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# Gravitational Motion

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## 1. INTRODUCTION

AMONG classical field theories, the theory of gravitation, also called general relativity theory, occupies a somewhat peculiar place. Unlike most other field theories, the field equations of relativity theory are nonlinear. This implies that many facts, well known in linear theories, have no analogs in general relativity theory, and conversely. The equations of motion of the gravitational field are contained in the field equations, a fact which does not apply for the motion of an electron in the electromagnetic field. Conversely, it is difficult to define the notion of a "wave" in relativity theory, for the linear principle of superposition is crucial for the existence of waves, at least in the sense that the notion of a "wave" is normally used.

Since the gravitational field manifests itself in the motion of its sources, the problem of finding the equations of motion is of fundamental importance. This problem has been puzzling theoretical physicists for a long time, and more or less convincing solutions have been given on several occasions. Today, the problem can be considered as solved, but the attempts and partial solutions are scattered over a variety of journals and over a long period of time. Thus, it is the object of this review article to summarize the methods that lead from the gravitational field equations to the equations of motion and to outline the physical implications of those methods. In addition, we shall discuss here some amplified aspects of the general methods which have grown mainly out of the personal association of the author with Dr. Infeld and the correspondence of the latter with Dr. Einstein. Although this article does not aim to consist entirely of original research, it will be found that many parts are presented in a way not published heretofore. In addition, the contents of Sec. 5 and of parts of Sec. 6 are believed to be original.

The plan for this review is as follows. After a brief introductory section on the notation and the principal contents of general relativity theory, the reader will find an exposition of the Einstein-Infeld-Hoffmann method. The aim was to give enough material of the calculations so that a reference to the unpublished notes deposited at the Institute of Advanced Studies[1] is no longer necessary. This is especially the case for the discussion of the two-body problem in the following section. Then the methods of integration of the differential equations of motion are studied and a new way is presented by which this aim can be achieved. The influence of the coordinate conditions and of general coordinate transformations upon the equations of motion are also investigated. Subsequently, the possibility of gravitational radiation of moving bodies in their own field and the possible reaction of such radiation upon their motion is discussed. Finally, a review of the attempts is given of generalizing the Einstein-Infeld-Hoffmann method to other field theories than that of gravitation.

It is hoped that this review will be found to be a useful summary of the work done on the question of gravitational motion during the last twenty years, and a suitable basis for carrying research into the yet unknown.

## 2. NOTATION

This paper will use extensively well-known facts of tensor calculus and general relativity. Unfortunately, no over-all accepted notation has been established so that it is necessary to list the notations and abbreviations which will be used.

We shall represent the four-dimensional time-space continuum by the coordinates

$$x^0,\ x^1,\ x^2,\ x^3,$$

where $x^0$ denotes time, the others the three space-coordinates. Since time often plays a different role in physics than the other three coordinates, we shall use a specific notation as follows. Wherever $x^\mu$ is written with a *Greek* index, it is understood to represent either time or one of the space coordinates. On the other hand, if we write $x^m$ with a *Latin* index, we mean by this that $x^m$ is one of the space coordinates only. This convention is extended to all indices such that a Greek index assumes the values 0–3, a Latin index 1–3.

We denote differentiation with respect to a world coordinate by a stroke. Thus, for any entity $T$ we have

$$T_{|\alpha} \equiv \partial T/\partial x^\alpha. \qquad (2.1)$$

We shall further denote the metric in the continuum of the $x^\alpha$'s by the ten functions

$$g_{\alpha\beta}(x^\mu) = g_{\beta\alpha}(x^\mu). \qquad (2.2)$$

Then the length of the world-line $x^\mu = \xi^\mu(t)$ is given by

$$s_{12} = \int_1^2 \left\{ \sum_{\alpha\beta} g_{\alpha\beta}[x^\mu = \xi^\mu(t)] \frac{d\xi^\alpha}{dt}\frac{d\xi^\beta}{dt} \right\}^{\frac{1}{2}} dt. \qquad (2.3)$$

Furthermore, we shall use the summation convention. In an expression containing entities with indices (any kind) or power exponents, it is understood that one has to sum over the whole range of any such symbol

[1] Einstein, Infeld, and Hoffmann, appendix to Ann. Math. **39**, 66 (1938), unpublished notes deposited at the Institute for Advanced Studies, Princeton, New Jersey.

that appears twice. In cases where we want to consider one single term only, although a symbol of the above kind appears twice, we shall use brackets in order to indicate that the summation convention does not apply.

There is no unique way for assigning to a particular world point a set of coordinates $x^\beta$. We could equally well take another set of numbers $x^{\beta *}$ (with the same properties as described above) for describing our world. The passing from one set of coordinates to another one is called a coordinate transformation and may be expressed mathematically in the following way:

$$x^{\beta *}=f^\beta(x^\alpha). \tag{2.4}$$

The starred and the unstarred "systems" of coordinates represent the same four-dimensional universe and in both the length of a line must be expressible according to (2.3).

Tensors are entities satisfying the following transformation law under a coordinate transformation (2.4):

$$t_\mu{}^{\sigma *}=\frac{\partial x^\alpha}{\partial x^{\mu *}}\frac{\partial x^{\sigma *}}{\partial x^\beta}t_\alpha{}^\beta. \tag{2.5}$$

The quantities $g_{\alpha\beta}$ form a tensor, the metric tensor of the world. We may define the contravariant metric tensor $g^{\alpha\beta}$ by the equation

$$g_{\alpha\beta}g^{\beta\gamma}=\delta_\alpha{}^\gamma=\begin{cases}1 & \text{for} \quad \alpha=\gamma \\ 0 & \text{for} \quad \alpha\neq\gamma\end{cases}. \tag{2.6}$$

With the help of the metric tensor we can rise or lower indices of any tensor:

$$g_{\alpha\beta}T^\beta=T_\alpha; \quad g^{\alpha\beta}T_\beta=T^\alpha. \tag{2.7}$$

A set of functions $y_A(x^\beta)$ defined in all points of the world is called a "field." $A$ is an index running from 1 to $N$, $N$ being the total number of algebraically independent "components" of the field at one world point. The components of the field (also called field variables) are subject to a set of functional relationships

$$L^B(y_A)=0; \quad B=1\cdots N \tag{2.8}$$

which are termed field equations. They can be expressed as the Euler Lagrangian equations of a variational principle

$$\delta\int_V Ld^4x=0 \tag{2.9}$$

taken in the domain $V$, where the field equations are to be satisfied.

The field variables of general relativity theory are the components $g_{\alpha\beta}$ of the metric tensor of the world. Instead of $g_{\alpha\beta}$ one could equally well use any linear combinations of them or also the components of the metric tensor density

$$\mathfrak{g}_{\alpha\beta}=(-g)^{\frac{1}{2}}g_{\alpha\beta}. \tag{2.10}$$

One also could use the contravariant entities instead of the covariant ones.

The field variables in relativity theory are subject to the following nonlinear field equations (see references 2, 3, 4)

$$G^{\alpha\beta}=0. \tag{2.11}$$

The notation is chosen as follows:

(a) Christoffel symbols

$$\begin{Bmatrix}\lambda \\ \iota\kappa\end{Bmatrix}=\tfrac{1}{2}g^{\lambda\sigma}(g_{\iota\sigma|\kappa}+g_{\kappa\sigma|\iota}-g_{\iota\kappa|\sigma})\equiv g^{\lambda\sigma}[\iota\kappa,\sigma]. \tag{2.12}$$

(b) Covariant derivatives

$$t_{\cdots\bullet\cdots,\sigma}^{\phantom{\cdots}\iota}=t_{\cdots\bullet\cdots|\sigma}^{\phantom{\cdots}\iota}+\cdots+\begin{Bmatrix}\iota \\ \rho\sigma\end{Bmatrix}t_{\cdots\bullet\cdots}^{\phantom{\cdots}\rho}+\cdots \tag{2.13}$$

$$t_{\cdots\kappa\cdots;\sigma}=t_{\cdots\kappa\cdots|\sigma}-\cdots-\begin{Bmatrix}\rho \\ \eta\sigma\end{Bmatrix}t_{\cdots\rho\cdots}-\cdots \tag{2.14}$$

(c) Curvature tensor

$$R_{\iota\kappa\lambda\cdot}{}^V=\begin{Bmatrix}\nu \\ \lambda\iota\end{Bmatrix}_{|\kappa}-\begin{Bmatrix}\nu \\ \lambda\kappa\end{Bmatrix}_{|\iota}-\begin{Bmatrix}\nu \\ \sigma\iota\end{Bmatrix}\begin{Bmatrix}\sigma \\ \lambda\kappa\end{Bmatrix}+\begin{Bmatrix}\nu \\ \sigma\kappa\end{Bmatrix}\begin{Bmatrix}\sigma \\ \lambda\iota\end{Bmatrix}, \tag{2.15}$$

$$R_{\alpha\beta\gamma\cdot}{}^\beta=R_{\alpha\gamma}; \quad R=g^{\alpha\beta}R_{\alpha\beta}; \tag{2.16}$$

$$G_{\alpha\beta}=R_{\gamma\delta}g^{\gamma\alpha}g^{\delta\beta}-\tfrac{1}{2}Rg^{\alpha\beta}. \tag{2.17}$$

The field equations (2.11) are equivalent to

$$R_{\alpha\beta}=0. \tag{2.18}$$

The field variables satisfy the Bianchi identities

$$R_{\alpha\beta\gamma\cdot}{}^\delta{}_{;\theta}+R_{\beta\theta\gamma\cdot}{}^\delta{}_{;\alpha}+R_{\theta\alpha\gamma\cdot}{}^\delta{}_{;\beta}=0. \tag{2.19}$$

The latter are a geometrical property of the metric tensor and thus are closely connected with the postulate of the general covariance of our theory.

A trivial rigorous solution of the field equations (2.18) is the flat Minkowski metric $\eta_{\alpha\beta}$. If we choose the units of space and time so that the velocity of light is equal to 1, the Minkowski metric may be represented by the following matrix:

$$\eta^{\alpha\beta}=\eta_{\alpha\beta}=\begin{bmatrix}+1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1\end{bmatrix}. \tag{2.20}$$

The Minkowski metric represents the empty space.

[2] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall Inc., New York, 1942).

[3] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, 1923).

[4] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, London, 1934).

### 3. THE DIFFERENTIAL EQUATIONS OF MOTION

It has been recognized some twenty years ago by Einstein and Grommer[5] that in general relativity theory the equations of motion follow from the field equations. Earlier, it was assumed that a "geodesic principle" governs the relativistic motion of bodies. By this one understood some generalization of the fact that a small body in the gravitational field of a big one moves along a geodesic line of the "external" field. This is a similar concept as in Newtonian mechanics where the field equations are completed by the equations of motion, that is by putting the acceleration of each particle equal to the negative gradient of the field due to the other particles present. Thus, the geodesic principle would play in general theory the role of the equations of motion.

When it became known that the relativistic equations of motion are contained in the field equations, it had to be questioned how the geodesic principle, whose validity for sufficiently small particles in a large field was all but established, could be fitted into the new scheme. This problem has been investigated by Infeld and Schild.[6] These authors were able to show that the geodesic principle (for sufficiently small particles) can be deduced from the field equations, and that it is not necessary to postulate it separately.

In order to obtain the relativistic equations of motion from the field equations, one has to recur to an approximation procedure. This is due to the fact that the motion of the sources cannot completely be determined unless such effects as spontaneous emission of radiation are specifically excluded. This exclusion is accomplished by assuming that all the motions are "slow" in the sense that differentiation with respect to $x^0$ reduces the order of magnitude of the term concerned at every stage of the approximation procedure.

An outline of this procedure was given for the first time by Einstein, Infeld, and Hoffmann in 1938.[1,7] Shortly thereafter there appeared an improved treatment by Einstein and Infeld.[8] In these first attempts there remained some ambiguities and logical difficulties unsolved. Later, however, the theory was developed further and a completely new treatment was given by Einstein and Infeld in 1949.[9] The method of Einstein and co-workers has been modified by Hu[10] and Papapetrou[11] so as to treat the masses present in the field as extended sources instead of as poles. Whereas this treatment saves some labor when it comes to the actual calculations of the equation of motion, it seems that the Einstein-Infeld-Hoffmann method is simpler from a logical standpoint, especially if the later amendments to that method are used.

In order to perform the approximation procedure, the field variables are split into a part representing the vacuum and another representing the deviation from it. Accordingly, they are written as follows:

$$g_{\alpha\beta}=\eta_{\alpha\beta}+h_{\alpha\beta}, \tag{3.1}$$

$$g^{\alpha\beta}=\eta^{\alpha\beta}+h^{\alpha\beta}, \tag{3.2}$$

where $h_{\alpha\beta}$ is not assumed to be small. It turns out to be convenient to replace the $h$'s by the following linear combinations:

$$\gamma_{\alpha\beta}=h_{\alpha\beta}-\tfrac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}h_{\gamma\delta}. \tag{3.3}$$

The original field equations are as given in (2.11), but one may take any linear combination hereof. According to the choice (3.3) of field variables, it will be convenient to choose the following field equations:

$$-2(R_{\gamma\theta}-\tfrac{1}{2}\eta_{\gamma\theta}\eta^{\alpha\beta}R_{\alpha\beta})=0. \tag{3.4}$$

Introducing everywhere the $\gamma$'s instead of the $h$'s, one can write Eq. (3.4) in the following way:

$$\left.\begin{aligned}\phi_{00}+2\Lambda_{00}&=0\\ \phi_{0n}+2\Lambda_{0n}&=0\\ \phi_{mn}+2\Lambda_{mn}&=0\end{aligned}\right\}, \tag{3.5}$$

where

$$\left.\begin{aligned}\phi_{00}&=-\gamma_{00|ss}\\ \phi_{0m}&=-\gamma_{0m|ss}+\gamma_{0s|0s}\\ \phi_{mn}&=-\gamma_{mn|ss}+\gamma_{ms|ns}+\gamma_{ns|ms}-\delta_{mn}\gamma_{rs|rs}\end{aligned}\right\} \tag{3.6}$$

and

$$\left.\begin{aligned}2\Lambda_{00}&=\gamma_{sr|sr}+2\Lambda'_{00}\\ 2\Lambda_{0m}&=\gamma_{ms|s0}-\gamma_{00|m0}+2\Lambda'_{0m}\\ 2\Lambda_{mn}&=-\gamma_{0m|0n}-\gamma_{0n|0m}+2\delta_{mn}\gamma_{0s|0s}+\gamma_{mn|00}\\ &\qquad-\delta_{mn}\gamma_{00|00}+2\Lambda'_{mn}\end{aligned}\right\}. \tag{3.7}$$

In these formulas, all the linear terms are written out explicitly, while $\Lambda'_{\mu\nu}$ stands for all the nonlinear terms in the $\gamma$'s.

