REVIEWS OF MODERN PHYSICS VOLUME 25, NUMBER 3 JULY, 1953

An Introduction to Spinors*

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CONTENTS

IN order to give a satisfactory account of the Lorentz-covariance of Dirac's equation for the electron, it is necessary to attribute certain transformation properties to the four-component wave function appearing there. In 1929, van der Waerden (20)[1] devised an *algebra of spinors* which plays a role in the transformation theory of such wave functions analogous to that played in special relativity theory by tensor algebra. Four years later, in 1933, van der Waerden and Infeld (14) presented a *spinor analysis* which liberated the formalism from the restrictions of special relativity theory and permitted the employment of spinors against the background of a general Riemannian metric. In addition, many other contributions to the subject appeared.

The purpose of the present article is to offer an exposition of spinor calculus which will be more elementary and more accessible than those which constitute the original literature.

I. INTRODUCTION

As a preliminary to description of the spinor formalism, the following brief discussion is intended to show how the concept of spinors arises. This discussion will also serve to define the notations to be used in this article.[2]

Schrödinger's representation in nonrelativistic quantum mechanics is characterized by the equations

$$H\psi=i\hbar\partial\psi/\partial t,\quad P_x\psi=-i\hbar\partial\psi/\partial x,\ \cdots. \tag{I 1}$$

Our choice for the metric of special relativity theory is

$$g_{00}=1,\quad g_{11}=g_{22}=g_{33}=-1,\quad g_{kl}=0\quad (k\neq l). \tag{I 2}$$

With the four-velocity $u^k=dx^k/ds$ we can form the free-particle world momentum

$$mcu_k=(E/c,\ -P_x,\ -P_y,\ -P_z).$$

We choose the sign of the four potential in such a way as to retain the conventional form of the field equations (IX 2) and (IX 4):

$$\phi_k=(-V,\ A_x,\ A_y,\ A_z). \tag{I 3}$$

When a charged particle is in an electromagnetic field, its total energy is $H=E+qV$. The world momentum of a particle in a field should have the property that its time-like component is H/c. This consideration[3] leads to

$$P_k=mcu_k-(q/c)\phi_k. \tag{I 4}$$

With the relativistic generalization of (I 1), $P_k\psi=i\hbar\partial_k\psi$ (where $\partial_k\equiv\partial/\partial x^k$), one can construct a single second-order wave equation:

$$g^{kl}[i\hbar\partial_k+(q/c)\phi_k][i\hbar\partial_l+(q/c)\phi_l]\psi=m^2c^2\psi, \tag{I 5}$$

by virtue of $g^{kl}u_ku_l=1$. Equation (I 5) is linearized, following Dirac, to[4]

$$\Gamma^k(\partial_k+i\epsilon\phi_k)\psi=\mu\psi, \tag{I 6}$$

with the conditions

$$\Gamma^k\Gamma^l+\Gamma^l\Gamma^k=-2g^{kl}1. \tag{I 7}$$

For abbreviation, we have set

$$\epsilon=-q/(\hbar c),\quad \mu=mc/h.$$

It is well known that (I 7) can be satisfied with matrices having four rows and columns, but not with matrices having fewer rows and columns.

* The authors wish to express their indebtedness to the University of Nebraska Research Council, which generously supported this work.

[1] Arabic numerals enclosed by parentheses and inserted into the text refer to items in the List of References at the end of this paper.
The fundamental mathematical discoveries underlying the theory of spinors were made by Cartan (13) and Weyl (12), Chap. III, Sec. 8c.

[2] The spinor notation used is taken from the paper of Infeld and van der Waerden (14). Gaussian units are used for electromagnetic quantities; the charge of a particle is denoted by q.
Spinor analysis necessitates the choice (I 2) for the signature of the special relativistic metric, unless one would load some of the formulas of Sec. VII with minus signs in a very ugly way. This would be most disturbing in Eqs. (VII 6) and (VII 11). The signature (I 2), however, implies the inconvenience that spatial contravariant and covariant vector components differ in sign, whereas in nonrelativistic quantum theory we work with a spatial metric $+++$ which makes it possible to equate a covariant to a contravariant vector: $mdx/dt=P_x=-i\hbar\partial/\partial x$. We make the transition to a relativistic metric (I 2) by identifying these P_x, P_y, P_z with the contravariant components.

[3] For a more precise way of arriving at (I 4), consult Corben and Stehle (3), p. 349.

[4] See Hill and Landshoff (43), and the numerous references given there.

However, the conditions

$$\sigma^{k*}\sigma^l+\sigma^{l*}\sigma^k=-g^{kl}1 \tag{I 8}$$

(asterisk denotes complex conjugate), which bear a certain resemblance to (I 7), can be satisfied with the two-row matrices

$$\sigma^0=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad \sigma^1=\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},$$
$$\sigma^2=\frac{1}{\sqrt{2}}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix},\quad \sigma^3=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{I 9}$$

Since (I 7) can be satisfied with

$$\Gamma^k=\sqrt{2}\begin{pmatrix}0 & \sigma^k\\ \sigma^{k*} & 0\end{pmatrix},$$

because of (I 8), Dirac's equation (I 6) can be written as a set of two two-component equations:

$$\begin{aligned}\sqrt{2}\sigma^k(\partial_k+i\epsilon\phi_k)\chi&=\mu\xi\\ \sqrt{2}\sigma^{k*}(\partial_k+i\epsilon\phi_k)\xi&=\mu\chi,\end{aligned} \tag{I 10}$$

where the ψ of (I 6) is written as[5]

$$\psi=\begin{pmatrix}\xi\\ \chi\end{pmatrix}.$$

Now, the covariance of Eqs. (I 8) and (I 10) can be viewed in either of two ways:

View (A). The quantities $\partial_k+i\epsilon\phi_k$ certainly transform as components of a covariant world vector. One can assume that the σ^k transform simply as components of a contravariant world vector. Then (I 8) and (I 10) are covariant with respect to Lorentz transformations. However, taking the σ^k as world vector components violates the spirit if not the letter of the relativity idea, for the following reason: As components of a world vector, the σ^k will have the especially simple values (I 9) only in certain frames of reference. In consequence, the four equations which appear when (I 10) is written out in full will not be the same in all Lorentz frames.

View (B). The alternative is to consider the σ^k as *fixed* matrices; then, to maintain the covariance of the Eqs. (I 10), it will be necessary to attribute some new transformation properties to the wave functions ξ, χ. When this procedure is carried out, with development of an appropriate mathematical formalism, the wave functions ξ and χ are renamed *spinors*.

The matrices appearing in (I 10) can be written out in the following way:

$$\chi=(\chi^\rho),\quad \xi=(\xi^{\dot\nu}),\quad \sigma^k=(\sigma^{k\dot\nu}{}_\rho),$$
$$\chi^*=(\chi^{\dot\rho}),\quad \xi^*=(\xi^\nu),\quad \sigma^{k*}=(\sigma^{k\nu}{}_{\dot\rho}).$$

Note should be taken of the following conventions:

1. Latin indexes run from 0 to 3.
2. Greek indexes take the values 1, 2 (or $\dot 1$, $\dot 2$).
3. The meaning of the "dotted indexes," as in $\xi^{\dot\sigma}$, will be discussed later in more detail; for the moment, it suffices to establish that adding a dot above every Greek index in a spin quantity means taking the complex conjugate of the quantity. (Two dots on the same index mean the same as none at all.)

Now (I 10) can be written

$$\begin{aligned}\sqrt{2}\sigma^{k\dot\nu}{}_\rho(\partial_k+i\epsilon\phi_k)\chi^\rho&=\mu\xi^{\dot\nu},\\ \sqrt{2}\sigma^{k\nu}{}_{\dot\rho}(\partial_k+i\epsilon\phi_k)\,\xi^{\dot\rho}&=\mu\chi^\nu.\end{aligned} \tag{I 11}$$

When we undertake to elaborate on the view (B) as to the covariance of (I 10) or (I 11) (namely, that the σ^k are constant matrices), the new transformation properties which must be attributed to ξ, χ will involve transformations in the "space" of the Greek indexes. These transformations are called "spin transformations." All quantities in this space which transform according to a linear homogeneous law of any sort are called "spin tensors" or "spinors."

In Sec. II to Sec. VI there is developed a formalism for the transformation theory of spinors. Sections VII and VIII deal with the connections between spin quantities and world quantities. In Sec. IX, the Maxwell-Lorentz equations and the Dirac equation for the electron are written out in spinor notation. Finally, in Sec. X we sketch Infeld and van der Waerden's spinor analysis and its application as a gauge-covariant formalism.

II. TRANSFORMATIONS OF TENSORS

The pattern after which the formalism of spin transformations is modeled is that of the transformations of tensors. To make the development of the spinor formalism more intelligible, we shall review this pattern briefly in the present section.

A contravariant world vector A^k transforms according to the rule

$$A'^l=a^l{}_kA^k,\quad A^l=a^{-1l}{}_kA'^k, \tag{II 1}$$

where the quantities $a^{-1l}{}_k$ are solutions of the set of equations $a^m{}_la^{-1l}{}_k=\delta^m{}_k$. To obtain the rule for transforming covariant world vectors, it is sufficient to require the invariance of the scalar product A^lB_l of such a vector B_l with an arbitrary contravariant vector. This requirement leads to the rule

$$B_l=B_k'a^k{}_l,\quad B_l'=B_ka^{-1k}{}_l. \tag{II 2}$$

Tensors of higher rank transform as outer products of world vectors.

[5] In the special case $\xi=\chi^*$, the first member of (I 10) reduces to the two-component equation $\sqrt{2}\sigma^k(\partial_k+i\epsilon\phi_k)\chi=\mu\chi^*$ discussed by H. Jehle, Phys. Rev. **75**, 1609 (1949) and J. Serpe, **76**, 1538 (1949). For comparison of our notation with that of Infeld and van der Waerden, see footnote 15.

Using the fundamental metric tensor g_{kl} one can generally construct invariants of the form $g_{kl}A^kA^l=\mathrm{inv}$, and can lower contravariant indexes, $A_l=A^kg_{kl}$. The tensor g_{kl} is symmetric, $g_{kl}=g_{lk}$; hence also $A_l=g_{lk}A^k$.

It is desirable further to have a tensor which raises covariant indexes, i.e., $A^k=A_lg^{lk}=A^mg_{ml}g^{lk}$. Hence we define g^{lk} by $g_{ml}g^{lk}=\delta_m{}^k$, so that g^{lk} is just the inverse of g_{lk}. As a further consequence, $g_{kl}A^kA^l=g^{kl}A_kA_l$.

III. TRANSFORMATIONS OF SPINORS

We establish for spinors the same form of transformation law as (II 1):

$$\psi'^\lambda=\Lambda^\lambda{}_\kappa\psi^\kappa,\quad \psi^\lambda=\Lambda^{-1\lambda}{}_\kappa\psi'^\kappa \qquad \text{(III 1)}$$

(Greek indexes$=1, 2$). Here ψ^λ is a spinor of first rank and $\Lambda^\lambda{}_\kappa$ is the matrix of transformation coefficients. For the present, we restrict the transformations $\Lambda^\lambda{}_\kappa$ only by requiring them to be nonsingular: $|\Lambda^\lambda{}_\kappa|\neq 0$. In general, any quantity with Greek indexes is liable to have complex components.

To obtain invariance of the form $\chi_\lambda\psi^\lambda$ with respect to spin transformations, it is necessary to establish the rule for transforming covariant spinors,

$$\chi_\lambda=\chi'_\kappa\Lambda^\kappa{}_\lambda,\quad \chi'_\lambda=\chi_\kappa\Lambda^{-1\kappa}{}_\lambda, \qquad \text{(III 2)}$$

the same form as (II 2) for covariant world vectors.

