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The Energy Levels and the Structure of Light Nuclei

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A theoretical interpretation of the energy level sequences of the light nuclei of the p shell is developed, in the spirit of an exploratory survey. Phenomenological assumptions concerning the nuclear forces are used and the secular problem of intermediate coupling (between the Russell-Saunders and (jj) extremes) is in most cases treated by means of rough interpolations, so no exact energy agreement can be expected, but the order of states of various quantum numbers in the p-shell configuration can be approximately established. This order and the general density of levels agree with the observations, so far as they have been made, in enough cases to give some evidence for the validity of the general method of approach, though further refinements are needed.

The presentation is made with a number of introductory sections, intended to orient an uninitiated reader in various relevant aspects of the theory of nuclear structure, preceding the discussion of the interpretation of the levels of the individual nuclei. Section 1, an introduction, discusses the meaning of nuclear models. Section 2 deals with the phenomenological approximations to the specific nuclear forces and suggests various possible sources of the spin-orbit coupling. Sections 3 and 4 are primarily pedagogical, in that they explain and formulate well-established wave-mechanical fundamentals of two-body (as an example of many-body) systems. Section 3 treats the relation between symmetry and energy for two electrons or two neutrons, and Sec. 4 presents in some detail the relation of the total isobaric spin quantum number T to the more directly physical dynamical variables of a system of nucleons. Section 5 surveys the experimental data for the various polyads, or groups of isobars, arranged in such a way as to make apparent the matching in energy of the states of the same T but different neutron excess, after a correction has been applied for Coulomb energy differences. Section 6 is again pedagogical, reviewing the theory of intermediate coupling in atoms, and introducing the application to nuclei. Section 7 discusses explicitly and in some detail the interpretation of the energy level sequences observed in the various polyads, from $A=5$ to $A=17$, as well as some of the methods that have been used for making the crucial experimental assignments of angular momentum and parity of the various states. The degree of partial success of the interpretation leaves the impression that the (jj)-coupling scheme of heavy nuclei gives way to intermediate coupling in the light nuclei, and that some of the higher order deviations from this scheme may usefully be described as a partial transition to the nucleon clustering of the alpha-model in a few nuclei whose atomic number A is especially favorable for this clustering. Section 8 presents some details of the Coulomb energy calculations. Section 9 explains the remarkable appearance of double levels (which look like atomic doublets but are explained differently) in some of the nuclei near $A=30$, beyond the formal scope of this paper, and shows that the explanation does not apply to the similar occurrence of apparently fortuitous double levels in some of the nuclei of the p shell. Section 10 discusses the possible role of the alpha-model in influencing the states of some of the light nuclei. Section 11 discusses beta-decay, particularly in the case of C^{14} where there is an order-of-magnitude effect in need of explanation. One of the appendices treats the molecular problem of quadrupole coupling particularly in relation to the measurement of the quadrupole moment of Li^7 in polar molecules.

1. INTRODUCTION—NUCLEAR MODELS

THE increasing wealth of new material on the identification of angular momenta and parities of the excited states of light nuclei brings ever nearer the prospect of a more complete understanding of nuclear structure in this region. The gratifying success of the (jj)-coupling shell model (M49, H49, K50)* in explaining many features of heavier nuclei encourages the hope of finding some understanding also of the light nuclei in terms of simple models, but one encounters numerous difficulties in trying to apply the (jj) model alone to all the states of the light nuclei, especially to those of the p-shell nuclei He^4 to O^{16} and immediately beyond.

One had no *a priori* right to expect any model to apply well to a nucleus consisting of many nucleons attracted by short-range interactions, even after the success of the Hartree model in the quite different case of an atom with many electrons. The motion of most of the electrons is largely dictated by the overwhelmingly strong field of a body fixed at the center. Lacking this, and with attractive rather than repulsive interactions, it is not clear, for example, that the nucleons would not have at least an appreciable tendency to cluster into small groups, such as alphas (W37). Even if this were only a weak trend it could greatly complicate their behavior. If it were a very strong trend, it might again simplify matters and make the alpha-model valid (W37, I41). It has long been empirically apparent from regularities among such phenomena as nuclear moments that nuclei behave more simply than we had any *a priori* right to expect (S37, I41).

The somewhat surprisingly successful (jj) coupling shell model is of course a special case of the central (or Hartree) model of the nucleus. Its success indicates that the interaction of a single nucleon with all the others may be fairly well represented by an average central field, at least in heavy nuclei. But the success of this particular (jj) version of the central model indicates also that the spin-orbit coupling energy (I36, B37) is surprisingly large in nuclei. The spin-orbit coupling energy attempts to orient the spin $\boldsymbol{s}$ of a nucleon relative to the orbital angular momentum $\boldsymbol{l}$ of the same nucleon to form a total angular momentum quantum number j of the individual nucleon, to give the (jj)-coupling scheme, while the "exchange integral" of the

* References are given at the end of this article.

specific nuclear interactions competes with it in trying to bring about the (LS)-coupling scheme (H37, Wi37, F37), or "Russell-Saunders Coupling" as it is known in the analogous atomic problem, in which the total spin angular momentum S, summed over all the particles, is a constant of the motion and also the total orbital angular momentum L. Thus the observed approximation of heavy nuclei to (jj) coupling means in general terms that the spin-orbit coupling energy is large compared to the "exchange integral" in those nuclei.

The failure, in the p shell, of the (jj)-coupling version of the central model may be due either to a failure of the "(jj)-coupling" aspect or of the "central model" aspect, or both, and we shall see some evidence of both. The central model includes as special cases not only the (jj)-coupling version and the (LS)-coupling version, each of which is an extreme case having specially simple features, but also the whole range between them known as intermediate coupling, as was once well known in the theory of complex atomic spectra. It applies to a wide range of possible ratios of the two parameters, giving a variation of the possible energy level scheme analogous to the Paschen-Back transitions which are still widely known because of their applicability to atomic-beam and microwave spectroscopy. Intermediate coupling seems to be responsible for many features of the energy level schemes encountered in the latter half of the p shell, from B^{10} to O^{16}, while the failure of the complexities of intermediate coupling to account for some of the features of the still lighter nuclei in the first half of the p shell seems to be associated with the tendency for alpha-clustering to become important in these rather loosely-bound nuclei (or to some other manifestation of higher order perturbation theory as will be discussed further below).

An extreme case of the tendency for the formation of clusters of nucleons such as alphas within nuclear matter is of course represented by the alpha-model, and good reasons have been given for expecting the applicability of this model to light nuclei (W37, H38), especially to those within which the energy that may be ascribed to the mutual binding energy between alpha-clusters is small compared to the internal binding of these clusters. One can conceive of a transition between the central model and the alpha-model analogous to the intermediate-coupling transition within the central model, though it is more difficult to formulate explicitly. Thus one might think of setting up a sort of two-dimensional variation principle in which the wave functions of the various states are formed as a compromise between the three extremes, the (jj)-coupling model, the (LS)-coupling model, and the alpha-model. Fortunately, some approximation to the behavior of most nuclei seems to be possible without making use of all this complexity.

2. NUCLEON INTERACTIONS

Nuclear spectroscopy aims on the one hand to help to develop, on the other hand to get along without, a detailed knowledge of the nature of nuclear "forces." This seeming contradiction arises from the great difficulty of uncovering the profound laws of nature, and the hope of doing so in a stepwise fashion. It is too much to hope in the near future to calculate the energy levels of B^{11}, say, on the basis of a completely satisfactory meson theory of nuclear structure. Instead, one hopes to divide the problem up into a phenomenological one in which the interactions between nucleons are assumed to have one of several possible simple forms, it being determined by trial which form seems to have the greatest empirical validity, and a second step in which the phenomenological interaction thus selected is to be understood on the basis of a theory of the structure of the nucleons themselves, akin to the present meson theories which as yet are not entirely free from divergences. This review of the energy spectra of the light nuclei is concerned entirely with the first or phenomenological step, though we shall discuss briefly the relation of our assumed spin-orbit coupling energy to the second or "meson theory" step.

Phenomenological Specific Nuclear Interactions

Most practical attempts to formulate the sequence of nuclear states have been based on the assumption of central interactions between two nucleons written as a function of their distance apart, $V(r_{ij})$, multiplied by an exchange operator O_{ij}, which is variously written as a linear combination of operators such as P_{ij} or simply P, the space-exchange or Majorana operator which exchanges the space coordinates of the two nucleons in a wave function following it, the unit or "Wigner" operator 1 which does nothing, the space-spin exchange or Heisenberg operator PQ, and the spin-exchange or Bartlett operator Q. This usage and the saturation requirements of the coefficients with which the operators are combined is adequately reviewed in Rosenfeld's book (R48), where it is suggested (p. 234) that the most nearly satisfactory version of O_{ij} is perhaps

$$\begin{aligned} O_{ij} &= \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j (0.1 + 0.23 \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) \\ &= (0.93P - 0.13 - 0.26PQ + 0.46Q). \end{aligned} \tag{1}$$

A simplified version that is almost equivalent to this in many rough calculations, because it keeps the large terms and contains the same proportion of terms involving spin exchange to those which do not, is

$$O_{ij} = 0.8P + 0.2Q. \tag{2}$$

As a similar part of the phenomenological step one usually assumes also a simple form of spin-orbit coupling operator, which we may write as a perturbation term to the Hamiltonian thus:

$$H' = \sum_i a_i \boldsymbol{l}_i \cdot \boldsymbol{s}_i = \sum a \boldsymbol{l} \cdot \boldsymbol{s}. \tag{3}$$

In most cases the parameter a will be the same for all nucleons under explicit consideration, they being all in

the same shell. This quantity a is the parameter which, by being large, can make the magnitude of the single-nucleon angular momentum vector $\boldsymbol{j}=\boldsymbol{l}+\boldsymbol{s}$ a constant of the motion and thus account for the approximate physical existence of the (jj)-coupling scheme in heavy nuclei. This approach is not sufficiently refined to account for the quadrupole moment of the deuteron.

Still within the framework of the first or phenomenological step, there is another approach involving the assumption of a tensor interaction, in which $V(r_{ij})$ is multiplied by an operator

$$D_{ij}=3(\boldsymbol{\sigma}_i\cdot\boldsymbol{r})(\boldsymbol{\sigma}_j\cdot\boldsymbol{r})-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j, \tag{4}$$

a term comparable in magnitude with the central interaction. In the treatment of the deuteron by Rarita and Schwinger (R41), which gives the quadrupole moment so nicely, neither term is given an exchange character and the lack of the saturation property has no dire consequences for this light nucleus or its immediate neighbors. The excitation energy, about 2.2 Mev, of the ^1S state of the deuteron is accounted for by the fact that (4) mixes some ^3D into the ^3S to make the ground state, but of course not into the ^1S. This separation, which is one of the two primary functions of the tensor interaction in this theory, is accomplished with the cruder interaction assumption, Eq. (2), as a result of the term in Q. If the tensor interaction were calculated for many cases in heavy nuclei in which Q provides energy splittings between certain states, the tensor interaction would presumably give somewhat similar splittings. In this sense we may think of the term in Q as an oversimplified sketch which preserves the main feature, a sort of caricature, of one aspect of the tensor force (which in turn is a caricature of the dim machinations of the mesons reserved for the second step). It is currently suspected that there may be a very short-range repulsive core to the nucleon interactions (J50), which could perhaps rescue the tensor-type interaction from its nonsaturating difficulties. If so, the saturation mechanism of the term of Eq. (2) in P could be looked on as a caricature, and it is to be hoped a useful caricature, of this aspect of the tensor interaction.

Spin-Orbit Interaction

There is still another aspect in which the phenomenological procedures implied by Eqs. (2) and (3) may be considered to have a significant parallelism to those implied by the use of the tensor interaction, Eq. (4), and this concerns the spin-orbit interaction. The tensor interaction (4) does contain the possibility of coupling spins $\boldsymbol{\sigma}_i$ to orbits which involve $\boldsymbol{r}_{ij}$, but in simple cases it does so only through the second-order and higher-order effects which involve excitation to other configurations. Feingold and Wigner (F50) have carried out extremely interesting calculations of this effect for He^5, and for Li^7, and for the effect which is not exactly spin-orbit coupling but closely analogous to the excitation of the deuteron plus effects arising from excitation of the core in Li^6. For the spin-orbit coupling with one and with three nucleons outside of an alpha-core, they obtain preliminary results which are somewhat small but of the right order of magnitude to account for the observed data. Very many extremely highly excited states of the alpha-core contribute to these results. The authors of this work have a feeling that in heavy nuclei analogous calculations will have the simplifying feature, as a result of statistical cancellations, that each nucleon outside a closed shell will because of its interaction with the nucleons of the closed shell have a spin-orbit coupling similar to that in He^5 and be very little affected by the coupling of other extra-shellular nucleons to the closed shell. These second-order contributions are thought to arise independently for the several nucleons in a manner somewhat analogous to the way in which the van der Waals forces, also of second order, are shown by London to add linearly for various pairs of molecules. Thus Feingold and Wigner feel that for heavy nuclei, Eq. (3) need not be an independent *ad hoc* hypothesis, but may be a higher-order consequence of Eq. (4).

It is desirable in exploratory work to use the simple assumptions implied by Eqs. (2) and (3) because of their easy tractability. We see that this may tentatively be justified by thinking of the term in Q in Eq. (2) as a caricature mainly of the first-order effects of the tensor interaction (4) (which has a greater claim to credibility because it accounts also for the quadruple moment of the deuteron), and by thinking of Eq. (3) as a manifestation of the second and higher order effects of the tensor interaction. The suspicion of the possible equivalence of Eq. (3) and the second-order aspects of Eq. (4) in heavy nuclei is analytically at such an extremely preliminary stage that it is not really clear that it exists, and least of all is it clear whether or how far it can be extended down from the heavier nuclei toward or into the p shell. For this reason we would be on extremely hazardous grounds in using Eq. (3) in the p shell if we felt dependent on this possible explanation for the adequacy of Eq. (3).

The most compelling reason for using Eq. (3) in the p shell is, of course, the empirical one: it is simple and has been successfully applied in heavy nuclei and also seems to correspond to the nuclear angular momentum values J for the ground states of the light nuclei. One wants to know how far down into the light nuclei it may be of use in accounting for all the available data. But there is also another possible source of spin-orbit coupling which may help to justify the exploratory use of Eq. (3), a direct manifestation of meson coupling to nucleon spin-orbit coupling without going through the intermediary of phenomenological nuclear forces. Both the Thomas-precession as a source of nuclear spin-orbit coupling (I36, B37), and the Feingold-Wigner higher order tensor-interaction effect (F50), have in common

the assumption that the coupling of the nucleons to the meson field contributes certain specific and intrinsic properties to the nucleons (their phenomenological interaction energies, their intrinsic magnetic moments) which may subsequently be introduced into the Hamiltonian for the quantum-mechanical calculation of spin-orbit coupling energies. In the case of the Thomas-precession, it is the acceleration of the nucleon as it circulates in its nebulous "orbit," a classical acceleration $(d^2\mathbf{r}/dt^2)=-\nabla V/M$ ascribed to the specific nucleon interactions such as Eq. (2) and so only indirectly to meson theory, that provides the spin-orbit coupling:

$$H'(r)=\hbar\mathbf{s}\cdot\boldsymbol{\omega}_T=\hbar\mathbf{s}\cdot(d\mathbf{r}/dt)\times(d^2\mathbf{r}/dt^2)/2c^2 =(\hbar^2/2M^2c^2)(r^{-1}dV/dr)\mathbf{l}\cdot\mathbf{s}. \quad (5)$$

Thus we have a coupling of the form, Eq. (3), with $(H'(r))_{\text{Av}}=H'$ and with the parameter

$$a=(\hbar^2/2M^2c^2)(r^{-1}dV/dr)_{\text{Av}} \quad (6)$$

proportional to $1/M^2$, M being of course the nucleon mass. One factor M arises from $(d^2\mathbf{r}/dt^2)$, that is, from an assumption concerning the interaction, and one from the expression for orbital angular momentum $M|\mathbf{r}\times(d\mathbf{r}/dt)|=l\hbar$. Here V is an effective (approximately central) potential energy $V(r_i)$ of one nucleon (i) in the field of all the others, an effective average of $V(r_{ij})$ summed over j, with exchange effects, Eq. (2), included in the averaging. The energy of the Thomas precession, Eq. (5), is now known to be inadequate in magnitude to account for more than a small fraction of the "observed" nuclear spin-orbit coupling: in atoms it is half as big as the magnetic spin-orbit coupling, but in light nuclei almost an order of magnitude larger than the magnetic coupling, the magnetic single-nucleon doublet splitting being of the order of 30 kev (I51), the Thomas-precession doublet splitting of the order of 100 kev, whereas the splittings needed to account for the empirical trends appear to be 1 Mev or considerably more, as we shall see in detail.

While Eq. (5) is thus inadequate in *magnitude*, its *form* still has considerable interest because this arises from requirements of relativistic and rotational invariance (F26, T26, D36, F36, B37) which are common to a larger class of assumptions than the particular phenomenological ones used in deriving Eq. (5), and in particular apply to the results of some meson theories. Thus a meson theory may give in the energy of the system a term of the form, Eq. (5), but with a different meaning for the symbols M and V. Unfortunately, meson theories are not yet far enough advanced in their application to all the appropriate phenomena that one can yet expect to find any agreement on the probable form of the final result. However, the hope that they will contribute mainly two-body interactions, and then only contribute to the spin-orbit coupling indirectly through the two-step arrangement we have discussed, seems to be an optimistic hope. The success of shell models as far as it goes does suggest that many-body interactions may, for a given three nucleons, say, be smaller than two-body forces for a given pair, to such an extent that the two-body forces predominate in determining the energy effects of orienting the vectors associated with the comparatively few nucleons in an unfilled shell. But when one counts among a large number of nucleons, there are enormously more many-body interactions than pair interactions. Thus it may be that the many-body interactions make a more important contribution to the total energy of the nucleus than to the energy separation of states within a configuration, which are determined by interactions within the unfilled shell.

These many-body interactions may perhaps be more properly described in terms of the coupling of the various nucleons to a common pool of mesons. In either description, they may be expected to be very important in determining the binding energy of one nucleon in an unfilled shell to the rest of the nucleus, the energy that determines the existence of the "shell." In the course of this complicated process, there would seem to be ample opportunity for the coupling between the single nucleon and the common meson field to give rise to a term of the properly invariant form, Eq. (5), directly, and not just through a two-step process involving specific nucleon properties such as M. The fact that M came in once through the nucleon angular momentum justifies its remaining as one factor in Eq. (5). The second factor M entered through the two-step treatment of the energy, and in a direct treatment is probably to be replaced by the π-meson mass, $\mu=278m\approx M/7$. Such a replacement of one factor M by μ has been suggested in early remarks of Teller and elsewhere (I49), then in connection with the particular assumption that the mesons concerned are pseudovector mesons (now out of favor on the basis of other phenomena) by Gaus (G49), and in a number of lectures by Heisenberg, who favored more specifically replacing M by $\mu/2$. With this replacement of M by μ, the question arises what should be done with V, and it has been argued in connection with the brief treatment by Gaus that V, although primarily the potential of the mesons in the field of the nucleon, or vice versa, as essentially the same thing as the potential of the nucleon in the nuclear potential well. While this point remains rather vague because it depends so sensitively on assumptions concerning the unformulated saturation properties of the meson field, it suggests that the main effect of a thorough meson theory may be to replace one factor M in Eq. (5) by μ or $\mu/2$, thus bringing the spin-orbit coupling parameter a in Eq. (3) up from the order of magnitude 100 kev to the order of 1 Mev or more. Such an effect could very well be additive both to the Thomas-precession coupling and to the second-order effect of tensor interactions, for which the Feingold-Wigner estimates have so far been somewhat smaller than needed, and might overwhelm them both. This possibility justifies proceeding in an

exploratory way on the basis of the attractively simple assumption, Eq. (3), with the parameter a quite large if need be.

To estimate a plausible dependence of the parameter a on atomic weight A one may assume that $\partial V/\partial r$ exists only on the surface of the nucleus, so that it may be replaced by $\Delta V/\Delta r$ in a surface layer Δr thick, or in a fraction $2\pi r^2\Delta r/(4\pi r^3/3)=(3/2)\Delta r/r$ of the nuclear volume. If this fraction be taken as the probability that the nucleon is in this layer, we have $(r^{-1}\partial V/\partial r)_{Av}=3(\Delta V/2r_0^2)A^{-2/3}$, where the nuclear radius is taken to be $r_0A^{1/3}$. For purposes of orientation concerning the magnitude of the spin-orbit coupling constant a in Eq. (3), we may replace one factor M by $\mu/2$, put $r_0=e^2/2mc^2$, and evaluate (6) thus:

$$a\approx(\hbar^2/M\mu c^2)(6m^2c^4\Delta V/e^4)A^{-2/3}$$
$$=(6\times137^2/1840\times278)\Delta VA^{-2/3}=0.22\Delta VA^{-2/3}. \quad (7)$$

For a nucleon entering a nucleus, ΔV is larger than the nucleon binding energy by a factor of about two or more because the kinetic energy is involved in the comparison. If we rather arbitrarily take $\Delta V=20$ Mev, we have $a=1.2$ Mev for Li^7, 0.125 Mev for Pb^{208}. The single-nucleon doublet splitting is $(l+\frac{1}{2})a$, which is 1.8 Mev for a ^2p splitting in Li^7, 0.75 Mev for a ^2i splitting in Pb^{208}. The latter is about a factor 3 smaller than needed to account for the energy jumps at the magic numbers in terms of the (jj)-coupling shell model.

There is another simple manner of taking the average (I36), which involves first the classical procedure of equating centripetal force to the mass times centripetal acceleration in a circular motion, and this yields an additional factor $l(l+1)r^{-2}$, leaving the doublet splitting approximately proportional to $l^3A^{-4/3}$ rather than to $lA^{-2/3}$, as would correspond roughly to the refinement that the probability distributions of orbits of higher l crowd toward the outer edge of the nucleus. This is mentioned only to emphasize the roughness of the above estimate, with its assumed uniform probability distribution among many other simplifications.

3. ENERGY CONTRIBUTIONS OF THE SPECIFIC NUCLEAR REACTIONS

Presuming, then, that some basis may eventually be found in meson theory for the validity of this procedure, we shall try to analyze the spacings of the energy states of light nuclei on the basis of the specific nuclear interactions with their exchange nature given by Eq. (2) and on the basis of an *ad hoc* spin-orbit coupling given by Eq. (3). The contribution of the spin-orbit coupling to the energies of nuclear states is in simple cases quite direct because the nucleons are involved singly: when the spin of a nucleon is "parallel" to its orbital angular momentum, the energy is thereby low (this being the sign empirically assumed, opposite to that for electrons in an atom). The specific nuclear interactions, Eq. (2), involve pairs of nucleons and for this reason we must invoke the distinction between direct integrals and exchange integrals, just as in the case of evaluating the effects of the Coulomb interaction between the electrons in an atom. Both types of integrals are altered by the exchange operators contained in them, but an important part of the separation between states is provided as in atoms by the fact that the sign with which the exchange integral enters depends on the symmetry of the wave function on interchange of the space coordinates of the particles. This effect puts triplets below singlets in an atom, an effect which may alternatively be described in the following well-known graphic way. The wave function of a two-electron system in a triplet state, being symmetric in spin, changes sign on interchange of the space coordinates of the two electrons, so is zero when they coincide and small if the two positions described by the space coordinates are near one another. Thus the probability of the electrons being close together is relatively small in the triplet state, which makes the average of the positive (repulsive) Coulomb interaction e^2/r_{12} relatively small, and the triplet lower than the singlet. In nuclei the origin of multiplet splittings is closely related to this: it would be the same with ordinary (Wigner) interactions except for a reversal in sign because we are dealing with attractive interactions (negative energy) between the nucleons, making singlets lie below triplets. This agrees with the observed tendency for the nuclear angular momenta J to be zero for the ground states of even-even nuclei, or for an even number of like nucleons in a shell. Because of the saturation property, nuclear forces are usually treated as exchange interactions, perhaps primarily space-exchange (Majorana) interactions as in Eq. (2). With a space-exchange interaction, the above explanation of a singlet-triplet separation is complicated by the fact that the quantity we are averaging is no longer just a function V but an operator VP which works on one pair of single-nucleon wave functions. The probability density is made up of the product of a wave function multiplied by itself, once before and once after the operator P. If the wave function is symmetric in interchange of the two particles, in the singlet, the operator P does not change anything and the attractive potential still gives a low energy (of large absolute magnitude). In the triplet, the wave function is antisymmetric and P has the effect of changing the sign of the average energy making the triplet energy small and positive rather than small and negative. Thus singlets continue to lie below triplets, but the separation is larger with a space-exchange interaction VP than with an ordinary interaction V. The same effect may be described more analytically in terms of the direct integral

$$L=\int\psi(r_1)\phi(r_2)V(r_{12})\psi(r_1)\phi(r_2)dv_1dv_2, \quad (8a)$$

and the exchange integral

$$K=\int \psi(r_1)\phi(r_2)V(r_{12})\phi(r_1)\psi(r_2)dv_1dv_2. \quad (8b)$$

Here ψ may in the simplest example be taken as a $1s$ wave function and ϕ a $2s$. With ordinary interactions the singlet and triplet energies are given by $L\pm K$, respectively, with L larger in magnitude than K, as is discussed further in Sec. 4. With anex change interaction the roles of L and K are interchanged, as may be seen from their definitions (8), making the energies $K\pm L$. The singlet-triplet separation is thus larger with the exchange operator than without it.

With several nucleons in the p shell, similar effects occur but, of course, with much more freedom of orientation of vectors, orbital as well as spin. In this case, only the p wave functions of the same shell are involved. We denote them by u_{m_l}, corresponding to the three projections $m_l=1, 0, -1$. In the definition, Eq. (8a), of L, both ψ and ϕ are replaced by u_0, with $m_l=0$. In K, one is u_0 and the other u_1 so as to give some meaning to the exchange operation. In terms of L and K thus defined, the energy separations of the various multiplets in the Russell-Saunders or (LS)-coupling scheme are given by Feenberg and Phillips (F37a) and the separations within the lowest (jj) configurations in the (jj)-coupling scheme by Kurath (K52). The physical basis for the separations in these cases is very much the same as in the simple case discussed above, though the computations are, of course, more involved.

In the approximation in which the range of $V(r_{12})$ is large compared to the nuclear radius, so that V may be considered constant throughout the region where the wave functions exist, K is zero because it reduces to this constant times the square of the integral expressing the orthogonality of ψ and ϕ. In the opposite extreme in which the range of $V(r_{12})$ is short compared to the size of the nucleus, known as the δ-function approximation, one obtains $K=L/3$. How much smaller K may be than L, within this range, depends on the details of the radial dependence of the wave functions and of the interaction. An evaluation based on oscillator wave functions and Gaussian radial dependence of V, with a reasonable estimate of their range parameters, yields a result (H51) near

$$L=6K, \quad (8c)$$

and we shall take this as the basis for numerical evaluations of the ratios of level spacings in the empirical comparisons made below.

Since the depth of the potential $V(r_{12})$ is known to be of the order of magnitude 20 Mev from general stability and scattering considerations, and L is reduced below this only by the failure of the two distributions to come within the range of V of one another, thus by perhaps a factor $\frac{1}{2}$ or $\frac{1}{3}$, it may be estimated that L is of the order of 6 to 10 Mev, and thus K of 1 Mev.

4. ISOBARIC SPIN

In the tabulations of multiplet spacings of Feenberg and Wigner and elsewhere (F37, F37a, K52) one finds the states listed according to the usual quantum numbers L and S of the (LS)-coupling scheme and also according to the isobaric spin quantum number T (or isotopic spin as it has been less aptly called since it was named long ago by Wigner, who agrees to this renaming of his child). This dynamical variable T, which was introduced by Wigner (Wi37) as a quantity which should be a constant of the motion or "good quantum number" with a charge-independent Hamiltonian, is attaining an increasing importance in experimental as well as theoretical physics with the discovery that it appears to help control reaction rates and provide valid selection rules in certain meson reactions as well as nuclear reactions. The usefulness of isobaric spin may be explained elegantly by saying that the Hamiltonian is invariant under rotations in isobaric-spin space (just as it is in ordinary space when there is no external field) if the nucleon isobaric spin vectors $\boldsymbol{\tau}_i$ appear only in the form of scalar products as in the first line of Eq. (1), and this invariance makes the total isobaric spin T a good quantum number (just as is S when $\boldsymbol{\sigma}_i$ appears no more than through $\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j$). A statement in this language is, however, apt to be too formal for those readers who want a more explicit and graphic description of how isobaric spin affects the energy of a nuclear state. A brief and completely nonmathematical statement of the usage of isobaric spin in these light nuclei is found in the introductory part of reference (Aj52).

Discussion of Ordinary Spin Preparatory to an Explanation of Isobaric Spin in a Simple Case

In preparation for a more complete explanation of the useful concept of isobaric spin in explicit analogy with the usual intrinsic spin of a system, we shall first formulate the dependence of the interaction energy on spin quantum number in an appropriate manner, more explicitly than was done in the more general remarks of Section 3. For the sake of clarifying the physical basis of the analogy, there are three aspects of the problem which we shall emphasize: (a) the formalism of single-particle "spin factors" of the wave function and the choice of the quantum numbers $\pm\frac{1}{2}$, (b) the Pauli principle in the case of "parallel spins," and (c) the Pauli principle in the more general case in which it selects four of the eight solutions of the two-body wave equation.

(a) A very essential feature of the discovery of electron "spin" by Goudsmit and Uhlenbeck was the recognition that there are in nature twice as many states as there are solutions of a one-electron equation of motion in three dimensions (wave equation in present terminology), and one ascribes to the electron another "degree of freedom" or coordinate or dynamical variable

(variously called $s_z=\frac{1}{2}\sigma_z$ or ω or σ) which can take on only two values rather than the continuous range of the three components of r. Just as the wave function $\psi(r)$ is in spherically symmetrical cases factorable into $R_{n,l}(r)Y_{l,m_l}(\theta,\phi)$ with factors specified by the three quantum numbers n, l, m_l, there is now a new factor of the wave function associated with the quantum number m_s, which can take on only two values, one for each of the Pauli spin wave functions α and β. In place of one wave function $\psi(r_i)$ we have two, $\psi(r_i)\alpha(s_{zi})$ and $\psi(r_i)\beta(s_{zi})$. The fact that there are two functions is important, and is the part which is most obviously convenient to carry over to the later discussion of the two charge states of a nucleon. Rosenfeld (R48) emphasizes the two-valuedness by calling the new coordinate a "dichotomic variable." Perhaps the simplest manifestation of electron spin is the observation that the energy given to an electron by a magnetic field H has two values, $\pm\mu_B H$, where μ_B is the Bohr magneton which in orbital motion is associated with the orbital angular momentum l of an electron in an atom, and now with the new coordinate is by analogy associated with "spin" angular momentum s. Because of this symmetry of the $\pm$ sign and because the proper values of a projection of an angular momentum jump by unit steps (in the unit $\hbar$), we come to say the two values of m_s and of s_z are $\pm\frac{1}{2}$. The "spin" factor of the wave function associated with $m_s=\frac{1}{2}$ we call $\alpha(s_z)$ and the one with $m_s=-\frac{1}{2}$ we call $\beta(s_z)$. Their dependence on the two-valued coordinate s_z is as follows: $\alpha(\frac{1}{2})=1$, $\alpha(-\frac{1}{2})=0$, $\beta(\frac{1}{2})=0$, $\beta(-\frac{1}{2})=1$, so that the wave function $\psi\alpha=\psi(r)\alpha(s_z)$ has a value with "spin up" ($s_z=\frac{1}{2}$), none with "spin down," etc., as is familiar in one notation or another in most books on quantum mechanics [for example, reference (C35), pages 55–56, and reference (P35)].

(b) Consider now the problem of two particles in a potential well, one with a $1s$ wave function ψ and the other with a $2s$ wave function ϕ. Before introducing the interaction $V(r_{12})$ between them, one has solutions of the wave equation composed of simple products of the one-particle wave functions, such as

$$\psi(\mathbf{r}_1)\alpha(s_{z1})\phi(\mathbf{r}_2)\beta(s_{z2})=\psi(1)\phi(2)\alpha(1)\beta(2)\equiv\psi\phi\alpha\beta. \quad (9a)$$

Here we define the convention that, in the product of a pair of similar functions such as ψ and ϕ, the first shall refer to particle 1 and the second to particle 2. The association of particle 1, rather than particle 2, with the function $\psi\alpha$ in that product is artificial and has no physical meaning. A wave function that avoids stating more than one knows about this is formed by taking a sum of such products which is symmetric or antisymmetric in the exchange of the indices 1 and 2. The simplest wave-mechanical form of the Pauli principle is that, for two identical particles, such as two neutrons with α (i.e., "spin up"), the wave function must be antisymmetric (not symmetric) in interchange of $\mathbf{r}_1$ and $\mathbf{r}_2$, so that it vanishes when $\psi=\phi$, and there is thus no wave function in which two identical particles have the same set of quantum numbers. With "both spins up," the function similar to Eq. (9a) is $\psi\phi\alpha\alpha$, but in a system in which electrons 1 and 2 interact it is not a satisfactory approximate solution of the wave equation, corresponding to the fact that it erroneously associates electron 1 with the state ψ and 2 with ϕ whereas they may exchange. The function $\phi\psi\alpha\alpha$, with the electrons exchanged, is another possibility with the same energy before $V(r_{12})$ is introduced, but only the sum and the difference of the two are (approximate) solutions of the wave equation with $V(r_{12})$, as we shall see in a more general case below. Of these two solutions, the difference

$$(\psi\phi-\phi\psi)\alpha\alpha$$

vanishes in the special case $\psi=\phi$, and this is the one selected by the Pauli principle in keeping with the simple statement that "no two identical particles shall have the same set of quantum numbers." This two-electron wave function changes sign when we change the order of the factors ψ and ϕ, which is equivalent to exchanging $\mathbf{r}_1$ with $\mathbf{r}_2$, so may be said to be antisymmetric in the exchange of the coordinates of the two electrons.

(c) With "one spin up and one down" there are four primitive products, corresponding to either the association of α or β, and either electron 1 or 2, with the function ψ (the other with ϕ). With "both spins up" we had two and with "both spins down" there are two more, or eight in all. Eight independent linear combinations may be made of them, and the combinations that are solutions of the wave equation with $V(r_{12})$ may be expected to separate into space and spin factors, since V does not contain the spin coordinates. The salient experimental fact demanding the Pauli principle, the example in this simple case of the general workings of the vector model, is that there are only four physical states corresponding to these eight solutions, three (the ^3S separable by a magnetic field) with one value of the energy and one with another (the ^1S). Correspondingly, four of the eight [including the one selected in (b) above] are selected by the more general form of the Pauli principle, which states that the functions shall be antisymmetric in exchange of the particle-labeling indices, 1 and 2, or antisymmetric in "all the coordinates, including spin coordinates s_{zi}." This leaves us with the following familiar functions:

	M_S	S	Spectroscopic notation	Interaction energy	
$\Psi_1=(\psi\phi-\phi\psi)\alpha\alpha$	1	1	^3S	$L-K$	(9b)
$\Psi_2=(\psi\phi-\phi\psi)(\alpha\beta+\beta\alpha)$	0				
$\Psi_3=(\psi\phi-\phi\psi)\beta\beta$	−1				
$\Psi_4=(\psi\phi+\phi\psi)(\alpha\beta-\beta\alpha)$	0	0	^1S	$L+K$.	

We see that they change sign when we interchange the space functions ψ with ϕ and the spin functions, such

as α with β, which is equivalent to interchanging 1 with 2. These combinations (as well as the four symmetric ones that have been rejected) avoid stating more than we want to claim to know about the interacting particles; they do not specify which electron is in which state and they do not specify (in the second and fourth lines) whether α is associated with ψ and β with ϕ, or vice versa. That these functions are the proper solutions of the wave equation with interaction and leave the energy matrix diagonal, we shall see in the following paragraph.

The labels $M_S=m_{s1}+m_{s2}$ are convenient, and the further label S, which may be associated with the length of a "total spin" vector having projections M_S, denotes whether the "spin factor" is symmetric ($S=1$) or antisymmetric ($S=0$), in which cases the more important space factors are antisymmetric or symmetric, respectively. Now we introduce an interaction energy $V(r_{12})$, which is negative for two neutrons or positive, e^2/r_{12}, for two electrons, and calculate the interaction energy

$$\frac{1}{2}\int(\psi\phi\pm\phi\psi)V(\psi\phi\pm\phi\psi)dv_1dv_2=L\pm K, \quad (10)$$

where the "direct integral" L and the "exchange integral" K are

$$L=\int\psi\phi V\psi\phi dv_1dv_2$$

$$=\int\rho_\psi(1)\rho_\phi(2)V(r_{12})dv_1dv_2, \quad \rho_\psi=\psi^2,$$

$$K=\int\psi\phi V\phi\psi dv_1dv_2$$

$$=\int\rho_0(1)\rho_0(2)V(r_{12})dv_1dv_2, \quad \rho_0(1)=\psi(1)\phi(1),$$

as in Eq. (8). The exchange integral, being a self-energy of an "overlap density" ρ_0, which may have positive and negative regions, is smaller than the direct integral. For a reasonable function $|V|$ having a maximum when the particles are close together, K has the sign of V, which is negative between nucleons, so the singlet state, ^1S, with $S=0$, lies below the triplet states, ^3S, with $S=1$. The singlet-triplet separation is $2K$. (With a space-exchange or "Majorana" interaction VP in place of V, the role of L and K would be interchanged, as has been already mentioned, and the singlet-triplet separation would be larger, $2L$.)

If $H_0\Psi_a=E_0\Psi_a$, as is true for any of the functions $\Psi_a(1, 2)$ given by (9b), then with the interaction V included in the Hamiltonian, $H=H_0(r_1, r_2)+V(r_{12})$, we still have $H\psi_a=(E_0+\epsilon_a)\psi_a$ if $V\psi_a=\epsilon_a\psi_a$ [with ϵ given by Eq. (10)], without any added term in ψ_b, that is, if $V_{ab}=\int\Psi_bV\Psi_ad\tau_1d\tau_2=0$ for $a\neq b$. Thus the vanishing of the nondiagonal matrix elements V_{ab} is essential for the Ψ's to be proper solutions of the wave equation (in the approximation in which we neglect other states remote in energy). That the V_{ab} vanish can be seen from the orthogonality of the spin factors alone in Eq. (9b), since V does not operate on the spins s_i. Between different values of M_S, this is a simple result of conservation of S_z. Between the two states with $M_S=0$, the vanishing of V_{24} expresses the conservation of S. In this case the spin orthogonality may be formulated

$$\sum_{s_{z1}=\frac{1}{2}}^{-\frac{1}{2}}\sum_{s_{z2}=\frac{1}{2}}^{-\frac{1}{2}}[\alpha(s_{z1})\beta(s_{z2})+\beta(s_{z1})\alpha(s_{z2})]$$

$$\times[\alpha(s_{z1})\beta(s_{z2})-\beta(s_{z1})\alpha(s_{z2})]=1-1=0.$$

Here we use the familiar generalization that with the spin factors in the wave function the integral over $d\tau_i$ means an integral over the space coordinates, dv_i, and a sum over the two values of the spin coordinate s_{zi}. The same vanishing of V_{24} may be demonstrated also by the space factor, since the first member of Eq. (10) vanishes if modified to have one + and one − sign. These are some of the elements of familiar atomic spectroscopy which apply also to a pair of neutrons.

Explanation of Isobaric Spin for Simplified Nucleons

Let us now imagine that we might have spinless nucleons, both protons and neutrons (or nucleons with "spin up" will do as well, with less violence to nature, and let us, in this case, simply omit the common spin factor $\alpha\alpha$). Protons and neutrons differ so little in all properties but charge that we make the very important assumption (H32) that we can treat them as two states of a single type of particle, the nucleon, as is done in the theory of beta-decay. We can put either a proton into ψ in the potential well, or a neutron, two single-nucleon states which we can distinguish from one another by application of an electric potential, which adds an energy e or 0 multiplied by the potential.

(a) This factor e or 0 does not have the symmetry of the factor $\pm\mu_B$ in the magnetic energy of an electron, but it does have the same two-valuedness. The two-valuedness leaves the naming of the two values arbitrary, and it is this symmetry in the case of the electron spin that makes it convenient to take the two values $\pm\frac{1}{2}$. This choice led incidentally to a formulation of the "$V(r_{12})$ problem" in which the magnetic energy with its special symmetry played no direct role, and the formulation had a convenient elegance involving simpler quantum numbers than might have been encountered if we had not been guided by the magnetic symmetry in our choice, namely that the quantum numbers $S=1$ and 0 are associated with the "triplet" and "singlet"

energies of the two-electron system (the numbers of degenerate states being easy to remember in terms of a "vector model" concept). Because this formulation is both elegant and familiar, it is convenient in the discussion of nucleon charge to exploit the arbitrariness by proceeding arbitrarily in close analogy to the discussion of spin. With any other choice of numbers such as 1 and 0 to denote the two arbitrary values of the charge, it would be more difficult to remember what we had done, and would lead to the same physical results. Thus the closeness of the analogy of isobaric spin, as we shall formulate it, to intrinsic spin is somewhat artificial; it is a result of an easy, but arbitrary, choice. Isobaric spin has a very insistent physical meaning, nevertheless, in terms of states differing in energy and characterized by some sort of proper values T of a new dynamical variable that is conserved under appropriate circumstances. It is the codification of the quantum numbers that is arbitrary.

