mathfrak{k}_1$ in the direction $\frac{1}{2}\pi$, φ_1 and $\mathfrak{k}_2$ in the direction $\frac{1}{2}\pi$, φ_2 is η_{12} while η_{21} corresponds to $\mathfrak{k}_1(\frac{1}{2}\pi,\ \varphi_2)$ and $\mathfrak{k}_2(\frac{1}{2}\pi,\ \varphi_1)$, the observed correlation would be

$$\bar{W}=\sum_n \frac{B_n}{1+(2n\omega_0\tau)^2} \times\left[\cos 2n\phi+\frac{\eta_{12}-\eta_{21}}{\eta_{12}+\eta_{21}}2n\omega_0\tau\sin 2n\phi\right]. \quad (129c)$$

Since the zero-field correlation should serve to determine the intermediate spin j, the magnetic moment in this excited state can also be measured. Such a measurement, first suggested by Brady and Deutsch (17) and by Sunyar *et al.* (74), has been carried out by the Swiss group for the case of the Cd^{111} cascade (2). It is clear that this method requires a measurement of the intermediate state lifetime but where this is not feasible it is not to be expected that any other method will be workable.

D. Paschen-Back Effect

In Sec. I reference was made to the fact that it would be highly desirable to be able to eliminate the magnetic (and quadrupole) interaction by using a strong field. In this paragraph we wish to discuss the requirements which such a procedure would have to fulfill. In doing so we assume that we deal with an isolated atom (ion) recognizing that in an actual experiment a different situation may apply in some cases. It will be sufficient to consider that the spin part of the zero-order Hamiltonian contains the following three terms (a) magnetic

[24] The time-dependent factor $e^{-t/\tau}$ is, or course, of no consequence for the present application. It is, however, needed for the following, see paragraph (2) below.

[25] This result can also be obtained from Eq. (117) using Eq. (125a).

interactions between external field and electron moment, (b) magnetic interaction between external field and nuclear moment, and (c) hyperfine coupling between electrons and nucleus. If the spins can be decoupled under these circumstances then, as will be clear from the following, it will follow *a fortiori* that any weaker couplings like the quadrupole-electric field gradient terms will cause no trouble.

The discussion is conveniently divided into two parts: First, we show that if there is a strong Paschen-Back effect on the electrons (electron Zeeman energy$\gg$hyperfine splitting) then there is no first-order effect on the matrix elements in Eq. (117). Starting from the strong field representation in which m and μ are good quantum numbers, the matrix elements are transformed by an infinitesimal unitary matrix

$$S=1+\sigma, \quad \sigma\ll 1,$$

where σ is antihermitian and has zero diagonal matrix elements. Also σ is diagonal in $m_F=m+\mu$. Clearly σ is of order $a/\mu_e H$ where the hyperfine energy is $a\mathbf{j}\cdot\mathbf{J}_e$ and μ_e is the magnetic moment of the electron shell.

Now the argument consists in showing that only diagonal matrix elements of σ enter in modifying Eq. (117). Since we are concerned with first-order effects we consider one matrix element at a time. Labeling the perturbed states by $\bar{m}$ and $\bar{\mu}$ and designating the perturbed and unperturbed wave functions by Ψ and Φ, respectively, the unitary transformation in question may be written in the form

$$\Psi(\bar{m}\bar{\mu})=\sum_{m\mu}(m\mu|\bar{m}\bar{\mu})\Phi_j{}^m\Phi_{J_e}{}^\mu.$$

Then,

$$(\alpha|H_1|\beta)=\sum_{m\mu}(\bar{m}_1|H_1|m)\delta(\bar{\mu}_1\mu)(m\mu|\bar{m}\bar{\mu}), \tag{130a}$$

$$(\beta|H_2|\gamma)=\sum_{m'\mu'}(m'|H_2|\bar{m}_2)\delta(\mu'\bar{\mu}_2)(m'\mu'|\bar{m}\bar{\mu}), \tag{130b}$$

$$(\alpha|H_1|\beta')=\sum_{m''\mu''}(\bar{m}_1|H_1|m'')\delta(\bar{\mu}_1\mu'')\times(m''\mu''|\bar{m}'\bar{\mu}'), \tag{130c}$$

$$(\beta'|H_2|\gamma)=\sum_{m'''\mu'''}(m'''|H_2|\bar{m}_2)\delta(\mu'''\bar{\mu}_2)\times(m'''\mu'''|\bar{m}'\bar{\mu}'), \tag{130d}$$

where the subscripts 1 and 2 on the magnetic quantum numbers refer to the initial and final states, respectively. If we consider the first term perturbed, the transformation coefficients in the remaining three matrix elements must reduce to the elements of the identity transformation. Thus

$$m'=\bar{m},\quad \mu'=\bar{\mu};\quad m''=\bar{m}',\quad \mu''=\bar{\mu}';\quad m'''=\bar{m}',\quad \mu'''=\bar{\mu}'.$$

From these results it follows that

$$\mu=\bar{\mu}_1=\mu''=\bar{\mu}'=\bar{\mu}.$$

Hence, since $m+\mu=\bar{m}+\bar{\mu}$,

$$m=\bar{m}, \tag{131a}$$

and only diagonal elements of $(m\mu|\bar{m}\bar{\mu})$ enter. In an entirely equivalent fashion one can show that

$$m'=\bar{m},\quad \mu'=\bar{\mu}, \tag{131b}$$

by considering the second matrix element (130b) to be transformed and the remaining ones unchanged, and similarly with Eqs. (130c) and (130d). It follows then that to first order there is no change in the matrix elements.