We shall denote by $\psi^{\dot\lambda}$ the contravariant spinor whose components are conjugate complex to those of ψ^λ, and which transforms according to the law

$$\psi'^{\dot\lambda}=\Lambda^{\dot\lambda}{}_{\dot\kappa}\psi^{\dot\kappa},\quad \psi^{\dot\lambda}=\Lambda^{-1\dot\lambda}{}_{\dot\kappa}\psi'^{\dot\kappa}. \qquad \text{(III 3)}$$

Here the transformation coefficients $\Lambda^{\dot\lambda}{}_{\dot\kappa}$ are conjugate complex to the $\Lambda^\lambda{}_\kappa$. Similarly, we shall denote by $\chi_{\dot\lambda}$ the spinor whose components are conjugate complex to those of χ_λ, and which transforms according to the law

$$\chi_{\dot\lambda}=\chi'_{\dot\kappa}\Lambda^{\dot\kappa}{}_{\dot\lambda},\quad \chi'_{\dot\lambda}=\chi_{\dot\kappa}\Lambda^{-1\dot\kappa}{}_{\dot\lambda}. \qquad \text{(III 4)}$$

A contravariant spin vector may transform either like ψ^λ [by (III 1)] or like $\psi^{\dot\lambda}$ [by (III 3)]; the transformation matrix may be either $\Lambda^\kappa{}_\lambda$ or $\Lambda^{\dot\kappa}{}_{\dot\lambda}$. Similarly, a covariant spin vector may transform either by (III 2) or by (III 4). The transformation rules for spinors of higher rank are defined by the requirement that such spinors transform like outer products of spin vectors. Quite generally, an undotted contravariant index transforms like ψ^λ, a dotted one like $\psi^{\dot\lambda}$. Thus, for example, $\eta^{\dot\mu\alpha}{}_{\dot\nu}$ transforms like $\psi^{\dot\mu}\xi^\alpha\chi_{\dot\nu}$, i.e.,

$$\eta'^{\dot\rho\beta}{}_{\dot\sigma}=\Lambda^{\dot\rho}{}_{\dot\mu}\Lambda^\beta{}_\alpha\eta^{\dot\mu\alpha}{}_{\dot\nu}\Lambda^{-1\dot\nu}{}_{\dot\sigma}.$$

The essential purpose of the "dotted index" notation is to distinguish these two transformation laws in spin space. Summation is to be permitted only over a dotted subscript and a dotted superscript, or over an undotted subscript and an undotted superscript. As a consequence of this restriction, the invariance of such expressions as $\psi^{\dot\lambda}\chi_{\dot\lambda}$ and $\alpha^{\dot\mu\nu}\chi_{\dot\mu}\xi_\nu$ becomes apparent on inspection, while non-invariant forms such as $\sum_\mu\psi^\mu\chi_{\dot\mu}$ are prevented from appearing.

It has sometimes been customary to let ψ^λ, $\psi^{\dot\lambda}$ represent arbitrary unrelated spinors transforming according to (III 1) and (III 3), respectively. In this case ψ^λ and $\psi^{\dot\lambda}$ are not necessarily conjugate complex to one another. However, the convention that $\psi^{\dot 1}=(\psi^1)^*$, $\psi^{\dot 2}=(\psi^2)^*$ (and analogously for all other spinors) is a convenient one, and we shall adhere to it.

In any second-rank spinor, as $\eta^{\dot\lambda}{}_\rho$, and in any spin transformation $\Lambda^\lambda{}_\kappa$, we make the further convention that the first index numbers the rows and the second index numbers the columns when such quantities are written out as matrices.

IV. THE FUNDAMENTAL SPINOR

In the preceding section, we have defined contravariant and covariant spinors in terms of their transformation properties. As yet, however, we have no means of finding the covariant components of a spinor when only its contravariant components are known, or conversely. In world space, the raising and lowering of indexes is accomplished by means of the metric tensor g_{kl} or g^{kl}. In the present section, we shall introduce an analogous "metric spinor"[6] for raising and lowering spin indexes.

We shall denote this fundamental spinor by $\gamma_{\mu\nu}$. With it, we can construct the invariant bilinear form

$$\chi^\mu\psi^\nu\gamma_{\mu\nu}=\chi'^\mu\psi'^\nu\gamma'_{\mu\nu}. \qquad \text{(IV 1)}$$

This form is naturally invariant with any covariant spin tensor $\gamma_{\mu\nu}$. However, we select $\gamma_{\mu\nu}$ to be an *antisymmetric* spinor

$$\gamma_{\mu\nu}=\begin{pmatrix} 0 & \gamma_{12} \\ -\gamma_{12} & 0 \end{pmatrix}. \qquad \text{(IV 2)}$$

The reason for this choice is the following one: In Sec. VII and VIII we shall establish a connection between spinors and spin transformations, on the one hand, and tensors and world transformations, on the other. Now, in world space, the case of special relativity is distinguished by unusual simplicity in that the metric tensor g_{kl} is invariant under Lorentz transformations. We should like the corresponding case in spin space to be similarly distinguished. This implies choosing $\gamma_{\mu\nu}$ to be invariant under unimodular spin transformations $|\Lambda^\rho{}_\kappa|=1$, since the entire connection between spin space and world space is based on the known isomorphism between the binary unimodular group and the restricted Lorentz group (see Sec. VIII). The antisymmetric tensor (IV 2) indeed satisfies this requirement:

$$\gamma'_{\mu\nu}=\gamma_{\lambda\kappa}\Lambda^{-1\lambda}{}_\mu\Lambda^{-1\kappa}{}_\nu=\gamma_{12}[(\Lambda^{-1})^1{}_\mu(\Lambda^{-1})^2{}_\nu-(\Lambda^{-1})^2{}_\mu(\Lambda^{-1})^1{}_\nu];$$

[6] This "metric" has the singular property that it assigns to all vectors in spin space the "length" zero; see Sec. V. Such an "Hermitizing matrix" was first introduced by Bargmann (30). See also van der Waerden (20), Pauli (27), and Kofink (47).

and since the second factor is zero for $\mu=\nu$, we obtain

$$\gamma'_{\mu\nu}=\gamma_{\mu\nu}|\Lambda^{-1\lambda}{}_{\kappa}|, \tag{IV 3}$$

so that with unimodular transformations, $\gamma'_{\mu\nu}=\gamma_{\mu\nu}$. A similar investigation shows that symmetric spinors are not invariant under such transformations.

We can now make use of the invariance of the expression (IV 1) to introduce a rule for lowering spin superscripts: Since covariant spin vectors are defined by the requirement that their scalar products with contravariant spin vectors be invariant,

$$\chi'_{\nu}\psi'^{\nu}=\chi_{\mu}\psi^{\mu}, \tag{IV 4}$$

it follows that we can define a covariant χ_{ν} by

$$\chi_{\nu}=\chi^{\mu}\gamma_{\mu\nu}=-\gamma_{\nu\mu}\chi^{\mu}. \tag{IV 5}$$

The second equation follows from the antisymmetry of $\gamma_{\mu\nu}$. It should be remarked that to lower a superscript without change of sign, the spin metric $\gamma_{\mu\nu}$ has to be written *after* the spinor whose index is to be lowered. (It is assumed that summation indices are placed adjacent, as in matrix multiplication.)

We establish the same rule for dotted indexes,

$$\chi_{\dot{\nu}}=\chi^{\dot{\mu}}\gamma_{\dot{\mu}\dot{\nu}},$$

with

$$\gamma_{\dot{\mu}\dot{\nu}}=(\gamma_{\mu\nu})^{*}.$$

It is now desired to define a $\gamma^{\mu\nu}$ which will raise subscripts. Because of (IV 5),

$$\chi_{\nu}\psi_{\lambda}=\chi^{\mu}\psi^{\kappa}\gamma_{\mu\nu}\gamma_{\kappa\lambda},$$

and a second-rank spinor $\gamma^{\nu\lambda}$ should have its indexes lowered by just this same rule. Thus

$$\gamma_{\nu\lambda}=\gamma^{\mu\kappa}\gamma_{\mu\nu}\gamma_{\kappa\lambda}=-\gamma_{\nu\mu}\gamma^{\mu\kappa}\gamma_{\kappa\lambda}.$$

From this equation it follows that

$$\left.\begin{aligned}\gamma_{\nu\mu}\gamma^{\mu\kappa}&=-\delta_{\nu}{}^{\kappa}=-\begin{pmatrix}1&0\\0&1\end{pmatrix},\\ \gamma^{\mu\kappa}\gamma_{\kappa\lambda}&=-\delta^{\mu}{}_{\lambda}=-\begin{pmatrix}1&0\\0&1\end{pmatrix};\end{aligned}\right. \tag{IV 6}$$

and so we define

$$\gamma^{\mu\nu}=\begin{pmatrix}0&1/\gamma_{12}\\-1/\gamma_{12}&0\end{pmatrix}. \tag{IV 7}$$

The formula for raising covariant spin indexes is now, because of (IV 5) and (IV 6),

$$\chi^{\mu}=\gamma^{\mu\lambda}\chi_{\lambda}=-\chi_{\lambda}\gamma^{\lambda\mu}. \tag{IV 8}$$

In the case without change of sign, the $\gamma^{\mu\lambda}$ is written *before* the spinor whose index is being raised.

The spinors $\gamma_{\lambda\mu}$, $\gamma^{\lambda\mu}$ are determined by a single complex number, say γ_{12}. The quantity

$$\gamma=\gamma_{12}\gamma_{\dot{1}\dot{2}} \tag{IV 9}$$

is real and positive. The forms

$$\gamma_{\lambda\mu}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\gamma^{\frac{1}{2}}e^{i\vartheta},\quad \gamma^{\lambda\mu}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\gamma^{-\frac{1}{2}}e^{-i\vartheta} \tag{IV 10}$$

are often useful.

The spinors $\delta_{\nu}{}^{\mu}$ and $\delta^{\mu}{}_{\nu}$ appearing in (IV 6) could be written $\delta_{\nu}{}^{\mu}=\delta^{\mu}{}_{\nu}=\gamma_{\nu}{}^{\mu}=-\gamma^{\mu}{}_{\nu}$, since it is evident from (IV 6) that they are just the forms of the fundamental spinor which have one covariant and one contravariant index.[7] These spinors $\gamma_{\nu}{}^{\kappa}$ and $\gamma^{\mu}{}_{\lambda}$ are invariant, since for example

$$\begin{aligned}\gamma'^{\kappa}{}_{\nu}&=-\delta'^{\kappa}{}_{\nu}=-\Lambda^{\kappa}{}_{\lambda}\delta^{\lambda}{}_{\mu}\Lambda^{-1\mu}{}_{\nu}\\&=-\Lambda^{\kappa}{}_{\lambda}\Lambda^{-1\lambda}{}_{\nu}=-\delta^{\kappa}{}_{\nu}=\gamma^{\kappa}{}_{\nu}.\end{aligned}$$

V. ALGEBRAIC PROPERTIES OF SPINORS

When performing calculations with spinors, it is essential to have at hand a number of relations which state the important algebraic properties of spinors. Some of these relations, which are frequently of use in manipulating expressions containing spinors, are written down in this section.

We first note some relations involving the fundamental spinor $\gamma^{\alpha\beta}$ (and arbitrary spinors $\eta_{\alpha\beta\gamma}$):

$$\gamma^{\alpha\beta}=-\gamma^{\beta\alpha}, \tag{V 1}$$

$$\gamma^{\alpha\beta}\gamma^{\gamma\delta}+\gamma^{\beta\gamma}\gamma^{\alpha\delta}+\gamma^{\gamma\alpha}\gamma^{\beta\delta}=0, \tag{V 2}$$

$$\gamma^{\alpha\beta}\eta_{\alpha\beta\gamma}+\gamma^{\alpha\beta}\eta_{\gamma\alpha\beta}+\gamma^{\alpha\beta}\eta_{\beta\gamma\alpha}=0. \tag{V 3}$$

They are easily verified by going through the possible numerical values of the indexes. Similar relations hold for $\gamma^{\dot{\alpha}\dot{\beta}}$, $\gamma_{\alpha\beta}$, $\gamma_{\dot{\alpha}\dot{\beta}}$.

Making use of these basic relations, we find the following formal properties of spinors:

A. Contraction proceeds just as with tensors, e.g.,

$$\varphi^{\dot{\lambda}\mu}\zeta_{\dot{\rho}\mu\sigma}{}^{\kappa}=\eta^{\dot{\lambda}}{}_{\dot{\rho}\sigma}{}^{\kappa}.$$

Summation can occur only over a dotted subscript and a dotted superscript, or over an undotted subscript and an undotted superscript.

B. Scalar multiplication of two spinors is "anticommutative" in the following sense:

$$\chi_{\nu}\psi^{\nu}=\chi_{\nu}\gamma^{\nu\kappa}\psi_{\kappa}=-\gamma^{\kappa\nu}\chi_{\nu}\psi_{\kappa}=-\chi^{\kappa}\psi_{\kappa}=-\psi_{\kappa}\chi^{\kappa}. \tag{V 4}$$

C. The absolute value of any spinor of *odd* rank is zero. For example,

$$\chi_{\nu}\chi^{\nu}=-\chi_{\nu}\chi^{\nu}=0,$$

from (V 4). But $\eta_{\lambda\nu}\eta^{\lambda\nu}=\eta^{\lambda\nu}\eta_{\lambda\nu}$ (by a similar raising and lowering of indexes) need not be zero.

D. The contraction of a symmetrical spinor is zero. Let $\eta_{\lambda\nu}=\eta_{\mu\lambda}$. The contraction is[8]

$$\eta_{\lambda}{}^{\lambda}=\gamma^{\lambda\mu}\eta_{\lambda\mu}=-\gamma^{\mu\lambda}\eta_{\lambda\mu}=-\gamma^{\mu\lambda}\eta_{\mu\lambda}=-\eta_{\mu}{}^{\mu}=0. \tag{V 5}$$

[7] Analogously, in world space $g_{k}{}^{l}=\delta_{k}{}^{l}$; but $g^{l}{}_{k}=\delta^{l}{}_{k}$.

