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Helmholtz Coils for Production of Powerful and Uniform Fields and Gradients*

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The magnet is an air core and water-cooled massive Helmholtz type with rectangular cross section. It consists of a field coil 10 cm in mean radius which can produce 35 000 oersteds at a power of 1700 kw with a uniformity of about 10 ppm at the center within a region of 10 percent of the mean radius, and a gradient coil 5.2 cm in mean radius which can produce a gradient of 100 oersteds per cm at a power of 2.5 kw with a uniformity of about 140 ppm within the same region. The fields and gradients are separately controlled.

INTRODUCTION

THE solenoid which is described below was designed for precise magnetic susceptibility determinations on metals and alloys over a wide range of temperature. For this purpose it is desirable to have a strong and uniform field of at least 30 000 oersteds and an accurately known and separately controllable uniform gradient. Another requisite is that there be enough room in the region of high field to accommodate Dewar flasks and more especially furnaces for high temperature observations. It has been our experience that one cannot achieve a high temperature accuracy of 1 °C in tiny furnaces in a severely restricted region. A secondary objective was to achieve a field uniformity of 10 parts per million or better over the sample volume so that the apparatus might be used for magnetic resonance studies. The equipment we have constructed meets all of these requirements.

GENERAL DESCRIPTION OF APPARATUS

The field is produced by massive water-cooled Helmholtz coils. With 1700 kw, it is possible to obtain about 35 000 oersteds with a field uniformity of 10 ppm over a distance of about 13 mm along the axis. The internal working space is about 4.4 inches in diameter. We have used part of this region to install a pair of water-cooled coils which can produce a gradient of 100 oersteds per cm with a uniformity of about 0.2 percent over 2 cm. The gradient is produced independently of the field so that it is possible to vary the gradient both in magnitude and direction as desired. It is thus possible to obtain the force on the specimen due to this known gradient, regardless of whether the main field is producing a small gradient or not. The gradient coils can also be used to compensate for variations in the main field, thus achieving a greater uniformity than is produced by the field coils alone. Inside the gradient coils, there is a space of about $3\frac{1}{8}$ inches diameter for working equipment. The coils have a vertical axis and are open at each end.

MAGNETIC DESIGN

Many workers have designed different types of coil systems for homogeneous fields but some of them, such as spherical harmonic current sheets, have just theoretical interest because they are too complicated to put into practice and others, such as Maxwell's three-loop combination[1] and McKeehan's four-loop system[2] are not suitable for high powered design because of the difficulty of obtaining homogeneous cooling and of eliminating the lead effects. On the other hand, the very simple and efficient single solenoids which are most suitable for high power construction fail to produce very homogeneous fields. It was decided to choose a system for our field coil which is a compromise between the highest possible uniformity and the highest engineering efficiency.

A. Field Coil

Our main objective was to obtain high uniformity of the magnetic field without sacrificing efficiency too much and, at the same time, to simplify the difficulties of construction. We decided to use a massive Helmholtz pair with rectangular cross section, as this system is simple to construct and requires only a single adjustable parameter to achieve a great uniformity.

Since the work of Maxwell and Grey, it has been well known that for an axially symmetric system the uniformity of the field along and off the axis is almost the same order of magnitude. The problem is how to obtain the correct separation of the Helmholtz pair to insure that the field along the axis is uniform. Many workers have calculated the best separation for a Helmholtz pair by different expansion series with different approximations.

It seemed that the most direct way to obtain this result was to use the exact formula for the field and

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[1] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Volume II. (Clarendon Press, Oxford, 1892).

[2] L. W. McKeehan, Rev. Sci. Instr. **7**, 178 (1936).

TABLE I. Uniformity of the magnetic field in parts per million of the central field. The values in the first column were calculated from Eq. (A3) and the rest values from Eqs. (1), (2), and (3).

Distance	$\theta=0°$	$\theta=0°$	15°	30°	45°	60°	75°	90°
$r=0.025r_0=2.5$ mm	+2.4	+2.4	+2.2	+1.7	+0.9	−0.2	−1.2	−1.5
$0.05r_0=5$ mm	+4.5	+4.5	+5.5	+6.8	+5.5	+0.5	−5.4	−8.0
$0.075r_0=7.5$ mm	−8.1	−8.4	−0.4	+15	+20	+6.5	−15	−25
$0.10r_0=10$ mm	−60	−61	−32	+25	+54	+25	−33	−62
$0.15r_0=15$ mm	−417	−435	−277	+50	+243	+142	−117	−251

by adjusting the separation to achieve the maximum uniformity in the region in which we are interested. With the exact formula there is no question of convergency and order of approximation as when series are used.

We desired, also, to detect the effect of inhomogeneous coil heating on the uniformity, so we derived in the Appendix the general formula including this effect.

Substituting in Eq. (A3) a set of values of coil dimensions r_0, b, d, or the corresponding values of k and y, and choosing a few appropriate values of x, one can find the proper S which determines the best separation. Usually only a few adjustments need be made until the desired uniformity of H at the chosen positions is obtained. For an ordinary Helmholtz pair the correct separation is about a few percent larger than for the Helmholtz filament of the same mean radius. With a good initial guess the proper S can be obtained by one or two interpolations after two trial values have been used.