We now consider the energy denominator in Eq. (117). If the matrix elements of the spin-dependent zero-order Hamiltonian are evaluated in the m, μ representation we need consider only diagonal matrix elements; off-diagonal matrix elements will all be of second order in $a/\mu_e H$ compared to the leading terms. Then the difference of two diagonal elements $\epsilon_\beta-\epsilon_{\beta'}$ will contain three terms corresponding to the three interaction energies listed above; electron and nuclear Zeeman energies and hyperfine energy. Among other terms there will be those for which $\mu=\mu'$, and the usually large electron Zeeman energy difference will vanish in this case. The spins will then be decoupled only if the nuclear Zeeman energy splitting is large compared to the hyperfine splitting. Clearly, this demands impracically large fields.

However, if one of the radiations is parallel to the field this requirement on the field is considerably relaxed. Then starting from state A (or C) with a fixed nuclear projection quantum number m_1 or m_2 and considering the fact that the radiation carries off ± 1 unit of angular momentum with respect to the field direction, the value of m in the intermediate state is fixed:

$$m=m_1\pm 1 \quad \text{or} \quad m=m_2\pm 1,$$

depending on which of the two radiations is parallel to the field. Since the two polarization states of the quantum do not interfere and are added incoherently only one of these possible values will occur. Consequently, we now have $m=m'$. Since the energy matrix is diagonal in $m+\mu$, it follows that $\mu=\mu'$ and $\epsilon_\beta=\epsilon_{\beta'}$ for all pairs of states. Consequently, the energy denominator reduces to unity and all first-order effects vanish. The magnitude of the second-order effects shows that the criterion for a strong field is that the electron Zeeman energy be large compared to the hyperfine splitting. That is, if the over-all hyperfine multiplet is $\Delta\nu$ in cm^{-1}, one must require that

$$(H/2\times 10^4\Delta\nu)^2\gg 1, \tag{132}$$

where H is in gauss (42). In many cases fields of order 5×10^3 to 10^4 gauss would presumably be adequate. Of course, if such a procedure were successful one would not have the opportunity to measure the intermediate state magnetic moment. The application of the Paschen-

Back effect would, of course, necessarily be confined to the $\gamma-\gamma$ correlation. The only attempt to carry out such an experiment was made by Frauenfelder *et al.* (36) for the Ni^{60} $\gamma-\gamma$ cascade. No effect of the magnetic field was observed, and it appears that condition (132) is not fulfilled in this case. *Note added in proof.*—A decoupling experiment with a metallic In^{111} source has also given a negative result. This problem has now been beautifully clarified by the experiments of Albers-Schönberg, Hänni, Heer, Novey, and Scherrer [Phys. Rev. **90**, 322 (1953)] and the work of A. Abragam and R. V. Pound (unpublished). The quadrupole coupling, which is in general quite significant, and in particular important in the In metal, will attenuate the correlation unless one of the radiations lies along the symmetry axis of the single crystal source. The importance of quadrupole coupling has since been discussed by A. Abragam and R. V. Pound [Eq. (1a) and Phys. Rev. **90**, 993 (1953)] and by Alder, Albers-Schönberg, Heer and Novey (to be published in Helv. Phys. Acta.).

Finally, we may inquire into the stringency of the parallelism requirement. How accurately must this be fulfilled? To investigate this point, we consider the relevant part of the correlation function; *viz.*,

$$\mathcal{S}\equiv\sum_{M\mu}\frac{Y_\nu^M(\mathbf{f}_1)Y_\nu^{M*}(\mathbf{f}_2)}{1+M(Z_1+Z_2)},\tag{133}$$

where $Z_1=i\mu_N g_N(H/\hbar)\tau$, $Z_2=2ia\mu\tau\equiv2\mu\zeta$ are the $i\omega t$ products for the nuclear Zeeman and the hyperfine energies. If we neglect the hyperfine energy, $\mathcal{S}$ would become

$$\mathcal{S}_0=\sum_{M\mu}\frac{Y_\nu^M(\mathbf{f}_1)Y_\nu^{M*}(\mathbf{f}_2)}{1+Z_1M}.$$

We therefore consider the difference

$$\Delta\mathcal{S}=\mathcal{S}-\mathcal{S}_0=-\sum_{M\mu}\frac{MZ_2Y_\nu^M(\mathbf{f}_1)Y_\nu^{M*}(\mathbf{f}_2)}{(1+Z_1M)[1+(Z_1+Z_2)M]}\equiv-\sum_{M\mu}\Delta_M\mathcal{S}.$$

Then for $J_e=\frac{1}{2}$, $\mu=\pm\frac{1}{2}$

$$\Delta_M\mathcal{S}=-\frac{Y_\nu^M(\mathbf{f}_1)Y_\nu^{M*}(\mathbf{f}_2)2M^2\zeta^2}{(1+Z_1M)[(1+MZ_1)^2-M^2\zeta^2]}.\tag{134a}$$

The case of interest is $|\zeta|\gg1$, $|Z_1|\sim1$ and then

$$\Delta_M\zeta\sim2\frac{Y_\nu^M(\mathbf{f}_1)Y_\nu^{M*}(\mathbf{f}_2)}{1+Z_1M}.\tag{134b}$$

Since $\Delta_0\mathcal{S}=0$, it follows that $\Delta_M\mathcal{S}\sim\Theta^M$ where Θ is the angle between the field and the (nearly parallel) radiation. Thus, at worst, $\Delta\mathcal{S}\sim\Theta$. It is seen that this conclusion is equally valid for arbitrary J_e. It is also valid from Eq. (134a) without the assumptions leading to Eq. (134b). Consequently, the parallelism requirement is not very stringent.

