The Mathematical Theory of Electrical Discharges in Gases

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The Boltzmann theory treatment of electrical discharges in gases has been one of the undeveloped fields in physics. The aim of this paper is to show what type of molecular model makes the mathematical treatment of the problem possible.

The fundamental processes of the molecular kinetic theory of electrical discharges concern the collisions between gas molecules and charged particles such as ions or electrons. The effective cross sections of the molecule for elastic, exciting, and ionizing collisions are the fundamental quantities regarding these phenomena. Two methods are considered, according to how these fundamental quantities are introduced, to establish the molecular kinetic theory.

One method is to adopt a proper molecular model for collision processes. The other method is to adopt the "exact" (i.e., observed or quantum-mechanically calculated) values of collision cross sections. This paper concerns itself with the first method, and the principal objective is to attain mathematical simplicity for the theory. The second method is abandoned because the statistical calculation on each step is complicated, and furthermore, the calculation must be achieved separately for each kind of gas. This second method may be adequately adopted after the treatment based on a model is accomplished.

CONTENTS

I. BASIS OF THE KINETIC THEORY OF CHARGED PARTICLES IN GASES

1. Introduction

BEFORE we proceed to the main subject of electrical discharges, let us summarize certain features of an ideal gas. Take the simple gas, i.e., a gas composed of one kind of molecule, and consider the distribution of molecular velocities, c_1. Let the number of molecules, whose x, y, z component of velocity c_1 is between u_1 and u_1+du_1, v_1 and v_1+dv_1, w_1 and w_1+dw_1, respectively, be per unit volume,

$$f_1(c_1)du_1dv_1dw_1 \equiv f_1(c_1)dc_1. \qquad (1.1)$$

f_1 is called the velocity-distribution function. The triple integration of this function with respect to the components of the molecular velocity,

$$N = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(c_1)du_1dv_1dw_1 \equiv \int f_1(c_1)dc_1, \qquad (1.2)$$

is the number-density of molecules.

Let $\phi(c_1)$ be any function of the molecular velocity; then the mean value of ϕ is defined by

$$\bar{\phi} = \frac{1}{N} \int \phi f_1 dc_1. \qquad (1.3)$$

In case the gas is uniform, f_1 takes the Maxwellian distribution,

$$f_1 = N(m_1/2\pi kT)^{\frac{3}{2}} \exp[-(m_1 c_1^2/2kT)] \quad c_1 = |c_1|, \qquad (1.4)$$

where m_1 denotes the mass of the molecule and k the Boltzmann constant. T is the absolute temperature of the gas, whose relation to the mean energy of translational motion of molecules is

$$\tfrac{3}{2}kT = \tfrac{1}{2}m_1\bar{c}_1^2, \qquad (1.5)$$

or

$$kT = m_1\bar{u}_1^2 = m_1\bar{v}_1^2 = m_1\bar{w}_1^2. \qquad (1.6)$$

Let P be the pressure of the gas, then the equation of state is

$$P = NkT, \qquad (1.7)$$

or

$$P = Nm_1\bar{u}_1^2 = Nm_1\bar{v}_1^2 = Nm_1\bar{w}_1^2. \qquad (1.8)$$

If the gas contains charged particles such as positive ions or electrons, the velocity-distribution function $f(c)$ of the charged particles of each kind is to be considered similarly.

$$n = \int f(c)dc \qquad (1.9)$$

is the number-density of the charged particles. For any function ϕ of the velocity c of the charged particles, the mean value is given by

$$\bar{\phi} = \frac{1}{n} \int \phi f dc. \qquad (1.10)$$

In the case when the density of the charged particles is uniform and no external force is acting, the velocity-distribution of the charged particles is also Maxwellian,

$$f = n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{mc^2}{2kT}\right) \quad c = |\mathbf{c}|, \quad (1.11)$$

characterized by the gas temperature T, m being the mass of the charged particle. The same relation as (1.5),

$$\tfrac{3}{2}kT = \tfrac{1}{2}m\bar{c}^2,$$

holds for the mean translational kinetic energy of the charged particles.

