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Acoustic Radiation Pressure of Plane Compressional Waves*

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Both electromagnetic and acoustic waves exert forces of radiation upon an obstacle placed in the path of the wave, the forces being proportional to the mean energy density of the wave motion. In electromagnetics the action of these forces is relatively easily understood through the concept of Maxwell's electromagnetic stress tensor.

The physical processes leading to these forces in a sound wave have been found to be considerably more complex; the difficulties belong to the fact that the acoustic wave equation is not linear and that a beam of finite cross section is subject to effects caused by the surrounding medium.

Though many papers have been devoted to the subject and though various theoretical approaches have been made, some difficulties still seem to stand in the way of a clear understanding of the physics of the problem.

The purpose of the present study is especially to throw light on the physical aspects. The approach adopted, which uses the momentum theorem, is believed to serve this purpose especially well. The expression for the radiation pressure is given in both Eulerian and Lagrangian coordinate systems.

Special consideration is given to liquids of constant compressibility, since in such media the processes involved can be dealt with mathematically in a simple manner. The general case of a plane reflector with arbitrary reflection coefficient is treated; the *modus operandi* of the forces at the interface between liquid and obstacle is explained for some special cases, including the radiation forces on the interface between two nonmiscible liquids.

Finally, a general relation is established between the energy density and the pressure caused by radiation falling normally upon a plane reflector, which, under certain assumptions, is valid in any fluid and at any amplitude.

1. INTRODUCTION

THE concept of radiation pressure originated in electrodynamics. According to Maxwell's equations and his concept of the electrodynamic stress tensor, a plane surface of perfect conductivity emitting normally a plane electromagnetic wave undergoes a reacting force per unit area equal to the total energy density E of the emitted wave. Also, such a plane wave, striking perpendicularly a plane and totally absorbing surface, exerts a radiation pressure equal to E.

Lord Rayleigh was the first to formulate a theory of radiation pressure resulting from compressional acoustic waves in fluids. He established a relation which gives the time-average value of the pressure produced upon a piston by a plane wave of infinite cross section in a fluid, and he showed that this mean pressure is proportional to the mean mechanical energy density $\bar{E}$ of the periodic wave motion. He found that the factor of proportionality was not in general unity, but dependent on the special law relating the pressure p to the density in the fluid under consideration.

The actual radiation pressure, however, as encountered in an acoustic beam under ordinary experimental conditions, is different from Rayleigh's "pressure of vibrations." Rayleigh's result,[1] which is often quoted, applies to a theoretical case rather than to what is usually measured.

The equations describing the motion of acoustic waves are nonlinear (with the one exception of the Lagrangian wave equation in a fluid of constant com-

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[1] Lord Rayleigh, Phil. Mag. 3, 338 (1902) and 10, 364 (1905). For example, in a gas under adiabatic conditions, Rayleigh's pressure amounts to $(1+\gamma_c)\bar{E}_i$ at a perfect reflector and at small amplitudes. γ_c = ratio of the specific heats and $\bar{E}_i$ = mean energy density of the incident wave = one-half of the total mean energy density in a standing wave.

pressibility). For many purposes, it is sufficient to "linearize" these equations and to retain only first-order terms of the velocities or particle displacements, regarding them as small quantities. Radiation pressure, however, is connected with energy densities, which are quadratic terms containing the squares of velocities or displacements. Any theory dealing with acoustic radiation pressure, therefore, must retain at least all second-order terms, to be valid even at small amplitudes.

The fact that radiation pressure is a second-order quantity and that the vibrational amplitudes are usually very small in comparison with the acoustic wavelength λ, explains its relatively small numerical value in comparison with the values of the periodically alternating first-order pressures encountered in acoustical waves. Whereas the first-order pressure amounts to maximal values up to kilograms per cm^2, acoustic radiation pressure only reaches values of the order of grams or dynes of force per cm^2. Nevertheless, radiation pressure is an important quantity in the experimental determination of acoustic intensity.

Since Rayleigh's investigations, a considerable number of papers have been devoted to the subject, many of

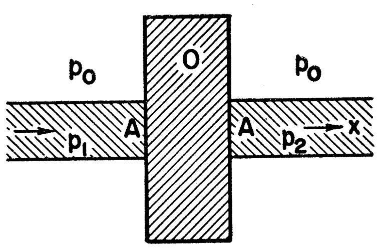

FIG. 1. Acoustic beam of finite cross section A incident upon a plane obstacle 0, and passing through undisturbed regions in which the hydrostatic pressure is p_0.

which, however, do not deal adequately with certain peculiar difficulties inherent in the problem.

L. Brillouin[2] seems to have been the first to give a comprehensive approach to the subject, especially in pointing out the tensorial character of what is usually called radiation "pressure." Indeed, this quantity is not a "pressure" in the sense ordinarily understood in hydrodynamics. Brillouin published his first paper on the subject as early as 1925; various authors, however, in later papers on the same subject, seem not to have paid proper attention to Brillouin's approach.

In what follows, a study is presented of the forces exerted upon a plane obstacle by plane compressional waves in a fluid, meeting the obstacle at normal incidence.[3]

Special attention is devoted to throwing light upon the actual physical processes involved. For this reason, a different approach from that used by Brillouin has been chosen; the study also goes further in the consideration of certain details than has been done hitherto.

[2] L. Brillouin, Ann. phys. (X) 4, 528 (1925).

[3] The case of oblique incidence can easily be solved, if the forces resulting from radiation pressure at normal incidence are known. See, for example, F. E. Borgnis, J. Acoust. Soc. Am. **24**, 468 (1952).

2. STATEMENT OF THE PROBLEM

In dealing with plane waves, some idealizing assumptions must be made. Plane waves cannot strictly be realized experimentally. Still, if the width of the acoustic beam is large in comparison with the wavelength, the concept of plane waves gives a good approximation, especially in the case of the high frequencies used in ultrasonics. Under most experimental conditions, the acoustic beam in any fluid is surrounded by regions of the same fluid. Reasoning as in geometrical optics, we assume that only the region inside the beam is subject to the acoustical wave motion, whereas the region surrounding the beam is assumed to be undisturbed; that is, we disregard diffraction effects at the edge of the beam and assume uniformity of wave motion over the cross section of the beam. The beam is supposed to fall normally upon a plane obstacle, entirely immersed and at rest in the fluid (Fig. 1). This obstacle causes reflection at its front surface; it may absorb partially or totally the penetrating wave energy. The transmitted part leaves the rear of the obstacle as a purely progressive wave.

The interaction of the obstacle with the plane wave motion can be described completely by an amplitude reflection coefficient γ, an amplitude transmission coefficient δ, and the phase angles θ and θ' of the reflected and transmitted waves with respect to the phase of the incident wave; a plane wave motion in the acoustic beam is thus uniquely determined.

Considerable complications in the measurement of radiation pressure are caused by the fact that the acoustic field sets up a steady streaming in the medium. Two effects have been made responsible for this mass flow: first, the so-called "pumping effect," which may occur at the source of radiation and by which fluid is sucked into the beam and set in motion in the direction of the wave propagation. Second, owing to viscous forces, the wave motion generates what is called the "hydrodynamic flow;" a steady vortical motion is set up in the fluid, causing a steady streaming along the beam in the direction of the incident wave and returning outside the beam. The phenomenon of hydrodynamic flow is inseparably associated with acoustic radiation in a fluid. Like the latter, it is an effect of second order; it cannot be treated without taking into account at least second-order terms in the hydrodynamic equations.[4]

In order to measure the forces due to radiation only, these forces must be separated from those caused by the steady mass flow of the fluid. Usually, some sort of screening is resorted to, consisting of thin films, transparent to radiation but preventing the flow from exerting forces upon the obstacle. Such screens are located close and parallel to the surface of the obstacle,

[4] C. Eckart, Phys. Rev. **73**, 68 (1948); J. J. Markham, Phys. Rev. **86**, 497 (1952); P. J. Westervelt, J. Acoust. Soc. Am. **25**, 60 (1953).

and it is commonly assumed that in this way the forces caused by acoustic radiation can be measured with sufficient accuracy.