Taking values for $k=0.961$ and $y=0.383$, we find the proper S to be 0.5286. The coil dimènsions we used were as follows:

mean radius $=r_0=10$ cm,
radial thickness $=2d=2yr_0=7.66$ cm,
axial breadth $=2b=2kyr_0=7.36126$ cm (each coil),
mean separation between tow coils $=2Sr_0=10.572$ cm.

The calculated uniformity for these dimensions is tabulated in the first column of Table I.

Alternative Calculation

We are deeply grateful to Professor M. W. Garrett[3] of Swarthmore College for showing us his elegant series method for calculating the field around coils. By using his method we had a chance to check our calculations.

For an axially symmetric system with a plane of symmetry the magnetic field can be expressed as zonal harmonics of even powers.

$$H_x=H_{x0}\left[1+k_2\left(\frac{r}{r_0}\right)^2P_2(\cos\theta)+k_4\left(\frac{r}{r_0}\right)^4P_4(\cos\theta)+\cdots\right], \quad (1)$$

where H_{x0} is the x component of the field at the origin, r_0 an arbitrary numerical value, and $k_2(r/r_0)^2P_2$ and $k_4(r/r_0)^4P_4$ are second- and fourth-order errors, respectively. According to Garrett, for thick solenoids k_2 and k_4 can be expressed as follows:

$$k_2=\frac{r_0^2}{2}\sum\frac{1}{x}(1-\sin^3\alpha)/\sum x\log_e(a+R), \quad (2)$$

$$k_4=\frac{r_0^4}{24}\sum\frac{1}{x^3}[2-\sin^3\alpha(15u^4+3u^2+2)]/\sum x\log_e(a+R), \quad (3)$$

where u denotes $\cos\alpha$ and $\sum$ denotes the summation over the four corners of the coil with alternate signs on adjacent corners, giving plus sign to the corner nearest the origin (see Fig. 1).

Substituting the above coil dimensions and the separation, we have found k_2 and k_4 to be 0.00445045 and −1.057581. The uniformity is indicated in Table I. The sixth-order error is negligibly small at the small distances in which we are interested. From the first and second columns of Table I is is clear that the calculated values from the series methods agree almost entirely with that from the exact formula. The other columns of the table give the field uniformity at points off the axis.

Effect of Coil Heating

In order to observe the heating effect, substitute the above coil dimensions in Eq. (A2) and take c equal to 0.004. This means the current density varies along the axis by 0.4 percent/cm of the central value of each coil, or the temperature varies by about 1°C/cm along the axis assuming that the temperature coefficient of re-

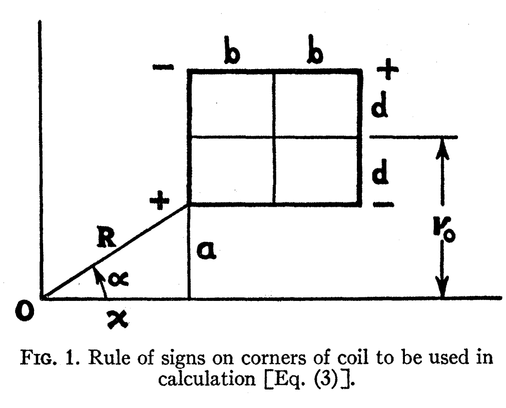

FIG. 1. Rule of signs on corners of coil to be used in calculation [Eq. (3)].

[3] Part of M. W. Garrett's work has been published in J. Appl. Phys. 82, 109 (1951).

TABLE II. Effect of water cooling on the uniformity of magnetic field. It is assumed that the temperature varies by 1°C per cm along the axis. The last column is compensated by a slope of 0.0055 percent per cm.

Distance from the center of the two coils	Uniformity in parts per million of the central field — Without compensation	With compensation
$0.15r_0=15$ mm	-356	-439
$0.1r_0=10$ mm	-7	-62
$0.075r_0=7.5$ mm	35	-7
$0.05r_0=5$ mm	35	8
$0.025r_0=2.5$ mm	18	4
0	0	0
$-0.025r_0=-2.5$ mm	-13	0.6
$-0.05r_0=-5$ mm	-25	2
$-0.075r_0=-7.5$ mm	-51	-10
$-0.1r_0=-10$ mm	-112	-57
$-0.15r_0=-15$ mm	-477	-395

sistivity is equal to 0.004. With these conditions the uniformity near the center of the system has been calculated and tabulated in Table II. It is interesting that in the neighborhood of the center the field varies almost linearly along the axis: the larger the c, the more prominent is the linearity. If we use the gradient from our gradient coil to compensate this small variation, a great uniformity can be obtained. The column 3 of Table II is the result of compensating the values of column 2 by a gradient of 0.0055 percent of the central field per cm, which is about 2 oersteds per cm at full power.