With any function $F_{st}$ one can form the integral

$$\oint_S F_{st}n_t dS \tag{3.8}$$

over an arbitrary closed surface $S$ that does not pass through any singularities of $F$. In Eq. (3.8)

$$n_t=\cos(x_t, \mathbf{n}) \tag{3.9}$$

are the components of the normal unit vector to the surface. The words "normal" and "unit" are used in the conventional sense to designate the corresponding

[5] A. Einstein and J. Grommer, Sitz. Berl. Akad. Wiss. 1927, 1
[6] L. Infeld and A. E. Schild, Revs. Modern Phys. 21, 408 (1949).
[7] Einstein, Infeld, and Hoffman, Ann. Math. 39, 66 (1938).
[8] A. Einstein and L. Infeld, Ann. Math. 41, 455 (1940).
[9] A. Einstein and L. Infeld, Can. J. Math. 1, 209 (1949).
[10] N. Hu, Proc. Roy. Irish Acad. 51A, 87 (1947).
[11] A. Papapetrou, Proc. Phys. Soc. (London) A64 57 (1951).

functions of the coordinates which are implied by these terms in Euclidian geometry.

If one takes as $F$ the left-hand sides of the field equations (3.5), one ends up with (since the surface integrals of $\phi$ vanish)

$$\oint_S \Lambda_{\beta k} n_k dS = 0, \tag{3.10}$$

which imples

$$\phi_{\alpha n|n} = 0; \quad \Lambda_{\alpha n|n} = 0. \tag{3.11}$$

The aim is to develop every function $f(x^\beta)$ into a power series in $1/c \equiv \lambda$:

$$f(x^\mu) = \lambda^0\, {}_0f + \lambda^1\, {}_1f + \lambda^2\, {}_2f + \cdots = \sum_{k=0}^{\infty} \lambda^k\, {}_kf. \tag{3.12}$$

The left-lower indices indicate the order of the term. If the function $f$ varies quickly in space, but slowly in $x^0$, then one is justified in not treating all its derivatives in the same fashion. The derivatives with respect to $x^0$ will be of a higher order than space derivatives. One can formalize this procedure by introducing an auxiliary time

$$\tau = x^0 \lambda, \tag{3.13}$$

so that derivatives with respect to $t$ can be treated on the same footing as the space derivatives

$$f_{|0} = \partial f/\partial x^0 = \partial f/\partial \tau \quad \lambda = \lambda f_{,0} \equiv \lambda \dot{f}. \tag{3.14}$$

In other words, the "stroke" differentiation of a quantity with respect to $x^0$ can be replaced by the "comma" differentiation with respect to $\tau$ if the power of $\lambda = 1/c$ with which this quantity is associated is simultaneously raised by one.

With this notation, the $\gamma$'s may be developed into a power series as follows:

$$\left.\begin{aligned} \gamma_{00} &= \lambda^2\, {}_2\gamma_{00} + \lambda^4\, {}_4\gamma_{00} + \lambda^6\, {}_6\gamma_{00} + \cdots && \text{(a)} \\ \gamma_{0m} &= \lambda^3\, {}_3\gamma_{0m} + \lambda^5\, {}_5\gamma_{0m} + \cdots && \text{(b)} \\ \gamma_{mn} &= \lambda^4\, {}_4\gamma_{mn} + \lambda^6\, {}_6\gamma_{mn} + \cdots && \text{(c)} \end{aligned}\right\} . \tag{3.15}$$

The start with different powers of $\lambda$ is an assumption which can be justified heuristically. It will be seen that Eq. (3.15) leads to a possible solution of the field equations. It is, however, possible to retain all the terms in Eq. (3.15), instead of only alternating powers of $\lambda$. This would lead to solutions analogous to those in electromagnetic theory representing radiation. It is for this reason that one calls the omitted terms in Eq. (3.15) "*radiation*" terms.

Going back to the field equations (3.5)–(3.7), one can introduce the $\gamma$'s into the latter in their power series development. Spontaneous radiation is excluded by the assumption that all the $\gamma$'s vary slowly in time and quickly in space; in other words, by assuming that all the $\gamma$'s are of such a type that Eq. (3.14) applies. Thus, according to Eq. (3.5)–(3.7), the field equations are split into the following equations for each approximation step

$$\left.\begin{aligned} {}_{2k-2}\phi_{00} + 2\,{}_{2k-2}\Lambda_{00} &= 0 && \text{(a)} \\ {}_{2k-1}\phi_{0m} + 2\,{}_{2k-1}\Lambda_{0m} &= 0 && \text{(b)} \\ {}_{2k}\phi_{mn} + 2\,{}_{2k}\Lambda_{mn} &= 0 && \text{(c)} \end{aligned}\right\} . \tag{3.16}$$

Let us now assume that ${}_2\gamma_{00} \cdots {}_{2k-4}\gamma_{00}$; ${}_3\gamma_{0m} \cdots {}_{2k-3}\gamma_{0m}$; and ${}_4\gamma_{mn} \cdots {}_{2k-4}\gamma_{mn}$ are all known. Then Eq. (3.16), if solved will yield ${}_{2k-2}\gamma_{00}$, ${}_{2k-1}\gamma_{0m}$, ${}_{2k}\gamma_{mn}$; and if such a procedure converges, one can determine the field to any approximation desired. The structure of the equations indicate that one really can set all the odd terms in $\gamma_{00}$, $\gamma_{mn}$ and the even ones in $\gamma_{0m}$ equal to zero. This amounts to taking "zero" as solutions of Eq. (3.16) for half-interger $k$'s.

There are, however, at every stage of the approximation procedure the conditions (3.10) and (3.11) to be satisfied. It can be shown that the divergence condition for $\phi$ (3.11) is automatically satisfied in each approximation step. On the other hand, the surface conditions (3.10) are not generally satisfied; they are the conditions which lead to the equations of motion.

To start the approximation procedure one has to solve the following equations:

$$\left.\begin{aligned} {}_2\gamma_{00,ss} &= 0, && \text{(a)} \\ -{}_3\gamma_{0m,ss} + {}_3\gamma_{0s,ms} &= {}_2\gamma_{00,0m}. && \text{(b)} \end{aligned}\right\} \tag{3.17}$$

The character of the entire solution will depend on the choice of the harmonic function one takes as a solution of (3.17a). The term ${}_2\gamma_{00}$ is the gravitational potential as one would find it if one were to use Newtonian theory instead of general relativity theory. Since one is interested in a solution representing $p$ particles in the Newtonian approximation, one has to write

$${}_2\gamma_{00} = 2\varphi; \quad \varphi = \sum_{A=1}^{p} (-2\, {}^Am\, {}^A\psi); \tag{3.18a}$$

$${}^A\psi = \{(x^k - {}^AX^k)(x^k - {}^AX^k)\}^{-\frac{1}{2}} = ({}^Ar)^{-1}. \tag{3.18b}$$

Here, ${}^Ar$ is the "distance" in space of a point from the $A$th singularity. Now, introducing ${}_2\gamma_{00}$ into (3.17b) and again obtaining three equations for the three functions ${}_3\gamma_{0m}$, one observes that the latter is only soluble if the ${}^Am$'s are constant in time. Only then the conditions (3.10) are satisfied for that stage of the approximation procedure.

Going on, one observes that at every step of the approximation one has to solve equations of the type of Eqs. (3.16). Since one has the surface condition (3.10), Eqs. (3.16) are consistent only if one has

$$\oint_{SA} 2\, {}_{2k-1}\Lambda_{0r} n_r dS \equiv {}_{2k-1}{}^AC_0 = 0, \tag{3.19}$$

$$\oint_{SA} 2\, {}_{2k}\Lambda_{mr} n_r dS \equiv {}_{2k}{}^AC_m = 0. \tag{3.20}$$

Herein, $S^A$ is representing a surface around the $A$th singularity. If $S^A$ would not enclose a singularity, then the surface integrals in (3.19/20) would be trivially zero. The $\Lambda$'s in Eqs. (3.19/20) are already known. Thus it is likely to happen that the $C$'s in (3.19) and (3.20) are not zero, so that Eqs. (3.16) cannot be integrated. However, by adding single poles to the *previous* $\gamma$'s one can insure the integrability of Eq. (3.16b), and by adding dipoles the integrability of Eq. (3.16c). It is easily seen that adding poles to the original solution ${}_{2k-2}\gamma_{00}$

$$ {}_{2k-2}\gamma_{00} \rightarrow {}_{2k-2}\gamma_{00} - 4\, {}_{2k-2}{}^{A}m\, {}^{A}\psi \tag{3.21}$$

changes ${}_{2k-1}{}^{A}C_0$ into

$$ {}_{2k-1}{}^{A}C_0 \rightarrow {}_{2k-1}{}^{A}C_0 - 4\,{}_{2k-2}{}^{A}\dot{m}. \tag{3.22}$$

Similarly, if one replaces ${}_{2k-2}\gamma_{00}$ (containing the additional poles) by

$$ {}_{2k-2}\gamma_{00} \rightarrow {}_{2k-2}\gamma_{00} - {}_{2k-2}{}^{A}S_r\, {}^{A}\psi_{,r} \tag{3.23}$$

(i.e. adding dipoles), ${}_{2k}{}^{A}C_m$ is changed into

$$ {}_{2k}{}^{A}C_m \rightarrow {}_{2k}{}^{A}C_m - {}_{2k-2}{}^{A}\ddot{S}_m. \tag{3.24}$$

Therefore, it can be made zero by choosing

$$ {}_{2k-2}{}^{A}\ddot{S}_m = {}_{2k}{}^{A}C_m. \tag{3.25}$$

By proceeding in this way, one accumulates single poles and dipoles; the additional expressions in $\gamma_{00}$ are

$$ \sum \lambda^{2k-2}(4\,{}_{2k-2}{}^{A}m\, {}^{A}\psi + {}_{2k-2}{}^{A}S_r\, {}^{A}\psi_{,r}). \tag{3.26}$$

However, since negative masses are not known, gravitational dipoles have no physical meaning. Thus, one has, at the end of the approximation procedure, to annihilate all these additional dipoles by taking

$$ \sum \lambda^{2k-2}\, {}_{2k-2}{}^{A}S_r = 0. \tag{3.27}$$

Differentiating this twice yields

$$ \sum \lambda^{2k}\, {}_{2k-2}{}^{A}\ddot{S}_m = \sum \lambda^{2k}\, {}_{2k}{}^{A}C_m = 0. \tag{3.28}$$

These are the $3p$ differential equations of motion of the $k$th approximation.

One may impose at every step of the approximation procedure four coordinate conditions in the form of four nontensorial equations involving the field variables. The fact that this is possible is due to the existence of four (Bianchi) identities between the field variables. The coordinate conditions if properly chosen, may be of considerable help if one is going to actually carry out the calculations of the approximation procedure.

One may notice that the field equations (3.16) would permit that arbitrary multipoles be added to the $\gamma_{00}$'s at every step of the approximation. The integrability conditions for the *subsequent* step, however, fix what multiples have to be chosen and make, for given coordinate conditions, the solution unique at every step of the procedure.

### 4. THE TWO-BODY PROBLEM

Let us now apply the procedure outlined in Sec. 3 to the task of finding the differential equations of motion of the two-body problem. For the actual calculations it will turn out to be convenient to assume the following coordinate conditions

$$\left.\begin{aligned} {}_{2k-1}\gamma_{0s,\,s} - {}_{2k-2}\gamma_{00,\,0} &= 0, && \text{(a)}\\ {}_{2k}\gamma_{mn,\,n} &= 0. && \text{(b)}\end{aligned}\right\}\tag{4.1}$$

The solution of the first approximation corresponding to Eq. (3.18) becomes for the two-body problem

$$\left.\begin{aligned} {}_{2}\gamma_{00} = 2\varphi;\quad \varphi &= -2\{{}^{1}m\,{}^{1}\psi - {}^{2}m\,{}^{2}\psi\}, && \text{(a)}\\ {}^{1}\psi &= [(x^k - Y^k)(x^k - Y^k)]^{\frac{1}{2}} = \frac{1}{{}^{1}r}, && \text{(b)}\\ {}^{2}\psi &= [(x^k - Z^k)(x^k - Z^k)]^{\frac{1}{2}} = \frac{1}{{}^{2}r}, && \text{(c)}\end{aligned}\right\}\tag{4.2}$$

$Y^k$, $Z^k$ being the coordinates of the two bodies. This solution determines the character of the entire solution for the field equations.