V. TRIPLE CORRELATION

The principal results of this paper are contained in the description of double-cascade process (Secs. II and III) where, when unpolarized radiations are observed, we deal with a two-vector problem. In the foregoing the only exception was the process of correlation in an external magnetic field wherein a third direction is introduced.

There is another important exception and this concerns the case in which three radiations are involved. These need not all be emitted radiations (thus, an absorption can equally well replace the first emitted radiation), and one need not observe all three radiations. For example, in the case of non s proton, or neutron, capture the compound nucleus formed after capture of the nuclear particle may decay by a cascade of two (or more) γ-rays, say. Then the correlation of the second γ-ray with the incident beam constitutes a special case of a three-vector problem or a triple correlation even though the correlation does not depend on the *direction* of the unobserved intervening radiation. Alternatively, there are several cases reported in the literature, [for example, (66)] in which there are three γ radiations observed in cascade. The correlation between all three (11) or between the first and third (8) will often be of relevance. The latter observation which, in some cases, may present no greater difficulty than the more usual type of measurement involved in double correlation, is useful in that it provides confirmation of assignments of angular momenta and parity obtained from observing the correlation between successive radiations taken in pairs. In a few cases the correlation of first and third radiations may resolve ambiguities resulting from the analysis of double correlation measurements (8).

The correlation between three propagation directions represents a complicated problem both from the experimental and theoretical point of view, and it is probable that only the special cases discussed below will be of major interest. In the following we present the formalism for the general case of three γ rays in cascade and then proceed to the special cases. The case in which one of the γ ray emissions is replaced by the absorption of a nucleon of nonvanishing orbital angular momentum has also been treated (11). Here the main complication arises from channel-spin degeneracy, and this complication would be aggravated if one considered a correlation such as $d-p-\gamma$ [proceeding by compound nucleus formation rather than stripping (14a)].

We consider the triple γ-cascade in which three pure multipoles are emitted. *Note added in proof.*—The extension to triple correlations with mixed multipoles is straightforward. (See R. K. Osborn and M. E. Rose, Oak Ridge National Laboratory Report No. 1560 and M. E. Rose, Oak Ridge National Laboratory Report No. 1555.) The nuclear angular momenta will be denoted by j_0, j_1, j_2, and j_3 in order of ascending energy (or the emission problem), and we consider emission of 2^{L_0}, 2^{L_1}, and 2^{L_2} poles in "temporal" order. The cascade is thus designated by $j_0(L_0)j_1(L_1)j_2(L_2)j_3$. In view of remarks made above consideration of the added complication with mixed radiations does not seem worth while at present. The envisaged process can then be interpreted in terms of the formalism already established in Sec. II by treating the intermediate transition $j_1(L_1)j_2$ as constituting a link between the first and third transitions. It thus provides a certain coupling coefficient of the form $S(m_am_a'; m_nm_n')$. The two end links $j_0(L_0)j_1$ and $j_2(L_2)j_3$ are characterized by density matrices of the form

$$\begin{aligned}\sum_{m_0P} C(j_0Lj; m_0m-m_0)&C(j_0Lj; m_0m'-m_0)\\ &\times D(L, m-m_0, P)D^*(L, m'-m_0, P)\\ =\sum_{m_0P\nu_0} &(-)^{m'-m_0-P}C(j_0Lj; m_0, m-m_0)\\ &\times C(j_0Lj; m_0, m'-m_0)C(LL\nu_0; m-m_0, m_0-m')\\ &\times C(LL\nu_0; P, -P)D(\nu_0, m-m', 0)\\ =2(2j+1)\sum_{\nu_0} &(-)^{m'+j_0-P}C(LL\nu_0; 1-1)\\ &\times C(jj\nu_0; -m'm)W(LLjj; \nu_0j_0)\\ &\times D(\nu_0, m-m', 0)\end{aligned}\quad(135)$$

by use of the Clebsch-Gordan series Eq. (9) and the usual Racah relations (see Appendix). In Eq. (135) ν_0 is even and the notation is, in part, descriptive of the first transition. For the first transition $j=j_1$, $L=L_0$ while for the third $j=j_3$, $L=L_2$ and j_0 is replaced by j_2. Of course, the arguments of the D matrices are $\mathbf{f}_0$ and $\mathbf{f}_2$, respectively. In all cases $P=\pm1$.

For the intermediate step, the second transition, one obtains the coupling coefficient

$$\begin{aligned}S(m_1m_1'; m_2m_2')\\ =\sum_P C(j_1L_1j_2; m_1m_2-m_1)&C(j_1L_1j_2; m_1'm_2'-m_1')\\ \times D(L_1, m_2-m_1, P; \mathbf{f}_1)&D^*(L_1, m_2'-m_1'; P; \mathbf{f}_1)\\ =2(-)^{1+m_1'-m_2'}\sum_{\nu_1} &C(L_1L_1\nu_1; 1-1)\\ \times C(L_1L_1\nu_1; m_2-m_1, m_1'-m_2')\\ \times C(j_1L_1j_2; m_1, m_2-m_1)&C(j_1L_1j_2; m_1'm_2'-m_1')\\ \times D(\nu_1, m_2-m_2'-m_1+m_1', 0; \mathbf{f}_1)\end{aligned}\quad(136)$$

and ν_1 is even.