Power Requirement

If the specific resistance of the conductor of which the coil is made is ρ ohms/cc and if only a fraction λ of the volume of the coil is occupied by the conductor, the total power required to maintain the current density I_0 in the Helmholtz pair is

$$\omega=4\pi\rho I_0^2\lambda\int_{r_1}^{r_2} r dr\int_{-b}^{b} dx=4\pi\rho I_0^2\lambda b(r_1^2-r_2^2), \quad (4)$$

where r_1 and r_2 are the inside and outside radius of the coil and the other notations are the same as before. Eliminating I_0 between Eq. (4) and (A4) multiplied by λ, we have

$$H=[\omega\lambda/\rho r_1]^{\frac{1}{2}}G,$$

where

$$G=[\pi(1-y)]^{\frac{1}{2}}g/20yK^{\frac{1}{2}}. \quad (5)$$

Substituting the coil dimensions in Eq. (5) and letting x equal to zero in factor g, we found G to be 0.144, which is about 20 percent less than the theoretical maximum value 0.179 calculated by Cockcroft[4] for a single solenoid with rectangular section. For a power of 1700 kw the magnet can produce a field of about 35 000 oersteds at the center. We have assumed that λ is equal to 0.57 and ρ equal to 2.58×10^{-6} at full power.

[4] J. D. Cockcroft, Trans. Roy. Soc. (London) **227**, 325 (1928).

B. Gradient Coil

As gradient power increases with r_0^3, the gradient coil was designed to be put inside the field coil. For the sake of obtaining great uniformity, we decided to use a long and a thin wall Helmholtz pair connected in opposition. The dimensions of the coils are as follows:

$$\begin{aligned}\text{mean radius} &= r_0=5.2 \text{ cm},\\ \text{axial breadth} &= 2b=5.3 \text{ cm},\\ \text{radial thickness} &= 2d=0.4191 \text{ cm}.\end{aligned}$$

The properties of Helmholtz gradient coils are just like those of field coils. The greatest uniformity of gradient can be obtained by proper separation of the pair of coils. For axially symmetric systems with a plane of antisymmetry the gradient can be expressed as follows:

$$\frac{dH_x}{dx}=\left(\frac{dH_x}{dx}\right)_0\left[1+k_2'\left(\frac{r}{r_0}\right)^2P_2(\cos\theta)+k_4'\left(\frac{r}{r_0}\right)^4P_4(\cos\theta)+\cdots\right], \quad (6)$$

where $(dH_x/dx)_0$ is the gradient at the origin, r_0 an arbitrary numerical value, and $k_2'(r/r_0)^2P_2$ and $k_4'(r/r_0)^4P_4$ are second- and fourth-order errors, respectively. According to Garrett, for thick solenoids

$$k_2'=\frac{r_0^2}{2}\sum\frac{1}{x^2}[1-\sin^3\alpha(3u^2+1)]/D, \quad (7)$$

$$k_4'=\frac{r_0^4}{8}\sum\frac{1}{x^4}[2-\sin^3\alpha(35u^6+0+3u^2+2)]/D, \quad (8)$$

where

$$D=\sum[\sin\alpha-\log_e(a+R)-1]:$$

The notations and the processes of taking the summation are the same as in Eqs. (2) and (3). The proper separation of the Helmholtz pair in terms of the x coordinate of the center of the solenoid can be found by adjusting its value until the k_2' in Eq. (7) vanishes.

Substituting the above coil dimensions we found the separation to be 11.21 cm, which made the k_2' to be a small positive number instead of zero in order to produce a prolate gradient. The values of k_2', k_4' and the uniformity along the axis are shown in Table III.

TABLE III. Uniformity of the gradient along the axis. $k_2'=0.0047208$; $k_4'=-1.70989$.

Distance from the center of the gradient coil	Uniformity in parts per million of the central gradient
$0.025r_0=1.3$ mm	-0.4
$0.05r_0=2.6$ mm	-10
$0.075r_0=3.9$ mm	-51
$0.1r_0=5.2$ mm	-165
$0.15r_0=7.8$ mm	-841[a]
$0.1923r_0=10$ mm	-2262[a]

[a] These two values are corrected for sixth-order errors.