[8] It should perhaps be remarked that $\eta_{\lambda}{}^{\lambda}$ is not the sum of the diagonal elements of either $\eta_{\lambda\mu}$ or $\eta^{\lambda\mu}$.

E. Let $\eta_{\lambda\mu}=-\eta_{\mu\lambda}$ be an antisymmetric spinor. Its contraction is, with reference to (IV 7),

$$\eta_\lambda{}^\lambda=\gamma^{\lambda\mu}\eta_{\lambda\mu}=(\eta_{12}-\eta_{21})/\gamma_{12}=2\eta_{12}/\gamma_{12}=-2\eta_{21}/\gamma_{12}.$$

Hence

$$\eta_{\lambda\mu}=\tfrac{1}{2}\eta_\alpha{}^\alpha\gamma_{\lambda\mu}$$

or

$$\eta_{\lambda\mu}-\eta_{\mu\lambda}=\eta_\alpha{}^\alpha\gamma_{\lambda\mu}. \tag{V 6}$$

Equation (V 6) holds for a *general* spinor of second rank, since the symmetrical part vanishes on both sides.

F. Equation (V 3) can be put into the special forms

$$\begin{gathered}\xi_\alpha\eta^\alpha\zeta_\gamma+\xi_\gamma\eta_\alpha\zeta^\alpha+\xi^\alpha\eta_\gamma\zeta_\alpha=0,\\ \eta_\alpha{}^\alpha{}_\gamma+\eta_{\gamma\alpha}{}^\alpha+\eta^\alpha{}_{\gamma\alpha}=0.\end{gathered} \tag{V 7}$$

Similarly, (V 2) gives

$$\eta_\alpha{}^\alpha{}_\gamma{}^\gamma+\eta_{\alpha\beta}{}^{\beta\alpha}+\eta^\gamma{}_{\beta\gamma}{}^\beta=0. \tag{V 8}$$

VI. SPECIAL SPIN TRANSFORMATIONS

So far no restrictions have been placed on the spin transformations $\Lambda^\lambda{}_\kappa$, except that they be nonsingular. Two kinds of specialization of $\Lambda^\lambda{}_\kappa$ will be of interest:

(A) $$\Lambda^\lambda{}_\kappa=\delta^\lambda{}_\kappa e^{i\varphi/2}, \tag{VI 1}$$

where φ is an arbitrary real function of $(xyzt)$. It will be shown in Sec. 10 that (VI 1) represents a gauge transformation.

B. *Unimodular* spin transformations

$$|\Lambda^\lambda{}_\kappa|=|\Lambda^{\dot\lambda}{}_{\dot\kappa}|=|\Lambda^{-1\lambda}{}_\kappa|=|\Lambda^{-1\dot\lambda}{}_{\dot\kappa}|=1. \tag{VI 2}$$

Such transformations will be shown in Sec. VIII to represent Lorentz transformations (not including inversions). In problems where special relativity theory is applicable one can restrict himself to unimodular spin transformations (VI 2) and because of (IV 3) can choose for the $\gamma_{\mu\nu}$ the simple invariant values

$$\gamma_{\mu\nu}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\gamma^{\mu\nu}. \tag{VI 3}$$

In this way one recovers, from the general theory which we have partially described, the special spinor algebra presented in van der Waerden's original 1929 paper.

In the case of this specialization, one can introduce the spin metric $\gamma_{\mu\nu}$ (VI 3) in a very simple way. Under unimodular spin transformations, both

$$\chi^1\psi^2-\chi^2\psi^1 \quad\text{and}\quad \chi_1\psi^1+\chi_2\psi^2 \tag{VI 4}$$

are invariant. Hence one can define covariant components χ_ν of a spinor χ^ν by identifying the two expressions (VI 4)

$$\chi_1=-\chi^2,\quad \chi_2=\chi^1.$$

This procedure immediately gives (VI 3) when one sets $\chi_\nu=\chi^\mu\gamma_{\mu\nu}$.

The possibility of establishing such a simple scheme rests upon the fact that $\gamma_{\mu\nu}$ and $\gamma^{\mu\nu}$ are invariant in respect to unimodular spin transformations, as is evident from (IV 3) and (IV 7). Section VIII of the present article will discuss this special spinor algebra in some detail.

VII. THE CONNECTION BETWEEN WORLD TENSORS AND SPINORS

An Hermitean[9] second-rank spinor

$$A_{\dot\lambda\mu}=A_{\dot\mu\lambda} \tag{VII 1}$$

is determined by four real numbers. One can therefore make correspond to every real world vector A_k, in a unique way, such an Hermitean spinor (VII 1), whose components are linear functions of the components of the vector

$$A_{\dot\lambda\mu}=\sigma^k{}_{\dot\lambda\mu}A_k. \tag{VII 2}$$

Here the "mixed quantities" $\sigma^k{}_{\dot\lambda\mu}$ must be Hermitean,

$$\sigma^k{}_{\dot\lambda\mu}=\sigma^k{}_{\dot\mu\lambda}, \tag{VII 3}$$

since the A_k are real and $A_{\dot\lambda\mu}$ in (VII 2) is assumed to be Hermitean. If (VII 2) is to have significance independently of particular coordinate systems, then $\sigma^k{}_{\dot\lambda\mu}$ must transform as a contravariant four-vector in respect to general world transformations and, independently, as a second-rank spinor in respect to spin transformations.

Let us assume that the metrics g_{kl} and $\gamma_{\mu\nu}$, in world space and in spin space, respectively, are given. The invariant quadratic form $g^{kl}A_kA_l$ in the world vector components A_k has the signature $+---$. The Hermitean spinor $A_{\dot\lambda\mu}$ has an invariant of just this same signature, namely,

$$A^{\dot\lambda\mu}A_{\dot\lambda\mu}=\gamma^{\dot\lambda\dot\rho}\gamma^{\mu\nu}A_{\dot\rho\nu}A_{\dot\lambda\mu}=(2/\gamma)|A_{\dot\lambda\mu}|, \tag{VII 4}$$

where we have used the form (IV 10) for $\gamma^{\mu\nu}$. To show that this expression has the desired signature, we can set $A_{\dot 11}=a+b$, $A_{\dot 22}=a-b$, $A_{\dot 12}=c+id$, $A_{\dot 21}=c-id$, from which

$$A_{\dot 11}A_{\dot 22}-A_{\dot 12}A_{\dot 21}=a^2-b^2-c^2-d^2.$$

The correspondence $A_k\rightarrow A_{\dot\lambda\mu}$ transforms the invariant expression (VII 4) into a quadratic form in the A_k. We arbitrarily identify this quadratic form with the fundamental metrical form $g^{kl}A_kA_l$:

$$g^{kl}A_kA_l=\gamma^{\dot\lambda\dot\rho}\gamma^{\mu\nu}A_{\dot\rho\nu}A_{\dot\lambda\mu}=\gamma^{\dot\lambda\dot\rho}\gamma^{\mu\nu}\sigma^k{}_{\dot\rho\nu}\sigma^l{}_{\dot\lambda\mu}A_kA_l$$

identically in A_k. Thus

$$\sigma^{k\dot\lambda\mu}\sigma^l{}_{\dot\lambda\mu}=\gamma^{\dot\lambda\dot\rho}\gamma^{\mu\nu}\sigma^k{}_{\dot\rho\nu}\sigma^l{}_{\dot\lambda\mu}=g^{kl}. \tag{VII 5}$$

Equation (VII 5) gives 10 conditions which must be satisfied by the 16 real parameters determining the Hermitean quantities $\sigma^k{}_{\dot\rho\nu}$. On the basis of (VII 5) we

[9] The transpose of $A_{\dot\lambda\mu}$ is $A_{\dot\mu\lambda}$, while the complex conjugate of the last spinor is $A_{\dot\mu\lambda}$. The dot in a dotted index belongs to the position of the index. Taking the transpose of a second-rank spinor interchanges rows and columns, but does not alter the fact that the first index in each component is dotted, the second not.

can solve (VII 2) for the A_k:

$$\sigma^{l\dot{\lambda}\mu}A_{\dot{\lambda}\mu}=\sigma^{l\dot{\lambda}\mu}\sigma^{k}{}_{\dot{\lambda}\mu}A_k=g^{lk}A_k;$$
$$A^l=\sigma^{l\dot{\lambda}\mu}A_{\dot{\lambda}\mu}. \tag{VII 6}$$

World tensors of higher rank can also be made to correspond to certain spinors by means of an obvious extension of the rules (VII 2), (VII 6). For example, if T_{kl} is a world tensor of second rank, we define

$$T^{\dot{\mu}\nu\dot{\rho}\sigma}=\sigma^{k\dot{\mu}\nu}\sigma^{l\dot{\rho}\sigma}T_{kl}. \tag{VII 7}$$

The inverse equation is

$$T^{kl}=\sigma^{k\dot{\mu}\nu}\sigma^{l\dot{\rho}\sigma}T_{\dot{\mu}\nu\dot{\rho}\sigma}. \tag{VII 8}$$

If one substitutes (VII 6) back into (VII 2), he obtains

$$A_{\dot{\lambda}\mu}=\sigma^{k}{}_{\dot{\lambda}\mu}g_{kl}\sigma^{l\dot{\rho}\sigma}A_{\dot{\rho}\sigma},$$

which can be satisfied identically only if

$$g_{kl}\sigma^{k}{}_{\dot{\lambda}\mu}\sigma^{l\dot{\rho}\sigma}=\delta_{\dot{\lambda}}{}^{\dot{\rho}}\delta_{\mu}{}^{\sigma}=\gamma_{\dot{\lambda}}{}^{\dot{\rho}}\gamma_{\mu}{}^{\sigma}. \tag{VII 9}$$

By raising indexes one obtains the equivalent relation

$$g_{kl}\sigma^{k\dot{\lambda}\mu}\sigma^{l\dot{\rho}\sigma}=\gamma^{\dot{\lambda}\dot{\rho}}\gamma^{\mu\sigma}. \tag{VII 10}$$

On the basis of (VII 5), it is clear that we can also equate the invariant *bilinear* forms in world space and in spin space:

$$A_{\dot{\lambda}\mu}B^{\dot{\lambda}\mu}=A_kB^k. \tag{VII 11}$$

The expression $\gamma_{\dot{\lambda}\dot{\rho}}(\sigma^{k\dot{\lambda}\mu}\sigma^{l\dot{\rho}\nu}+\sigma^{l\dot{\lambda}\mu}\sigma^{k\dot{\rho}\nu})$ is antisymmetric in the indexes μ and ν, and hence is equal to a multiple of $\gamma^{\mu\nu}$, say $h^{kl}\gamma^{\mu\nu}$. If one multiples by $\gamma_{\nu\sigma}$ and considers the rule for lowering indexes, it follows that

$$\sigma^{k\dot{\lambda}\mu}\sigma^{l}{}_{\dot{\lambda}\sigma}+\sigma^{l\dot{\lambda}\mu}\sigma^{k}{}_{\dot{\lambda}\sigma}=h^{kl}\delta^{\mu}{}_{\sigma}.$$

In order to determine h^{kl}, we set $\mu=\sigma$, sum over μ and compare with (VII 5). The result is $h^{kl}=g^{kl}$. Thus

$$\sigma^{l\mu\dot{\lambda}}\sigma^{k}{}_{\dot{\lambda}\sigma}+\sigma^{k\mu\dot{\lambda}}\sigma^{l}{}_{\dot{\lambda}\sigma}=g^{kl}\delta^{\mu}{}_{\sigma}. \tag{VII 12}$$

VIII. LORENTZ TRANSFORMATIONS AND SPIN TRANSFORMATIONS

In this section, and in the next, we restrict ourselves to the special metric (I 2) and to Lorentz transformations. As already mentioned in Sec. VI, these restrictions characterize the special spinor calculus presented by van der Waerden in his original 1929 paper.