Thus with two charge states of a nucleon distinguishable in an electric potential the situation is almost exactly similar to the previous one wherein we had states $\psi\alpha$ and $\psi\beta$ distinguishable in a magnetic field (H32). With a neutron in the space-state ψ we shall call the wave function $\psi\nu$, or with a proton, $\psi\pi$, and similarly for the next lowest state $\phi\nu$ or $\phi\pi$. Here $\nu=\nu(t_z)$ is a function of the two-valued charge coordinate which we (arbitrarily) give the values $\pm\frac{1}{2}$, ν being associated with the quantum number $m_t=\frac{1}{2}$, etc., as for spin above.

(b) For two neutrons without spin, the situation is the same as for two electrons or two neutrons with "spin up," as discussed in (b) above. The solution allowed by the simple form of the Pauli principle is $(\psi\phi-\phi\psi)\nu\nu$.

(c) If we have a proton and a neutron and try to find simple product solutions before introduction of $V(r_{12})$, and if we treat the proton and neutron as quite different particles, not states of the nucleon, or if we claim to know that nucleon 1 is a proton, etc., there are only two states, a proton in ψ and a neutron in ϕ, or vice versa. Not only from old ideas about beta-decay and about the saturation of nuclear forces, but also even more convincingly from newer observations of exchange scattering at high energies, there are insistent suggestions of the nucleon concept, and of the exchange of identity of protons and neutrons when they interact. The possible existence of nucleon number 1 as either a proton or neutron, and of number 2 then as neutron or proton, makes the number of states of the neutron-proton system twice as great, or four. Of these we can make four independent linear combinations, two symmetric and two antisymmetric in the exchange of 1 and 2, and each a solution of the wave equation including an interaction $V(r_{12})$ that does not depend on the charge of the nucleons (in the same way that we made linear combinations of the spin functions that we showed to be solutions when V did not depend on spin). Because the simple statement of the Pauli principle applies to two neutrons in (b) just above, and because the simple generalization of the Pauli principle from (b) to (c), in the case of electrons, leads to the observed number of states, we here *assume* that the similar generalization from (b) to (c) may be made in the case of nucleons. The Pauli principle then states that the wave function shall be antisymmetric in interchange again of the particle-labeling indices 1 and 2, that is, of "all coordinates, including the charge coordinate or isobaric spin coordinate." All of this is independent of the choice of indices discussed in (a) above. Here the charge coordinate need only represent the two-valuedness, with π and ν vanishing or not depending on which of its two arbitrary values it takes, but from now on we shall let them be the convenient values $t_z=\pm\frac{1}{2}$.

On the basis of this new assumption concerning the scope of the Pauli principle, we discard the two symmetric states of the neutron-proton system. The physical necessity for the corresponding step for two electrons with spin in atoms could be verified by counting the number of states in multiplets. Unfortunately, the groupings of states in light nuclei are more complex and their identifications less complete, as we shall see, so this type of verification that states like the two discarded symmetric states (c) are really missing can come only with the success of a rather extensive interpretation of the various energy levels, but some fairly clear examples of it can be found in the cases discussed below. Another type of verification for the assumption that isobaric spin is a coordinate entering in the statement of the Pauli principle is found in the experimental evidence for conservation of isobaric spin T in reactions.

On this basis we select the four antisymmetric states of the system of two spinless s nucleons:

	M_T	T	Interaction energy	
$(\psi\phi-\phi\psi)\nu\nu$	1	1	$L-K$	(11)
$(\psi\phi-\phi\psi)(\nu\pi+\pi\nu)$	0			
$(\psi\phi-\phi\psi)\pi\pi$	-1			
$(\psi\phi+\phi\psi)(\nu\pi-\pi\nu)$	0	0	$L+K$.	

Again for convenience we use the labels M_T and T, analogous to M_S and S, and may use a vector concept in remembering how many states M_T are associated with each value of the isobaric spin T, as we shall call it, of the system. M_T may then be called isobaric spin projection. $M_T=1$ means two ν's, or the dineutron in this simplified potential-well model, $M_T=0$, the deuteron, and $M_T=-1$ means He^2, so M_T denotes which isobar we have. The state $T=1$ has substates in the three isobars, all with the same energy $L-K$ if we have just one $V(r_{12})$, or more generally if we have a "charge-independent Hamiltonian," independent of the distinction π and ν. With these peculiar spinless nucleons in this model, the ground state of the "deuteron" would have both nucleons in the lowest state ψ, and then in

our first excited configuration with ψ and ϕ populated, there would in this isobar be two excited states, one with $T=0$ and one with $T=1$. The physically significant fact would be that this $T=1$ ($M_T=0$) state would coincide in energy with the ground state of the dineutron ($T=1$, $M_T=1$) and of He^2 ($T=1$, $M_T=-1$), in this model. The reason for this fact is that the three $T=1$ states each have a minus sign in the space part of the wave function $(\psi\phi-\phi\psi)$. In the isobar $M_T=0$, the state with $T=1$ has a wave function of such symmetry that the Pauli principle would permit the two nucleons to be the same (both neutrons, e.g.). This is, in the simplest terms, how the $T=1$ state differs from the $T=0$ state in this isobar, and this symmetry difference affects the energy.

So far, in this simple example, the only difference between spin and isobaric spin is that states differing in spin projection may be distinguished by a magnetic field and those differing in isobaric spin projection by an electric field. When we go to a case in which the single-particle wave functions depend on angle and introduce orbital angular momentum operators, the projection quantum numbers of these may also be distinguished by a magnetic field in a way with which the behavior of spin operators may be put in close analogy, and in a way to preserve the classical property of conservation of angular momentum if the spin operators are associated with an intrinsic angular momentum of the particle. Thus the spin operators become associated with the Euclidian space of the laboratory, whereas the isobaric spin operators do not. One sometimes makes this distinction by referring to an isobaric spin projection operator as T_ζ rather than T_z. For a single nucleon one would have t_ζ or τ_ζ, analogous to s_z or σ_z, and m_t analogous to m_s.

Spin and Isobaric Spin Combined in a Simple Example

Now to get back to real, healthy nucleons, let us see how spin and isobaric spin are handled at the same time in this same simple example, at the same time introducing a more nearly realistic nuclear interaction, Eq. (2). It is convenient to list the states of the two-nucleon system by use of that form of the Pauli principle which states that no two nucleons may have the same set of quantum numbers, as is done in Table I. (States with negative M_S or M_T are omitted since they introduce nothing new.) The first line has the wave function

$$(\psi\phi-\phi\psi)\alpha\alpha,\ \nu\nu \qquad (9')$$

which combines the properties† $S=1$, $T=1$, of the first lines of Eqs. (9b) and (11), an antisymmetric space function of two neutrons with "spin up." The second and third lines are combined in the familiar way into the two antisymmetric combinations (one antisymmetric in space and the other in spin)

$$(\psi\phi\pm\phi\psi)(\alpha\beta\mp\beta\alpha)\nu\nu \qquad (9'')$$

which correspond to the second and fourth lines of Eq. (9b) for two neutrons, with $T=M_T=1$, $M_S=0$, and $S=0$ or 1, respectively. The fifth and sixth lines similarly have

$$(\psi\phi\pm\phi\psi)\alpha\alpha(\nu\pi\mp\pi\nu) \qquad (11')$$

corresponding to the second and fourth lines of Eq. (11) for a neutron and a proton with "spin up," with $S=M_S=1$, $M_T=0$ (deuteron), and $T=0$ or 1, respectively. From Eq. (10) we see that for an ordinary interaction V we have various states with $S=1$, $T=1$, all with energy $L-K$, whereas with $(S, T)=(1, 0)$ or $(0, 1)$ the energy is $L+K$, the difference again arising from the symmetry of the space factor.

The degeneracy arising from the fact that $(S, T)=(1, 0)$ or $(0, 1)$ have the same energy with this simple interaction V is slightly confusing, so we list in Table I the energies derived from the assumed specific nuclear interaction, Eq. (2), specialized according to Eq. (8c). Through the spin-exchange operator, this differentiates between spin and isobaric spin, and thus lifts the degeneracy. These energies are calculated from the usual prescription $\int\Psi^*H\Psi dv$, taking care of the normalization, and with Ψ given by Eq. (11′) the interaction energy is

$$0.8\int(\psi\phi\pm\phi\psi)^*V(\phi\psi\pm\psi\phi)dv/2$$
$$+0.2\int(\psi\phi\pm\psi\phi)^2Vdv/2$$
$$=0.8(\pm L+K)+0.2(L\pm K); \qquad (12)$$

the + sign applying to $T=0$ and the − to $T=1$. In such a calculation one sees that the space-exchange

TABLE I. States allowed for two nonequivalent s nucleons.

ψ m_s	ψ m_t	ϕ m_s	ϕ m_t	M_S	M_T	S	T	Energy calculated with (2) and (8c)
+	+	+	+	1	1	1	1	$-3K$
+	+	−	+	0	1	1	1	$-3K$
−	+	+	+	0	1	0	1	$4.2K$
+	+	+	−	1	0	1	1	$-3K$
+	−	+	+	1	0	1	0	$7K$
+	+	−	−	0	0	1	1	$-3K$
−	+	+	−	0	0	0	1	$4.2K$
+	−	−	+	0	0	1	0	$7K$
−	−	+	+	0	0	0	0	$-5K$

† The demonstration that $S=1$ in this case, or that $(\mathbf{s}_1+\mathbf{s}_2)^2=S(S+1)=2$, is known from almost any textbook in quantum mechanics. It is simply that $(\mathbf{s}_1+\mathbf{s}_2)^2=\mathbf{s}_1^2+\mathbf{s}_2^2+2\mathbf{s}_1\cdot\mathbf{s}_2=\mathbf{s}_1^2+\mathbf{s}_2^2+2m_{s1}m_{s2}=\frac{3}{4}+\frac{3}{4}+\frac{1}{2}=2$. In the case $M_S=0$ of Eq. (10) the evaluation of $\mathbf{s}_1\cdot\mathbf{s}_2$ involves nondiagonal matrix elements, but the result is the same if the spin function is symmetric, or $S=0$ if antisymmetric. Exactly the same analysis applies to T in terms of the m_t's.

operator P operates to exchange ψ and φ in the second space factor only, and the summation (implied in the integral) over the other functions, being unaffected by any operators, comes in just as in the normalization and gives a factor unity. In the second term the spin-exchange operator Q in this case has no effect because of the symmetry of $\alpha\alpha$. Thus, we see that the two states differing in T have quite different energies. Let us see that one and only one of these energies corresponds to that of a state of another isobar, because of a similarity in space and spin symmetry.

The states of the other isobar, the dineutron, say, are obtained similarly by use of Eq. (9″):

$$0.8\int(\psi\varphi\pm\varphi\psi)V(\varphi\psi\pm\psi\varphi)dv/2$$

$$+0.2\int(\psi\varphi\pm\varphi\psi)^2Vdv(\alpha\beta\mp\beta\alpha)(\beta\alpha\mp\alpha\beta)/4$$

$$=0.8(\pm L+K)\mp0.2(L\pm K), \quad (13)$$

the upper sign applying to $S=0$, the lower to $S=1$, both with $T=M_T=1$. With the lower choice of signs the energies expressed by Eqs. (12) and (13) agree, but with the upper they do not, which means that the energy of the state $S=1$, $T=1$ calculated with the charge-independent interaction, Eq. (2) is the same for the isobaric spin projection $M_T=1$ or 0. That is, there is a state in the deuteron which should have the same energy as a state in the dineutron because the two states have the same dependence on space and spin coordinates.‡

Thus we may answer the question, "What is the physical property of a state with $T=1$ to distinguish it from a state with $T=0$ in the same nucleus?" by saying that it has a type of symmetry that would still be allowed by the Pauli principle even if one of the protons of the nucleus were a neutron. We have seen an example in which the calculation of the energy depends just on this type of symmetry. More generally, if the Hamiltonian is charge independent, that is, if it does not depend on the proportion of neutrons and protons among the nucleons involved, then the states of the same value of T in different isobars (that is, different values of M_T) have the same energy for much the same reason that makes the states of an atom with the same S but different M_S have the same energy when there is no external magnetic field and the Hamiltonian is thus orientation-independent. (In these cases, the operators S_z and T_z, of which M_S and M_T are quantum numbers, commute with the Hamiltonian.)

The main point of this explanation is that it shows in detail how the quantum number $T=1$ or 0 determines the symmetry that the Pauli principle will allow for the space and spin factors and through them has an influence on the energy of the system. That is why it is possible to write the physical operator in the first line of Eq. (1) in terms of the scalar product of the single-nucleon isobaric spin vectors $\boldsymbol{\tau}_i=2\mathbf{t}_i$, which have their physical effect only indirectly through the effects of symmetry on the evaluation of the space function (as indeed do also the spin vectors $\boldsymbol{\sigma}_i=2\mathbf{s}_i$ in this formulation or in the atomic singlet-triplet problem). A more elegant evaluation of an exchange interaction is possible in this form, since $\boldsymbol{\tau}\cdot\boldsymbol{\tau}$ or $\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}$ is 1 for T or S equal to 1, -3 for T or S equal to 0, but this form is no more useful because the convenient tabulations of references (F37) and (K52) are presented in terms of the exchange operators.

With many particles, the symmetry is not as clear-cut as in the two-particle case. A function is in general neither completely symmetric in the exchange of the space coordinates of all pairs of particles, nor completely antisymmetric, but perhaps symmetric in interchange of some pairs, antisymmetric in others. The energy is still dependent on the degree of symmetry and the isobaric spin formalism provides more help to distinguish between the various symmetries than in the two-particle case where it is easier to explain.

Conservation of Spin and Isobaric Spin in Reactions

One aspect of the conservation of isobaric spin is then seen in the existence of states of a symmetry denoted by $T=1$ in isobars $M_T=1, 0$, and perhaps -1 as demonstrated by the fact that their energy differences may be estimated by treating the charge dependence of the Hamiltonian as a first-order perturbation, as is discussed further, from an empirical point of view, in the following section. Another way in which a quantity such as an angular momentum may appear as conserved is by remaining with its initial value, or within a group of initial values through a scattering process, or through a reaction involving the intermediate formation of a compound nucleus. For example, when only one orbital angular momentum is involved in the incident beam and one in the emergent beam, conservation of parity is assumed, and of course conservation of total angular momentum. When spin-orbit coupling is so small as to be small compared to the natural width of a compound state, it would be expected that spin angular momentum S would be conserved, and this process could be analyzed by expressing the various contributions s_i in

‡ To make a similar comparison for the $S=0$, $T=1$ and 0 states it is necessary to complete the discussion of Table I by considering the four $M_S=0$, $M_T=0$ states, in the last four lines. Note that lines six and seven match lines two and three in the m_s's and match line four in the m_t's, and similarly for the last two lines (matching line 5 in m_t). Thus the last four lines are antisymmetrized by taking the space-spin factor of Eq. (9″) multiplied by the isobaric spin factor of Eq. (11′), with three independent $\pm$ signs. Successive choice of one of the $\pm$ signs as $-$ gives the first three states (S, T) listed, and the last, (0, 0) is given by the remaining antisymmetric possibility with all three negative. The energies for the four choices of signs, with interaction, Eq. (2) specialized as in (8c), are listed in the last column of Table I, three of them agreeing, of course, with those obtained earlier because they agree with choices of signs made earlier in the space and spin factors.

terms of the spin operators, and analyzing the rate of change $\dot{S}$ in terms of these operators and the spin factors of the wave function. This $\dot{S}$ would be found to be small and S to be conserved to the extent that the dependence of the Hamiltonian on the s_i is small. Our choice of the isobaric spin operators and factors in the wave function in close analogy with those for ordinary spin provides us with a convenient dictionary with which to translate this expectation to isobaric spin, and leads us to expect T to be conserved in certain reactions. We shall see examples where the corresponding isobaric spin selection rule (A52a) forbids a transition that is indeed observed to be very weak. The conservation of the formal vector T is thus explained by comparing the formal operations with those associated with the physically familiar vector S of which we have physical reason to expect conservation, but here, just as with the energies, the physical meaning of the combination of formal vectors to make up T is to be understood in terms of the combination of space-spin symmetries of the parts of the system which is implied by the formal vectors and is difficult to calculate more directly.

Charge Independence As a Special Case of Charge Symmetry

We have seen that the conservation of isobaric spin follows from the assumption that the Hamiltonian is charge-independent, that is, that the specific nuclear interaction (neglecting the Coulomb interaction) between two nucleons is independent of their nature as neutrons or protons, the same for the three cases $n-n$, $n-p$, and $p-p$. Some of the consequences of charge independence also follow from the less restrictive assumption of charge symmetry, that is that the $n-n$ and $p-p$ interactions are identical, but not necessarily the same as the $n-p$ interaction, so that the Hamiltonian may be said to be symmetric in the interchange of all protons with all neutrons. Charge independence is a special case of charge symmetry. The great similarity of the energy separation in Li^7 and Be^7, each of which has four nucleons of one charge and three of the other, follows from the more general assumption of charge symmetry. There are other less obvious cases, particularly among the nuclei with $A=6$, 10, and 14, where some of the consequences of charge independence may not be considered to verify this special assumption because they are also consequences of the more general hypothesis of charge symmetry (Kr52). Because evidence for charge independence as quite a good approximation is found, as we shall see, particularly in the comparison of energies of states of the same T but different $|M_T|$, we shall not attempt to pursue the distinction further in the present survey. The Coulomb interaction between protons, which is considered to be the principal deviation from charge independence, constitutes just as large a deviation from charge symmetry, so the distinction does not appear to be very fruitful.

5. SURVEY OF THE EXPERIMENTAL DATA FOR THE POLYADS OF THE p SHELL, WITH COULOMB ENERGY ELIMINATED

Polyads

To bring into evidence the way in which the isobaric spin appears to be a fairly good quantum number, and to prepare for an interpretation in terms of isobaric spin, it is useful to present the data, not as associated with individual nuclei, but with the nuclei grouped into isobaric polyads after applying the "Coulomb correction," as has been done in Fig. 1. Isobaric spin T is, as we have seen, a dynamical variable not of a single nucleus, but of an isobaric set of nuclei, all with the same mass number A, and such a set of isobars we call a polyad, denoted by the symbol Py^A. Thus, Py^6 means $\cdots He^6+Li^6+Be^6+\cdots$. (The name "triad" or "isobaric triad" has been used for such a set of three isobaric nuclei in a recent review article by Lauritsen (L52), but in principle there are not only the states with $T=1$ spread over the three isobars, but also still higher groups of states with $T=2$ and so on up to $T=A/2$ spread over all $A+1$ isobars, so the name "polyad" seems more appropriate.) A polyad may alternatively be defined as a mechanical system of A nucleons, of which isobaric spin is a dynamical variable.

An extremely valuable compilation and digest of the far-flung experimental data on energy levels of light nuclei has been made by Lauritsen and his collaborators (H50, Aj52), and the level diagrams which they have tabulated form the starting point for any analysis such as this. Figure 1 is intended to give a survey of these data in a form to facilitate comparison and convey general impressions at a glance, and for this reason the proton-rich isobars are omitted except where their data contribute significantly to the identification of excited states. In Py^{10}, for example, only Be^{10} and B^{10} are shown, and the relative heights of their ground states are determined by making a "Coulomb correction," including a correction for the neutron-hydrogen mass difference.

Elimination of the Coulomb Energy and the Charge Independence of the Remaining Interaction

The correction is made in a simple manner on the basis of the assumption that the change in Coulomb energy on changing from four to five protons is the same in the difference $B^{10}-Be^{10}$ as in the difference of the other isobaric pair of the same elements, B^9-Be^9. (This assumption neglects the small effects of symmetry differences, and approximates a more careful theoretical correction about as well as the two steep theoretical curves in Fig. 18 are approximated by straight lines.) If we write the total nuclide ground-state mass energy, Mc^2, as $E(Z,A)=E_o(Z,A)+E_c(Z,A)+(ZM_H+NM_N)c^2$, the energy of binding (kinetic included) provided by the specific nuclear interactions being E_o, and the Coulomb

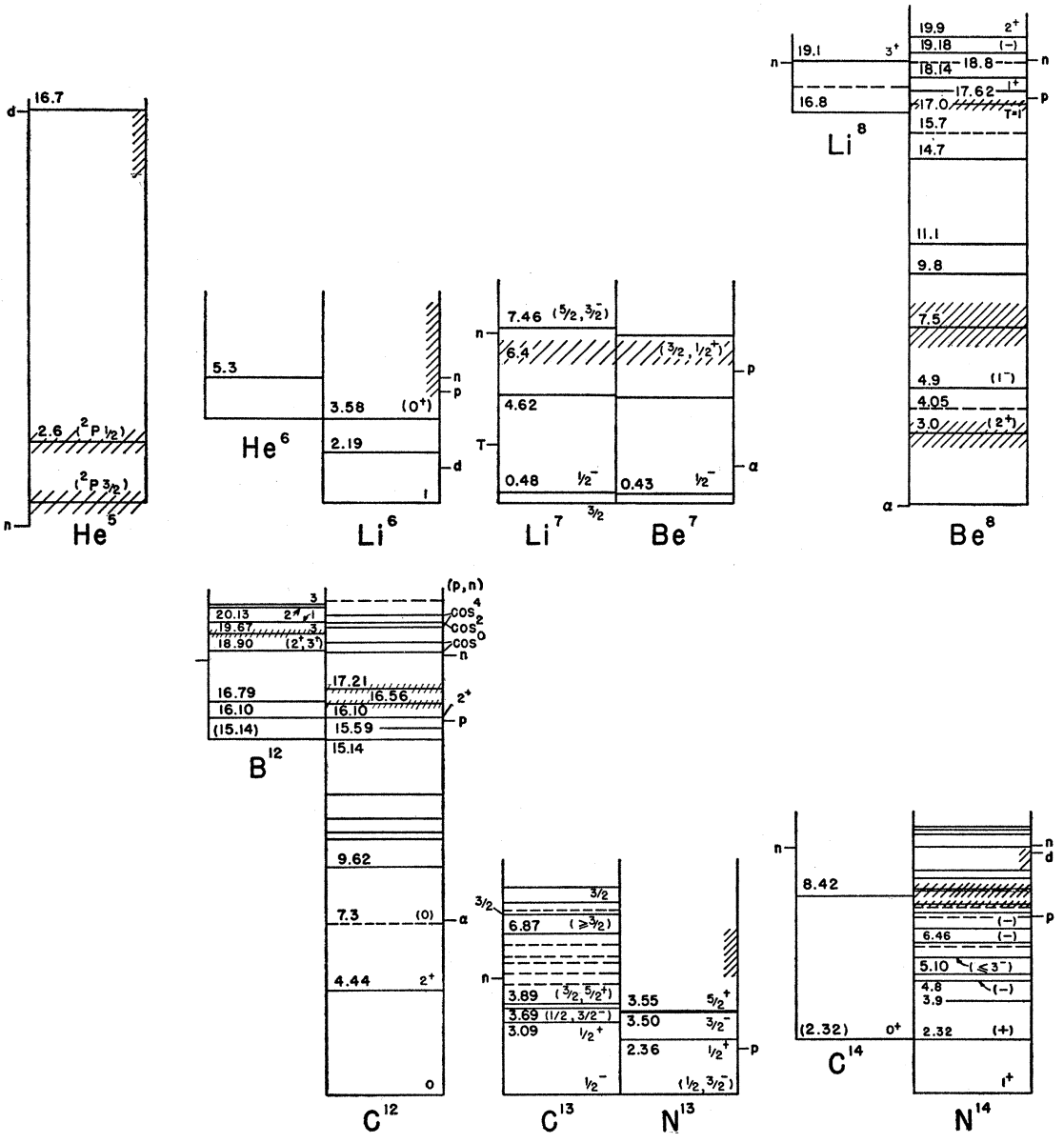


FIG. 1. Energy levels

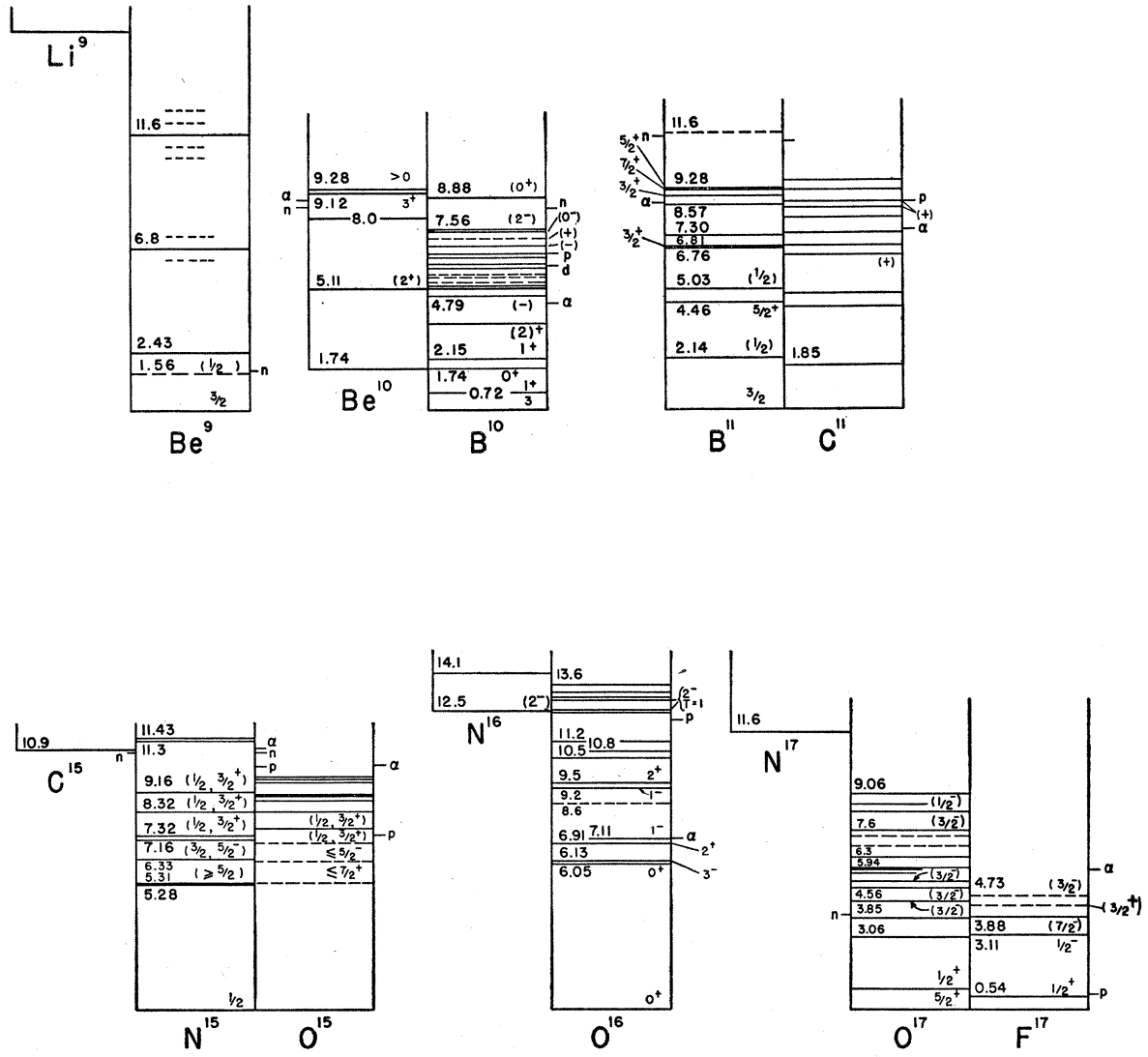
energy E_c, then the ground-state energy difference $Be^{10}-B^{10}$ is

$$\Delta_{10}=E(4,10)-E(5,10)=E_o(4,10)-E_o(5,10)+E_c(4,10)-E_c(5,10)+(M_n-M_H)c^2=\Delta_{o,10}+\Delta_{c,10}+\Delta_{n,H}. \quad (14)$$

A similar equation holds with the 10 replaced by 9, with the differences Δ similarly defined. We assume that the specific nuclear interactions are symmetric (that is, the same between two protons as between two neutrons), which may be stated as $\Delta_{o,9}=0$, since taking this difference implies interchanging neutron number $N=5$ or 4 with proton number $Z=4$ or 5, and we further assume as just mentioned that $\Delta_{c,10}=\Delta_{c,9}$. We then have the $Be^{10}-B^{10}$ difference in binding energies provided by the specific nuclear Hamiltonian:

$$\Delta_{o,10}=\Delta_{10}-\Delta_9=E_o(4,10)-E_o(5,10). \quad (15)$$

If now we further assume that the specific nuclear interactions are charge-independent (that is, the same between a neutron and proton as between two neutrons or two protons) so as to expect isobaric spin T to be a good quantum number, we have this as an estimate of the excitation energy of the first state with $T=1$ above the ground state of the polyad, with the Coulomb energy differences of the charge projections $M_T=1, 0$ eliminated. Thus from the observed differences in the mass-energy of the ground states of the two isobars the



of the p-shell polyads.

excitation energy of the state $T=1$, $M_T=1$ is estimated to be 1.625 Mev in a hypothetical idealized polyad having no Coulomb term in its Hamiltonian. There is an observed level in B^{10} at 1.74 Mev, and this is assumed to be the $T=1$, $M_T=0$ state. A measure of the exactness of the assumption of a charge-independent Hamiltonian is provided by the near equality of these two numbers, together with the plausibility of this identification, as dependent on the fact that the nearest other level is at 2.1 Mev and also in this case on the very spectacular failure of inelastic deuteron scattering (the deuteron has $T=0$) to excite the 1.74-Mev level (B53, A52a).

The data for the positron-emitting isobar C^{10} are in this case not so reliable (the positron energy end point being given to ± 100 kev) but indicate that the $M_T=-1$ component of the $T=1$ state lies at about 1.9 Mev, as indicated by the lines with δ in place of Δ in Table II. The regular progression from 1.625 to 1.74 to 1.9 Mev with added Coulomb energy may be a secondary effect of the Coulomb repulsion which, by expanding the nucleus, may slightly reduce the average of the specific nuclear interactions. A still more favorable indication of the charge independence of the interactions may then be obtained by comparing the average of 1.625 and 1.9 (or better 1.9_4) Mev, that is about 1.78 Mev, with 1.74 Mev, the average for $M_T=\pm 1$ with $M_T=0$.

A similar treatment of other polyads of the p shell is given in Table II. The data there used are taken from reference (L51) where possible, and otherwise from (L52) and (De52) and may in some cases involve an experimental uncertainty of the order of magnitude of

TABLE II. Differences in energies of binding, in Mev. Coulomb correction (including neutron-H, mass difference), and matching of states of isobaric spin $T=1$ in the even polyads.

A	4	6	8	10	12	14	16
Z	2	3	4	5	6	7	8
$\Delta_A=E(Z-1, A)-E(Z, A)$		3.55	15.985	0.556	13.370	0.155	(10)
$-\Delta_{A-1}=-E(Z-1, A-1)+E(Z, A-1)$	0.0185	(1.32)	0.863	1.069	1.98	2.221	2.705
$\Delta_A-\Delta_{A-1}$		(4.9)	16.848	1.625	15.35	2.38	(12.7)
$\delta_A=E(Z+1, A)-E(Z, A)$		(4.9)	(18.0)	(3.9)	17.6	5.1	15.5
$\delta_{A+1}=E(Z+1, A+1)-E(Z, A+1)$	(1.32)	0.863	1.069	1.98	2.221	2.705	2.749
$\delta_A-\delta_{A+1}$		(4.0)	(17)	1.9	15.4	2.4	12.8
Next higher level		—		2.15	15.59	3.7	12.95
Matched level in Z^A		3.58		1.74	15.14	2.32	12.51
Next lower level		2.17		0.72	(11.9)	0.0	12.44

100 kev. They are mainly beta-decay differences. It is seen that the only very good evidence of this sort for the charge independence of the Hamiltonian is found in Py^{10} and Py^{14}. In Py^{6} the fit is not very good (partly because of special uncertainties concerning the virtual states of Py^{5} with which it is compared) and in the others the first state with $T=1$ is quite high, in a region where the density of states with $M_T=0$ is so great that an approximate match is not very significant. It should be remarked that a fit within about 100 kev indicates a very high degree of charge independence (to the order of 1 percent) since the interaction between the pair of nucleons which changes, e.g., from neutron-neutron to neutron-proton contributes something like -10 Mev to the potential energy of the system. The details of the Coulomb energy as it varies through the p shell are discussed further in Sec. 8.

Figure 1 is constructed for the purpose of convenience of interpretation, not for the purpose of displaying the comparisons which test the exactness of the conserva-

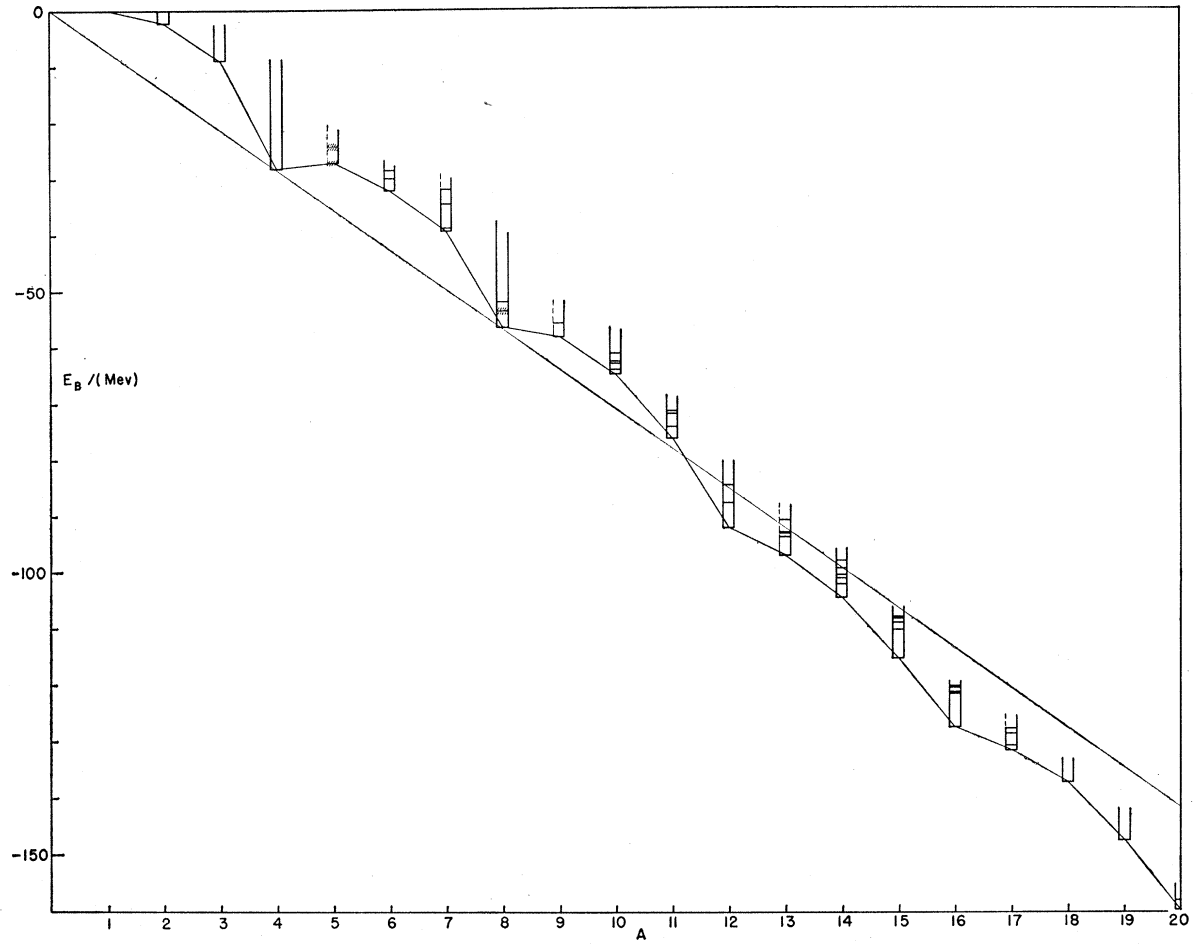


FIG. 2. Energies of binding of the states of the polyads.

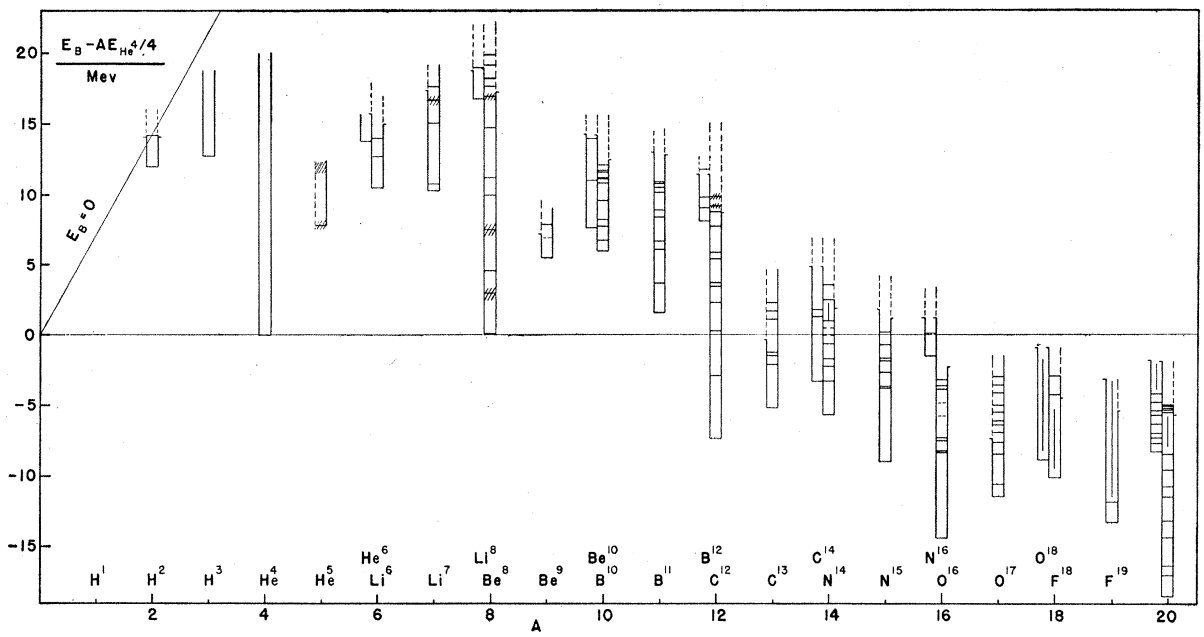


FIG. 3. Energies of binding relative to the binding per nucleon in an alpha.

tion of isobaric spin. In constructing Fig. 1 exact equality has been assumed of the $M_T=1$ with the $M_T=0$ charge projection of the lowest $T=1$ state after identification on the basis of the approximate equality found in Table II. (The identifications were actually made first in reference (L52) on the basis of a slightly different estimate of the Coulomb energies.) This makes Fig. 1 convenient for use in discussions of the interpretation of the energy spacings and sequence of levels, and facilitates the attempt to identify the higher $T=1$, $M_T=0$ states on the basis of the assumption that the Coulomb difference is the same for all $T=1$ states. In a similar way in the odd polyads, the ground states of the two isobars, with $M_T=\frac{1}{2}$ and $-\frac{1}{2}$, are placed on the same level in those cases in which the proton-rich isobar is shown. Thus we may say that an attempt has been made to display the data in Fig. 1 in such a way as to make it apply as well as possible to hypothetical polyads having no Coulomb terms in their Hamiltonians, although the spacings of levels within each isobar are obtained experimentally from a real nucleus in which it is assumed that the Coulomb terms contribute approximately equally to all states.

Trends in the Data

In Fig. 1 we see for each nucleus a number of horizontal lines representing observed energy levels, the excitation energy in the polyad being indicated, in Mev, by a number near the left side for many of the levels. In some cases a symbol appears near the right side indicating an experimental angular momentum and/or parity assignment. In cases in which this assignment is considered uncertain, it appears in parentheses ( ). Uncertain energy levels are indicated with broken lines. Levels observed to be broad are cross-hatched. Most of the spectral regions which have not been experimentally explored for the existence of levels are indicated by a vertical strip of cross-hatching at the right side. Outside the "boxes" containing the energy spectra are short lines labeled p, n, or α, indicating the level above which the nucleus is unstable relative to emission of these particles.

The ground-state angular momenta $\frac{3}{2}$, $\frac{3}{2}$, 3, $\frac{3}{2}$, for Li^7, Be^9, B^{10}, and B^{11} are what one expects from the compounding of $j=\frac{3}{2}$ vectors, as expected in the (jj)-coupling shell model in this first part of the p shell, and the values $\frac{1}{2}$, 1, $\frac{1}{2}$ for C^{13}, N^{14}, and N^{15} likewise may arise from the compounding of the $j=\frac{1}{2}$ vectors in the last part of the p shell. These are also the angular momenta expected on the basis of (LS) coupling (F37, R37) with the exception of B^{11} for which the low ^2P should be inverted because it is past the middle of the p shell.