For going on, one needs the explicit forms of all the occurring $\Lambda$'s, $g$'s, etc. It follows from Eqs. (3.1) and (3.2) that

$$\left.\begin{aligned} g_{00} &= 1 + \lambda^2\, {}_{2}h_{00} + \lambda^4\, {}_{4}h_{00} + \cdots, && \text{(a)}\\ g_{0m} &= \lambda^3\, {}_{3}h_{0m} + \lambda^5\, {}_{5}h_{0m} + \cdots, && \text{(b)}\\ g_{mn} &= -\delta_{mn} + \lambda^2\, {}_{2}h_{mn} + \lambda^4\, {}_{4}h_{mn} + \cdots, && \text{(c)}\end{aligned}\right\}\tag{4.3}$$

and similarly

$$\left.\begin{aligned} g^{00} &= 1 + \lambda^2\, {}_{2}h^{00} + \lambda^4\, {}_{4}h^{00} + \cdots, && \text{(a)}\\ g^{0m} &= \lambda^3\, {}_{3}h^{0m} + \lambda^5\, {}_{5}h^{0m} + \cdots, && \text{(b)}\\ g^{mn} &= -\delta_{mn} + \lambda^2\, {}_{2}h^{mn} + \lambda^4\, {}_{4}h^{mn} + \cdots. && \text{(c)}\end{aligned}\right\}\tag{4.4}$$

Therefore, from Eq. (4.2)

$$ {}_{2}h_{00} = \varphi, \tag{4.5}$$

$$ {}_{2}h_{mn} = \varphi\delta_{mn}. \tag{4.6}$$

The general property (2.6) of the metric tensor allows one to express all the contravariant $h$'s by the covariant ones.

Thus, one obtains

$$ {}_{2}h^{00} = -\varphi;\quad {}_{4}h^{00} = -\,{}_{4}h_{00} + \varphi\varphi, \tag{4.7}$$

$$ {}_{3}h^{m0} = {}_{3}h_{m0};\quad {}_{5}h^{m0} = {}_{5}h_{m0}, \tag{4.8}$$

$$ {}_{2}h^{mn} = -\delta_{mn}\varphi;\quad {}_{4}h^{mn} = -\,{}_{4}h_{mn} - \varphi\varphi\delta_{mn}. \tag{4.9}$$

Moreover, one observes that generally ${}_{k}h^{0m}$ contains ${}_{k}h_{0m}$ only linearly, so that one may write

$$ {}_{k}h^{0m} = {}_{k}h_{0m} + \text{terms not containing } {}_{k}h_{0m}. \tag{4.10}$$

The next step is to write down the field equations explicitly in terms of the $h$'s. They are already split into terms denoted by $\Lambda$ and $\phi$. The $\phi$'s are already found explicitly, and the $\Lambda$'s may be represented in the following way

$$\left.\begin{aligned}\Lambda_{00}&=\tfrac{1}{2}L_{00}+\tfrac{1}{2}L_{ss}, &&\text{(a)}\\ \Lambda_{0m}&=L_{0m}, &&\text{(b)}\\ \Lambda_{mn}&=L_{mn}-\tfrac{1}{2}\delta_{mn}L_{ss}+\tfrac{1}{2}\delta_{mn}L_{00}, &&\text{(c)}\end{aligned}\right\}\quad(4.11)$$

with

$$\begin{aligned}2L_{00}&=2h_{0s|0s}-h_{ss|00}\\ &\quad-2(h^{\alpha\beta}[00,\beta])_{|\alpha}+2(h^{\alpha\beta}[0\alpha,\beta])_{|0}\\ &\quad+2\begin{Bmatrix}\alpha\\0\beta\end{Bmatrix}\begin{Bmatrix}\beta\\\alpha 0\end{Bmatrix}-2\begin{Bmatrix}\alpha\\00\end{Bmatrix}\begin{Bmatrix}\beta\\\alpha\beta\end{Bmatrix},\end{aligned}\quad(4.12)$$

$$\begin{aligned}2L_{0n}&=-2(h^{\alpha\beta}[0n,\beta))_{|\alpha}+2(h^{\alpha\beta}[m\alpha,\beta])_{|0}\\ &\quad+2\begin{Bmatrix}\alpha\\m\beta\end{Bmatrix}\begin{Bmatrix}\beta\\\alpha 0\end{Bmatrix}-2\begin{Bmatrix}\beta\\m0\end{Bmatrix}\begin{Bmatrix}\alpha\\\beta\alpha\end{Bmatrix},\end{aligned}\quad(4.13)$$

$$\begin{aligned}2L_{mn}&=-h_{0n|0m}-h_{0m|0n}+h_{mn|00}\\ &\quad-2(h^{\alpha\beta}[mn,\beta])_{|\alpha}+2(h^{\alpha\beta}[m\alpha,\beta])_{|n}\\ &\quad+2\begin{Bmatrix}\alpha\\m\beta\end{Bmatrix}\begin{Bmatrix}\beta\\\alpha n\end{Bmatrix}-2\begin{Bmatrix}\alpha\\mn\end{Bmatrix}\begin{Bmatrix}\beta\\\alpha\beta\end{Bmatrix}.\end{aligned}\quad(4.14)$$

The next step is to insert the power series for all the $h$'s into the above equations. Then one can split the latter into corresponding equations for every power of $\lambda$.

The result is, if one uses the coordinate conditions,

$$2\,{}_3\Lambda_{0m}=0, \quad(4.15)$$

$$\left.\begin{aligned}2\,{}_4\Lambda_{mn}&=-{}_3\gamma_{0m,0n}-{}_3\gamma_{0n,0m}+2\delta_{mn}\,\varphi_{,00}\\ &\quad-2\varphi\varphi_{,mn}-\varphi_{,m}\,\varphi_{,n}+\tfrac{3}{2}\delta_{mn}\varphi_{,s}\,\varphi_{,s}\end{aligned}\right\},\quad(4.16)$$

$$2\,{}_4\Lambda_{00}=-\tfrac{3}{2}\varphi_{,s}\,\varphi_{,s}, \quad(4.17)$$

$$2\,{}_5\Lambda_{0m}=\varphi_{,s}\,{}_3\gamma_{0s,m}-\varphi_{,sm}\,{}_3\gamma_{0s}-3\varphi_{,0}\,\varphi_{,m}, \quad(4.18\text{a})$$

$$\left.\begin{aligned}2\,{}_6\Lambda_{mn}&=-{}_5\gamma_{0m,0n}-{}_5\gamma_{0n,0m}+\delta_{mn}\,{}_4\gamma_{00,00}\\ &\quad+{}_4\gamma_{mn,00}-\varphi\,{}_4\gamma_{00,mn}-\varphi\,{}_4\gamma_{ss,mn}\\ &\quad-\varphi_{,mn}\,{}_4\gamma_{00}-\varphi_{,mn}\,{}_4\gamma_{ss}+\varphi_{,ms}\,{}_4\gamma_{ns}\\ &\quad+\varphi_{,ns}\,{}_4\gamma_{ms}-\delta_{mn}\varphi_{,sr}\,{}_4\gamma_{sr}-2\varphi_{,s}\,{}_4\gamma_{mn,s}\\ &\quad+\varphi_{,s}\,{}_4\gamma_{ms,n}+\varphi_{,s}\,{}_4\gamma_{ns,m}-\tfrac{1}{2}\varphi_{,m}\,{}_4\gamma_{ss,n}\\ &\quad-\tfrac{1}{2}\varphi_{,n}\,{}_4\gamma_{ss,m}-\tfrac{1}{2}\varphi_{,n}\,{}_4\gamma_{00,m}-\tfrac{1}{2}\varphi_{,m}\,{}_4\gamma_{00,n}\\ &\quad+\tfrac{3}{2}\delta_{mn}\varphi_{,s}\,{}_4\gamma_{rr,s}+\tfrac{3}{2}\delta_{mn}\varphi_{,s}\,{}_4\gamma_{00,s}\\ &\quad-{}_3\gamma_{0s}\,{}_3\gamma_{0n,ms}-{}_3\gamma_{0s}\,{}_3\gamma_{0m,ns}\\ &\quad+2\,{}_3\gamma_{0s}\,{}_3\gamma_{0s,mn}+\tfrac{1}{2}\delta_{mn}\,{}_3\gamma_{0s,r}\,{}_3\gamma_{0r,s}\\ &\quad-\tfrac{3}{2}\delta_{mn}\,{}_3\gamma_{0s,r}\,{}_3\gamma_{0s,r}+{}_3\gamma_{0s,m}\,{}_3\gamma_{0s,n}\\ &\quad+{}_3\gamma_{0m,s}\,{}_3\gamma_{0n,s}-\varphi_{,0n}\,{}_3\gamma_{0m}-\varphi_{,0m}\,{}_3\gamma_{0n}\\ &\quad+2\delta_{mn}\,{}_3\gamma_{0s}\varphi_{,0s}-\varphi_{,0}\,{}_3\gamma_{0m,n}\\ &\quad-\varphi_{,0}\,{}_3\gamma_{0n,m}-\varphi_{,n}\,{}_3\gamma_{0m,0}-\varphi_{,m}\,{}_3\gamma_{0n,0}\\ &\quad+2\varphi\,{}_3\gamma_{0m,0n}+2\varphi\,{}_3\gamma_{0n,0m}-2\delta_{mn}\varphi\varphi_{,00}\\ &\quad+2\varphi\varphi\varphi_{,mn}-\varphi\varphi_{,m}\,\varphi_{,n}+\tfrac{3}{2}\delta_{mn}\,\varphi\varphi_{,s}\,\varphi_{,s}\\ &\quad+\tfrac{1}{2}\delta_{mn}\varphi_{,0}\,\varphi_{,0}.\end{aligned}\right\}.\quad(4.18\text{b})$$

Starting now the calculations of the field variables, one has to solve the following equations:

$$\left.\begin{aligned}{}_2\gamma_{00,ss}&=0, &&\text{(a)}\\ {}_3\gamma_{mn,ss}&=0, &&\text{(b)}\\ {}_4\gamma_{mn,ss}&=2\,{}_4\Lambda_{mn}. &&\text{(c)}\end{aligned}\right\}\quad(4.19)$$

It is convenient to introduce here a slightly new notation by putting

$$\varphi=f+g;\quad f=-2\,{}^1m\,{}^1\psi;\quad g=-2\,{}^2m\,{}^2\psi. \quad(4.20)$$

Equation (4.19a) has been solved already to yield (4.2). Using the coordinate condition (4.1) one obtains therefore

$${}_3\gamma_{0n}=4({}^1m\,{}^1\psi\dot{Y}^n+{}^2m\,{}^2\psi\dot{Z}^n). \quad(4.21)$$

The surface integral (3.19) vanishes; thus the next step is to determine

$$\left.\begin{aligned}{}_4{}^1C_m&=\frac{1}{4\pi}\oint_1 2\,{}_4\Lambda_{ms}n_s dS,\\ {}_4{}^2C_m&=\frac{1}{4\pi}\oint_2 2\,{}_4\Lambda_{ms}n_s dS.\end{aligned}\right\}\quad(4.22)$$

If one wishes to finish the approximation procedure here, the differential equations of motion of the 4th approximation are

$${}_4{}^1C_m=0;\quad {}_4{}^2C_m=0. \quad(4.23)$$

Therefore, all one has to do is to calculate those surface integrals. The result of this calculation is

$$\left.\begin{aligned}{}_4{}^1C_m(\tau)&=4\,{}^1m(\ddot{Y}^m+\tfrac{1}{2}\tilde{g}_{,m})=0,\\ {}_4{}^2C_m(\tau)&=4\,{}^2m(\ddot{Z}^m+\tfrac{1}{2}\tilde{f}_{,m})=0,\end{aligned}\right\}\quad(4.24)$$

where

$$\tilde{g}_{,m}\equiv g_{,m}\quad\text{for}\quad x^s=Y^s;\qquad \tilde{f}_{,m}\equiv f_{,m}\quad\text{for}\quad x^s=Z^s.$$

These are the equations of the Newtonian approximation of general relativity theory.

In order to go beyond the Newtonian approximation, one has to calculate ${}_4\gamma_{mn}$. It is to be found from the equation

$${}_4\gamma_{mn,ss}=2\,{}_4\Lambda_{mn}. \quad(4.25)$$

These equations are integrable only if one assumes Newtonian motion in the lower approximation. Otherwise one would have to add dipoles. Yet if one wishes to proceed only to the sixth approximation, one may ignore these additional dipoles since they do not influence the surface integrals. Thus, the solution for ${}_4\gamma_{mn}$ is in the neighborhood of the *first* singularity,

$$\begin{aligned}{}_4\gamma_{mn}&=\{f[(x^n-Y^n)\dot{Y}^m+(x^m-Y^m)\dot{Y}^n\\ &\quad-\delta_{mn}(x^s-Y^s)\dot{Y}^s]\}_{,0}+\{g[(x^n-Z^n)\dot{Z}^m\\ &\quad+(x^m-Z^m)\dot{Z}^n-\delta_{mn}(x^s-Z^s)\dot{Z}^s]\}_{,0}\\ &\quad+(7/4)\,{}^1r^2f_{,m}f_{,n}+(7/4)\,{}^2r^2g_{,m}g_{,n}\\ &\quad+\alpha_{mn}f+\beta_{mn}g.\end{aligned}\quad(4.26)$$

The last two expressions are determined by the coordinate conditions and are found to be

$$\alpha_{mn}=2\dot{Y}^m\dot{Y}^n+\delta_{mn}\tilde{g};\quad \beta_{mn}=2\dot{Z}^m\dot{Z}^n+\delta_{mn}\tilde{f}; \qquad (4.27)$$
$$\tilde{f}=f(r);\quad \tilde{g}=g(r);\quad r^2=(Y^s-Z^s)(Y^s-Z^s).$$

Moreover, ${}_4\gamma_{ss}$ can be calculated rigorously. The result is

$${}_4\gamma_{ss}=-2\,{}^1m\,{}^1r_{,00}-2\,{}^2m\,{}^2r_{,00}+(7/4)\varphi^2+\alpha f+\beta g \qquad (4.28)$$

with

$$\alpha=2\dot{Y}^s\dot{Y}^s+\tfrac{1}{2}\tilde{g};\quad \beta=2\dot{Z}^s\dot{Z}^s+\tfrac{1}{2}f. \qquad (4.29)$$