The theme of Sec. VII was the possibility of establishing, quite generally, correspondences between world tensors of any rank and spinors with certain special symmetry properties. The present section deals with the further possibility, which arises in the case of the specialization to Lorentz frames and transformations, of establishing a coordination between world transformations and unimodular spin transformations.[10]

We choose for the spin metric $\gamma_{\mu\nu}$ the simple form

$$\gamma_{\mu\nu}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{VI 3}$$

Then (VII 10) is satisfied with

$$\sigma^{0\dot{\rho}\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \sigma^{1\dot{\rho}\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$
$$\sigma^{2\dot{\rho}\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma^{3\dot{\rho}\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{VIII 1}$$

as can be verified by direct calculation. By lowering indexes, using (VI 3), one finds[11]

$$\sigma^{0\dot{\rho}}{}_{\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad \sigma^{1\dot{\rho}}{}_{\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},$$
$$\sigma^{2\dot{\rho}}{}_{\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix},\quad \sigma^{3\dot{\rho}}{}_{\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{VIII 2}$$

and

$$\sigma^{0}{}_{\dot{\rho}\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \sigma^{1}{}_{\dot{\rho}\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix},$$
$$\sigma^{2}{}_{\dot{\rho}\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma^{3}{}_{\dot{\rho}\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{VIII 3}$$

Here our exposition has, so to speak, returned to its starting point. The set of matrices $\sigma^{k\dot{\rho}}{}_{\sigma}$ (VIII 2) is identical with the set σ^k (I 9), the study of whose transformation properties initiated the entire development. Moreover, the relations (I 8) are completely equivalent to (VII 12), from which they can be obtained by raising and lowering indexes. As a matter of fact we are now prepared to carry to completion the program sketched at the end of Sec. I.

This program consisted essentially in considering the matrices σ^k and the wave functions ξ, χ in (I 10) as spinors, and in seeking to discover how the wave functions must transform with a Lorentz transformation if the σ^k are to remain constant while the equations remain covariant. Since it was assumed in Sec. VII that the mixed quantities $\sigma^{k\dot{\rho}}{}_{\sigma}$ transform like world vector components in respect to world transformations and like spin tensor components in respect to spin transformations, these quantities can remain constant when a Lorentz transformation is made only if an appropriate spin transformation is made at the same time. That is to say, the $\sigma^{k\dot{\rho}}{}_{\sigma}$ can be held constant only by coordinating a suitably chosen spin transformation with the Lorentz transformation. Moreover, we desire to

[10] This possibility is founded essentially on the existence, in world space and in spin space, of quadratic forms having the same signature. See Sec. VII.

[11] Note that the matrices (VIII 1) and (VIII 3) are all Hermitean, while some of those of the set (VIII 2) are not.

maintain the choice (VI 3) of $\gamma_{\mu\nu}$ when such spin transformations are made.

We therefore determine conditions that the $\sigma^{k\dot{\rho}}{}_{\sigma}$ and $\gamma_{\mu\nu}$ have fixed values, for example (VIII 2) and (VI 3), independent of particular frames in spin space and world space. It is clear from (IV 3) that $\gamma_{\mu\nu}$ is left invariant only by unimodular spin transformations

$$|\Lambda^{\sigma}{}_{\nu}|=1. \qquad \text{(VI 2)}$$

Equation (VI 2) determines two of the eight real parameters specifying $\Lambda^{\sigma}{}_{\nu}$. The remaining six parameters permit us to coordinate $\Lambda^{\sigma}{}_{\nu}$ to a six-parameter Lorentz transformation. Let $a^{k}{}_{l}$ be a Lorentz transformation. Then $a^{k}{}_{l}$ and the spin transformation $\Lambda^{\sigma}{}_{\nu}$, applied in concert, leave the $\sigma^{k\dot{\rho}}{}_{\sigma}$ invariant if

$$\sigma^{k\dot{\rho}}{}_{\sigma}=\sigma'^{k\dot{\rho}}{}_{\sigma}=a^{k}{}_{l}\Lambda^{\dot{\rho}}{}_{\dot{\mu}}\sigma^{l\dot{\mu}}{}_{\nu}\Lambda^{-1\nu}{}_{\sigma},$$

i.e., if

$$a^{k}{}_{l}\sigma^{l\dot{\mu}}{}_{\nu}=\Lambda^{-1\dot{\mu}}{}_{\dot{\rho}}\sigma^{k\dot{\rho}}{}_{\sigma}\Lambda^{\sigma}{}_{\nu}. \qquad \text{(VIII 4)}$$

When a Lorentz transformation $a^{k}{}_{l}$ is given, these equations can be solved in a straightforward way, if they can be solved at all; together with (VI 2), they suffice to determine $\Lambda^{\sigma}{}_{\nu}$ up to algebraic sign. Thus there are exactly two spin transformations corresponding to every Lorentz transformation which is represented at all in spin space, namely $\pm\Lambda^{\sigma}{}_{\nu}$, where $\Lambda^{\sigma}{}_{\nu}$ is a unimodular solution of (VIII 4).

On account of the invariance of $\gamma_{\mu\nu}$ and $\gamma^{\mu\nu}$ in respect to unimodular transformations, the $\Lambda^{\sigma}{}_{\nu}$ determined by (VIII 4) and (VI 2) will, when coordinated with the given Lorentz transformation, hold constant not only $\sigma^{k\dot{\rho}}{}_{\sigma}$ but also $\sigma^{k\dot{\rho}\sigma}$ and $\sigma^{k}{}_{\dot{\rho}\sigma}$.

All the Lorentz transformations constitute a group. From (I 2) and the defining equation

$$g_{kl}=g'_{mn}a^{m}{}_{k}a^{n}{}_{l}=g'_{kl},$$

it is clear that the determinant of every Lorentz transformation is either $+1$ or -1. The transformations with determinant $+1$ constitute a subgroup of the full group, the so-called *restricted* Lorentz group. The transformations with determinant -1 are called inversions.

If one undertakes to solve (VIII 4) for the spin transformation corresponding to an inversion in world space, in place of a solution he gets a self-contradictory set of equations. The following argument shows that, in general, there is no unimodular spin transformation which will hold the σ's constant when a Lorentz transformation of determinant -1 is made in world space: The identity transformation in spin space corresponds to the identity Lorentz transformation. All the spin transformations of determinant $+1$ can be produced in a continuous way from the identity transformation, i.e., can be arrived at by application of infinitesimal transformations of determinant $+1$. An infinitesimal transformation in spin space corresponds to an infinitesimal Lorentz transformation. Hence the Lorentz transformations corresponding to unimodular spin transformations can be reached continuously from the identity transformation, which has determinant $+1$. But every Lorentz transformation has determinant $+1$ or -1, and the ones with determinant -1 are not infinitesimal (they are inversions). Thus only the positive Lorentz transformations correspond to unimodular spin transformations.

It is important for the purpose of discussing the reflection covariance of equations that we be able to represent inversions, somehow, in spin space. We should therefore like to coordinate with each negative Lorentz transformation some sort of spin transformation, such that when the two are performed together the matrices $\sigma^{m\dot{\rho}}{}_{\sigma}$ are left invariant. This can be accomplished as follows:

An arbitrary inversion may be regarded as being composed of the reflection

$$c^{l}{}_{m}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix} \qquad \text{(VIII 5)}$$

and a positive Lorentz transformation $a^{k}{}_{l}$. We coordinate with the latter, as usual, a spin transformation $\Lambda^{\sigma}{}_{\nu}$ which is a unimodular solution of (VIII 4).

With the special reflection (VIII 5) we shall coordinate a new kind of transformation which is defined in the following way: To each spinor $\eta^{\mu}{}_{\nu\dot{\sigma}}$ (for example) we make correspond a new spinor $\eta^{C\mu}{}_{\nu\dot{\sigma}}$, whose components are conjugate complex to the corresponding components of $\eta^{\mu}{}_{\nu\dot{\sigma}}$, and which obeys the same transformation law as does $\eta^{\mu}{}_{\nu\dot{\sigma}}$. Thus $\eta^{C\mu}{}_{\nu\dot{\sigma}}$ has the same components as $\eta^{\dot{\mu}}{}_{\dot{\nu}\sigma}$, but transforms quite differently. The new transformation, which we shall coordinate with (VIII 5), is then defined by

$$\eta^{\mu}{}_{\nu\dot{\sigma}}\rightarrow\eta^{C\mu}{}_{\nu\dot{\sigma}}$$

for all spinors η. It is clear from this definition that

$$(\eta^{C\mu}{}_{\nu\dot{\sigma}})^{C}=\eta^{\mu}{}_{\nu\dot{\sigma}}.$$

When this C transformation and the reflection (VIII 5) are performed together, the $\sigma^{m\dot{\rho}}{}_{\sigma}$ are left invariant:

$$(c^{l}{}_{m}\sigma^{m\dot{\rho}}{}_{\sigma})^{C}\equiv c^{l}{}_{m}\sigma^{Cm\dot{\rho}}{}_{\sigma}=\sigma^{l\dot{\rho}}{}_{\sigma}. \qquad \text{(VIII 6)}$$

This result is easy to verify by reference to (VIII 2); for $l=2$,

$$(c^{2}{}_{m}\sigma^{m\dot{\rho}}{}_{\sigma})^{C}=(-\sigma^{2\dot{\rho}}{}_{\sigma})^{C}=\sigma^{2\dot{\rho}}{}_{\sigma},$$

since the components of $\sigma^{2\dot{\rho}}{}_{\sigma}$ are pure imaginary and the C transformation does not affect the Λ-transformation properties. It is similarly clear that

$$(c^{l}{}_{m}\sigma^{m\dot{\rho}\sigma})^{C}=\sigma^{l\dot{\rho}\sigma}, \quad (c^{l}{}_{m}\sigma^{m}{}_{\dot{\rho}\sigma})^{C}=\sigma^{l}{}_{\dot{\rho}\sigma}.$$

An arbitrary inversion can be expressed in the form $a^{k}{}_{l}c^{l}{}_{m}$, where $|a^{k}{}_{l}|=1$ and $c^{l}{}_{m}$ is (VIII 5). With this inversion we coordinate the spin transformation which consists in first performing the C transformation, then performing the unimodular Λ-transformation corresponding to $a^k{}_l$ by (VIII 4). With this coordination, the σ's are evidently kept invariant.

Using (VIII 6), we can now formulate the covariance of equations with respect to the reflection (VIII 5), and hence with respect to all negative Lorentz transformations. For example, (VII 6) becomes

$$A'^l = c^l{}_m A^m = c^l{}_m \sigma^{m\dot{\rho}\sigma} A_{\dot{\rho}\sigma} = \sigma^{Cl\dot{\rho}\sigma} A_{\dot{\rho}\sigma}$$

or

$$A'^l = \sigma^{l\dot{\rho}\sigma} A^C{}_{\dot{\rho}\sigma},$$

by virtue of the reality of the components of A'^l. Just as with the positive Lorentz transformations, we are able to express the transformed world vector components in terms of the transformed spinor components by means of the *original* σ's and an equation of the same form as (VII 6). This means that (VII 6) is covariant with respect to the reflection $c^l{}_m$ (VIII 5). But evidently (VII 6) is also covariant with respect to positive Lorentz transformations $a^k{}_l$. Hence it is covariant with respect to any inversion $a^k{}_l c^l{}_m$.

The covariance of the Dirac set (I 11) with respect to inversions is proved in a similar way. We substitute into (I 11) the expression

$$\partial_k + i\epsilon\phi_k = c^l{}_k(\partial'_l + i\epsilon\phi'_l),$$

where $c^l{}_m$ is given by (VIII 5). The result is, by (VIII 6),

$$\sqrt{2}\sigma^{Cl\dot{\nu}}{}_\rho(\partial'_l + i\epsilon\phi'_l)\chi^\rho = \mu\xi^{\dot{\nu}},$$
$$\sqrt{2}\sigma^{Cl\nu}{}_{\dot{\rho}}(\partial'_l + i\epsilon\phi'_l)\xi^{\dot{\rho}} = \mu\chi^\nu.$$

In order to recover the original σ's, we perform the C transformation on these equations:

$$\sqrt{2}\sigma^{l\dot{\nu}}{}_\rho(\partial'_l - i\epsilon\phi'_l)\chi^{C\rho} = \mu\xi^{C\dot{\nu}},$$
$$\sqrt{2}\sigma^{l\nu}{}_{\dot{\rho}}(\partial'_l - i\epsilon\phi'_l)\xi^{C\dot{\rho}} = \mu\chi^{C\nu}.$$

Taking complex conjugates so as to return to a form similar to that of (I 11), we obtain finally

$$\sqrt{2}\sigma^{l\nu}{}_{\dot{\rho}}(\partial'_l + i\epsilon\phi'_l)\chi^{C\dot{\rho}} = \mu\xi^{C\nu},$$
$$\sqrt{2}\sigma^{l\dot{\nu}}{}_\rho(\partial'_l + i\epsilon\phi'_l)\xi^{C\rho} = \mu\chi^{C\dot{\nu}}.$$

Upon comparison with (I 11), this is seen to be the same set of equations. The correspondence between the transformed and the untransformed spinors χ, ξ is

$$\chi^\rho \to \xi^{C\rho}, \quad \xi^{\dot{\rho}} \to \chi^{C\dot{\rho}}.$$

As these various spinors are only determined as solutions of (I 11) and the transformed set of equations, this interchange of names is not significant. Thus the Dirac set (I 11) is covariant with respect to (VIII 5), and hence with respect to all inversions.[12]

[12] The two-component equation mentioned in footnote 5 is not reflection covariant: (VIII 5) carries it into

$$\sqrt{2}\sigma^{k\dot{\nu}}{}_\rho c^l{}_k(\partial'_l + i\epsilon\phi_l')\chi^\rho = \mu\chi^{\dot{\nu}},$$

i.e.,

$$\sqrt{2}\sigma^{k\dot{\nu}}{}_\rho(\partial'_l - i\epsilon\phi'_l)\chi^{C\rho} = \mu\chi^{C\dot{\nu}},$$

which is the charge-conjugate of the original equation.