Among the even polyads, there is a striking alternation in the excitation of the first state with $T=1$: in the $A=4n$ polyads Py^8, Py^{12}, and Py^{16} these excitation energies have large values in the range 12.5 to 16.7 Mev, while in the $A=4n+2$ polyads Py^6, Py^{10}, and Py^{14} they are much smaller, in the range 1.7 to 3.6 Mev. This appears to be an aspect of the "four structure" which seems to be associated with the symmetries available in (LS) coupling as clarified long ago by Wigner, Feenberg, and Phillips (W37, F37, F37a) and most apparent in the familiar "four structure" of the stability curve shown in Fig. 2. In exciting the ground state of Be^8 to the ground state of Li^8, for example, one has to break up a "four-group" of two neutrons and two protons

TABLE III. Number of states in the atomic configuration p^2, (LS) coupling.

m_l	m_s	m_l	m_s	M_L	M_S	M_L	$M_S=1$	0
1	+	1	−	2	0	2		1
1	+	0	+	1	1	1	1	2
:	:	:	:	:	:	0	1	3

TABLE IV. Number of states of two p electrons, (jj) coupling.

j_1	j_2	m_{j1}	m_{j2}	M	J	(jj) configuration	E	J
3/2	3/2	3/2	1/2	2	2	$p_{\frac{3}{2}}^2$	a	2, 0
		3/2	−1/2	1	2	$p_{\frac{3}{2}}p_{\frac{1}{2}}$	$-a/2$	2, 1
		3/2	−3/2	0	2	$p_{\frac{1}{2}}^2$	$-2a$	0
		1/2	−1/2	0	0			
3/2	1/2	3/2	1/2	2	2			
:	:	:	:	:	:			

whose spins cancel and whose interactions saturate in a particularly stable way, and make instead a group of three neutrons and one proton which cannot attain the same high symmetry relative to interchange of nucleons. (In the alpha-model, one has to excite an alpha, but much the same sort of effect prevails strongly in the central model in (LS) coupling, only weakly in (jj) coupling.) The odd polyads have relatively high excitation energies of the first $T=\frac{3}{2}$ states, presumably for this same reason in some cases and for reasons of shell structure in others.

Figure 2 is plotted in such a way as to display not only the "four structure" in the curve showing the stability of the ground states, but also to show in the same perspective the relative heights and thus the energies of binding of the excited states of the various polyads. Thus the energy spectra in Fig. 2 are small-scale versions of the energy spectra displayed in Fig. 1, standing on the stability curve for the ground states. One sees that the same four-structure is not in evidence for the energies of binding of the low states of higher isobaric spin. The stability curve of Fig. 2 is inverted relative to that shown by Feenberg and Phillips (F37a, Fig. 1), because they plotted binding energy, which is a positive quantity, the absolute magnitude of the negative "energy of binding" plotted in Fig. 2.

The straight line in Fig. 2 is drawn through the origin and the ground state of He^4. It thus represents the number of nucleons multiplied by the energy per nucleon in an alpha. In Fig. 3 the differences between the energy levels and this line are shown on a magnified scale, so as to exaggerate the four-structure and display the energy spectra on a larger scale. To see on this figure that Be^9 is stable, for example, one draws through the ground state of Be^8 a line parallel to the line $E_B=0$.

The most abundantly known energy levels are, of course, those of the more stable isobars of the various polyads, those with $M_T=0$ or $\pm\frac{1}{2}$. Among these, there is a striking variation in the density of the low states. In Py^7 one finds a low "doublet" and then a gap about ten times as wide as the "doublet" before the second excited state at about 4.6 Mev. The excited states of Be^8 start off with a broad state at 3 Mev and bear no resemblance to any other spectrum. Be^9 has two or perhaps three states below 2.5 Mev, then no others until at least 5 Mev, above which only a couple of broad maxima have been observed with poor resolution, which might be broad states or groups of states. B^{10} contrasts with the other light nuclei in having a more dense spectrum of low states, starting with a group of four below 2.2 Mev, of which one has $T=1$. Such a high density of low states in this odd-odd nucleus is somehow associated with the lack of exceptional symmetry in any one of them, to pull it down from the rest, or in (jj) coupling with the freedom to orient two $j=\frac{3}{2}$ vectors (nucleon "holes") in several ways. The contrast is most striking between B^{10} and the nearby even-even nucleus C^{12}, which has only a few widely spaced excited states up to 15 Mev, the first at $4\frac{1}{2}$ Mev. In this $A=4n$ case the wide spacing does appear to arise from an exceptional symmetry of a few low states, which pulls them down below the rest and is associated with the dip in the stability curve of Fig. 2 at $A=12$. The spectrum looks much more like (LS) coupling than (jj) coupling. In (jj) coupling, one would expect the first excited states to occur in a group of four, two of them with $T=1$, corresponding to the excitation of one $p_{\frac{3}{2}}$ nucleon to a $p_{\frac{1}{2}}$ state. (There are four states because there are two relative orientations of these two vectors; and either a proton or a neutron may be excited, which freedom of choice we may count in another way by saying that T may be 0 or 1.) The low states of Py^{13}, which are fairly dense after about 3 Mev, appear on the other hand to agree very nicely with the expectations of the (jj) model (K51). N^{14} has its first excited state at 2.3 Mev, and this has $T=1$ as expected for the $J=0$ orientation of two $p_{\frac{1}{2}}$ nucleons in the (jj) model, the other orientation $J=1$ being the ground state. Beginning at about 4 Mev there are numerous other states. The next nuclei N^{15} and O^{16} are very remarkable for having no excited states at all below 5.3 Mev and 6 Mev, respectively, and for then starting off in each case with a remarkably closely spaced pair of states. The large gap before the first excited state is doubtless associated with the end of the p shell, but in the case of N^{15} also seems to indicate that spin-orbit coupling is large. Beyond the closing of the shell at $A=16$, there are again many low states in O^{17}.

6. INTERMEDIATE COUPLING

From this survey we see that there is no consistent conformity of the energy spectra to the expectation of either (LS) or (jj) coupling. In (LS) coupling the various multiplets are characterized by values of the quantum numbers L and S, and the spacing between them is measured by the exchange integral K, as we have seen. In a graphic but still legitimate sense, K may be said to be a measure of the stiffness with which the

vectors $\boldsymbol{s}_i$ are coupled together to make a definite vector sum $\mathbf{S}$, and similarly the $\mathbf{l}_i$ to make $\mathbf{L}$. The separation of the states of various values of $\mathbf{J}=\mathbf{L}+\mathbf{S}$ within the multiplet is provided by a spin-orbit coupling perturbation which may be expressed in the form

$$H'=\Sigma a\boldsymbol{l}\cdot\boldsymbol{s}=A\boldsymbol{L}\cdot\boldsymbol{S} \tag{16}$$

just as in Eqs. (3) and (5) so far as the individual nucleons are concerned, but now with a collective or average spin-orbit parameter A characteristic of the multiplet. For a given multiplet, A is a multiple of a, with a positive constant near the beginning of a shell, a negative constant (for "holes") near the end of a shell, as is well known in atomic spectra. The approximation, Eq. (16) is a result of first-order perturbation theory, valid as long as the splittings within the multiplet, caused by a, are small compared to the separation of multiplets, caused by K. Thus we expect (LS) coupling to prevail if $a\ll K$, and similarly (jj) coupling if $a\gg K$, since a then measures the stiffness with which each $\boldsymbol{s}_i$ is coupled to $\boldsymbol{l}_i$ to make $\boldsymbol{j}_i$. In the central model, in which individual-nucleon wave functions are assumed to have an approximate meaning as a starting point, there is a broad region between the (LS) and (jj) extremes known as *intermediate coupling*, in which it may, in general, be said that a is of the same order of magnitude as K.

Derivation of the Intermediate-Coupling Transition for a Simple Case

Since the concept of intermediate coupling is taken over directly from the theory of complex atomic spectra (C35), it may be well to introduce our discussion of it by giving a simple example in an atomic case. It is, of course, just one of a number of effects in which a set of secular equations contains a parameter expressing a competition between two types of perturbing energy, such as the Paschen-Back effect in which external and internal fields compete in trying to orient the angular momentum vectors.

Let us consider the intermediate-coupling transition of an atom having two electrons in the same p shell, outside of closed shells, the configuration p^2. In (LS) coupling one counts the states allowed by the Pauli exclusion principle by constructing a little table starting as in Table III, which when completed gives the numbers of states for each M_L, M_S shown at the right side of Table III, enough to account for the multiplets ^1D, ^3P, ^1S. In atoms, triplets lie below singlets as we have seen, and the separations are given (S29) by $^1S=5K$, $^1D=2K$, $^3P=0$. By calculating Σam_lm_s in line 2 one sees that the spin-orbit energy, Eq. (3), of the $^3P_2$ is $a/2$ [or $A=a/2$ in Eq. (16)] from which one has by the Landé interval rule or by the usual vector-coupling formulas (or by further lines of Table III) that for $^3P_1$ it is $-a/2$ and for $^3P_0$ it is $-a$. At the risk of being redundant, one may count the states also in (jj) coupling, as in Table IV. The spin-orbit energies E listed for the configurations are calculated from Eq. (3), being $a/2$ for $p_{\frac{3}{2}}$ and $-a$ for $p_{\frac{1}{2}}$. In both extremes one finds two states with $J=2$, two with $J=0$, and one with $J=1$, so the energies in intermediate coupling are solutions of two quadratic secular equations, and one straight line $E=-a/2$ for $J=1$. For $J=2$ the quadratic equation which has as asymptotic values the (LS) and (jj) coupling energies we have listed is

$$E^2-(2K+a/2)E+aK-a^2/2=0, \tag{17}$$

and for $J=0$ we likewise have

$$E^2-(5K-a)E-5aK-2a^2=0. \tag{18}$$

In Fig. 4 we have plotted E/K as a function of a/K to show the solutions of these equations (after dividing through by K^2). The asymptotes showing the linear variation near (LS) coupling are shown at the left side, where $a\ll K$, and (jj) coupling is approached at the right side, where $a\ll K$. There are four energy differences between the five levels, and only two parameters, so the system is overdetermined and a verification of the adequacy of the theory by comparison with experiment is possible. Three of the levels may be used to determine a and K, after which the expected position of the two remaining levels may be calculated. Such comparisons in atomic spectra work out fairly well, but there are often discrepancies of the order of 10 percent or more to be ascribed to second and higher-order perturbations involving states of excited configurations (known as "configuration interaction").

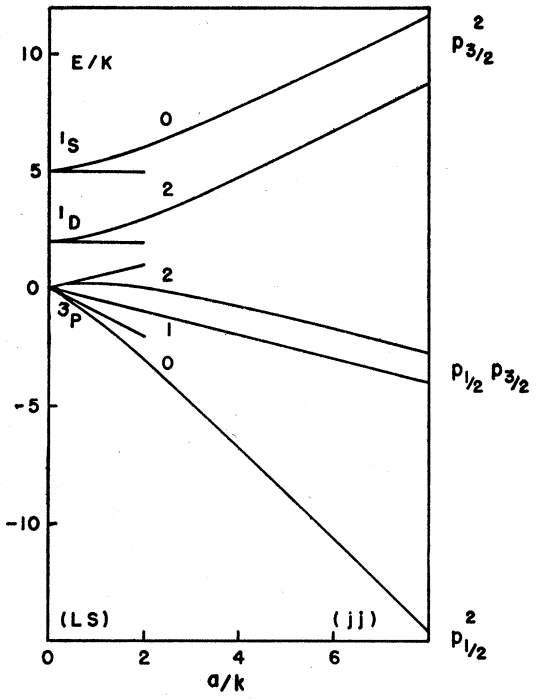


FIG. 4. The intermediate-coupling transition for the atomic configuration p^2.

Variation of the Parameters in Nuclei

It has recently been suggested that at least one important reason why the p-shell nuclei show no simple regularity of level spacing is the prevalence of intermediate coupling among such light nuclei (I52, H51). The fact that (jj) coupling seems to apply to the heavy nuclei does not necessarily imply that it should to the light nuclei, as the parameters a and K (or its more complicated equivalent) may vary, perhaps in a fairly regular manner, through the periodic table as suggested for a in Eq. (7). For K a similar variation may be estimated in the large-nucleus approximation by noting that K consists of summing the probability that an element of one "overlapping-density" function $\psi(r_1)\varphi(r_1)$ fall within the range of V of an element of another such function, which would make

$$K\sim R^{-3}\sim A^{-1}, \tag{19}$$

A here being atomic number again. Here we have again assumed that nuclear density is approximately constant from one nucleus to another, and this is shown to be valid even within the p shell, not only in the comparison of light with heavy nuclei, by the variation of Coulomb energies within the p shell, as discussed in Section 8. In the short-range approximation, Eq. (19), K thus varies more rapidly with atomic number than does the spin-orbit coupling parameter a in (7), which means that K might grow relative to a as one goes down from heavy nuclei, where $K\ll a$ as indicated by the success of (jj) coupling, to the lighter nuclei, and this would justify the existence of intermediate coupling in the latter. In the small-nucleus approximation (discussed so extensively by Wigner in his original paper on supermultiplets, Wi37) the wave-function volume elements are all within range of one another and K is independent of nuclear size (in fact, it is asymptotically zero), so that, in the actual case with range comparable with nuclear radius, K may vary among the light nuclei more slowly than indicated by Eq. (19). According to Eq. (7), one might expect the change ΔV in the average potential energy of a nucleon on coming within range of the other nucleons to affect its spin-orbit coupling, as well as the factor $A^{-\frac{2}{3}}$. ΔV does not vary as drastically as does the binding energy, so its variation may not have much effect, but of course the theoretical basis for Eq. (7) is sufficiently tenuous that no definite statement may be made about this. Since a variation of K a little less rapid than A^{-1} may be quite similar to that given to a by the factor $A^{-\frac{2}{3}}$, we shall for simplicity explore and discuss the validity of the assumption that the ratio a/K does not vary very much across the p shell.

7. INTERPRETATION OF THE STATES OF THE INDIVIDUAL POLYADS IN TERMS OF THE CENTRAL MODEL IN INTERMEDIATE COUPLING

In reference (I52) it was shown that the general spacing and sequence of the low levels of Py^{12}, which resembles (LS) coupling to the extent we have discussed above, and Py^{14}, which shows a closer resemblance to (jj) coupling, may both be understood in a rough sort of way as examples of intermediate coupling with approximately the same value of a/K, namely $a/K\approx 5$. The resemblance of one to one extreme and the other to the other extreme may be accounted for by refining the criterion for the validity of either extreme coupling. While (LS) coupling is valid when $K\gg a$, a less stringent criterion is this: the spacings of the first few states show some resemblance to (LS) coupling when the multiplet separations calculated in terms of K in (LS) coupling are large compared to a, for then A of Eq. (16) (or a in second order) does not have a chance to make states from the different multiplets come close to one another and thus disturb the general grouping. There could be some semblance to the general grouping into multiplets even if the spacing and order within a multiplet were disturbed, but in Py^{12} even this does not occur among the low states because the first three multiplets are singlets. The large spacing of these singlets is, as we have suggested, the result of the exceptional symmetry attainable in singlets in $A=4n$ nucleus, and this is what makes it look like (LS) coupling. In the polyad P^{14}, the order of states, with only slightly larger a/K, is the order given by (jj) coupling because in (LS) coupling K does not supply such large multiplet separations to compete so effectively with a.

Levels Arising from Excitation to a Higher Shell

One aspect of the energy spectra which was made clear by the examples just cited is the large extent of overlapping of the (jj) configurations (such as $p_{\frac{3}{2}}^7p_{\frac{1}{2}}$ with $p_{\frac{3}{2}}^6p_{\frac{1}{2}}^2$) on the one hand and of the (LS) multiplets on the other. Another aspect of the complexity is the overlapping of configurations, such as p^{10} with p^9d, for evidence of such inter-shell excitation was found in Py^{14} as low as about 4 Mev. This compares in order of magnitude with the evidence from the last two polyads of the p-shell, Py^{15} and Py^{16}, the high first excitation energy of which indicates, as we have seen above, that the inter-shell excitation there enters at about 5 or 6 Mev. It is apparent that the ground configuration spreads out over many Mev, and the configuration arising from inter-shell excitation is expected to do likewise, so we can expect no simple regularity in the excitation energy of the lowest state of the excited configuration, such as one might naively expect by postulating that the excitation energy of the lowest state obtained by exciting a p nucleon to the d shell is simply a definite amount of energy required to excite a p nucleon to the d shell, a single-nucleon or shell property that might carry over from one polyad to another in an oversimplified single-nucleon model.

A general survey of the data is enough to dissuade one from seeking any very simple regularities among the energy spectra of the light nuclei. What has been said of Py^{12} and Py^{14} suggests that at least the com-

plexity of intermediate coupling will have to be taken into account. Most polyads contain in their ground configurations so many p nucleons that it is not practicable to solve secular equations and draw exact curves giving the manner in which the energies of the various states should vary with the ratio a/K in intermediate coupling, in the basis of any definite interaction assumption. Our knowledge of the interactions is so meager that this is just as well. As was done (I52) for Py[12], it is perhaps sufficient to use a rather rough characterization of the specific nuclear interaction, Eq. (2), and spin-orbit coupling, Eq. (3), and then to treat the intermediate-coupling variation in a suitably rough manner by drawing smooth curves between the asymptotes which represent what is known in the relatively simple (LS) and (jj) extremes. We cannot of course hope to reproduce experimental data thus in any fine detail, but merely to understand general features, such as roughly the spacing of levels in different energy regions and perhaps their order, where identifications to distinguish the states from one another have been made. Future identifications will provide verifications or discrepancies, on the basis of which refinements in the interaction assumptions or general procedure may become justified.

We proceed now to discuss the individual polyads in that exploratory spirit. We discuss them in order of increasing mass as is customary for a review article, in spite of the fact that the first cases thus encountered are the most flagrant exceptions to the success of the intermediate coupling procedure, and will require a separate discussion later on.

He^5+Li^5

In this case both of the known low states, are not only virtual, being unstable with respect to single-nucleon emission, but are presumably very well described as single-particle states with all the excitation energy contained in a single nucleon, because of the exceptionally great stability of the alpha-particle which constitutes the s-shell core. We may say that one nucleon always has enough energy to escape, and under these circumstances we have no good approximation to a stationary state of the system, so that differences between different descriptions of what we mean by a "state" become especially severe. There seems to be general agreement that the low state is a $^2p_{\frac{3}{2}}$ and that the state several Mev above it may be interpreted, so far as it seems to exist, as a $^2p_{\frac{1}{2}}$, thus providing the fundamental instance of nuclear spin-orbit coupling, with the sign as anticipated (I36, B37) from the comparison with Li^7, a in Eq. (3) being negative. But the reaction and scattering methods of observation do not seem to agree very well at present on the doublet splitting: the broad peaks observed (Le51a, Ti51) in the reactions $T(T, n)He^5$ and $Li^6(\gamma, n)Li^5(p)He^4$ seem to indicate a doublet splitting of 2.5 or 2.6 Mev. Analysis (Hu52) of the $n+He^4$ scattering data shows the $P_{\frac{3}{2}}$ phase shift passing steeply up through 90° at 1.04 Mev (c.m.) and continuing to rise at higher energies, while the $P_{\frac{1}{2}}$ phase shift rises gently through 90° at 2.8 Mev, the ^2P splitting being thus only 1.76 Mev by this criterion. (See also A51, A52.) It may be that this $P_{\frac{1}{2}}$ shift in He^5 comes down again through 90° at a somewhat higher energy and that the broad reaction peak represents an average over this region of phase shift near 90°, or the discrepancy between the apparent 1.76 and 2.6 Mev doublet splittings may be an indication of the indefiniteness of the concept of a short-lived state in He^5 which would be even worse in Li^5. In Li^5 a $P_{\frac{1}{2}}$ phase shift is observed (Kre52, C49) of only about 35° at about 4.65 Mev contrasting with 115° at 3.4 Mev, which is about the equivalent energy, in He^5. It is enigmatic that the resonance peaks can be so similar (Le51, Ti51) in He^5 and Li^5 and the $P_{\frac{1}{2}}$ phase shifts apparently so different.

The methods of estimating a spin-orbit coupling parameter in terms of an assumption about ΔV, as in Eqs. (5) and (7), are not the same as in stable nuclei, and a special treatment is needed. There is, thus, no expectation of any semblance of continuity of a as it varies from He^5 to Li^6, even on the basis of this simple assumption.

With only one nucleon beyond the tightly closed s shell, intermediate coupling is, of course, meaningless. There seems to be no evidence of p-to-d shell excitation, even the states of the ground configuration being so broad as to be difficult to recognize. The single known state near 17 Mev probably arises from s-to-p excitation of the alpha-core, and should be one of several. Here the excitation energy is divided between nucleons, so the nucleus does not seem ready to come apart quite so quickly and the state is not so broad.

He^6+Li^6

Here the ground configuration, p^2, is simple enough and its states few enough that the secular equations can be written out explicitly and solved, with nothing more complicated than a cubic. The secular problem is only a slight extension of the atomic p^2 problem presented above in Eqs. (17) and (18). With two p nucleons rather than two p electrons we have the added coordinate of isobaric spin, which is handled very much as for the configuration $1s2s$ in Table I and Eqs. (9) above. Instead of the two-dimensional table at the right side of Table III, with M_L and M_S as its two dimensions, one has a three-dimensional assemblage of blocks, with M_T as the new dimension, consisting of the figure at the right side of Table III as one layer, for $M_T=1$, since the problem for two neutrons is the same as for two electrons, and another layer for $M_T=0$. With $M_T=0$ one obtains all integral vector sums that are possible with the vectors l_1, l_2, s_1 and s_2, not only the ^1S, ^3P, and ^1D which were obtained for $M_T=1$, and

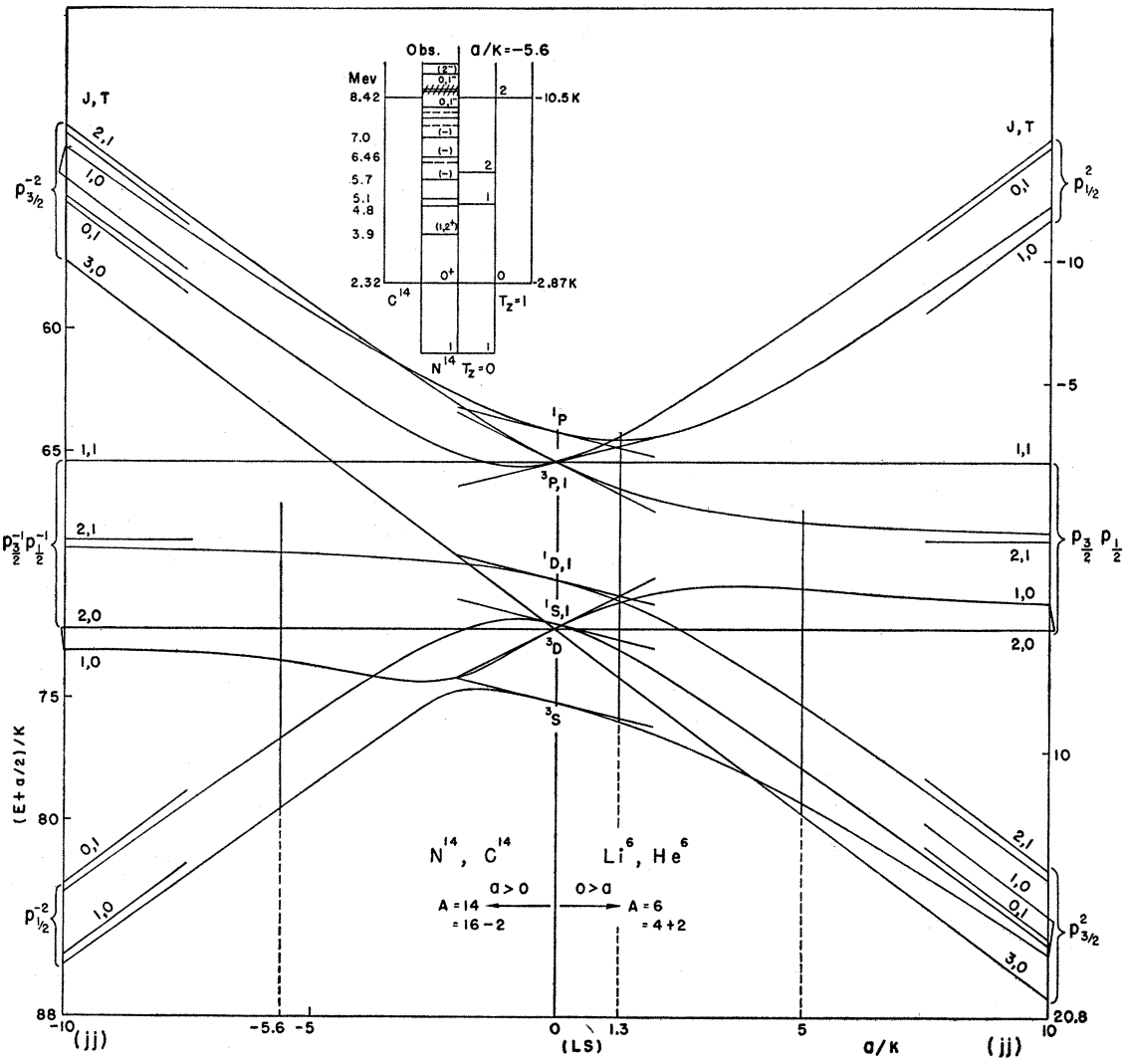


FIG. 5. Intermediate-coupling transition for the nuclear configuration p^2 and p^{-2} applying to the polyads Py^6 and Py^{14}.

thus have $T=1$, but also ^3S, ^1P, and ^3D which exist only for $M_T=0$, and thus have $T=0$. Among these latter multiplets we find three states with $J=1$, and thus have a cubic secular equation for their energies, besides two more linear equations. With the specialization of the specific nuclear interaction expressed by Eqs. (2) and (8c), the equations are to be found written explicitly in reference (I52), and the solutions of interest in Py^6 are portrayed in the right half of Fig. 5. The left half of Fig. 5 is intended for the discussion found below of Py^{14}, and is entered on the same graph because of the simple relationship between the states of two nucleons and the states of two "holes" in the p shell: the inversion of the spin-orbit coupling for holes means that the states in the configuration p^{-2} arise from the same secular equations as those of p^2, but with the opposite sign of the parameter a, so the energies for the two cases join continuously at $a=0$.

We see that all the states with $T=1$ on the right half of Fig. 5 correspond closely with those that are obtained by inverting Fig. 4, top for bottom. Figure 4 applies to two electrons, for which K is positive because of the repulsive interaction and a is positive because of the predominance of the magnetic term over the Thomas term (I36), and for a nuclear system with $T=1$, such as two neutrons or two protons, it applies with the ordinate inverted because of the change of sign of K for attractive forces, but with the abscissa unchanged because a too is here negative so a/K is the same as for electrons. The slope of the lines has been altered in Fig. 5, relative to Fig. 4, for the sake of symmetry and compactness of the figure, by plotting $(E+a/2)/K$ in place of E/K. The multiplet separations are taken from the paper of Feenberg and Phillips (F37a); the ratio between them for the $T=1$ multiplets is of course different from the ratio in the electron

case because of the different shape of the interaction function $V(r_{12})$, but after inversion the order of the multiplets is the same.

With the $T=0$ levels also on the figure, one sees that the ground (jj) configuration contains the four states that may be made by compounding two $j=\frac{3}{2}$ vectors, and that in the extreme (jj) coupling represented by the asymptotes they come in the order $(J, T)=(3, 0)$, $(0, 1)$, $(1, 0)$, and $(2, 1)$, a result derived also by a very different method, along with similar results for the other p-shell polyads, by Kurath (K52). Even with the approximation to (jj) coupling represented by the right side of the figure, where there is still a distinct division of the levels into groups according to their (jj) configurations, there has been a cross-over within the ground (jj) configuration making the lowest $T=1$ state the third excited state. This is indeed true experimentally (in contrast to the situation in Py^{14}), but still another cross-over is required, for the ground state is known to have $J=1$, whereas according to this theory $J=3$ is lowest near (jj) coupling and well into the region of intermediate coupling, down to $a/K\approx 4$. To match the observed spectrum with this scheme one has, therefore, to go down to rather low values of a/K, much lower than for the heavier polyads discussed below. Determining the parameters a and K requires using the two intervals between three levels, and the three lowest levels are used to determine the values $a/K=1.3$, $K=-1.23$ Mev, $a=-1.6$ Mev. The comparison between theory and experiment implied by these values is shown in Fig. 6(a). Unfortunately, the spectrum of Li^6 has not been explored much beyond the third level at 3.58 Mev, so there is only one further datum with which comparison may be made, the second $T=1$ level which experimentally is at about 5.3 Mev, theoretically at -4.9 $K=6$ Mev. The search for a couple of other low states in Li^6 arising from the ^3D will provide a crucial test of this straightforward but perhaps oversimplified interpretation.

Because smaller values of K and larger values of a seem to prevail in the somewhat heavier polyads, an attempt is made in Fig. 5 to see what can be done in the way of interpretation of the Py^6 spectrum in the neighborhood of $a/K\approx 5$, but here, of course, some other mechanism must be called upon in a rather *ad hoc* manner to explain the depression of the $J=1$ state below the $J=3$ state. This $J=1$ state is the only state that has a plausible stable counterpart in the alpha-model (alpha plus deuteron), so one might be tempted to ascribe the formation of this as the ground state to a variation theory in which some alpha-model wave function is mixed with a central-model wave function. The possible influence of the alpha-model in several polyads is discussed separately below. It may not be necessary to go to such arbitrary complexity because this rather special behavior of Py^6 (and perhaps a similar one in Py^7) may arise from their special place in the tensor-interaction theory of spin-orbit coupling of Feingold and Wigner (F52).

Li^7+Be^7

In this polyad, Py^7, the ground configuration is p^3, and the low states in (LS) coupling are provided by the multiplets ^2P, ^2F, ^4P, etc., as shown on the left side of Fig. 7(a), the separations again being taken from reference (F37), specialized for the interaction, Eq. (2), with (6). The multiplet splittings arising from spin-orbit coupling may, in this case, be calculated by the relatively simple method of trace invariance, without calculation of specific wave functions for the multiplet states, with the one exception that only the sum of the energies for the two doublets ^2P with $T=\frac{1}{2}$ are thereby given. The doublet splitting energy for one of these doublets, the one which includes the ground state, has been worked out in detail by Breit and Stehn (B38), so no further (LS) wave functions need be constructed. The calculation by trace invariance is outlined in Appendix 1. The result is that $A=a/3$ for each multiplet having $T=\frac{1}{2}$, and $A=0$ for the two doublets having $T=3/2$. This result is used to give the slopes of the (LS) asymptotes on the left side of Fig. 7(a).

In (jj) coupling the ground "(jj) configuration" is $p_{\frac{3}{2}}^3$, and the separation between the states of this (jj) configuration have been calculated by Kurath (K52). The next (jj) configuration is $p_{\frac{3}{2}}^2 p_{\frac{1}{2}}$, and because it lies above the region of present experimental interest, no great care has been used in plotting its (jj) asymptotes. [They are most of them drawn in broken lines because the energies expressible in K have not been used in plotting them, though these too have very recently been calculated (K52a).] The single-nucleon spin-orbit energies $a/2$ for $p_{\frac{3}{2}}$ and $-a$ for $p_{\frac{1}{2}}$ in this case give the total spin-orbit energy $3a/2$ for the ground (jj) configuration $p_{\frac{3}{2}}^2$ and zero for the next, $p_{\frac{3}{2}}^2 p_{\frac{1}{2}}$. Correspondingly, one set of (jj) asymptotes for E/K slopes sharply downward to the right (a being negative, as is K) in

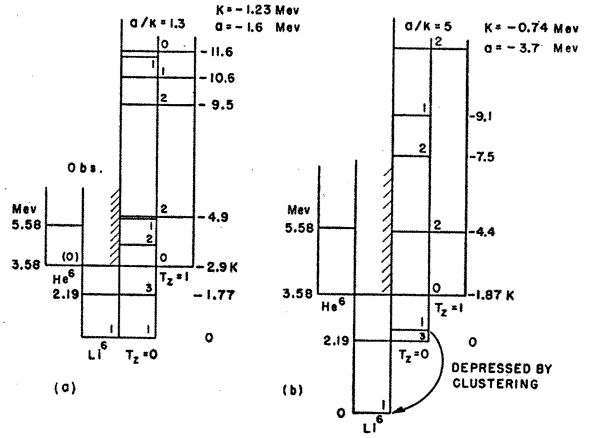


FIG. 6. Alternative interpretations of the energy levels of Py^6. (The energy 5.58 Mev is reduced to 5.3 Mev in the newer data as shown in Fig. 1.)

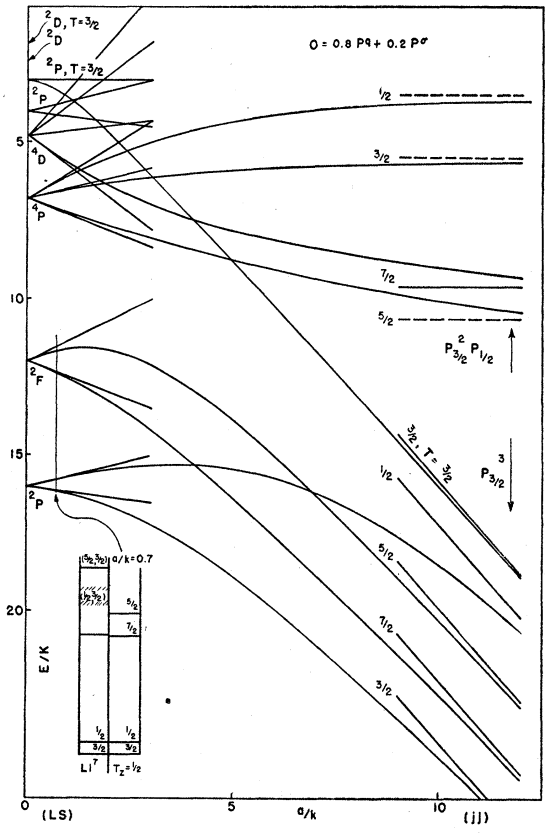


FIG. 7(a). Intermediate-coupling transition for the configuration p^3 applying to Py^7, with the interaction assumption (2). The insert shows that, even if one goes near (LS) coupling, it is not possible to obtain agreement between the ^2P and ^2F splittings.

Fig. 7(a), and the higher set is horizontal. Of the excited (jj) configuration, only a few of the lower states are shown in the figure.

An early discussion of intermediate coupling in nuclei (H51) was concerned with this configuration p^3. It was shown there that neither with a non-exchange (Wigner) nor with a space-exchange (Majorana) interaction alone can one obtain a low isolated pair of states similar to those observed except in the near neighborhood of (LS) coupling. With this true for two such very different simple interactions, it was felt that it would remain true for other more complicated interactions of a plausible nature. [Because only two states had then been observed, and there was contradictory evidence as to the identification of the first excited state, which has since been shown (Ph51, C52, Bu52) to have $J=\frac{1}{2}$, special emphasis which is now obsolete was put on the possibility of explaining the scarcity of low states by exploiting a degeneracy between three of the $p_{\frac{3}{2}}^3$ states with a non-exchange interaction (T52).] Now that three much higher excited states are known (Aj52, Tho52), it is of interest to explore this point in a little more detail. The interpolated intermediate-coupling pattern shown in Fig. 7(a) is that for the simple interaction, Eq. (2), which forms a basis for our discussion of other polyads. For the sake of indicating the sort of change that may be expected from a change in the interaction assumption, a similar intermediate-coupling scheme is shown in Fig. 7(b) for the somewhat different and perhaps more elegant interaction, Eq. (1), which also obeys the saturation requirements. It is seen that the main difference is that with Eq. (1) in place of Eq. (2) the first $T=\frac{3}{2}$ state is brought below the $J=\frac{1}{2}$ state in the ground (jj) configuration. The general pattern for the first four states in the intermediate-coupling region is altered very little. For this reason it seems satisfactory to proceed with the following rather preliminary discussion of other polyads on the basis of the interaction, Eq. (2), alone, for the sake of simplicity, realizing that it may ultimately be possible to improve the fit with experiment by introducing another interaction, and perhaps even to provide a selection of the most nearly adequate interaction law in this manner.

The attempt at an (LS) coupling interpretation is shown in the insert in Fig. 7(a). For the first three states one finds a match between theory and experiment with $a/K\approx 0.7$ but finds the ^2F much too narrow to account for the separation between the two sharp states at 4.6 and 7.5 Mev. The ratio of this separation to the low doublet splitting is 2.9 Mev/0.48 Mev=6.0. The ^2F splitting in the (LS) limit is only 7/3 times as great as the ^2P splitting, since $A=a/3$ for each in the doublet splitting formula $\Delta E=(L+\frac{1}{2})A$, and the curvatures away from the (LS) limit increase this discrepancy. Since the wave functions are relatively simple so near (LS) coupling, this discrepancy is based mainly on the assumption of the form of spin-orbit coupling, Eq. (3). It would be interesting to know whether the Feingold-Wigner treatment (F50) of the tensor interaction, even though a bit vague on the magnitude of the ^2P splitting itself, could give the ratio of the ^2F to ^2P splitting with sufficient accuracy for a significant comparison with these data.

Quite aside from the discrepancy based on Eq. (3), so small a value of a/K (which incidentally implies $a=-0.85$ Mev, $K=-1.2$ Mev) when compared with the much larger values found in the heavier polyads of the p-shell, seems incompatible with any simple assumption leading to Eq. (3). One might naively think that it is also incompatible when compared with the larger value of the doublet splitting in Py^5, which seems to be 2.6 Mev or more, since this might lead one to expect in Py^7 a ^2P splitting of 2.6 Mev/3=0.9 Mev or more, rather than about 0.48 Mev. The naiveté here lies in forgetting that the states of Py^5 are virtual so that their separation must be treated in a rather special way, as is being investigated by H. Hummel.

While the possible role of the alpha-model is to be discussed separately at greater length below, it may be pointed out here that the discrepancy does not necessarily throw one into a dependence upon the particular type of complexity of the second-order effects of the

tensor interaction. The type of alpha-model complexity discussed for Py^6 in connection with Fig. 6(b) may be operative here also. In this case one might identify the 4.6 and 7.5 Mev levels with 7/2 and 5/2 levels in intermediate coupling, perhaps in the neighborhood of $a/K \approx 4$ or 5, which would leave the 3/2 and 1/2 levels far apart and not much below these in the central model, but we might assume that they are physically much modified and depressed by a transition most of the way to the alpha-model, that is, by a strong admixture of alpha-model wave functions. The small separation of the two levels arising from the ^2P would then be a characteristic of the alpha-model, in which they are called in molecular terminology $^2\Sigma_u$ states ("doublet sigma ungerade"). The smallness of their splitting in first order presumably has something to do with the slow rotation of the "molecular framework" because the orbital angular momentum is divided between seven nucleons (I39, R48). In second order it is described in molecular discussions as "rho-type doubling" which is concerned with an interplay between conventional spin-orbit coupling as in Eq. (3) and the coriolis effects of the rotation in second-order perturbation theory. It has some reason to be small, although the separation of the rotational and other parts of the problem is not nearly as well justified in nuclei as in molecules where the electron mass is really very small compared to the mass factor in the moment of inertia.

The other attractive aspect of the possible participation of the alpha-model in Py^7 is the existence of a broad state at 6.4 Mev, identified as either $\frac{1}{2}^+$ or $\frac{3}{2}^+$, for the alpha-model has just one further low state besides the $^2\Sigma_u$ doublet, before going to higher rotational states which are apt to be too short-lived to be recognized, and that is one known as a $^2\Sigma_g$ in which the missing nucleon from two alphas is represented by a "hole" wave function which is even in the interchange of direction along the figure axis, rather than odd as in the $^2\Sigma_u$, and this lone excited state would indeed be a broad $\frac{1}{2}^+$ state. A state from the central model with this angular momentum and symmetry is found only among higher configurations such as p^2d, so would probably be intermixed only very weakly with this alpha-model state.

The difference between the doublet splitting 0.48 Mev in Li^7 and 0.43 Mev in Be^7 has been discussed about equally successfully in the central model with (LS) coupling (I51), in which case the classical magnetic term appears to play a somewhat larger role than the difference in size of the two nuclei, and in the alpha-model (F51). This discussion of the alpha-model is so general that it avoids any detailed specification of the source of the coupling and the difference in size of the nuclei arising from the Coulomb repulsion provides the entire spin-orbit coupling difference.