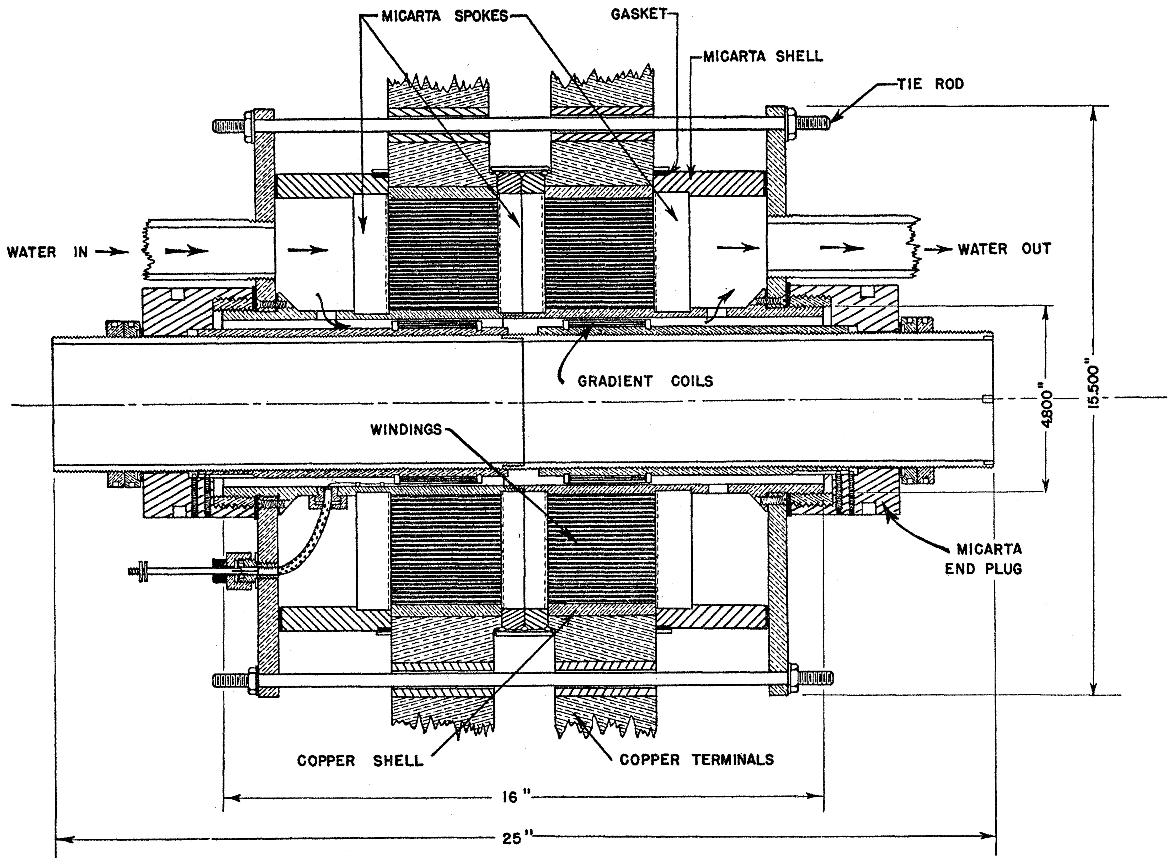

FIG. 2. Cross-sectional view of magnet.

The present coil with a power of about 2.5 kw will produce a gradient at the center of the coil of about 100 oersteds/cm. Another thick coil is under design which is more efficient and can produce a gradient of about 550 with a power of 6 kw and 1000 with 20 kw.

The gradient coil was designed to serve double purposes. First, it could be used for compensating the very small variation of the main field near the center of the magnet. This variation usually happens either due to inaccuracy of construction or the inhomogeneous temperature. Secondly, it could be used as a separate gradient in conjunction with the main field for accurate determination of magnetic susceptibilities. The separation of the two solenoids of the gradient coil can be adjusted to the correct value, and the whole coil can move along its axis in order to let its center coincide with that of the field coil.

ENGINEERING DESIGN

A. Field Coil

The available laboratory facilities provide 10 000 amperes at 170 volts and a heat exchanger system for water cooling. To best meet these conditions it was decided to wind the Helmholtz coil in layers with one turn per layer, as in a roller bandage, and with water flowing between each layer. In an effort to minimize leakage currents it was decided to use enamel insulation over the copper. With these basic conditions prescribed, the detailed design was worked out as follows:

1. Over-All Dimensions

An internal diameter of about 5 in. was selected as giving a satisfactory compromise between large internal working space and stronger field. The choice of this diameter also makes the heat removal conditions less severe than with a smaller coil. The cross section of the windings is almost square and was determined largely by the dimensions of commercially available flat wire and the necessary heat removal area. The spacing to give maximum uniformity over a distance of about one centimeter was then calculated as previously described.

The two coils of the Helmholtz pair were wound on two sections of heavy walled brass pipe as shown in Fig. 2. The two pieces of pipe were then threaded in the region between the coils and screwed onto each other. This construction was used in order that the distance between coils might be varied after testing in order to obtain the best spacing.

2. Current Conductors

It was decided to use commercially available enamel coated copper wire for the winding. The axial width of