In Part I the motion of charged particles in gases is investigated generally. The treatment of this part is restricted, however, to the phenomena in which only elastic collisions occur.

2. The Boltzmann Equation

The velocity-distribution of charged particles in a gas with uniform temperature and pressure is not necessarily Maxwellian if their density is not uniform or if an external electric field exists. In cases of this nature, their velocity-distribution function, f, would depend not only on the velocity, $\mathbf{c}$, but also on the position, $\mathbf{r}$, and the time, t. The fundamental equation which determines this function $f(\mathbf{c}, \mathbf{r}, t)$ is that of Boltzmann,

$$\frac{\partial f}{\partial t} + \mathbf{c}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{e}{m}\mathbf{E}\cdot\frac{\partial f}{\partial \mathbf{c}} = \frac{\partial_e f}{\partial t}, \quad (2.1)$$

where the second term on the left side is the scalar product of $\mathbf{c}$ and the gradient of f,

$$\mathbf{c}\cdot\frac{\partial f}{\partial \mathbf{r}} \equiv u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}.$$

The third term is the scalar product of the external electric field $\mathbf{E}$ and the gradient of f in the velocity space, multiplied by e/m, e and m being the charge and the mass of the charged particles, respectively (Chapman-Cowling's[1] notations are used throughout this
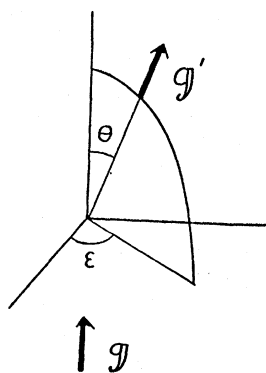

FIG. 1. Change of direction of the relative velocity as a result of an encounter.

[1] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1939).

treatise). The right side denotes the change of f caused by encounters which we will consider later.

When a magnetic field, $\mathbf{H}$, and an electric field, $\mathbf{E}$, exist, the force acting on charged particles is $(e/m)(\mathbf{E}+\mathbf{c}\times\mathbf{H})$, where e, $\mathbf{E}$, and $\mathbf{H}$ are all expressed in ane lectrostatic unit. (If we measure e and $\mathbf{E}$ in cgs electrostatic units, the value of $\mathbf{H}$ equals its values in Gauss units divided by light velocity 3×10^{10}.) In this case Boltzmann's equation is generalized into

$$\frac{\partial f}{\partial t} + \mathbf{c}\cdot\frac{\partial f}{\partial \mathbf{c}} + \frac{e}{m}(\mathbf{E}+\mathbf{c}\times\mathbf{H})\cdot\frac{\partial f}{\partial \mathbf{c}} = \frac{\partial_e f}{\partial t}. \quad (2.2)$$

The right side of Boltzmann's equation is the time variation of f as a result of encounters with other charged particles and gas molecules. When the density of charged particles is much smaller than that of the gas molecules, collisions of the charged particles with each other can be neglected in contrast to their collisions with gas molecules. We assume this condition, since in this case Boltzmann's equation becomes linear for the velocity-distribution of charged particles, and furthermore, we can disregard the diverging cross section of collisions between charged particles.

Under this condition $\partial_e f/\partial t$ is expressed in the form

$$\frac{\partial_e f}{\partial t} = \iiint (f'f_1' - ff_1)gI(g, \theta)\,\sin\theta\, d\theta\, d\epsilon\, d\mathbf{c}_1, \quad (2.3)$$

