Magnetic Energy Formulas and their Relation to Magnetization Theory

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Present theories of magnetization are based largely on energy formulas. These are incomplete because they rely on a calculation of magnetization work in the absence of strain and of strain work in the absence of magnetization. The present paper summarizes and relates to magnetization theory a calculation, already published elsewhere, in which magnetization and strain are assumed from the start to be present simultaneously; furthermore forces are calculated directly, without use of energy arguments until the properties special materials are considered. An important result is that the separation of the force on part of a body into a "magnetic" force and a force derivable from "stresses" can be accomplished in more than one way. The theory confirms the traditional results for fluids; but for elastic solids, it yields terms not present in the formulas of the traditional theory. These terms may be important in the magnetization process. For a uniformly magnetized ellipsoid they lead to a nonuniform magnetostriction "form effect" with a mean strain equal to the strain calculated by Becker.

THE theory of domain formation, magnetization, and magnetostriction is based on the concept of spontaneous magnetization, on symmetry requirements, and on energy formulas. The energy formulas are usually quoted as if they were a part of everybody's background of knowledge. Unfortunately this knowledge is not in a very satisfactory state. Many of the textbook treatments assume a constant permeability; the results are therefore useless for ferromagnetic purposes. Other treatments introduce other restrictive assumptions, which are often not stated explicitly.

The basic formulas used in magnetization and magnetostriction theory may be summarized as follows. The work of magnetizing unit volume is calculated for a body with no strains:

$$dw_m = \mathbf{H}\cdot d\mathbf{M} \tag{1}$$

($\mathbf{H}$ = magnetizing force, $\mathbf{M}$ = magnetization). The work of straining unit volume is calculated for a body with no magnetization:

$$dw_s = X_x de_{xx} + \cdots + Y_z de_{yz} + \cdots \tag{2}$$

(the stress and strain notation is that of Love[1]). These two expressions are added to get a formula for the change of free energy density when magnetization and strain both occur:

$$dF = dw_m + dw_s. \tag{3}$$

The free energy density is expressed as a series in the magnetization and strain components, in conformity with symmetry requirements and to as many terms as seem necessary. Other formulas are derived from this one by simple applications of differential calculus:

$$\partial F/\partial M_x = H_x, \quad \partial F/\partial e_{xx} = X_x, \quad \partial F/\partial e_{yz} = Y_z = Z_y. \tag{4}$$

Occasionally a magnetician becomes uneasy about the effect of strains on magnetic formulas. Stoner[2] in 1937 recognized that his formulas applied strictly only to a rigid body; but he found no way of improving on them. Becker[3] in 1933 took account of magnetization and strain simultaneously, in his calculation of the magnetostriction of an ellipsoid. He concluded that the magnetostriction is shape-dependent, and he verified this conclusion experimentally. Nevertheless this "form effect" is scarcely mentioned in Becker and Döring's book;[4] and it has been ignored by recent authors. Furthermore, Becker's calculation does not tell us what happens to the form effect when the specimen is not an ellipsoid or the magnetization is not uniform.

Occasionally, also, an elastician becomes uneasy about the effect of magnetization on elastic formulas. Brillouin,[5] in his book on tensors, remarks that in a polarized body the stress tensor is no longer symmetrical: that is, Y_z, the y component of force across unit area perpendicular to z, is no longer equal to Z_y, the z component of force across unit area perpendicular to y. Sokolnikoff[6] makes a similar statement in his book on elasticity. If Brillouin and Sokolnikoff are right, then what does one get by differentiating a free energy density with respect to a shearing strain?

The traditional formula for magnetization work has various "derivations." One of them is the following. When the magnetization is changed by controlling the current I through a coil, work is done against the induced electromotive force E_c. The rate at which work is done can be expressed as a volume integral:

$$dW/dt = -IE_c = -\int \mathbf{J}\cdot\mathbf{E}d\tau, \tag{5}$$

where $\mathbf{J}$ is the current density and $\mathbf{E}$ the electric field

[1] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Cambridge University Press, New York, 1934), fourth edition.

[2] Edmund C. Stoner, Phil Mag. **23**, 833 (1937).

[3] R. Becker, Z. Physik **87**, 547 (1933).

[4] R. Becker and W. Döring, *Ferromagnetismus* (Julius Springer, Berlin, 1939).