Now, the next field equations are

$$\left.\begin{aligned} &{}_4\gamma_{00,rr}=2_4\Lambda_{00}=-\tfrac{3}{2}\varphi_{,s}\varphi_{,s}, && \text{(a)}\\ &{}_5\gamma_{0m,ss}=2_5\Lambda_{0m}=\varphi_{,s}\,{}_3\gamma_{0s,m} \\ &\qquad -\varphi_{,sm}\,{}_3\gamma_{0s}-3\varphi_{,0}\varphi_{,m}, && \text{(b)}\\ &{}_6\gamma_{mn,ss}=2_6\Lambda_{mn}. && \text{(c)}\end{aligned}\right\} \qquad (4.30)$$

The solution of (4.30a) is simply

$${}_4\gamma_{00}=-\tfrac{3}{2}\varphi^2-4_4{}^1m\,{}^1\psi-4_4\,{}^2m{}^2\psi. \qquad (4.31)$$

One knows from the general theory that the arbitrary harmonic functions have to be determined in such a way as to make (4.30b) self-consistent, that is, the corresponding surface integral must vanish:

$$\frac{1}{4\pi}\oint(2_5\Lambda'-{}_{0m}-{}_4\gamma_{00,0m})n_m dS=0. \qquad (4.32)$$

The next step is to calculate the ${}_5\gamma_{0s}$. Including only relevant terms that can enter into the equations of motion of the sixth approximation, one obtains near the first singularity

$$\begin{aligned}{}_5\gamma_{0m}=&-(7/4){}^1rf_{,m}f_{,s}\dot{Y}^s+\tfrac{3}{4}f^2\dot{Y}^m\\ &+\tfrac{3}{2}(x^s-Y^s)(\dot{Y}^s-\dot{Z}^s)f\tilde{g}_{,m}\\ &-(x^m-Y^m)(\dot{Y}^s-\dot{Z}^s)f\tilde{g}_{,s}\\ &+\tfrac{1}{2}(x^s-Y^s)f_{,m}\dot{Z}^s[g+\tilde{g}_{,r}(x^r-Y^r)]\\ &+(x^s-Y^s)(f\tilde{g}_{,s}\dot{Z}^m+f_{,m}\tilde{g}\dot{Z}^s)+\alpha_{0m}f. \end{aligned} \qquad (4.33)$$

Again $\alpha_{0m}$ is determined from the coordinate condition. The result is

$$\alpha_{0m}=-\dot{Y}^s\dot{Y}^s\dot{Y}^m+\tilde{g}\dot{Y}^m-\tilde{g}\dot{Z}^m. \qquad (4.34)$$

One has to insert all these values for the field variables into the expressions for ${}_6\Lambda_{mn}$ and to calculate the surface integrals

$${}_6{}^1C_m=\frac{1}{4\pi}\oint 2_6\Lambda_{mn}n_n dS. \qquad (4.35)$$

The result of this calculation is

$$\left.\begin{aligned}{}_6{}^1C_m=-4\,{}^1m\,{}^2m\Bigg\{\Bigg(&\dot{Y}^s\dot{Y}^s+\tfrac{3}{2}\dot{Z}^s\dot{Z}^s\\ &-4\dot{Y}^s\dot{Z}^s-4\frac{{}^2m}{r}-5\frac{{}^1m}{r}\Bigg)\frac{\partial(1/r)}{\partial Y^m}\\ &+[4\dot{Y}^s(\dot{Z}^m-\dot{Y}^m)+3\dot{Y}^m\dot{Z}^s-4\dot{Z}^s\dot{Z}^m]\\ &\times\frac{\partial(1/r)}{\partial Y^s}+\frac{1}{2}\frac{\partial^3 r}{\partial Y^s\partial Y^n\partial Y^m}\dot{Z}^s\dot{Z}^n.\end{aligned}\right\}. \qquad (4.36)$$

Thus, the differential equations of motion of this stage of the approximation procedure are

$$\lambda^4\,{}_4{}^1C_m+\lambda^6\,{}_6{}^1C_m=0. \qquad (4.37)$$

It is possible to absorb the parameter $\lambda$ by changing the units of mass and time. The final result of the 6th approximation is, then,

$$\begin{aligned}\ddot{Y}^m-{}^2m\frac{\partial(1/r)}{\partial Y^m}&\\ ={}^2m\Bigg\{\Bigg(&\dot{Y}^s\dot{Y}^s+\tfrac{3}{2}\dot{Z}^s\dot{Z}^s-4\dot{Y}^s\dot{Z}^s-4\frac{{}^2m}{r}-5\frac{{}^1m}{r}\Bigg)\frac{\partial(1/r)}{\partial Y^m}\\ &+[4\dot{Y}^s(\dot{Z}^m-\dot{Y}^m)+3\dot{Y}^m\dot{Z}^s+3\dot{Z}^s\dot{Z}^m]\frac{\partial(1/r)}{\partial Y^s}\\ &+\frac{1}{2}\frac{\partial^3 r}{\partial Y^s\partial Y^n\partial Y^m}\dot{Z}^s\dot{Z}^n.\Bigg\}\end{aligned} \qquad (4.38)$$

The equations of motion for the other body are obtained from that one above by an obvious substitution. These differential equations can be integrated as will be shown in the next section of this paper.

## 5. INTEGRATION OF THE DIFFERENTIAL EQUATIONS OF MOTION

If one is to integrate the differential equations of motion as found in Sec. 4, one has to keep in mind that the latter were found by an approximation procedure. Thus an *exact* integration of the differential Eq. (4.38) does *not* yield exact equations of motion. It is, therefore, reasonable to assume a similar power series development with respect to the parameter $\lambda$ for the $Y$, $Z$ as this was done for the $\gamma$'s. Then, in order to solve the differential equations of the sixth approximation, one needs to take into account in $Y$, $Z$ only terms up to the power $\lambda^6$, for the higher order terms are not contained in the differential equations anyway.

Robertson[12] has integrated the differential equations of motion for the two-body problem (4.38). He ac-

[12] H. P. Robertson, Ann. Math. **39**, 101 (1938).

counted for the fact that the solution can be accurate only up to the 6th order by simply omitting in the left-hand side of (4.38) all terms of any order higher than Newtonian. The result is that up to the sixth order in $\lambda$ the orbit of a double star in general relativity theory differs in its secular behavior from the classical orbit only in an advance of perihelion equal to that which an infinitesimal planet, describing the same relative orbit, would undergo in the field of a star whose mass is the sum of those of the two components of the double star.

The fact that the differential equations of motion are solved by an expansion of the coordinates of the particles into a power series of the parameter $\lambda$ (i.e., the same parameter with respect to which Einstein, Infeld, and Hoffmann's approximation procedure is performed) suggests that it might be possible to tie together the two approximation procedures. The notion of accuracy "up to" a certain order, intuitively conceived above, can be formulated mathematically by a specific notation. Let us take any field expression

$$ {}_{2k}S\cdots \tag{5.1}$$

(e.g., $S=\phi$, $C$) where the dots stand for indices. The left subscript indicates that one considers the $2k$th approximation of $S$. $S$ may depend on terms containing $Y$, $Z$ (both together denoted by $X$), i.e., the coordinates of the two sources in the two-body problem. If one introduces into such a field quantity motion up to some approximation, ${}_{2k}S$ will not be of the $2k$th order; for one observes

$$ {}_{2k}S({}_0X+{}_2X)={}_{2k}S({}_0X)+\frac{\partial}{\partial_0 X}({}_{2k}S)_2X+\cdots, \tag{5.2}$$

which shows that ${}_{2k}S({}_0X+{}_2X)$ is really an expression containing terms of the $2k$th and *higher* orders.

However, if one wants to keep all the terms of the $2k$th order in a certain equation involving $S$'s, then these do not only originate from ${}_{2k}S$, since the lower orders of $S$ combine with the higher order of $X$ to yield terms of the order $2k$. Therefore, it is impossible to split an equation containing such $S$'s depending on the $X$'s into $\lambda$-terms *before* the development for the $X$'s is inserted. Thus, if one desires to have an accuracy up to all terms of order $2k$, one has to write the expression for $S$ as follows:

$$ {}_{[2k]}S({}_{[2k]}X), \tag{5.3}$$

the square brakets having the meaning that one has to take the sum of all the terms up to the denoted order.

It is possible to reformulate the approximation procedure by use of this notation. The field equations are up to the $2k$th order

$$\begin{aligned} &{}_{[2k-2]}\phi_{00}({}_{[2k-4]}X)+2{}_{[2k-2]}\Lambda_{0C}({}_{[2k-4]}X)=0, && \text{(a)} \\ &{}_{[2k-1]}\phi_{0m}({}_{[2k-4]}X)+2{}_{[2k-1]}\Lambda_{0m}({}_{[2k-4]}X)=0, && \text{(b)} \\ &{}_{[2k]}\phi_{mn}({}_{[2k-4]}X)+2{}_{[2k]}\Lambda_{mn}({}_{[2k-4]}X)=0. && \text{(c)} \end{aligned} \tag{5.4}$$

For $k=2$ those equations are identical with those for $k=2$ in Sec. 3, since there is no difference between the equations "up to" and "of" the second order. Thus, the beginning of the procedure is identical with that in Sec. 3.

Let us now assume that one has solved the Eqs. (5.4) up to the order $2k-1$. The next equations to be solved are then Eqs. (5.4) as they stand. These equations are solvable only if one assumes that the surface conditions (3.19) and (3.20) are satisfied. Thus, the integrability condition for (5.4b) leads to the adding of poles in the previous $\gamma$, whereas the integrability condition for Eq. (5.4c) determines the new corrective terms in the series expansion for the motion; for, these integrability conditions are

$$ \oint_S 2{}_{[2k]}\Lambda_{mn}({}_{[2k-4]}X)dS={}_{[2k]}{}^SC_m({}_{[2k-4]}X)=0. \tag{5.5}$$

Written out more explicitly this is

$$\begin{aligned} &{}_4{}^SC_m({}_0X+\cdots+{}_{2k-4}X) \\ &\quad+{}_6{}^SC_m({}_0X+\cdots+{}_{2k-6}X)+\cdots+{}_{2k}{}^SC_m({}_0X)=0. \end{aligned} \tag{5.6}$$

Since the corrective functions ${}_{2k-4}X$ appear only in ${}_4C$, Eqs. (5.6) are of the second order. One has thus a procedure whereby the motion is determined step by step to a higher accuracy.

The version of Einstein, Infeld, and Hoffmann's approximation procedure presented here has been suggested by Dr. Infeld. It is from a technical standpoint considerably simpler than that of Sec. 3 involving the dipoles. However, it should be noted that theoretically this new version tells one less than that of Sec. 3. The method of Sec. 3 uses physical notions such as dipoles whose annihilation, again a physical procedure, yields the equations of motion. The present method exemplified by Eq. (5.6), however, adjusts the corrective terms in the motion so as to make the *equations* consistent. Thus, it is seen that one is completely tied up with the representation of the motion in a particular coordinate system, which is not so satisfactory.

## 6. COORDINATE CONDITIONS

We have mentioned on several previous occasions that one can change the gravitational equations of motion in form by changing the coordinate conditions. It must be expected that the different solutions of the field equations thus obtained can be transformed into each other by simple coordinate transformations. This question has been studied by Infeld and Scheidegger.[13] Earlier it had been shown by Einstein and Infeld[9] that the most general solutions of the field equations that can be obtained by rejecting the coordinate condi-

[13] L. Infeld and A. E. Scheidegger, Can. J. Math. 3, 195 (1951).

tions are

$$ {}_{k-2}\gamma_{00}{}^{*} = {}_{k-2}\gamma_{00} \quad \text{(a)} $$

$$ {}_{k-1}\gamma_{0m}{}^{*} = {}_{k-1}\gamma_{0m} + {}_{k-1}a_{0,m} \quad \text{(b)} \quad (6.1) $$

$$ {}_{k}\gamma_{mn}{}^{*} = {}_{k}\gamma_{mn} + {}_{k}a_{m,n} + {}_{k}a_{n,m} - \delta_{mn}\, {}_{k}a_{r,r} + \delta_{mn}\, {}_{k-1}a_{0,0} \quad \text{(c)} $$

and

$$ {}_{k}\gamma_{mn}{}^{*} = {}_{k}\gamma_{mn} + {}_{k}b_{m,n} + {}_{k}b_{n,m} - \delta_{mn}\, {}_{k}b_{r,r} \quad \text{(a)} $$

$$ {}_{k}\gamma_{00}{}^{*} = {}_{k}\gamma_{00} + {}_{k}b_{r,r} \quad \text{(b)} \quad (6.2) $$

$$ {}_{k+1}\gamma_{0m}{}^{*} = {}_{k+1}\gamma_{0m} + {}_{k+1}b_{0,m} + {}_{k}b_{m,0}, \quad \text{(c)} $$

where the functions $a_0$, $a_m$ and $b_0$, $b_m$ are arbitrary. We shall prove here that it is possible to obtain the solutions $\gamma^*$ from the $\gamma$'s by a suitable coordinate transformation.

A general coordinate transformation is given as follows:

$$ x^\beta = x^\beta(x^{\alpha*}) = T^\beta(x^{\alpha*}). \quad (6.3) $$

Thus, the task is to calculate the transformed $\gamma$'s which can be done straightforwardly starting from the transformation law of the metric tensor

$$ g_{\mu\nu}{}^{*} = T^\rho{}_{|\mu} T^\sigma{}_{|\nu}\, g_{\rho\sigma}. \quad (6.4) $$

When applying Eqs. (6.4) one has to be careful that one takes the same world point as argument in all the functions which occur. Thus, in addition to the tensorial transformation (6.4) one has to perform inside the $g$'s a substitution of the variables $x^\beta$ by $x^{\beta*}$ according to Eq. (6.3).