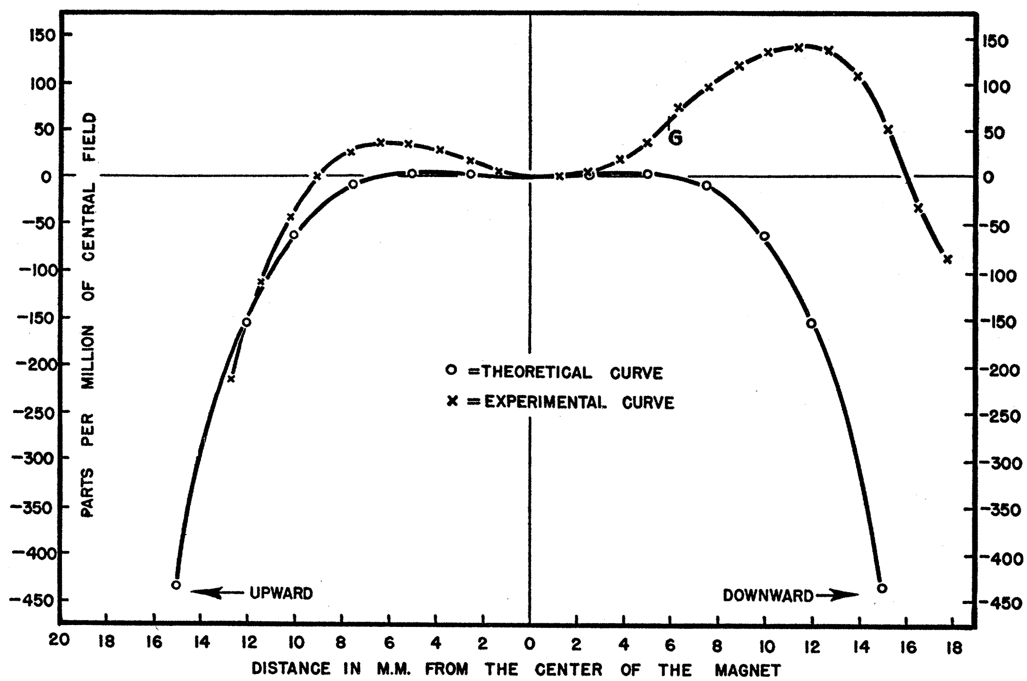

FIG. 3. Uniformity of magnetic field.

2.898 in. was achieved by winding in parallel seven flat wires each of which was 0.044 in. thick by 0.414 in. wide including insulation. The insulation consisted of a single layer of Formex§ 0.00075 in. thick.

The flat wire was carefully wound under tension into helical coils of 40 layers each with space between each layer as described below. Current enters the outermost layer of one coil, flows across the brass tube on which the coils are wound, and then leaves from the outer layer of the second coil. It is conceivable that the flow of current through the brass tube at the center might lead to a distortion of the field on the axis. Measurement of this effect on a model indicated the least distortion of field when the coil connections were made along the same axial line.

3. *Cooling*

Heat is removed by water flowing axially between each layer of the coils as shown in Fig. 2. Water is brought into and out of the coils by means of eight $1\frac{1}{4}$-in. pipes fitted into each end plate. With this arrangement the downstream coil will be hotter than the other but will not affect the current flow since the two coils are in series. Each coil will also experience an axial variation in temperature within each layer due to heating of the water. Because of the temperature coefficient of resistivity this effect will lead to a nonuniform current density within each coil. It is calculated, as mentioned before, that this will cause a field gradient of about 2 oersteds per cm at full power. This distortion of the field was not considered intolerable since compensation can be effected with the gradient coils.

The choice of wire thickness mentioned above alowed a space of 0.027 in. between layers. It was determined experimentally that the available water pressure (100 psi) would allow ample cooling water to flow through the coils with this spacing. The heat removal rate at 1700 kw is about 25 watts per sq cm. This was considered a conservative figure since other solenoids designed by Professor F. Bitter[5] have operated at 200 watts per cm^2. We hope that the moderate heat removal rate will allow us to get away with poor heat transfer conditions at the points where the windings are supported.

The layer of insulation causes the copper to operate at a higher temperature than if water were in direct contact with the metal. This problem was studied by passing current through water-cooled wires having various thicknesses of Formex insulation. By a combination of measurements and calculations it was found that with a single layer (0.00075 in. thick) the copper would be about 40°C hotter than the water at full power. It was felt that the insulating value of the Formex more than offset the disadvantage of operating the copper at this temperature.

4. *Support of the Windings*

An accurate space of 0.027 in. between layers was achieved by inserting axially oriented micarta strips between each layer. These strips were obtained by milling slices 0.027 in. thick from a sheet of 0.050-in.

§ Sold by the General Electric Company.

[5] F. Bitter, Rev. Sci. Instr. **7**, 482 (1936).