A separation of the forces in question may also be achieved by a low-frequency intensity modulation of the beam and measuring the radiation pressure by a device which responds to the modulation but not to the steady forces of the mass flow nor to the unmodulated wave.[5]

Since we are only concerned with forces caused by radiation, viscous forces are not taken into account in this paper.

3. GENERAL FORMULA FOR THE ACOUSTIC RADIATION PRESSURE

In order to obtain the force exerted upon 0 in Fig. 1, we apply Newton's theorem of equivalence of the time rate of change in momentum to the forces acting upon 0. To keep 0 in time average at rest, we have to apply a force F, equal and opposite to that caused by the quantity which we call radiation pressure. Denoting by P this radiation pressure, that is, the force *per unit area* of the beam of total area A, F amounts to PA, since we are assuming uniformity over the cross section A.

The other force acting on 0 is due to the "dynamic" pressure p_d and is given by $\oint p_d \mathbf{da}$ over the entire surface of 0. In absence of acoustic waves, the pressure p_d is the undisturbed hydrostatic pressure p_0; when variation with depth is ignored, p_0 is constant over the whole surface and $\oint p_d\mathbf{da}=p_0\oint \mathbf{da}=0$.

If an acoustic wave motion is present, the hydrostatic pressure p_0 becomes "modulated" over the affected parts of 0 by the "excess pressure" p of the acoustic waves; the resultant pressure may be called the "dynamic pressure" $p_d=p_0+p$. Forces are counted positive in the positive x direction (Fig. 1), which is the direction of propagation of the incident wave; the force upon 0 due to the action of p_d is $\oint p_d\mathbf{da}=\oint p\mathbf{da}=(p_1-p_2)A$.

Next we establish the expression for the time rate of change of momentum of 0. As we have to assume continuity of particle displacement at the interface between the fluid and the surface of 0, mechanical motion is transferred to 0. Let ρ be the density and $\mathbf{u}$ the velocity vector of mass elements in 0; the "momentum-density" then is given by $\rho\mathbf{u}$. The quantities p, ρ, and $\mathbf{u}$ are considered as functions of x, y, z, and t, the coordinates belonging to a system fixed in space. 0 may be regarded as having a large inertia, so that its center of gravity may be assumed to be practically at rest, although parts of 0 undergo small and rapid mechanical movements.

In order to compute the change of momentum and to express it by values of the density and the velocity of the *fluid*, we visualize an imaginary surface S stationary in space and in close neighborhood of the surface of 0, but at any time entirely within the fluid. Owing to the law of action and reaction, it is permissible to extend the integrations involved over the surface S instead of over the actual surface of 0.

The momentum of 0 can change in two ways: (a) by a "local" change *in time* of $\rho\mathbf{u}$ within volume elements of 0; the rate of change of this momentum is given by $(d/dt)\int_V \rho\mathbf{u}dV$; (b) by a "convectional" change of $\rho\mathbf{u}$ in the course of the displacement *in space* of mass elements of 0.†

The volume integral of the convectional change in

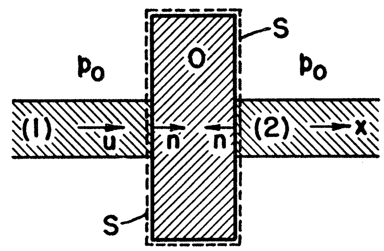

FIG. 2. Imaginary surface S enclosing 0 and stationary in space. n is the inner normal, u the (positive) particle velocity.

momentum of 0 can be transformed into the surface integral over the entire surface of S in Fig. 2 of the "flux of momentum" crossing S per unit time; the integral amounts to $\oint_S (\rho\mathbf{u})(\mathbf{u}\cdot\mathbf{n})da$,[6] Indeed, during a time element dt the "flux of momentum" crossing a surface element da of S is given by $(\rho\mathbf{u})(\mathbf{u}\cdot\mathbf{n})dadt$, $\rho\mathbf{u}$ being the density of momentum and $(\mathbf{u}\cdot\mathbf{n})$ the normal velocity across S. The surface integral results from a gain or loss in momentum of mass elements in the course of their displacements within 0.

In our one-dimensional problem, the total flux of momentum *entering* S per unit time is given by $(\rho_1u_1^2-\rho_2u_2^2)A$; the minus sign of $\rho_2u_2^2$ expresses the fact that this quantity is the flux *leaving* S, when u is positive in the positive x direction. The parts of S out-

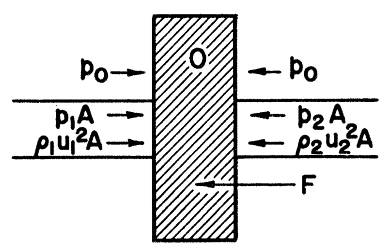

FIG. 3. Directions of the forces acting upon 0.

side the beam give no contribution, as the fluid there is assumed to be at rest.

It may be noted that ρu^2 is always a positive quantity, independent of the sign of u. Owing to the equivalence between force and rate of change of momentum, the

[5] A. Barone and M. Nuovo, Ricerca sci. 21, 516 (1951).

† The *total* change of a quantity q is $Dq/Dt=\partial q/\partial t+(\mathbf{u}\cdot\nabla)q$; the first term is called the local change, the latter term the convectional change of q.

[6] See, for example, L. M. Milne-Thomson, *Theoretical Hydrodynamics* (The Macmillan Company, New York, 1950), p. 72, or *Handbuch der Physik* (Julius Springer, Berlin, 1925), Vol. VII, p. 22.

terms $(\rho_1 u_1^2)$ and $-(\rho_2 u_2^2)$, representing the flux of momentum entering and leaving S per unit time, may be treated as if they represented forces acting upon S and directed always in the sense of the *inner* normal of S (Fig. 2), whatever the direction of motion of the particles crossing S may be. Nothing, however, indicates that such forces *really* act on the front and rear surfaces of S; only their *sum* has a physical meaning, as it integrates the convectional changes of momentum per unit time of all mass elements within S.

Summing up all the forces acting upon 0 (Fig. 3) and assuming that they are balanced by the external force F necessary to keep the center of gravity of 0 in equilibrium,‡ we arrive at the equation

$$\oint_S p_0 \mathbf{da} + (p_1+\rho_1 u_1^2 - p_2 - \rho_2 u_2^2)A - F - (d/dt) \times \int_V \rho u dV = 0, \quad (1)$$

where the last integral is taken over the volume bounded by S.

Since we are interested only in the time average of the radiation pressure, we average Eq. (1) over a full period. Assuming a periodic character of the wave motion and periodicity in time of $\int_V \rho u dV$, the time average of $(d/dt)\int_V \rho u dV$ vanishes; also $\oint_S p_0 \mathbf{da}=0$, disregarding variation of p_0 with depth. From Eq. (1) we obtain, after replacing F by $P_t A$,

$$\bar{P}_t = \bar{p}_1 + \langle \rho_1 u_1^2 \rangle - \bar{p}_2 - \langle \rho_2 u_2^2 \rangle, \quad (2)$$

where the angular parenthesis denotes the average value.

This is the general expression for the *total* mean radiation pressure $\bar{P}_t$ exerted upon 0 in the present case.

Equation (2) suggests regarding $\bar{P}_t$ as consisting of two separate parts $\bar{P}_1$ and $\bar{P}_2$, belonging to the wave motions in front and at the rear of 0. $\bar{P}_1$ can be thought of as being caused by the wave motion in front of 0, $\bar{P}_2$ by an "emitted" wave leaving 0 at its rear. It may be recalled that F was defined as the force necessary to keep 0 in equilibrium; the direction of F is opposite to that of radiation pressure. We can, therefore, define a mean radiation pressure $\bar{P}$ per unit area of a plane compressional wave by

$$\bar{P} = \bar{p} + \langle \rho u^2 \rangle. \quad (3)$$

As so defined, $\bar{P}$ is always directed in the sense of the *inner* normal of the surface interacting with the acoustic wave, whether the surface is thought of as receiving or as emitting radiation.

