On the Theory of the Ising Model of Ferromagnetism*

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IN the Ising model of ferromagnetism a scalar "spin" variable, σ, is associated with each lattice point. The possible values of σ are chosen to be ± 1, and the interaction between the jth and kth lattice points is postulated to be

$$E_{jk}=\begin{cases} -J\sigma_j\sigma_k & \text{if } j \text{ and } k \text{ are nearest neighbors,} \\ 0 & \text{otherwise.} \end{cases}$$

A magnetic moment μ is assigned to each spin so that the interaction energy of the jth particle with an external field H is chosen to be

$$E_j=-\mu H\sigma_j.$$

Although the Ising model lacks several important characteristics of a real ferromagnet, it is of considerable interest because it is one of the simplest examples of a system of interacting particles which still has some features of physical reality left in it. Also, it is equivalent to a very good model of a binary substitutional alloy and an interesting model of a gas or liquid.

The partition function of the Ising model can be expressed conveniently in terms of either of the following two quantities

(a) the largest characteristic value of a matrix whose elements $P(\Sigma, \Sigma')$ are

$$\exp-\{V(\Sigma)+V(\Sigma, \Sigma')\}/kT=P(\Sigma, \Sigma'),$$

where Σ and Σ' represent spin configurations on two adjacent layers of a crystal (or two adjacent rows in a two dimensional lattice), $V(\Sigma)$ is the potential energy of interaction between nearest neighbors on a layer in the Σth configuration, and $V(\Sigma, \Sigma')$ is the potential energy of interaction between spins in neighboring layers.

(b) The number of ways a closed path of a given length can be constructed on a lattice of interest.

Both of these formulations of the partition function have been successfully applied in the investigation of two-dimensional lattices. The first was used by Onsager in his derivation of the thermodynamic properties of a square lattice, and the second has been examined recently by Kac and Ward. Both formulations are discussed in our review.

The specific heat of a two-dimensional Ising model has a logarithmic infinity at the temperature at which long range order disappears. The spontaneous magnetization, which has been calculated exactly by Yang, drops very rapidly to zero at the Curie point. To date no one has found exact expressions for thermodynamic and magnetic properties of the two-dimensional Ising model in the presence of an external magnetic field, or in a lattice in which interactions between next nearest neighbors contribute to the interaction energy. The effect of lattice vibrations on spin interactions has not been determined.

No exact formulas for the partition function of a three-dimensional Ising model have been discovered. However, a rather large number of terms have been calculated in series expansions which are valid at very high or very low temperatures. These are discussed in the review.

* This is an abstract of an extensive review (with the same title) of recent developments in the theory of the Ising model of ferromagnetism. Because of its length, the publication of the full text will be postponed until the next issue of the *Reviews of Modern Physics.*

DISCUSSION

W. OPECHOWSKI, *University of British Columbia, Vancouver, Canada:* All calculations on the two-dimensional Ising crystal involve the assumption that $N\rightarrow\infty$ (N is the number of atoms composing the crystal). The results so obtained will, in general, give an excellent approximation to the actual physical situation, for which N is very large but necessarily finite. However, it may not be so for a range of temperatures that includes the Curie point. The range would certainly be very narrow, but perhaps not so narrow as one might think at first sight. The reason for these doubts is as follows. In the one-dimensional case rigorous formulas can be obtained easily for any finite N. The Curie point is, in this case, of course identical with the absolute zero. It turns out that, for the susceptibility *vs* temperature curve, deviations of the kind mentioned above occur for temperatures T for which $T<J/k\log N$, where J is the interaction constant (note the logarithmic dependence on N).[1]

[1] For more details, see Opechowski and Bryan, Can. J. Phys. **29,** 236 (1951).