Discarding scale factors we obtain for the correlation function

$$\begin{aligned}W(\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2)\\ =\sum(-)^{m_2+m_2'}&C(L_0L_0\nu_0; 1-1)C(L_1L_1\nu_1; 1-1)\\ \times C(L_2L_2\nu_2; 1-1)&W(L_0L_0j_1j_1; \nu_0j_0)\\ \times W(L_2L_2j_2j_2; \nu_2j_3)&C(j_1j_1\nu_0; -m_1'm_1)\\ \times C(j_2j_2\nu_2; -m_2'm_2)&C(L_1L_1\nu_1; m_2-m_1, m_1'-m_2')\\ \times C(j_1L_1j_2; m_1m_2-m_1)&C(j_1L_1j_2; m_1'm_2'-m_1')\\ \times D(\nu_0, m_1-m_1', 0; \mathbf{f}_0)\\ \times D(\nu_1, m_2-m_2'-m_1+m_1', 0; \mathbf{f}_1)\\ \times D^*(\nu_2, m_2-m_2', 0; \mathbf{f}_2),\end{aligned}\quad(137)$$

and the sum is over ν_0, ν_1, ν_2, (all even) and m_1, m_1', m_2, m_2'.

We define

$$\mu_1=m_1-m_1'$$
$$\mu_2=m_2-m_2'$$

and perform the sums over m_1 and m_2 keeping μ_1 and μ_2 fixed. This summation is performed by repeating application of the Racah relations (Appendix Eq. (A5)) and after a somewhat lengthy but straightforward calculation we obtain for the triple correlation function the result

$$\begin{aligned}W(\mathbf{f}_0\mathbf{f}_1\mathbf{f}_2)=\sum F_{\nu_0}(L_0j_0j_1)&F_{\nu_2}(L_2j_3j_2)\\ \times\Gamma(j_1L_1j_2; \nu_0\nu_1\nu_2)&\Lambda_{\nu_0\nu_1\nu_2}(\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2),\end{aligned}\quad(138)$$

where the sum is over ν_0, ν_1, ν_2 and the notation of Eq. (69b) has been introduced. Also

$$\begin{aligned}\Gamma(j_1L_1j_2; \nu_0\nu_1\nu_2)\\ ==(2\nu_2+1)^{-\frac{1}{2}}C(L_1L_1\nu_1; 1-1)\sum_\lambda (2\lambda+1)\\ \times W(\nu_0\nu_1j_2j_2; \nu_2\lambda)W(\nu_1\lambda L_1j_1; j_2L_1)\\ \times W(j_1j_1\lambda j_2; \nu_0L_1),\end{aligned}\quad(138a)$$

and

$$\begin{aligned}\Lambda_{\nu_0\nu_1\nu_2}(\mathbf{f}_0\mathbf{f}_1\mathbf{f}_2)=\sum_{\mu_1\mu_2} C(\nu_1\nu_0\nu_2; \mu_2-\mu_1, \mu_1)\\ \times Y_{\nu_0}^{\mu_1*}(\mathbf{f}_0)Y_{\nu_1}^{\mu_2-\mu_1*}(\mathbf{f}_1)Y_{\nu_2}^{\mu_2}(\mathbf{f}_2).\end{aligned}\quad(138\text{b})$$

The limits on the λ sum in Eq. (138a) are given by the triangular inequalities defined by the Racah coefficient (see Appendix). Equation (138) is the general form for the triple correlation of three pure multipole γ rays. Aside from the fact that one can choose the polar axis along any one of the unit vectors $\mathbf{f}_0$, $\mathbf{f}_1$, or $\mathbf{f}_2$ (which eliminates one of the summations in Eq. (138b) the result is clearly rather cumbersome and involves laborious calculations before numerical results can be obtained. For this purpose the tabulation of reference (13) is helpful. In the following we consider certain special cases which may be of interest.

A. Correlation with Intermediate Radiation Unobserved (8)

If the correlation between only the first and third radiation is observed, the appropriate correlation function is obtained from Eq. (138) by integration over $\mathbf{f}_1$. Then, from Eq. (138b), we find $\nu_1=0$, $\mu_2=\mu_1$ and, using the addition theorem for spherical harmonics, $\Lambda_{\nu_0\nu_1\nu_2}(\mathbf{f}_0\mathbf{f}_1\mathbf{f}_2)$ reduces to

$$\Lambda=\frac{2\nu+1}{(4\pi)^{\frac{1}{2}}}P_\nu(\mathbf{f}_0\cdot\mathbf{f}_2)\delta(\nu_0\nu_2)\delta(\nu_10)\qquad(\nu=\nu_0).\quad(139)$$

Using this result in (138a) one obtains

$$\Gamma=(-)^{L-1}[(2\nu+1)(2j_2+1)(2L_1+1)]^{-1}\times W(j_1j_1j_2j_2; \nu L_1),\quad(140)$$

TABLE XI(a). Values of the coefficient a_2 [Eq. (142)] for the case in which either the first and/or third radiations are dipole. The triad $\Delta_3\Delta_2\Delta_1$ are defined in Eq. (142a). The double entry at the top of column 3, for example, means that a_2 is the same for the decay schemes j_3+1 (1) j_3+1 (1) j_3(1) j_3 and j_3 (1) j_3 (1) j_3+1 (1) j_3+1, as explained in the text. In general the number in parentheses following $\Delta_3\Delta_2\Delta_1$ is δ, the difference in spins of the lowest state—see Eq. (142b).