As a generalization of the expression (3), the mean radiation pressure upon an obstacle of arbitrary shape in an acoustic field can obviously be written

$$\langle \mathbf{P} \rangle = \bar{p}\mathbf{n} + \langle \rho \mathbf{u}(\mathbf{u}\cdot\mathbf{n}) \rangle, \quad (4)$$

where $\mathbf{u}$ now stands for the vector of the velocity $\mathbf{u}(x, y, z)$, and $\mathbf{n}$, as before, is the vector of the inner normal on the surface element da.[7] In the general case, the acoustic radiation may also exert shearing forces upon the obstacle, resulting in a torque upon 0.

The procedure of attributing to each surface element of S a radiation force of amount $\bar{P}$, as defined by Eqs. (3) or (4), has its analogy in electrodynamics, where Poynting's radiation vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ is assigned to each surface element of an irradiated surface. What is, in fact, derived in both cases is the total force or the total electromagnetic radiation belonging to a closed surface; from the surface integral, however, no rigorous conclusion can be drawn with regard to the distribution over the elements of the surface of derived quantities such as $\langle \mathbf{P} \rangle$ or $\mathbf{S}$. Still the concept of a radiation pressure $\bar{P}$ as defined in Eqs. (3) and (4) is very useful and leads, if properly applied, to correct results.

Incidentally, an expression analogous to Eq. (4) is well known in hydrodynamics in connection with what has been called "Euler's momentum theorem."§

4. BRILLOUIN'S STRESS TENSOR IN FLUIDS

From the foregoing considerations, Brillouin's concept of a tensor describing the dynamical stresses acting upon a fluid under acoustic motion can easily be obtained. The general result of Eq. (4) may equally well be applied to a volume V belonging to the *fluid*. We consider a unit volume in the fluid and describe the force exerted upon it in the x direction by the component T_{xx} of a general stress tensor; the tensor components T_{xx}, T_{yy}, T_{zz} are here positive in the sense of the inner normal, that is, in the sense of the positive pressure. Equation (4), then, gives in the one-dimensional case

$$\oint_S \bar{P} da = (\bar{T}_{xx})_1 - (\bar{T}_{xx})_2 = -\int_1^2 (\partial \bar{T}_{xx}/\partial x)dx = \oint_S (\langle p + \rho u^2 \rangle) da,$$

whence by use of a well-known integral transformation

$$-\int_1^2 \frac{\partial \bar{T}_{xx}}{\partial x} dx = -\int_V \frac{\partial}{\partial x}(\langle p+\rho u^2 \rangle) dx dy dz = -\int_1^2 \frac{\partial}{\partial x}(\langle p+\rho u^2 \rangle) dx,$$

since $(\langle p+\rho u^2 \rangle)$ is a function of x only and $dydz=1$. The fluid, therefore, can be thought of as subjected in the

‡ Viscous forces at the interface between 0 and the fluid are excluded. Any viscous forces inside 0 cancel out.

[7] See P. J. Westervelt, J. Acoust. Soc. Am. **23**, 312 (1951), Eq. (8).

§ See L. M. Milne-Thomson, reference 6, p. 74.

x direction to a dynamical stress component

$$\bar{T}_{xx}=p_0+\langle(p+\rho u^2)\rangle=\bar{p}_d+\langle\rho u^2\rangle. \quad (5)$$

Since the particle velocities in the y and z directions are zero, the pressure $\bar{p}_d=p_0+\bar{p}$ is the only force acting in those directions. Thus,

$$\bar{T}_{yy}=\bar{T}_{zz}=\bar{p}_d. \quad (6)$$

These forces acting in the fluid can be described by use of the dynamical stress tensor introduced by Brillouin:[8]

$$\begin{vmatrix} \bar{p}_d+\langle\rho u^2\rangle & 0 & 0 \\ 0 & \bar{p}_d & 0 \\ 0 & 0 & \bar{p}_d \end{vmatrix}. \quad (7)$$

In the absence of acoustic wave motion $\langle\rho u^2\rangle=0$ and $\bar{p}_d=p_0$; the liquid is under the hydrostatic pressure p_0 only, which is the same in every direction. If the acoustic wave is present, the fluid undergoes a non-isotropic state of stress: the stress component $\bar{T}_{xx}$ differs from $\bar{T}_{yy}$ and $\bar{T}_{zz}$ by $\langle\rho u^2\rangle$, that is by the mean flux of momentum density through a stationary area normal to the x direction. Moreover, as is shown later, the pressure p_0 is changed in time average to the mean dynamic pressure $\bar{p}_d=p_0+\bar{p}$, where $\bar{p}$ is found to be proportional to the energy density of the wave motion.

If the fluid is bounded by a plane material surface normal to x, the mean normal stress $\bar{T}_{xx}=\bar{p}_d+\langle\rho u^2\rangle$ is transferred to unit area of this surface; thus, we are led back to the expression for $\bar{P}$, as defined in Eq. (3). At oblique incidence the radiation tensor transforms like the tensor in Eq. (7). In order to attain the final value of $\bar{P}$, the expressions for the tensor components must be completed; since $\bar{p}_d$ will turn out to be different from the pressure p_0 in the undisturbed medium, the interaction between the two regions inside and outside the beam must be considered.

5. A GENERAL RELATION FOR THE MEAN EXCESS PRESSURE, AND FOR THE "RAYLEIGH-PRESSURE"

The mean excess pressure $\bar{p}$ depends upon the properties of the fluid under consideration, that is, upon the relation $p(\rho)$ between pressure and density. $\bar{p}$ finally follows from the solution for the wave motion. Since $\bar{p}$ is related to an area fixed in space, the solution of the so-called *Eulerian* equation of motion is indicated. In this equation the quantities p, ρ, and $\mathbf{u}$ are regarded as functions of x, y, z, and t, the coordinates belonging to a system at rest. A velocity $\mathbf{u}(x, y, z, t)$, for example, means the velocity that would be observed at the point (x, y, z) at the time t; since the fluid is in motion, different particles of the fluid are found at the same point (x, y, z) at a different time t'. In other words, p, ρ, and $\mathbf{u}$ are *not* associated with *particles*, but with a fixed point in space.

Often, however, a solution for the wave motion can more easily be established by using the *Lagrangian* equation of motion. Here the quantities p, ρ, and $\mathbf{u}$ are related to special particles and the solution describes the change in time of p, ρ, and $\mathbf{u}$, which would be noted by an observer at the instantaneous positions of the particles. In order to mark the difference between Eulerian and Lagrangian quantities, we shall denote the latter by p^*, ρ^*, and $\mathbf{u}^*$. The particles themselves are characterized by their original positions at rest, the coordinates of which may be a, b, c; their instantaneous positions are denoted by the displacements ξ, η, ζ (which are functions of a, b, c, t) from their original positions. The *actual* position of a particle with respect to a fixed system is therefore given by $x=a+\xi$; $y=b+\eta$, $z=c+\zeta$. For example, a solution for $\mathbf{u}(a, b, c, t)$ in Lagrangian coordinates indicates the velocity found at the actual position $a+\xi$, $b+\eta$, $c+\zeta$ of the particle, the rest position of which was a, b, c.

The relation between a Lagrangian quantity q^* and the corresponding Eulerian quantity q is expressed by

$$q^*(a, b, c, t)=q\{a+\xi(t), b+\eta(t), c+\zeta(t), t\} = q\{x(t), y(t), z(t), t\} \quad (8)$$

since the positions and velocities of particles are given by

$$\begin{aligned} x(t)&=a+\xi(a, b, c, t); & dx/dt&=u_x=\partial\xi/\partial t \\ y(t)&=b+\eta(a, b, c, t); & dy/dt&=u_y=\partial\eta/\partial t \\ z(t)&=c+\zeta(a, b, c, t); & dz/dt&=u_z=\partial\zeta/\partial t. \end{aligned} \quad (9)$$

If ξ, η, ζ are known functions of (a, b, c, t), the inversion of the system (9) allows one to express a, b, c as functions of (x, y, z, t). By insertion of a, b, c, so found, into q^* of Eq. (8), one obtains the corresponding quantity q in Eulerian coordinates. The inversion of the system (9) cannot ordinarily be accomplished by functions of closed form; one has rather to resort to power developments.‖

Our further considerations will be limited to *small amplitudes* of the acoustic wave motion; that is, terms of third and higher order of amplitudes will be neglected. In compressible liquids, the amplitudes that are so far experimentally obtainable in plane waves are always small; in gases, owing to the nonlinearity of the wave equation, the mathematical difficulties in dealing with finite amplitudes are beyond the scope of the present paper.