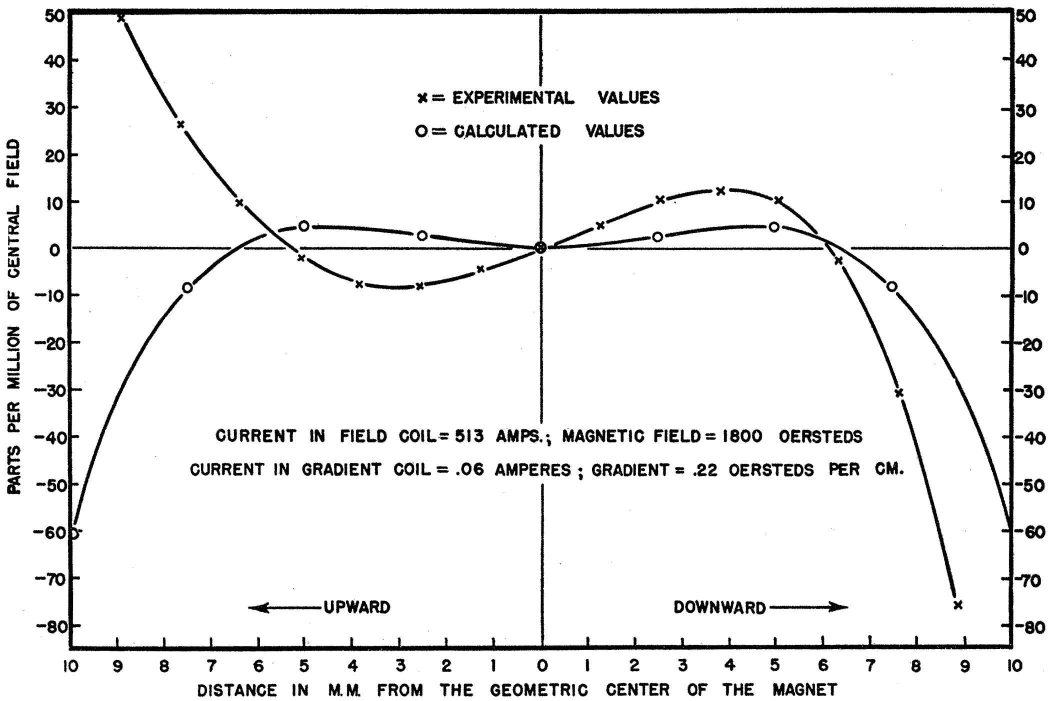

FIG. 4. Uniformity of magnetic field when compensated by gradient coil.

thickness. There are 42 equally positioned spacers around each turn of the helix, and the successive layers of spacers lie over each other exactly. This positioning was achieved by having the spacers slightly longer than the coil width and then having the projecting ends fit into radial slits which were cut into radial micarta spokes. There are 42 spokes and these are in contact with the side of the coils as shown in Fig, 2. The ends of the spokes are inserted into slots which are recessed into the brass pipe at the center and the micarta casing at the outside. In order to achieve a perfect helix for the innermost layer to avoid a hump at every layer and a perfect circle for the outermost spacers which must make good contact everywhere with the 5/16 in. thick copper ring for terminal connection, the micarta spacers at these layers were machined to a series of increasing thicknesses.

The windings are not in contact with flowing water over the 0.050-in. width of each spacing strip. It is believed that conduction of heat through the copper will avoid overheating at these points. The faces of the spokes in contact with the sides of the winding are 3/16 in. wide. These faces were not tapered since support of the coil against axial forces was considered more important than avoidance of eddys in the water flow.

The radial forces acting on the conductors were given some consideration. These forces are transmitted through the spacers, and at the outermost turn the force per spacer is estimated to be about 1500 lb at 10 000 amperes. It was determined by means of compression tests that the flat spacers described above did not appreciably indent the copper or damage the Formex insulation at this load.

5. *Terminal Connections*

Slabs of copper 5/16 in. thick and of coil width were shaped into rings which were placed around the outer turn of each coil in good contact against the outer layer of spacers. The ends of the slabs were tapered and then overlapped to form a passage which was a continuation of the coil helix. The flat wires were brought out through this passage and were then soldered and riveted to the copper ring over a distance of about two inches in the region of overlapping ends. The ring was attached to the micarta casing by means of brass keys in order that there could be no unwinding of the coil, especially when the winding tension on the wires was released. The final step was to cut the wires and then carefully machine the outside of the copper ring to a perfect surface. The wires of both coils were terminated in about the same angular position with respect to the coil axis.

Electrical contact was made by clamping massive slabs of copper around each of the coils. The slabs were accurately machined to make a light mechanical contact with the copper rings when the bolts were tightened in order not to crush the coil. These bolts were calculated to be strong enough to resist the bursting forces of the coil at full current.

6. *Miscellaneous*

The outer casing of the apparatus consists of micarta shells in order to minimize insulation and leakage cur-

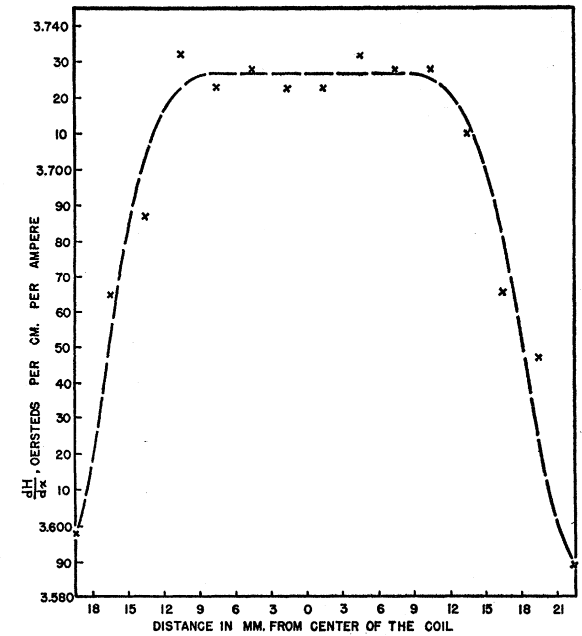

FIG. 5. (dH/dx) along axis of gradient coil.

rent problems. The end plates consist of brass which are bolted to a flange on the central pipe as shown in Fig. 2 and are held together with axial tie rods on the outside.