[5] L. Brillouin, *Les Tenseurs en Mécanique et en Elasticité* (Dover Publications, New York, 1946), pp. 11, 216–217.

[6] I. S. Sokolnikoff, *Mathematical Theory of Elasticity* (McGraw-Hill Book Company, Inc., New York, 1946), p. 43.

intensity. The integral can be transformed by use of Maxwell's equations. The result is

$$\frac{dW}{dt}=\frac{d}{dt}\left[\frac{1}{2\gamma}\int \mathbf{H}^2 d\tau\right]+\int \mathbf{H}\cdot\frac{\partial \mathbf{M}}{\partial t}d\tau \tag{6}$$

plus terms that are unimportant in magnetostatics. The constant γ is 4π in Gaussian units. Up to this point the derivation is completely general. In the next step the quantity $(\partial\mathbf{M}/\partial t)dt$, the change of magnetization at a fixed point of space, is identified with the change $d\mathbf{M}$ of magnetization at a definite point in a body:

$$dW=\text{perfect differential}+\int \mathbf{H}\cdot d\mathbf{M}d\tau. \tag{7}$$

This step is valid only for rigid bodies and, under special conditions, for fluids. In the final step a transition is made from a whole body to a volume element, by removing the integral sign (Eq. (1)). This step has no mathematical justification, for the preceding formula has been established only when the integral is extended over all space. The final formula is therefore not a logical consequence of the preceding argument; it is an additional postulate suggested by it.

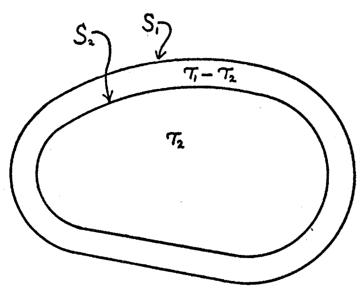

FIG. 1. Definition of the *magnetic* or *long-range* force on an arbitrary part of a magnetized body. First calculate, by macroscopic methods, the force exerted on the matter inside S_2 by the matter outside S_1; then let $S_1 \rightarrow S_2$. Deviations of the actual force from the force thus calculated are interpretable in terms of *stresses*. Such deviations are due partly to the presence of nonmagnetic short-range forces and partly to inaccuracy of the macroscopic magnetic formulas at molecular distances. In a macroscopic theory, the two terms cannot be distinguished, and even the separation of the total force into a long-range term and a stress term can be made in several ways.

Other derivations use other postulates. Some of these postulates concern the nature of the "effective field" inside a magnetized body. Some concern the localization of energy in the electromagnetic field, or the derivability of forces from tensors. The literature in this field is full of speculations and controversy. Livens[7] at his death in 1950 was still publishing polemics on the subject. Sommerfeld[8] shortly before his death published a paper on the subject, and Diesselhorst,[9] Gans,[10] and Döring[11] promptly rushed into print to refute it.

Are such postulates necessary, or can the general problem of magnetic forces and energies be handled by using only the laws of mechanics and the formula for the forces between dipoles? I once thought that additional postulates were necessary, because the dipole formula does not lead to any unique expression for the "effective field intensity" inside a magnetized body. Several years ago, however, I decided that despite this fact, the dipole formula should be sufficient, without additional postulates. My reasoning is illustrated by Fig. 1.

The basic problem is to calculate the magnetic force and torque on an arbitrary part of a magnetized body. To do this, let us first calculate the force and torque exerted on the matter inside a surface S_2 by the matter outside a slightly larger surface S_1; and then let us find the limit as the surfaces come together. For this calculation we do not need to know anything about an effective field inside S_2; all we need to know is the part of the field inside S_2 that is due to magnetized matter outside S_1, and this we know how to calculate.

The result can be written in several equivalent forms; here is one of them:

$$\mathbf{F}=\int \mathbf{M}\cdot\nabla\mathbf{H}d\tau+\tfrac{1}{2}\gamma\int \mathbf{n}M_n{}^2 dS, \tag{8}$$

$$\mathbf{L}=\int \mathbf{r}\times[\mathbf{M}\cdot\nabla\mathbf{H}]d\tau +\tfrac{1}{2}\gamma\int \mathbf{r}\times[\mathbf{n}M_n{}^2]dS+\int \mathbf{M}\times\mathbf{H}d\tau. \tag{9}$$

These formulas are rather distressing because each contains a surface integral that cannot be transformed into a volume integral. The force is not just so much per unit volume; it also contains a term that depends on the shape of the volume considered. Such forces are quite different from the "body forces" allowed in the standard theory of elasticity. Therefore, how can we justify applying the elasticity formulas to a magnetized body?