Returning to the one-dimensional case under consideration, we will now establish a relation between p and p^*, that is, between the mean pressure p at a fixed position, as needed in Eq. (3) for P, and the mean pressure p^* related to an oscillating particle. Such a relation is very useful, because in acoustics solutions

[8] L. Brillouin, reference 2. Also, "*Les Tenseurs en Mécanique et en Élasticité*" (Dover Publications, New York, 1946), pp. 290, 302. It is proved by Brillouin that the forces can actually be represented by a tensor, that is, that they transform like a tensor.

‖ See F. E. Borgnis, Technical Report No. 1A, March 10, 1953, under U. S. Office of Naval Research Contract Nonr-220(02).

for p^* are often easier to find than solutions for p. According to Eq. (8), $p\{a+\xi(a,t)\}=p^*(a,t)$: by substituting a for $a+\xi$, we have $p(a,t)=p^*\{a-\xi(a-\xi,t)\}$. Regarding ξ, p, and p^* as small first-order quantities and neglecting terms of order higher than second, we may write

$$p(a)=p^*(a-\xi)=p^*(a)-\xi(\partial p^*/\partial a). \tag{10}$$

The Lagrangian equation of motion in one dimension is known to be

$$\rho_0\frac{\partial^2\xi(a,t)}{\partial t^2}=-\frac{\partial p^*(a,t)}{\partial a}, \tag{11}$$

where ρ_0 is the *undisturbed* density. This equation is rigorous, with nothing neglected.[9]

From Eqs. (10) and (11), we find, using customary abbreviations denoting partial derivatives,

$$\bar{p}=\bar{p}^*-\langle\langle\xi p_a^*\rangle\rangle=\bar{p}^*+\rho_0\langle\langle\xi\xi_{tt}\rangle\rangle.$$

Now we can write $\xi\xi_{tt}\equiv(\xi\xi_t)_t-(\xi_t)^2$; taking the time average and assuming ξ and its derivative as periodic in time, we find $\langle\langle\xi\xi_{tt}\rangle\rangle=-\langle\langle(\xi_t)^2\rangle\rangle=-\langle u^{*2}\rangle$, since $u^*=\xi_t$. Thus, we obtain

$$\bar{p}(a)=\bar{p}^*(a)-\rho_0\langle u^{*2}\rangle(a). \tag{12}$$

Denoting first- and second-order terms by the subscripts 1 and 2, we have, excluding a continuous particle velocity u_0,

$$\begin{aligned} &u=u_1+u_2, && p=p_1+p_2, && \rho=\rho_0+\rho_1+\rho_2 \\ &u^*=u_1^*+u_2^*, && p^*=p_1^*+p_2^*, && \rho^*=\rho_0+\rho_1^*+\rho_2^*. \end{aligned} \tag{13}$$

To the second order twice the kinetic energy is $2E_{\text{kin}}=\rho u^2=\rho_0u_1^2=E_{\text{kin}}^*=\rho^*u^{*2}=\rho_0u^{*2}$ as seen from Eq. (13), because u^2 differs from u^{*2} in terms higher than the second. Also, in this approximation $\bar{E}_{\text{kin}}^*=\bar{E}_{\text{pot}}^*$, and the total energy density $\bar{E}^*=\bar{E}_{\text{kin}}^*+\bar{E}_{\text{pot}}^*=2\bar{E}_{\text{kin}}^*=\langle\rho u^{*2}\rangle=\langle\rho_0u^{*2}\rangle$. Consequently, we have at small amplitudes

$$\bar{E}=\bar{E}^*=\langle\rho u^2\rangle=\langle\rho u^{*2}\rangle=\rho_0\langle u^2\rangle=\rho_0\langle u^{*2}\rangle, \tag{14}$$

and hence, from Eq. (12)

$$\bar{p}(a)=\bar{p}^*(a)-\bar{E}^*(a)=\bar{p}^*(a)-\bar{E}(a). \tag{15}$$

This general relation between the mean pressure $\bar{p}$ at a fixed coordinate a, and the mean pressure $\bar{p}^*$ which would be observed by moving with the particle around a, is independent of the special function $p(\rho)$.[10] Equation (15) enables us to find the mean Eulerian excess pressure p, when p^* and u^* in Lagrangian coordinates are known.

Inserting Eq. (15) in Eq. (3), with consideration of Eq. (14), we obtain

$$\bar{P}=\bar{p}+\langle\rho u^2\rangle=\bar{p}^*. \tag{16}$$

Thus, the radiation pressure can equally well be expressed by the mean pressure averaged in time over a unit area *moving with the particles* at the interface between fluid and obstacle.

Some authors dealing with the present subject identify radiation pressure with the mean Eulerian excess pressure $\bar{p}$ only. Most problems in acoustics are usually treated only to the first order of approximation; within this approximation the second-order term $\langle\rho u^2\rangle$ is neglected and $\bar{p}$ becomes identical with $\langle p^*\rangle$. Radiation pressure, however, is a second-order quantity. If Eulerian quantities are used, the term $\langle\rho u^2\rangle$ (as well as at least second-order terms in $\bar{p}$) are essential; only in the case of a perfectly rigid reflector ($u=0$) does the identification of the radiation pressure with $\bar{p}$ lead to accurate results.[11] If Lagrangian quantities are used, at least second-order terms in $\langle p^*\rangle$ have to be taken into account.

From the physical point of view, it appears natural to reason that the radiation force exerted upon a material surface which (unless it belongs to a perfectly stiff reflector) is in periodic motion, should be obtained by averaging in time the excess pressure on the moving surface, as indicated by Eq. (16). Indeed, as Eq. (16) shows, this is perfectly correct in dealing with a beam of infinite width, or of finite width but not in communication with undisturbed regions of static pressure p_0; in these cases both the Lagrangian quantity $\bar{p}^*$ and the Eulerian quantity $\langle\langle p+\rho u^2\rangle\rangle$ constitute a correct expression for the radiation force. Nevertheless, the case of infinite width is, of course, purely theoretical; the case of finite width could be conceived as represented by a beam filling completely a closed cylindrical tube with perfectly rigid walls. Experimentally, however, it seems hardly feasible to measure radiation pressure by such a device.

G. Hertz and H. Mende[10] introduced the notation "*Rayleigh pressure*" for the Lagrangian quantity $\bar{p}^*$, that is, the excess pressure averaged in time over a moving surface. Therefore, according to the statements above, the Rayleigh pressure can be identified with the radiation pressure in an *infinitely extended* beam, not in communication with undisturbed regions.

Practically, however, the acoustic beam is of finite width and interacts under almost all conditions with undisturbed regions. This interaction plays a fundamental part in producing the actual radiation forces, because it changes the dynamic pressure in the beam in a way which will be treated below. It is this change in pressure that leads to the actual expression for the radiation pressure. The expressions for $\bar{P}$ in Eq. (16),

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[9] See, for example, H. Lamb, *Theory of Sound* (Edwin Arnold, London, 1910), p. 176; also *Hydrodynamics* (University Press, Cambridge, 1936), p. 479.

[10] See G. Hertz and H. Mende, Z. Physik **114**, 354 (1939); also F. Borgnis, Z. Physik **134**, 363 (1953), where it is shown that Eq. (15) holds also for finite amplitudes. In the present paper, it is sufficient to know its validity at small amplitudes.

[11] F. Bopp, Ann. Physik **38**, 495 (1940).

therefore, have to be modified, because they fail to take into account this phenomenon of interaction.