Li^7 is a nucleus that has attracted an extraordinary amount of interest, partly because it is, with its mirror Be^7, the lightest and thus, in one sense perhaps, the simplest stable nucleus in which the ground state in (LS) coupling is complicated enough to involve the problem of coupling an orbital moment L to a spin S. Its properties have therefore been scrutinized relatively carefully, and there have been among others some rather unfruitful excursions into speculation. One of these (I50a) arose from the fact that the $B^{10}(n, \alpha)Li^7$ reaction with thermal neutrons goes to the 480 kev state about 17 times as strongly as to the ground state. Since B^{10} has a large J-value, $J=3$, this anomalous ratio could be explained by the false assumption that the 480 kev state also has a large J, but it appears instead to arise from a not very unlikely, fortuitous cancellation in a matrix element, since this state has more recently been shown to have $J=\frac{1}{2}$. Another excursion arose from an "experimental determination" of the quadrupole moment Q of Li^7 which recently turns out to have been false even as to sign. All the simple models give a negative theoretical Q for Li^7, essentially because an orbit flattened in a plane normal to its angular momentum implies negative Q, and this sign persists in the various coupling schemes of nucleons here encountered. Attempts to find a theory (Av50, P50) that would be compatible with a positive Q serve now only to illustrate the "stability" and hence the reliability

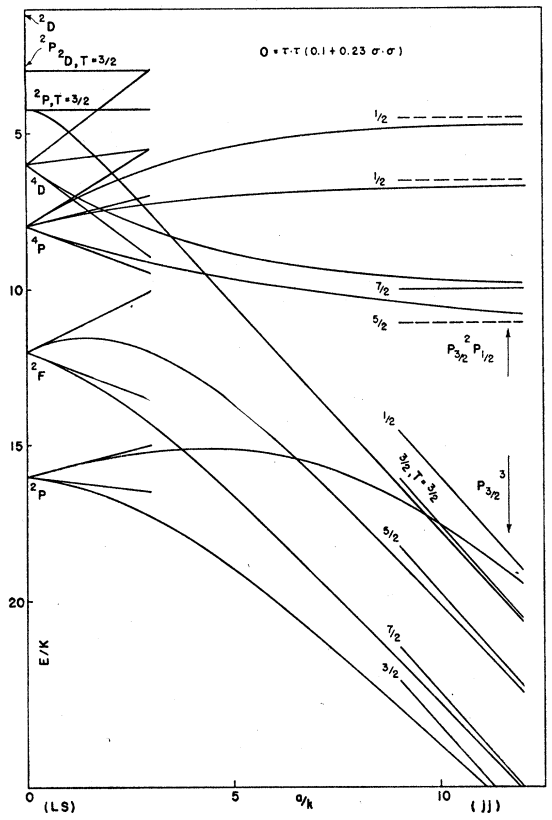


FIG. 7(b). Intermediate-coupling transition for the configuration p^3, with interaction assumption (1).

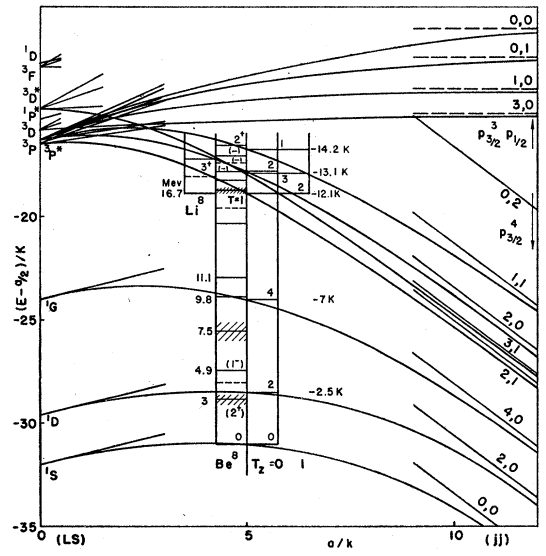


FIG. 8. Intermediate-coupling transition for the configuration p^4 applying to Py[8]. In the notation for the multiplets of (LS) coupling, the asterisk (*) is used to denote $T=1$.

of some theoretical results based on models, for one had to go to great lengths of artifice to obtain a positive Q.

The weak link in the first "experimental determination" of Q of Li^7 was the molecular-theory calculation of the molecular quadrupole coupling constant q, for only qQ is measured. There have been two independent recent demonstrations that the first calculation of q gave the wrong sign. First, the measurement in Kusch's laboratory of qQ for Li^7 and other alkali nuclei in a number of polar as well as homopolar molecules (Lo52 and Table X below) showed that qQ had the same sign for Li^7 in all molecules for which it was measured, and the opposite sign for Na^{23} in each of a series of similar molecules, suggesting very strongly that Q has the opposite sign for Li^7 and Na^{23}, and further an atomic measurement in Bitter's laboratory (communication from P. Kusch) of Q for Na^{23} shows it to be positive, from which we may infer that Q for Li^7 is negative. (The sign of the atomic coupling constant may be calculated much more easily and reliably because it has contributions from electrons only, and mainly one electron, with no competition from a positive charge.) It is also possible to understand the molecular origin of the sign of q more easily in polar molecules than in homopolar molecules, because of a predominant term at a given nucleus arising from the charge on the other ion, and this shows that the experimental results are all compatible with each other and with a negative Q for Li^7, as is explained in Appendix II. Second, there has very recently become available (Har52) the result of a new calculation of q for the homopolar molecule Li_2 involving better wave functions, and this time the sign is altered from the earlier calculation, q being negative as a result of a more accurate representation of the lumping of the electron charge in the region between the nuclei because of the antisymmetry, as is discussed further in Appendix II. Thus Q_{Li^7} is now found to be negative, in agreement with the oblate charge distribution expected from the models.

Be^8+Li^8

A similar treatment of Py^8 is displayed in Fig. 8. The interaction, Eq. (2), is assumed, and the multiplet splittings are taken arbitrarily to be given by $A=a/5$. This represents interpolation between $A=a/3$ for Py^7 and $A=0$ for Py^{10}, at the middle of the p-shell where a "hole" is the same as a nucleon and the spin-orbit coupling constant A must according to the results of "hole theory" be the negative of itself, that is, zero. *Note added in proof.*—Recent calculation (C53) shows that the values of A/a should be 3/8, 3/8, and 1/12 for the $^3P^*$, ^3P, and ^3D in Fig. 8, respectively, rather than 1/5. The order of the first four levels is the same in (LS) coupling as in (jj) coupling, and remains the same in intermediate coupling. There is thus a great deal of latitude in the choice of the parameter a/K to obtaiu a rough comparison with the experimental spectrum, and the value $a/K=5$ has been chosen for this purpose, largely because about this value will be found to apply elsewhere, in the region Py^{10} to Py^{16}. In Py^8, a small value $a/K\approx 2$ could not be used because the density of states just beyond the first state with $T=1$ would be much greater than observed. Since Py^8 is thus not very near (LS) coupling, it is not very plausible to attribute the isolation of the low doublet in Li^7 to (LS) coupling in that adjacent nucleus.

At $a/K=5$ the energy scale has been chosen to fit the first $T=1$ state, $(J, T)=(2, 1)$, as indicated by the position of the ground state of Li^8, and we see in the overlaid insert in Fig. 8 that the broad states at 3 and 7.5 Mev lie considerably below the (2, 0) and (4, 0) theoretical levels. The 3-Mev state has been tentatively identified experimentally (T50, T51; see, however, W41) as (2, 0), and the fact that it is broad makes this assignment very likely because this is also the first excited state of the alpha-model, analogous to a molecular rotation state with Bose statistics of the constituent alphas. Such a state of the alpha-model is above the Coulomb-plus-centrifugal barrier (see Section 10), so it would be expected to be extremely short-lived and probably too broad to be recognized as a level at all. It is reasonable to assume that the 3-Mev level is a mixture of central-model wave functions, as given by the (2, 0) curve of Fig. 8, and alpha-model wave functions, the latter component accounting for the level width. The next rotational state of the alpha-model is a (4, 0) state, and it is reasonable to assume that the state corresponding to the (4, 0) curve in Fig. 8 is depressed and broadened by a similar admixture, giving rise to the 7.5-Mev state.

The prediction (of (LS), (jj), or intermediate coupling) is that the ground state of Li^8 is (2, 1), which is in

keeping with the fact that the beta-transition goes to the 3-Mev state of Be^8, but not to the ground state (or 7.5-Mev state). A further prediction of (jj) or intermediate coupling is that the first excited state of Li^8 is (3, 1), which corresponds to the identification 3^+ of a state at about 19 Mev in Py^8. There is a possibility that this is the second excited state of Li^8. The possible intervening state is shown by a broken line, and, if it exists, is high enough that it might have odd parity, arising from an excited configuration not shown on this graph. The experimental (3, 1) state lies above the theoretical, and this discrepancy would be more pronounced for larger values of a/K, nearer (jj) coupling. This is not to be considered a serious discrepancy, however, because this is the sort of separation which, in (jj) coupling for instance, is very sensitive to the choice of exchange operators, as we have seen in the comparison of Figs. 7(a) and 7(b). Thus it cannot be excluded that a/K is considerably larger than 5 in Py^8.

The existence of a state at 4.9 Mev (and perhaps one at 4.05 Mev) is rather surprising in this interpretation, because it does not correspond to any central-model state of the configuration p^4, and if it were purely a state of the alpha-model, in addition to those already mentioned, such as the vibration state discussed in Section 10 below (see Table V), it would be expected to be extremely broad (and probably for this reason unobservable). It lies at about the excitation energy at which the first state arising from the (jj) model excitation $p_{\frac{3}{2}} \rightarrow d_{5/2}$ appears in the nuclei C^{13} to O^{16}, but in Be^8 the corresponding excitation $p_{\frac{3}{2}} \rightarrow d_{5/2}$ would be expected to involve an excitation energy higher by something like the single-nucleon p-doublet splitting $-3a/2$. By tentatively taking $a/K=5$ we imply that $K=-1.38$ Mev, $a=-6.9$ Mev, $-3a/2=10.4$ Mev, so on an oversimplified single-nucleon picture we might expect the first level from the excited configuration $p_{\frac{3}{2}}{}^3 d_{5/2}$ to appear about 10 Mev higher than it seems to. However, there are also involved very great energies of interaction of the nucleons, of the sort indicated by the fact that the spread within the (jj) configuration $p_{\frac{3}{2}}{}^4$, after passing into intermediate coupling, is about 20 Mev, or even more if we include the (0, 2) state of higher isobaric spin, so that such predictions neglecting the variations in these interactions from one configuration to another cannot perhaps be expected to be accurate to much better than 10 Mev. It thus seems not entirely unreasonable to attribute the level at 4.9 Mev, and others such as those at 9.8 and 11.1 Mev, to the odd-parity configuration $p_{\frac{3}{2}}{}^3 d_{5/2}$. The 4.9-Mev state does indeed appear§ (Th51) to be odd, (1^-), and an experimental determination of the parities of the others will also be of interest.

§ The (n, γ) angular correlation following a (d, n) reaction (in the plane normal to the deuteron beam) is about of the form $(1-\frac{1}{2}\cos^2\theta)$, with rather poor statistics.

A similar situation is found in B^{10}, where the first odd state occurs at about 5 Mev.

The states of two alphas have J even, even parity, and $T=0$, and such states in Py^8 may be expected in general to be broad because of the rapidity of a possible break-up into alphas, others sharp even above the high thresholds for neutron and proton emission because the excitation energy is in this interpretation divided between many particles, and no admixture of alpha-model states is possible. The fact that there are, as far as known, just two broad excited states in the region below 15 Mev to correspond to the two theoretical excited states of the ground configuration, (2, 0) and (4, 0), is thus very satisfactory. The most recent compilation (Aj52) shows a third at 10 Mev, but it is not apparent that it is reliably established as a broad level.

The ground state of Li^8 is estimated to lie at 16.7 Mev in Py^8, and very recent evidence (T53, see also Ra52) of the preferential formation of the broad 16.9-Mev state of Be^8 resulting from conservation of T in the high-energy photodisintegration of C^{12} indicates that this lone nearby $T_z=0$ state is indeed the lowest one with $T=1$. This match in energy is an example of the importance of isobaric spin T as a fairly good quantum number as discussed in Section 4, above. The selection of the high state in competition with the lower states shows that it (or, at least, the more general charge symmetry) provides a significant selection rule, but the breadth of the state (presumably 2^+) caused by break-up into two alphas (competing only with gammas) indicates that the conservation is not perfect.

Note added in proof.—Very recent observations of the reaction $B^{10}(\gamma, d)Be^{8*}(2\alpha)$ and $B^{11}(\gamma, t)Be^{8*}(2\alpha)$ in photographic emulsions in Zuerich (S53) have shown that the Be^8 level previously observed only as a broad level at about 3 Mev may be resolved into three levels at 2.2 Mev, 2.9 Mev, and 3.4 Mev, and that these and the 4.0-Mev, 4.9-Mev, 6.9-(our 7.5) Mev, 9-(our 10) Mev, and 14.7-Mev levels have even J and even parity. (The statistics leave the resolution of the 3.4-Mev level quite doubtful.) The 2.2-Mev level has also been observed with doubtful statistics in the $Li^7(d, n)$ reaction (Tru52). A recent magnetic analysis (G53) of the reaction $Be^9(d, t)Be^8$ shows the 2.9-Mev level as broad [definitely broader than the central peak in (S53)], with a sharp peak attributed to an impurity that might hide the 3.4-Mev level, but the complete lack of a peak corresponding to the 2.2-Mev level may suggest that this state, if it exists, has too high J to be observed at the low energies involved, $E_d=1$ Mev, $E_t=2.2$ Mev. Some apparent confirmation of the suggestion that the alpha-model plays an important role in the Li and Be nuclei is found in the rather surprising fit (S53) of all but two of these levels to the alpha-model vibration-rotation scheme

$$E_{j,k}=(0.36J(J+1)+2.8K) \text{ Mev},$$

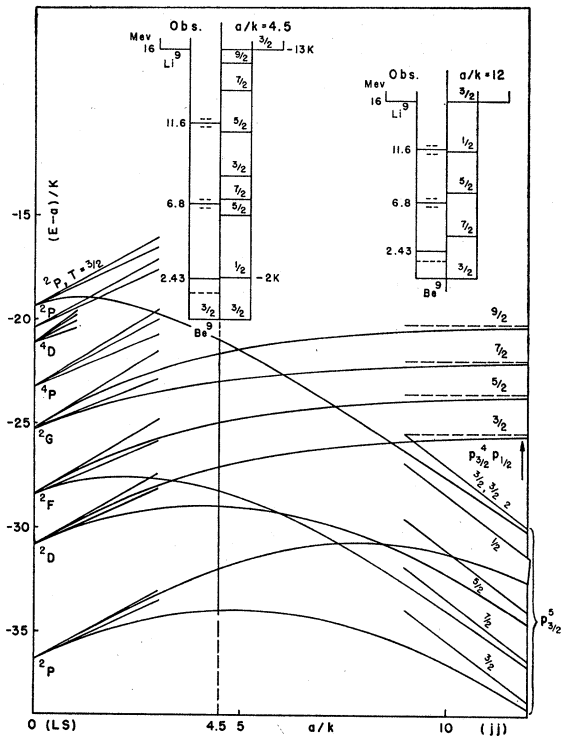


FIG. 9. Intermediate-coupling transition for the configuration p^5 applying to Py^9. The isobaric spin is $T=\frac{1}{2}$ except where indicated as $T=\frac{3}{2}$.

as in "Alpha-Model I":

Experimental energy	Alpha-Model I			Alpha-Model II		
Mev	Mev	J	K	Mev	J	K
14.7	14.7	6	0			
11.1 (odd?)						
9.7 (10)	9.8	4	1			
6.9 (7.5)	7.0	4	0	7.0	4	0
4.9	4.9	2	1	4.8	2	4
4.0				4.1	2	3
3.4				3.4	2	2
2.9	2.8	0	1	2.7	2	1
2.1	2.1	2	0	2.1	2	0
0	0	0	0	0	0	0

This scheme leaves the interpretation of the first excited state and the 7.5-Mev level about as above but with more complete participation of the alpha-model, and adds 14.7 Mev to the sequence. There are two unexplained states in the scheme "Alpha-Model I" and three in the central model with these new data. In the central model their even parity means that they would have to belong to doubly-excited configurations, analogous to the first excited state of O^{16} but involving greater spin-orbit energy. In the alpha-model, the presence of the lowest $T=1$ state at 17 Mev requires some explaining, for which the existence of the suspected excited state of the alpha at about 22 Mev would be helpful. The evidence for the latter is, however, conflicting and its existence is on the whole very doubtful (Ar50, Al51, Be51). In Section 10, there is a discussion of "The Low Barrier in the Alpha-Model of Be^8," in which it is estimated that the barrier for $J=0$ is less than about 0.9 Mev, on the assumption that the interaction between alphas provided by the specific nuclear forces is a simple well, attractive at large distances with a repulsive core. "Alpha-Model I" requires that these forces themselves provide a barrier at least about 2 Mev high, a nuclear repulsion between alphas beyond the attractive region. The large number and fairly regular spacing of the states from 2.1 to 4.9 Mev suggests the interpretation "Alpha-Model II," with a very small curvature at the bottom of the potential well and no need for a high barrier, but the lack of states $(J, K)=(0, 1)$, $(0, 2)$, $(4, 1)$, etc., is unexplained. There is still another possibility, involving $K=2$ and requiring a very high barrier, which accounts roughly for all but the doubtful 3.4-Mev level. Experimental assignments for the new states are needed to clarify the puzzle.

Li^9+Be^9

On the (jj)-coupling side of Py^9 we have first the configuration $p_{\frac{3}{2}}^5$, which is the same as $p_{\frac{3}{2}}^{-3}$ or three $p_{\frac{3}{2}}$ holes, and the order and separation of the states of this configuration are the same as in the low (jj) configuration $p_{\frac{3}{2}}^3$ of Py^7, as given by Kurath (K52). On the (LS) side there is no such similarity, since we are not yet quite halfway across the entire p shell, and the low ^2p is here followed by a ^2D before the ^2F. Because this is just one nucleon short of the middle of the shell (where multiplet splittings vanish), the multiplet splittings are expected to be very small, and we arbitrarily assume $A=a/10$ for all multiplets, and thus plot the (LS) asymptotes as very narrow doublets and quartets. The intermediate coupling transition is again drawn entirely by interpolation, Fig. 9, with smooth curves that approach the asymptotes in a reasonable fashion.

The observations on Be^9 are more incomplete than on most of the other light nuclei (it is much used as a target because of its availability in thin foils but infrequently attained as a final nucleus!). It is usually considered to have three low states, below 2.5 Mev, and no further states have been found (V51) in careful exploration up to 5 Mev with $B^{11}(d, \alpha)Be^9$. Beyond this, inelastic scattering of protons observed by NaI pulse-height analysis (Br51) shows broad maxima in a curve that does not dip down near zero between them, and indicates prominent excited states in Be^9 at 6.8 and 11.6 Mev, or groups of states clustering about these energies.

Of the states below 5 Mev, only two have been observed directly in the $B^{11}(d, \alpha)$ and $Be^9(p, p')$ data. The third, though absent in these investigations, is inferred from an analysis (Gu49) of the photo-excitation of Be^9 on the basis of a very crude model. The same analysis also infers that the 2.42-Mev level is a superposition of two states, an unresolved ^2D, though it appears experimentally to be very sharp. The model

used for Be^9 is a core of two alphas with a single neutron circulating in their field, and is subsequently treated for mathematical convenience as though the two alphas were coincident. Now that we seem to be forced in other light nuclei to the complexity of an intermediate-coupling interpretation, this model seems so crude that we may consider the photodisintegration not to have been adequately investigated, and we may, thus, be reasonably skeptical of the inference of a state at 1.56 Mev and of an associated preliminary assignment of a J to the state at 2.43 Mev.

One sees in Fig. 9 that there seems to be no possibility of obtaining in the intermediate-coupling scheme for the configuration p^5 just three low states, followed by a wide gap. For this reason, we must assume either that the inferred, unobserved state at 1.56 Mev does not exist or, as seems improbable but might be possible with such a large spread within each configuration, that one of the first two excited states belongs to an excited configuration such as p^4d. With this in mind, we have drawn the 1.56-Mev state with broken lines and ignore it in the comparison with experiment.

Because the slightly heavier nuclei suggest a ratio $a/K\approx 5$, we may attempt to match the 2.43-Mev state and the lowest states with $T=0$ and $T=1$ in this region. We thus find $a/K\approx 4.5$ (a value which is rough, *inter alia*, because it depends on the interpolation curves). In one of the inserts of Fig. 9 we see that this very satisfactorily implies first a wide gap extending from 2.43 Mev up beyond 5 Mev, and then several states which might cluster about the observed broad maxima (or some of them be lost in the background) in the higher region which has been explored only with poor resolution. If it should subsequently be found that there are only single states corresponding to these broad maxima, one might have to go to (jj) coupling to find so few widely spaced levels, as indicated in the second insert of Fig. 9, drawn for $a/K=12$, but this seems unlikely.

$Be^{10}+B^{10}$

Here we come to a polyad in which there are a considerable number of identifications of J's and parities, especially among the rather numerous low states (Aj52a, R52). The separations of the states, with the interaction given by Eqs. (2) and (6), are in (LS) coupling again taken from (F37), and in the ground (jj) configuration, $p_{\frac{3}{2}}^6$, they are the same as plotted in Fig. 5 for Py^6 and for the high states of Py^{14} in (jj) coupling. The separations of the (jj) asymptotes of many of the fairly low states in the excited configuration $p_{\frac{3}{2}}^5p_{\frac{1}{2}}$ have very recently been calculated and kindly made available by Kurath (K52a), and make it possible to interpolate a fairly complete set of intermediate coupling curves between known asymptotes. This is done in Fig. 10, which is plotted with (jj) coupling on the left side (as for N^{14} in Fig. 5), and with the ordinate $[E-(3/2)a]/K$, in such a way that the (jj) asymptotes of the numerous states in the first excited (jj) configuration are horizontal.

The first excited configuration $p_{\frac{3}{2}}^5p_{\frac{1}{2}}$ may be considered to be compounded by coupling the $p_{\frac{1}{2}}$ nucleon to the various levels of the $p_{\frac{3}{2}}^5$ partial configuration. This is, of course, artificial from a strict point of view, for the coupling between all nucleons is apt to be about equally strong, but the energy of the partial configuration does constitute a large contribution to the energy of the complete state, and there is a strong correlation between the lowest states of the excited configuration and those of the partial configuration. The calculated energies of the partial configurations (not including spin-orbit energy) are the same as already used in Py^7 and Py^9 above and are indicated by the horizontal lines in the upper left part of Fig. 10. Each of the states with $J=1$, 2, or 3 is associated with two partial configurations, as is indicated by broken lines in the figure, and the quantum numbers of the partial configuration, of course, do not remain constants of the motion after coupling on the $p_{\frac{1}{2}}$ nucleon. They do help in counting the number of states expected, and one sees that not all the states of the excited configuration have been plotted (the indication from the "Majorana" term alone being that the states omitted are too high to be of much interest).

This polyad is at the very middle of the p shell; its ground configuration may as well be called p^6 as p^{-6}. From the theory of holes, one knows that the spin-orbit

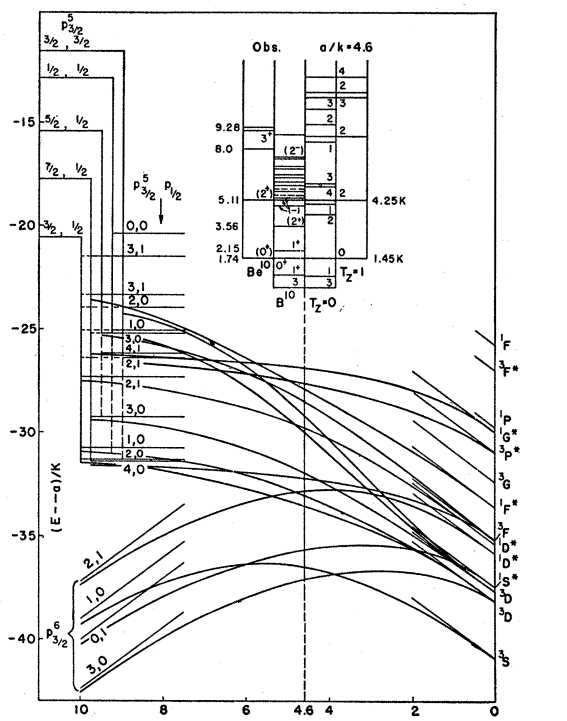


FIG. 10. Intermediate-coupling transition for the configuration p^6 applying to Py^{10}. The asterisk again denotes $T=1$. The ordinate is $(E-\frac{3}{2}a)/K$.

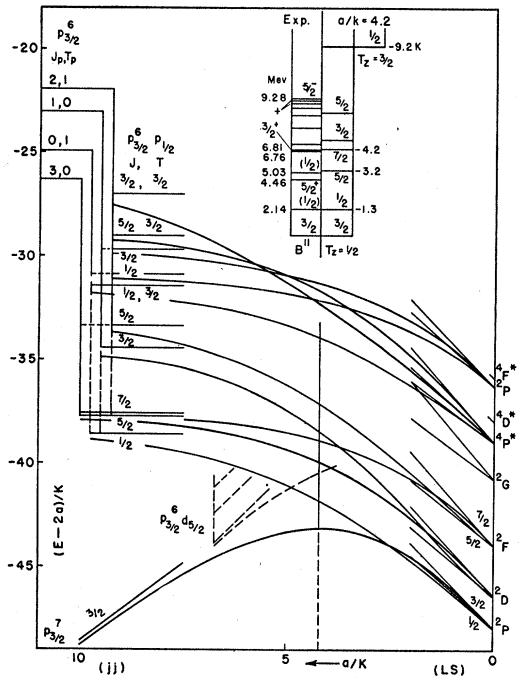


FIG. 11. Intermediate-coupling transition for the configuration p^5 applying to Py^{11}. The asterisk here denotes $T=\frac{3}{2}$, T being $\frac{1}{2}$ for the states not so marked.

coupling in any multiplet must be the negative of itself, and, therefore, the spin-orbit coupling parameter A may here be put equal to zero, as has been pointed out above. Thus the triplets are drawn, like the singlets, each with a single (LS) asymptote. Going from this side over into intermediate coupling, as shown in Fig. 10, the splittings within the multiplets are caused by what in (LS) coupling may be considered second-order effects, by the differences in curvature of the interpolation lines as they start to deflect toward the (jj) asymptotes.

The value $a/K=4.6$ is chosen for the comparison with experiment, because it makes the first $T=1$ state the third state of the polyad and places the second state at about the right level below it, as shown in the insert of Fig. 10. The energy scale is determined by the interval between the lowest $T=0$ and $T=1$ states. This then places the second $T=1$ state at the observed energy and makes it the sixth even state of the polyad as seems to be observed, a result strongly influenced by (LS) coupling, while leaving the ground state the one with $J=3$, which is a (jj)-coupling result, emphasizing the importance of the intermediate-coupling interpretation to this polyad. There is agreement with the isolated experimental observations that the second $T=1$ state (Co52, Th52) probably has $J=2^+$ and that the fourth $T=1$ state has $J=3$, though its estimated energy is too high. The first five states are observed to have even parity so should belong to this configuration, and of these the fourth and fifth together have the theoretical J's 1 and 2 as observed but in the wrong order, the experimental $J=1$ state being much lower than the theoretical level and the only sharp discrepancy encountered in this interpretaion. It does not seem possible to remove this discrepancy by reasonable use of the arbitrariness in drawing the interpolated energy curves. It is still a valuable accomplishment of the greater complexity of the intermediate-coupling interpretation that it may be made to fit the general features of level density and more special features of most of the assignments, including especially the existence of just one $T=1$ state among the first five, but the discrepancy shows its limitations and detracts from its persuasiveness. Whether it indicates inadequacy of the interaction assumption (2) or capricious intervention of configuration interaction, the discrepancy shows that we can claim no more than an incomplete understanding of general features in terms of a rather complicated scheme.

It is consistent with our treatment of the lighter polyads if we introduce here the further complexity of partial participation of alpha-model states to try to account for the discrepancy, for the state that appears to have been depressed by this or some other sort of configuration interaction does indeed have the J and parity of the ground state of the alpha-model, 1^+, corresponding to a nonrotating structure of two alphas and a deuteron or of three alphas and a deuteron hole. The use of this explanation is complicated by the fact that only the second 1^+ state appears to be strongly depressed by such an admixture of an alpha-model state. It is possible that further investigation would show that the second 1^+ state consists mostly of ^3S and thus resonates most strongly with the deuteron state (the pure deuteron having only about 4 percent of ^3D, though in the alpha-model of B^{10} this proportion might be altered). It is pointed out in the discussion of Py^{14} below that the ground state energy of N^{14} over most of the range of a/K plotted in Fig. 5 is near the extension of the (LS)-coupling asymptote of the $^3D_1$ state, and correspondingly that the wave function of this state consists mostly of $^3D_1$, as shown in Fig. 20. One may note that the second 1^+ state at $a/K=4.6$ in Fig. 10 is fairly near the extension of the (LS)-coupling asymptote of the ^3S, and there might correspondingly be the required preponderance of ^3S in the wave function, though it would require a more difficult calculation than in Py^{14} to show it.

There is a rather high density of B^{10} states at energies near 5 Mev and higher. This high density presumably corresponds to the contribution of excited configurations at these energies, and most of the lowest of the states so contributed would be expected to have odd parity, from the configurations p^5d and p^5s. The first three of this group of states, those at 4.79, 5.11, and 5.17 Mev, have indeed been tentatively identified as odd. If these identifications are all correct, we have the difficulty that the $T_z=1$ state observed at 5.11 Mev

would be expected to correspond to one of these, which would mean that the $T=1$ state in this region would be odd and would not correspond to the theoretical even (2, 1) state at $4.25K$. It would be surprising to have the first $T=1$ state of the odd configuration lie so low in the configuration. It is possible that the 5.11-Mev state in the Be^{10} column and the next higher state of B^{10} which is at 5.58 Mev (or possibly the uncertain state at 5.37 Mev) together form the components of the $T=1$ state, but the difference between 5.58 and 5.11 Mev would constitute a slightly larger deviation of apparent conservation of isobaric spin than is usual. It seems more likely either that one of the states at 5.11 or 5.17 Mev has even parity and $T=1$ or that there is another such state unobserved because of being practically coincident with one of them, or that the uncertain state indicated by the broken line at 5.37 Mev is the one in question. The possibility that the closely spaced pair at 5.11 and 5.17 Mev should arise from the coupling of an s nucleon to the p shell is discussed in Sec. 9.

In a shell with many nucleons in intermediate coupling, no single pair of states gives any indication of the magnitude of the spin-orbit parameter a, as may be obtained from a doublet splitting in a single-nucleon case such as O^{17}. By finding a rough fit in intermediate coupling with $a/K=4.6$, $K=-1.2$ Mev, we thus have a less direct estimate that $a=-5\frac{1}{2}$ Mev. The corresponding single-nucleon ^2p splitting in this part of the periodic table, or at least in this polyad, would be about 8 Mev.

$B^{11}+C^{11}$

In this case and in Py^{12} and Py^{13} the ground (jj) configuration contains only one state so the published calculations (K52) on the energies of the states of the ground (jj) configuration in (jj) coupling give no information concerning the separations of the (jj) asymptotes and left quite a lot of latitude in the drawing of the intermediate-coupling curves from known (LS) asymptotes to (jj) asymptotes of which only the slope was known. For the sake of filling in the information in the (jj) extreme, Dr. Kurath has very recently calculated the energies of the states in the first excited (jj) configuration for these polyads (K52a). This enables us to interpolate the intermediate-coupling curves between definite asymptotes, determined here again for the interaction assumptions, Eqs. (2) and (6), as in Fig. 11. This is an example of the sort of improvement in the theoretical basis of the intermediate-coupling interpretation that was anticipated in the introduction, and it enables us to present in Fig. 11 a second stage in the development of the interpretation. [It differs from the earlier, more imaginative, stage mainly in having the $J=7/2$ level considerably depressed, and in a similar evolution of Fig. 10 for Py^{10}, the state with the high value $J=4$, and $T=0$, also lies lower than was at first imagined, this depression of the high J's apparently being a characteristic of (jj) coupling.] At the same time a further step in the experimental clarification of the situation has become available in the J and parity determinations of Jones and Wilkinson (J52) for B^{11}, which are shown in Fig. 1 and in the insert of Fig. 11. Their assignment of even parity to the 4.46-Mev state means that there is, at most, one state of the odd ground configuration p^7 in the neighborhood of 5 Mev and forces us to go to a rather small value of a/K (and large value of K) in comparing theory with experiment. The comparison shown in the insert of Fig. 11 is for $a/K=4.2$ ($K=-1.65$ Mev, $a=-6.9$ Mev). The energy scale is determined to match the 2.14-Mev state in addition to the ground state. The experimental assignments at 4.46, 6.81, 8.93, 9.19, and 9.28 Mev are reported (J52) to have been made "with fair certainty," but those at 2.14 and 5.03 Mev involve further "reasonable assumptions," so we have indicated them in parentheses. In particular, the assignment $J=\frac{1}{2}$ at 5.03 Mev rather than, say, $J=5/2$ apparently leans very heavily on the failure to observe a gamma-transition from 9.19 to the 5.03 level with NaI pulse-height analysis. If the 5.03-Mev level should on further investigation be found to have $J=5/2$, there would be close agreement in the insert of Fig. 11. It seems more likely, however, that the state at 5.03 Mev is $\frac{1}{2}^+$ belonging to an excited configuration along with the 4.46-Mev state, and that the third and higher p^7 states are to be found at 6.76 Mev and above, in which case one might go to somewhat smaller a/K to obtain a closer fit.

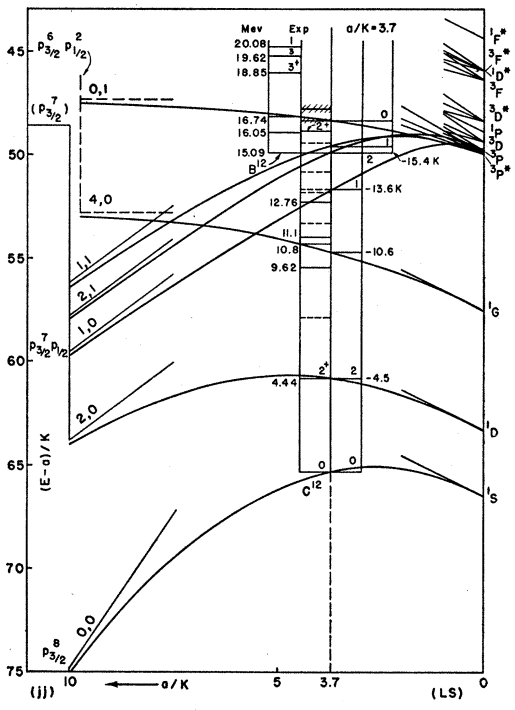


FIG. 12. Intermediate-coupling transition for the configuration p^6 applying to Py^{12}.

The "parentage" of the $p_{\frac{3}{2}}^6 p_{\frac{1}{2}}$ states in terms of the partial quantum numbers J_p, T_p of the $p_{\frac{3}{2}}^6$ "core" is shown in the upper left corner of Fig. 11. The spacings between the "core" states are the same as in Py^{10}. Cases of dual parentage are indicated by dotted lines (in these cases Kurath's results were obtained by solving secular equations, the low J, $T=\frac{1}{2}, \frac{1}{2}$ state having been "repelled" downward by the higher one, e.g.). In reference (J52) a frankly exploratory attempt was made to interpret the excited even levels in terms of a single-particle model, as though the $(J_p, T_p)=(0, 1)$ state were the only state of the "core," but because B^{10} has so many low states, the lowest with $J=3$, such a simplified explanation is not tenable. The more complex situation in the configuration $p_{\frac{3}{2}}^6 d_{5/2}$, for example, is suggested by the broken lines in Fig. 11, where an intermediate-coupling deviation from one of several (jj) asymptotes is schematically indicated at a position to account for the 4.46-Mev state.

The appearance (V51a) of two pairs of closely-spaced levels in B^{11}, one with a separation of 50 kev at 6.8 Mev and one with a separation of 90 kev at 9.2 Mev, does not very pressingly call for causal explanation, both because they are in fairly dense parts of the spectrum and because the lower one is not reproduced as a narrow double-level in the energy-level spectrum of the mirror nucleus C^{11}. (An explanation in terms of the configuration $p_{\frac{3}{2}}^6 s_{\frac{1}{2}}$, such as is discussed in Sec. 10 below, does not apply to the upper pair if the determination is correct that these two closely spaced levels have opposite parity, and if applied to the lower one would require special consideration of the complexities of this configuration.)

$B^{12}+C^{12}$

In this case the high degree of symmetry available in (LS) coupling (because of the $4n$ structure, with the periodicity 2×2 arising from multiplicities of spin and isobaric spin orientation) separates the three low states in (LS) coupling from one another and from the rest of the spectrum by such large energy intervals that the introduction of a spin-orbit coupling similar to that found in the adjacent nuclei has only a minor effect on their relative spacing. Thus we have a semblance of (LS) coupling even though the ratio a/K is appropriate for intermediate coupling in the other nuclei (I52), as remarked above.

The recent calculation (K52a) of the separations in the first excited configuration $p_{\frac{3}{2}}^7 p_{\frac{1}{2}}$ [for the interaction assumptions (2) and (6)] make it possible to draw the intermediate-coupling curves with more assurance than was possible in our earlier discussion of this polyad (I52), by interpolation between calculated asymptotes on both sides (except for the uncertainty in A on the LS side), as in Fig. 12. The most significant changes are that the state $(J, T)=(2, 1)$ is now depressed below $(1, 1)$, and that it is necessary to go to a lower value of (a/K) to fit the first two $T=0$ states and the first $T=1$ state. This is done in Fig. 12 at $(a/K)=3.7$, although this value is of course dependent on the indefinite drawing of the curves as well as on the interaction assumption.

An unsatisfactory feature of this set of curves as thus fitted is that this would make the ground state of B^{12} have $J=2$, whereas $J=1$ seems to be required by the existence of an allowed beta-transition to the ground state (and apparently also to the $J=2$ first excited state) of C^{12}. There is also very recent evidence, including some from photo-excitation and the $\Delta T=\pm 1$ selection rule for dipole radiation (Ra52, T53), that the 15.09-Mev level of C^{12} has $(J, T)=(1^+, 1)$. (This energy level was called 15.14 Mev on an earlier energy scale which survives in Fig. 1.) With the ground state of B^{12} assumed to have $J=1^+$ and matched with this level, there is also a good match between the first excited state of B^{12} at 16.05 Mev and a 2^+ state of C^{12} at 16.10 Mev. Thus the first two $T=1$ states have $J=1$ and 2 both experimentally and theoretically, but they are close together theoretically in Fig. 12, and experimentally they are about 1 Mev apart in the reverse order. It would be possible to obtain the reverse order (and thus $J=1$ for the ground state of B^{12}) but not the observed separation by reasonable modifications in the drawing of the curves between the asymptotes shown in Fig. 12. This is high enough in the energy spectrum of the polyad that reasonable changes in the interaction assumption, Eq. (2) may make enough difference to invert the order of the (2, 1) and (1, 1) states also in the (jj) limit [compare Figs. 7 and 7(a)], and this would perhaps be an appropriate place for an investigation of such changes to start.

The experimental data now include several states at about 10 Mev and immediately above, and it is not yet clear which are the (4, 0) and (1, 0) states of the ground configuration p^8. The negative-parity states of the excited configurations $p^7 d$ and $p^7 s$ apparently start in this region of quite high energy (unless the doubtful state at 7.3 Mev exists), in contrast with the finding of the first even state of B^{11} as low as 4.46 Mev. Here again the exceptional symmetry attainable in the ground configuration of C^{12} seems to have pulled the low states down.

On the (LS) side, it has been assumed that the multiplets are regular (with the lowest J lowest) corresponding to having a more than half-filled shell, although it is not necessarily the case for multiplets of such complex parentage as these. The great density of multiplets above the first triplet indicate that there should be many states just above the first $T=1$ state, but no attempt has been made to draw in the curves beyond the first triplet [except one for which the (jj) asymptote is known]. Further experimental identifications of these high states will be required before their order can begin to make sense. It is apparent at least that the five available j identifications of high states of B^{12} are present among the $T=1$ multiplets.

The striking lack of resemblance to any simple result of (jj) coupling is epitomized by the fact that the first three states arise from three different (jj) configurations, the $J=0$ state of course from $p_{\frac{3}{2}}^8$, the $J=2$ state from $p_{\frac{3}{2}}^7 p_{\frac{1}{2}}$, and the $J=4$ state from $p_{\frac{3}{2}}^6 p_{\frac{1}{2}}^2$. They accordingly approach asymptotes of three different slopes.

The experimental assignment 2^+ for the 4.44-Mev state (Ha51, Th52, H52, Le52) is based on recent observations of the angular correlation in the reaction $B^{11}(p;\gamma_1,\gamma_2)C^{12}$. Any other assignment would have seemed anomalous, for this agrees with all the models.

$C^{13}+N^{13}$

This is another case in which the ground (jj) configuration contains only one state and the recent unpublished energies (K52a) for the five states of the first excited (jj) configuration have been used to provide definite (jj) asymptotes for the drawing of the curves in Fig. 13. These go over into states in as high as the sixth multiplet in (LS) coupling. All the states for which curves are not drawn in these multiplets would curve up to states of higher (jj) configurations, and so would contribute only very high states to the energy spectrum.