In the end I decided that there was only one way out of this difficulty: to abandon the standard theory of elasticity, and to develop, instead, a theory that would take account of the peculiar nature of magnetic forces.

It turned out that the modifications needed were rather minor. One is the asymmetry of the stress tensor mentioned by Brillouin and Sokolnikoff:

$$Y_z-Z_y=(\mathbf{M}\times\mathbf{H})_x; \tag{10}$$

this comes from the existence of a couple per unit

[7] G. H. Livens, Phil. Mag. **36**, 1 (1945) and **38**, 453–479 (1947); Phys. Rev. **71**, 58–63 (1947); Proc. Cambridge Phil. Soc. **44**, 534–545 (1948). Editorial note, Proc. Cambridge Phil. Soc. **47**, 450 (1951).

[8] A. Sommerfeld and F. Bopp, Ann. Physik **8**, 41–45 (1950).

[9] H. Diesselhorst, Ann. Physik **9**, 316–324 (1951).

[10] Richard Gans, Ann. Physik **9**, 337–340 (1951).

[11] W. Döring, Ann. Physik **9**, 363–372 (1951).

volume, $\mathbf{M}\times\mathbf{H}$. Then there are some extra terms in the surface traction formula:

$$X_\nu=[X_x+\tfrac{1}{2}\gamma(M_x^2-M_n^2)]l+X_y m+X_z n. \quad (11)$$

Finally, there are extra terms in the equations of motion:

$$(\partial/\partial x)(X_x+\tfrac{1}{2}\gamma M_x^2)+(\partial X_y/\partial y) +(\partial X_z/\partial z)+\mathbf{M}\cdot\nabla H_x+\rho X=\rho f_x. \quad (12)$$

Once these formulas were derived, the special case of a fluid was easy to treat. All the electrostrictive or magnetostrictive properties of a fluid in equilibrium came out of the theory directly, without introduction of thermodynamic arguments as such. But the results were very perplexing. All the final formulas for observable quantities were identical with those derived by the energy method and given, for instance, in Abraham-Becker.[12] Yet some of the intermediate formulas were incompatible with those of Becker. For instance, according to Becker a fluid is a substance in which "there is only one kind of elastic stress, namely equal pressure in all directions"; but according to my formulas the pressure in a polarized fluid depends on the orientation of the surface element across which the pressure is being computed. This same anisotropy of of pressure had been found earlier by Guggenheim.[13]

Eventually it became clear that these differences were differences only of definition: my "pressure" and Becker's "pressure" were different quantities. The principle involved here is the following. There is no unique way of separating the total force into a long-range or "magnetic" force and a short-range force described in terms of "stresses." The reason is that "long-range" forces, such as magnetic forces, do not act exclusively over large distances; they are also effective at small distances. The expression for the magnetic force can always be changed by adding a term that affects only the short-range behavior; for the change can be neutralized by making a corresponding change in the definition of the stresses. Whether such a term is to be considered part of the magnetic force or part of the stress system is purely a matter of definition, and it happens that my definition is different from Becker's. The reason that this nonuniqueness has escaped attention so long is that most of the calculations have been done by energy methods, rather than by direct calculation of the forces. In an energy method, it is difficult to determine what definition of magnetic force is implied by the equations.

Once this nonuniqueness principle has been grasped, most of the controversy in the literature becomes pointless. The question is not "Which of these formulas is right?" It is "What definition of magnetic force and stress is implied by each of these formulas?" It is not important which definitions are used, but it is essential that the stress formulas used be the ones appropriate to the force definition adopted.