6. MEAN EXCESS PRESSURE AND ENERGY DENSITY IN LIQUIDS OF CONSTANT COMPRESSIBILITY

The case of liquids offers the least complicated analytical access to the quantities involved and, therefore, affords the most perspicuous physical insight into the problem. This advantage is due to the fact that in liquids we can assume *constant compressibility*. This concept introduces a simple analytical relation between the hydrodynamic pressure p_d and the relative change in volume $\Delta V/V$ of a volume element having the original volume V, on which a pressure Δp_d is exerted. The compressibility is by definition

$$\beta = -\frac{\Delta V}{V}\frac{1}{\Delta p_d}. \quad (17)$$

For acoustic waves one uses normally the adiabatic value of β, though the processes are certainly not strictly adiabatic. Still, the exact value of β will not be very much affected, even if the process is not strictly adiabatic; this conclusion follows from the fact that the difference between the adiabatic and isothermal compressibility amounts to only a few percent for liquids forming drops and under normal conditions of temperature and pressure. Over *large* ranges of pressure the compressibility is not constant. Still, within the range of excess pressures encountered in acoustic waves, which in plane waves rarely exceed about 30 atmos, the compressibility for both positive and negative pressures can be regarded as practically constant.[12]

In the one-dimensional case, the relative change of a unit volume element is $\Delta V/V = \partial\xi/\partial a$, where ξ denotes the particle displacement in the Lagrangian sense. Hence, we have from Eq. (17) with $\Delta p_d = p^*$,

$$p^* = -(1/\beta)\cdot(\partial\xi/\partial a). \quad (18)$$

In a plane wave, the volume element is stretched and compressed only in the x direction; this causes, however, a dynamic pressure which is a scalar and, therefore, the same in every direction. The relation (18) is exact by definition for media with constant compressibility; no higher terms in ξ are involved.

The exact one-dimensional Lagrangian equation (11) gives, with Eq. (18),

$$\rho_0\frac{\partial^2\xi(a,t)}{\partial t^2} = \frac{1}{\beta}\frac{\partial^2\xi(a,t)}{\partial a^2}. \quad (19)$$

The well-known solution for plane acoustic waves generated by a simple harmonic piston motion of angular frequency ω and located, for example, at any $a = \pm 2\pi n/k$, is

$$\xi(a,t) = \xi_0\{\sin(\omega t - ka) + \gamma\sin(\omega t + ka + \theta)\}, \quad (20)$$

where $\omega/k = c = (1/\beta\rho_0)^{\frac{1}{2}}$ (c = phase velocity). This solution is rigorous, including finite amplitudes, subject only to the limitation that the solution (20) remains unique. This limitation excludes amplitudes so large that one particle can overtake the one in front of it. The amplitude ξ_0 is thus limited to values smaller than $\lambda/4\pi$.

From Eq. (20) we obtain immediately the velocity $u^*(a,t) = \partial\xi/\partial t$ as

$$u^*(a,t) = \omega\xi_0\{\cos(\omega t - ka) + \gamma\cos(\omega t + ka + \theta)\}. \quad (21)$$

The excess pressure, according to Eq. (18), is

$$p^*(a,t) = (k\xi_0/\beta)\{\cos(\omega t - ka) - \gamma\cos(\omega t + ka + \theta)\}. \quad (22)$$

The constants γ and θ are the amplitude reflection coefficient and the phase angle of the reflected wave with respect to the incident wave.

The mean energy density follows from Eqs. (14) and (21):

$$\bar{E}^*(a) = (\rho_0/\tau)\int_0^\tau u^{*2}(a,t)dt, \quad (\tau = 2\pi/\omega),$$

whence

$$\bar{E}^*(a) = \bar{E}_i\{1 + 2\gamma\cos(2ka + \theta) + \gamma^2\}, \quad (23)$$

where the mean energy density of the *incident wave* is denoted by

$$\bar{E}_i = \tfrac{1}{2}\rho_0\omega^2\xi_0^2. \quad (24)$$

Since $\bar{p}^*(a) = 0$ according to Eq. (22) in liquids of constant compressibility, Eq. (15) becomes, by use of Eq. (23),

$$\bar{p}(x) = -\bar{E}_i[1 + 2\gamma\cos(2kx + \theta) + \gamma^2], \quad (25)$$

where x is now written for the Eulerian coordinate; it is a matter of notation only whether we call this variable x or a.

With the same change in notation we have, according to Eq. (23), $\langle\rho u^2\rangle = \bar{E} = \bar{E}_i\{1 + 2\gamma\cos(2kx + \theta) + \gamma^2\}$ and with Eq. (25) we obtain $\bar{P} = \bar{p} + \langle\rho u^2\rangle = 0$. This relation, which holds for an infinitely extended beam or a beam not communicating with undisturbed regions, shows that the mean decrease of pressure due to the acoustic field is just compensated by the flux of momentum density at any x; the tensor component $\bar{T}_{xx}$ in Eq. (7) becomes p_0 in this case. Thus, no radiation pressure would be found in liquids of constant compressibility, when the beam does not interact with outside regions. This is obviously in contradiction to experimental results, because, as already pointed out, the radiation pressure actually encountered is essentially due to the interaction of the beam with the surrounding medium; this effect will be treated in the following section.

[12] See, for example, the tables in N. E. Dorsey, *Properties of Ordinary Water-Substance* (Reinhold Publishing Corporation, New York, 1940).

7. INTERACTION BETWEEN ACOUSTIC BEAM AND SURROUNDING UNDISTURBED MEDIUM, AND THE RESULTANT RADIATION PRESSURE

Equation (25) indicates that the mean dynamic excess pressure produced by the periodic wave motion varies periodically along x, except when the incident wave falls upon a perfect absorber ($\gamma=0$). Along its circumference the beam is bordered by the parts of the fluid unaffected by the wave motion. The boundary conditions require continuity of stress, that is of $\bar{T}_{yy}$ and $\bar{T}_{zz}$; both amount to $\bar{p}_d=p_0+\bar{p}$ as seen from the stress tensor (7). The instantaneous stresses T_{yy} and T_{zz} have been replaced by their time averages $\bar{T}_{yy}$ and $\bar{T}_{zz}$, since it is sufficient to postulate that the balance of stresses has to be maintained in time average. Since the pressure outside is assumed to be p_0, the continuity of stress requires that $p_0=\bar{T}_{yy}=\bar{T}_{zz}=\bar{p}_d=p_0+\bar{p}$, or: $\bar{p}=0$.

Thus, if no reflected wave is present, $\bar{p}(x)=-\bar{E}_i$ according to Eq. (25), the total dynamic pressure being $\bar{p}_d=p_0-\bar{E}_i$ inside the beam and p_0 outside. In order to establish the same amount of pressure at the boundary of the beam, the beam will be compressed and its mean density raised by the outside pressure p_0 until the dynamic pressure inside the beam equals p_0.

If a reflected wave is present, the case is more complicated, owing to the second-order periodic change of $\bar{p}$ along x, as expressed by Eq. (25). It is not possible to fulfill exactly the boundary conditions at the interface between beam and undisturbed medium, since the pressure is independent of x in the medium, while periodic in x within the beam; this dilemma results from our idealized assumption of a sharp boundary between the two regions. Actually, the periodic variation of p inside the beam will change gradually into the constant pressure p_0 in the undisturbed medium. However, it is reasonable to assume that this change is essentially performed within a region of transition which is small in comparison with the width of the beam, if the latter is large compared with the acoustic wavelength. Theoretically, this region of transition might extend to infinity. Within this "edge region" of the beam a more complicated (vortical) motion of particles will occur; a closer theoretical investigation of this effect is beyond our scope.