For the theory of charge-conjugation, see Kramers (48) and (7), Sec. 63 and 64. Charge conjugation in the Dirac theory is best discussed on the basis of (IX 31).

To every positive Lorentz transformation there correspond two unimodular spin transformations, which are negatives of one another and are solutions of (VI 2) and (VIII 4). This correspondence maintains product relations: i.e., if

$$a^k{}_l \to \Lambda^\rho{}_\mu, \quad b^m{}_k \to \Omega^\kappa{}_\rho$$

then

$$\sigma^{\kappa\dot{\rho}}{}_\sigma = a^k{}_l \Lambda^{\dot{\rho}}{}_{\dot{\mu}} \sigma^{l\dot{\mu}}{}_\nu \Lambda^{-1\nu}{}_\sigma$$

and

$$\sigma^{m\dot{\kappa}}{}_\lambda = (b^m{}_k a^k{}_l)(\Omega^{\dot{\kappa}}{}_{\dot{\rho}} \Lambda^{\dot{\rho}}{}_{\dot{\mu}}) \sigma^{l\dot{\mu}}{}_\nu (\Lambda^{-1\nu}{}_\sigma \Omega^{-1\sigma}{}_\lambda),$$

so that

$$b^m{}_k a^k{}_l \to \Omega^\kappa{}_\rho \Lambda^\rho{}_\mu.$$

When to every element of a group there is made correspond some linear transformation of an n-dimensional vector space (over a specified field) onto itself, such that to the product of two group elements there always corresponds the product of the transformations representing those elements, this correspondence is called a representation of the group, of degree n. Spin space, as already presented, is a two-dimensional vector space over the field of complex numbers. It is now apparent that the correspondence of unimodular spin transformations to positive Lorentz transformations gives a representation (of degree 2) of the restricted Lorentz group. This is a two-valued representation, since to each positive Lorentz transformation there correspond two spin transformations.

As an example of the coordination of spin transformations to Lorentz transformations, consider the special Lorentz transformation,

$$x' = x\cosh\phi - ct\sinh\phi,$$
$$ct' = ct\cosh\phi - x\sinh\phi,$$

from an original frame K to a new frame K' moving along the x axis of K with velocity

$$V = c\tanh\phi$$

relative to K. By solving Eqs. (VIII 4) in this case, one finds the corresponding spin transformation to be either of the pair

$$\Lambda^\mu{}_\sigma = \pm\begin{pmatrix}\cosh(\phi/2) & \sinh(\phi/2)\\ \sinh(\phi/2) & \cosh(\phi/2)\end{pmatrix}.$$

Another kind of special Lorentz transformation is a rotation in space. In this case the Lorentz transformation coefficients satisfy the conditions

$$a^0{}_0 = 1, \quad a^0{}_k = a^m{}_0 = 0 \quad (k, m \neq 0).$$

Then the first equation of the set

$$\sigma^{m\dot{\mu}\nu} = a^m{}_k \Lambda^{\dot{\mu}}{}_{\dot{\rho}} \Lambda^\nu{}_\sigma \sigma^{k\dot{\rho}\sigma}$$

[see (VIII 4), (III 1), (III 3)] reads

$$\begin{pmatrix}\Lambda^{\dot{1}}{}_{\dot{1}} & \Lambda^{\dot{1}}{}_{\dot{2}}\\ \Lambda^{\dot{2}}{}_{\dot{1}} & \Lambda^{\dot{2}}{}_{\dot{2}}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Lambda^1{}_1 & \Lambda^2{}_1\\ \Lambda^1{}_2 & \Lambda^2{}_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \text{(VIII 7)}$$

because of the form of $\sigma^{0\dot{\mu}\nu}$ in (VIII 1). Equation (VIII 7) is just the *unitary* condition, often written $\Lambda^*\tilde{\Lambda}=1$ or $\Lambda^\dagger\Lambda=1$.

Thus spatial rotations can be represented by spin transformations which are not only unimodular but also unitary. The matrix elements of such spin transformations are called the Cayley-Klein parameters representing the rotation.

It may be of some utility to have the spin transformations representing a general spatial rotation written down in terms of the Euler angles.[13] The spin transformations corresponding to a rotation φ about the z axis are

$$\Phi^\rho{}_\nu=\pm\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2}\end{pmatrix};$$

those corresponding to a rotation θ about the new x axis are

$$\Theta^\sigma{}_\rho=\pm\begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2)\end{pmatrix};$$

and those representing a rotation ψ about the new z axis have the same form as $\Phi^\rho{}_\nu$, but with ψ substituted for φ. The spin transformation $\Lambda^\mu{}_\nu$ representing the entire rotation in space which is specified by the Euler angles φ, θ, ψ is obtained by multiplying the matrices representing the separate rotations

$$\Lambda^\mu{}_\nu=\Psi^\mu{}_\sigma\Theta^\sigma{}_\rho\Phi^\rho{}_\nu$$
$$=\pm\begin{pmatrix} e^{-i(\varphi+\psi)/2}\cos(\theta/2) & -ie^{i(\varphi-\psi)/2}\sin(\theta/2) \\ -ie^{-i(\varphi-\psi)/2}\sin(\theta/2) & e^{i(\varphi+\psi)/2}\cos(\theta/2)\end{pmatrix}. \tag{VIII 8}$$

The following rule is a convenient mnemonic device: The spin transformation corresponding to a rotation through angle θ about the kth Cartesian coordinate axis ($k=1, 2, 3$) is

$$\Lambda_k=1\cos(\theta/2)-i\sigma^k\sin(\theta/2),$$

where σ^k are the last three of the set of matrices $\sigma^{k\dot{\rho}\sigma}$ (VIII 1). One can verify this result by direct calculation. Its significance is discussed by Corben and Stehle (3) in relation to the representation of rotations by quaternions.

IX. THE MAXWELL-LORENTZ AND DIRAC EQUATIONS

In Sec. I it was seen that the concept of spinors arises in a study of the transformation properties of the wave functions in Dirac's equation for the electron. Now that a complete spinor algebra is at hand, it is of interest to see how the Dirac equation can be written entirely in spinor notation. At the same time, it will also be useful to have the fundamental equations of the theory of electromagnetism written out in terms of spinors.[14] In this section we shall assume that we have to deal only with Lorentz frames and transformations, so that the special spinor calculus presented in Sec. VIII will be applicable.

Because of the continuity equation

$$\partial_k j^k=0, \tag{IX 1}$$

the Maxwell-Lorentz equations for the antisymmetric field tensor F^{kl},

$$\partial_l F^{kl}=(4\pi/c)j^k, \tag{IX 2}$$

$$\partial_m F_{kl}+\partial_k F_{lm}+\partial_l F_{mk}=0, \tag{IX 3}$$

give only $3+3=6$ independent relations. Equation (IX 3) is satisfied identically if the electromagnetic field tensor is written in terms of the four-potential ϕ_k (I 3):

$$F_{kl}=\partial_k\phi_l-\partial_l\phi_k. \tag{IX 4}$$

The remaining three Eqs. (IX 2) do not suffice to determine the four components of ϕ_k; this ambiguity signifies that we can subject ϕ_k to a gauge transformation without altering F_{kl}. If we fix the gauge by means of the Maxwell-Lorentz condition

$$\partial_k\phi^k=0, \tag{IX 5}$$

Eqs. (IX 2) become

$$g^{kl}\partial_k\partial_l\phi^m=-(4\pi/c)j^m. \tag{IX 6}$$

It is immediately clear that (IX 1) and (IX 5) can be written in terms of Hermitean spinors of second rank by means of (VII 11). We define

$$\begin{aligned}\partial_{\dot{\mu}\nu}&=\sigma^k{}_{\dot{\mu}\nu}\partial_k,\\ j^{\dot{\mu}\nu}&=\sigma^{k\dot{\mu}\nu}j_k,\\ \phi^{\dot{\mu}\nu}&=\sigma^{k\dot{\mu}\nu}\phi_k,\end{aligned} \tag{IX 7}$$

in accordance with (VII 2). Then (IX 1) becomes

$$\partial_{\dot{\mu}\nu}j^{\dot{\mu}\nu}=0, \tag{IX 8}$$

because of (VII 11). Similarly, (IX 5) becomes

$$\partial_{\dot{\mu}\nu}\phi^{\dot{\mu}\nu}=0. \tag{IX 9}$$

Equation (IX 6) can also be written over into spinor form without difficulty. Because of (VII 11), the d'Alembertian becomes

$$g^{kl}\partial_k\partial_l=\partial_{\dot{\mu}\nu}\partial^{\dot{\mu}\nu}, \tag{IX 10}$$

and since the $\sigma^{k\dot{\rho}\sigma}$ are constants so long as we deal only with special relativity theory, it follows that (IX 6) can be written

$$\partial_{\dot{\mu}\nu}\partial^{\dot{\mu}\nu}\phi_{\dot{\rho}\sigma}=-(4\pi/c)j_{\dot{\rho}\sigma}. \tag{IX 11}$$

[13] To avoid ambiguity, let us state here that we have chosen the Euler angles as given in Goldstein (6), p. 107. The same book contains an instructive discussion of the Cayley-Klein parameters.

[14] The spinor form of the Maxwell equations was presented by Laporte and Uhlenbeck (15). In this same paper the Dirac equations were derived from a spinor variational principle, and some new relations following from the Dirac equations were given.

In part of our derivation of the spinor form of Maxwell's equations, we follow unpublished lectures of Dirac.

The field tensor F^{kl} can, of course, be expressed in terms of a fourth-rank spinor by means of (VII 8):

$$F^{kl}=\sigma^{k\dot\mu\nu}\sigma^{l\dot\rho\sigma}F_{\dot\mu\nu\dot\rho\sigma}. \qquad \text{(IX 12)}$$

However, F^{kl} is antisymmetric, and this fact permits us to express it in terms of a spinor of only second rank, which is not Hermitean but symmetric. The proof of this theorem will be given for the field tensor F^{kl}, but it will naturally remain valid if F^{kl} is replaced by an arbitrary antisymmetric contravariant world tensor of second rank.

From $F^{kl}=-F^{lk}$ and (IX 12),

$$F_{\dot\rho\sigma\dot\mu\nu}=-F_{\dot\mu\nu\dot\rho\sigma}. \qquad \text{(IX 13)}$$

Because the $\sigma^{k\dot\mu\nu}$ are Hermitean, it also follows that

$$F_{\dot\mu\nu\dot\rho\sigma}=F_{\nu\dot\mu\sigma\dot\rho}. \qquad \text{(IX 14)}$$

The reduction to second rank is to be carried out by using

$$\eta_{\lambda\mu}-\eta_{\mu\lambda}=\eta_{\alpha}{}^{\alpha}\gamma_{\lambda\mu}. \qquad \text{(V 6)}$$

Using (IX 13), we write

$$F_{\dot\mu\nu\dot\rho\sigma}=\tfrac{1}{2}(F_{\dot\mu\nu\dot\rho\sigma}-F_{\dot\rho\sigma\dot\mu\nu})$$
$$=\tfrac{1}{2}(F_{\dot\mu\nu\dot\rho\sigma}-F_{\dot\mu\sigma\dot\rho\nu})+\tfrac{1}{2}(F_{\sigma\dot\mu\nu\dot\rho}-F_{\sigma\dot\rho\nu\dot\mu}),$$

by adding and subtracting a term and using (IX 14). Finally, applying (V 6) to this result,

$$F_{\dot\mu\nu\dot\rho\sigma}=\tfrac{1}{2}(F_{\dot\mu\dot\rho}\gamma_{\nu\sigma}+F_{\sigma\nu}\gamma_{\dot\mu\dot\rho}), \qquad \text{(IX 15)}$$

where

$$F_{\sigma\nu}\equiv F_{\sigma\dot\alpha\nu}{}^{\dot\alpha}. \qquad \text{(IX 16)}$$