With this apparent discrepancy explained, the next step was to calculate the work done, in an arbitrary change of magnetization and deformation, upon an arbitrary mass m. One form of the result is the following. The work done can be separated into two terms:

$$dW=dU_m+\int_m dF dm. \quad (13)$$

The first term is the differential of

$$U_m=-\frac{1}{2}\int_\tau \mathbf{M}\cdot\mathbf{H}_1 d\tau; \quad (14)$$

here τ is the volume instantaneously occupied by the mass m, and $\mathbf{H}_1$ is the contribution to $\mathbf{H}$ from magnetization in τ. The value of U_m for two parts of a body combined is not the sum of its values for the separate parts. The other term of dW does have this additivity property but is not in general a perfect differential:

$$dF=\mathbf{H}\cdot D\mathbf{M}+V[(X_x+\tfrac{1}{2}\gamma M_x^2)de_{xx}+\cdots +\tfrac{1}{2}(Y_z+Z_y)de_{yz}+\cdots]. \quad (15)$$

Here $\mathbf{M}$ is the moment and V the volume of unit mass, and the symbol D is used for a change computed with respect to axes attached to the mass element. In this formula the strains are not necessarily infinitesimal: de_{xx} etc. are defined not as differentials of strains but as products of dt by velocity strains.

The separation into two terms is not unique; it can be made in various ways, corresponding to choice of $\mathbf{H}$, of $\mathbf{B}$, or of something else as independent field variable. The ultimate formulas for observable quantities will be independent of this choice.

For reversible isothermal processes, the term with the additivity property becomes the differential of a free energy. Since the formula holds for an arbitrary part of a material body, and not just for all space, the integrand is the differential of a free energy per unit mass. This differential is given by an expression (Eq. (15)) more complicated than that of the traditional theory.

For elastic strains, the free energy per unit mass can be expressed as a series in the strains, through degree 2, with coefficients that are functions of the components of specific magnetization $\mathbf{M}$ in axes fixed in the mass element. Symmetry can be used, high order terms neglected, and so on, as in the traditional theory. In this way the theory of magnetization and magnetostriction can be put on a sound basis. If the quantity being calculated in shape-dependent, that fact will emerge automatically from the calculation, because everything relevant has been put into the basic equations. No restrictive assumptions have been made, and no arbitrary postulates have been introduced.

[12] Max Abraham, *The Classical Theory of Electricity and Magnetism*, revised by Richard Becker, translated by John Dougall (G. E. Stechert and Company, New York, 1932), Chapter V.

[13] E. A. Guggenheim, Proc. Roy. Soc. (London) **A155**, 49–70 and 70–101 (1936).

I have worked out the details of such a calculation for an ellipsoidal specimen of isotropic material, in a state of uniform magnetization and strain. The problem is to find the applied tractions necessary to maintain such a strain, and in particular to find whether the assumed uniform strain is compatible with zero surface traction. It turns out that it is not. The magnetostriction is therefore not uniform, as was assumed by Becker in his form effect calculation. The actual strain can be calculated approximately by the following method: first find the surface tractions necessary to maintain zero strain; then imagnine the negative of these tractions applied to an ordinary elastic, nonmagnetic body, and calculate the resulting strains. The second step is difficult to carry out completely, except for a sphere; but the *average* strain is easy to compute. It turns out to be equal to the strain that Becker computed by assuming uniform strain and minimizing the energy.

The details of this special calculation are given in an appendix to this paper. The general theory appeared in the *American Journal of Physics* in 1951.[14,15] I hope that it will prove useful in other problems in magnetization and magnetostriction.

APPENDIX. MAGNETOSTRICTION OF AN ELLIPSOID

In Voigt's[16] notation ($x_1=e_{xx},\cdots,\ x_4=e_{yz},\cdots$), for small elastic strains

$$F=A+\textstyle\sum_i B_i x_i+\frac{1}{2}\sum_i\sum_j C_{ij}x_ix_j, \tag{A1}$$

where A, B_i, and $C_{ij}=C_{ji}$ are functions of $\mathfrak{m}_x$, $\mathfrak{m}_y$, and $\mathfrak{m}_z$. The strain and specific magnetization components are referred to axes that rotate with the mass element (reference 14, Sec. 4.4).

Suppose first that $\mathbf{M}$ is everywhere along the local z axis: then for isotropic material, by symmetry, $B_1=B_2$, $B_4=B_5=B_6=0$, and the nonvanishing C's are $C_{11}=C_{22}$, C_{33}, $C_{44}=C_{55}$, C_{66}, $C_{12}=C_{11}-2C_{66}$, and C_{13} (reference 1, p. 160, Sect. 110, (2)). Since $M_x=M_y=0$, the conditions that dF in Eq. (15) be a perfect differential give