Experimentally, this polyad has been explored only for fairly low states in the two low mirror nuclei. The fact that it is not very stable relative to single-nucleon emission seems to induce an unusually large displacement (Th50, Th52a) between the mirror levels which lie between the proton threshold for N^{13} and the neutron threshold for C^{13} (see Fig. 1), so the comparison is made with the actual energies of C^{13} for which these levels are stable in this respect. There are some fine new possibilities for making unique or almost unique assignments of angular momentum from angular distribution of (d,p) or (d,n) stripping reactions near the forward direction (B51, Bh52, R51) and from elastic scattering resonances when the target nucleus, in this case C^{12}, has $J=0$, as is explained further a few paragraphs below, and quite a lot of information is available concerning the identification of the states.

By comparing the unique assignments of the resonant-scattering method in N^{13} with the not quite unique assignments of the stripping-reaction method in C^{13}, one may on the basis of the expected correspondence of mirror levels reasonably select the assignments in C^{13} as follows: $\frac{3}{2}^-$ at 3.69 Mev (3.50 Mev in N^{13}) and $5/2^+$ at 3.89 Mev (3.55 in N^{13}), in addition to the unique assignment $\frac{1}{2}^+$ at 3.09 Mev (2.36 in N^{13}), as has already been assumed above. The close juxtaposition of the $\frac{3}{2}^-$ and $5/2^+$ levels in N^{13} is here seen to be accidental.

Only one of the first three excited states of C^{13}, the one at 3.69 Mev, has the parity characteristic of the ground configuration. The energy scale in the insert of Fig. 13 is determined by matching the interval between this and the ground state, after arbitrarily selecting $a/K=5$ on the basis of experience with other, more

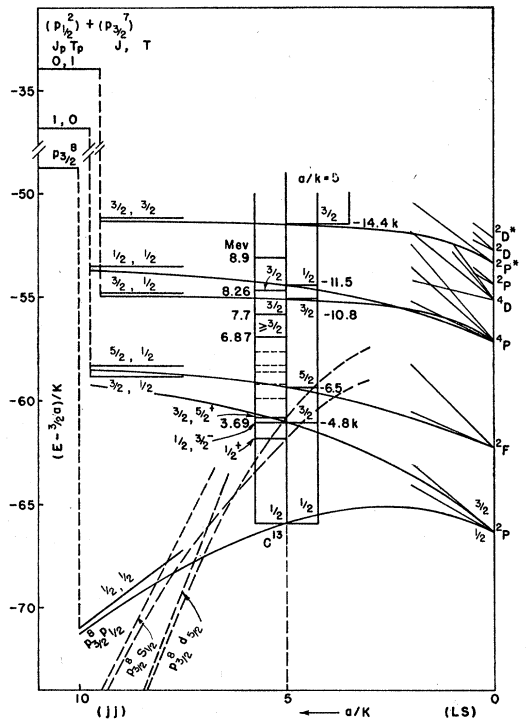


FIG. 13. Intermediate-coupling transition for the configuration p^7 applying to Py^{13}. The asterisk denotes $T=\frac{3}{2}$.

revealing, polyads. The theoretical values $J=\frac{1}{2}$ and $\frac{3}{2}$ for the first two states of this configuration then agree with the experimental assignments, the two states arising from the ^2P of (LS) coupling, inverted in the nuclear sense because of being in the last half of the p shell, but much distorted by the transition to intermediate coupling and too far apart to look like a doublet.

From the (jj)-coupling point of view, the mistake is sometimes made of thinking that, since the ground state consists of a single nucleon outside of a closed (jj) shell, the interval from the ground $J=\frac{1}{2}$ state to the first $J=\frac{3}{2}$ state is a measure of the single-nucleon doublet splitting, $\frac{3}{2}a$. This excitation is actually from the (jj) configuration $p_{\frac{3}{2}}^8 p_{\frac{1}{2}}$ to $p_{\frac{3}{2}}^7 p_{\frac{1}{2}}^2$, and thinking in a very simple (and still unjustifiably simple) way, one must at least consider that he is breaking up a pair of $p_{\frac{3}{2}}$ nucleons and forming a pair of $p_{\frac{1}{2}}$ nucleons (in keeping with the Mayer δ-function treatment (M50) which is valid only for large nuclei), and that the two pairing energies are different so that the observed interval is no measure of the single-nucleon $p_{\frac{3}{2}} \rightarrow p_{\frac{1}{2}}$ excitation energy. More properly, one notes that either a neutron or a proton may be excited, and in either case one can make a state with $(J,T)=(\frac{3}{2},\frac{1}{2})$, as may be described also by coupling the $p_{\frac{3}{2}}$ hole to the $(J_p,T_p)=(1,0)$ or $(0,1)$ states of the partial configuration $p_{\frac{1}{2}}^2$. Thus even in extreme (jj) coupling the mixing of these two states as solutions of the same secular equation tends to push

the lower one down toward the ground state and complicates the interpretation of the $J=\frac{1}{2}$ to $\frac{3}{2}$ excitation, so that it is not a simple measure of single-nucleon spin-orbit coupling.

The additional complexity encountered in intermediate coupling is more pronounced for these odd-parity states of the configuration p^9 than for the even-parity states p^8d and p^8s. The ground state is derived from the (jj)-coupling state $p_{\frac{3}{2}}{}^6p_{\frac{1}{2}}$ and other (jj)-coupling states formed by simple excitation of the least bound nucleon are $p_{\frac{3}{2}}{}^6d_{5/2}$ and $p_{\frac{3}{2}}{}^6s_{\frac{1}{2}}$, each with a full $p_{\frac{3}{2}}$ subshell so that the states have the characteristics of single-nucleon states in (jj) coupling. The single-nucleon state of the "extra nucleon" is distinguishable from the other nucleon states, only by its spin-orbit energy if it is $p_{\frac{1}{2}}$ but also by its radial wave function if it is $d_{5/2}$ or $s_{\frac{1}{2}}$. In (LS) coupling one may think of forming C^{13} by adding a nucleon to a C^{12} core. When a p nucleon is added, there is no energetic or other distinction of the added-nucleon state from the others, and we have the formation of a ^2P characterized by a communal $L=1$ and $S=\frac{1}{2}$, the added nucleon amalgamated with the rest and the spin-orbit parameter A being calculable by the appropriate vector-model composition of all the $\mathbf{l}_i$ with each other, etc. In intermediate coupling, most of this thorough mixing of the roles of the p nucleons persists, and these states of the system do not begin to resemble single-nucleon states until one gets to (jj) coupling, where the integrals L and K are too small compared to a to mix the (jj) configurations.

When instead a d nucleon (or an s nucleon) is added to the "C^{12} core," there remains a distinction between the added-nucleon state and the others in terms of the number of nodes of the radial wave function, for example, and in the problem of coupling the added nucleon to the "core," there appear integrals which we might call L_d and K_d, analogous to L and K except that they contain p and d wave functions where L and K contain only p functions, and which are expected to be considerably smaller than L and K because of the more complete cancellation of the positive and negative contributions to the integrals arising from the rapidly oscillating products of the different wave functions. It is this fact that in intermediate coupling tends to preserve the single-nucleon nature of the wave functions of the system. In (jj) coupling we have the single-nucleon semblance, as before, because in this case L_d and K_d are much too small to mix in the excited-core (jj) configurations, but as we pass to intermediate coupling, the $J_p=0$ to $J_p=2$ excitation of the core remains large, as we see in Fig. 12, of a magnitude determined both by K and by the spin-orbit parameter a (which make contributions of comparable magnitude in intermediate coupling) and the contributions of the K_d (and L_d) are presumably small in comparison, so the low $5/2^+$ state is composed almost entirely of the core state $J_p=0$ plus the $d_{5/2}$, with fairly little admixture of the core state $J_p=2$ plus the $d_{5/2}$, for example. Thus in this approximation the state of the system is expected to resemble a single-nucleon state, and the complexities of intermediate coupling are hidden within the core where they have no outside influence because $J_p=0$. In (LS) coupling the core again makes no contribution to the external properties of the low D and S states, because the core is in a ^1S state, and the parameter a is too small to mix in anything else.

In the course of making assignments to the states of N^{13} by examination of scattering resonances, the $\frac{3}{2}^-$ state, doubtless the mirror of the C^{13} state at 3.69 Mev, is found to have a "reduced width" only about a tenth as great as that of the $\frac{1}{2}^+$ and $5/2^+$ states which are presumably the mirror states of the 3.09- and 3.89-Mev states, respectively (J51). This small reduced width in single-nucleon scattering indicates that this $\frac{3}{2}^-$ state cannot be so well approximated as the others by a single-nucleon wave function, and this seems to be in accord with our intermediate-coupling interpretation. The same thing is indicated by the observation (R51) that the $C^{12}(d, p)C^{13}$ stripping reaction, in which a single neutron is added to the "core," gives the $\frac{3}{2}^-$ state with only a few percent of the yield of the $5/2^+$ state.

The order of the states associated with the $d_{5/2}$ and $s_{\frac{1}{2}}$ orbits in Py^{13} and in F^{19}, with $\frac{1}{2}^+$ below $5/2^+$, is opposite to that in Py^{17} in which the ground state $5/2^+$ is more than $\frac{1}{2}$ Mev below the $\frac{1}{2}^+$, and this seems like an anomaly if one tries to push the single-nucleon interpretation too far. The states of the system are probably most nearly like single-particle states in Py^{17}, in which the O^{16} "core" is in its ground state a filled p shell, not just a filled $p_{\frac{3}{2}}$ shell of (jj) coupling. As is discussed further below, the excited states of O^{16} not only begin at the high level 6 Mev, but the first of them is a O^+ state which would not change some properties if it were admixed and probably involves two-nucleon excitation which would greatly inhibit its being admixed, and the second excited state of the core has opposite parity which would probably inhibit its being admixed. Thus we may judge that the single-electron state $d_{5/2}$ lies below the $s_{\frac{1}{2}}$, at least in the potential well encountered in Py^{17}. It is possible that the potential well of the others would differ enough from this to change the order of the single-nucleon states, but it does not seem necessary to conclude that this is so, for the deviations from resemblance to single-particle states encountered in intermediate coupling, even if not very pronounced, could cause such an inversion of order. In Py^{13}, the resemblance of the $5/2^+$ and $\frac{1}{2}^+$ states to single-nucleon states in intermediate coupling depends on the integrals L_d and K_d being small. While no calculations have yet been made on the subject, it seems plausible that they could be of such a magnitude as to make enough admixture of excited states of the core to invert the order, and yet leave enough appearance of single-particle states to contrast strongly in the scattering and stripping results with the $\frac{3}{2}^-$ state in which

there is no distinction between core and added particle in the very thoroughly amalgamated situation of intermediate coupling.

If the single-nucleon state $d_{5/2}$ lies a little below $s_{\frac{1}{2}}$ (by an amount considerably less than the spin-orbit contribution to $d_{5/2}$) at the actual strength of spin-orbit coupling parameter a, as we may judge it does from O^{17} (neglecting any difference between shapes of potential wells), then it lies much lower than $s_{\frac{1}{2}}$ in extreme (jj) coupling attained by making a large and higher than $s_{\frac{1}{2}}$ in (LS) coupling with a small. Since also in C^{13} in both extremes the even nuclear states resemble single-nucleon states, the nuclear $5/2^+$ state would thus likewise lie above the $\frac{1}{2}^+$ state of C^{13} in (LS) coupling and below it in (jj) coupling. There is thus a cross-over of the intermediate-coupling curves somewhere between the two extremes, and it appears from the observed order of the levels in C^{13} that the cross-over occurs at a value of a/K higher than the actual one, as suggested by the broken lines in the lower part of Fig. 13 (where only the slopes of the (jj) asymptotes have been calculated). These sketched intermediate-coupling transitions for the $5/2^+$ and $\frac{1}{2}^+$ states are purely schematic. They are not properly functions of a/K but of a with K fixed, since otherwise the relation between K and the single-nucleon s, p, d energy differences would come into play.

The $\frac{3}{2}^-$ state of N^{13} has been interpreted elsewhere in papers which place too much emphasis on the closed-shell nature of the "$p_{\frac{3}{2}}^8$ core" (K51, J51), as arising from an anomalous depression of the next higher single-nucleon $p_{\frac{3}{2}}$ state, the $2p_{\frac{3}{2}}$ which in the oscillator model belongs to the same degenerate group as the first ^2f. The intermediate-coupling interpretation seems far preferable to pushing the single-nucleon model so far as to tolerate such a glaring exception to the usual order, especially in view of the small natural width of the $\frac{3}{2}^-$ level.

In this polyad there are only two known states in the ground configuration, so there are no verifiable relations between energy intervals to confirm the intermediate coupling scheme and one can make few predictions beyond the value $J=\frac{3}{2}$ for the first $T=\frac{3}{2}$ state, but the assignment data available do fit nicely into the intermediate-coupling scheme.

Two Methods of Assigning J's and Parities in C^{13}, N^{13}, and Other Nuclei

There are two new methods, one of which is conceptually new and the other of which has been only recently applied, to obtain angular momentum and parity assignments especially effectively in nuclei formed by adding one nucleon to a nucleus having $J=0$. The first is the analysis of the forward portion of the angular distribution of a (d, p) or (d, n) reaction, first made by Butler (B51). Attempts made earlier to obtain information concerning nuclear states from the angular distribution of (d, p) reactions failed to achieve unique assignments because the solidly backed targets used (H48) did not permit observation near the forward direction, and the analysis of the data (Di50) was too greatly complicated by the necessity of simultaneously making assignments to the rather numerous states of the compound nucleus contributing to the reaction.

It was found in this connection by Rotblat and his coworkers (R51) that at fairly high bombarding energies the (d, p) and (d, n) reactions have a strong forward or near forward peak in their angular distribution which may plausibly be attributed to a stripping process similar to that discussed earlier by Oppenheimer and Phillips in the case of heavy nuclei where barrier penetrability is more important. The process is pictured as the entry of one of the nucleons of the deuteron into the target nucleus while the other nucleon of the deuteron flies on its way, influenced in its direction and energy of flight both by the center-of-mass motion and the internal motion of the deuteron which it formerly shared. The way in which the internal motion is shared is determined by the fact that the final nucleus has very strong preferences for the energy and angular momentum of the nucleon with which it may be formed, and these preferences reflect the properties of the states which one wishes to ascertain. The beauty of this process is that it does not involve formation of a compound nucleus and the attendant complexity of trying to determine its properties at the same time.

The peak observed for a given reaction near the forward direction is so sharp that its analysis in spherical harmonics involves partial waves of several units of angular momentum, so high as to represent passage of the incident deuteron with quite a large collision parameter (or extrapolated distance of closest approach). This is possible only because of the large size of the deuteron and permits entry of only one of the nucleons into the target nucleus. Close enough collision for formation of a compound nucleus involves only about two or less units of angular momentum and consequently displays rather broad angular maxima spread out over all angles with rather low intensity.

The powerful analysis of Butler treats the various angular partial waves of the incoming beam, and determines the fate of each by joining incoming and outgoing wave functions in an annular region just outside the nucleus, before superposing the effects of the various outgoing waves. The results are fairly insensitive to the choice of the radius at which the joining is carried out, corresponding to the expectation that there is a region where the nucleon about to enter the nucleus is outside the influence of the deuteron potential, in the wings of the deuteron wave function, and not yet strongly attracted by the nuclear potential. The result of the analysis is to relate the shape and position of the near forward or forward peak to the angular momentum l_n with which the incoming nucleon is captured by the target nucleus. If the target nucleus is in a 0^+ state, as

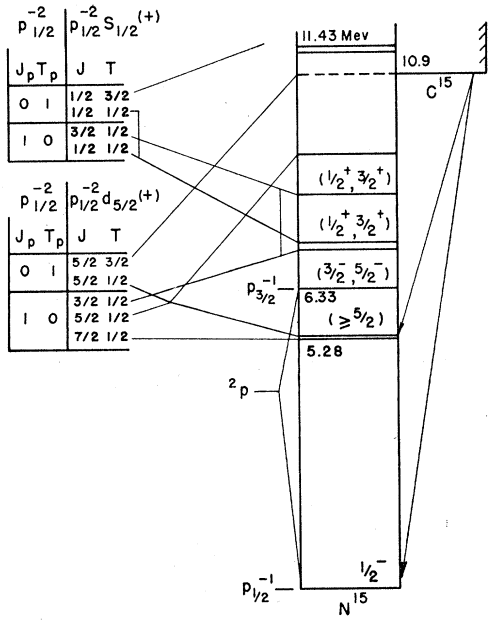


FIG. 14. (jj) configuration assignments to the states of Py^{15}. The "partial configurations" from which they arise are indicated in the left-hand columns, J_p and T_p being "partial quantum numbers" (of the two $p_{\frac{1}{2}}$ nucleons) which do not remain good quantum numbers in the complete system.

in the case of $C^{12}(d, p)C^{13}$, then the parity of the final nucleus, in the state being observed by selection of proton energy, is odd or even as l_n is odd or even, and the angular momentum of the final state is $l_n \pm \frac{1}{2}$. Thus in this most favorable case, J is not assigned uniquely but merely limited to a choice of two values, as one sees them listed in Fig. 1 and elsewhere.

A similar analysis based on the Born approximation, and thus differing radically from Butler's in the nature of its simplifying assumptions, has recently become available (Bh52, D52). It agrees completely with Butler's in providing assignments based on the position of the near forward peak, and seems to agree better with experimental results concerning the amplitude of the secondary maximum further from the forward direction.

There is a very similar phenomenon of pick-up of a nucleon from a nucleus by an incident particle in such reactions as $Be^9(d, t)Be^8$, characterized by a similar near forward peak, to which a similar analysis applies (B51, B52).

The second method which has only recently been extensively used to make assignments to the states of such nuclei as N^{13} is the detailed observation and analysis of the shape of elastic scattering resonances, as is possible now with the very nearly monoenergetic proton beams of a modern, well-controlled statitron. Most of the low states of light nuclei are far enough apart that the scattering may be analyzed in terms of the single-level scattering formula. The phase shift of the effective angular partial wave depends, in its Coulomb part on the orbital angular momentum l of the partial wave, in its resonant part on the "width" of the resonance, and the amplitude associated with this phase shift depends also on j. These properties of the scattered proton determine the parity and angular momentum of the state of the compound nucleus causing the resonance, in the simple case in which the target nucleus is 0^+. Probably because in a real "close-shell-plus-one-nucleon" nucleus such as O^{17} this also determines the value of the total orbital angular momentum vector L for the nucleus, these resonances have customarily been designated with symbols for nuclear multiplet levels, such as $P_{\frac{3}{2}}$, with a capital "P" which in atomic spectra implies that $L=1$, but these symbols are meant strictly merely to indicate parity and angular momentum. Thus we use, for example, $\frac{3}{2}^-$ in place of the $^2P_{\frac{3}{2}}$ of the experimental papers.

$C^{14}+N^{14}$

For this polyad we return to the left side of Fig. 5, since the configuration p^{-2}, or two nucleon "holes" in the p shell, has some similarity to the configuration p^2 discussed for Py^6. The order of the two low states of Py^{14} is that given by (jj) coupling, and the ratio of this interval to the interval up to the second state with $T=1$ is given by the transition curves at $a/K=-5.6$. Here a is the spin-orbit coupling parameter for a "hole," everywhere else in this paper it is for a single nucleon, since only here have we plotted the results for nucleons and holes on the same graph and used the "hole" nomenclature in labeling the (jj) configurations.

The prediction concerning energy levels is that there should be two even levels from this configuration in N^{14} somewhere in the region near 5 or 6 Mev, having $J=1$ and 2 in that order. Since odd parity has tentatively been assigned to most of the newly observed levels in that region and above (Be52), it appears that these two even levels may be the ones at 3.96 and 5.1 Mev, each about 1 Mev lower than given by the theoretical treatment. It would not be at all surprising if there should be another even level in the neighborhood of 6 Mev corresponding to the 6.05-Mev state in O^{16} as discussed below, arising from the doubly-excited configuration p^8d^2, and this might be the one at 5.7 Mev. Otherwise the levels in this region and immediately above are expected to arise from the singly-excited configurations p^9d and p^9s. The fact that they start at 4.8 Mev, so far as is well established, is consistent with the energies at which they begin to appear in the neighboring nuclei, with which no close agreement can be expected because of the complexities of intermediate coupling. There is no obvious reason that there should not be an odd $T=1$ level as low as or lower than the one we have assumed to be even at 8.42 Mev, and indeed recent observations (privately communicated by T. Lauritsen) suggest that there may be a $T=1$ level, which may be 0^- or 0^+ and is not shown in Fig. 1, at about 7 Mev.

One of the most puzzling facts known in light nuclei is the archeologically useful but apparently anomalous long life of C^{14}. Until recently the hope was entertained by those most interested in the theory of beta-decay that the puzzle could be relegated to those interested in nuclear structure by assuming (G51) that the C^{14} and N^{14} ground states have opposite parity, or else (F49) that the C^{14} state is a pure ^1S and the N^{14} ground state a pure ^3D. Both assumptions have been disproved by recent observations. The former possibility has vanished with the experimental proof, by means of the stripping-reaction technique, that the $T=0$ and 1 states have the same parity (Br52), and the latter possibility would require an isolated case of extreme (LS) coupling, with the order of the two low multiplets reversed from that in Fig. 5, and would lead one to expect two very low excited ^3D states of N^{14} that have escaped careful observation (Va52). The lifetime is about 10^6 times longer than expected. Either one must expect to find here some new principle of nuclear structure or of beta-theory to explain this, or one must accept it as chance cancellation in a matrix element to an accuracy of perhaps one part in 10^3. A calculation of the nuclear matrix element in intermediate coupling is discussed in detail in Sec. 11. It is there shown that a factor of about $10^{-\frac{1}{2}}$ in the matrix element, or 10 in the lifetime, appears naturally from the magnitude of the intermediate-coupling coefficients in the N^{14} wave function, but that the fortuitous cancellation required to give the remaining factor $10^{-5/2}$ in the matrix element does not occur in the ground configuration and must in this interpretation arise from a moderate amount of configuration mixing.

$C^{15}+N^{15}$

Here the ground (jj) configuration $p_{\frac{1}{2}}^{-1}$ has only one state $(J, T)=(\frac{1}{2}, \frac{1}{2})$ and the only excited (jj) configuration within the general configuration p^{-1}, namely, $p_{\frac{3}{2}}^{-1}$, has only one state, these two states being identical with the (LS) description of the two states $^2P_{\frac{1}{2}}$ and $^2P_{\frac{3}{2}}$ of a nucleus with a single p-nucleon "hole." Thus for this polyad the intermediate-coupling transition of the ground configuration consists merely of two straight lines. Experimental assignments of parity and not quite unique assignments of J have been made (B51) but they should be considered to some extent tentative because the observation of the near forward peaks did not cover the angles 0–15°. The (unresolved) pair of states at 5.3 Mev is exceptional in showing an approximately isotropic distribution with no forward peak, presumably indicating large angular momentum making possible a very large cross section for compound-nucleus formation.

A set of assignments of excited (jj) configurations consistent with these tentative experimental assignments, is shown in Fig. 14, and a very schematic representation of the intermediate-coupling transition of these states is shown in Fig. 15. The curvature of the lines suggests the direction of energy repulsion as the pairs of states with the same (J, T) are intermixed. The excited configurations are divided up into partial configurations mainly for the sake of convenience in counting the states by vector addition. For example, $p_{\frac{1}{2}}^{-2}(0)s_{\frac{1}{2}}$ in Fig. 14 indicates the $J=0$ state of the partial configuration $p_{\frac{1}{2}}^{-2}$, compounded with an $s_{\frac{1}{2}}$ nucleon. Since for $p_{\frac{1}{2}}^{-2}$, $J=0$ is associated with $T=1$ (as in the first excited state of Py^{14}), the addition of an $s_{\frac{1}{2}}$ nucleon with its $t=\frac{1}{2}$ gives states of the system with $T=\frac{3}{2}$ and $T=\frac{1}{2}$, each of course with $J=\frac{1}{2}$, as indicated in the upper left part of Fig. 15. The right column of Fig. 15 lists these assignments of angular momentum and parity, which have been arranged to agree with the experimental possibilities indicated in parentheses above the levels in Fig. 1. The only excited state observed to have negative parity, at 6.33 Mev, is, of course, assigned to the $p_{\frac{3}{2}}^{-1}$. This state does not mix in higher order with the states of opposite parity listed in the figure, but only with higher states obtained by double excitation. This 6.33-Mev excitation is thus the most direct measure we have of the doublet splitting of a p nucleon in a stable nucleus. Of the $T=\frac{3}{2}$ states, it is not clear from the available configuration assignments whether $J=5/2$ or $J=\frac{1}{2}$ should lie lower. The C^{15} beta-transition to the ground state is probably to be considered first forbidden ($\log ft=5.3$) (Fe51) and there is evidence (Hu50) for a transition to the 5.3-Mev or 6.33-Mev level, which to compete with the ground-state transition must be allowed. This would favor $J=5/2$ for the lowest $T=\frac{3}{2}$ state, as drawn in Fig. 14.

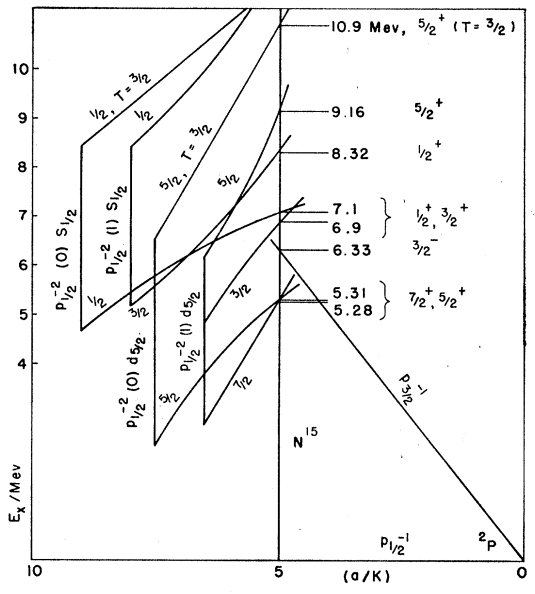


FIG. 15. Schematic intermediate-coupling transition for Py^{15}, including the first excited configurations. The numbers in parenthesis, (0) or (1), denote J_p.

The Closely Spaced Pairs in N^{15} *and Other Nuclei*

The grouping of the first two excited states into a very closely-spaced pair, with a separation of only 30 kev after a gap of 5.3 Mev and followed by a gap of 1 Mev, is very striking. It is particularly striking in combination with a similar close pairing of the first two excited states in the neighboring nucleus, O^{16}. Similar close pairing, though not of first excited states, has been noted in B^{10} and in B^{11}, where it was considered to be fortuitous and not very unlikely because of the density of nearby levels, and in N^{13}, where the accidental nature of the pairing is attested by the wider spacing of the levels in the mirror nucleus C^{13}. In the interpretation of Fig. 15 the close pairing in N^{15} is fortuitous. The chance that the first two levels should be only 30 kev apart in a spectrum starting off with averaging spacings of $\frac{1}{2}$ Mev is about 1/15. The chance in O^{16}, where the spacing is 80 kev, is about $\frac{1}{6}$. The chance that this should happen in two specified successive nuclei of which we have high resolution data is about 10^{-3}. The chance that it should happen in any two successive light nuclei out of about eight for which we have high resolution data is about 10^{-2}. But it is strange that it happens in the two with the highest first excitation energy—the probability of this happening fortuitously is about 10^{-3}. The unpleasantness of this is not mitigated by evidence to be cited below, concerning observed parities, that the close pairing in O^{16} is indeed fortuitous. Under these circumstances, one grasps at straws.

The most obvious assumption (I50) would be that the close pairs are caused by the coupling of an excited s-nucleon to the angular momentum vector of an almost filled p-shell, and that this coupling is very weak for some reason associated with the inadequacy of the phenomenological Hamiltonian (2). But in this case the two paired levels would have the same parity, so it does not apply to O^{16}. In N^{15}, this would mean modifying Fig. 15 to make the two states with $J=\frac{3}{2}$ and $\frac{1}{2}$ of the partial configuration $p_{\frac{1}{2}}^{-2}(1)s_{\frac{1}{2}}$ lie lowest and, in spite of their curvature which indicates their impurity, very close together. These J's do not agree with the tentative experimental assignment, $J\geqslant 5/2$. An artificial way to satisfy this assignment and explain the pairing is to assume weak intershell coupling in the configuration $d_{5/2}^{2}(5)p_{\frac{3}{2}}$ arising from two-nucleon excitation, in spite of the fact the partial configuration $d_{5/2}^{2}(5)$ does not contribute to the low excited states of either O^{16} or O^{17}, as indicated in Figs. 16 and 17. The experimental situation is too uncertain to justify pursuing such conjecture.

While nature seems to be trying to teach us something by the frequency of closely spaced pairs, we have not yet learned it and tentatively ascribe the pairs in N^{15} and O^{16} to chance. The relation between these close pairs for which we have no causal explanation within the framework of the present analysis, and the close pairs observed in the odd-odd nuclei Na^{24}, Al^{28}, and P^{32}, for which an explanation is available involving an expected weak coupling of an s-nucleon spin to another j is discussed further in Sec. 9.

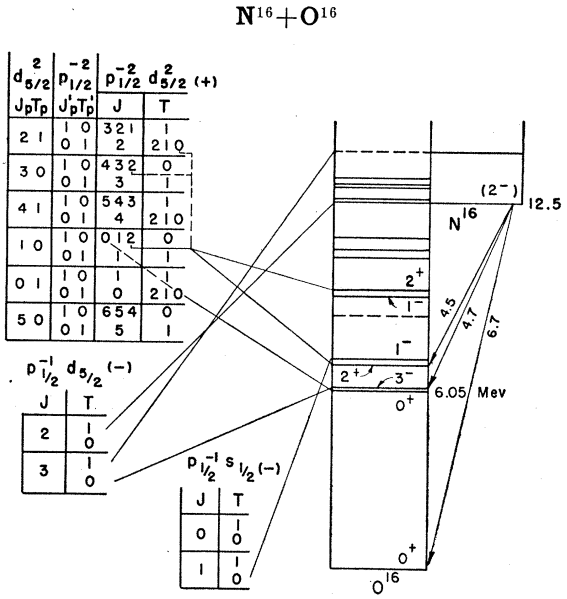


FIG. 16. (jj) configuration assignments to the states of Py^{16}.

Both here and in Py^{15}, the exceptionally high first excitation energy attests the influence of shell structure at the closing of the p shell. Another remarkable feature of the spectrum is the clustering of the first four excited states into two closely spaced pairs, the first pair separated by 80 kev (only a little more than 1 percent of the excitation energy) and the second pair by 200 kev (the same as between two states at the same excitation in N^{15}!) after a gap of about 800 kev.

As a result of recent careful studies of angular correlations we are fortunate in having experimental assignments of these first four excited states. The first has long been known to be 0^+ because of the fact that it does not emit gammas ($0^+\rightarrow 0^+$ forbidden, the ground state of course being 0^+), and thus has time to emit electron-positron pairs. The second excited state has been shown by independent work in two laboratories (A50, B50) to be 3^-. Recent work at Cambridge (S52), mainly on alpha-gamma correlations, has identified the third and fourth excited levels as 2^+ and 1^-, respectively [although an earlier report (Fr51) on this work had given the preliminary result 2^+ and 2^- for these states]. It is possible that experimental clarification of further details will show that there is here very strong evidence for the validity of the alpha-model in this particular nucleus, which is particularly suited to this because it can be built as a compact cluster of four alphas (D40). This possibility is discussed further in Sec. 10. Pending further evidence, we wish, however, to view this nucleus here as a consistent part of a

sequence of nuclei to which the central model seems to apply.

Despairing of the hope of finding in the central model any simple cause for the close juxtaposition of states of opposite parity, as mentioned above, we regard the pairing as accidental and make the configuration assignments indicated in Fig. 16. The great complexity of the situation is typified by the fact that the first four excited states are ascribed to three different (jj) configurations, either with a $p_{\frac{1}{2}}$ nucleon excited to a $d_{5/2}$ state or to an $s_{\frac{1}{2}}$ state or with two $p_{\frac{1}{2}}$ nucleons excited to $d_{5/2}$ states. The perhaps rather surprising suggestion that the first excited state should result from two-nucleon excitation was first made by M. G. Mayer, who based it on the result of the short-range or δ-function approximation (M50) which states that two equivalent nucleons with large j have a large "pairing energy" when their j's add up to $J=0$. While the δ-function is not a good approximation (K50) in so small a nucleus as O^{16}, this result reflects the fact that this state has a high symmetry (upon nucleon exchange), and a high symmetry is doubtless one of the reasons why the $J=0$ and $J=2$ states of the two-nucleon excitation configuration have such low energy as to appear among the first four excited states. Another reason is that there is for each of these J's more than one state with the same (J, T), as indicated in Fig. 16, and the intermixing of these states pushes the lowest one down. It is a characteristic of our intermediate-coupling interpretation of the spectra of other light nuclei that a configuration is spread out over a wide energy region, and this doubly-excited configuration is so complicated, including very many states, that it is spread out over a very wide region and some of its states are low, one of them even lower than any of the states of the simpler singly excited configurations.

Note added in proof.—One would hope for some simplicity in the excited configuration $p_{\frac{1}{2}}^2 d_{\frac{5}{2}}^2$, arising from the expectation that the p–d exchange integrals are smaller than the p–p or d–d integrals. In this approximation one would expect to be able to estimate which states have the lowest energy from the energies of the two-nucleon partial configurations $p_{\frac{1}{2}}^2$ and $d_{\frac{5}{2}}^2$ separately, as is indeed implied in the above remarks of pairing energy in the δ-function approximation. Recent calculations of the energies of these partial configurations separately, by Flowers and Edmonds (Fl52), show that for reasonable ranges instead of the δ-function approximation, the next-to-lowest line of the two-nucleon excited configuration in Fig. 16 should lie lowest, that is, the first states of this configuration should be a group of three with $J=4$, 5, 6, $T=0$. We are left with no central-model explanation of the O^+ first excited state of O^{16}, which enhances the attractiveness of the alpha-model as an alternative. The possibility needs further investigation that the p–d exchange integrals might be responsible. The calculated result just quoted is dependent on the phenomenological representation (2) of the nuclear interactions. It is closely related to the explanation of the low state $J=3$ of $p_{\frac{3}{2}}^{-2}$ in B^{10}, and it is not apparent how to alter the interaction assumption to keep that experimentally satisfactory result and still obtain $J=0$ (rather than $J=5$) lowest in $d_{\frac{5}{2}}^2$. The intervention of the $(J, T) = (5, 0)$ state as the lowest state is due to the interplay of an excited proton and neutron along with a pair of either. If we were concerned with excitation only of a pair of neutrons or a pair of protons, it would have been (0, 1), as was assumed above. There are other phenomena which would be clearer if neutron and proton excitation should be understood to act more nearly separately than we here expect them to. These phenomena are mentioned at the end of the discussion of the (jj) double levels of Na^{24}, about four pages below [see also (SG53)]. Some of these difficulties suggest that consideration of the deformability of the potential well which shapes the single-nucleon orbits (Ga53, W53) may provide a fruitful extension of this discussion of light nuclei.

The beta-decay to each of the (unresolved) pairs of excited levels is allowed, while that to the ground state is forbidden. (The log ft values appear on the transition lines in Fig. 16.) This suggests 2^- for the ground state of N^{16}, and such a state is available from one of the singly excited configurations, as indicated in the figure. It is satisfactory that the configuration $p_{\frac{1}{2}}^{-1} d_{5/2}$ should appear below $p_{\frac{1}{2}}^{-1} s_{\frac{1}{2}}$ among the $T=1$ states (as well as the $T=0$ states), both because one expects it to cover a wider energy spread and because the single-nucleon state $d_{5/2}$ appears according to the evidence of O^{17} to lie below $s_{\frac{1}{2}}$. It is further satisfactory that the $(J, T) = (2, 1)$ state should lie below the (3, 1) state of $p_{\frac{1}{2}}^{-1} d_{5/2}$, because it agrees with a general rule (K52b) that, when a neutron hole, for instance, is coupled to a proton, the lowest state formed has $J=j_1+j_2-1$, that is, a total angular momentum quantum number one less than the maximum possible. Most examples are found with neutrons and protons in the same shell (for which the result follows for both the long-range and short-range extreme theoretical calculations, and presumably between), but the case of $_{19}K^{40}$ is very similar to the present case, configuration $d_{5/3}^{-1} f_{7/2}$, with two different shells involved. That the lowest state should be $J=j_1+j_2-1=4$, in agreement with observation, in this case with reasonable assumptions has been shown by Kurath (K52b), and similarly $J=2$ for N^{16}. There are only two states of this configuration in N^{16}, so the (jj)-coupling statement of which should be the lower should probably have significance also in intermediate coupling.

As we have suggested in connection with Be^8 above, the existence of states of the same symmetry in the alpha-model may influence the spacings and properties of the states encountered in the intermediate-coupling scheme, and O^{16} is a favorable place for this to happen because the first four excited states just discussed have

their counterparts among the low excited states of the alpha-model (Sec. 10).

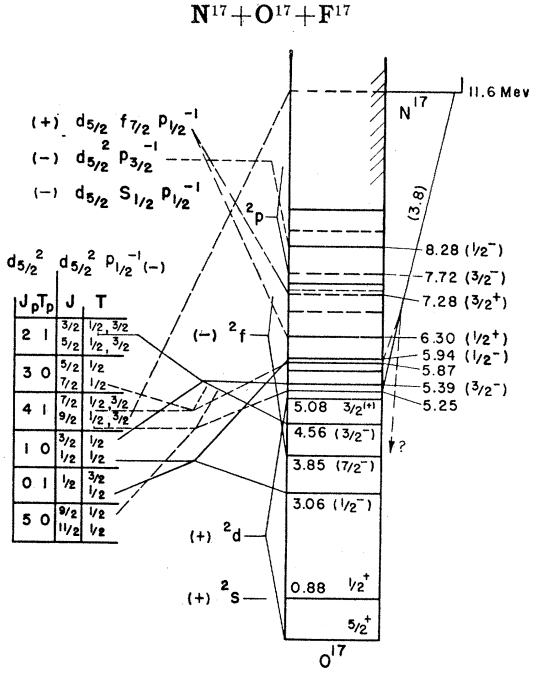


FIG. 17. (jj) configuration assignments to the states of Py^{17}.

In most of the low states of this polyad we have essentially a single nucleon outside of a closed p shell, and the existence of several states for this relatively loosely bound nucleon contributes several levels of moderate excitation energy, as shown in Fig. 17, in strong contrast to the closed-shell appearance of the O^{16} spectrum. The second excited state at about 3 Mev, however, does not fit into this single-nucleon scheme, and must be ascribed to an extra excitation from one shell to another, $p_{\frac{1}{2}}$ to $d_{5/2}$, similar to that responsible for the first excited state of O^{16}, the resultant low energy again resulting from "pairing" in the partial configuration $d_{5/2}^2$ (as was also suggested by Mrs. Mayer). In this case the excitation energy is lower because excitation of only one nucleon is necessary, to pair with a $d_{5/2}$ nucleon that is already there in the ground state. It is not so remarkable here as in O^{16} that this state lies low because here it arises from single-nucleon excitation, as do the other low excited states. It is more complicated than the others because the possibilities of vector addition give two $\frac{1}{2}^-$ states in this configuration and they are mixed to give the 3.09-Mev state as indicated in Fig. 17.

Note added in proof.—This interpretation, too, is cast in doubt by the calculated result (F152) that $(J_p, T_p) = (5, 0)$ lies lowest. Sufficient repulsion of the two $(J, T) = (\frac{1}{2}, \frac{1}{2})$ states again requires large p–d exchange integrals.

Partly by comparison with a state in F^{17} identified by a proton scattering resonance, the O^{17} state at 3.85 Mev is identified as the $7/2^-$ state corresponding to the single-nucleon state $f_{7/2}$, which it is very satisfactory to find here because it is expected to be the next single-nucleon state after the $d_{5/2}$ ground state and the $s_{\frac{1}{2}}$ just above it. Above this, the identifications are made by observation of the scattering resonances of neutrons on O^{16}. In the case of the 5.08-Mev state, there is some question whether it should be interpreted as an exceptionally strong $\frac{3}{2}^+$ resonance [with γ^2 about $\frac{1}{4}$ of the theoretical limit, according to Table I(17) of reference (Aj52)] or a $\frac{3}{2}^-$ resonance about as strong as the next few states above it (about 0.03 of the limit). The former possibility, $\frac{3}{2}^+$, is favored, partly because there is another level identified as $\frac{3}{2}^-$ only 50 kev below it and it does not seem likely that two such levels from the first excited configuration would occur so close together, but mainly because an exceptionally strong level $\frac{3}{2}^+$ is expected somewhere, namely, the single-nucleon-like $d_{\frac{3}{2}}$ state which is needed to join the ground state, essentially a $d_{5/2}$, to complete a ^2D. The single-nucleon-like level is expected to correspond to a much stronger resonance than the levels involving excitation of the "core," and this is the only level with the appropriate characteristics in the region up to above 8 Mev that has been properly explored.