In conformity with the general methods of the approximation procedure one has to expand the metric tensor $g_{\alpha\beta}$ into a power series with respect to the parameter $\lambda$. However, now *all* the terms instead of only alternating ones have to be kept.

Furthermore, it was assumed in the original solution that the motion of the particles is "slow" which made it necessary to introduce the "comma-differentiation" for $g_{\mu\nu}$. It is natural to require that the motion remains "slow" in the starred coordinate system. Thus one has to assume that the derivatives of $T^\mu$, too, are subject to the "comma-differentiation." Therefore, the transformation of the metric tensor represents itself as follows:

$$ g_{mn}{}^{*} = T^r{}_{,m} T^s{}_{,n} g_{rs} + T^0{}_{,n} T^s{}_{,n} g_{0s} + T^r{}_{,m} T^0{}_{,n} g_{r0} + T^0{}_{,m} T^0{}_{,n} g_{00}, $$

$$ \frac{1}{\lambda} g_{m0}{}^{*} = T^r{}_{,m} T^s{}_{,0} g_{rs} + T^0{}_{,m} T^s{}_{,0} g_{0s} + T^r{}_{,m} T^0{}_{,0} g_{r0} + T^0{}_{,m} T^0{}_{,0} g_{00}, \quad (6.5) $$

$$ \frac{1}{\lambda^2} g_{00}{}^{*} = T^r{}_{,0} T^s{}_{,0} g_{rs} + 2T^0{}_{,0} T^s{}_{,0} g_{0s} + T^0{}_{,0} T^0{}_{,0} g_{00}. $$

Equations (6.5) apply quite independently from whether for $T$ an expansion in $\lambda$ is used or not.

Furthermore, one observes that in the usual solution the lowest terms different from zero are of the order $\lambda^2$, apart from constant ones ${}_0g_{\alpha\beta} = \eta_{\alpha\beta}$. It will be convenient to confine oneself to coordinate systems where this same property holds. This means that in every coordinate system which is admitted for consideration, one has the flat Miskowskian metric as a first approximation of the gravitational field. This restriction of the coordinate transformations is justified by a remark of Papapetrou[14] who has proved that a change of the coordinate conditions will never affect the Newtonian approximation of general relativity. Thus, one is justified to assume that the coordinate transformation $T^\mu$ differs from the identity transformation only by terms proportional to $\lambda^2$:

$$ T^\mu = x^\mu + \lambda^k\, {}_k f^\mu; \quad k \geqslant 2. \quad (6.6) $$

As the indices 0 and $k$ have to be treated differently, one has to distinguish between two cases.

(a) Only the space coordinates are transformed by a single transormation of the order $\lambda^k$,

$$ x^r = T^r = x^{r*} + \lambda^k\, {}_{(k)}T^r(x^{\beta*}), \quad \text{(a)} \quad (6.7) $$

$$ x^0 = T^0 = x^{0*}. \quad \text{(b)} $$

If this is inserted into Eqs. (6.5) and everything expressed in $\gamma$'s, one obtains

$$ {}_{k}\gamma_{mn}{}^{*} = {}_{k}\gamma_{mn} + \delta_{mn}\, {}_kT^s{}_{,s} - {}_kT^m{}_{,n} - {}_kT^n{}_{,m}, \quad \text{(a)} $$

$$ {}_{k}\gamma_{00}{}^{*} = {}_{k}\gamma_{00} - {}_kT^s{}_{,s}, \quad \text{(b)} \quad (6.8) $$

$$ {}_{k+1}\gamma_{0m}{}^{*} = {}_{k+1}\gamma_{0m} - {}_kT^s{}_{,0}. \quad \text{(c)} $$

One observes that only the $k$th and higher approximations are influenced.

(b) Only the time coordinates are changed by a single transformation of the order $\lambda^k$,

$$ x^0 = T^0 = x^{0*} + \lambda^k\, {}_{(k)}T^0(x^{\beta*}), \quad \text{(a)} \quad (6.9) $$

$$ x^m = x^{m*}. \quad \text{(b)} $$

In a similar way the following occurs:

$$ {}_{k+1}\gamma_{mn}{}^{*} = {}_{k+1}\gamma_{mn} + \delta_{mn}\, {}_kT^0{}_{,0}, \quad \text{(a)} $$

$$ {}_{k}\gamma_{0n}{}^{*} = {}_{k}\gamma_{0n} + {}_kT^0{}_{,n}, \quad \text{(b)} \quad (6.10) $$

$$ {}_{k+1}\gamma_{00}{}^{*} = {}_{k+1}\gamma_{00} + {}_kT^0{}_{,0}. \quad \text{(c)} $$

By combining the cases (a) and (b) arbitrarily one obtains the most general coordinate transformation within the restrictions imposed here. Thus one may, for instance, consider the following combination:

$$ x^m = x^{m*} + \lambda^k\, {}_{(k)}T^m, \quad \text{(a)} \quad (6.11) $$

$$ x^0 = x^{0*} + \lambda^{k-1}\, {}_{(k-1)}T^0. \quad \text{(b)} $$

[14] A. Papapetrou, Proc. Phys. Soc. (London) **A64**, 302 (1951).

This yields the following transformation of the $Y$'s:

$$\begin{aligned} {}_{k-1}\gamma_{0m}^{*} &= {}_{k-1}\gamma_{0m} + T^{0}{}_{,m}, & \text{(a)} \\ {}_{k}\gamma_{mn}^{*} &= {}_{k}\gamma_{mn} + \delta_{mn}\, {}_{k}T^{s}{}_{,s} - {}_{k}T^{m}{}_{,n} \\ &\quad - {}_{k}T^{n}{}_{,m} + \delta_{mn}\, {}_{k-1}T^{0}{}_{,0}, & \text{(b)} \\ {}_{k}\gamma_{00}^{*} &= {}_{k}\gamma_{00} - {}_{k}T^{s}{}_{,s} + {}_{k-1}T^{0}{}_{,0}, & \text{(c)} \end{aligned} \tag{6.12}$$

which produces the formulas of Einstein and Infeld quoted in Eqs. (6.1) if one sets

$$\begin{aligned} {}_{k}T^{m} &= -{}_{k}a_{m}, & \text{(a)} \\ {}_{k-1}T^{0} &= {}_{k-1}a_{0}. & \text{(b)} \end{aligned} \tag{6.13}$$

Similarly, one can obtain the other set of Einstein and Infeld's formulas, quoted as Eqs. (6.2), by choosing a somewhat different combination of the coordinate transformations, (6.7) and (6.9), namely

$$\begin{aligned} x^{s} &= x^{s*} + \lambda^{k}{}_{(k)}T^{s}, & \text{(a)} \\ x^{0} &= x^{0*} + \lambda^{k+1}{}_{(k+1)}T^{0}. & \text{(b)} \end{aligned} \tag{6.14}$$

This yields the following transformation of the $\gamma$'s

$$\begin{aligned} {}_{k}\gamma_{mn}^{*} &= {}_{k}\gamma_{mn} + \delta_{mn}\, {}_{k}T^{s}{}_{,s} - {}_{k}T^{m}{}_{,n} - {}_{k}T^{n}{}_{,m} & \text{(a)} \\ {}_{k+1}\gamma_{0m}^{*} &= {}_{k+1}\gamma_{0m} - {}_{k}T^{s}{}_{,0} + {}_{k+1}T^{0}{}_{,m}, & \text{(b)} \\ {}_{k}\gamma_{00}^{*} &= {}_{k}\gamma_{00} - {}_{k}T^{s}{}_{,s}, & \text{(c)} \end{aligned} \tag{6.15}$$

which is identical with (6.2) if one puts

$$\begin{aligned} {}_{k}T^{s} &= -{}_{k}b_{s}, & \text{(a)} \\ {}_{k+1}T^{0} &= {}_{k+1}b_{0}. & \text{(b)} \end{aligned} \tag{6.16}$$

The results contained in Eqs. (6.13) and (6.16) show that coordinate transformations produce all the changes in the $\gamma$'s which Einstein and Infeld[9] found possible by rejecting the coordinate conditions in the $k$th step of the approximation procedure. Thus, having the usual solution, all the different solutions which result from the arbitrariness in the approximation procedure can be obtained simply by an appropriate coordinate transformation, and conversely.

## 7. COORDINATE TRANSFORMATIONS

The next question to be investigated is the possible influence of the coordinate transformations which were under consideration in Sec. 6, upon the equations of motion. This question has been studied previously up to the Newtonian approximation by Papapetrou,[14] and generally by Infeld and Scheidegger.[13] However, a new approach will be used here, which appears somewhat simpler.

Assume that the field equation be solved up to the order $\lambda^{2k+1}$. Thus, one knows the following quantities:

$$\begin{aligned} &{}_{2}\gamma_{00}\cdots{}_{2k}\gamma_{00}, & \text{(a)} \\ &{}_{3}\gamma_{0m}\cdots{}_{2k+1}\gamma_{0m}, & \text{(b)} \\ &{}_{4}\gamma_{mn}\cdots{}_{2k}\gamma_{mn}. & \text{(c)} \end{aligned} \tag{7.1}$$

Furthermore, the equations of motion of the corresponding order are

$$\begin{aligned} \lambda^{4}{}_{4}{}^{1}C(Y,Z) + \cdots + \lambda^{2k}{}_{2k}{}^{1}C(Y,Z) &= 0, & \text{(a)} \\ \lambda^{4}{}_{4}{}^{2}C(Y,Z) + \cdots + \lambda^{2k}{}_{2k}{}^{2}C(Y,Z) &= 0. & \text{(b)} \end{aligned} \tag{7.2}$$

One has to consider now two cases where in the first, one leaves all the Eqs. (7.1–2) unaltered, but in the second, one performs a coordinate transformation. The aim is to compare the equations of motion of higher approximation in those two cases.

To simplify the calculations involved, it turns out to be convenient to make some special assumptions. Firstly, the transformation $T$ be of the form

$$T^{\beta} = x^{\beta*} + \lambda^{2k}{}_{(2k)}T^{\beta} \tag{7.3}$$

as before. Secondly, noting that one needs the behavior of the expressions for ${}_{2k}T^{\beta}$ only in the neighborhood of the world lines of the particles, and thus, that they can be developed near the world lines into a Taylor series, one assumes that the occurring space derivatives near those lines shall vanish up to a 4th order. One may call such a coordinate transformation an infinitesimal one; because of all the group property all others can be obtained by repetitions of such infinitesimal transformations. Thirdly, one assumes that only ${}_{2k}T^{m}$ is different from zero, whereas ${}_{2k}T^{0}$ vanishes. These assumptions restrict, of course, the transformations considered to a large extent; it will be seen, however, that the admitted ones are still general enough for all purposes.

Thus, one has near the first world line the following coordinate transformation:

$$x^{m} = x^{m*} + \lambda^{2k}{}_{(2k)}{}^{1}T^{m}(x^{\beta*}). \tag{7.4}$$

An example for a choice of ${}_{2k}{}^{1}T^{m}$ satisfying all the above requirements would be near the world line $Y$

$${}_{2k}{}^{1}T^{m} = F^{m}(\tau) + [(x^{s}-Y^{s})(x^{s}-Y^{s})]^{5/2} f^{m}(\tau). \tag{7.5}$$

Then, the only $\gamma$ which is influenced in form up to the order $2k+1$ is ${}_{2k+1}\gamma_{0k}$. It becomes near the first world line, according to Eqs. (6.8),

$${}_{2k+1}\gamma_{0k}^{*} = {}_{2k+1}\gamma_{0m} - {}_{2k}{}^{1}T^{m}{}_{,0}. \tag{7.6}$$

When calculating the higher approximations, one is interested only in terms which contain $T^{m}$; the other ones are the terms which one would have got without the coordinate transformation. Thus, keeping only terms containing $T$'s, one obtains the formal difference betweeen the equations of motion in the old and in the new coordinate systems. We may note that one can use the standard coordinate conditions (4.1) throughout; for, (7.6) satisfies these conditions and for the later steps one is free to choose any coordinate conditions one likes.

To proceed with the approximation, one has to calculate the quantities denoted by ${}_{2k+2}L_{mn}$ and ${}_{2k+2}L_{00}$

in Eqs. (4.12)–(4.14). An inspection of the terms shows that a contribution containing $T$'s could only come from the linear terms. Those, however, vanish because they all contain space-derivatives of ${}_{2k}T$. From these statements we conclude that the additional terms in ${}_{2k}\Lambda_{mn}$ are just zero, so that we find

$$ {}_{2k+2}\Lambda_{mn}{}^{*} = {}_{2k+2}\Lambda_{mn}(x^{*}). \tag{7.7}$$

The corresponding surface integrals, therefore, remain unchanged; a fact which shows that an alteration of terms at the $2k$th step within the prescriptions of the approximation procedure never affects the differential equations of motion of the $2k+2$nd step. This is in conformity with the cited statement of Papapetrou[14] who observed that an infinitesimal coordinate transformation will not change the Newtonian approximation.