$$\begin{aligned}VX_x&=\partial F/\partial x_1=B_1+C_{11}x_1+C_{12}x_2+C_{13}x_3,\\ VY_y&=\partial F/\partial x_2=B_1+C_{12}x_1+C_{11}x_2+C_{13}x_3,\\ V(Z_z+\tfrac{1}{2}\gamma M^2)&=\partial F/\partial x_3=B_3+C_{13}(x_1+x_2)+C_{33}x_3;\end{aligned} \tag{A2}$$

$$\begin{aligned}V\cdot\tfrac{1}{2}(Y_z+Z_y)&=\partial F/\partial x_4=C_{44}x_4,\\ V\cdot\tfrac{1}{2}(Z_x+X_z)&=\partial F/\partial x_5=C_{44}x_5,\\ V\cdot\tfrac{1}{2}(X_y+Y_x)&=\partial F/\partial x_6=C_{66}x_6.\end{aligned} \tag{A3}$$

Now suppose further that the rotation vanishes; that the only nonvanishing strains are an extension x_1 along the fixed x and y axes and an extension x_3 along the fixed z axis; and that x_1, x_3, and $\mathfrak{m}=\mathfrak{m}_z$ are all independent of (x, y, z). Then since the magnetization is along and the strain symmetric about the z axis, $H_x(=\partial F/\partial\mathfrak{m}_x)$ and $H_y(=\partial F/\partial\mathfrak{m}_y)$ must vanish; and the relation $H_z=\partial F/\partial\mathfrak{m}_z$ requires that H_z be independent of (x, y, z). Since $\mathfrak{m}$ and the dilatation are independent of (x, y, z), so too is M. If, therefore, the specimen is an ellipsoid, the assumed state of magnetization and strain can be maintained by a uniform applied field and by suitable surface tractions, which will now be determined.

Under the assumed conditions $\mathbf{M}\times\mathbf{H}=0$, so that $Y_z-Z_y=0$, etc., by Eq. (10); and since $x_4=x_5=x_6=0$, $Y_z+Z_y=0$, etc. by Eqs., (A3). Therefore $Y_z=Z_y=0$, etc. The remaining stress components are given by Eqs. (A2). To the first order in the strains,

$$1/V=(1/V_0)(1-2x_1-x_3), \tag{A4}$$

where V_0 is the specific volume in the unstrained state, and therefore

$$\begin{aligned}X_x=Y_y&=b_1+(c_{11}+c_{12})x_1+c_{13}x_3,\\ Z_z+\tfrac{1}{2}\gamma M^2&=b_3+2c_{31}x_1+c_{33}x_3,\end{aligned} \tag{A5}$$

with

$$b_i=B_i/V_0,\quad c_{ij}=(C_{ij}-B_i)/V_0. \tag{A6}$$

(Note that if $B_3\neq B_1$, then $c_{13}\neq c_{31}$.)

The tractions on the surface of the ellipsoid are, by Eqs. (11),

$$\begin{aligned}X_\nu&=(X_x-\tfrac{1}{2}\gamma M_n^2)l,\quad Y_\nu=(Y_y-\tfrac{1}{2}\gamma M_n^2)m,\\ Z_\nu&=[Z_z+\tfrac{1}{2}\gamma(M^2-M_n^2)]n.\end{aligned} \tag{A7}$$

Substitution of Eqs. (A5) in Eqs. (A7) gives

$$\begin{aligned}X_\nu/l=Y_\nu/m&=(c_{11}+c_{12})x_1+c_{13}x_3+b_1-\tfrac{1}{2}\gamma M^2n^2,\\ Z_\nu/n&=2c_{31}x_1+c_{33}x_3+b_3-\tfrac{1}{2}\gamma M^2n^2.\end{aligned} \tag{A8}$$

Consistently with the symmetry already assumed, suppose the ellipsoid to be one of revolution about the z axis. Then if the ellipsoid is finite, the quantities on the right vary with position on the surface because of the term $-\frac{1}{2}\gamma M^2n^2$. Consequently, no values of x_1 and x_3 will make X_ν, Y_ν, and Z_ν vanish at all points of the surface. The assumed uniform state is not a possible one for an ellipsoid with its surface free; such a state can be maintained only by applying forces to the surface.

The tractions required to maintain zero strain are

$$\begin{aligned}X_{\nu0}&=(b_1-\tfrac{1}{2}\gamma M^2n^2)l,\\ Y_{\nu0}&=(b_1-\tfrac{1}{2}\gamma M^2n^2)m,\\ Z_{\nu0}&=(b_3-\tfrac{1}{2}M^2n^2)n.\end{aligned} \tag{A9}$$

The actual magnetostrictive strains can be calculated approximately by finding the strains produced in a

[14] William Fuller Brown, Jr., Am. J. Phys. 19, 290–304 and 333–350 (1951).