A reasonable way to satisfy the boundary condition at the edge of the beam is the assumption that by reaction of the surrounding medium *the average value in space* of the dynamic pressure $\bar{p}_d=p_0+\bar{p}$ is brought to p_0; inside the beam $\bar{p}_d$ then varies periodically along the x axis around the value p_0.¶

If the average value of p in *time and space* is denoted by $\langle\langle p\rangle\rangle$, the boundary condition at the edge of the beam thus leads us to the condition $\langle\langle p\rangle\rangle=0$ inside the beam. However, if there is no interaction with the surrounding undisturbed medium, it is found from Eq. (25) that $\langle\langle p\rangle\rangle=1/\lambda\int_0^\lambda\bar{p}(x)dx=-\bar{E}_i(1+\gamma^2)$, where λ denotes the acoustic wavelength. The beam, therefore, will undergo a mean compression, which raises the pressure by the opposite amount $\langle\langle p_0'\rangle\rangle=+\bar{E}_i(1+\gamma^2)$ in order to bring the total *space and time average* of p in the beam to p_0. The mean "effective dynamic pressure" in the beam can then be expressed as

$$(\bar{p}_d)_{\text{eff}}=p_0+\bar{p}+p_0'=p_0-2\gamma\bar{E}_i\cos(2kx+\theta), \quad (26)$$

from which expression we indeed obtain $\langle\langle p_d\rangle\rangle_{\text{eff}}=p_0$. The resultant mean stress tensor in the fluid according to Eq. (7), upon introducing $\langle p_d\rangle_{\text{eff}}$ and $\langle\rho u^2\rangle=\bar{E}(x)$ from Eq. (23), becomes

$$\begin{vmatrix} p_0+\bar{E}_i(1+\gamma^2) & 0 & 0 \\ 0 & p_0-2\gamma\bar{E}_i\cos(2kx+\theta) & 0 \\ 0 & 0 & p_0-2\gamma\bar{E}_i\cos(2kx+\theta) \end{vmatrix}. \quad (27)$$

The stress tensor averaged in time *and space* becomes

$$\begin{vmatrix} p_0+\bar{E}_i(1+\gamma^2) & 0 & 0 \\ 0 & p_0 & 0 \\ 0 & 0 & p_0 \end{vmatrix}, \quad (27a)$$

thus satisfying the boundary conditions for $\langle\langle T_{yy}\rangle\rangle$ and $\langle\langle T_{zz}\rangle\rangle$, both of which now equal p_0.

Finally, we calculate the value of the radiation pressure, taking into account the additional pressure $\bar{p}_0'=\bar{E}_i(1+\gamma^2)$ resulting from the interaction between beam and undisturbed medium.

From Eqs. (3) and (23) we have

$$\bar{P}=\bar{p}+\langle\rho u^2\rangle=\bar{p}+\bar{E}_i\{1+2\gamma\cos(2kx+\theta)+\gamma^2\}. \quad (28)$$

Replacing $\bar{p}$ by the "effective excess pressure,"

$$\bar{p}_{\text{eff}}=\bar{p}+\bar{p}_0'=\bar{p}+\bar{E}_i(1+\gamma^2)=-2\gamma\bar{E}_i\cos(2kx+\theta), \quad (29)$$

we find for the radiation pressure from Eq. (28):

$$\bar{P}=\bar{E}_i(1+\gamma^2). \quad (30)$$

This result agrees, as it should, with the tensor component $\bar{T}_{xx}$ in Eq. (27).

For a perfect absorber ($\gamma=0$), $\bar{P}=\bar{E}_i$; for a perfect reflector ($\gamma^2=1$), $\bar{P}=2\bar{E}_i$. The radiation pressure is independent of the phase angle between incident and reflected wave.

One might construe $\bar{P}$ in Eq. (30) as consisting of two parts, $\bar{P}_i=\bar{E}_i$ and $\bar{P}_r=\gamma^2\bar{E}_i$, $\bar{P}_i$ caused by the incident wave only and $\bar{P}_r$ by the reflected wave, whose mean energy density is $\gamma^2\bar{E}_i$. In adopting this view,[8] one might say that the incident wave of energy density

¶ F. Bopp (reference 11) uses the same assumption, basing it on the premise that a nonvanishing average value in space of $\bar{p}$ would be neutralized by a lateral flow of fluid into or out from the beam.

$\bar{E}_i$ is completely absorbed by the obstacle, leading to a radiation pressure $\bar{P}_i=\bar{E}_i$, while at the same time, the obstacle re-emits a reflected wave of energy density $\gamma^2\bar{E}_i$, the obstacle undergoing a *reactional* radiation pressure $\bar{P}_r=\gamma^2\bar{E}_i$. This concept yields the right numerical value for the radiation pressure; it does not, however, afford an insight into the physical background of the forces really acting at the surface of the obstacle. That mechanism will become apparent from the following section.

8. JOINT ACTION OF THE DYNAMIC PRESSURE AND THE FLUX OF MOMENTUM IN PRODUCING RADIATION PRESSURE IN LIQUIDS

Considering the interface between liquid and obstacle at $x=0$, we find the following Eulerian components acting at the interface by applying Eqs. (23) and (25) to the plane x (or a)$=0$:

$$\bar{p}=-\bar{E}_i(1+\gamma^2+2\gamma\cos\theta), \tag{31}$$

$$\bar{p}_0'=\bar{E}_i(1+\gamma^2), \tag{32}$$

$$\langle\rho u^2\rangle=\bar{E}_i(1+\gamma^2+2\gamma\cos\theta), \tag{33}$$

the sum of which amounts to the radiation pressure $\bar{P}=\bar{E}_i(1+\gamma^2)$ actually measured. The joint *modus operandi* of these forces may be demonstrated by a discussion of certain examples.

(a) Perfect Absorber ($\gamma=0$)

An incident wave is absorbed completely by an obstacle, the surface particles of which follow exactly the movement of the fluid particles in the pure progressive wave at the interface. No reflected wave is set up in this case and $\gamma=0$.**

The mean excess pressure $\bar{p}(x)$ caused by the periodic particle movement is $-\bar{E}_i$ and is constant throughout the beam [Eq. (25)]; by interaction with the surrounding medium $\bar{p}$ is exactly compensated throughout the beam by $\bar{p}_0'=\bar{E}_i$ [Eq. (32)], so that the total mean pressure equals p_0 in the beam. The radiation pressure $\bar{P}$ is solely a consequence of the flux of momentum ρu^2 and therefore equals $\bar{P}=\langle\rho u^2\rangle=\bar{E}_i$.

(b) Perfect Reflector ($\gamma=1$)

If the obstacle does not absorb any energy, the entire energy of the incident wave is returned as a reflected wave of the same amplitude. By interference, the two waves cause a standing wave (in a nonviscous liquid), with periodic variation in excess pressure $\bar{p}(x)$, as well as in $\langle\rho u^2\rangle(x)$ along the axis of the beam.

Perfect reflection can be achieved by both a perfectly *stiff* and a perfectly *soft* reflector. The particles at the surface of the perfectly stiff reflector are considered as absolutely immovable; the boundary condition at the interface between fluid and reflector is, therefore, expressed as $u(0,t)=u^*(0,t)=0$ at any t, u being the velocity component normal to the boundary. From Eq. (21) we find that the boundary condition $u^*(0,t)=0$ requires $\gamma=1$ and $\theta=\pi$ in the one-dimensional case. On the other hand, a perfectly soft reflecting surface is characterized by $p^*(0,t)=0$ at any t; a reflection of this kind occurs at the plane free surface of a liquid, where the condition must be satisfied that the pressure shall be continuous as we pass from liquid to air. Since the pressure in air can be assumed to be constant and equal to p_0 (disregarding the negligible wave motion transmitted into the air), the Lagrangian excess pressure p^* must vanish at the free surface of the liquid. The boundary condition $p^*(0,t)=0$ is satisfied by $\gamma=1$ and $\theta=0$, as seen from Eq. (22).

First we consider the *perfectly stiff* reflector: The mean flux of momentum $\langle\rho u^2\rangle$ vanishes at the interface because here $u=0$; the radiation pressure is now due entirely to the dynamic pressure in the liquid. According to Eq. (31), $\bar{p}=0$, since $\gamma=1$ and $\theta=\pi$. However, by interaction between beam and surrounding medium, all values of $\bar{p}$ distributed along the x axis are raised by an amount $\bar{p}_0'=2\bar{E}_i$ [Eq. (32)], which leads to the actual radiation pressure $\bar{P}=2\bar{E}_i$.