Thus, an infinitesimal coordinate transformation of the order $\lambda^{2k}$ will affect only terms from ${}_{2k+4}C$ onward in the equations of motion. The object is to calculate the first term which is affected. In the old coordinate system the equations of motion are

$$\begin{aligned} &\lambda^4\,{}_4{}^1C_m(Y,Z)+\lambda^6\,{}_6{}^1C_m(Y,Z) + \cdots + \lambda^{2k+4}{}_{2k+4}{}^1C_m = 0, \quad \text{(a)}\\ &\lambda^4\,{}_4{}^2C_m(Y,Z)+\lambda^6\,{}_6{}^2C_m(Y,Z) + \cdots + \lambda^{2k+4}{}_{2k+4}{}^2C_m = 0. \quad \text{(b)} \end{aligned} \tag{7.8}$$

Transforming the coordinate system at the $2k$th step by an infinitesimal transformation (7.4) will change somehow Eqs. (7.8). If one performs the *substitution* (7.4) in Eqs. (7.8) one obtains that same *old* condition for the motion of the particles concerned, but now expressed in the *new* coordinate system. The substitution (7.4) is equivalent to saying that one has to replace in (7.8)

$$\begin{aligned} Y^n \quad &\text{by} \quad Y^{n*}+\lambda^{2k}{}_{(2k)}T^n(Y^*), \quad \text{(a)}\\ Z^n \quad &\text{by} \quad Z^{n*}+\lambda^{2k}{}_{(2k)}T^n(Z^*). \quad \text{(b)} \end{aligned} \tag{7.9}$$

It will be seen that the substitution (7.4) does not only yield the old equations of motion in the new coordinate system, but the new equations of motion altogether. For, one observes that generally the new equations of motion will contain terms with $\lambda^j$ where $j>2k+4$. However, one has made the assumption that ${}_{2k}T$ is of the order $\lambda^0$ in the total domain where it is defined so that one can be sure that all the terms involving $T$ in the transformed equations of motion are actually of the order indicated by the power of $\lambda$ by which they are multiplied. If one keeps in mind that the method for solving the differential equations is by making a similar development for the solution, namely

$$Y^* = {}_0Y^* + \lambda^2\,{}_2Y^* + \cdots$$

and that in the latter in any case only terms up to the order $\lambda^{2k+4}$ are reliable, then one can discard all the terms of higher order than $2k+4$th arising from the coordinate substitution,—which proves the sufficiency of the substitution. The argument may not seem quite legitimate, but it would certainly be legitimate if the old $C$'s are of such a type that in performing the substitution terms of higher order than $2k+4$ just do not occur. This is actually the case in one significant application which will be given below as follows:

Consider the equations of motion of the 6th order; i.e., in the old coordinate system

$$\begin{aligned} \lambda^4\,{}_4{}^1C_m(Y,Z)+\lambda^6\,{}_6{}^1C_m(Y,Z)=0, \quad \text{(a)}\\ \lambda^4\,{}_4{}^2C_m(Y,Z)+\lambda^6\,{}_6{}^2C_m(Y,Z)=0. \quad \text{(b)} \end{aligned} \tag{7.10}$$

All the occurring $C$'s have been calculated by Einstein, Infeld, and Hoffmann[5] and are listed in the earlier sections of this paper as Eqs. (4.24) and (4.36). Perform now the coordinate transformation (7.4) which alters (7.10) by the corresponding substitution

$$\begin{aligned} &\lambda^4\,{}_4{}^1C_m(Y^*+\lambda^2\,{}_2T(Y^*), Z^*+\lambda^2\,{}_2T(Z^*)) + \lambda^6\,{}_6{}^1C_m(Y^*,Z^*)=0, \quad \text{(a)}\\ &\lambda^4\,{}_4{}^2C_m(Y^*+\lambda^2\,{}_2T(Y^*), Z^*+\lambda^2\,{}_2T(Z^*)) + \lambda^6\,{}_6{}^2C_m(Y^*,Z^*)=0. \quad \text{(b)} \end{aligned} \tag{7.11}$$

For the calculations it is very convenient to have assumed that

$${}_2T(Y^*)={}_2T(Z^*)={}_2T_{,k}(Y^*)={}_2T_{,k}(Z^*) = {}_2T_{,0}(Y^*)={}_2T_{,0}(Z^*)=0, \tag{7.12}$$

as this effects that one has $T^m$ equal to zero on the first world line. Thus, one finds near the first world line

$$\begin{aligned} {}_2T^m &= (x^s-Y^s)(x^s-Y^s)f^m(\tau), \quad \text{(a)}\\ {}_2T^m{}_{,k} &= 2(x^k-Y^k)f^m(\tau), \quad \text{(b)} \end{aligned} \tag{7.13}$$

and

$${}_2T^m{}_{,0} = -2(x^k-Y^k)\dot{Y}^k + (x^s-Y^s)(x^s-Y^s)\dot{f}^m(\tau), \tag{7.14}$$

which are all zero on the line; but

$${}_2T^m{}_{,00} = 2\dot{Y}^k\dot{Y}^k f(\tau) + \textit{terms zero on the line.} \tag{7.15}$$

Thus, the last term can be put equal to any arbitrary function of time.

Under these assumptions ${}_4C^m$ changes into

$$\begin{aligned} {}_4{}^1C_m \quad &\text{into} \quad {}_4{}^1C_m + 4\,{}^1m T^m{}_{,00}(Y^*)\lambda^2, \quad \text{(a)}\\ {}_4{}^2C_m \quad &\text{into} \quad {}_4{}^2C_m + 4\,{}^2m T^m{}_{,00}(Z^*)\lambda^2. \quad \text{(b)} \end{aligned} \tag{7.16}$$

Hence, one obtains the following equations of motion of the 6th order:

$$\begin{aligned} \lambda^4\,{}_4{}^1C(Y^*,Z^*) + \lambda^6\{4\,{}^1m\,{}_4T^m{}_{,00}(Y^*) + {}_6{}^1C_m\} = 0, \quad \text{(a)}\\ \lambda^4\,{}_4{}^2C(Y^*,Z^*) + \lambda^6\{4\,{}^2m\,{}_4T^m{}_{,00}(Z^*) + {}_6{}^2C_m\} = 0. \quad \text{(b)} \end{aligned} \tag{7.17}$$

It is readily seen that it is possible to choose the transformation $T$ in such a way that the coefficient of $\lambda^6$ in the equations of motion of the 6th order vanishes. Considering the expression (7.13) for $T$ near the first world line, one observes that one can annihilate the 6th order terms in Eq. (7.17a) simply by putting

$$f(\tau)=-{}_6{}^1C_m\{1/(4\dot{Y}^k\dot{Y}^k\,{}^1m)\}, \tag{7.18}$$

which, at the same time, insures one that ${}_2T$ is actually of the order $\lambda^0$ as required (except for $\dot{Y}^k=0$, a singular case which can easily be avoided by a translation of the origin). A similar procedure leads to the annihilation of the 6th order term in (7.17b).

Thus, we obtain equations of motion of the 6th order which do not contain any 6th order terms at all. This argument can be generalized for transformations of the $2j$th order

$$x^m=x^{m*}+\lambda^{2j}{}_{(2j)}T^m(x^*). \tag{7.19}$$

All the calculations are identical to those above; one has simply to apply them everywhere to the corresponding order of the equations. Thus, it is again possible to construct a coordinate transformation such that the equations of the order $i$ are transformed so as to contain no terms of the order $2j\leq i$. One can do that a sufficient number of times and thus finally end up with Newtonian equations of motion.

Thus we may shortly summarize the results of this section as follows.

It is always possible to construct such a coordinate transformation that the differential equations of motion have Newtonian form. In other words, the Newtonian form is a standard form to which the differential equations of motion can be reduced. As to the physical significance of this formalism, one has to observe, first of all, that it does *not* mean that the motion is just the same as it would be nonrelativistically in such a specially chosen coordinate system. Only the *form* of the differential equations is Newtonian; one must keep in mind, however, that the *metric* is by no means Galilean. Thus, the relativistic corrections of the motion are only transferred into the metric of the universe. Infeld[15] has demonstrated this recently explicitly for the case of the two-body problem.

So far, this does not yield any new conceptions. One may note, however, that the above statement about the possible Newtonian form of the equations of motion can be formulated in a slightly different way. For, one observes that it is the same as saying that, at every step of the approximation procedure, one can reach the vanishing of the surface integrals concerned by choosing the coordinate system in an appropriate way. This shows that one really has a new version of Einstein, Infeld, and Hoffmann's method for solving Einstein's field equations, which is equivalent to that introducing and annihilating dipoles.

[15] L. Infeld, Can. J. Math. **5**, 17 (1953).

## 8. THE PROBLEM OF GRAVITATIONAL RADIATION

In every field theory the problem of radiation is very closely connected with the problem of motion of a particle in that field. This is best seen in the electromagnetic case where the motion of a particle may be slowed down on account of a radiation damping effect by the field.

The gravitational field in space and time is in many respects very analogous to the electromagnetic field. From this analogy one might expect that moving bodies radiate gravitational waves and undergo a damping of their motion in a manner similar to moving charged particles. However, one will have to be very careful in defining what one means by "radiation of gravitational waves." In the usual way of speaking about "waves" the linear superposition principle is crucial for their existence. Since this superposition principle does not hold in nonlinear relativity theory one will have to study the analogy between electrodynamics and relativity in great detail so as to see whether the notion of a "wave" makes sense in the latter. To do this, one needs a formulation of electrodynamics which resembles the approximation procedure leading to the relativistic equations of motion of masses. This formulation was given by Infeld[16] as follows.

One can write down the field equations of electrodynamics in the following form

$$\begin{aligned} \gamma_{0|ss}-\gamma_{0|00}&=-\rho, && \text{(a)}\\ \gamma_{m|ss}-\gamma_{m|ss}&=\rho\dot{X}^m, && \text{(b)}\\ \gamma_{m|ss}&=\gamma_{0|0}. && \text{(c)} \end{aligned} \tag{8.1}$$

Here $\gamma_0$ is the electric, $\gamma_m$ the magnetic potential, $\rho$ the charge density and $\dot{X}^m$ the velocity vector of the charges. If the velocity of all the charged particles is small compared with the velocity of light, one may assume $\gamma_0$ as of the order $\lambda^2$ and $\gamma_m$ as of the order $\lambda^3$. Under these circumstances, the field variables $\gamma$ vary slowly in time but quickly in space. This compels one to introduce the "comma-differentiation" as before. If one assumes that all the charges are concentrated in particles and the latter described by singularities of the field, then Eq. (8.1) takes the following form outside of the singularities:

$$\begin{aligned} \gamma_{0,ss}&=\lambda^2\gamma_{0,00}, && \text{(a)}\\ \gamma_{m,ss}&=\lambda^2\gamma_{m,00}, && \text{(b)}\\ \gamma_{m,n}&=\lambda\gamma_{0,0}. && \text{(c)} \end{aligned} \tag{8.2}$$

If one expands the $\gamma$'s in power series of $\lambda$, taking the lowest powers of expansion to be of the order indicated by the assumptions about the $\gamma$'s, the Eqs. (8.2) split into

$$\begin{aligned} {}_k\gamma_{0,ss}&={}_{k-2}\gamma_{0,00}, && \text{(a)}\\ {}_{k+1}\gamma_{m,ss}&={}_{k-1}\gamma_{m,00}, && \text{(b)}\\ {}_{k+1}\gamma_{m,m}&={}_k\gamma_{0,0}. && \text{(c)} \end{aligned} \tag{8.3}$$

[16] L. Infeld, Phys. Rev. **53**, 836 (1938).

The fact that (8.3) connects only ${}_k\gamma_0$ with ${}_{k+2}\gamma_0$ and ${}_{k+1}\gamma_m$ with ${}_{k-1}\gamma_m$ permits one to set all the odd ${}_k\gamma_0$ and all the even ${}_k\gamma_m$ equal to zero.

Consider now the case of one point-singularity with its motion represented by $X^s$. (It is obvious how to generalize the ensuing statements for many particles.) The Eqs. (8.3) in the lowest order lead to the solution (choosing the particle at the origin and at rest for $t=0$)

$$\begin{aligned} {}_2\gamma_0&=e/r, &&\text{(a)}\\ {}_3\gamma_m&=-e\dot{X}/r, &&\text{(b)}\quad (8.4)\\ r^2&=(x^s-X^s)(x^s-X^s). &&\text{(c)}\end{aligned}$$

The nature of the whole solution is determined by this initial choice of the harmonic functions for ${}_2\gamma_0$ and ${}_3\gamma_m$, if one agrees not to introduce any arbitrary harmonic functions in the further approximation steps. One obtains then

$$\begin{aligned} {}_{2k}\gamma_0&=\frac{e}{(2k-2)!}\frac{d^{2k-2}}{d\tau^{2k-2}}(r^{2k-3}), &&\text{(a)}\\ {}_{2k+1}\gamma_m&=\frac{-e}{(2k-2)!}\frac{d^{2k-2}}{d\tau^{2k-2}}(r^{2k-3}\dot{X}^m). &&\text{(b)}\end{aligned}\qquad (8.5)$$

This corresponds to a standing wave, that is to $\frac{1}{2}$ advanced $+\frac{1}{2}$ retarded potential, as discussed long ago by Nordstrøm[17] and Page.[18]

If one wants to retain in the expressions for $\gamma$ all the powers of $\lambda$, then one has to make an arbitrary choice for ${}_3\gamma_0$, ${}_4\gamma_m$. The odd and even powers of $\lambda$ do not mix since the electrodynamic field Eqs. (8.1) are linear and of the second order. Take for ${}_3\gamma_0$ and ${}_4\gamma_m$ the following simple harmonic functions satisfying (8.3) for $k=3$:

$$\begin{aligned} {}_3\gamma_0&=0, &&\text{(a)}\\ {}_4\gamma_m&=ed^2X/d\tau^2. &&\text{(b)}\end{aligned}\qquad (8.6)$$

This leads to

$$\begin{aligned} {}_{2k+1}\gamma_0&=-\frac{e}{(2k-1)!}\frac{d^{2k-1}}{d\tau^{2k-1}}(r^{2k-2}), &&\text{(a)}\\ {}_{2k+2}\gamma_m&=\frac{e}{(2k-1)!}\frac{d^{2k-1}}{d\tau^{2k-1}}(r^{2k-2}\dot{X}^m). &&\text{(b)}\end{aligned}\qquad (8.7)$$

The general power series for the $\gamma$'s, if (8.5) and (8.7) are inserted, represent a retarded potential. Therefore one is justified to call the terms of Eqs. (8.7) "radiation terms." They change the standing potential, if added, into a solution with radiation.