Several levels next above this one have moderately weak resonances, apparently because they arise from excited configurations, and possible configuration assignments for them are suggested in Fig. 17. Above the $^2f_{7/2}$ single-nucleon state one expects to find the ^2p levels (with radial quantum number 2 corresponding to one radial node), first the $^2p_{\frac{3}{2}}$ and above it the $^2p_{\frac{1}{2}}$, and if one were to ignore the weakness of the resonances one might think to find the corresponding O^{17} states at 5.39 and 8.38 Mev, which would make a ^2p splitting of about 3 Mev related to the 5 Mev ^2d splitting roughly in the ratio of their $(2l+1)$. Because the single-nucleon-like levels corresponding to the ^2p are expected to be exceptionally strong, it seems preferable to recognize that these single-nucleon levels may be spread quite far apart in this nucleus, and to identify the strong level at 7.72 Mev with the $^2p_{\frac{3}{2}}$, as is done in the figure.

The beta-decay of N^{17} is observed under conditions of high energy bombardment and consequent strong contamination of other beta-decays, through the unusual circumstance that it gives rise to simultaneous neutron emission ("delayed" because it follows the beta) from excited states of O^{17}, the betas and neutrons being observed in coincidence (A49). The neutron energy measurements (H49) seem to indicate an intense transition to an O^{17} level at about 5.26 Mev ($E_n \approx 1.05$ Mev) and a weaker transition to a level at about 5.89 Mev ($E_n \approx 1.65$ Mev), which we take to be the levels at 5.25 and 5.87 Mev for which no identifications have been given because they have not been resolved in the neutron scattering experiments. It is not clear

to what extent they may have contributed to the neutron scattering and whether they may have falsified the identifications of the adjacent states from which they were not resolved, but it is possible that these states may have high J's and their contributions to the scattering may have been suppressed thereby. There are many other states of both parities and low J's in this region through which the $(\beta+n)$ process does not seem to go (though the neutron energy resolution is not good enough to make this certain). Therefore, it seems likely that these two states differ from the others in having high J's, and that the ground state of N^{17} also has high J. Either the $(J, T)=(7/2, \frac{3}{2})$ or $(9/2, \frac{3}{2})$ state of the configuration $d_{5/2}{}^2p_{\frac{1}{2}}{}^{-1}$, assigned as the ground state of N^{17}, could accomplish this selection, and one could distinguish between these possibilities if it could be observed whether or not the beta-transition also goes to the $5/2^+$ ground state of O^{17}. The latter has been indicated among the tentative configuration assignments suggested in Fig. 17, along with similar suggested assignments from the same configuration for the presumed delayed-neutron states at 5.25 and 5.87 Mev. The favored beta-transition indicated by the value $\log ft=3.8$ is valid only if there is no transition to lower states, and is consistent with this assignment of initial and final states of the beta-decay to the same configuration.

8. COULOMB ENERGY IN THE CENTRAL MODEL

The central model, of which the (jj) model is a special case, is a model in which states of the system are formed of individual-nucleon wave functions which are solutions of a wave equation with a spherically symmetrical potential. The state of the system does not necessarily possess central symmetry. (With a statistical distribution of the direction of J, the over-all state has central symmetry, but not the magnetic substate.) This lack of central symmetry, which implies a correlation between the angular coordinates of two nucleons, is in part responsible for the multiplet separations in (LS) coupling and the separations within a (jj) multiplet in (jj) coupling. The Pauli asymmetry, which also implies a correlation of a different sort between the positions of the nucleons, is also and usually even more heavily responsible for these separations.

The lack of central symmetry may be described in terms of a flattening of the orbits of two nucleons almost into one plane when their orbital angular momenta $\boldsymbol{l}$ are nearly parallel or antiparallel (such as when $\boldsymbol{L}=\boldsymbol{l}_1+\boldsymbol{l}_2$ is as large or small as possible), as compared to a closer approximation to spherical symmetry in states with intermediate values of L [or of J in (jj) coupling]. For example, the ^1S lies below the ^1D in the middle of Fig. 5, although they both have the Pauli antisymmetry in the spin factor, and this energy difference may be associated with the picture that the antiparallelism in the ^1S is exact, the vector sum $[L(L+1)]^{\frac{1}{2}}$ is exactly zero,

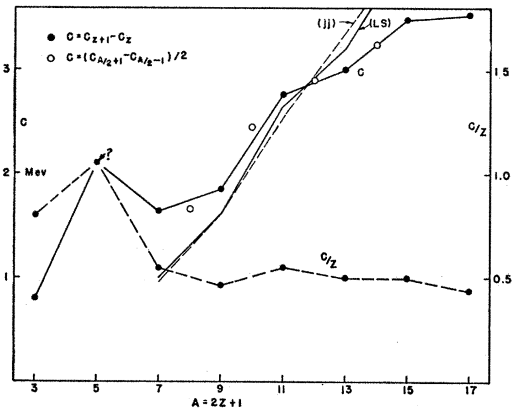


FIG. 18. The Coulomb energy differences of the mirror isobars. The curves are drawn between the points for the odd isobars only. The Coulomb energies differences per unit charge difference for the even isobars are indicated by open circles. The theoretical curves for (LS) and (jj) coupling assume a constant Coulomb exchange integral K_c.

whereas the corresponding parallelism in the ^1D is not exact, the two vectors of "length" $[l(l+1)]^{\frac{1}{2}}$ adding up vectorially to less than their arithmetic sum.

The Coulomb energy differences between the mirror nuclei are also affected by these considerations of nuclear "shape" and other correlations between the positions of the nucleons, and for this reason it is only an approximation to consider simply the electrostatic energy of a spherically symmetric charge cloud. The effect of the correlation considerations is, however, quite small. The Coulomb energy differences of the neighboring mirror pairs of isobars have been calculated both in (LS) coupling (F37) and in (jj) coupling (K52), and they are plotted in Fig. 18. (There the scale is quite arbitrary, the differences taken from Kurath's Table III (K52) being plotted with $17K_c=1$ Mev, K_c being the Coulomb exchange integral.) The experimental curve for the Coulomb difference C has a meaning for the p-shell only for $A=7$ and up, since $A=5$ is an anomalous case in which the ground states of both isobars are virtual. This portion of the experimental curve shows a rather pronounced four-structure, an alternation in slope with the points for $A=4n+1$ lying relatively low. The calculated (LS)-coupling curve is plotted in the same figure and shows a much weaker four-structure of the same sort. The (jj)-coupling curve is even more nearly a straight line. The fact that the theoretical curves are much too steep to follow the trend of the experimental curve indicates that K_c should not be taken constant across the p shell, as is done in Fig. 18. Instead it should decrease with increasing A, and a very good fit to the general trend (but with inadequate four-structure) may be obtained by assuming $K_c\sim A^{-1}$. This is exactly the variation one expects on the usual simple assumption of constant nuclear density, and range of interactions small compared to nuclear size, but it is perhaps somewhat surprising that it should

apply so well in the p shell. It indicates that the p shell is to be thought of as a collection of angular wave functions, but that the radial wave functions going with them are determined separately for each nucleus (or even each state), and that for the ground states they are determined in such a way as to keep nuclear density nearly constant, so far as the general trend is concerned.

Thus the four-structure is not explained by the correlations, but may be attributed to small deviations from the constancy of nuclear density, as dependent on the four-structure of the binding energy. The four structure of the Coulomb energy, Fig. 18, consists in the fact that C is abnormally low for the three polyads $A=9$, 13, and 17. Of these, $A=17$ is beyond the p shell, and may be low because a d orbit is larger than a p orbit. That and much more questionably the ground state of $A=13$ are cases where it is perhaps a better approximation than elsewhere to think of a "core" plus one nucleon, and this corresponds to the simple spherically symmetric model used by Bethe in discussing this effect (Be38, E41). The Coulomb difference C is a property of the last nucleon in its interaction with the "core," and its unusually small binding energy in Py^{13} leads one to expect it to have an abnormally diffuse probability distribution and thus an abnormally small value of C. In $A=9$ the experimental "binding energy of the last nucleon" is particularly low, being positive for the neutron in Be^9 but negative (-0.2 Mev) for the proton in B^9. Here it is probably not a good approximation to think of this low binding energy as belonging to any one nucleon, but even when shared among several it may have quite an effect on the nuclear size (contributing to a cooperative phenomenon). The virtual nature of B^9 makes this polyad an atypical case something like Py^5, though the influence of the virtual nature is probably much reduced by the sharing of energy between nucleons in B^9 (and indeed it seems in Fig. 18 that the sign of the influence is the opposite of that in Py^5).

9. THE NEUTRON-PROTON (jj)-COUPLING DOUBLE LEVELS

(jj) Double Levels in Atoms

In atomic spectra the only examples of real (jj) coupling are found in the configurations p^5s and d^9s involving the coupling of an s-electron spin to the j of an almost closed shell. Examples of (LS) coupling and quite frequently of intermediate coupling are found among the cases of coupling of two equivalent electrons. For example, the configuration p^2 shows a nice intermediate-coupling transition in Fig. 4[11] of reference (C35), going from GeI $4p^2$ with a moderately low principle quantum number and displaying a clear separation into the triplet and two singlets of (LS) coupling, and SnI $5p^2$ in which the multiplets may still be recognized, over to PbI $6p^2$ with a high principle quantum number, which is well out into intermediate coupling and even begins to show a grouping into the double-levels characteristic of (jj) coupling. But extreme (jj) coupling occurs only when one of the j's is an s-electron spin, and when the s electron has a high radial quantum number so that its charge distribution overlaps little with the almost closed shell. In the spectrum of un-ionized neon, for example, the four levels of the configuration $2p^53s$ are arranged in a singlet and triplet, with some deviation from the "interval rule" of (LS) coupling, and the $2p^54s$ and successively higher excitations progress across the region of intermediate coupling to (jj) coupling, until finally $2p^511s$ has two narrow (jj) double levels whose splitting is less than one percent of their separation.

(jj) Double Levels in Na, Al, and P

The reason that a high principle quantum number is required in atoms is that the exchange integral, arising from the Pauli antisymmetry of the two electrons, tends to separate the two levels $J=j\pm\frac{1}{2}$. For a neutron and a proton in a similar situation this is not so if the interaction between them does not involve the spin. Beautiful examples of (jj) double levels are for this reason found in the odd-odd nuclei in the region of filling of the second s orbit, in Na^{24}, Al^{28}, and P^{32} (as was recently pointed out by Maria G. Mayer at a conference in Pittsburgh). In the two latter nuclei (E51, E52, V52a), the ground states belong to double levels, the ground configuration of ${}_{13}Al^{28}$ being $d_{5/2}^{-1}s_{\frac{1}{2}}$ while in ${}_{15}P^{32}$ we have $s_{\frac{1}{2}}^3d_{\frac{3}{2}}$, since the $d_{5/2}$ orbit is filled at 14 nucleons and the s orbit at 16. The point is that the nuclear interactions as we treat them phenomenologically are only about one-fifth spin-exchange forces, and the rest does not involve the spin. This factor 0.2 from (2), say, is enough when combined with the fact that the two nucleons do not have the same radial wave function, to explain the small splitting of the double levels. Let us show that the large term not involving the spin gives no splitting. This is most easily done (as was suggested by Mrs. Mayer) by considering the problem of constructing the wave functions for the two values of J in the magnetic sublevel $M=0$, since the two simple product functions with which we start then have the same diagonal matrix element and the splitting is made only by the nondiagonal element between them. For the configuration $s^3d_{\frac{3}{2}}$, for example, let us define $d_{\frac{3}{2},\frac{1}{2}}=a$, $d_{\frac{3}{2},-\frac{1}{2}}=b$ (where the subscripts mean j, m_j) and write the two $M=0$ product wave functions for three neutrons and one proton of P^{32}:

$$\begin{aligned}&\langle a^+\alpha^+\beta^+\beta^-\rangle,\\ &\langle b^+\alpha^+\beta^+\alpha^-\rangle.\end{aligned}\tag{20}$$

The superscripts indicate isobaric spin projection, + for a neutron and − for a proton, α and β imply s-wave functions including these spin functions for spin up and down, the four factors are functions of the coordinates

of the four nucleons, respectively, and the pointed brackets mean "antisymmetrized sum." The matrix element between them is

$$H_{12}=(a^+\beta^-|\,V\,|b^+\alpha^-)-(a^+\beta^-|\,V\,|\alpha^-b^+). \quad (21)$$

The first term is zero because of spin orthogonality for the last nucleon if V does not operate on σ, and the second term is zero because of isobaric spin orthogonality (or another way of saying it is that "we don't have to antisymmetrize between a neutron and a proton"). Thus the levels $J=1$ and $J=2$ are degenerate so far as the main term $0.8P$ of (2) is concerned, and are split only by the small term in the spin-exchange operator Q which spoils the spin orthogonality. (Presumably it hereby represents what a tensor force would do, but this should be calculated.)

The same argument applies with the $d_{\frac{3}{2}}$ excess neutron replaced in excited states by $f_{7/2}$, $f_{5/2}$, $p_{\frac{3}{2}}$ and $p_{\frac{1}{2}}$, each giving an excited double level in P^{32}. In Al^{28} an excited double level may arise similarly from $d_{5/2}\rightarrow d_{\frac{3}{2}}$ excitation of the odd proton, leaving the spin of the s neutron the only neutron vector free to be oriented. In $_{11}Na^{24}$ the ground configuration does not contain an s nucleon [and the observations (Sp52) show the ground state to be single], and in the excited states of this nucleus the formation of the double levels is not quite so simple because there is the possibility of exciting either a proton or a neutron. Here the neutron excess comes into play in a somewhat different way from the way it does in P^{32} to avoid interaction between the states formed by these different types of excitation and leave intact any pairing into double levels that might be caused in some way similar to the same simple mechanism explained above. If we excite a $d_{5/2}$ neutron to the s orbit, we have a partially filled d shell $(d_{5/2})_{\text{prot}}^3(d_{5/2})_{\text{neut}}^4$ which possesses a low state of partial isobaric spin $T_P=\frac{1}{2}$, whereas proton excitation leaves the d shell $(d_{5/2})_{\text{prot}}^2(d_{5/2})_{\text{neut}}^5$, with its lowest $T_P=\frac{3}{2}$ and hence considerably higher energy that cannot be brought down by the weak coupling with an s nucleon. Hence the two excitations may be expected to act fairly independently, and each might give rise to a double level. In such a case we have, however, the coupling of an s-nucleon to a complex partial wave function of a d shell formed by the orientation of the vectors of both protons and neutrons, and it is not at all clear from simple considerations that an exchange integral of the Majorana term should be absent from the splitting $J=J_P\pm\frac{1}{2}$. If the excited nucleon is a neutron, to take the example just discussed, the d shell has an odd number of protons and an even number of neutrons. The fact that double levels appear in Na^{24} suggests that the s neutron is coupled only to the protons, as though the even number of neutrons were somehow suppressed from sharing in the complexity of the wave function of the d shell. This would be analogous in a simpler case to having the two $p_{\frac{3}{2}}$ neutrons of the configuration $p_{\frac{3}{2}}^3$ in the ground state of Li^7 form a partial resultant $J_P=0$ and ascribe all of the angular momentum to the $p_{\frac{3}{2}}$ proton, which is not in keeping with the way the energy of the ground state on the (jj) side of Fig. 6 was calculated. (It would give a Schmidt-line magnetic moment, whereas the magnetic moment calculated with proper neutron participation is close to that observed.)

Thus it is not clear why the even numbers of like nucleons should pair up to make their total angular momentum zero in such cases of only partly filled shells, but the double levels of Na^{24} suggests that they do. There are other indications that this happens in the intermediate and heavy nuclei, a set of probably related but enigmatic "queer facts." There are several cases in which the addition of two neutrons to go from one odd-even isotope to another changes the magnetic moment of a nucleus by only a small fraction of a percent. There are also several cases where the addition of two protons to an even-even nucleus changes the first excitation energy of a nucleus almost none at all.

A (jj) double level in a heavy nucleus. The ground state of the nucleus $_{81}Tl^{208}_{127}$ is a member of a double-level with a splitting of only 40 kev. This nucleus, though very heavy, is simple enough to discuss because of differing from a double closed shell by only one nucleon and one hole, and Pryce (Pr52) attributes the double-level to the configuration $(s_{\frac{1}{2}})_\pi^{-1}(g_{9/2})_\nu$. Its reason for being only 40 kev wide, as compared with about 300 kev for the quadruple-level attributed to $(d_{\frac{3}{2}})_\pi^{-1}(g_{9/2})_\nu$ is the same as for the double-levels in the much lighter nuclei here discussed.

(jj) Triple Levels in Si^{29} and Al^{28}

In $_{14}Si^{29}$ there is a triple level consisting of states at 4.840, 4.897, and 4.934 Mev, separated from the nearest neighbor by $\frac{3}{4}$ Mev (V52a). This may be ascribed to the excited configuration $d_{5/2}^{-1}s^2$, which is obtained from the ground configuration $d_{5/2}^{12}s$ by excitation of one $d_{5/2}$ nucleon into an s orbit. It is consistent with other examples that this excitation should "cost" about 5 Mev. The partial configuration s^2 can have a spin $S_p=1$ which couples to the $d_{5/2}^{-1}$ "hole" to give three states with $J=\frac{3}{2}$, 5/2, and 7/2 (a prediction of which verification would be of interest). Since the partial configuration $d_{5/2}^{-1}$ has only one state, it is consistent with this interpretation that there should be only one such triple level in Si^{29}. In $_{13}Al^{28}$ two triple levels (or more—the density of states is too high to be sure) have been observed at about this elevation, corresponding apparently to the excited configuration $d_{5/2}^{-2}s^2$, which also requires single-nucleon $d_{5/2}\rightarrow s$ excitation, and which could give rise to as many as four triple levels. Above the three fairly low double levels in $_{11}Na^{24}$ there is a region of high level density near 4 Mev which may be considered to contain as many as three triple levels, the narrowest with a total splitting of only 35 kev. In this case two-nucleon excitation to the configuration

$d_{5/2}s^2$ is required to obtain a partial spin $S_p=1$ which might give a triple level.

The reason for the small splitting of these triple levels is not as clear as in the case of the double levels, because the excited configurations required may be attained by either proton or neutron excitation, and the $S_p=1$ state of the partial configuration s^2 has $T=0$. Thus it necessarily involves both protons and neutrons, to be coupled to either protons or neutrons or both in the d shell. Lacking a more elegant suggestion, it is perhaps to be inferred that the different arrangement of nodes of the s and d orbits makes the coupling between them (an integral like K) quite small, suppressing the contribution of the leading term of Eq. (2) to the splitting.

The Possibility of (jj) Double Levels in Li^6, B^{10}, and N^{14}

The situation for the excited configuration $p_{\frac{3}{2}}{}^5s_{\frac{1}{2}}$ in B^{10} has a slight similarity to that in Na^{24}, but differs from it drastically in the fact that this is a "self-mirrored" nucleus with no neutron excess. There is, thus, no energetic tendency to separate the states formed by neutron excitation from those formed by proton excitation. Even if we should invoke a not understood tendency for even numbers of nucleons to pair their j's to a zero resultant, as is suggested by the Na^{24} double levels, we should, because of the interference between neutron and proton excitation, still not expect double levels in B^{10} any more than in Li^6.

The lack of double levels caused by s excitation in Li^6+He^6 is a matter of easy calculation. We now use the notation a and b for the $m_j=\pm\frac{1}{2}$ projections of $p_{\frac{3}{2}}$:

$$\begin{aligned} a&=(2/3)^{\frac{1}{2}}z\alpha+(1/3)^{\frac{1}{2}}[(x+iy)/2^{\frac{1}{2}}]\beta,\\ b&=(2/3)^{\frac{1}{2}}z\beta+(1/3)^{\frac{1}{2}}[(x-iy)/2^{\frac{1}{2}}]\alpha, \end{aligned} \tag{22}$$

where $x\pm iy$ and z indicate the angular part of the p orbital. Then the four $M=0$ states are

$$a^+\beta^-,\quad b^+\alpha^-,\quad \beta^+a^-,\quad \alpha^+b^- \tag{23}$$

the superscripts again indicating the sign of the isobaric spin projection m_t. The secular determinant made from the matrix between these states, for an interaction not involving spin, is

$$\begin{vmatrix} -E & 0 & A & B \\ 0 & -E & B & A \\ A & B & -E & 0 \\ B & A & 0 & -E \end{vmatrix} = \begin{vmatrix} 0 & 0 & A & (B-E) \\ 0 & 0 & (B-E) & A \\ A & (B+E) & -E & 0 \\ (B+E) & A & 0 & -E \end{vmatrix} = 0, \tag{24}$$

where

$$\begin{aligned} B=(a\beta|V|\alpha b)&=(2/3)(zs|V|sz)\\ &=2(a\beta|V|\beta a)=2A \end{aligned} \tag{25}$$

since, because of the vanishing of integrals of odd powers and the equivalence of the three Cartesian coordinates,

$$\begin{aligned} &[(x+iy)^*s|V|s(x+iy)]/2\\ &\quad=[(xs|V|sx)+(ys|V|sy)]/2=(zs|V|sz). \end{aligned} \tag{26}$$

The diagonal elements have been omitted because they differ only in orientation or in the proton-neutron difference and are all equal. The factorization of the determinant into the product of two quadratics, as in the second member of Eq. (24), is accomplished by adding rows and subtracting columns, the third with the second and the fourth with the first. Thus we have the four distinct roots $\epsilon=\pm A, \pm 3A$, a ladder with three equal steps. (One might speculate as to whether the equal spacing between the first, second, and third $T=0$ excited states of B^{10} may indicate that they come from $p\rightarrow s$ excitation, were it not that they are observed to have even parity, and that the four odd theoretical levels have alternately $T=0$ and 1.) Thus the four levels do not fall into double levels. The same analysis applies to the configuration $p_{\frac{3}{2}}s$ of Py^{14} except for an interchange of the factors (2/3) with (1/3), and a sign of the coefficients which does not affect the result.

In B^{10} the two levels at 5.11 and 5.17 Mev appear to form a double level, the splitting between them being only 60 kev and the separation from the nearest other level being 220 kev. The ratio between these is not in itself very convincing. In such a densely populated part of the energy spectrum it is not very unlikely that levels will fall close together by chance. These two levels are both observed, and with comparable intensity, in the forward yield of low energy neutrons from the $Be^9(d, n)$ reaction as the deuteron energy is raised past their respective thresholds (Bo51). Since these are so strong that the first of them constitutes practically the threshold for the reaction, somewhat below 1 Mev, they are thought to involve capture of an s-proton, $l_p=0$, which would give them odd parity and require that their wave functions contain at least a major portion from the configuration p^5s. Thus it is very tempting to suppose that they correspond to the two orientations of an s-proton spin relative to another j or J_P. The fact that the states are prepared in such a way that the odd nucleon is a proton is not reason enough to expect that it remains so, without interference from the state formed by neutron excitation. As long as the interaction with the excited neutron state is represented by an exchange integral capable of making a splitting of the order of $\frac{1}{2}$ Mev or more, as would seem to be the most likely result of a complicated calculation on the basis of what we have said, the

prepared "excited proton state" could not remain undisturbed long enough to define an energy as sharply as 60 kev. Their juxtaposition may be accidental, or it may be related to the smaller than expected coupling suggested by the triple levels.

The Enigmatic Closely Spaced Pair in N^{15}

The double levels here discussed have been in odd-odd nuclei. Their occurrence and explanation does not seem to throw any light on the occurrence of closely spaced pairs in the odd-even nucleus N^{15} (even if we ignore the uncertain identification $J \gtrsim 5/2$ for the lower pair). Here one may excite a neutron to the s state and leave the rest of the nucleus in the $J_P=1$, $T_P=0$ state corresponding to the ground state of N^{14}, and this is not matched by proton excitation, but then J_P is made of a neutron as well as a proton and one would expect exchange splittings from the Pauli antisymmetry between the neutrons. Alternatively, one may form the $J_P=0$, $T_P=1$ state by either proton or neutron excitation to the s state, but then there is only one resultant angular momentum $J=\frac{1}{2}$, and the two isobaric spin states $T=\frac{1}{2}$ and $\frac{3}{2}$ resulting from the interference of the two excitations cannot form a $T=\frac{1}{2}$ double level such as observed at 5.3 Mev in N^{15} (far below the ground state of C^{15} at 10.9 Mev).

10. THE ALPHA-MODEL AS AN ALTERNATE POSSIBILITY AND AS AN OCCASIONAL ADMIXTURE

The intermediate-coupling interpretation presented above is a complex answer to a complicated experimental situation, in which there is no simple grouping of energy levels into consistent patterns of suggestive multiplets, but rather a variety of types of spacings as one compares the various polyads. This does not satisfy a physicist's natural desire for simplicity and elegance. It is probably true but disappointing. In accepting it with hesitance, one wishes to examine any alternative which might present the possibility of a simpler interpretation of the same data. Partly for this reason we discuss here the alpha-model, for which some rather persuasive arguments have been given in the past, before so many excited energy levels of the light nuclei were known, and before the success of the (jj)-coupling version of the central model became apparent in heavy nuclei. We discuss it also because it may actually contribute to the physical properties of some of the states of a few favorable nuclei and modify the properties that they would have in a pure central model, in the sense that the wave functions of the two models may in nature be, as we have already mentioned, intermixed. As an introduction to discussing the alpha-model as an admixture, we discuss it as an alternative, and show that it does not provide a simple and adequate source of the observed energy-level patterns.

TABLE V. Vibrational constants and excitation energies in the alpha-model, on the assumption of a standard harmonic "bond."

	Be^8	C^{12}	O^{16}
c	2	$(3)^{\frac{1}{2}}$	$2(2/3)^{\frac{1}{2}}$
b	1	3	6
n	2	3	4
$(c^2/nb)^{\frac{1}{2}}$	$2^{\frac{1}{2}}$	1/3	1/9
$\omega_1 \sim (bc^2/n)$	$2^{\frac{1}{2}}$	$3^{\frac{1}{2}}$	2
$\omega_2 \sim$		$(3/2)^{\frac{1}{2}}$	1
$\omega_3 \sim$			$2^{\frac{1}{2}}$
$(\omega_1+2\omega_2+3\omega_3)/2^{\frac{1}{2}}$	1	2.96	5.83
0^+	4.38	5.24	6.05 Mev
3^-		7.66	6.13
2^+	6.13	6.13	6.09
2^-			6.09
1^-		5.24	6.45

The Alpha-Model of O^{16}

The simplest and most hopeful application of the alpha-model is to nuclei which could consist solely of alphas, the even-even $A=4n$ nuclei, with no extra nucleons or "holes." Among the light nuclei these are the "dumbbell" or "diatomic molecule" Be^8, the equilateral triangle C^{12}, and the regular tetrahedron O^{16}, and we consider them in reverse order.

Dennison (D40) has shown that the first two excited states of O^{16} in the alpha-model should be the 0^+ state of uniform dilational vibration (which he identified with the well-known 6.05 Mev pair-emitting state) and the 3^- state corresponding to the rotation of an equilateral triangle, which is the base of the tetrahedron, about its threefold axis. This agrees with the experimental identification of the first two excited states. The difference in parity of the first two excited states is a natural consequence of the alpha-model, arising from the difference between dilational vibration and rotation of a body with threefold symmetry. In the central model we were forced instead to the interpretation, which might at first seem somewhat artificial but is probably correct, that the first excited state arises from two-nucleon excitation. Dennison estimated that the 3^- state of the alpha-model should come at about 4 or 5 Mev, dependent of course on an estimated moment of inertia to which the energy is inversely proportional. The mean distance of the alphas from the center was taken to be the same as that of the extra nucleon in O^{17} and F^{17}, as indicated by their Coulomb energy difference. It seems very reasonable to assume instead that this loosely bound extra nucleon has a considerably larger mean radial displacement than do the alphas, which revises the estimate of the rotational energy upward, in a direction to agree with the energy 6.13 Mev of the observed 3^- state.

The next two states in O^{16} also fall close together, though the 200-kev separation is great enough that theirs might more easily be a chance juxtaposition. It is a curious fact that the alpha-model predicts that the third and fourth excited states should indeed fall close

together. They arise from a mode of vibration in which the pairs of corners, like two dumbbells, both stretch and move so their centers of mass approach one another, or vice versa. There are two such modes corresponding to the possibility of reflecting one corner through the opposite face of the tetrahedron, and the levels are split by only the energy corresponding to the frequency of this inversion, which should be small. They have angular momentum and parity 2^+ and 2^-. The recent experimental result of French *et al.* (Fr52) is that they are actually 2^+ at 6.91 Mev and 1^- at 7.11 Mev. This experimental assignment, if final as it seems to be, precludes the possibility that the 200 kev splitting of the levels near 7 Mev is to be explained as the energy of the 2^+-2^- inversion frequency. It is to be noted, however, that this energy is expected by Dennison to be very much smaller than the other separations among the first five excited states, and might easily be too small to have been resolved. There is a third mode of vibration, in which one of the "dumbbells" shortens while the other lengthens, whose first excitation energy is expected to fall rather close to the 2^+ and 2^-, and indeed, is expected according to Dennison's analysis to fall quite near 7.1 Mev if the 2^+ and 2^- both fall very close to 6.9 Mev, as we shall show in more detail. It is, therefore, important for the question of the validity of the alpha-model to determine experimentally whether it might not be possible that the 2^+ and 2^- are very nearly degenerate at 6.91 Mev. Although the angular correlation experiments indicate that this level is 2^+, it must be considered whether it may not merely be predominantly 2^+ in a set of reactions, in which there may be some selection.

Dennison (D40) analyzed the frequencies of vibration in terms of the elastic constants a, for stretching, and b and c for types of bending. In making a comparison with the preliminary data available a dozen years ago, he pointed out that b and c must be much smaller than a and set $c=0$. In fitting the modern data, we determine the moment-of-inertia parameter A by putting $6\hbar^2/A=6.13$ Mev and the first vibration frequency from $\omega_1\hbar=6.05$ Mev. From the estimate $\epsilon_0\approx\epsilon_1/25$, we put $\epsilon_0=0$, which means assuming the 2^+ and 2^- degenerate, and have $\omega_2\hbar=6.91$ Mev$-3\hbar^2/A=3.85$ Mev. If we here too put $c=0$, we find from Dennison's relations $M\omega_1^2=4a+16b$ and $M\omega_2^2=a-2b$ that b is indeed small, $b=-a/13.8$, and from $M\omega_3^2=2a$ that $\omega_3\hbar=4.24$ Mev. After using the estimate $\epsilon_1\approx0.04\omega_3\hbar$ we evaluate the energy of the 1^- state as $2.125\hbar^2/A+0.96\omega_3\hbar=7.07$ Mev. Thus the assumption $c=0$ leads one to expect the separation between the 1^- and 2^+ (degenerate with 2^-) to be 160 kev, which compares very favorably with the observed 200 kev. (The value of c required to give an exact fit is about $-b/3$, the other numbers being not much changed.) From the estimate $\epsilon_0\approx0.003\omega_3\hbar$ one expects the nearly degenerate pair to be separated by about 13 kev, which should be capable of being resolved by a high resolution magnetic analyzer (and we intend to have a try at it).

Beyond 7.11 Mev, there appears to be a gap without levels for at least 1.5 Mev, with the next dependably observed level at 10.5 Mev. The next level listed by Dennison is a 4^+ at (10/6) 6.13 Mev$\approx$10.2 Mev. Thus again, if one can find a fifth level hiding in the observed group of four near 6 or 7 Mev, the grouping of these followed by a gap seems strongly indicative of some validity of the alpha-model, since no counterpart is found in merely taking note of the wide spread within excited configurations in the central model. Further experimental investigation of the possibility of states at 8.6 and 9.5 Mev will help clear up this point.

Comparison of C^{12} and Be^8 with O^{16}

One of the strongest empirical arguments for the applicability of the alpha-model to the even-even $4n$ nuclei is the one given by Hafstad and Teller when they showed that the mutual binding energy of the alphas per "inter-alpha bond" is quite nearly constant for a series of these nuclei. Their Fig. 2 shows that the linear relation holds for $n=3$, 4, 6, 7, and 8 while Ne^{20} ($n=5$) lies about ten percent low and Be^8 is the more serious exception, with about 2 Mev less binding energy than required. Among the light nuclei of special interest to this review, the significant part of this is that the energy of binding "relative to alphas" shown in Fig. 3 above lies twice as low for O^{16} (at -14.42 ± 0.06 Mev) as it is for C^{12} (at -7.28 ± 0.05 Mev), corresponding to the fact that a tetrahedron has twice as many edges as does a triangle, or that there are six bonds in the alpha-model of O^{16}, three in C^{12}.

This and the considerable degree of success of the alpha-model in accounting for assignments among the first few excited states of O^{16} make it tempting to apply the alpha-model at least to C^{12} on the assumption that the bonds between alphas have an individual reality such that they are simply additive and that their strength is not affected by whether or not the alphas are subject to the forces of other bonds at the same time. For simplicity we, of course, also represent the bonds in terms of the parabolic potential of a harmonic oscillator. In treating only the simple dilational mode of vibration, we may consider a system of n alphas equidistant, at a distance r, from the center of mass in a geometry such that there are b bonds between alphas at a distance $a=cr$ from each other. Then the potential energy of the system in dilational vibration is

$$V=(b/2)K(\delta a)^2=(bc^2/2)k(\delta r)^2, \tag{27}$$

and the kinetic energy

$$T=(n/2)M\dot{r}^2\rightarrow(n\hbar^2/2M)\partial^2/\partial r^2, \tag{28}$$

so that the normal frequency is $(bc^2/n)^{\frac{1}{2}}(K/M)^{\frac{1}{2}}$, the vibrational energy $(n+\frac{1}{2})$ times that, and the mean square amplitude of the zero-point vibration

$$(\delta a^2)_{\text{Av}}=(c^2/nb)^{\frac{1}{2}}(\hbar/2)(Mk)^{-\frac{1}{2}}. \tag{29}$$

These quantities are listed in Table V. There are other modes of vibration (W37, D40), the two we have discussed above for O^{16} and a similar one for C^{12}, and in the simple approximation, Eq. (27), which amounts to putting Dennisons constants $b=c=0$, ω_2 and ω_3 are related to the dilational frequency ω_1 as shown in Table V.

The fact that the mutual binding energy of the alphas is closely proportional to the number of bonds in C^{12} and O^{16} becomes an impressive argument for the use of the alpha-model with the same potential for each bond only after one notes that the vibrational zero-point energy in these structures is also closely proportional to the number of bonds, as was pointed out by Hafstad and Teller (H38) and is shown in the eighth line of Table V. In that line is given a number proportional to the sum of the vibrational frequencies multiplied by the number of modes of each frequency, and these numbers are almost equal to b in line 2. This indicates that Be^8 should also have this same binding energy per bond if the bonds were so simple. Actually, the bond potential is expected to be anharmonic, and the simplest deviation from our assumptions which could account for the low binding energy of Be^8 might be that the anharmonicity is more important in determining the ground state energy of Be^8 than in the other nuclei because the mean-square amplitude of the zero-point bond vibrations in the dilational mode is considerably greater in Be^8, as indicated in the fourth line of Table V. In C^{12} and O^{16}, the zero-point energy is divided between several bonds.

The rotations that go with these vibrations were discussed for O^{16}, and for C^{12} there is the similar symmetry requirement that the vibration ω_2 is accompanied by one unit of angular momentum about an axis normal to the figure axis (W37). If the alpha-model of C^{12} is considered to be a plane figure, the moment of inertia about this axis is half of that about the figure axis, which is the same as that in O^{16}, if a is the same as we here assume. Thus the rotation energy (H38) is proportional to $[J(J+1)-K^2/2]$. The consequent excitation energies are also listed in Table V, in units Mev, with the elastic constant of the bond determined by matching the 6.05-Mev state and the moment of inertia by matching the 6.13-Mev state of O^{16}. Some of the refinement of our earlier discussion of the states of O^{16} is here lost by having put Dennison's constant $b=0$, but this suffices to give a rough indication of where the levels in the other nuclei would be expected to lie if the bonds were similar in all of these nuclei. We run into the same difficulty that is encountered in trying to apply the central model in (jj) coupling to C^{12}: there are simply too many theoretical states accompanying the first excited state. There is one experimental level at 4.44 Mev and the possibility of another (that does not show up in all reactions (M51)) at 7.3 Mev, at most two in this general region. This is a qualitative discrepancy of a sort that is not apt to be removed by a more refined assumption concerning the nature of the bonds. It therefore does not seem possible to apply the alpha-model in any simple way to C^{12}.

The alpha-model might, however, be used to get us out of a possible difficulty with the intermediate-coupling interpretation of C^{12}. The possible level at 7.3 Mev has recently been shown to be probably a pair-emitting 0^+ level if it exists (Ha52, G52) .The evidence is confusing and is based on very meagre observations because in the (α, n) reaction in which it is questionably observed polonium alphas on Be^9 seem to excite the level very weakly if at all. If the 0^+ level exists, it does not seem to arise in any simple way from the central model (double excitation from the $p_{\frac{3}{2}}$ shell in C^{12} being much more expensive than that from the $p_{\frac{1}{2}}$ shell giving rise to the 6.05-Mev state in O^{16}). It might possibly be that the other states are mostly central model (in intermediate coupling) with rather little tendency to cluster into alphas, but that the 7.3-Mev state is mostly the dilational-vibration 0^+ state of the alpha-model. The excitation energy 5.24 Mev listed for this state in Table V is what would be expected above a pure-alpha-model ground state and might instead be 7.3 Mev above the actual (more complicated) ground state. This happens to be just below the energy required for break-up into alphas. The other excited alpha-model states listed in Table V might be absent because of falling above this energy.

In Be^8 it is not clear, as it is in C^{12}, that there are too many fairly low excited states to be ascribed to the alpha-model if one simply assumes a harmonic bond. The number of states in the region 4 to 5 Mev is not experimentally certain—different experiments each give one state at a different energy, so there may be one, two, or perhaps three states in this region (where one solid and one broken line are shown in Fig. 1). If there were only one, this would be qualitatively compatible with the number of states in the alpha-model, the two states in Table V being at about 4.9 and 3 Mev, respectively, and the 4^+ state being at 7.5 Mev. One trouble with this supposition is that the harmonic assumption is expected to fail on the outside in the form of too low a potential barrier.

The Low Barrier in the Alpha-Model of Be^8

In making a simple estimate of the barrier height we consider the "rotational potential," or the term which enters the radial wave equation of the 2^+ state in the way a potential does in a one-dimensional equation,

$$V_{\text{rot}}=L(L+1)h^2/4Ma^2=(15.3/x^2)mc^2, \qquad (30)$$

the Coulomb potential

$$V_c=4e^2/a=(4/x)mc^2, \qquad (31)$$

and the vibrational potential (harmonic term of the bond)

$$V_{\text{vib}}=(k/2)(\Delta a)^2=6.8(\Delta x)^2mc^2. \qquad (32)$$

Here we have put the distance between the alphas $=a=x(e^2/mc^2)$, $L=2$, and, by assuming the stiffness of the bonds the same as indicated by the 6.05-Mev state of O^{16}, $(k/M)=(11.8mc^2/h)^2$, M being the nucleon mass. If we equate the kinetic energy of rotation at a fixed separation to the excitation energy of the 2.9-Mev state, we have $V_{\text{rot}}(x_0)=5.7mc^2$, or $x_0=1.64$, some sort of

average separation which we may take as an approximate equilibrium distance. From this the effective potential is assumed to rise according to Eq. (32) until it joins on to $V_{\text{rot}}+V_c$, which gives the potential outside the cut-off point. As the width of the potential hole V_{vib} we take the value where it is equal to the zero-point energy, which is at $\Delta x=0.66$, and take the cutoff roughly at $x_B=1.64+0.66=2.3$. This gives $V_c(x_B)=0.89$ Mev, $V_{\text{rot}}=1.48$ Mev, or a total barrier height of 2.37 Mev.

A slightly more refined estimate is made by taking into account the centrifugal stretching of the bond, which is one aspect of vibration-rotation interaction. Inside the top of the barrier, the bond energy is given V_0+V_{vib}, which includes V_c and applies to both nonrotating and rotating states. By equating the energy of the ground state, $0.17mc^2$, to V_0 plus the zero-point energy, $(11.8/2^{\frac{1}{2}})mc^2$, we have $V_0=-4mc^2$. In the 2^+ state the effective potential is $V_0+V_{\text{vib}}+V_{\text{rot}}$, and it has a minimum at a separation x_2 greater than has the minimum without rotation by an amount $\Delta x_2=2.25/x_2^3$. Thus at the equilibrium separation in the rotating state the contribution of the bond energy to the total energy is greater than in the ground state by $V_{\text{vib}}(\Delta x_2)=34.4/x_2^6$. The two states also differ by the rotational energy $(V_{\text{rot}})_{\text{Av}}$ of the 2^+ state, and by the difference of the zero-point energies of the radial motion, both of which depend on wave functions in a detailed way. To avoid the use of wave functions, we assume that $(V_{\text{rot}})_{\text{Av}}\approx V_{\text{rot}}(x_2)$, since it is not clear whether this gives too large or too small an estimate, and that the radial zero-point contributions of V_{vib} plus kinetic energy are about the same in the two states. We then equate the energy difference to the excitation energy $5.7mc^2$:

$$15.3/x_2^2+34.4/x_2^6=5.7, \tag{33}$$

which determines $x_2=1.81$. This includes only the simplest terms whose sign is clear without recourse to wave functions, and is obviously only a rough estimate. From this we have $\Delta x_2=0.38$ and thus the minimum of the potential energy curve representing the bond at $x_0=1.81-0.38=1.43$. If now we equate the expression for the effective potential inside, $V_0+V_{\text{vib}}+V_{\text{rot}}$, to that outside, $V_{\text{rot}}+V_c$, we find that they are equal at $\Delta x_B=0.92$, or $x_B=2.35$, the position of the top of the barrier. The barrier height is 2.29 Mev, composed of $V_c(x_B)=0.87$ Mev and $V_{\text{rot}}(x_B)=1.42$ Mev.