[15] The work reported here grew out of the author's activities as a member of the Coulomb's Law Committee of the American Association of Physics Teachers. The Committee, under the chairmanship of Professor E. C. Kemble, was appointed in 1944 and presented its report in Am. J. Phys. 18, 1–25 and 69–88 (1950).

[16] Woldemar Voigt, *Lehrbuch der Kristallphysik* (B. G. Teubner, Leipzig, 1910 and 1928), p. 563.

nonmagnetic elastic ellipsoid by the tractions $(-X_{\nu 0}, -Y_{\nu 0}, -Z_{\nu 0})$. In the limiting case of a long needle-ellipsoid, the term $-\frac{1}{2}\gamma M^2 n^2$ becomes negligible except near the ends, and the strains are those due to tractions $(-b_1 l, -b_1 m, -b_3 n)$. This part of the strain is independent of position and corresponds to uniform stresses $X_x = Y_y = -b_1$, $Z_z = -b_3$ in the nonmagnetic ellipsoid. With respect to it as standard, the "form effect" strain is that produced by normal tension $\frac{1}{2}\gamma M^2 n^2$.

The strains produced by such a normal tension can be computed for a sphere by methods described in reference 1, Chapter XI. For a nonspherical ellipsoid of revolution, the complete solution would require an extension of methods at present available.[17] However, the *volume average* strains can be computed by formula (13) of reference 1, p. 175:

$$\bar{x}_1 = (\gamma M^2/2EV)\int [lx - \sigma(my+nz)]n^2 dS,$$
$$\bar{x}_3 = (\gamma M^2/2EV)\int [nz - \sigma(lx+my)]n^2 dS. \quad \text{(A10)}$$

Here E is Young's modulus, σ is Poisson's ratio, and V is the ellipsoid volume. For a prolate spheroid, the parametric equation of the surface in cylindrical coordinates is

$$z = a\cos\theta, \quad \rho = b\sin\theta, \quad \text{(A11)}$$

and the volume is $\pi ab^2/3$. By integration over ϕ and introduction of the eccentricity $e = (1-b^2/a^2)^{\frac{1}{2}}$, the problem can be reduced to the evaluation of the two integrals

^{17} M. A. Sadowsky and E. Sternberg, J. Appl. Mech. **14**, A191–A201 (1947).

$$J_1 = \int_0^1 u^2(1-e^2u^2)^{-1}du = e^{-3}\{\tfrac{1}{2}\ln[(1+e)/(1-e)] - e\}$$
$$= N/[4\pi(1-e^2)], \quad \text{(A12)}$$

$$J_2 = \int_0^1 u^4(1-e^2u^2)^{-1}du = e^{-2}(J_1 - \tfrac{1}{3}). \quad \text{(A13)}$$

Here N is the longitudinal demagnetizing factor.

The strains given by Eq. (A10) consist of a pure dilatation

$$x_1' = x_3' = \gamma M^2(1-e^2)J_1/6k = (\gamma/4\pi)NM^2/6k \quad \text{(A14)}$$

(k = bulk modulus) and an equivoluminal strain

$$-2x_1'' = x_3'' = \gamma M^2(1-e^2)(3J_2 - J_1)/4G$$
$$= (\gamma/4\pi)[(3-e^2)N - 4\pi(1-e^2)]M^2/4e^2G \quad \text{(A15)}$$

(G = rigidity). These agree with Becker's formulas for the form effect, which he assumed to be uniform.

This method of evaluating the strains is an approximation, not merely because it neglects the variation of "elastic constants" (c_{ij}) with magnetization, but also because it neglects in all the equations some terms that no longer vanish when the strain varies with position. A rigorous calculation would require simultaneous solution of the equations of elastic equilibrium and of Poisson's equation and would probably show that the magnetization as well as the strain varies with position.

My crystal calculation of 1945,[18] which led to a paradox, was valid for a rigid crystal; but for a crystal capable of deformation, the derivation holds only if the approximations of the traditional theory are accepted. The theory outlined here provides a method of carrying out a rigorous calculation for a deformable crystal.

[18] William F. Brown, Jr., Revs. Modern Phys. **17**, 15–19 (1945).