Water leakage is prevented with an assortment of rubber gaskets and O rings at various points. These gaskets were located in such a way that dead water does not occur at any place across an electrical potential.

The assembled magnet is supported along the length of the heavy copper terminals. Thus the solenoid casing hangs from these terminals and is free to expand or contract without pulling apart at the gasketed joints.

B. Gradient Coils

The gradient coils were located and dimensioned as shown in Fig. 2.

There are three layers with 55 turns per layer of No. 19 Formex-covered wire with a 0.026-in. space between layers. A stream of water in parallel with the main circuit effects cooling. The coils are wound on threaded brass bushings in order that the distance of separation may be accurately varied. Likewise, the two gradient coils together may be moved axially by means of the threaded end caps. The No. 19 wire is fastened to heavier copper rod inside the end plates and this rod comes out through insulated Wilson seals.

EXPERIMENTAL RESULTS

A. Test of Magnet

The apparatus was connected up to the water and power source on a temporary basis and operated at 5000 amperes for several hours. At this current which corresponds to about 300 kw the field strength was measured to be 18 000 oersteds, which agrees well with our calculated value using the measured resistivity of the coil. Everything went according to expectation, and therefore we believe that no troubles will develop at full power.

B. Test of Uniformity

The uniformity of the main field has been tested at low power by measuring the force acting on an annealed steel ball bearing suspended in the field from an analytical balance. Then the field was kept constant, a known gradient applied from the gradient coil, and the total force on the sample measured again. The small gradient of the main field was calculated in terms of the known gradient. By measuring the gradient at successive positions near the center in small intervals along the axis of the magnet, the fields, and hence their deviations from the central value in this region, were calculated. The results are shown in Fig. 3. It is to be noted that the curve is not symmetrical and that the geometrical center is at position G. This imperfection may result from the construction. For purposes of comparison we put the minimum of the calculated curve at that of the experimental curve. It is clear from Fig. 3 that in a small region near the center the actual uniformity is a little worse, but in larger regions better than the calculated values, and that the field near the geometrical center is almost linear making a small slope with the axis. If the small slope is compensated by means of the gradient coil, an almost entirely flat region of about 5 mm can be obtained. For a little under compensation a uniformity of about 10 parts per million in a distance of 13 mm can be reached, as shown in Fig. 4.

The uniformity of the gradient has been roughly tested by quickly shifting a search coil successively along the axis in definite, small intervals and observing

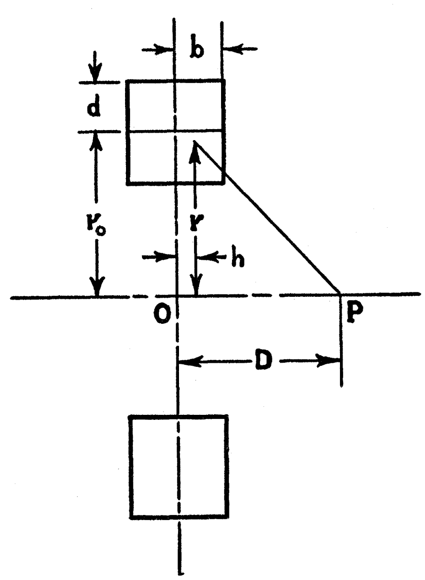

FIG. 6. Current element of radius r.

the change of field by means of a ballistic galvanometer. Figure 5 shows this rough determination. The scattered points showing the experimental error are not a measure of the true uniformity since this is beyond the sensitivity of the apparatus.

ACKNOWLEDGMENTS

We wish to express our appreciation of the assistance of Mr. Francis Vidito and Mr. Oscar James Van Sant, Jr., who supplied many valuable ideas for the actual construction of the solenoid.

APPENDIX

Derivation of Field Equation

If I is the current density in amp/cm^2 of a circular current element of radius r (see Fig. 6), the field at a point P on the axis at a distance D from the center of the cylindrical coil is

$$\frac{2\pi I r^2 dr dh}{10\{(D-h)^2+r^2\}^{\frac{3}{2}}}.$$

If the cooling water comes in from one end of the coil and goes out at the other, the heating effect and hence the current density I may be, assumed linear along the axis of the coil and can be expressed as $I=I_0(1-ch)$, where I_0 is the current density at the middle cross section of the coil and c is a constant depending upon the temperature gradient along the axis. The total field at point P, due to the whole cylinder of mean radius r_0, axial width $2b$, and radial thickness $2d$, is given by

$$H=\frac{2\pi}{10}\int_{-b}^{b}\int_{r_0-d}^{r_0+d} I_0(1-hc)\frac{r^2 dr dh}{\{(D-h)^2+r^2\}^{\frac{3}{2}}}.$$