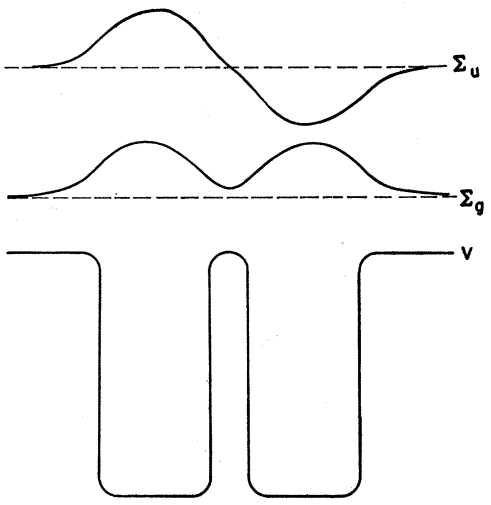


FIG. 19. Schematic potential and nucleon wave functions of an alpha-model of Be^8.

Though this estimate is rough, it makes it unlikely that the effective barrier in the 2^+ state is as high as 3 Mev. There are two phenomena that might make it possible to have a broad virtual state approximated by an alpha-model but slightly above the barrier: (1) the partial reflection of a particle just above a barrier (C29); (2) a possible division of the energy associated with vibration into effective energy of vibration in the one-dimensional problem and some other form such as "internal" energy of the "fluid" that makes up and binds the clusters. The partial reflection (1) is very weak at energies more than a very few percent above the barrier. It thus appears quite unlikely that a reasonable alpha-model would, without quite drastic and complicated modification, contain a 2^+ state near 3 Mev, and even less likely that it would have an excited vibration state 0^+ at this energy or above.

In the most recent previous treatment of these problems in this journal, Haefner (Ha51) has discussed the alpha-model both for Be^8 and C^{12}, as well as some other light nuclei. He does not make the questionable assumption of a standard harmonic bond, for which the principal support is the constancy of binding energy per bond, and thus does not try to compare different nuclei. In C^{12} he considers only rotational states and thus, by neglecting vibration, fails to find too many fairly low excited states. He also does not try to relate the spacing of states to the excitation of the first state of higher isobaric spin. When so few demands are made on the alpha-model, it looks more hopeful, even in C^{12}. In Be^8, he gives a new treatment of the radial motion, based on a bond potential which has no similarity to an harmonic approximation, but instead includes a short-range repulsion and a sudden cutoff to a Coulomb potential, which emphasizes the artificiality of the harmonic form in a problem with so much uncertainty of position. The reader is also referred to his discussion of the experimental situation. More recent experimental results (T51) on the photodisintegration of C^{12} show evidence for a long-range and a short-range peak in the alphas from the secondary break-up of Be^{8*} from the 3-Mev state, corresponding to forward and backward motion relative to the recoil direction, and this implies an angular correlation such that the 3-Mev state cannot be 0^+ (as it seems to be on the basis of alpha-alpha-scattering), thus providing further evidence that it must be 2^+. This is, however, not evidence for the alpha-model, since this assignment is expected in the (LS) model and in the (jj) model, so also in intermediate coupling as in the interpretation of Fig. 8. As was remarked above, a partial admixture of the alpha-model may account for the width of the state.

The Nucleon Coordinates in the Alpha-Model of Be^8

In those simple applications of the alpha-model to the vibration and rotation energies of the $4n$ nuclei,

we treat the alpha-model in terms of the coordinates of the alphas only. The coordinates of the individual nucleons are there ignored except in so far as they cause the alphas to obey Bose statistics. When we now turn to the discussion of the possible intermixture of alpha-model wave functions and central-model wave functions, it is necessary to look in more detail into the alpha-model, and to formulate it in terms of the same nucleon coordinates which appear also in the central model. Be^8 provides an appropriate example.

In the alpha-model we make assumptions similar to those made in the quantum treatment of molecular mechanics, in spite of the lack in the nucleus of a physical analog of the nuclei in a molecule. It is assumed that the angular correlations of the positions of the nucleons assemble them in clusters to a sufficient extent that there is a preferred "body-fixed" coordinate system whose motion may be treated as "adiabatic" in the sense that it is much slower than the motions of the nucleons within this coordinate system. The "self-consistent" field of the other nucleons is assumed to provide an effective ("Hartree") potential defined in this coordinate system, and such a potential V for Be^8 with two minima is indicated in Fig. 19. The distance between the two minima is assumed to vary so slowly as to be "adiabatic," corresponding to the vibration of the simple alpha-model discussed above, as is familiar in discussions of molecules.

The alpha-model wave functions in Be^8 are similar to "molecular orbitals." They may be assumed to have local maxima or minima near the potential minima, and may be either symmetric or antisymmetric in the change of direction of the figure axis, or z axis. The symmetric one, Σ_g, may be assumed to have lower energy because its smaller slope near the center of mass contributes less kinetic energy than does the greater slope in this part of the antisymmetric function Σ_u. These two functions, each with two spin and two isobaric spin projections, may accommodate eight nucleons, and are thus filled in Be^8.

Instead of starting a calculation with "molecular orbitals," one may start with "atomic orbitals" such as

$$\begin{aligned}\psi_1 &\sim \exp[-(\beta/2)(x^2+y^2)]\exp[-(\alpha/2)(z+a)^2],\\ \psi_2 &\sim \exp[-(\beta/2)(x^2+y^2)]\exp[-(\alpha/2)(z-a)^2].\end{aligned} \tag{34}$$

Gaussian functions each having a maximum at the center of an alpha on the z axis at a distance a from the center of mass. Then the "molecular orbitals" may be made as the sum and difference of these in the familiar Heitler-London fashion:

$$\begin{aligned}\Sigma_g &\sim \psi_1+\psi_2,\\ \Sigma_u &\sim \psi_1-\psi_2.\end{aligned} \tag{35}$$

An Elliptical Model of Be^8

In discussing the possibility that the central-model wave functions may be mixed with alpha-model wave functions to form a good approximation to the actual wave function of a system, it is desirable at least to point out some of the difficulties of formulating the calculation of such an admixture. In doing so we shall first describe a very interesting calculation which has been carried out by Wergeland (We41) and has some slight similarity to the sort of calculation that would be necessary.

TABLE VI. Schematic wave functions for Be^8, indicated by occupation numbers of single-nucleon states.

	Eliptical model					(jj) model: s shell		m_j of p shell			
	ψ_3^+	ψ_3^-	ψ_4^+	ψ_4^-		s^+	s^-	3/2	1/2	−1/2	−3/2
proton	1	1	1	1	a	1	1		1	1	
neutron	1	1	1	1		1	1		1	1	
					b	1	1	1			1
						1	1	1			1
					c	1	1	1			1
						1	1		1	1	
					d	1	1	1	1		
						1	1			1	1
					e	1	1	1		1	
						1	1		1		1

Wergeland discusses the problem of the interaction of two alphas by means of an application of the variation principle in which the alpha-model wave function is modified by an admixture of another wave function which we may call an elliptical-model wave function. Thus he takes

$$\Psi = \Psi_H + \lambda\Psi_\alpha, \tag{36}$$

where Ψ_α is an antisymmetric sum of products of eight "molecular orbitals," Eq. (35), that is, an alpha-model wave function which may be written as a "Slater determinant," and Ψ_H is a similar wave function for the elliptical model, an antisymmetric sum of products of the anisotropic Gaussian functions

$$\begin{aligned}\psi_3 &\approx \exp[-(\beta/2)(x^2+y^2)]\exp[-(\alpha/2)z^2],\\ \psi_4 &\approx z\psi_3.\end{aligned} \tag{37}$$

These functions have an ellipsoidal anisotropy, having two extension parameters α and β where one would suffice in a central model. Aside from this they are identical in form with the wave functions for the s shell and the particular $m_1=0$ one of the three wave functions for the p shell. The actual identity does not go so far, however, because they are written in terms of the coordinates of the body-fixed coordinate system of the alpha-model. The elliptical model thus is a special type of "Hartree model," but bears very little resemblance to the central model as discussed in the rest of this paper, since in the latter all three of the p-shell wave functions are populated, and in such a way as to provide special correlations between spin and orbital orientations. The elliptical model is sufficiently similar to the alpha-model [compare Eqs. (34) with (37)] to make the calculation of the mixing straightforward, and yet

contains enough of the concentration about the center of mass characteristic of the central model to exploit the feeling that an improvement of the energy may be obtained by mixing because an actual Be^8 nucleus may be intermediate between an alpha-model and something more concentrated at the center. It may be noted that ψ_3 and ψ_4 have the same symmetries as Σ_g and Σ_u, respectively, of the alpha-model. The main difference is that ψ_3 has a maximum at the center of mass where Σ_g has been assumed to have a minimum (Fig. 19).

The variation calculation is carried out (We41) by varying simultaneously λ, α, and β, with a fixed space-exchange (Majorana) interaction capable of providing enough binding for two separated alphas, and with various fixed values of the inter-alpha spacing $2a$. The resultant minimum energy as a function $2a$ then gives an interaction potential of the two alphas, which indeed has a nice dip inside of the Coulomb barrier, but a dip with a minimum a little above zero energy (the energy of two separated alphas), and thus not deep enough to give, after addition of the zero-point energy, a barely unstable ground state of Be^8. This still constitutes a very interesting treatment of the alpha-interaction problem. Its failure to provide a strong enough interaction with a simple one-term nucleon interaction is hardly surprising in the light of the failure of more elaborate second-order calculations (with more general nucleon interactions adjusted to the scattering and saturation demands) to provide enough binding in the central model of nuclei like Li^6, e.g., (I37). The value of λ found by Wergeland is $\lambda \approx 1.3$, suggesting that the nucleus prefers to remain somewhat closer to the alpha-model than to the elliptical model. The admixture is substantial, but the elliptical model resembles the alpha-model so much more than the central model that this result in itself may not be considered to show that the alpha-model approach is invalid.

The Ground-State Wave Function of Be^8 in the (jj) Version of the Central Model

In order to emphasize how small a part of the problem of the mixture of the central model and the alpha-model was carried out by Wergeland, we wish to compare his elliptical-model wave function with the central-model wave function of the ground state of Be^8 in the relatively simple case of pure (jj) coupling (not intermediate coupling). On the left side of Table VI, the complete occupancy of the single-nucleon functions ψ_3 and ψ_4 in the elliptical model wave function Ψ_H, for instance, is indicated by the occupation numbers "1" for both proton and neutron, for both ψ_3 and ψ_4 orbitals, and for both + and − spin orientations. The alpha-model is similar with Σ_g and Σ_u. Partial wave functions for the (jj) model are similarly indicated in Table VI. In terms of them, with a prime (′) indicating interchange of proton and neutron occupation numbers, the complete wave function of the ground state of Be^8 in the (jj) model (in the configuration $S^4p_{\frac{3}{2}}^4$) is

$$\Psi_{(jj)}=(3a+3b-2c-2c'+d+d'-e-e')/(30)^{\frac{1}{2}}. \quad (38)$$

With this more complicated wave function in place of Ψ_H, one could conceivably pursue a calculation similar to Wergeland's in order to calculate the expected degree of mixing of the central model and the alpha-model of the ground state of Be^8, as a simple example of a typical nuclear state. One might again allow an ellipsoidal distortion of the single-nucleon orbitals, but there might be more energetic tendency to resist this distortion, at least when one is near the central-model side, both because the $m_l=\pm 1$ states are occupied as much as the $m_l=0$ state [within the $\psi_{mj}^{\frac{3}{2}}$ states given by Eq. (22), etc.] and because the spin-orbit term (3) in the Hamiltonian is an important contributor to the energy. This is the general sort of procedure that we imagine when we think of a partial admixture of alpha-model wave functions to modify the properties of specific nuclear states. It may be expected to be energetically advantageous only when a fairly low state of the alpha-model is available with the same symmetry (that is, J, T, and parity) as has the state of the central model under consideration.

In so complicated a calculation, it is not entirely clear that there would be only one local minimum as one varies λ. It might, for example, be possible to find a local minimum near the central model with little ellipsoidal distortion, and one near the alpha-model with much more, though this would seem a strange result of a variation procedure. The rise between them would then perhaps act as a potential barrier to isolate these as two essentially separate almost stationary states of the system, one resembling the central model and another at a different energy resembling more closely a state of the same symmetry of the alpha-model. The possibility of such a two-model interpretation of a nuclear energy spectrum has been discussed (I51) for Li^7 as a result of the preliminary identification (P52) of the 7.4 Mev state as having $J=\frac{3}{2}$, the same as that of the ground state. (It seemed that the only place one can find another $J=\frac{3}{2}$ state is from the other model.) While the relevant experimental measurements have not been completed, it appears on the basis of the available data that this excited state may instead have $J=5/2$, so that such an unorthodox interpretation may not be required.

If, as seems more likely, there is effectively only one minimum for each such problem, then it may turn out that some states of a given nucleus resemble the central model rather closely, and others of other symmetries resemble the alpha-model, depending largely on which model provides the lower state of a given symmetry. In this case, each variation problem provides either a central model state (approximately) or an alpha-model state, but not both; or it might provide a state about midway between the models, as we presume to be the case among the low even-J states of Be^8.

The Alpha-Model States of Li^7

The alpha-model of a nucleus not containing $4n$ nucleons is described (H38) in terms of incomplete occupancy of molecular orbitals such as those shown in Fig. 19. The lowest state is described by saying that Σ_g is occupied by four nucleons and Σ_u by three, or alternatively that there is a Σ_u "hole." The wave function changes sign by interchanging the positive and negative z axis, which is physically the same as rotating the body-fixed coordinate system by 180° in space, so single valuedness requires that the rotational function be odd, with rotational quantum number 1, having a projection 0 along the body-fixed z axis, a $^2\Sigma_u$ as it is called in molecular spectroscopy. Such a doublet has two components depending on the relative orientation of spin and orbital moments, analogous to a ^2P in atomic spectra. In the interpretation given above these are supposed to account for the ground state and 0.48-Mev state, the "isolated low doublet," whose splitting is then largely an alpha-model characteristic. The other fairly low state in this model is the one described similarly as a Σ_g "hole," called a $^2\Sigma_g$, but like a ^2S it is a "doublet" having only one state. If this is at about 6.5 Mev as is supposed above, it has enough energy (and appropriate symmetry) to break up quickly into an alpha and a triton, so is expected to have short life and be broad, as is the state observed at that energy.

An alternative alpha-model of Li^7 would consist simply of an alpha and a triton, if they would attract each other. Its ground state would be a ^2S, and higher states similar to ^2P, etc., would arise from rotation. In view of the instability of He^5 and Be^8 and the large size of the triton, it seems unlikely that the alpha and triton would attract each other enough for this model to stick together. The model discussed above, consisting of Be^8 with a "hole," is favored by the importance of the exchange process, with resonance between the two positions of the "hole."

11. BETA-DECAY AND THE SURPRISING LONGEVITY OF C^{14}

General Situation of Beta-Decay in the p Shell

The squared matrix elements encountered in beta-decays of the various elements, or their reciprocals the ft values, vary so widely in order of magnitude that one might at first sight expect to find little meaning to a refined factor, such as $3\frac{1}{2}$, by which a calculated matrix element might differ between two models, or between (LS) and (jj) coupling. Yet there is a similar variation of gamma-lifetimes, and once these transitions are sorted out according to their magnetic or electric multipole order, there is, at least among the very numerous magnetic 2^4-pole transitions, a very surprising exactitude with which they may be plotted on a straight line on the usual logarithmic plot (Go51). This phenomenon among the moderately heavy nuclei encourages the hope that matrix elements may have a fairly exact meaning in beta-decay, if ever we can calculate them. Attempts to compare the experimental matrix elements in the p-shell nuclei with those calculated with (LS) coupling on the one hand and with (jj) coupling on the other, always using the wave functions appropriate to a symmetric Hamiltonian, have indicated no uniform preference for either coupling scheme, which fact, in itself, is suggestive of the prevalence of intermediate coupling or other such complexity among these light nuclei (Fe50). Yet certain facts do appear particularly clearly in these light nuclei. The occurrence of several pairs of self-mirrored nuclei in this region gives rise to the highly favored or "super-allowed" transitions in which the nuclear matrix elements are expected to be unity because of the great similarity of the ground-state matrix elements of two mirror nuclei. Among the polyads centered about odd-odd nuclei, the allowed transition $He^6 \rightarrow Li^6$, with J changing from 0 to 1, shows that there must be a strong component of the Gamow-Teller operator,

$$V = g \sum_{\text{nucleons}} \tau_x \boldsymbol{\sigma}, \tag{39}$$

effective in the transition, and the two examples of $0 \rightarrow 0$ transitions, $C^{10} \rightarrow B^{10}$ (O^+ state at 1.74 Mev) and $O^{14} \rightarrow N^{14}$ (0^+ state at 2.32 Mev), show that the Fermi operator, which is similar to Eq. (39) but lacks the factor σ, must also appear. Comparison of these cases and others shows that the two operators must be present with at least very roughly equal coefficients (Mo51, Bl52). [Those discussions are based on the assumption that in Li the contribution of the spin-orbit term, Eq. (3), is not large enough to cause a strong deviation from (LS) coupling, which does not seem to be quite assured in the light of the above review of the energy levels, so the equality of the coefficients may be only very rough.] The existence of a strong Gamow-Teller term demonstrated by the He^6 decay brings about what has probably been the most persistent puzzle in the study of beta-decay, the long life of C^{14}, which also involves a $0 \rightarrow 1$ transition. While this long life has made C^{14} a very useful radiological tool for archeology because of its cosmic-ray origin in the atmosphere (L49), it has been considered so anomalous that some rather farfetched *ad hoc* assumptions have at times been reluctantly introduced in an attempt to account for it, but have been disproved by subsequent experimental results, as has been mentioned in Sec. 7.

The Lifetime of C^{14} as Calculated in Intermediate Coupling

It has been recognized that the alternative to those unsuccessful *ad hoc* assumptions would be to admit a chance cancellation within a complicated matrix element, but the factor 10^6 by which the transition rate appears to be suppressed is so large as to make this explanation seem very unlikely, *a priori*. It happens

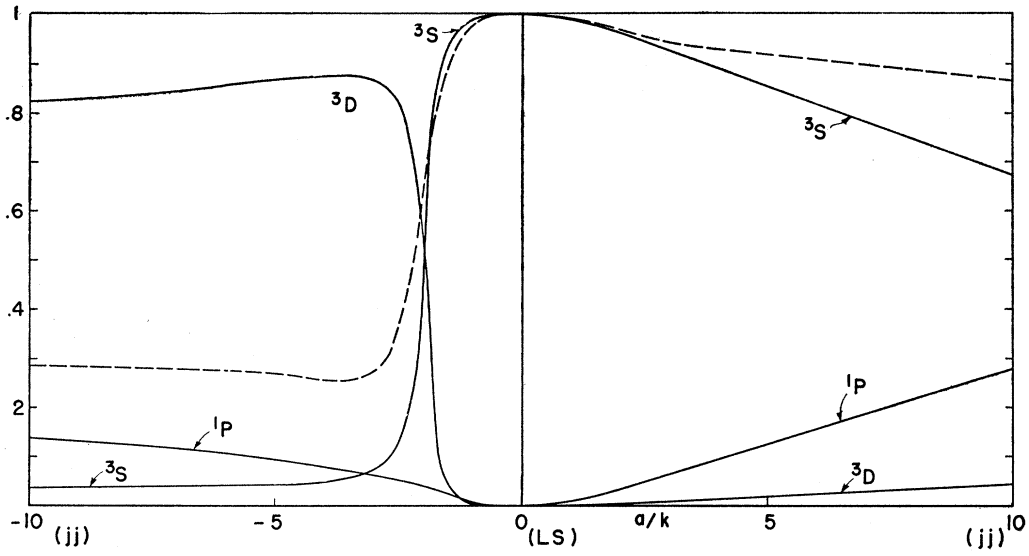


FIG. 20. The broken curve gives the nuclear matrix element for the beta-decay of C^{14} on the left and of He^6 on the right. The solid curves give the fractional constitution of the ground state of N^{14} (and Li^6), that is, the squares of the coefficients in the wave function, in terms of ^3S, ^1P, and ^3D.

that C^{14} is a part of the one polyad, Py^{14}, for which we have been able to present an analytic solution to the intermediate-coupling problem in the ground configuration. This makes it possible at least to calculate whether or not complete cancellation is possible with a reasonable value of the parameter a/K. We shall show that it is not possible within the ground configuration, but that the amount of configuration mixing required to supply the cancellation is not exorbitant.

The energies of the three $(J, T)=(1, 0)$ states plotted in Fig. 5 are derived from a cubic equation (4) of reference (I52), and from the energy matrix there given we may write the corresponding secular equations

$$\begin{aligned}(H_{11}-\epsilon_1)C_1+H_{12}C_2&=0,\\ H_{21}C_1+(H_{22}-\epsilon_1)C_2+H_{23}&=0,\\ H_{32}C_2+(H_{33}-\epsilon_1)&=0,\end{aligned}\tag{40}$$

with $H_{23}=H_{32}=(2/3)^{\frac{1}{2}}a$ [note misprint in (I52) on this point], $H_{33}=g+a/2$, etc. Similarly for the two $(0, 1)$ states we have the secular equations

$$\begin{aligned}(H_{44}-\epsilon_2)C_4+H_{45}&=0,\\ H_{54}C_4+H_{55}-\epsilon_2&=0,\end{aligned}\tag{41}$$

with $H_{45}=-2^{\frac{1}{2}}a$, $H_{44}=-a/2$, $H_{55}=e+a/2$. Here e and g are energies arising from the specific nuclear interaction, Eq. (2), and when we specialized for $L=6K$, they became $e=6.6K$, $f=-8K$, $g=3K$. The C's are coefficients in the expansions of the wave functions

$$\begin{aligned}\psi(1, 0)&=N_1[C_1\psi(^3D_1)+C_2\psi(^1P_1)+\psi(^3S_1)],\\ \psi(0, 1)&=N_2[C_4\psi(^3P_0)+\psi(^1S_0)].\end{aligned}\tag{42}$$

With the Gamow-Teller interaction, Eq. (39), we have the matrix element for the beta-transition

$$\begin{aligned}\int\psi^*(1, 0)V\psi(0, 1)dv&=(1, 0|\,V|\,0, 1)\\ &=N_1N_2[C_2C_4(^1P_1|\,V|\,^3P_0)+(^3S_1|\,V|\,^1S_0)]\\ &=N_1N_2V_{35}\{(H_{45}/H_{32})[(H_{33}-\epsilon_1)/(H_{44}-\epsilon_2)]\\ &\qquad\times(V_{24}/V_{35})+1\}.\end{aligned}\tag{43}$$

This sum has only two terms because of L orthogonality, since the interaction operates on the spin and isobaric spin only, not on the space coordinates of the nucleons. There is a similar reduction of the number of terms, because of orthogonality in M_L, when we evaluate the two terms of the matrix of V in Eq. (43) using the expansions:

$$\begin{aligned}\psi_2&=\psi(^1P_1)=P^1(\alpha\beta-\beta\alpha)(\nu\pi-\pi\nu)/2,\\ \psi_3&=\psi(^3S_1)=S^0(\alpha\alpha)(\nu\pi-\pi\nu)/2^{\frac{1}{2}},\\ \psi_4&=\psi(^3P_0)=[P^1\beta\beta-P^0(\alpha\beta+\beta\alpha)/2^{\frac{1}{2}}\\ &\qquad+P^{-1}\alpha\alpha]\nu\nu/3^{\frac{1}{2}},\\ \psi_5&=\psi(^1S_0)=S^0(\alpha\beta-\beta\alpha)\nu\nu/2^{\frac{1}{2}}.\end{aligned}\tag{44}$$

Here we write two-nucleon wave functions to represent two holes in the p shell, on the assumption, to be justified in Appendix III, that holes behave as do nucleons in beta-decay as well as in energy matrices. We use a spectroscopic notation such as P^1 for L^{M_L} or $\psi_{L^{M_L}}$, the α and β are the usual Pauli spin functions, ν and π are analogous functions for isobaric spin, and within each product of two of these the first factor is associated with the first nucleon and the second with the second. There are three equal matrix elements of the form, Eq. (43) corresponding to the three states

$M_J=1$, 0, or -1 associated with $J=1$. It suffices to calculate with any one of them, and we select $M_J=1$ in writing and working with Eq. (44).

The wave functions, Eq. (44), are used to calculate both the matrix elements H_{ij} and V_{ij}, and the arbitrary choice of phase thus does not, of course, affect the final result. The same is true of the sign convention in the well-known operator

$$\hbar^{-1}(J_x\pm iJ_y)\psi^m=[(j\mp m)(j+1\pm m)]^{\frac{1}{2}}\psi^{m\pm1},$$

which is used with $\hbar^{-1}J_z\psi^m=m\psi^m$. Examples of this operator as we use it are $l^+c=2^{\frac{1}{2}}b$, $l^+b=2^{\frac{1}{2}}a$, $\frac{1}{2}\sigma^+\beta=s^+\beta=\alpha$, $t^+\pi=\nu$, etc., with positive square roots. We abbreviate $J_x\pm iJ_y=J^\pm$, $\boldsymbol{i}\pm i\boldsymbol{j}=\boldsymbol{i}^\pm$, and the combinations of them with which we operate on the wave functions are

$$(\boldsymbol{l}\cdot\boldsymbol{s})=\tfrac{1}{2}(l^+s^-+l^-s^+)+l_zs_z \tag{45}$$

from Eq. (3), and from Eq. (39)

$$\tau_x\boldsymbol{\sigma}=(t^++t^-)(s^+\boldsymbol{i}^-+s^-\boldsymbol{i}^++2s_z\boldsymbol{k}). \tag{46}$$

The relevant beta-decay matrix elements are then

$$\begin{aligned}\boldsymbol{V}_{24}&=(^1P_1|\,\boldsymbol{V}\,|^3P_0)=3^{-\frac{1}{2}}(g/8)\\&\quad\times[(\alpha\beta-\beta\alpha)(\nu\pi-\pi\nu)\,|\,\textstyle\sum\sigma^+\tau^-\boldsymbol{i}^-\,|\,\beta\beta\nu\nu]\\&=3^{-\frac{1}{2}}(g/8)[(\alpha\beta-\beta\alpha)(\nu\pi-\pi\nu)\,|\,\alpha\beta\pi\nu-\beta\alpha\nu\pi]\boldsymbol{i}^-\\&=-3^{-\frac{1}{2}}g\boldsymbol{i}^-,\end{aligned} \tag{47}$$

and similarly

$$\boldsymbol{V}_{35}=(^3S_1|\,\boldsymbol{V}\,|^1S_0)=g\boldsymbol{i}^-. \tag{48}$$

In verifying the spin-orbit elements H_{32} and H_{45} as given above, one further specifies $S^0=3^{-\frac{1}{2}}(ac+ca-bb)$, $P^1=2^{-\frac{1}{2}}(ab-ba)$, etc., in Eq. (44). Thus in the first term of the last number of Eq. (43) we find

$$(H_{45}/H_{32})(V_{24}/V_{35})=1, \tag{49}$$

the sign being definitely positive. This is multiplied in the first term by the ratio of two expressions, each of the form (ϵ_i-H_{jj}), and they both have the same sign because in solving secular equations of this sort one always obtains a lowest root lower than the lowest diagonal matrix element used, and we are here dealing with two lowest roots (ground states), ϵ_i. Thus there is no opportunity for the two terms in Eq. (43) to cancel one another, so it appears from this calculation that the long life of C^{14} may not be attributed to accidental cancellation within the matrix element here considered. Since there appears to be no other possibility within the framework of our present interpretation of nuclear structure and beta-decay (in the approximation in which we neglect configuration interaction), it is important to examine carefully all the assumptions involved in the determination of the relative sign of the two terms in Eq. (43). This is done in Appendix III. One must be careful because questions of phase sometimes arise when transferring from wave functions of particles to those of holes, and it is shown there explicitly that there is no change of relative sign in transferring the beta-decay matrix elements from particles to holes if the wave functions are those which give the usual change of sign in the more familiar transfer of the matrix of $\boldsymbol{l}\cdot\boldsymbol{s}$ from particles to holes.

Although there is no cancellation between the two terms in ground-configuration matrix element, the detailed nature of the coefficients in the wave functions do provide a little tendency to make the lifetime of C^{14} at least slightly longer than would without calculation be expected, by a factor of about 10 which is only a very small part of the factor 10^6 needed. This comes about because the ground state of N^{14}, well out in intermediate coupling to the left of the value $a/K\approx-3$, contains almost 90 percent of ^3D, only a little more than 10 percent of ^3S and ^1P combined. The behavior of the squares of the coefficients in the wave function is shown by the three full lines in Fig. 20. The interesting sudden cross-over in the vicinity of $a/K=-2$ is to be compared with the close approach of the two low (1, 0) energy curves in Fig. 5.

The cross-over of the ^3S and ^3D contributions is to be understood this way: without the influence of nondiagonal matrix elements, the ^3S and ^3D energy curves in Fig. 5 would follow their sloping (LS)-coupling asymptotes as straight lines which cross each other near $a/K=-2$, and further to the left the lowest (1, 0) state would be a pure ^3D following this straight line a little above the actual line drawn in Fig. 5 and a little above the (jj) asymptote, while to the right of $a/K=-2$ the lowest (1, 0) state would be a pure ^3S. The influence of the nondiagonal terms is to mix the two (LS)-coupling states quite completely where they came quite close together, in the region of the cross-over, and to mix the states only to a limited extent elsewhere, almost none at all in the vicinity of $a/K=0$ where the nondiagonal elements vanish.

The effect that this suppression of the beta-decaying ^3S and ^1P components has on the matrix element is indicated by the broken line in Fig. 20, which gives $(1,0|\,V\,|0,1)/V_{35}$ from Eq. (43) with Eq. (49), that is, the nuclear matrix element relative to the $^1S\rightarrow{}^3S$ transition. For most of the region of a/K that is expected to apply to Py^{14} it has about one-third of the value that it has in the region appropriate to Py^6, and the square of this nuclear matrix element gives a factor 10 in the lifetime of C^{14} relative to He^6.

Other Possible Sources of the Cancellation

Since the required cancellation does not occur in the ground-configuration matrix element with charge-independent central interactions, and there remains a factor 10^5 in the lifetime of C^{14} to be explained, the question arises what modification of our assumptions could permit sufficient cancellation to occur, and whether the modification would invalidate our intermediate-coupling interpretation of the ground-configuration states as a plausible or useful approximation for other nuclear properties.

TABLE VIII. Spin-orbit energies of the electron configuration p^3 in the M_L, M_S representation.

m_l	m_s	m_l	m_s	m_l	m_s	M_L	M_S	$\Sigma m_l m_s = \Sigma(l \cdot s)$	AM_LM_S	A/a	M_L \ M_S	1/2	3/2
1	+	1	−	0	+	2	$\frac{1}{2}$	0	$^2D, 0$	0	2	1	0
1	+	1	−	−1	+	1	$\frac{1}{2}$	0	$^2D, 0$	0	1	2	0
1	+	0	−	0	+	1	$\frac{1}{2}$		$\therefore {}^2P, 0$	0	0	3	1
1	+	0	+	−1	+	0	$\frac{3}{2}$	0	$^4S, 0$				

If we depart from the charge-independent Hamiltonian by introducing the Coulomb interaction we do not introduce any new contribution to this matrix element, for this still leaves J, L, and S conserved. The only state with the same J that could be admixed is the $(J, T)=(1, 1)$ state from the ^3P, but this is not admixed because it has a different (L, S) from any of the (LS)-coupling states contained in the ground state. The introduction of a tensor interaction also fails to admix any effective states of the ground configuration. Because a two-nucleon or two-hole wave function is purely symmetric or antisymmetric in space and the tensor interaction is symmetric, there is no mixture of the P multiplets with S or D, so the only nondiagonal matrix element of a tensor interaction is between the $^3S_1$ and the $^3D_1$. This would appear in the first equation of (40) but we used only the third in showing there is no cancellation, that is, it is only the ratio of the terms in the wave function ^3S to ^1P and not of these to ^3D that matters, so this tensor matrix element does not affect this result. (The influence of that nondiagonal tensor matrix element on ϵ_1 affects the detailed curves of Fig. 20, but not the lack of cancellation in the ground configuration p^{-2}.)

Thus we are forced to look to configuration mixing as a source of the apparently fortuitous cancellation that makes the lifetime of C^{14} so long. We may conveniently formulate the configuration mixing thus:

$$\begin{aligned}\psi(1, 0)&=\textstyle\sum_\alpha [c_{S\alpha}\psi_\alpha(^3S)+c_{P\alpha}\psi_\alpha(^1P)+c_{D\alpha}\psi_\alpha(^3D)],\\ \psi(0, 1)&=\textstyle\sum_\alpha [d_{S\alpha}\psi_\alpha(^1S)+d_{P\alpha}\psi_\alpha(^3P)],\end{aligned} \tag{49}$$

where the summation index α indicates successively each of the configurations p^{-2}, $p^{-3}f$, $p^{-4}d^2$, $p^{-4}s^2$, etc., including many higher configurations. The beta-decay matrix element that must so nearly vanish is then

$$\begin{aligned}(1, 0|\,V\,|0, 1)=\textstyle\sum[&c_{S\alpha}d_{S\alpha}(^3S|\,V\,|^1S)_\alpha\\ &+c_{P\alpha}d_{P\alpha}(^1P|\,V\,|^3P)_\alpha].\end{aligned} \tag{50}$$

We have seen from Fig. 20 that c_{S1} and c_{P1} are each roughly equal to $c_{D1}/4$, and that their signs do not permit the two leading terms of Eq. (50) to cancel one another. If, for example, only one other configuration, $\alpha=2$, should contribute, it would be possible for Eq. (50) to vanish with $|c_{S2}|\approx c_{S1}/2\approx c_{D1}/8|d_{S2}|\approx d_{S1}/2$, and similarly for the subscripts P, or, as a second possibility, with $|c_{S2}|\approx c_{S1}\approx c_{D1}/4$, $d_{S2}\approx d_{S1}/4$, if the matrix elements like $(^3S|\,V\,|^1S)_\alpha$ do not depend strongly on the subscript. In the second possibility the mixture from the excited configuration appears with an amplitude $\frac{1}{4}$ or a probability $\frac{1}{16}$ relative to the leading term of each wave function. It is the probability rather than the amplitude that measures the modification of some properties of the state, such as magnetic moments.

Thus we see that it is possible for configuration mixing to provide the fortuitous cancellation without destroying the validity of the ground-configuration representation of the state as a valid approximation. Whether the fortuitous cancellation is not *a priori* so unlikely as to make it seem to be an unacceptable way to account for the long lifetime is another question. A factor 1/300 is needed in the matrix element. If the configuration mixing is just of the order of magnitude required, the probability that the cancellation should be so exact is somewhat more than 1/300. There are of the order of perhaps several dozen beta-transitions with J-values well enough known that an accident of this sort would have been recognized as troublesome. The probability that such an exact cancellation should have happened in any one of them is of course much greater than 1/300, and, indeed, not so improbable as to make implausible an otherwise acceptable interpretation of nuclei which requires this to have happened.

CONCLUSION

From the foregoing review we see that the situation in the p shell is more complicated than we might have hoped. An attempt has been made to outline the nature of the complexities as far as this can, at present, be done, with the expectation that this beginning may serve as a useful guide to further investigation, both experimental and theoretical. The type of agreement found between theory and experiment is such as to provide some indication but not very convincing proof that the intermediate-coupling interpretation is correct. The fact that the intermediate-coupling parameter a/K has been freely varied to fit the empirical needs of the various polyads, and that the range of variation about the rough mean value $|a/K|=5$ has become rather wide (from 3.7 in Py^{12} to 5.6 in Py^{14}) as slight refinements have become possible, leaves the uncomfortable feeling that such an array of rough fits might have been arranged even if the theory were not essentially correct. The variation of the energy parameters a and K individually is also disturbing. Surely the theory must be

TABLE IX. Further spin-orbit energies in the nucleon configuration p^3.

m_l	m_s	m_t	m_l	m_s	m_t	m_l	m_s	m_t	M_L	M_S	M_T	$\Sigma m_l m_s = \Sigma(l\cdot s)$	$(A/a)M_L M_S$	A/a
1	+	+	0	+	+	1	+	−	2	$\frac{3}{2}$	$\frac{1}{2}$	1	$^4D, 1$	$^4D, \frac{1}{3}$
1	+	+	−1	+	+	1	+	−	1	$\frac{3}{2}$	$\frac{1}{2}$		$^4D, \frac{1}{2}$	
1	+	+	0	+	+	0	+	−				1	$^4P, \frac{1}{2}$	$^4P, \frac{1}{3}$
1	+	+	1	−	+	1	+	−	3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$^2F, \frac{1}{2}$	$^2F, \frac{1}{3}$
1	+	+	1	−	+	0	+	−	2	$\frac{1}{2}$	$\frac{1}{2}$		$^2D,^*0$	
1	+	+	0	+	+	1	−	−					$^4D, \frac{1}{3}$	
1	+	+	0	−	+	1	+	−					$^2F, \frac{1}{3}$	
1	−	+	0	+	+	1	+	−				1	$^2D, \frac{1}{3}$	$^2D, \frac{1}{3}$

looked upon as exploratory and subject to verification or disproof by the accumulating evidence of further experimental assignments of quantum numbers as they increase in number and certainty. A propitious beginning of such verification has been made. The best that can be hoped is that the theory will merely have to be modified, and that the necessary modifications will be indicative of the more detailed nature of nuclear interactions, both specific and spin-orbit.

The reason for the peculiar behavior of the lightest p-shell nuclei, and the nature of the transition from these to the heavy nuclei, is here left in urgent need of further study and more detailed understanding. The intermediate-coupling interpretation is the most natural extension down toward light nuclei from the successful (jj)-coupling interpretation of the heavier nuclei, but its lack of spectacular and detailed success, even if that could hardly be expected, leaves one with some feeling of reservation concerning the general approach. Just as is sometimes said of other unfinished endeavors (such as the "majority plan" in the United Nations for the international control of atomic energy), we should perhaps stick by the approach we have already developed until something better is presented; but may its existence and its shortcomings stimulate the search both for improvements and for something better and more clearly acceptable.

In conclusion, there comes the opportunity to thank many colleagues for the help provided by their remarks on these varied subjects. They include Maria G. Mayer, Dieter Kurath, and Eugene P. Wigner for discussions and for communication of theoretical results before publication, Louis A. Turner for suggestions concerning parts of the presentation, Thomas Lauritsen and Fay Ajzenberg for sending preliminary sheets of the latest edition of their very valuable review of the experimental data, and William W. Buechner, Anthony P. French, Harry W. Fulbright, and Denys H. Wilkinson for communication and discussion of recent experimental findings.

APPENDIX I

Multiplet Splittings in Py[7]

Without the slight complication of isobaric spin, the calculation of multiplet splitting energies by trace invariance is familiar. In the three-electron configuration p^3, by way of example, the permitted multiplets are first found to be ^2D, ^2P, and ^4S by counting the numbers of sets of electron quantum numbers for each M_L, M_S, as listed in Table VIII, and the spin-orbit coupling parameter A for each multiplet is then calculated by trace invariance as in the right-hand columns of this table. It is remarkable that in this particular configuration all multiplet splittings vanish in first order. For three nucleons, when they each have the same charge with, say, $m_t=\frac{1}{2}$, the same calculation applies. Thus for the nucleon configuration p^3 there are, among others, three $T=\frac{3}{2}$ multiplets (which we denote by an asterisk), $^2D^*$, $^2P^*$, and $^4S^*$, each with zero multiplet splitting.

In Table IX a start is made of listing the allowed states having $M_T=\frac{1}{2}$. Aside from the $T=\frac{3}{2}$ multiplets already mentioned, these also require the $T=\frac{1}{2}$ multiplets ^2F, ^4D, ^2D, ^4P, ^2P, ^2P, and ^2S. In Table IX it is calculated by trace invariance that each of the first four of these have $A=a/3$, which suffices to show the method, and the table carried one step further shows that the sum of the A's for the two ^2P's is $(2/3)a$, which is all that this method of trace invariance can show. A detailed calculation of the ground-state wave functions (B38) shows that for the ground ^2P we have $A=a/3$, so this must be true for the remaining ^2P as well. Of course, A has no meaning for the ^4S and ^2S for which $L\cdot S=0$.