where $\mathbf{c}_1$ and f_1 denote the velocity and velocity-distribution function of gas molecules, respectively; $f'f_1' - ff_1$ is the abbreviation of $f(\mathbf{c}')f_1(\mathbf{c}_1') - f(\mathbf{c})f_1(\mathbf{c}_1)$; $\mathbf{c}'$ and $\mathbf{c}_1'$ are the final velocities of the charged particle and gas molecule encountering with initial velocities $\mathbf{c}$ and $\mathbf{c}_1$ with deviation angle θ for the orbit of relative motion; $I(g, \theta)\,\sin\theta\, d\theta\, d\epsilon$ is the differential cross section for the relative speed, $g = |\mathbf{c} - \mathbf{c}_1| = |\mathbf{c}' - \mathbf{c}_1'|$, to be scattered into the solid angle $\sin\theta\, d\theta\, d\epsilon$ (see Fig. 1). ($f'f_1' - ff_1$ is proportional to the difference in the frequencies between direct and inverse encounters, see Fig. 2.)

3. Auxiliary Theorem

The right side of Boltzmann's equation (2.2) for the velocity-distribution function $f(\mathbf{c}, \mathbf{r}, t)$ of the charged particles, can be treated as follows. If the electric field, $\mathbf{E}$, and the magnetic field, $\mathbf{H}$, were zero, and the density of the particles were uniform, f would be Maxwellian (1.11),

$$f^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{mc^2}{2kT}\right), \quad (3.1)$$

and the condition of the detailed balancing,

$$f^{(0)'}f_1' - f^{(0)}f_1 = 0, \quad (3.2)$$

would hold. Generally let

$$f = f^{(0)}(1 + \Phi), \quad (3.3)$$

where Φ is not necessarily a small quantity. Inserting (3.3) into (2.3) and considering Eq. (3.2), we have

$$\frac{\partial_e f}{\partial t} = -Nf^{(0)}J\Phi. \tag{3.4}$$

$$J\Phi = \frac{1}{N}\int\int\int f_1(\Phi - \Phi')gI(g, \theta)\,\sin\theta d\theta d\epsilon d\mathbf{c}_1, \tag{3.5}$$

where J is a linear operator operating on any function $\phi(\mathbf{c})$ of the velocity of the charged particle; it is defined generally as

$$J\phi = \frac{1}{N}\int\int\int f_1(\phi - \phi')gI(g, \theta)\,\sin\theta d\theta d\epsilon d\mathbf{c}_1. \tag{3.6}$$

Here N is the number-density of the gas molecules, ϕ' denotes $\phi(\mathbf{c}')$, $\mathbf{c}'$ being the velocity of the charged particle after encounter.

Let us define the inner product of two functions $\phi(\mathbf{c})$ and $\psi(\mathbf{c})$ by

$$(\phi, \psi) = \frac{1}{n}\int f^{(0)}\phi\psi d\mathbf{c}, \tag{3.7}$$

and consider the following expression,

$$(\psi, J\phi) = \frac{1}{n}\int f^{(0)}\psi J\phi d\mathbf{c}$$

$$= \frac{1}{nN}\int\int\int\int f^{(0)}f_1(\phi - \phi')\psi gI(g, \theta)$$

$$\times \sin\theta d\theta d\epsilon d\mathbf{c}_1 d\mathbf{c}. \tag{3.8}$$

Considering the inverse encounters, and taking into account Eq. (3.2) and the relation $d\mathbf{c}_1'd\mathbf{c}' = d\mathbf{c}_1 d\mathbf{c}$, we can in the integral,

$$\int\int\int\int f^{(0)}f_1\phi'\psi gI(g, \theta)\,\sin\theta d\theta d\epsilon d\mathbf{c}_1 d\mathbf{c},$$

exchange ϕ' for ϕ and ψ for ψ'. The expression (3.8), therefore, is equal to

$$\frac{1}{nN}\int\int\int\int f^{(0)}f_1\phi(\psi - \psi')gI(g, \theta)\,\sin\theta d\theta d\epsilon d\mathbf{c}_1 d\mathbf{c},$$

and we have the symmetric property of the operator J,

$$(\psi, J\phi) = (\phi, J\psi). \tag{3.9}$$

Similarly, expression (3.8) is equal to

$$\frac{1}{2}\cdot\frac{1}{nN}\int\int\int\int f^{(0)}f_1(\phi - \phi')(\psi - \psi')gI(g, \theta)$$