j_3	$\Delta_3\Delta_2\Delta_1$ 0 0 0	0 1 0 0 −1 0(−1)	0 2 0 0 −2 0(−2)	0 0 1 −1 0 0(−1)	0 1 1 −1 −1 0(−2) 0 2 1(0) −1 −2 0(−3)
0	...	...	...	...	...
1/2	0	0	0	0	0
1	−0.0625	0.0875	0.0750	0.0125	−0.0250
3/2	0.0320	0.1280	0.1143	−0.0080	−0.0400
2	0.0875	0.1500	0.1375	−0.0250	−0.0500
5/2	0.1202	0.1633	0.1524	−0.0376	−0.0571
3	0.1406	0.1719	0.1625	−0.0469	−0.0625
7/2	0.1542	0.1777	0.1697	−0.0540	−0.0667
4	0.1636	0.1820	0.1750	−0.0595	−0.0700
9/2	0.1704	0.1851	0.1790	−0.0639	−0.0727
5	0.1755	0.1875	0.1821	−0.0675	−0.0750
11/2	0.1794	0.1893		−0.0705	−0.0769
6	0.1824	0.1908		−0.0730	−0.0786
13/2	0.1848			−0.0751	
7	0.1868			−0.0769	
∞	0.2000	0.2000	0.2000	−0.1000	−0.1000

j_3	$\Delta_3\Delta_2\Delta_1$ 1 −1 1 −1 1 −1 (−1)	−1 1 1 −1 −1 1(−1) −1 2 1(0) −1 −2 1(−2)	0 0 2 −2 0 0(−2)	2 0 0 0 0 −2(−2)	0 1 2 −2 −1 0(−3) 0 2 2(0) −2 −2 0(−4)
0	0	...	...	0.1250	...
1/2	0	...	0	0.1502	0
1	0.0175	0	−0.0179	0.1607	0.0357
3/2	0.0280	0	0.0114	0.1652	0.0571
2	0.0343	0.0050	0.0357	0.1670	0.0714
5/2	0.0383	0.0100	0.0536	0.1674	0.0816
3	0.0409	0.0143	0.0670	0.1671	0.0893
7/2	0.0428	0.0179	0.0771	0.1666	0.0952
4	0.0441	0.0208	0.0850	0.1658	0.1000
9/2	0.0451	0.0233	0.0913	0.1650	0.1039
5	0.0459	0.0255	0.0964	0.1642	0.1071
11/2	0.0465	0.0274	0.1007		0.1099
6	0.0470	0.0288	0.1042		0.1123
13/2		0.0302	0.1073		
7		0.0314	0.1099		
∞	0.0500	0.0500	0.1429	0.1429	0.1429

j_3	0 1 −1 1 −1 0(0)	0 −1 1 −1 1 0(0)	1 0 0 0 0 −1(−1)	1 1 0 0 −1 −1(−2) 1 2 0(1) 0 −2 −1(−2)	0 2 −1 1 −2 0(−1)
0	...	...	0.1250	−0.1750	...
1/2	0	—	−0.0400	−0.1600	0
1	−0.0875	0	−0.0875	−0.1500	−0.0600
3/2	−0.1120	0	−0.1051	−0.1429	−0.0857
2	−0.1200	−0.0175	−0.1125	−0.1375	−0.0982
5/2	−0.1224	−0.0320	−0.1156	−0.1333	−0.1047
3	−0.1228	−0.0429	−0.1169	−0.1300	−0.1083
7/2	−0.1222	−0.0510	−0.1172	−0.1273	−0.1104
4	−0.1213	−0.0573	−0.1170	−0.1250	−0.1114
9/2	−0.1203	−0.0622	−0.1166	−0.1231	−0.1118
5	−0.1193	−0.0662	−0.1161	−0.1214	−0.1121
11/2	−0.1183	−0.0694	−0.1155		
6	−0.1174	−0.0721	−0.1149		
13/2		−0.0744			
7		−0.0763			
∞	−0.1000	−0.1000	−0.1000	−0.1000	−0.1000

j_3	2 −1 0 0 1 −2(−1)	0 −1 2 −2 1 0(−1)	2 1 0 0 −1 −2(−3) 0 −2 −2(−3) 2 2 0(1)	1 0 2 −2 0 −1(−3) −1 0 −2(−2) 2 0 1(1)	2 0 −1 1 0 −2(−1)
0	0.1250	...	0.2143	0.0357	−0.1250
1/2	0.1600	...	0.2041	−0.0143	−0.1314
1	0.1714	0	0.1964	−0.0357	−0.1286
3/2	0.1749	0	0.1905	−0.0470	−0.1238
2	0.1754	0.0250	0.1857	−0.0536	−0.1193
5/2	0.1746	0.0457	0.1819	−0.0579	−0.1151
3	0.1733	0.0613	0.1786	−0.0607	−0.1114
7/2	0.1719	0.0729	0.1759	−0.0627	−0.1083
4	0.1704	0.0819	0.1734	−0.0643	−0.1056
9/2	0.1690	0.0889		−0.0654	−0.1031
5	0.1677	0.0946		−0.0663	−0.1010
11/2		0.0991		−0.0670	
6		0.1030		−0.0676	
13/2		0.1063			
7		0.1090			
∞	0.1429	0.1429	0.1429	−0.0714	−0.0714