A proof that $F_{\sigma\nu}$ is symmetric goes

$$F_{\sigma\dot\alpha\nu}{}^{\dot\alpha}=-F_{\sigma\dot\alpha\nu\dot\beta}\gamma^{\dot\beta\dot\alpha}=+F_{\nu\dot\beta\sigma\dot\alpha}\gamma^{\dot\beta\dot\alpha}=-F_{\nu\dot\beta\sigma\dot\alpha}\gamma^{\dot\alpha\dot\beta}=+F_{\nu\dot\beta\sigma}{}^{\dot\beta}$$

by using (IV 8), (IX 13), (IV 7), and (IV 8) (in that order). Note that

$$2F_{kl}F^{kl}=F_{\sigma\nu}F^{\sigma\nu}+F_{\dot\rho\dot\mu}F^{\dot\rho\dot\mu}. \qquad \text{(IX 16a)}$$

When ϕ_k and ∂_k are expressed in terms of the spinors $\phi_{\dot\mu\nu}$ and $\partial_{\dot\mu\nu}$ introduced in (IX 7), Eq. (IX 4) becomes

$$F^{kl}=\sigma^{k\dot\mu\nu}\sigma^{l\dot\rho\sigma}(\partial_{\dot\mu\nu}\phi_{\dot\rho\sigma}-\partial_{\dot\rho\sigma}\phi_{\dot\mu\nu}).$$

By comparison with (IX 12),

$$F_{\dot\mu\nu\dot\rho\sigma}=\partial_{\dot\mu\nu}\phi_{\dot\rho\sigma}-\partial_{\dot\rho\sigma}\phi_{\dot\mu\nu}.$$

Then, from (IX 16), with use of (IX 14),

$$F_{\nu\sigma}=\partial_{\nu\dot\alpha}\phi_{\sigma}{}^{\dot\alpha}-\partial_{\sigma}{}^{\dot\alpha}\phi_{\nu\dot\alpha}=\partial_{\nu\dot\alpha}\phi_{\sigma}{}^{\dot\alpha}+\partial_{\sigma\dot\alpha}\phi_{\nu}{}^{\dot\alpha}. \qquad \text{(IX 17a)}$$

This result can be simplified still further by remarking that

$$\partial_{\nu\dot\alpha}\phi_{\sigma}{}^{\dot\alpha}-\partial_{\sigma\dot\alpha}\phi_{\nu}{}^{\dot\alpha}=\partial_{\beta\dot\alpha}\phi^{\beta\dot\alpha}\gamma_{\nu\sigma}=0$$

because of (V 6) and the Maxwell-Lorentz condition (IX 9). Thus

$$F_{\sigma\nu}=2\partial_{\sigma\dot\alpha}\phi_{\nu}{}^{\dot\alpha}. \qquad \text{(IX 17)}$$

Now we can write the field Eqs. (IX 2) over into spinor form. Consider the expression

$$\partial^{\sigma}{}_{\dot\mu}F_{\sigma\nu}=2\partial^{\sigma}{}_{\dot\mu}\partial_{\sigma\dot\alpha}\phi_{\nu}{}^{\dot\alpha},$$

where (IX 17) has been used. The differential operators on the right can be combined to form the d'Alembertian (IX 10) as follows: Since

$$\partial^{\sigma}{}_{\dot\mu}\partial_{\sigma\dot\alpha}\phi_{\nu}{}^{\dot\alpha}=-\partial_{\sigma\dot\mu}\partial^{\sigma}{}_{\dot\alpha}\phi_{\nu}{}^{\dot\alpha},$$

we have

$$2\partial^{\sigma}{}_{\dot\mu}\partial_{\sigma\dot\alpha}\phi_{\nu}{}^{\dot\alpha}=(\partial_{\sigma\dot\alpha}\partial^{\sigma}{}_{\dot\mu}-\partial_{\sigma\dot\mu}\partial^{\sigma}{}_{\dot\alpha}]\phi_{\nu}{}^{\dot\alpha}=\partial_{\sigma\dot\rho}\partial^{\sigma\dot\rho}\phi_{\nu\dot\mu}=\partial_{\sigma\dot\rho}\partial^{\sigma\dot\rho}\phi_{\dot\mu\nu}$$

by (V 6). Thus

$$\partial^{\sigma}{}_{\dot\mu}F_{\sigma\nu}=-(4\pi/c)j_{\dot\mu\nu}, \qquad \text{(IX 18)}$$

by comparison with (IX 11).

Now let us write the Dirac equation (I 6) entirely over into spinor form. By raising and lowering indexes in (I 11), which may conveniently be taken as a starting point, we obtain

$$\begin{aligned}\sqrt{2}\sigma^{k}{}_{\dot\mu\rho}(\partial_k+i\epsilon\phi_k)\chi^{\rho}&=\mu\xi_{\dot\mu}\\ \sqrt{2}\sigma^{k\nu\dot\sigma}(\partial_k+i\epsilon\phi_k)\xi_{\dot\sigma}&=-\mu\chi^{\nu}.\end{aligned} \qquad \text{(IX 19)}$$

When (IX 7) is taken into account, these equations become

$$\begin{aligned}\sqrt{2}(\partial_{\dot\mu\rho}+i\epsilon\phi_{\dot\mu\rho})\chi^{\rho}&=\mu\xi_{\dot\mu},\\ \sqrt{2}(\partial^{\nu\dot\sigma}+i\epsilon\phi^{\nu\dot\sigma})\xi_{\dot\sigma}&=-\mu\chi^{\nu}.\end{aligned} \qquad \text{(IX 20)}$$

which is a spinor form of Dirac's equation.[15]

The current is

$$j^{k}=q\sigma^{k\dot\mu\nu}(\chi_{\dot\mu}\chi_{\nu}+\xi_{\dot\mu}\xi_{\nu}). \qquad \text{(IX 21)}$$

This quantity transforms as a world vector. Its components are real, since the combinations $\chi_{\dot\mu}\chi_{\nu}$ and $\xi_{\dot\mu}\xi_{\nu}$ are Hermitean. Let

$$j_{\dot\mu\nu}=q(\chi_{\dot\mu}\chi_{\nu}+\xi_{\dot\mu}\xi_{\nu}). \qquad \text{(IX 22)}$$

Then, by (VII 11),

$$\begin{aligned}\partial_k j^k=\partial^{\dot\mu\nu}j_{\dot\mu\nu}&=q(\partial_{\dot\mu\nu}\chi^{\dot\mu}\chi^{\nu}+\partial^{\dot\mu\nu}\xi_{\dot\mu}\xi_{\nu})\\ &=q\{(\partial_{\dot\mu\nu}\chi^{\dot\mu})\chi^{\nu}+\chi^{\dot\mu}(\partial_{\dot\mu\nu}\chi^{\nu})\\ &\quad+(\partial^{\dot\mu\nu}\xi_{\dot\mu})\xi_{\nu}+\xi_{\dot\mu}(\partial^{\dot\mu\nu}\xi_{\nu})\}.\end{aligned}$$

Taking Dirac's set (IX 20) into account, one sees that this quantity is identically zero:

$$\partial_k j^k=\partial^{\dot\mu\nu}j_{\dot\mu\nu}=0. \qquad \text{(IX 23)}$$

The Schwarzschild-Darwin variational principle takes the form derived by Laporte and Uhlenbeck, with a Lagrangian density

$$\begin{aligned}L=i\{&\chi^{\lambda}\partial_{\lambda\dot\mu}\chi^{\dot\mu}-\xi^{\lambda}\partial_{\lambda\dot\mu}\xi^{\dot\mu}-\chi_{\dot\mu}\partial^{\dot\mu\lambda}\chi_{\lambda}+\xi_{\dot\mu}\partial^{\dot\mu\lambda}\xi_{\lambda}\}\\ &+\sqrt{2}\mu i(-\chi^{\lambda}\xi_{\lambda}+\chi^{\dot\lambda}\xi_{\dot\lambda})+2\epsilon\phi^{\dot\mu\lambda}(\chi_{\dot\mu}\chi_{\lambda}+\xi_{\dot\mu}\xi_{\lambda})\\ &-(1/16\pi\hbar)[(\partial^{\mu}{}_{\dot\lambda}\phi^{\nu\dot\lambda}+\partial^{\nu}{}_{\dot\lambda}\phi^{\mu\dot\lambda})(\partial_{\mu}{}^{\dot\sigma}\phi_{\nu\dot\sigma}+\partial_{\nu}{}^{\dot\sigma}\phi_{\mu\dot\sigma})\\ &+(\partial^{\dot\mu}{}_{\lambda}\phi^{\dot\nu\lambda}+\partial^{\dot\nu}{}_{\lambda}\phi^{\dot\mu\lambda})(\partial_{\dot\mu}{}^{\sigma}\phi_{\dot\nu\sigma}+\partial_{\dot\nu}{}^{\sigma}\phi_{\dot\mu\sigma})].\end{aligned} \qquad \text{(IX 24)}$$

When forming the Euler-Lagrange equations, it is necessary to recognize that this Lagrangian can be considered as a function

$$L(\phi^{\dot\mu\lambda},\Sigma\partial^{\dot\mu}{}_{\lambda}\phi^{\dot\nu\lambda},\ldots;\chi^{\lambda},\textstyle\sum\partial^{\dot\mu\lambda}\chi_{\lambda};\ldots) \qquad \text{(IX 25)}$$

[15] Our choice of the σ's coincides with that of Infeld and van der Waerden; our χ^{ρ} and $\xi_{\dot\sigma}$ are their ψ and $-i\chi$.

of the indicated spinors of first and second rank, as a function

$$\mathfrak{L}(\phi^{\dot\mu\lambda},\ \partial^{\dot\mu}{}_{\rho}\phi^{\dot\nu\lambda};\ \chi^{\lambda},\ \partial^{\dot\mu\rho}\chi_{\lambda};\ \ldots) \qquad \text{(IX 26)}$$

of the much more numerous spinors of higher rank, or as a function

$$\mathrm{L}(\phi^{\dot\mu\lambda},\ \partial_k\phi^{\dot\mu\lambda};\ \chi^{\lambda},\ \partial_k\chi^{\lambda};\ \ldots).$$

By a simple transformation using (VIII 3), (IX 7), the Euler-Lagrange equations can be put into the spinor form

$$0=\partial_k\frac{\partial \mathrm{L}}{\partial(\partial_k\chi^{\dot\lambda})}-\frac{\partial \mathrm{L}}{\partial\chi^{\dot\lambda}}=\partial_{\rho\dot\sigma}\frac{\partial\mathfrak{L}}{\partial(\partial_{\rho\dot\sigma}\chi^{\dot\lambda})}-\frac{\partial\mathfrak{L}}{\partial\chi^{\dot\lambda}}.$$

Because of the particular form of L, these equations become

$$\begin{aligned}0=-\frac{\partial\mathfrak{L}}{\partial\chi^{\dot\lambda}}+\partial_{\rho\dot\sigma}\delta^{\dot\sigma}{}_{\dot\lambda}\frac{\partial\mathfrak{L}}{\partial(\partial_{\rho\dot 1}\chi^{\dot 1})}\\ =-\frac{\partial\mathfrak{L}}{\partial\chi^{\dot\lambda}}+\partial_{\rho\dot\lambda}\frac{\partial\mathfrak{L}}{\partial(\partial_{\rho\dot 2}\chi^{\dot 2})}\\ =-\frac{\partial L}{\partial\chi^{\dot\lambda}}+\partial_{\rho\dot\lambda}\frac{\partial L}{\partial(\Sigma\partial_{\rho\dot\alpha}\chi^{\dot\alpha})}.\end{aligned} \qquad \text{(IX 27)}$$

In this last equation, the summation over $\dot\alpha$ is to be taken before the differentiation is performed, contrary to the usual rule that a doubly occurring index indicates summation of the entire expression.

By simple algebraic transformations, observing the somewhat strange-looking identity

$$\partial_k\frac{\partial \mathrm{L}}{\partial(\partial_k\eta)}=\partial^{\dot\sigma\lambda}\frac{\partial\mathfrak{L}}{\partial(\partial^{\dot\sigma\lambda}\eta)}\equiv+\partial_{\dot\sigma}{}^{\lambda}\frac{\partial\mathfrak{L}}{\partial(\partial_{\dot\sigma}{}^{\lambda}\eta)}, \qquad \text{(IX 28)}$$

we obtain

$$\begin{aligned}&-\frac{\partial L}{\partial\chi^{\dot\lambda}}+\partial_{\rho\dot\lambda}\frac{\partial L}{\partial(\Sigma\partial_{\rho\dot\alpha}\chi^{\dot\alpha})}=0,\\ &-\frac{\partial L}{\partial\xi_{\lambda}}+\partial^{\dot\sigma\lambda}\frac{\partial L}{\partial(\Sigma\partial^{\dot\sigma\alpha}\xi_{\alpha})}=0,\\ &-\frac{\partial L}{\partial\phi_{\nu\dot\sigma}}+\partial_{\mu}{}^{\dot\sigma}\frac{\partial L}{\partial(\Sigma\partial_{\mu}{}^{\dot\alpha}\phi_{\nu\dot\alpha})}\\ &\qquad+\partial_{\dot\mu}{}^{\nu}\frac{\partial L}{\partial(\Sigma\partial_{\dot\mu}{}^{\alpha}\phi_{\dot\sigma\alpha})}=0.\end{aligned} \qquad \text{(IX 29)}$$

These equations are the Dirac equations (IX 20) and the Maxwell-Lorentz equations (IX 18) with current (IX 22).