APPENDIX II

Nuclear Quadrupole Coupling in Polar Molecules

The sign and magnitude of a nuclear quadrupole moment Q are in most instances measured only indirectly through the effect on molecular energy levels which depend on the product of Qq, where q is the quadrupole coupling constant determined by the distribution of electric charge in the molecule. The calculation of q is difficult and in many cases unreliable because it depends on taking a difference, and sometimes a relatively small difference, between positive and negative contributions representing the competing effects of electronic charge and the charge of the other nucleus.

Although molecular binding forces are expressed in terms of "exchange forces," it must be remembered

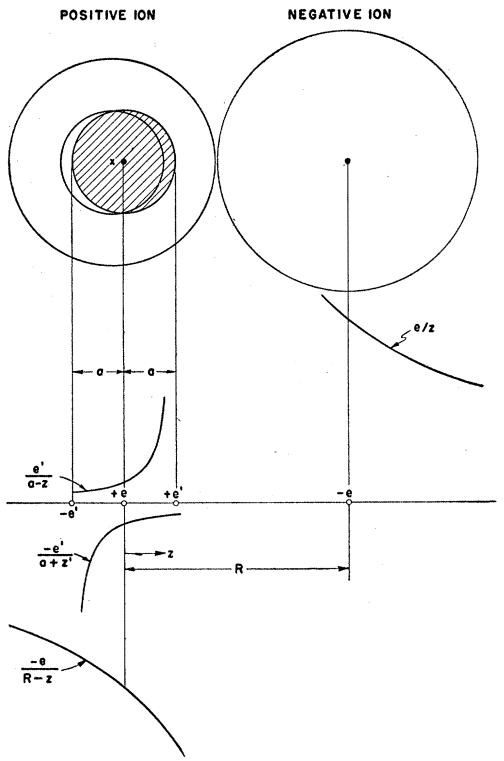


FIG. 21. Schematic charge redistribution in one of the ions of a polar molecule, and the terms of the potential which contribute to the calculation of the quadrupole coupling constant q at its nucleus.

that these are nothing but the result of making a quantum-mechanical average of the Coulomb interactions of the electrons and nuclei which constitute a molecule, and that the forces acting on a nucleus within the molecule are entirely electrical forces (aside from very small magnetic forces). It is important not to confuse the question of stability of a molecule at a given internuclear distance R with the question of the force acting on a nucleus, or the potential effective on the parts of the nucleus as it alters its orientation. In making this distinction, let us neglect zero-point vibration and consider a nonrotational state, so the molecule may be assumed to be at rest and in equilibrium at a constant internuclear distance. The equilibrium depends on the fact that, if R should be altered, the energy of the molecule would be higher after the electrons had readjusted their rapid motions to fit this adiabatic change. The total energy of the calculated ground state of the molecule, plotted against R, has a minimum at the equilibrium value of R. This statement says nothing about the electric potential in the vicinity of the nucleus. It may perfectly well be true that the nucleus is sitting at the top of a potential hill, but never slides off because the hill changes its height every time the nucleus moves a little. The electrons which determine the height of the hill are governed in their motions by the position of the nucleus. In a first-order treatment of hyperfine structure, which is a relatively small effect, we assume that the electrons are not appreciably affected in their motions by the orientation of the nonspherical nucleus, and thus use a potential function in this calculation. The "hill" is level on top, or the valley is level at the bottom, as the case may be (that is, there is no electric field acting on the nucleus, so that it may remain unaccelerated in its equilibrium position) but the sign of the quadrupole coupling constant q is negative or positive depending on whether the potential function is a hill or a valley, in the profile made by cutting along the internuclear axis z.

The charge distribution of a diatomic molecule has rotational symmetry about the z axis, and the potential in the neighborhood of a given nucleus resulting from all charges outside of (an arbitrarily small sphere surrounding) that nucleus may be described, in the light of the Laplace equation, by the derivatives

$$\partial^2V/\partial x^2=\partial^2V/\partial y^2=-\tfrac{1}{2}\partial^2V/\partial z^2,$$

that is, essentially by $\partial^2V/\partial z^2$, the first derivatives being zero because of the equilibrium. The nonspherical nature of the nucleus is defined by its quadrupole moment

$$Q=\sum_{\text{protons}}(3z_p^2-r_p^2)_{\text{Av}},$$

where z_p and r_p are proton coordinates in the nucleus, with the z_p axis the preferred one along the nuclear angular momentum, which in this section we traditionally call I (elsewhere J). With I oriented along the z axis of the molecule, the two coordinate systems coincide and the orientation energy is

$$\begin{aligned}\epsilon&=\tfrac{1}{2}e\sum_{\text{protons}}(x^2\partial^2V/\partial x^2+y^2\partial^2V/\partial y^2+z^2\partial^2V/\partial z^2)_{\text{Av}}\\&=\tfrac{1}{4}e\sum(2z^2-x^2-y^2)_{\text{Av}}\partial^2V/\partial z^2\\&=\tfrac{1}{4}eQ\partial^2V/\partial z^2\equiv\tfrac{1}{4}eQq,\end{aligned}$$

e being proton charge. In the orientation normal to this, with z_p along x, one obtains $\epsilon=-eQq/8$ (after using the symmetry of the nucleus about z_p). The quadrupole coupling constant q as here defined, $q=\partial^2V/\partial z^2$, agrees with the definition usually used in the experimental analysis (F45),

$$eq=\int de_j(3z_j^2-r_j^2)/r_j^5,$$

which is equal to

$$\int de_j\partial^2r_j^{-1}/\partial z_j^2.$$

In the case of a diatomic polar molecule, such as LiCl and the other alkali halides, the molecule may be assumed to consist essentially of two ions, and an important contribution to the electric potential at the

Li nucleus, say, is the Coulomb term $V_1=-e/r$ arising from the charge of the negative Cl ion. We shall show that one may plausibly conclude that this is the preponderant term in determining the sign of q.

If the ions remained unpolarized (spherically symmetrical), V_1 would be the only potential acting on the nucleus and the nucleus would be accelerated by an electric field E_1. The field to annul this must be supplied by polarization, and, since polarization forces are comparatively short-ranged, this must come primarily from the ion in which the nucleus is located. The simplest model for polarization would be to consider that the nucleus moves away from the center of a uniformly filled sphere of electron charge, but this is too simple. In the Thomas-Fermi approximation to atomic theory, for example, it is known that the electron density increases toward the center, and correspondingly, there is a tendency for the nucleus, if displaced, to carry the electron density in its immediate vicinity along with it. As a simple model intended to describe this effect, with a discontinuity replacing a gradual change, let us consider that the polarization of the positive ion (for example) is accomplished by moving the nucleus and all the electron charge within a sphere of radius a about it bodily to the right, as sketched in Fig. 21, by application of an imaginary constraint. In this step we have introduced no new electric force acting on the nucleus: the total force of all the electrons in the positive ion on its nucleus remains zero, because the nucleus is at the center of a spherical distribution of charge and, at the same time, inside of a spherical shell, each of which makes zero field. In this step, we have, however, piled up the charge in the region of overlap and thus made it necessary to occupy high energy parts of momentum space in one crescent-shaped segment, and have left another empty. If we relax the constraint, the most marked readjustment to this inequitable situation is expected to be that some of the charge from the overlapped segment will settle in the empty segment. Electrically, this readjustment is roughly equivalent to placing a positive charge, let us say e', at about a distance a to the right of the nucleus, and a negative charge $-e'$ at about the same distance to the left. The corresponding equivalent charges and potentials are shown in the lower part of Fig. 21.

The electric potential caused by all other charges (the electron cloud and the other nucleus) at and near the nucleus of the positive ion is the sum of three terms, which along the z axis are

$$V_1(z)=-e/(R-z),$$
$$V_2(z)=e'/(a-z),$$
$$V_3(z)=-e'/(a+z).$$

The z components of the corresponding electric fields are $E_i=-dV_i/dz$, and from the vanishing of the total electric field at the nucleus we have $E_2=E_3=-E_1/2$,

TABLE X. Observed sign of qQ for various nuclei in various molecules (Lo52), and the sign of Q deduced.

Nucleus	Homo-polar	Other ion F	Cl	I	Br	K	Tl	Q	Config.
Li^7	+	+	+	+	+			−	$p_{3/2}^3$
Na^{23}	−	−	−	−	−			+	$(\pi d_{5/2})^3(\nu d_{5/2})^4$
$_{19}K_{20}^{39}$	−	−						+	$(\pi d_{3/2})^{-1}$
$_{37}Rb_{48}^{85}$	−	−						+	$(\pi f_{5/2})^{-1}$
$_{55}Cs_{78}^{133}$	+	+						−	$(\pi g_{7/2})^{-3}$
$_{17}Cl^{35,37}$						−	−	−	$(\pi d_{3/2})$

in the approximation in which the similar polarization of the other ion is here neglected. Thus the two polarization terms in the electric field have the same sign and combine to annul the ionic term.

Among the corresponding contributions $q_i=\partial^2 V_i/\partial z^2$ to the quadrupole coupling constant q, the two polarization terms annul one another in this approximation, for $q_2=-q_3$, corresponding to the fact that the curve resulting from e' is concave upward and the curve resulting from $-e'$ is concave downward in Fig. 21. This leaves the quadrupole coupling $q=q_1$ with the sign determined simply by the sign of the charge on the other ion, corresponding to the downward curvature of the curve due to $-e$ in Fig. 21. Thus q is in this approximation expected to be negative at the Li nucleus and a similar treatment of the polarization of the other ion makes it positive at the Cl nucleus in the molecule LiCl, for example.

This oversimplified approximation is of course only a beginning of a calculation, intended to give some initial insight into the quadrupole coupling process, and it is gratifying that it does seem to have a simple correspondence with the observed results. The next higher terms in such a calculation may be of two types. First, there are those involving a difference in the curvatures q_2 and q_3, which might be troublesome, being individually larger in magnitude than the presumably dominant term q_1 by a factor $R/2a$, because of the shorter range a in the equations $q_2=E_2/a$, etc. These are essentially higher approximations in the treatment of the polarization of the ion caused by the application of the constant field E_1. Second, there are the effects of the ionic distortion caused by the derivatives of E_1, that is, by q_1 applied to the electrons, and their subsequent effect on the nuclear orientation. Since E_1 curves downward in Fig. 21, the corresponding potential energy for electrons curves upward and the electrons tend to shun the z axis, concentrating slightly near the xy plane. This makes the electric potential acting on the bulges of the nucleus lower in the xy plane, opposing the direct effect of the downward curvature of E_1 to make it lower along the z axis. The electrons thus tend to shield the nucleus from q_1.

Note added in proof.—The extent of this shielding is being investigated by Foley and Sessler (according to a private communication). By a spherical-harmonic

expansion of the potential applied to an atom and a perturbation-theory expansion of the statistical atom, they estimate that the shielding can be as high as 90 percent.

The consistency of the experimental results seems to indicate that these higher order effects still leave q_1 dominant in determining the sign of q. The measured signs (Lo52) of the products qQ are shown in Table X. The sign of Q, deduced from the rule that the sign of q is the sign of the charge on the other ion in the case of the alkali halides, is also listed. The (jj) configurations are listed. In the case of Li^7, where the (jj) model apparently does not apply, Q is expected to be negative on any reasonably simple model (essentially because the configuration p^3 is near the beginning of the p shell), and it is very gratifying to have the experimental evidence in favor of this expectation, since this was not clear earlier when the evidence depended on an uncertain calculation of q in Li_2.

Recent results (F52) show that q at the Cl nucleus in KCl is very sensitive to vibrational quantum numbers, so presumably to internuclear distance. This suggests (Du52) that the molecule is not quite completely ionic, and that a small admixture of a nonspherical $2p^{-1}$ shell in Cl contributes very significantly to q, so that the variation of the admixture with internuclear distance may account for the variation of q. The p electrons in Cl tend to overlap with the other atom to contribute some valence binding, and the corresponding sign of the quadrupole polarization of the atom makes a negative contribution to q, opposite to the contribution of the charge of the positive K ion. The small atomic admixture to the K ion (or the Li ion in LiCl, etc.) involves an s electron so does not appreciably affect q at its nucleus, so our conclusions concerning Q_{Li} are unaffected.

The signs of Q listed for K, Rb, and Cl are also in keeping with expectations based on the theory of nuclear structure, negative at the beginning of a subshell corresponding to an orbit spread out in a flat distribution near a plane normal to J, and positive for proton "holes" near the end of a subshell. In the cases of Na^{23} and Cs^{133} we do not have such a simple possibility with just one proton or "hole" in addition to closed subshells, so the shell-model expectation is not so clear. The Na^{23} configuration is complex enough that anything could happen, but in this case Q has been measured (by F. Bitter *et al.*, private communication from P. Kusch) to be positive by an atomic method not involving the uncertainty of a molecular q. This gives us our only direct verification of the signs here deduced. The Cs^{133} configuration listed should have positive Q, being nearer the end of a subshell, but the higher pairing energy (M50) of the competing nucleon level $h_{11/2}$ would favor the configuration $h_{11/2}{}^4g_{7/2}$, which woulp have a negative Q as observed, if the h protons pair to give zero.

For the polar molecules, a more physical argument may be given leading to the same picture of polarization of the ions, and not involving the temporary application of imaginary constraints. When an ion containing only closed electron shells begins to overlap with another ion, one may expect the angular factor of the electron distribution, originally isotropic, to begin to be affected by admixture of higher configurations (electrons being partially excited to higher shells) in such a way as to move electrons away from the region of overlap and over to the far side of the ion. This suggests that in case of modest overlap, the angular redistribution is most important near the outside of the ion and the parameter a in Fig. 21 should be approximately equal to $R/2$. In this case the curvature of the individual curves involving e' is about equal to that of the ionic term and higher approximations (such as the existence of radial as well as angular redistribution) are not apt to be more important than the terms already discussed.

Purely classically, one may describe the same effect by thinking of the balance of electric forces. The two ions must attract one another electrically, either in the unpolarized or polarized approximation. Thus some of the matter in each ion is accelerated toward the other ion, while the average charge distribution remains fixed. There is a reciprocal flux of electrons between the ions, which in their effective z motion come to rest on the far side of the ion and start back. It is their relatively low probability density in the region between the nuclei, where their z motion is rapid, and high density in the region where they come to rest that provides the polarization of the ion which we have described.

The signs of qQ for the homopolar molecules Li_2, etc., are also listed in Table X. Here the calculation of the sign of q appears to be more difficult because q is zero in the approximation corresponding to the one we have discussed, there being no simple Coulomb term from the charge on the "other ion." The deviation from spherical symmetry of the separate atoms is influenced by the symmetry of the electron wave function in the space coordinates and the consequent tendency for the electrons to concentrate in the region of atomic overlapping between the two nuclei. This effect appears to predominate, for it gives the negative sign observed for q in the alkali molecules, but it has to compete with the effect of the penetration of one nucleus into the initially spherical screening charge distribution of the other atom. It is perhaps gratifying that the effect predominates which sets in first as the atoms begin to overlap, but in an actual molecule the result might be expected to be a fairly delicate balance between the positive and negative contributions of electron and nuclear charge. That the balance is not extremely delicate is suggested by the consistency with which the homopolar value of q agrees with the value at the nucleus of the positive ion in Table X.

APPENDIX III

Transfer from Particles to Holes in the Theory of Beta-Decay

A convenient notation for calculations of beta-decay and other matrix elements is to list terms in a wave function according to entries of nucleons as 1, rather than 0, in a little table with rows and columns labeled with the quantum numbers of the substates of the p shell as follows:

$m_l=$	1		0		−1				a		b		c	
$m_s=$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$			α	β	α	β	α	β
m_t $\frac{1}{2}$	1	0	0	0	0	0	or	ν	1	0	0	0	0	0
$-\frac{1}{2}$	0	0	0	1	0	0		π	0	0	0	1	0	0.

Of course one need not bother to write the labels of the rows and columns in calculating. A "term" indicated thus is understood to be antisymmetric in exchange of the nucleons, and normalized. In our previous notation, the "term" given in this example is

$$(ab\alpha\beta\nu\pi - ba\beta\alpha\pi\nu)/2^{\frac{1}{2}} = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}.$$

This is a two-nucleon "term" applying to Li^6, and the ten-nucleon "term" of N^{14} associated with this one according to a generalization for nucleons of the systematization used by Condon and Shortley (C35, p. 284) in their treatment of the matrix of $l\cdot s$, is

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1, \end{vmatrix}$$

which has the same value of M_L, M_S, and M_T. The nucleons are numbered in standard order consecutively from left to right across the upper row and then across the lower row, in the first term of the antisymmetric sum of products implied by this symbol.

The first line of Eq. (44) can be written

$$\psi_2 = \psi(^1P_1^1) = 8^{-\frac{1}{2}}(ab-ba)(\alpha\beta-\beta\alpha)(\nu\pi-\pi\nu)$$

$$= \tfrac{1}{2}\left\{ \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{vmatrix} \right\},$$

and for ten nucleons we have for this $^1P_1$ state and correspondingly for the $^3S_1$, the $^3P_0$ and the $^1S_0$ states, respectively, the wave functions

$$\psi_2 = \tfrac{1}{2}\left\{ - \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{vmatrix} \right\},$$

$$\psi_3 = 3^{-\frac{1}{2}}\left\{ \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{vmatrix} \right\},$$

$$\psi_4 = -6^{-\frac{1}{2}}\left\{ 2^{\frac{1}{2}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{vmatrix} + 2^{\frac{1}{2}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{vmatrix} \right\},$$

$$\psi_5 = 3^{-\frac{1}{2}}\left\{ \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{vmatrix} \right\}.$$

Here the sign has been changed (C35, p. 285) of each ten-nucleon "term" having odd M_L, from the sign of its two-nucleon counterpart, for reasons which we shall see presently.

Thus we have, for example,

$$\tfrac{1}{2}\sum l^+s^- \begin{vmatrix} 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \end{vmatrix} = 0, \quad \tfrac{1}{2}\sum l^-s^+ \begin{vmatrix} 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \end{vmatrix} = 2^{-\frac{1}{2}} \begin{vmatrix} 1&0&0&0&0&0 \\ 0&0&0&0&1&0 \end{vmatrix},$$

with only one term because the operator s^+ on the spin function α of the first nucleon gives zero. The corresponding examples for ten nucleons are

$$\tfrac{1}{2}\sum l^+s^- \begin{vmatrix} 1&1&1&1&1&0 \\ 1&1&0&1&1&1 \end{vmatrix} = 0, \quad \tfrac{1}{2}\sum l^-s^+ \begin{vmatrix} 1&1&1&1&1&0 \\ 1&1&0&1&1&1 \end{vmatrix} = 2^{-\frac{1}{2}} \begin{vmatrix} 1&1&1&1&1&0 \\ 1&0&1&1&1&1 \end{vmatrix}.$$

Here the number of effective operations in the sum is limited because the antisymmetry gives zero if more than one nucleon is recorded in the same space, and the only effective operation is one that changes the seventh nucleon from $a\beta$ to $b\alpha$, with no change from the standard order of nucleon numbers. Thus the relation between the corresponding terms is the same in the two-nucleon and ten-nucleon cases in these examples and continues similarly (C35) for all the operations involving l^+s^- and l^-s^+, which change M_L by one unit. The change of sign introduced into the wave functions for "terms" of odd M_L, as mentioned above, makes the contribution to the matrix element have opposite sign in the two cases, in keeping with the familiar fact that the spin-orbit coupling energy for two holes is the opposite of that for two particles. In the operations involving $\sum l_z s_z$ the summation gives the opposite sign in the ten-nucleon case from the two-nucleon case, a familiar fact which constitutes the most elementary part of the theory of holes. Thus the matrix of $\sum \mathbf{l}\cdot\mathbf{s}$ has opposite sign for particles and holes.

The beta-decay operator works somewhat differently. We have, for example,

$$t^+s^- \begin{vmatrix} 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \end{vmatrix} = 0; \quad t^+s^- \begin{vmatrix} 0&0&0&1&0&0 \\ 1&0&0&0&0&0 \end{vmatrix} = - \begin{vmatrix} 0&1&0&1&0&0 \\ 0&0&0&0&0&0 \end{vmatrix},$$

$$t^+s^- \begin{vmatrix} 1&1&1&1&1&0 \\ 1&1&0&1&1&1 \end{vmatrix} = - \begin{vmatrix} 1&1&1&1&1&1 \\ 1&1&0&1&0&1 \end{vmatrix}; \quad t^+s^- \begin{vmatrix} 1&1&0&1&1&1 \\ 1&1&1&1&1&0 \end{vmatrix} = 0.$$

In the two-nucleon case which has a nonvanishing result, a negative sign appears because the second nucleon has been moved by the operation into a position where the first nucleon should appear in the standard order, and a permutation introducing a negative sign must be affected before the symbol implying standard order may be used. Similarly in the ten-nucleon case, the effective operation moves a nucleon, in this case the ninth, upward toward the right to put it ahead of an odd number of other nucleons, in this case three, placing it where nucleon number six should be, and an odd permutation is required to get it back where it belongs in the standard order (so as to match a term in the other wave function appearing in the calculation of the matrix element). The two two-nucleon terms operated upon in this example differ from one another only in an interchange of neutrons and protons, and always appear together in the wave functions we consider, the same being true of the two ten-nucleon terms, so the operator acting on the sum of the two terms has just the corresponding effect, including the same negative sign, in the two cases. The generalization of the theorem here illustrated involves showing that the permutation to compensate the operator always has the same "parity" in the two cases, essentially because one fills in places in the table in adjacent pairs when filling up the shell from the particle case to the hole case.

While this theorem is sufficient, we may complete the explicit demonstration of a lack of cancellation by listing the results of the operations involved in calculating the four crucial matrix elements:

$$\sum l\cdot s\psi_2 = (\tfrac{1}{4})\Bigg\{ \begin{vmatrix} 1&1&1&1&1&0 \\ 1&1&0&1&1&1 \end{vmatrix} + \begin{vmatrix} 1&1&0&1&1&1 \\ 1&1&1&1&1&0 \end{vmatrix} + \begin{vmatrix} 1&1&1&1&0&1 \\ 1&1&1&0&1&1 \end{vmatrix} + \begin{vmatrix} 1&1&1&0&1&1 \\ 1&1&1&1&0&1 \end{vmatrix}$$

$$-2^{\frac{1}{2}} \begin{vmatrix} 1&1&1&1&1&0 \\ 1&0&1&1&1&1 \end{vmatrix} - 2^{\frac{1}{2}} \begin{vmatrix} 1&0&1&1&1&1 \\ 1&1&1&1&1&0 \end{vmatrix} + 2^{\frac{1}{2}} \begin{vmatrix} 1&1&1&0&1&1 \\ 1&1&1&0&1&1 \end{vmatrix} \Bigg\},$$

$$\sum l\cdot s\psi_5=-3^{-\frac{1}{2}}\left\{\begin{vmatrix}1&1&1&1&1&1\\&&&&&\\0&1&1&1&1&0\end{vmatrix}+\begin{vmatrix}1&1&1&1&1&1\\&&&&&\\1&0&1&1&0&1\end{vmatrix}+2^{\frac{1}{2}}\begin{vmatrix}1&1&1&1&1&1\\&&&&&\\1&1&0&1&0&1\end{vmatrix}+2^{\frac{1}{2}}\begin{vmatrix}1&1&1&1&1&1\\&&&&&\\1&0&1&0&1&1\end{vmatrix}\right\},$$

$$\sum\tau_x\sigma\psi_2=i^+\begin{vmatrix}1&1&1&1&1&1\\&&&&&\\1&1&0&1&0&1\end{vmatrix}+\cdots,$$

$$\sum\tau_x\sigma\psi_3=3^{-\frac{1}{2}}i^+\left\{\begin{vmatrix}1&1&1&1&1&1\\&&&&&\\0&1&1&1&1&0\end{vmatrix}-\begin{vmatrix}1&1&1&1&1&1\\&&&&&\\1&0&1&1&0&1\end{vmatrix}-\begin{vmatrix}1&1&1&1&1&1\\&&&&&\\1&1&0&0&1&1\end{vmatrix}+\cdots\right\}.$$

Here the dots indicate that terms have been omitted having the wrong M_T. The matrix elements are

$$({}^3S_1|H'|{}^1P_1)/a=(\psi_3,\sum l\cdot s\psi_2)=-(2/3)^{\frac{1}{2}},$$
$$({}^3P_0|H'|{}^1S_0)/a=(\psi_4,\sum l\cdot s\psi_5)=2^{\frac{1}{2}},$$
$$({}^3P_0|\mathbf{V}|{}^1P_1)/g=(\psi_4,\sum\tau_x\boldsymbol{\sigma}\psi_2)=-3^{-\frac{1}{2}}\mathbf{i}^+,$$
$$({}^1S_0|\mathbf{V}|{}^3S_1)/g=(\psi_5,\sum\tau_x\boldsymbol{\sigma}\psi_3)=\mathbf{i}^+.$$

The ratio of the first two divided by the ratio of the last two is $+1$, just as it appeared in Eq. (47) as calculated for the two-nucleon case, and this is the sign which gives no cancellation in the ground-configuration beta-decay matrix element to the ground state of Py^{14}. With the arbitrary phases of the ψ's chosen so that the nondiagonal as well as the diagonal matrix elements of H' have the opposite sign, those of $\mathbf{V}$ have the same sign, for holes as for particles.

BIBLIOGRAPHY

A49 L. W. Alvarez, Phys. Rev. **75**, 1127 (1949).
A50 Wayne Arnold, Phys. Rev. **79**, 170 (1950); **80**, 34 (1950).
Ar50 Argo, Gittings, Hemmendinger, Jarvis, and Taschek, Phys. Rev. **78**, 691 (1950).
Al51 J. C. Allred, Phys. Rev. **84**, 695 (1951).
Av50 R. Avery and C. H. Blanchard, Phys. Rev. **78**, 704 (1950).
A51 R. K. Adair, Phys. Rev. **82**, 750 (1951); Bashkin, Petree, and Mooring, Phys. Rev. **82**, 378 (1951); H. Staub and H. Tatel, Phys. Rev. **58**, 820 (1940).
Aj51 F. Ajzenberg, Phys. Rev. **82**, 43 (1951).
Aj51a F. Ajzenberg, Phys. Rev. **83**, 693 (1951).
Ar51 Arfken, Biedenharn, and Rose, Phys. Rev. **84**, 89 (1951).
A52 R. K. Adair, Phys. Rev. **86**, 155 (1952).
A52a R. K. Adair, Phys. Rev. **87**, 1041 (1952).
Aj52 F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **24**, 321 (1952); see (H50). It is a matter for apology to the observers that most of the extensive data on which this interpretation is based are referred to only indirectly through these very convenient reviews.
Aj52a F. Ajzenberg, Phys. Rev. **88**, 298 (1952).
B37 G. Breit, Phys. Rev. **51**, 248 (1937).
B38 G. Breit and G. R. Stehn, Phys. Rev. **53**, 459 (1938).
Be38 H. A. Bethe, Phys. Rev. **54**, 436 (1938).
B50 Barnes, French, and Devons, Nature **166**, 145 (1950).
B51 S. T. Butler, Proc. Roy. Soc. (London) **208A**, 559 (1951); N. Austern and S. T. Butler, Phys. Rev. **90**, 365(A) (1953).
Be51 J. Benveniste and B. Cork, Phys. Rev. **83**, 894 (1951).
Bo51 T. W. Bonner and J. W. Butler, Phys. Rev. **83**, 1091 (1951).
Br51 R. Britten, Phys. Rev. **82**, 295 (1951); U. S. Atomic Energy Commission Report NYO 970.
B52 S. T. Butler and E. E. Salpeter, Phys. Rev. **88**, 133 (1952).
Be52 R. E. Benenson, Phys. Rev. **87**, 207(A) (1952).
Bh52 Bhatia, Huang, Huby, and Newns, Phil. Mag. **7**, 485 (1952).
B152 J. M. Blatt, Phys. Rev. **89**, 83 (1953). It was in a preliminary communication that (*LS*) coupling in Py^6 was used as implied in the text.
Br52 D. A. Bromley and L. M. Goldman, Phys. Rev. **86**, 790 (1952).
Bu52 W. H. Burke and J. R. Risser, Phys. Rev. **87**, 294 (1952).
B53 Bockelman, Browne, Buechner, and Sperduto, Phys. Rev. **90**, 340(A) (1953).
C29 Condon and Morse, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1929).
C35 Condon and Shortley, *Atomic Energy States* (Cambridge University Press, Cambridge, 1935).
C49 C. L. Critchfield and D. C. Dodder, Phys. Rev. **76**, 602 (1949); Frier, Lampi, Sleator, and Williams, Phys. Rev. **75**, 1345 (1949).
C50 C. D. Curling and J. O. Newton, Nature **166**, 339 (1950).
C52 C. M. Class and S. S. Hanna, Phys. Rev. **87**, 247 (1952).
Co52 Cohen, Shefroth, Class, and Hanna, Phys. Rev. **87**, 205 (1952).
C53 E. A. Crosbie, private communication. Some of the ground-configuration results of reference (K52), and further second-order deviations from extreme-coupling asymptotes, are implied in a recent summary, Phys. Rev. **90**, 138 (1953).
D36 S. Dancoff and D. R. Inglis, Phys. Rev. **51**, 784 (1936); see errata Phys. Rev. **82**, 182 (1951).
D40 D. M. Dennison, Phys. Rev. **57**, 454 (1940).
D50 S. M. Dancoff, Phys. Rev. **77**, 775A; **78**, 382 (1950).
Di50 L. Diesendruck, dissertation, The Johns Hopkins University, 1950.
D52 P. B. Daitch and J. B. French, Phys. Rev. **87**, 900 (1952).
De52 Dewan, Pepper, Allen, and Almquist, Phys. Rev. **86**, 416 (1952).
Du52 J. Duchesne, J. Chem. Phys. **20**, 1804 (1952).
E41 D. Elliott and L. D. P. King, Phys. Rev. **60**, 489 (1941).
E51 Enge, Buechner, Sperduto, and Van Patter, Phys. Rev. **83**, 31 (1951).
E52 Enge, Buechner, and Sperduto, Phys. Rev. **88**, 963 (1952).
F26 J. Frenkel, Z. Physik **37**, 243 (1926).
F36 W. H. Furry, Phys. Rev. **51**, 784 (1936).
F37 E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937); E. Feenberg and M. Phillips, Phys. Rev. **51**, 597 (1937).
F45 B. T. Feld and W. E. Lamb, Jr., Phys. Rev. **67**, 15 (1945); Eq. II(5).
F49 E. Feenberg and K. C. Hammack, Phys. Rev. **75**, 1877 (1949).
F50 A. M. Feingold and E. P. Wigner, Phys. Rev. **79**, 221A (1950), and valued expository remarks thereon by Professor Wigner.
Fe50 E. Fermi and M. G. Mayer, private communication.
Fee50 E. Feenberg and G. Trigg, Revs. Modern Phys. **22**, 399 (1950).
F51 E. Feenberg, Phys. Rev. **81**, 644 (1951).

Fe51 A. M. Feingold, Revs. Modern Phys. **23**, 10 (1951).
Fr51 O. R. Frisch, International Conference, Chicago, September 1951, Proc., p. 54.
F52 Fabricand, Carlson, Lee, and Rabi, Phys. Rev. **86**, 607 (1952).
Fl52 B. H. Flowers and A. R. Edmonds, Proc. Roy. Soc. (London) **A215**, 120 (1952).
G49 H. Gauss, Z. Naturforsch. **4A**, 721 (1949).
Gu49 E. Guth and C. J. Mullin, Phys. Rev. **76**, 234, 682 (1949).
G51 E. Gerjuoy, Phys. Rev. **81**, 62 (1951).
Go51 M. Goldhaber and A. W. Sunyar, Phys. Rev. **83**, 906 (1951).
G52 Guier, Bertini, and Roberts, Phys. Rev. **85**, 426 (1952).
G53 R. W. Gelinas and S. S. Hanna, Phys. Rev. **89**, 483 (1953).
Ga53 S. Gallone and C. Salvetti, Nuovo cimento **10**, 145 (1953).
H32 W. Heisenberg, Z. Physik **77**, 1 (1932).
H37 F. Hund, Z. Physik **105**, 202 (1937).
H38 L. R. Hafstad and E. Teller, Phys. Rev. **54**, 681 (1938).
H48 N. P. Heydenburg and D. R. Inglis, Phys. Rev. **73**, 230 (1948).
H49 Haxel, Jensen, and Suess, Phys. Rev. **75**, 1766 (1949); Naturwiss. **36**, 155 (1949).
H50 Hornyak, Lauritsen, Morrison, and Fowler, Revs. Modern Phys. **22**, 291 (1950). The next revision, "Energy Levels IV" (Aj52), appeared just as the present paper was about to go to press, and complete references to the experimental data are to be found there. Professor Lauritsen and Dr. Ajzenberg have very kindly communicated prepublication copies of many of the valuable charts and tables. In spite of this, there remain some trivial differences in the numbers given for the energies in Fig. 1 and in (Aj52).
Hu50 Hudspeth, Swann, and Heydenburg, Phys. Rev. **77**, 736 (1950).
H51 H. H. Hummel and D. R. Inglis, Phys. Rev. **81**, 910; **83**, 1078 (1951).
Ha51 R. R. Haefner, Revs. Modern Phys. **23**, 228 (1951).
H52 Hubbard, Nelson, and Jacobs, Phys. Rev. **87**, 378 (1952).
Ha52 G. Harries and G. W. Davies, Proc. Phys. Soc. (London) **65A**, 564 (1952).
Har52 E. G. Harris and M. A. Melanoff, Bull. Am. Phys. Soc. **27**, 5 (1952).
Hu52 P. Huber and E. Baldinger, Helv. Phys. Acta **25**, 435 (1952).
I36 D. R. Inglis, Phys. Rev. **50**, 783 (1936).
I37 D. R. Inglis, Phys. Rev. **51**, 531 (1937); H. Margenau and K. G. Carrol, Phys. Rev. **54**, 705 (1938); W. J. Kroeger, Phys. Rev. **54**, 1048 (1938).
I39 D. R. Inglis, Phys. Rev. **56**, 1175 (1939).
I41 D. R. Inglis, Phys. Rev. **60**, 837 (1941).
I49 D. R. Inglis, Phys. Rev. **75**, 1767 (1949).
I50 D. R. Inglis, Phys. Rev. **78**, 616 (1950).
I50a D. R. Inglis, Phys. Rev. **81**, 914 (1950).
I51 D. R. Inglis, Phys. Rev. **82**, 181 (1951).
I51a D. R. Inglis, Phys. Rev. **81**, 914 (1951).
I52 D. R. Inglis, Phys. Rev. **87**, 915 (1952).
I52a D. R. Inglis, Phys. Rev. **85**, 492 (1952).
J50 R. Jastrow, Phys. Rev. **81**, 165 (1950).
J51 H. L. Jackson and A. I. Galonsky, Phys. Rev. **84**, 401 (1951).
J52 G. A. Jones and D. H. Wilkinson, Phys. Rev. **88**, 423 (1952).
K50 D. Kurath, Phys. Rev. **80**, 98 (1950).
K51 Koester, Jackson, and Adair, Phys. Rev. **83**, 1250 (1951).
K52 D. Kurath, Phys. Rev. **88**, 804 (1952).
K52a D. Kurath, recent unpublished calculation.
K52b D. Kurath, Phys. Rev. **87**, 218 (1952).
Kr52 N. M. Kroll and L. L. Foldy, Phys. Rev. **88**, 1177 (1952).
Kre52 Kreger, Kerman, and Jentschke, Phys. Rev. **86**, 593 (1952).
Ki53 King, Dismuke, and Way, Table of log ft values: October, 1952 (Oak Ridge National Laboratory Report ORNL 1450.)
L49 Libby, Anderson, and Arnold, Science **109**, 227; **110**, 679 (1949).
L51 Li, Whaling, Fowler, and Lauritsen, Phys. Rev. **83**, 512 (1951).
La51 H. H. Landon, Phys. Rev. **83**, 1081 (1951).
Le51 L. T. Leland and H. M. Agnew, Phys. Rev. **82**, 559 (1951).
L52 T. Lauritsen, Annual Reviews of Nuclear Science **1**, 67 (1952).
Le52 G. M. Lewis, Phil. Mag. **43**, 690 (1952).
Lo52 Logan, Cote, and Kusch, Phys. Rev. **86**, 280 (1952).
M49 M. G. Mayer, Phys. Rev. **75**, 1969 (1949); **78**, 16 (1950).
M50 M. G. Mayer, Phys. Rev. **78**, 22 (1950).
M51 R. Malm and W. W. Buechner, Phys. Rev. **81**, 519 (1951).
Mo51 S. A. Moszkowski, Phys. Rev. **82**, 118 (1951).
P35 L. Pauling and E. B. Wilson, *Introduction to Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1935), pp. 209–218.
P50 R. D. Present, Phys. Rev. **80**, 43 (1950).
Ph51 Phillips, Heydenburg, and Cowie, Phys. Rev. **85**, 742(A) (1951).
P52 Murray Peshkin and A. J. F. Siegert, Phys. Rev. **87**, 735 (1952).
Pr52 M. H. L. Pryce, Proc. Phys. Soc. (London) **A65**, 773 (1952).
R41 W. Rarita and J. Schwinger, Phys. Rev. **59**, 209, 436, 556 (1941).
R48 L. Rosenfeld, *Nuclear Forces* (North-Holland, Amsterdam 1948).
R49 Rasmussen, Hornyak, and Lauritsen, Phys. Rev. **76**, 581 (1949); Chao, Lauritsen, and Rasmussen, Phys. Rev. **76**, 582 (1949).
R51 J. Rotblat, Nature **167**, 1027 (1951); Phys. Rev. **83**, 1271 (1951).
R52 Hugh T. Richards, Remarks at the Pittsburgh Conference on Nuclear Physics, p. 33, June, 1952 (to be published).
Ra52 L. A. Radicati, Phys. Rev. **87**, 521 (1952).
S29 J. C. Slater, Phys. Rev. **34**, 1293 (1929).
S37 T. Schmidt, Z. Physik **106**, 358 (1937).
S52 J. Seed and A. P. French, Phys. Rev. **88**, 1007 (1952).
Sp52 A. Sperduto and W. W. Buechner, Phys. Rev. **88**, 574 (1952).
S53 P. Stoll, private communication; Erdös, Stoll, and Scherrer, to appear in Helv. Phys. Acta. See also W. E. Burcham and J. M. Freeman, Phil. Mag. **41**, 921 (1950); O. Rochat and P. Stoll, Helv. Phys. Acta **25**, 451 (1952); H. Glättli and P. Stoll, Helv. Phys. Acta **25**, 455 (1952); R. Müller and P. Stoll, Nuovo cimento **9**, 1232 (1952), J. D. Jackson and D. I. Wanklyn, Phys. Rev. **90**, 381(A) (1953).
SG53 G. Scharff-Goldhaber, Phys. Rev. **90**, 587 (1953).
T26 L. H. Thomas, Phil. Mag. **3**, 1 (1926).
Th50 R. G. Thomas, Phys. Rev. **80**, 136 (1950).
T51 V. L. Telegdi, Phys. Rev. **84**, 600 (1951); V. L. Telegdi and W. Zunti, Helv. Phys. Acta **23**, 744 (1950).
Th51 J. Thirion and T. Muller, Compt. rend. **232**, 1093 (1951).
Ti51 E. W. Titterton and T. A. Brinkley, Proc. Phys. Soc. (London) **64A**, 212 (1951).
T52 I. Talmi, Helv. Phys. Acta **25**, 185 (1952).
Th52 R. G. Thomas and T. Lauritsen, Phys. Rev. **88**, 966 (1952).
Th52a R. G. Thomas, Phys. Rev. **88**, 1109 (1952).
Tho52 D. M. Thompson, Phys. Rev. **88**, 954 (1952).
Tr52 L. Trainor, Phys. Rev. **85**, 962 (1952).
Tru52 Trumpy, Grotdal, and Graue, Nature **170**, 1118 (1952).
T53 V. Telegdi and M. Gell-Mann, private communication.
V51 Van Patter, Serduto, Huang, Strait, and Buechner, Phys. Rev. **81**, 233 (1951).
V51a Van Patter, Buechner, and Sperduto, Phys. Rev. **82**, 248 (1951).
V52 Van Patter, Endt, Sperduto, and Buechner, Phys. Rev. **86**, 502 (1952).
V52a D. M. Van Patter and W. W. Buechner, Phys. Rev. **87**, 51 (1952).
Va52 Van de Graaff, Sperduto, Buechner, and Enge, Phys. Rev. **86**, 966 (1952).
W37 J. A. Wheeler, Phys. Rev. **52**, 1088, 1107 (1937); J. A. Wheeler and E. Teller, Phys. Rev. **53**, 778 (1938).
We37 W. Wefelmeier, Z. Physik **107**, 332 (1937).
Wi37 E. Wigner, Phys. Rev. **51**, 106 (1937). Anticipated in the concept T by B. Cassen and E. U. Condon, Phys. Rev. **50**, 846 (1936).
W41 J. A. Wheeler, Phys. Rev. **59**, 16 (1941).
We41 Harald Wergeland, Norske Videnskabers Selskabs Skrifter, 1 (1941).
W50 S. D. Warshaw, Phys. Rev. **80**, 111 (1950).
W53 J. A. Wheeler and D. L. Hill, Phys. Rev. **90**, 366(A) (1953); K. W. Ford, Phys. Rev. **90**, 366(A) (1953).