j_3	−1 2 0 0 −2 1(−1)	1 0 1 −1 0 −1(−2)	1 0 −1	−1 0 1	1 1 −1 1 −1 −1(−1) 1 2 −1(1) 1 −2 −1(−1)
0	...	−0.0250	−0.2500	...	0.1750
1/2	...	0.0100	0.0500	...	0.1400
1	0	0.0250	0.0875	...	0.1200
3/2	0	0.0329	0.0920	0	0.1071
2	−0.0150	0.0375	0.0900	−0.0025	0.0982
5/2	−0.0286	0.0405	0.0867	0.0020	0.0917
3	−0.0393	0.0425	0.0835	0.0071	0.0867
7/2	−0.0476	0.0439	0.0806	0.0117	0.0827
4	−0.0542	0.0450	0.0780	0.0156	0.0795
9/2	−0.0594	0.0458	0.0758	0.0189	0.0769
5	−0.0636	0.0464	0.0739	0.0216	0.0747
11/2	−0.0671	0.0469	0.0722	0.0240	
6	−0.0700	0.0473	0.0707	0.0260	
13/2				0.0277	
7				0.0292	
∞	−0.1000	0.0500	0.0500	0.0500	0.0500

j_3	1 1 −2 2 −1 −1(0) 1 −1 −2(−1) 2 1 −1(1) 1 −2 −2(−1) 2 2 −1(2)	1 −1 2 −2 1 −1(−2) −1 1 −2(−1) 2 −1 1(1)	−1 0 2 −2 0 1(−1)	−1 −1 2 −2 1 1(0) −1 1 2(1) −2 −1 1(−1) −1 2 2(1) −2 −2 1(−2)
0	−0.2500	0	...	...
1/2	−0.2000	0	...	...
1	−0.1714	−0.0250	...	...
3/2	−0.1530	−0.0400	0	...
2	−0.1403	−0.0490	0.0036	0
5/2	−0.1310	−0.0547	−0.0029	0
3	−0.1239	−0.0584	−0.0101	−0.0071
7/2	−0.1181	−0.0611	−0.0167	−0.0143
4	−0.1136	−0.0630	−0.0223	−0.0204
9/2	−0.1099	−0.0644	−0.0270	−0.0256
5	−0.1067	−0.0656	−0.0309	−0.0297
11/2		−0.0664	−0.0343	−0.0333
6		−0.0671	−0.0371	−0.0364
13/2			−0.0396	−0.0391
7			−0.0417	−0.0411
∞	−0.0714	−0.0714	−0.0714	−0.0714

and the correlation functions assume the simple form

$$W(\beta)=N\sum_{\nu}F_{\nu}(L_0j_0j_1)F_{\nu}(L_2j_3j_2)\times W(j_1j_1j_2j_2;\,\nu L_1)P_{\nu}(\cos\beta),\quad(141)$$

where β is the angle between $\mathfrak{k}_0$ and $\mathfrak{k}_2$. In Eq. (141) N is a normalizing factor fixed so that $\langle W\rangle_{\text{Av}}=1$.

$$N=(-)^{L_1-j_1-j_2}[(2j_1+1)(2j_2+1)]^{-\frac{1}{2}}.\quad(141a)$$

From the Racah inequalities we find that upper limit on ν is

$$\nu_{\max}=2\min(L_0, L_2, j_1, j_2), \qquad (141b)$$

the multipolarity of the unobserved radiation does not affect $\nu_{\max}$.

We write the correlation function (141) in the form

$$W(\beta)=\sum_{\nu} a_\nu P_\nu(\cos\beta); \quad a_0=1. \qquad (142)$$

Tables XI(a) and (b) contain numerical results for the coefficients a_ν in the case of cascades for which L_0, L_1, $L_2 \leqslant 2$ (dipole, quadrupole only), and the only cases omitted are those in which a dipole crossover would be competing with a quadrupole transition. In such cases, it is assumed, the triple cascade would be improbable. In Tables XI we have adopted the notation

$$\Delta_i=j_{i-1}-j_i, \qquad (142a)$$

and $L_i=1$ when $\Delta_i=0, \pm 1$ while $L_i=2$ when $\Delta_i=\pm 2$. In Tables XI the coefficients a_2 and a_4 have been given for various possible j_3 (usually ground state angular momentum) and a particular set of Δ's, represented in the order $\Delta_3\Delta_2\Delta_1$. The tables have been made more compact by noting that different sets of Δ's correspond to the same a_2 (or a_4) if the spin of the lowest state is j_3' (instead of j_3). The triad of numbers $\Delta_3\Delta_2\Delta_1$ at the head of column are used when the lowest state has a spin given by the first column. For the additional entries $\Delta_3'\Delta_2'\Delta_1'(\delta)$ appearing immediately below $\Delta_3\Delta_2\Delta_1$ the results for $a_2(a_4)$ apply with the lowest state spin given by

$$j_3'=j_3-\delta. \qquad (142b)$$

Table XI(a) refers to transitions in which either L_0 or $L_2=1$ so that $\nu_{\max}=2$. The entries are the a_2 coefficients. In Table XI(b) both L_0 and $L_2=2$ and a_2 and a_4 are given. The extension of these numerical results is easily obtained by using Table I and the Racah coefficient tabulation of reference (13).