Evaluating the integrals we obtain

$$\begin{aligned}
H=\frac{2\pi I_0}{10}\Bigg[&\left\{1-\frac{c(D-b)}{2}\right\}(D+b)\log_e\frac{r_0+d+[(D+b)^2+(r_0+d)^2]^{\frac{1}{2}}}{r_0-d+[(D+b)^2+(r_0-d)^2]^{\frac{1}{2}}}-\left\{1-\frac{c(D+b)}{2}\right\}(D-b)\\
&\times\log_e\frac{r_0+d+[(D-b)^2+(r_0+d)^2]^{\frac{1}{2}}}{r_0-d+[(D-b)^2+(r_0-d)^2]^{\frac{1}{2}}}\Bigg]+\frac{2\pi I_0 c}{10}\Bigg[\frac{r_0+d}{2}[(D-b)^2+(r_0+d)^2]^{\frac{1}{2}}+\frac{r_0-d}{2}[(D+b)^2+(r_0-d)^2]^{\frac{1}{2}}\\
&-\frac{r_0+d}{2}[(D+b)^2+(r_0+d)^2]^{\frac{1}{2}}-\frac{r_0-d}{2}[(D-b)^2+(r_0-d)^2]^{\frac{1}{2}}\Bigg]. \qquad \text{(A1)}
\end{aligned}$$

Taking account of the two identical coils separated at a distance $2\,Sr_0$ and substituting the coil dimensions in terms of the mean radius r_0, i.e., $d=yr_0$, $b=kd=kyr_0$, and $D=Sr_0$, we find the field on the common axis at a distance xr_0 from the center of the Helmholtz pair to be

$$\begin{aligned}
H=\frac{2\pi I_0 r_0}{10}\Bigg[&\left\{1-\frac{cr_0(S+x-ky)}{2}\right\}\left\{(S+x+ky)\log_e\frac{1+y+[(S+x+ky)^2+(1+y)^2]^{\frac{1}{2}}}{1-y+[(S+x+ky)^2+(1-y)^2]^{\frac{1}{2}}}\right\}\\
&-\left\{1-\frac{cr_0(S+x+ky)}{2}\right\}\left\{(S+x-ky)\log_e\frac{1+y+[(S+x-ky)^2+(1+y)^2]^{\frac{1}{2}}}{1-y+[(S+x-ky)^2+(1-y)^2]^{\frac{1}{2}}}\right\}\\
&+\left\{1+\frac{cr_0(S-x-ky)}{2}\right\}\left\{(S-x+ky)\log_e\frac{1+y+[(S-x+ky)^2+(1+y)^2]^{\frac{1}{2}}}{1-y+[(S-x+ky)^2+(1-y)^2]^{\frac{1}{2}}}\right\}\\
&-\left\{1+\frac{cr_0(S-x+ky)}{2}\right\}\left\{(S-x-ky)\log_e\frac{1+y+[(S-x-ky)^2+(1+y)^2]^{\frac{1}{2}}}{1-y+[(S-x-ky)^2+(1-y)^2]^{\frac{1}{2}}}\right\}\Bigg]\\
&+\frac{\pi c I_0 r_0^2}{10}\Big[(1+y)\{[(S+x-ky)^2+(1+y)^2]^{\frac{1}{2}}-[(S-x-ky)^2+(1+y)^2]^{\frac{1}{2}}\}\\
&+(1-y)\{[(S+x+ky)^2+(1-y)^2]^{\frac{1}{2}}-[(S-x+ky)^2+(1-y)^2]^{\frac{1}{2}}\}\\
&-(1+y)\{[(S+x+ky)^2+(1+y)^2]^{\frac{1}{2}}-[(S-x+ky)^2+(1+y)^2]^{\frac{1}{2}}\}\\
&-(1-y)\{[(S+x-ky)^2+(1-y)^2]^{\frac{1}{2}}-[(S-x-ky)^2+(1-y)^2]^{\frac{1}{2}}\}\Big]. \qquad \text{(A2)}
\end{aligned}$$

It is expected that the main field is superimposed by an auxiliary field which is proportional to the parameter c.

If c is zero, which means the coil temperature is uniform and the current density is constant throughout the Helmholtz pair, the above formula is reduced to the following simple form:

$$
\begin{aligned}
H=\frac{2\pi I_0 r_0}{10}\Bigg[&\left\{(S+x+ky)\log_e\frac{1+y+[(S+x+ky)^2+(1+y)^2]^{\frac{1}{2}}}{1-y+[(S+x+ky)^2+(1-y)^2]^{\frac{1}{2}}}\right\}\\
&-\left\{(S+x-ky)\log_e\frac{1+y+[(S+x-ky)^2+(1+y)^2]^{\frac{1}{2}}}{1-y+[(S+x-ky)^2+(1-y)^2]^{\frac{1}{2}}}\right\}\\
&+\left\{(S-x+ky)\log_e\frac{1+y+[(S-x+ky)^2+(1+y)^2]^{\frac{1}{2}}}{1-y+[(S-x+ky)^2+(1-y)^2]^{\frac{1}{2}}}\right\}\\
&-\left\{(S-x-ky)\log_e\frac{1+y+[(S-x-ky)^2+(1+y)^2]^{\frac{1}{2}}}{1-y+[(S-x-ky)^2+(1-y)^2]^{\frac{1}{2}}}\right\}\Bigg] \qquad \text{(A3)}
\end{aligned}
$$

$$
=2\pi I_0 r_0 g/10, \qquad \text{(A4)}
$$

where g is the square bracket of Eq. (A3).