The conservation laws for the current and the energy-momentum tensor can be derived directly from the Lagrangian. We refer the reader to the original literature and former review articles for discussions of these topics.[16]

We wish to close this section with a brief discussion of the relation between the form (I 6) of the Dirac equation (in which the indexes 0, 1, 2, 3 appear in a symmetrical way) and the more commonly quoted form. When

$$i\Gamma^k(-i\partial_k+\epsilon\phi_k)\psi-\mu\psi=0 \qquad \text{(I 6)}$$

is multiplied by $-i\Gamma^0$, it becomes

$$\begin{aligned}\{-\Gamma^0\Gamma^0(i\partial_0-\epsilon\phi_0)+\Gamma^0\Gamma^1(-i\partial_1+\epsilon\phi_1)\\ +\Gamma^0\Gamma^2(-i\partial_2+\epsilon\phi_2)+\Gamma^0\Gamma^3(-i\partial_3+\epsilon\phi_3)+i\mu\Gamma^0\}\psi=0.\end{aligned}$$

If each Γ or the whole operator $\{\ \}$ is subjected to a similarity transformation $\Lambda^{-1}\Gamma\Lambda$ with

$$\Lambda=\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&i&0&-i\\-i&0&i&0\end{bmatrix}, \qquad \text{(IX 30)}$$

this equation is brought into the form[17]

$$\{(i\partial_0-\epsilon\phi_0)+\rho_1\sum_{k=1}^{3}\sigma_k(-i\partial_k+\epsilon\phi_k)+\rho_3\mu\}\Psi=0. \qquad \text{(IX 31)}$$

In this nonrelativistic notation, the Dirac equation is more convenient for the description of spin; and the four components of Ψ fall into a pair of big Ψ functions and a pair of small ones.

X. SPINOR ANALYSIS

In tensor analysis, one considers covariant derivatives of tensors in affinely connected manifolds. We shall pursue a similar procedure in spinor analysis, following in detail the excellent exposition of Infeld and van der Waerden (14).

Covariant differentiation is introduced in world space by means of an affine connection $\Gamma^k{}_{rs}$:

$$A_{r;s}=\partial_sA_r-A_k\Gamma^k{}_{rs},\quad A^r{}_{;s}=\partial_sA^r+A^k\Gamma^r{}_{ks}. \qquad \text{(X 1)}$$

In a similar way, we now define covariant derivatives of spinors

$$\psi_{\alpha;s}=\partial_s\psi_\alpha-\Gamma^\rho{}_{\alpha s}\psi_\rho,\quad \psi^\alpha{}_{;s}=\partial_s\psi^\alpha+\Gamma^\alpha{}_{\rho s}\psi^\rho, \qquad \text{(X 2)}$$

in which we consider the spinors ψ_α, ψ^α as functions of the space-time point specified by the world coordinates x^s. Note that the "product rule" holds for covariant differentiation:

$$(\psi_\alpha\chi^\alpha)_{;s}=\partial_s(\psi_\alpha\chi^\alpha)=\psi_{\alpha;s}\chi^\alpha+\psi_\alpha\chi^\alpha{}_{;s}. \qquad \text{(X 3)}$$

The requirement that $\psi_{\alpha;s}$ and $\psi^\alpha{}_{;s}$ transform as spin vectors in respect to spin transformations leads to the

[16] See Pauli (62), Møller and Rosenfeld (61), Rosenfeld (65), and Belinfante (51).

[17] See Dirac (4) and Hill and Landshoff (43). Here ρ_1, ρ_3, σ_k denote the four-row matrices used by Dirac. These σ_k are, of course, not the σ's of the present paper.

following rule for transforming $\Gamma^{\alpha}{}_{\rho s}$:

$$\Lambda^{\rho}{}_{\sigma}\Gamma'^{\alpha}{}_{\rho s}=\Lambda^{\alpha}{}_{\rho}\Gamma^{\rho}{}_{\sigma s}-\partial_s\Lambda^{\alpha}{}_{\sigma}. \qquad \text{(X 4)}$$

In order to have a consistent formalism, we desire that the complex conjugates of $\psi_{\alpha;s}$ and $\psi^{\alpha}{}_{;s}$ be, respectively, $\psi_{\dot\alpha;s}$ and $\psi^{\dot\alpha}{}_{;s}$. Then the covariant derivatives of the complex conjugate spinors $\psi_{\dot\alpha}$, $\psi^{\dot\alpha}$ are given by placing a dot above every Greek index in (X 2), where $\Gamma^{\dot\rho}{}_{\dot\alpha s}=(\Gamma^{\rho}{}_{\alpha s})^*$. Moreover, an arbitrary spin tensor is differentiated covariantly like an outer product of spin vectors, for example

$$\eta^{\dot\lambda}{}_{\mu;s}=\partial_s\eta^{\dot\lambda}{}_{\mu}+\Gamma^{\dot\lambda}{}_{\dot\rho s}\eta^{\dot\rho}{}_{\mu}-\Gamma^{\sigma}{}_{\mu s}\eta^{\dot\lambda}{}_{\sigma}. \qquad \text{(X 5)}$$

In Sec. VII there was established a correspondence between arbitrary world vectors A_k and Hermitean spinors $A_{\dot\lambda\mu}$ by means of the mixed quantities $\sigma^k{}_{\dot\lambda\mu}$:

$$A^k=\sigma^{k\dot\lambda\mu}A_{\dot\lambda\mu}.$$

The quantities $A^k{}_{;s}$ and $A_{\dot\lambda\mu;s}$ transform as covariant world tensors on the index s. It is natural to require that the connection between these quantities should be as between any other quantities $A^k{}_s$ and $A_{\dot\lambda\mu s}$, *viz.*,

$$A^k{}_{;s}=\sigma^{k\dot\lambda\mu}A_{\dot\lambda\mu;s}. \qquad \text{(X 6)}$$

Covariant differentiation of any tensor or spinor equation obeys the "product rule," as in (X 3). Thus (X 6) has as a consequence

$$\begin{aligned}\sigma^{k\dot\lambda\mu}{}_{;s}&=0\\ &=\partial_s\sigma^{k\dot\lambda\mu}+\Gamma^k{}_{rs}\sigma^{r\dot\lambda\mu}+\Gamma^{\dot\lambda}{}_{\dot\rho s}\sigma^{k\dot\rho\mu}+\Gamma^{\mu}{}_{\sigma s}\sigma^{k\dot\lambda\sigma}.\end{aligned} \qquad \text{(X 7)}$$

We assume that the $\Gamma^k{}_{rs}$ are the Christoffel symbols $\left\{\begin{matrix}rs\\k\end{matrix}\right\}$, i.e., that the notion of parallel transfer is introduced[18] in connection with the metric g_{kl}. The metric establishes geodesics defined by parallel transfer of dx^k/ds. At the same time, $g_{kl}(dx^k/ds)(dx^l/ds)=1$ is invariant, so that g_{kl} also goes into itself when subjected to parallel transfer along a geodesic. Hence

$$g^{kl}{}_{;s}=0. \qquad \text{(X 8)}$$

Christoffel's choice

$$\Gamma^k{}_{rs}=\Gamma^k{}_{sr}, \qquad \text{(X 9)}$$

together with (X 8), determines the $\Gamma^k{}_{rs}$ uniquely in terms of the metric. Formula (X 9) represents 4.6 real equations.

We now proceed to study the spinor affinities $\Gamma^{\alpha}{}_{\rho s}$, considering the $\gamma_{\mu\nu}$ and the $\sigma^{k\dot\lambda\mu}$ to be given as functions of space and time. Because of

$$g^{kl}=\sigma^{k\dot\lambda\mu}\sigma^l{}_{\dot\lambda\mu}=\sigma^{k\dot\lambda\mu}\sigma^{l\dot\rho\sigma}\gamma_{\dot\rho\dot\lambda}\gamma_{\sigma\mu} \qquad \text{(VII 5)}$$

and (X 7), (X 8) we take

$$0=(\gamma_{\dot\rho\dot\lambda}\gamma_{\sigma\mu})_{;s}=\gamma_{\dot\rho\dot\lambda;s}\gamma_{\sigma\mu}+\gamma_{\dot\rho\dot\lambda}\gamma_{\sigma\mu;s}. \qquad \text{(X 10)}$$

This equation has only one distinct nonvanishing component,

$$\begin{aligned}0=\partial_s(\gamma_{\dot1\dot2}\gamma_{12})-\Gamma^{\dot1}{}_{\dot1s}\gamma_{\dot1\dot2}\gamma_{12}-\Gamma^{\dot2}{}_{\dot2s}\gamma_{\dot1\dot2}\gamma_{12}&\\ -\Gamma^1{}_{1s}\gamma_{\dot1\dot2}\gamma_{12}-\Gamma^2{}_{2s}\gamma_{\dot1\dot2}\gamma_{12}.&\end{aligned} \qquad \text{(X 11)}$$

With the abbreviation (IV 9), $\gamma_{\dot1\dot2}\gamma_{12}=\gamma$, this becomes

$$\begin{aligned}\partial_s\gamma-(\Gamma^{\alpha}{}_{\alpha s}+\Gamma^{\dot\alpha}{}_{\dot\alpha s})\gamma&=0,\\ \Gamma^{\alpha}{}_{\alpha s}+\Gamma^{\dot\alpha}{}_{\dot\alpha s}&=\partial_s\ln\gamma.\end{aligned} \qquad \text{(X 12)}$$

Equations (X 10) and (X 11) can be interpreted to signify that covariant differentiation preserves volumes in spin space. (Compare with the Infeld-van der Waerden paper cited above.)

The Eqs. (X 12) are 4 real equations for the $\Gamma^{\alpha}{}_{\rho s}$. The symmetry conditions (X 9), when we insert $\Gamma^k{}_{rs}$, $\Gamma^k{}_{sr}$ from (X 7), are 24 more real conditions on the $\Gamma^{\alpha}{}_{\beta s}$. These 28 conditions are not sufficient to determine the 8.4 real and imaginary parts of the 2.2.4 complex quantities $\Gamma^{\alpha}{}_{\beta s}$. Four real parameters remain arbitrary. As a matter of fact, if Eqs. (X 7) and (X 12) are satisfied, they remain satisfied when one leaves the $\Gamma^k{}_{rs}$ unchanged but replaces the $\Gamma^{\mu}{}_{\rho s}$ and $\Gamma^{\dot\mu}{}_{\dot\rho s}$ by

$$\Gamma^{\mu}{}_{\rho s}+i\epsilon\Phi_s\delta^{\mu}{}_{\rho} \quad\text{and}\quad \Gamma^{\dot\mu}{}_{\dot\rho s}-i\epsilon\Phi_s\delta^{\dot\mu}{}_{\dot\rho}.$$

There thus appears here an arbitrary real quantity ϕ_s which is defined by

$$\Gamma^{\alpha}{}_{\alpha s}-\Gamma^{\dot\alpha}{}_{\dot\alpha s}=4i\epsilon\phi_s. \qquad \text{(X 13)}$$

We investigate the transformation property of this quantity. In respect to world transformations, ϕ_s behaves like a covariant world vector. With transformations in spin space, one can set $|\Lambda^{\rho}{}_{\sigma}|=\Delta e^{i\varphi}$ (Δ positive real) and obtain

$$\phi'_s=\phi_s-\partial_s\varphi/2\epsilon. \qquad \text{(X 14)}$$

This follows from the transformation Eq. (X 4):

$$\Gamma'^{\alpha}{}_{\alpha s}=\Gamma^{\alpha}{}_{\alpha s}-\partial_s\ln|\Lambda^{\rho}{}_{\sigma}|$$

and its complex conjugate. ϕ_s thus transforms with a transformation of the form

$$\Lambda^{\rho}{}_{\sigma}=\delta^{\rho}{}_{\sigma}e^{i\varphi/2} \qquad \text{(X 15)}$$

exactly as the four-potential is supposed to do according to Weyl's principle of gauge invariance. We therefore identify ϕ_s with the electromagnetic potential vector.