It is interesting to note that in at least two cases an observation of the first and third radiations removes an ambiguity which would be present if only successive radiations were observed. For example, consider the pair of decay schemes j_0, j_1, j_2, $j_3=0$, 1, 1, 2 and 0, 2, 1, 1, respectively. For each of these $a_2=-0.250$ for the correlation of the first two γ-rays and -0.025 for the correlation of the last two. However, for the correlation of the first and third γ rays $a_2=-0.0250$ and 0.1250 for the two levels schemes, respectively. A similar resolution of ambiguity applies in the comparison of the levels with j_0, j_1, j_2, $j_3=\frac{1}{2}$, 3/2, 3/2, 5/2 and $\frac{1}{2}$, 5/2, 3/2, 3/2.

B. Parallel Radiations

An alternative procedure, whereby corroborative evidence may be obtained for the angular momentum assignment in the triple cascade, is to observe the radiations with one pair of them parallel, or antiparallel.

TABLE XI (b). Values of the coefficients a_2 and a_4 for both first and third radiations quadrupole. The notation is the same as in Table XI(a).

j_3	$\Delta_3\Delta_2\ \Delta_1$ 2 0 2 $-2\,0\,-2(-4)$ a_2	a_4
0	0.0510	−0.0060
1/2	0.0671	−0.0013
1	0.0765	0.0015
3/2	0.0826	0.0033
2	0.0867	0.0045
∞	0.1020	0.0091

Let the two parallel (or antiparallel) radiations be designated by the indices p, q and the other radiation by the index r. Then taking the polar axis along $\mathfrak{k}_p$ or $\mathfrak{k}_q$ we find that $\mu_1=\mu_2=0$ in all cases and hence

$$\Lambda=\left[\frac{(2\nu_0+1)(2\nu_1+1)(2\nu_2+1)}{(4\pi)^3}\right]^{\frac{1}{2}} \times C(\nu_1\nu_0\nu_2; 00)P_{\nu_r}(\cos\beta_r) \qquad (143)$$

for both parallel and antiparallel radiation (since all ν are even). Here β_r is the angle between $\mathfrak{k}_r$ and $\mathfrak{k}_p$ or $\mathfrak{k}_q$ and ν_r is ν_0 if $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are parallel, $\nu_r=\nu_1$ if $\mathfrak{k}_0$ and $\mathfrak{k}_2$ are parallel, $\nu_r=\nu_2$ if $\mathfrak{k}_0$ and $\mathfrak{k}_1$ are parallel. No further simplification is possible. Apart from scale factors the angular correlation function is now

$$W(\beta_r)=\sum_{\nu_0\nu_1\nu_2} F_{\nu_0}(L_0j_0j_1) \times F_{\nu_2}(L_2j_3j_2)\Gamma(j_1L_1j_2;\nu_0\nu_1\nu_2) \times[(2\nu_0+1)(2\nu_1+1)(2\nu_2+1)]^{\frac{1}{2}} \times C(\nu_1\nu_0\nu_2; 00)P_{\nu_r}(\cos\beta_r). \qquad (144)$$

The *complexity* of the correlation function depends on which pair of radiations are made parallel (or antiparallel) since $\nu_{\max}$ may be different for ν_0, ν_1, ν_2. If the first two radiations are parallel $(\nu_r=\nu_2)$

$$(\nu_2)_{\max}=2\min[\min(L_0, j_1)+L_1, j_2, L_2].$$

If the first and third radiations are parallel $(\nu_r=\nu_1)$

$$(\nu_1)_{\max}=2\min[\min(L_0, j_1)+\min(L_2, j_2), L_1]$$

as is shown in reference (8). Also, if the second and third radiations are parallel $(\nu_r=\nu_0)$

$$(\nu_0)_{\max}=2\min[L_0, j_1, \min(L_2j_2)+L_1].$$

If one of the intermediate spins j_1, j_2 is 0 or $\frac{1}{2}$, then isotropy results whenever the 0 or $\frac{1}{2}$ spin "joins" the radiations $\mathfrak{k}_r$ and $\mathfrak{k}_p$ (or $\mathfrak{k}_q$). Thus if $j_2=0$ or $\frac{1}{2}$ one obtains isotropy when the first and second radiations are parallel and if $j_1=0$ or $\frac{1}{2}$ the correlation is isotropic for the second and third radiations parallel. However, unless *both* j_1 and $j_2=0$ or $\frac{1}{2}$, the correlation with first and third radiations parallel will not, in general, be isotropic.

It is clear that the triple correlation process in which any one of the γ rays is replaced by another particle is obtained from the above by the usual procedure of insertion of the particle parameters: $b_{\nu 0}(x)$ if the first γ ray is replaced by another particle, and similarly for the other transitions. If one of the particles is a proton or neutron, the procedure described in Sec. III(B) should be followed. Some special cases for the $p-\gamma-\gamma$ correlation have been worked out in reference (8).

APPENDIX: VECTOR ADDITION AND RACAH COEFFICIENT RELATIONS

Throughout the text repeated use is made of a number of symmetry relations and other properties of the vector addition coefficients $C(j_1j_2j_3;\, m_1m_2)$ and the Racah coefficients $W(abcd;\, ef)$. These are given below.