At the *perfectly soft* reflector we find from Eq. (31), with $\gamma=1$ and $\theta=0$, that $\bar{p}=-4\bar{E}_i$; from Eq. (32), $\bar{p}_0'=2\bar{E}_i$; and from Eq. (33), $\langle\rho u^2\rangle=4\bar{E}_i$. Three effects act jointly in this case: First, the mean pressure $\bar{p}$ amounts to $-4\bar{E}_i$ per unit area of the free surface; second, by interaction between beam and surrounding medium $\bar{p}$ is raised by an amount $\bar{p}_0'=2\bar{E}_i$, that is, to $-2\bar{E}_i$; third, the mean flux of momentum equals $4\bar{E}_i$, because, owing to the reflection, the resultant velocity amplitude $u(0,t)$ is twice as large as that belonging to the incident wave alone [Eq. (21)]. Thus, in the sum the radiation pressure amounts to $\bar{P}=4\bar{E}_i-2\bar{E}_i=2\bar{E}_i$, just as was found for the perfectly stiff reflector.

Figure 4 illustrates the distribution of pressure and flux of momentum for the two cases of perfect reflection.

(c) General Case ($0\leq\gamma\leq1$; $-\pi\leq\theta\leq\pi$)

In general, where the incident energy is partially absorbed and partially reflected, the force upon the obstacle is due to both effects: the effective excess pressure $\bar{p}_{\text{eff}}=\bar{p}+\bar{p}_0'$, and the quantity $\langle\rho u^2\rangle$. Each of the quantities $\bar{p}$ and $\langle\rho u^2\rangle$ depends on γ and θ, as indicated by Eqs. (31) and (33), but their *sum* does not; it cancels out in liquids of constant compressibility,

** One practicable approach to a perfect absorber is the acoustic "hohlraum," that is, a cavity with acoustically insulating walls, filled with an absorbing medium, and provided with a small window through which the acoustic beam is admitted. Such a device has been applied for measuring acoustic intensities in water in the form of a cylindrical tube; for frequencies in the megacycle range, the absorption of energy is practically complete in a tube that is not excessively long. The plane of the window, therefore, serves as a totally absorbing surface. Another practicable solution is the use of a 90° wedge (F. Borgnis, J. Acoust. Soc. Am. **24**, 468 (1952).

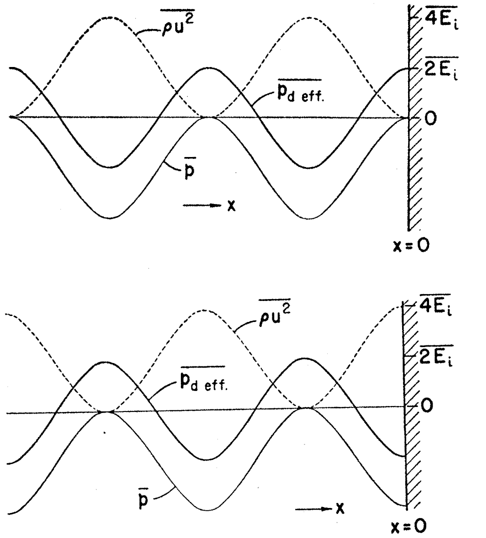

FIG. 4. Distribution of the mean excess pressure $\bar{p}$ resulting from the acoustic wave motion alone; the mean effective dynamic pressure $(\bar{p}_d)_{\text{eff}}=\bar{p}+\bar{p}_0'$ resulting from both the wave motion and the interaction of the beam with the surrounding medium; and the mean flux of momentum $\langle\rho u^2\rangle$, in the neighborhood of the boundary of a perfectly stiff reflector (above), and a perfectly soft reflector (below), in a liquid of constant compressibility.

reducing the Rayleigh pressure to zero. The radiation pressure encountered in a beam of finite width is a mere consequence of the mean compression of the beam by the outside medium, that is, of the term $\bar{p}_0'=\bar{E}_i(1+\gamma^2)$.

The Rayleigh pressure $\bar{p}+\langle\rho u^2\rangle$ does not vanish, however, if the compressibility is not constant, as in gases. For example, in a purely progressive wave in a gas under adiabatic conditions and in the neighborhood of the source $\bar{p}^*=\bar{p}+\langle\rho u^2\rangle=(1+\gamma_c)\bar{E}_i/4$ at small amplitudes (γ_c = ratio of the specific heats).

As an example of a general case, we treat in the following section the radiation pressure produced at the interface between two nonmiscible liquids.

9. RADIATION PRESSURE UPON A PLANE INTERFACE BETWEEN TWO NONMISCIBLE LIQUIDS

Let an acoustic beam of plane waves fall normally upon the plane interface between two nonmiscible liquids 1 and 2. The static pressure p_0 outside the beam may be regarded as constant and the same in both liquids. The "obstacle" is reduced in this case to the interface between the two liquids. In liquid 1 we assume a progressive wave of energy density $\bar{E}_{i1}$; this incident wave causes, in general, a reflected wave in liquid 1 and a transmitted wave in liquid 2. The waves in 1 produce a radiation pressure $\bar{P}_1=\bar{E}_{i1}(1+\gamma^2)$, while the transmitted wave causes a reactional radiation pressure $\bar{P}_2=-\bar{E}_{t2}$, where $\bar{E}_{t2}$ is the mean energy density of the progressive wave transmitted into liquid 2. The total radiation pressure, according to the result of Eq. (2), is then given by

$$\bar{P}=\bar{P}_1+\bar{P}_2=\bar{E}_{i1}(1+\gamma^2)-\bar{E}_{t2}. \qquad (34)$$

Since the interface does not absorb energy, the balance of the power transmitted requires $c_2\bar{E}_{t2}=c_1(1-\gamma^2)\bar{E}_{i1}$. Hence, we obtain from Eq. (34):

$$\bar{P}=E_{i1}\{1-c_1/c_2+\gamma^2(1+c_1/c_2)\}. \qquad (35)$$

At normal incidence the coefficient of reflection is known to be $\gamma=(1-m)/(1+m)$, where $m=\rho_2c_2/\rho_1c_1$, ρ and c being the undisturbed densities and velocities of sound in the respective media.[13]

Equation (35) shows that for special values of c_2/c_1 and ρ_2/ρ_1, the pressure $\bar{P}$ at the interface becomes zero. Inserting the above value of γ in Eq. (35), we obtain the following condition for vanishing radiation pressure:

$$c_2/c_1=(\rho_1/\rho_2)\{2(\rho_2/\rho_1)-1\}^{\frac{1}{2}}. \qquad (36)$$

According to this, $\bar{P}$ becomes zero only if the two liquids are arranged so that $\rho_1<2\rho_2$; moreover, c_2/c_1 has to obey Eq. (36). If c_2/c_1 is *smaller* than indicated by Eq. (36), $\bar{P}$ is found from Eq. (35) to become *negative*. In this case, the direction of $\bar{P}$ is *opposite* to the direction of propagation of the incident wave. This effect is caused by the fact that c_2/c_1 is now small enough to make the energy density $\bar{E}_{t2}$ exceed $\bar{E}_{i1}(1+\gamma^2)$ in Eq. (35).[14]

These two quantities represent the additional pressures $\bar{p}_0'$ on the two sides of the interface because of the compression of the beam by the surrounding medium. The mean Lagrangian pressure $\bar{p}^*$ on both sides of the interface is zero when the liquids have constant compressibility, and therefore does not contribute to the radiation pressure. The actual physical forces to which the interface is subjected result from the difference in compression of the acoustic beam on both sides of the partition.

10. GENERAL REMARKS ON THE RADIATION PRESSURE IN GASES

The relation between pressure and density in gases leads to a nonlinear Lagrangian wave equation, the rigorous solution of which can be obtained only by series development. As is well known, the wave form in gases becomes distorted in the course of the propagation. This fact is expressed by a variation in space of the amplitudes of higher terms in the series developments; more and more wave energy is transferred from the fundamental mode, which holds in the immediate neighborhood of the source, to higher harmonics in the course of the wave propagation.[15]

It is difficult to obtain a strict solution of the wave

[13] H. Lamb, *Theory of Sound* (Edwin Arnold, London, 1910), p. 169.