From (IV 10), namely $\gamma_{12}=\gamma^{\frac12}e^{i\theta}$, it follows that $\partial_s\theta$ transforms just as $2\epsilon\phi_s$ does,

$$\partial_s\theta'=\partial_s\theta-\partial_s\varphi.$$

[Apply (IV 3) with the general expression $|\Lambda^{-1\rho}{}_{\sigma}|=\Delta^{-1}e^{-i\varphi}$ to $\theta=-\frac12 i(\ln\gamma_{12}-\ln\gamma_{\dot1\dot2})$.] Therefore $2\epsilon\phi_s-\partial_s\theta$ is a vector which is invariant with respect to gauge transformations.

From (X 12), (X 13) one can calculate that

$$\begin{aligned}\Gamma^{\alpha}{}_{\alpha s}&=2i\epsilon\phi_s+\partial_s\ln\gamma^{\frac12},\\ \Gamma^{\dot\alpha}{}_{\dot\alpha s}&=-2i\epsilon\phi_s+\partial_s\ln\gamma^{\frac12}.\end{aligned} \qquad \text{(X 16)}$$

[18] See Schrödinger (9), Chap. IX.

Hence, also taking (X 5) and (IV 10) into account, the differential equations for the fundamental spinors are

$$\begin{aligned}\gamma^{\lambda\mu}{}_{;s}&=i\gamma^{\lambda\mu}(2\epsilon\phi_s-\partial_s\theta),\\ \gamma_{\lambda\mu;s}&=-i\gamma_{\lambda\mu}(2\epsilon\phi_s-\partial_s\theta).\end{aligned}\tag{X 17}$$

The Riemann-Christoffel tensor is

$$R^r{}_{kps}=-\partial_s\Gamma^r{}_{kp}+\partial_p\Gamma^r{}_{ks}-\Gamma^h{}_{kp}\Gamma^r{}_{hs}+\Gamma^h{}_{ks}\Gamma^r{}_{hp}.\tag{X 18}$$

Similarly, we can form, in spin space, the mixed tensor

$$P^\mu{}_{\lambda ps}=-\partial_s\Gamma^\mu{}_{\lambda p}+\partial_p\Gamma^\mu{}_{\lambda s}-\Gamma^\rho{}_{\lambda p}\Gamma^\mu{}_{\rho s}+\Gamma^\rho{}_{\lambda s}\Gamma^\mu{}_{\rho p},\tag{X 19}$$

and the corresponding one with dotted Greek indices. The contraction of the former gives, by (X 16), the electromagnetic field intensity

$$-P^{\dot\rho}{}_{\dot\rho ps}=P^\rho{}_{\rho ps}=2i\epsilon(\partial_p\phi_s-\partial_s\phi_p)=2i\epsilon F_{ps}.\tag{X 20}$$

By definition of the curvature tensors, we have

$$\chi^\rho{}_{;ps}-\chi^\rho{}_{;sp}=\chi^\sigma P^\rho{}_{\sigma sp}\tag{X 21}$$

and hence also

$$\sigma^{k\dot\lambda\mu}{}_{;ps}-\sigma^{k\dot\lambda\mu}{}_{;sp}=\sigma^{k\dot\rho\mu}P^{\dot\lambda}{}_{\dot\rho sp}+\sigma^{k\dot\lambda\rho}P^\mu{}_{\rho sp}+\sigma^{r\dot\lambda\mu}R^k{}_{rsp}.\tag{X 22}$$

Because of (X 7), the left-hand side vanishes. Hence we arrive at a relation between R and P. This relation can be solved for R or for P, e.g.,

$$P^\lambda{}_{\rho sp}=\tfrac{1}{2}R_{krsp}\sigma^{k\lambda\dot\nu}\sigma^r{}_{\dot\nu\rho}+i\epsilon F_{sp}\delta^\lambda{}_\rho.\tag{X 23}$$

If the world space under consideration is flat, the theory can be reduced to a gauge-covariant modification of the 1929 spinor calculus presented in Sec. VIII. In this case, the conditions (VII 5), (X 7), (X 9), (X 12), and (X 13) can be satisfied by taking (I 2) and (VIII 1) for g_{kl} and $\sigma^{k\dot\rho\sigma}$, respectively, and taking

$$\gamma_{\mu\nu}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}e^{i\theta},\tag{X 24}$$

$$\Gamma^k{}_{rs}=\left\{\begin{matrix}rs\\k\end{matrix}\right\}=0,\tag{X 25}$$

$$\Gamma^\beta{}_{\alpha s}=i\epsilon\phi_s\delta^\beta{}_\alpha.\tag{X 26}$$

These forms are all maintained when one carries out in world space an arbitrary positive Lorentz transformation and at the same time applies in spin space the unimodular spin transformation corresponding thereto by (VIII 4). They are also maintained under gauge transformations (X 15).

With the special metrics (I 2) and (X 24) given, the $\sigma^{k\dot\rho\sigma}$ appearing in (VIII 1) are not the only ones which satisfy (VII 5). One still has the freedom to apply an arbitrary (positive or negative) Lorentz transformation on the index k, or to apply an arbitrary unimodular spin transformation on the indices $\dot\rho$, σ. That is, however, the *only* freedom which one has. For the condition (VII 5) means, according to its derivation, that with the correspondence $A_{\dot\rho\sigma}\rightarrow A_k$ the quadratic form $\gamma^{\dot\lambda\dot\rho}\gamma^{\mu\sigma}A_{\dot\lambda\mu}A_{\dot\rho\sigma}$ is brought over into $g^{kl}A_kA_l$, and such a linear correspondence is completely determined up to a linear transformation which leaves the latter quadratic form invariant, i.e., up to a Lorentz transformation.

In Infeld and van der Waerden's spinor analysis, gauge transformations are represented as spin transformations of form (X 15). As a consequence, any generally covariant equation between tensors, spinors, and their covariant derivatives is *ipso facto* gauge-covariant. We have adopted a specialization to the case of flat space-time which has the advantage of retaining this property. It may be instructive to contrast this specialization with other possible specializations and modifications of Infeld and van der Waerden's spinor analysis.

One such modification would be achieved by replacing (X 10) by the stronger assumption $\gamma_{\mu\nu;s}=0$. This procedure would have the advantage that it would permit the raising and lowering of Greek indexes in covariant derivatives by the natural-looking rule $\xi_{\nu;s}=\xi^\mu{}_{;s}\gamma_{\mu\nu}$, which is *not* valid in Infeld and van der Waerden's formalism. This stronger assumption would, however, eliminate from the theory the four arbitrary real quantities ϕ_s, which were interpreted as components of the electromagnetic potential. In Eqs. (X 13), (X 16), and (X 17), $2\epsilon\phi_s$ would be replaced by $\partial_s\theta$. Thus the possibility of using spinor analysis as a gauge-covariant formalism would be lost. In the case of flat space-time, the selection of (I 2), (VI 3), (VIII 1), and (X 25) for g_{kl}, $\gamma_{\mu\nu}$, $\sigma^{k\dot\rho\sigma}$, and $\Gamma^k{}_{rs}$ would reduce this modification exactly to the 1929 spinor calculus of Sec. VIII, since the choice $\gamma=1$, $\theta=0$ (VI 3) would entail $\Gamma^\beta{}_{\alpha s}=0$.

Reverting to Infeld and van der Waerden's form of spinor analysis (that is, rejecting the assumption $\gamma_{\mu\nu;s}=0$), we note that in specializing to the case of flat space-time we could have achieved a closer approach to the 1929 theory by selecting (VI 3) in place of (X 24). Had we made this choice, the resulting specialization would differ from the 1929 spinor calculus only by having (X 26) in place of $\Gamma^\beta{}_{\alpha s}=0$. This is another enticing simplification which, if adopted, would entail the loss of gauge covariance. The choice (VI 3) could not be maintained under gauge transformations, since these are not unimodular [see (IV 3)].

Let us now write the equations of the Dirac theory over into the gauge-covariant formalism provided by the specialization [(I 2), (VIII 1), (X 24), (X 25), (X 26)] which we actually chose to adopt. In our previous discussions of the Dirac theory (Sec. I and Sec. IX), we have worked within the framework of the 1929 spinor calculus, which provides for Lorentz covariance but not gauge covariance. In making the transition to a gauge-covariant theory, we subject Eqs. (I 11) to manipulations admissible under the Lorentz group and the unimodular spin group, in order to bring them into a form in which the differential operators ∂_k and potentials ϕ_k appear only in the gauge-covariant combinations produced by covariant differentiation. Equations (IX 19) have such a form and were obtained by subjecting (I 11) to just such manipulations; they can at once be written

$$\begin{aligned}\sqrt{2}\sigma^k{}_{\dot{\mu}\rho}\chi^\rho{}_{;k}=\sqrt{2}(\sigma^k{}_{\dot{\mu}\rho}\chi^\rho)_{;k}=\mu\xi_{\dot{\mu}}\\ \sqrt{2}\sigma^{k\nu\dot{\sigma}}\xi_{\dot{\sigma};k}=\sqrt{2}(\sigma^{k\nu\dot{\sigma}}\xi_{\dot{\sigma}})_{;k}=-\mu\chi^\nu,\end{aligned} \tag{X 27}$$

by using (X 2), (X 7), and (X 26). Equations (X 27) are the Dirac equations in the spin frame with $\gamma=1$ and $\theta=0$, and have gauge-covariant form; hence they are the Dirac equations in any spin frame with $\gamma=1$, θ arbitrary.

The conservation theorem for the current $j_{\dot{\mu}\nu}$ (IX 22) takes the form $j^k{}_{;k}=(\sigma^{k\dot{\mu}\nu}j_{\dot{\mu}\nu})_{;k}=\sigma^{k\dot{\mu}\nu}j_{\dot{\mu}\nu;k}=0$, and can easily be derived in gauge-covariant fashion using the Dirac equations (X 27) and their complex conjugates.

The second-order equation can be derived in the following way: From (X 27),

$$[\sigma^{k\nu\dot{\sigma}}(\sigma^l{}_{\dot{\sigma}\rho}\chi^\rho)_{;l}]_{;k}=\sigma^{k\nu\dot{\sigma}}\sigma^l{}_{\dot{\sigma}\rho}\chi^\rho{}_{;lk}=-\tfrac{1}{2}\mu^2\chi^\nu.$$

Using (VII 12), one obtains

$$g^{lk}\chi^\nu{}_{;lk}-\sigma^{l\nu\dot{\sigma}}\sigma^k{}_{\dot{\sigma}\rho}\chi^\rho{}_{;lk}=-\tfrac{1}{2}\mu^2\chi^\nu.$$

Taking account of the fact that l and k are summation indexes, and using (X 21),

$$g^{lk}\chi^\nu{}_{;lk}+\sigma^{l\nu\dot{\sigma}}\sigma^k{}_{\dot{\sigma}\rho}P^\rho{}_{\alpha lk}\chi^\alpha=-\mu^2\chi^\nu.$$

This is the second-order equation for χ^ν. If we use (X 2), (X 26), and (X 23), setting $R_{krsp}=0$ since we are working in a flat world space, we obtain

$$g^{lk}(\partial_k+i\epsilon\phi_k)(\partial_l+i\epsilon\phi_l)\chi^\nu+i\epsilon F_{lk}\sigma^{l\nu\dot{\sigma}}\sigma^k{}_{\dot{\sigma}\rho}\chi^\rho=-\mu^2\chi^\nu,$$

which is the Schrödinger-Klein-Gordon equation including the spin interaction term. A similar equation holds for $\xi_{\dot{\sigma}}$.

The Lagrangian density (IX 24) can be written in the form

$$\begin{aligned}L=i[\chi^\lambda(\sigma^k{}_{\lambda\dot{\mu}}\chi^{\dot{\mu}})_{;k}-\chi^{\dot{\mu}}(\sigma^k{}_{\dot{\mu}\lambda}\chi^\lambda)_{;k}-\xi_\lambda(\sigma^{k\lambda\dot{\mu}}\xi_{\dot{\mu}})_{;k}\\ +\xi_{\dot{\mu}}(\sigma^{k\dot{\mu}\lambda}\xi_\lambda)_{;k}]+\sqrt{2}\mu i(-\chi^\lambda\xi_\lambda+\chi^{\dot{\lambda}}\xi_{\dot{\lambda}})\\ +(1/16\pi\hbar)(F^{\mu\nu}F_{\mu\nu}+F^{\dot{\mu}\dot{\nu}}F_{\dot{\mu}\dot{\nu}}),\end{aligned}$$

by use of (IX 17), (X 2), (X 7), and (X 26). The last term is equal to $(1/8\pi\hbar)F^{kl}F_{kl}$, by (IX 16a). The invariance of this Lagrangian density with respect to general spin transformations implies the covariance of the whole theory, in particular with respect to gauge transformations.

LIST OF REFERENCES

We cannot attempt to give a comprehensive literature list for spinor calculus and its related fields. Instead we shall mention, beneath each of the following subheadings, a few of the important publications which might be of interest or of aid to the reader.

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