1. Vector Addition Coefficient

Interchange of the angular momentum quantum numbers is made by use of the symmetry properties

$$\begin{aligned} C(j_1j_2j_3;\, m_1m_2) &= (-)^{j_1+j_2-j_3}C(j_2j_1j_3;\, m_2m_1) \\ &= (-)^{j_1+j_2-j_3}C(j_1j_2j_3;\, -m_1-m_2) \\ &= (-)^{j_1-m_1}[(2j_3+1)/(2j_2+1)]^{\frac{1}{2}} \times C(j_1j_3j_2;\, m_1,\, -m_3) \\ &= (-)^{j_2+m_2}[(2j_3+1)/(2j_1+1)]^{\frac{1}{2}} \times C(j_3j_2j_1;\, -m_3m_2) \\ &= (-)^{j_1-j_3+m_2}[(2j_3+1)/(2j_1+1)]^{\frac{1}{2}} \times C(j_2j_3j_1;\, m_2-m_3). \end{aligned} \quad \text{(A1)}$$

The explicit form for $C(ll'\nu;\, 00)$ (with $l+l'+\nu=2g$ = even integer) is

$$C(ll'\nu;\, 00) = (-)^{g+\nu}(2\nu+1)^{\frac{1}{2}} \frac{\Delta(ll'\nu)g\,!}{(g-l)\,!(g-l')\,!(g-\nu)\,!}, \quad \text{(A2)}$$

where

$$\Delta(ll'\nu) = \left[\frac{(l+l'-\nu)\,!(l-l'+\nu)\,!(l'+\nu-l)\,!}{(l+l'+\nu+1)\,!}\right]^{\frac{1}{2}}.$$

The vector addition coefficient which occurs in the γ-ray correlations is

$$C(ll'\nu;\, 1-1) = \frac{1}{2}\,\frac{\nu(\nu+1)-l(l+1)-l'(l'+1)}{[l(l+1)l'(l'+1)]^{\frac{1}{2}}} \times C(ll'\nu;\, 00). \quad \text{(A3)}$$

In all cases

$$C(j_10j_3;\, m_10) = \delta_{j_1j_3}.$$

2. Racah Coefficients

The Racah coefficients vanish unless the triads *abe*, *cde*, *bdf*, and *acf* form a triangle. The symmetry properties are

$$\begin{aligned} W(abcd;\, ef) &= W(badc;\, ef) = W(cdab;\, ef) \\ &= W(acbd;\, fe) = (-)^{e+f-a-d}W(ebcf;\, ad) \\ &= (-)^{e+f-b-c}W(aefd;\, bc), \end{aligned} \quad \text{(A4)}$$

and others obtained by combining the operations indicated.

The Racah recoupling is accomplished by means of

$$\begin{aligned} C(j_1j_2j;\, m_1m_2)C(jj_3j_4;\, m_1+m_2,\, m_3) &= \sum_s (2s+1)^{\frac{1}{2}}(2j+1)^{\frac{1}{2}}C(j_2j_3s;\, m_2m_3) \\ &\quad \times C(j_1sj_4;\, m_1,\, m_2+m_3)W(j_1j_2j_4j_3;\, js) \end{aligned} \quad \text{(A5)}$$

and variants of (A5) obtained by using (A1) and/or (A4). In Section III(C) we make use of the relations

$$\begin{aligned} [(2l+1)(2l'+1)]^{\frac{1}{2}}C(ll'\nu;\, 00)W(ljl'j';\, \tfrac{1}{2}\nu) &= [(2L+1)(2L'+1)]^{\frac{1}{2}}C(LL'\nu;\, 00) \\ &\quad \times W(LjL'j';\, \tfrac{1}{2}\nu), \end{aligned} \quad \text{(A6)}$$

where $l=l(\kappa)=|\kappa|+\frac{1}{2}[S(\kappa)-1]$, ($S(\kappa)$ = sign of κ) and $l'=l(\kappa')$ with

$$\kappa=-L \quad \text{and} \quad L+1$$

and

$$\kappa'=-L' \quad \text{and} \quad L'+1$$

In (A6) ν is an even integer and $j=|\kappa|-\frac{1}{2}$, $j'=|\kappa'|-\frac{1}{2}$

$$\begin{aligned} C(LL'\nu;\, 1-1) &= -2\left(\frac{L'+1}{L'}\right)^{\frac{1}{2}S(\kappa')}\left(\frac{L}{L+1}\right)^{\frac{1}{2}S(\kappa)} \\ &\quad \times[(2L+1)(2L'+1)(2l+1)(2l'+1)]^{\frac{1}{2}} \\ &\quad \times S(\kappa)S(\kappa')C(ll'\nu;\, 00)W(ljl'j';\, \tfrac{1}{2}\nu) \\ &\quad \times W(LjL'j';\, \tfrac{1}{2}\nu), \end{aligned} \quad \text{(A7)}$$

where the notation is as above except that now

$$\kappa=L \quad \text{and} \quad -L-1$$

$$\kappa'=-L' \quad \text{and} \quad L'+1.$$

Also

$$W^2(LLL-\tfrac{1}{2}L-\tfrac{1}{2};\, \nu\tfrac{1}{2}) = \frac{(2L-\nu)(2L+\nu+1)}{4L^2(2L+1)^2}, \quad \text{(A8)}$$

$$W^2(LLL+\tfrac{1}{2}L+\tfrac{1}{2};\, \nu\tfrac{1}{2}) = \frac{(2L+1-\nu)(2L+2+\nu)}{4(L+1)^2(2L+1)^2}, \quad \text{(A9)}$$

$$W^2(LLL+\tfrac{1}{2}L-\tfrac{1}{2};\, \nu\tfrac{1}{2}) = \frac{\nu(\nu+1)}{4L(L+1)(2L+1)^2}. \quad \text{(A10)}$$

If any parameter in W vanishes the value of the Racah coefficient is

$$W(abcd,\, 0f) = (-)^{b+c-f}[(2b+1)(2c+1)]^{-\frac{1}{2}}\delta_{ab}\delta_{cd}. \quad \text{(A11)}$$

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