[14] G. Hertz and H. Mende (reference 10) have demonstrated this effect. If $\rho_1\geqslant 2\rho_2$, $\bar{P}$ in Eq. (35) is always positive, that is, in the direction of the incident wave.

[15] Reference 13, p. 174.

equation even for small amplitudes, if a reflected wave is present. A rigorous treatment would require consideration of absorption, because the amplitude of each higher harmonic is determined by its particular rate of absorption.[16]

A procedure for computing the radiation pressure for the general case of reflection, such as was applied in the previous sections for compressible liquids, is hardly feasible for gases, owing to the mathematical difficulties involved. Still, the insight into the physical processes that has been gained from the treatment of liquids can be used to establish a very general expression for the radiation pressure in all fluids. It will be shown that the formula $\bar{P}=\bar{E}_i(1+\gamma^2)$ holds in any fluid, that is, also in gases, at small amplitudes, at least under the idealized assumptions introduced in the present treatment.

11. A GENERAL EXPRESSION FOR THE RADIATION PRESSURE IN FLUIDS

By multiplying the one-dimensional Eulerian equation of continuity $\rho_t+(\rho u)_x=0$ by u and adding the equation so obtained to the Eulerian equation of motion $\rho(u_t+uu_x)+p_x=0$, a well-known form of the equation of motion in one dimension is obtained:

$$(\rho u)_t+(\rho u^2)_x+p_x=0. \tag{37}$$

In Eqs. (37) to (39), p may be regarded as representing either the excess pressure or the total dynamic pressure p_d. Considering a purely harmonic motion of the acoustic source and assuming that Eq. (37) has solutions periodic in time, we find by averaging Eq. (37) in time and integrating with respect to x,

$$\langle\rho u^2\rangle+\bar{p}=C, \tag{38}$$

where C is a constant independent of x and t, but not, in general, of the wave amplitude. Next, averaging Eq. (38) also in *space*, we obtain

$$\langle\langle\rho u^2\rangle\rangle+\langle\langle p\rangle\rangle=C. \tag{39}$$

Regarding for the moment p as the *total* dynamic pressure, we apply the same conclusions concerning p that were used in Sec. 7, namely, that owing to the interaction between beam and surrounding medium, the total mean Eulerian pressure averaged in *time and space* along the beam may reasonably be assumed equal to the undisturbed outside pressure p_0, or in other words, that $\langle\langle p\rangle\rangle=p_0$. Inserting this condition in Eq. (39) and substituting C so obtained from Eq. (39) in Eq. (38), we find

$$\langle\rho u^2\rangle+(\bar{p}-p_0)=\langle\langle\rho u^2\rangle\rangle. \tag{40}$$

Now $(\bar{p}-p_0)$ is what we previously called the mean excess pressure $\bar{p}$, and $\langle\langle\rho u^2\rangle\rangle=2\langle\langle E\rangle\rangle_{\text{kin}}=\langle\langle E\rangle\rangle_{\text{total}}+(\langle\langle E\rangle\rangle_{\text{kin}}-\langle\langle E\rangle\rangle_{\text{pot}})$. Therefore, from Eq. (40) and according to the definition of radiation pressure in Eq. (3), we have††

$$\bar{P}=2\langle\langle E\rangle\rangle_{\text{kin}}=\langle\langle E\rangle\rangle_{\text{total}}+(\langle\langle E\rangle\rangle_{\text{kin}}-\langle\langle E\rangle\rangle_{\text{pot}}). \tag{41}$$

Equation (41), which includes the interaction between beam and surrounding medium, gives the radiation pressure for a beam of finite width in any fluid and on any plane reflecting surface. With the assumption $\langle\langle p\rangle\rangle=p_0$, which led to Eq. (40), Eq. (41) is valid for finite amplitudes, since no restriction was introduced in this respect in the derivation. Whether this assumption is valid or not at *finite* amplitudes is an open question.

At small amplitudes, where we limit ourselves to terms up to the second order, it is sufficient to know the *first*-order solution in u, as already mentioned in Sec. 5. This is correct, at least, within a distance not too far from the origin of the wave motion; or, if absorption is assumed to be exactly zero, within a not too large time interval after the wave motion started. Owing to the transfer of wave energy from the fundamental to higher harmonics, as mentioned in Sec. 10, the amplitudes of these harmonics increase with distance from the origin (Earnshaw's solution), and also with time in absence of absorption. At larger distances, therefore, the terms of higher order may no longer be negligible. On the other hand, absorption is always present, limiting the amplitudes of the harmonics. It is only in a medium with constant compressibility that the fundamental wave is propagated without producing higher harmonics, preserving its original shape everywhere.

To the first order the solution for u of the Lagrangian wave equation is given in any fluid by Eq. (21). Moreover, $\langle\langle E\rangle\rangle_{\text{pot}}=\langle\langle E\rangle\rangle_{\text{kin}}$ at small amplitudes, and therefore $2\langle\langle E\rangle\rangle_{\text{kin}}=\langle\langle E\rangle\rangle_{\text{total}}=\bar{E}_i(1+\gamma^2)$, as seen from Eq. (23). Consequently, we find from Eq. (41) that the expression

$$\bar{P}=2\bar{E}_{\text{kin}}=\bar{E}_i(1+\gamma^2)=\tfrac{1}{2}\rho_0\omega^2\xi_0^2(1+\gamma^2) \tag{42}$$

is valid both in liquids and gases, when terms of third and higher order in $k\xi_0=2\pi\xi_0/\lambda$ are excluded. Eq. (42) agrees with Eq. (30), which was found to hold for liquids of constant compressibility.

Since $\bar{E}_i(1+\gamma^2)$ is the mean total energy density encountered at the surface of the plane reflector, Eq. (42) states that *at small amplitudes the radiation pressure of a finite beam of plane compressional waves equals the mean total energy density at the reflecting surface.* This result is independent of the special law connecting pressure and density in the fluid under consideration. At larger amplitudes, according to Eq. (41), the total energy density must be replaced by $2\langle\langle E\rangle\rangle_{\text{kin}}$, that is twice the average in time and space of the kinetic energy density.

[16] See P. J. Westervelt, J. Acoust. Soc. Am. 22, 319 (1950), Sec. VI, and F. E. Borgnis, Technical Report No. 1A, March 10, 1953, under U. S. Office of Naval Research Contract Nonr-220(02).

†† An analogous derivation was applied to the special case of a perfectly stiff reflector by F. Bopp (reference 11).

It may be recalled that the expressions for the radiation pressure given in Eqs. (41) and (42) represent only the force that may be attributed to one side of the reflector. A force of the same kind, equal to the energy density behind the reflector, but opposite in sign, has to be attributed to the opposite side.‡‡ Only the *sum* of these two forces has a physical significance; it represents both the change in momentum per unit time to which the wave motion is subjected in passing through a partition, and the effects of interaction between beam and surrounding medium on both sides of the partition.

The *total* radiation pressure $\bar{P}_t$ exerted upon a partition by a finite beam is, therefore, under the assumption leading to Eq. (41), given by

$$\bar{P}_t = (\langle\langle \rho u^2 \rangle\rangle)_1 - (\langle\langle \rho u^2 \rangle\rangle)_2 = 2(\langle\langle E \rangle\rangle_{\mathrm{kin}})_1 - 2(\langle\langle E \rangle\rangle_{\mathrm{kin}})_2, \quad (43)$$

where, as before, the indices 1 and 2 denote the two sides of the partition (Fig. 2). At small amplitudes, $2\langle\langle E \rangle\rangle_{\mathrm{kin}}$ can be replaced on both sides by the total energy density $\langle\langle E \rangle\rangle$; the total radiation pressure then equals the difference in energy densities on both sides of the partition. A similar result, namely, that the radiation pressure is equal to the difference between two energy densities, is also well known in electrodynamics.

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‡‡ In the special cases of a perfect absorber or of a perfect reflector this force at the rear is zero.

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