Now, one observes that the electromagnetic equations (8.3) as discussed here are analogous to a corresponding set in general relativity theory, namely (3.16) and (4.1). The two sets of equations differ only by the additional index "0" in the gravitational case. In both cases the right-hand sides are known and determined by the previous approximation steps, for the $\Lambda$'s in the gravitational case are known functions of the $\gamma$'s.

This analogy with electrodynamics induces one to seek in the gravitational case a solution which corresponds to the retarded potential in electrodynamics. Such a solution might properly be expected to represent gravitational radiation. The analogy suggests to choose for ${}_3\gamma_{00}$ and ${}_4\gamma_{0m}$ the following expressions in the case of two particles:

$$\begin{aligned} {}_3\gamma_{00}&=0, &&\text{(a)}\\ {}_4\gamma_{0m}&=-4\,{}^1m\,{}^m\ddot{Y}-4\,{}^2m\ddot{Z}^m. &&\text{(b)}\end{aligned}\qquad (8.8)$$

One should note, however, that in electrodynamics the "radiation terms" initiated by the choice (8.7) can be calculated by themselves and simply inserted into the power series for the $\gamma$'s. In relativity theory the assumption of (8.8) will not only initiate "radiation terms," but will also alter all the $\gamma$'s as calculated without the radiation.

The introduction of radiation in relativity theory by analogy with electrodynamics as given above, may seem somewhat artificial. However, it is also possible to arrive at gravitational radiation terms by a more physical argument. True enough, one cannot speak of radiation or waves in relativity theory in the usual way, due to its nonlinear field equations, but one can set up a linear approximation of the theory as shown for instance by Eddington[3] where the notion of waves is sensible. Setting as usual

$$g_{\alpha\beta}=\eta_{\alpha\beta}+h_{\alpha\beta} \qquad (8.9)$$

and assuming $h_{\alpha\beta}$ to be *small*, one can neglect all terms involving the $h$'s more than linearly. Then one has correct to the first order, outside of the singularities of the field (reference 3, p. 128 ff),

$$\eta^{\alpha\beta}(h_{\beta\mu}-\tfrac{1}{2}\eta_{\beta\mu}\eta^{\sigma\rho}h_{\sigma\rho})_{|\alpha}=0, \qquad (8.10)$$

$$\Box\,\eta^{\alpha\beta}(h_{\beta\mu}-\tfrac{1}{2}\eta_{\beta\mu}\eta^{\rho\sigma}h_{\rho\sigma})=0, \qquad (8.11)$$

where $\Box$ denotes the d'Alembertian operator. Introducing the $\gamma$'s instead of the $h$'s yields

$$\begin{aligned} \gamma_{\alpha0|0}-\gamma_{\alpha s|s}&=0, &&\text{(a)}\\ \Box\gamma_{\alpha\beta}&=0. &&\text{(b)}\end{aligned}\qquad (8.12)$$

Setting $\alpha=0$ yields

$$\begin{aligned} \gamma_{00|0}&=\gamma_{0s|s}, &&\text{(a)}\\ \Box\gamma_{0\beta}&=0. &&\text{(b)}\end{aligned}\qquad (8.13)$$

These equations are precisely identical with (8.1), if one omits in (8.13) the additional index 0 and sets $\rho=0$ which means that the particles are represented as singularities of the field. Hence, it follows that one may deduce from (8.13) exactly the same solutions as one did from (8.1) (see for instance Infeld and Wallace[19]).

[17] C. Nordstrøm, Proc. Acad. Amsterdam **22**, 145 (1920).

[18] L. Page, Phys. Rev. **24**, 296 (1924).

[19] L. Infeld and P. R. Wallace, Phys. Rev. **57**, 797 (1940).

One can take over all the formulas from (8.4) to (8.7). Thus one obtains the result that the standing as well as the retarded potential is an exact solution of the linearized field equations. Gravitational radiation exists in the linearized theory in the conventional sense.

It is customary to regard the linearized theory as a first approximation to exact general relativity theory. Therefore, if one wants to retain radiation terms in the development for the field variables, then it is reasonable to request that the first term starting the radiation terms is equal to that which one had found starting radiation terms in the linear approximation. Thus, one is again led to the assumption that this term must be of the form given in Eqs. (8.8). Taking this term and inserting it into Einstein, Infeld and Hoffmann's approximation procedure should yield a physical effect which represents ordinary radiation in the limiting case of a weak field represented by linearized equations.

Generalizing the argument above, one is not forced to start the "radiation terms" with the choice of ${}_4\gamma_{0m}$ as this was done in (8.8). One can ask whether it would be possible to start the omitted terms in the original development for the $\gamma$'s at any stage of the approximation procedure, say in the $2k$th.

The prescriptions of Einstein, Infeld, and Hoffmann imply that one never must add arbitrarily to a field variable any additional poles or higher harmonic functions. On the other hand, the first equation starting the radiation terms is (keeping the usual coordinate conditions) one of the following:

$$\begin{aligned} &{}_{2k+1}\gamma_{00,\,ss}=0 && \text{(a)}\\ \text{or}\quad &{}_{2k}\gamma_{0m,\,ss}=0 && \text{(b)}\\ \text{or}\quad &{}_{2k+1}\gamma_{mn,\,ss}=0. && \text{(c)} \end{aligned} \tag{8.14}$$

If one wishes to take for one of these $\gamma$'s a solution $\neq 0$ which is nowhere singular in space (including infinity), one observes that the only possibility is $\gamma$ equal to a function of $\tau$. Thus, one can start "radiation terms" only with one of the following possibilities:

$$\begin{aligned} &{}_{2k+1}\gamma_{00}=f_{00}(\tau) && \text{(a)}\\ \text{or}\quad &{}_{2k}\gamma_{0m}=f_{0m}(\tau) && \text{(b)}\\ \text{or}\quad &{}_{2k+1}\gamma_{mn}=f_{mn}(\tau). && \text{(c)} \end{aligned} \tag{8.15}$$

It is readily seen that the particular choice (8.8) suggested by the electromagnetic analogy is indeed of the form (8.15) since $Y$, $Z$ are functions of $\tau$ only.

Concluding, we may summarize the contents of this section by stating that gravitational radiation in connection with gravitational motion can formally be introduced by analogy with electrodynamics. Whether this formal definition of radiation will represent any physical effects (as which one would expect radiation damping of the moving bodies) remains to be seen.

## 9. RADIATION DAMPING OF GRAVITATIONAL MOTION

All early attempts to obtain radiation damping effects in general relativity theory started from the electromagnetic analogy as expressed by Eq. (8.8). In particular, starting with the assumption (8.8), Infeld[16] has calculated the equation of motion up to the order $\lambda^7$.

It is readily seen that with this assumption one obtains

$${}_5\Lambda_{mn}=0. \tag{9.1}$$

Therefore, the equations of motion are not changed up to the fifth order by assuming a radiation term.

If one desires to go on in the approximation schedule, the next equations that one is faced with are

$$\begin{aligned} &{}_5\gamma_{00,\,ss}=0, && \text{(a)}\\ &{}_6\gamma_{0m,\,ss}=0, && \text{(b)}\\ &{}_7\gamma_{mn,\,ss}={}_5\gamma_{00,\,0}. && \text{(c)} \end{aligned} \tag{9.2}$$

As solutions of these equations, one can conveniently take

$${}_5\gamma_{00}=\frac{2}{3}\frac{d^3}{d\tau^3}\{{}^1m\,{}^2r^2+{}^2m\,{}^2r^2\}, \tag{9.3}$$

$${}_6\gamma_{0m}=-4\frac{d}{d\tau}\{{}^1m\dot{Y}^m+{}^2m\dot{Z}^m\} -\frac{2}{3}\frac{d^3}{d\tau^3}\{{}^1m\,{}^1r^2\,\dot{Y}^m+{}^2m\,{}^2r^2\,\dot{Z}^m\}, \tag{9.4}$$

which is in analogy with the corresponding electrodynamic equations. Then, one can calculate ${}_7\Lambda_{mn}$ in a similar way as this has been done before. The result is

$$\begin{aligned} 2{}_7\Lambda_{mn}=&-{}_6\gamma_{0m,\,0n}-{}_6\gamma_{0n,\,0m}+\delta_{mn}\,{}_5\gamma_{00,\,00}\\ &-\varphi\,{}_5\gamma_{00,\,mn}-{}_5\gamma_{00}\varphi_{,\,mn}-\tfrac{1}{2}{}_5\gamma_{00,\,m}\,\varphi_{,\,n}\\ &-\tfrac{1}{2}{}_5\gamma_{00,\,n}\,\varphi_{,\,m}+\tfrac{3}{2}\delta_{mn}\,{}_5\gamma_{00,\,s}\,\varphi_{,\,s}, \end{aligned} \tag{9.5}$$

which leads to

$${}_7{}^1C_m=\oint {}_7\Lambda_{mn}n_n dS=-(4/3){}^1m\{{}^1m\dddot{Y}^m+{}^2m\dddot{Z}^m\}. \tag{9.6}$$

But again, because of the Newtonian equations of motion, the right-hand side is not of the 7th, but at least of the 9th order; i.e., if one considers the equations of motion up to the 7th or 8th order, one obtains *no* contribution from inserting the radiation terms (8.8) into the usual approximation schedule.

Taking this result, Hu[10] went on in the approximation procedure and calculated the equations of motion up to the 9th order. This involves a tremendous amount of calculations which will not be reproduced here. The result is $({}^1m={}^2m=m)$

$$ {}_9{}^1C_n=-\frac{17}{60}Y^s\frac{d^5}{d\tau^5}\{m^2Y^sY^n\}+\frac{8}{3}Y^s\frac{d^4}{d\tau^4}\{m^2Y^s\dot{Y}^n\}. \quad (9.7) $$

Carrying out the differentiation with respect to time, and eliminating after each stage of differentiation the second derivatives by the Newtonian equations of motion, one obtains finally

$$ {}_9{}^1C_n=\frac{84}{5}m^4\dot{Y}^n/r^4 \quad (9.8) $$

and a similar equation for the second particle. From this position Hu calculated the "loss" of energy per unit time as given by

$$ -\dot{E}=-\frac{84}{5}m^4v^2/r^4=-(42/5)m^5/r^5 \quad (9.9) $$

with

$$ v^2=\dot{Y}^s\dot{Y}^s=m/2r \quad (9.10) $$

for a circular orbit. The sign in Eq. (9.9) means that the total energy defined in Newtonian mechanics as

$$ E=mv^2-2m^2/r \quad (9.11) $$

is *increased* by the radiation "damping" force. This result is rather strange from the point of view of Newtonian mechanics, according to which the energy can only be radiated out at the loss of the total energy. Physically, this result, if correct, has the meaning that the radiation damping force makes the particles move spirally away from each other, which is rather unbelievable.

Such are the results obtained by direct calculations. The most natural thing of taking the term ${}_4\gamma_{0m}=(8.8)$ for starting the radiation expansion, as suggested by electrodynamic analogy, leads obviously to an unbelievable result when physically interpreted in the conventional way, as shown above.

Thus, one is strongly induced to look for another approach to the problem of gravitational radiation damping. In this instance, it has been observed by Scheidegger[20] that a clue for the proper interpretation of the "radiation terms" might be provided by the remarks of Sec. 6 on coordinate conditions. There it was shown that one can create additional terms in the usual solutions of Einstein, Infeld, and Hoffmann's approximation procedure according to Eqs. (6.1) and (6.2) by a mere infinitesimal coordinate transformation. This statement gives us the clue for the proper interpretation of the radiation terms. It is easily seen that the terms which were chosen to start the radiation expansion can be obtained from the usual solution by putting in (6.7),

$$ {}_3T^m=4\,{}^1m\dot{Y}^m+4\,{}^2m\dot{Z}^m. \quad (9.12) $$

[20] A. E. Scheidegger, Proc. Second Can. Math. Congr. 218 (1949).

Thus, the starting term (8.8) is just of such a form that it can be created by an infinitesimal coordinate transformation. Therefore, it can also be wiped out by the concerning inverse transformation. But after this inverse transformation, one has no radiation terms, the metric tensor is that which one had before radiation terms were inserted, the equations of motion are the original ones (without the radiation), and thus the old solutions of the relativistic field equations without the radiation terms are regained.

It may be noted that the coordinate system containing the radiation terms with the particular assumption (9.12) does not even require a departure from the usual coordinate conditions, since ${}_4\gamma_{0m,\,0}={}_3\gamma_{00,\,0}=0$.

One may ask now whether there are other possibilities for inserting radiation terms. It was seen generally that a term starting the radiation expansion is of the form (8.15). Let us take first the following possibility:

$$ {}_{2k}\gamma_{0m}=f_m(\tau). \quad (9.13) $$

This yields formally an even $\gamma_{0m}$ which is different from zero. However, we note again that (9.13) can be created by a coordinate transformation of the usual solution by putting in Eq. (6.7),

$$ {}_{2k-1}T^m=-\int f_m d\tau. \quad (9.14) $$

Hence, it also may be destroyed by the corresponding inverse coordinate transformation.

The second possibility to start a radiation expansion is with ${}_{2k+1}\gamma_{mn}$. Thus, one assumes

$$ {}_{2k+1}\gamma_{mn}=f_{mn}(\tau), \quad (9.15) $$

where again $f_{mn}$ is an arbitrary function of time. However, this term, too, can be obtained from the usual solution by a coordinate transformation. It is of the form (6.1) if one sets

$$ {}_{2k}a_m=\tfrac{1}{2}f_{ms}x^s, \quad \text{(a)} \qquad (9.16) $$

$$ {}_{2k-1}a_0=\tfrac{1}{2}\int f_{ss}d\tau. \quad \text{(b)} $$

It has been shown in Sec. 6 that all such terms can be obtained or annihilated by an infinitesimal coordinate transformation.

Finally, it remains to investigate what happens if a radiation term is inserted for $\gamma_{00}$ at a certain stage of the approximation procedure

$$ {}_{2k+1}\gamma_{00}=f(\tau). \quad (9.17) $$

In order to annihilate it, one may construct the following coordinate transformation:

$$ {}_{2k}T_0=\tfrac{1}{4}\int f(\tau)d\tau, \quad \text{(a)} \qquad (9.18) $$

$$ {}_{2k+1}T^s=-\tfrac{1}{4}f(\tau)x^s. \quad \text{(b)} $$

According to Eq. (6.12) one finds indeed that the old ${}_{2k+1}\gamma_{00}$ is changed into (9.17), whereas all the other field variables are unaffected up to the considered approximation. Therefore, the corresponding inverse coordinate transformation again effects the annihilation of (9.17).

These results provide now an easy explanation of the radiation terms obtained by direct calculations.

First, one observes that all the additional terms of the form (6.1) or (6.2) added to the usual solutions do not affect the equations of motion in the next stage of the approximation. All the radiation terms can be brought into a linear combination of (6.1)and (6.2). Hence, one obtains the result that adding a radiation term at a certain stage of the approximation procedure leaves unaltered the equations of motion in the next one.

Second, having explicitly shown that all the radiation terms whatsoever can be destroyed by coordinate transformations, one observes that the terms that had been found by straightforward calculations must be entirely due to the particular choice of the coordinate system. Thus, there is no radiation damping of gravitational motion.

Actually, part of the results of this section could have been deduced directly from the structure of the gravitational field equations as observed by Scheidegger,[21] for it is well known (see Bergmann, reference 2, p. 188) that in the linearized theory only the waves where $\gamma_{st}\neq 0$ (so-called transverse-transverse waves) have a physical meaning, as the others can be annihilated by coordinate transformations. Now, it should be noted that the linearized theory should be identical to the first approximation of the exact theory, thus implying that radiation terms of the type $\gamma_{00}\neq 0$ or $\gamma_{0m}\neq 0$ can be wiped out by a suitable coordinate transformation in the lowest approximation of Einstein, Infeld, and Hoffmann's method. Hence, it follows that such terms cannot represent any physical effects. Thus, the only gravitational waves for which the possibility of annihilation by coordinate transformation is not obvious beforehand are the transverse-transverse waves. They have to be treated in the manner demonstrated earlier in this section.

In conclusion, it should be remarked that some theoretical physicists do not entirely agree with all the conclusions given in this and the last sections. Whereas it is undisputed that there is no physical reality to gravitational radiation *as defined in Section 8*, it has been questioned whether this is the *only* possible way to define gravitational radiation. The difficulty is that so far no satisfactory definition of "real" (as against apparent) radiation is to be found in literature. Until such a definition can be given, one will have to contend with giving (by analogies) reasonable definitions of gravitational radiation and to test whether those definitions will lead to physically real effects. This is the approach used by Infeld and Scheidegger.[13] It is the opinion of these authors that this is also the *only* way to obtain the radiation of a two-body system if there is any. The argument for this is largely one of principle, namely, if the gravitational motion of a system of two bodies in otherwise empty space is to be determined by the initial conditions implying position and velocities of the bodies only, then there is no room for additional boundary conditions that would determine in which of several choices of "stages" ("with" or "without" radiation) the system would be at a later instant. It seems to the writer logically impossible that a system whose behavior is thought to be completely determined by the initial conditions (according to the causalistic relativity theory) should have the freedom of several "behaviors." Since a solution fitting the initial conditions has been found, this is not only a solution, but the *only* solution of the problem. Nothing, of course, is being said of the possibility of gravitational radiation if the system is subjected to additional forces and not only to the gravitational ones originating from itself.

Nevertheless, the argument given in the last paragraph has been and still is being questioned by noted experts in relativity[22] who maintain that a different approach from that of Einstein, Infeld, and Hoffmann to the two-body problem and a different definition of "radiation" might lead to a different behavior of the system from that presented in the previous sections of this paper. To this date, however, nothing of such a different approach is to be found in the literature.

## 10. THE EINSTEIN-INFELD-HOFFMANN FORMALISM; IN GENERAL COVARIANT NONLINEAR FIELD THEORIES

In the last section the deduction of the equations of motion in general relativity theory was studied. It would be interesting to know whether the procedure outlined there is restricted to relativity theory or whether it might be possible to generalize it in such a way that it provides one with a scheme allowing to find equations of motion in any given nonlinear field theory.

Recently Bergmann[23] has made an attempt to set up a general covariant scheme for many types of field theories. Then, relativity theory would appear as a special case in this formalism. The principal aim in Bergmann's[23] investigations was to bring all these field theories into a Hamiltonian form so that their quantization would be possible (see Bergmann and Brunings[24]). In the classical part of his work, however, he points out some features which are connected with the equations of motion. One of his statements is that it should be possible to deduce in a covariant nonlinear theory the equations of motion of the sources from the field equations. If this is true, then it must be possible

[21] A. E. Scheidegger, Phys. Rev. **82**, 883 (1951).

[22] Private communications, notably from Dr. P. G. Bergmann, Syracuse University, Syracuse, New York.

[23] P. G. Bergmann, Phys. Rev. **75**, 680 (1949).

[24] P. G. Bergmann and J. Brunings, Revs. Modern Phys. **21**, 480 (1949).

to find Lorentz's equations of motion of an electron in a generalized nonlinear theory of the electromagnetic field, (see the attempts of Born,[25] Born and Infeld,[26] Infeld,[27,28] Infeld and Wallace[19]) in the same way as the equations of motion of stars are found in general relativity theory. Up to now it has not yet been possible to deduce the equations of motion of an electron from an electrodynamic field theory.

We denote the field variables by $y_A$, $(A=1\cdots N)$, where $N$ is the number of algebraically independent components. It will be assumed that the field equations can be deduced from a variational principle of the form

$$\delta I=\delta\int_V L(y_A,\, y_{A|\beta})d^4x=0, \tag{10.1}$$

satisfied in the four-dimensional domain $V$. $L$ is an algebraic function of $y_A$ and $y_{A|\beta}$ only. The field equations which result from the (infinitesimal) variation of the field variables in the interior of $V$ are

$$L^A\equiv\frac{\partial L}{\partial y_A}-\left(\frac{\partial L}{\partial y_{A|\beta}}\right)_{|\beta}=0. \tag{10.2}$$

With respect to an infinitesimal coordinate transformation,

$$x^\mu=x^\mu-\epsilon\xi^\mu, \tag{10.3}$$

the field variables are assumed to transform according to a law having the form

$$\bar{\delta}y_A\equiv y_A{}^*-y_A=\epsilon(F_{A\mu}{}^{B\nu}\xi^\mu{}_{|}\nu-y_{A|\mu}\xi^\mu)+0(\epsilon^2). \tag{10.4}$$

The $F_{A\alpha}{}^{B\beta}$ are numbers characteristic for the type of field variables used. It should be noted that $\bar{\delta}y_A$ are the changes of the $y_A$ as functions of their *arguments* (and not as functions of the world point only); hence, the second term in (10.4).

The field Eqs. (10.2) must be covariant. If the Lagrangian, in the face of an infinitesimal coordinate transformation, adds a divergence, then that condition is sufficient (though possibly not necessary) to assure covariant field equations. For, in this case, the infinitesimal change in the integral $I$ can be expressed by means of a surface integral, and it must be assumed that the coordinate transformation is the identity on the boundary of our volume of integration. Thus, one has

$$\bar{\delta}L=\epsilon Q^\mu{}_{|\mu}+0(\epsilon^2), \tag{10.5}$$

where the $Q^\beta$ are functions of the $\xi^\beta$ and their derivatives. Then, the transformation laws of the field equations can be computed straightforwardly to be

$$\bar{\delta}L^B=\epsilon\{-F_{A\mu}{}^{B\nu}\xi^\mu{}_{|}\nu L^A-(L^B\xi^\mu)_{|\mu}\}+0(\epsilon^2). \tag{10.6}$$

[25] M. Born, Proc. Roy. Soc. (London) **A143**, 410 (1934).
[26] M. Born and L. Infeld, Proc. Roy. Soc. (London) **A144**, 485 (1934).
[27] L. Infeld, Proc. Cambridge Phil. Soc. **32**, 127 (1936).
[28] L. Infeld, Proc. Cambridge Phil. Soc. **33**, 70 (1937).

A further remarkable property of this scheme is that the field variables must fulfill 4 identities, namely,

$$(F_{A\mu}{}^{B\nu}y_BL^A)_{|}\nu+y_{A|\mu}L^A=0 \tag{10.7}$$

(generalized Bianchi identities).

The expressions $L^A$ contain the field variables and their first and second derivatives. The second derivatives occur only linearly, and their coefficients can be represented in the form

$$L^A=L^{AB\alpha\gamma}y_{B|\alpha\gamma}+\cdots. \tag{10.8}$$

Thus, the third derivatives occur only linearly in the identities (10.7) and those must cancel each other. It follows that the coefficients $L^{AB\alpha\gamma}$ must satisfy the identities

$$\{F_{A\beta}{}^{B\gamma}L^{AC\alpha\theta}+F_{A\beta}{}^{B\alpha}L^{AC\theta\gamma}+F_{A\beta}{}^{B\theta}L^{AC\gamma\alpha}\}y_B=0. \tag{10.9}$$

To deduce the equations of motion from the above scheme, one has to apply an approximation procedure. The solutions of the field equations are to be obtained as a power series in a certain parameter $\lambda$

$$y_A={}_0y_A+\lambda\,{}_1y_A+\lambda^2\,{}_2y_A+\cdots=0. \tag{10.10}$$

The time differentiation is thought to raise the order of the differentiated term; thus, the "comma" differentiation is used in the same sense as before.

The zeroth approximation of the approximation procedure shall be a "trivial" solution, a rigorous solution of the field equations in which all field variables are constants. The first approximation will yield

$${}_0L^{ABrs}{}_1y_{B,\,rs}=0. \tag{10.11}$$

The solutions ${}_1y_B$ will generally not be defined throughout space. At each instant of time there will be certain three-dimensional domains in which the field equations have no bounded solutions. However, only such solutions will be considered in which each one of these singular regions can be surrounded by a closed surface $S$ on which the field equations are satisfied. Proceeding to the next approximation, the equations have the form

$${}_0L^{ABrs}{}_2y_{B,\,rs}=-{}_2L^A({}_0y+\lambda\,{}_1y). \tag{10.12}$$

The right-hand sides will be the $L^A$ of the second order formed from the first-order solutions. The left-hand sides satisfy identically the four relationships

$$F_{A\alpha}{}^{Ct}{}_0y_C({}_0L^{ABrs}{}_2y_{B,\,rs})_{,\,t}=0 \tag{10.13}$$

irrespective of the choice of the second-order field variables. On a surface $S$ surrounding a singularity, on which (10.12) are to be satisfied, one has then

$$0\equiv\oint F_{\alpha\mu}{}^{Ct}{}_0y_C\,{}_0L^{ABrs}{}_2y_{B,\,rs}n_tdS$$

$$=-\oint_S F_{A\mu}{}^{Ct}{}_0y_C\,{}_2L^A({}_0y+\lambda\,{}_1y)n_tdS, \tag{10.14}$$

which represent for that singularity inside $S$ the three equations of motion and the law of conservation of "mass."

To investigate the possibility of the existence of equations of motion in different nonlinear field theories, one has to apply Eq. (10.14) systematically to certain types of possible field variables. The results quoted here have been deduced by Scheidegger.[29]

Starting with field theories using one scalar field variable only, one sets

$$y_A = \phi. \tag{10.15}$$

Then, the transformation of the field variables is given by just the last term in Eq. (10.14) so that the coefficients $F_{A\alpha}{}^{B\beta}$ all vanish

$$F_{A\alpha}{}^{B\beta} = 0. \tag{10.16}$$

The surface conditions, therefore, become trivial identities, and one gets the result that no equations of motion can be deduced in any scalar field theory whatsoever.

The next more complicated type of field variables are vectors. Thus,

$$y_A = A_\beta. \tag{10.17}$$

Using the infinitesimal transformation

$$x^{\beta *} = x^\beta - \epsilon\xi^\beta, \tag{10.18}$$

one obtains the following change of the field variables (except for the term originating in the coordinate substitution):

$$A_\mu{}^* = (\delta_{\rho\mu} - \epsilon\xi^\rho{}_{|\mu})A_\rho = \delta_{\rho\mu}A_\rho - \epsilon\xi^\eta{}_{|\lambda}\delta_{\rho\eta}\delta_{\mu\lambda}A_\rho. \tag{10.19}$$

Hence,

$$F_{\mu\eta}{}^{\rho\lambda} = -\delta_{\rho\eta}\delta_{\mu\lambda}. \tag{10.20}$$

The calculation of the surface condition with these coefficients (denoting the "trivial" solution, constant in space and time, by $a_\alpha$) yields

$$0 = -\oint F_{\alpha\mu}{}^{\gamma t} a_\gamma L^\alpha n_t dS = a_\mu \oint L^t n_t dS. \tag{10.21}$$

This shows that the four conditions of Eq. (10.14) reduce to only one in every vector theory whatsoever. This one condition, however, cannot be sufficient for determining the motion of the singularity to which it refers; it is just the conservation law for the pole strength of that singularity.

Thus, one concludes that the analogous procedure to that of Einstein, Infeld, and Hoffmann cannot be set up for a covariant vector theory. Electromagnetic theory is a covariant vector theory; the field variables are the components of the potential vector $\phi_\alpha$. Even in a nonlinear generalization such as that of Born and Infeld[25–28] there are, therefore, no equations of motion.

Thus one may state that a covariant field theory has to be at least a tensor field theory so that it is possible to obtain equations of motion from the field equations.

[29] A. E. Scheidegger, Helv. Phys. Acta **23**, 740 (1950).