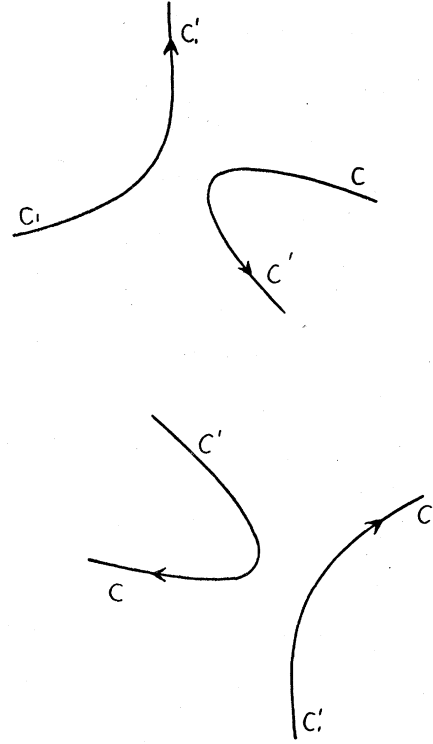$$\times \sin\theta d\theta d\epsilon d\mathbf{c}_1 d\mathbf{c}$$

FIG. 2. The direct and the inverse encounter.

and, when we let $\phi = \psi$,

$$(\phi, J\phi) \geqq 0, \tag{3.10}$$

i.e., J is a positive-definite operator.

Now, the Boltzmann equation (2.2) is expressed as follows

$$\frac{\partial f}{\partial t} + \mathbf{c}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{e}{m}(\mathbf{E}+\mathbf{c}\times\mathbf{H})\cdot\frac{\partial f}{\partial \mathbf{c}} = -Nf^{(0)}J\Phi. \tag{3.11}$$

Multiplying $\phi(\mathbf{c})d\mathbf{c}$ on both sides and integrating, we have

$$\frac{\partial}{\partial t}\int f\phi d\mathbf{c} + \frac{\partial}{\partial \mathbf{r}}\cdot\int f\mathbf{c}\phi d\mathbf{c}$$

$$+ \frac{e}{m}\int (\mathbf{E}+\mathbf{c}\times\mathbf{H})\cdot\frac{\partial f}{\partial \mathbf{c}}\phi d\mathbf{c} = -N\int f^{(0)}\phi J\Phi d\mathbf{c}.$$

On the left side we integrate by parts the third term and consider the identity

$$\partial/\partial\mathbf{c}\cdot(\mathbf{c}\times\mathbf{H}) = 0.$$

On the right side we use the relation $J(1+\Phi) = J\Phi$ and the symmetric property of J; then we get

$$\frac{\partial}{\partial t}\int f\phi d\mathbf{c} + \frac{\partial}{\partial \mathbf{r}}\cdot\int f\mathbf{c}\phi d\mathbf{c}$$

$$- \frac{e}{m}\int f(\mathbf{E}+\mathbf{c}\times\mathbf{H})\cdot\frac{\partial\phi}{\partial\mathbf{c}}d\mathbf{c} = -N\int fJ\phi d\mathbf{c},$$

i.e., with the mean value symbols, $\langle\ \rangle_{Av}$ or a bar over letter,

$$\frac{\partial}{\partial t}(n\bar{\phi})+\frac{\partial}{\partial r}\cdot(n\langle c\phi\rangle_{Av})$$

$$-\frac{en}{m}\left\langle(E+c\times H)\cdot\frac{\partial\phi}{\partial c}\right\rangle_{Av}=-Nn\langle J\phi\rangle_{Av}. \quad (3.12)$$

This is the fundamental equation to be used in the following sections.

Let $\phi=1$, then we have the equation of continuity,

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial r}\cdot(n\langle c\rangle_{Av})=0, \quad (3.13)$$

where $n\langle c\rangle_{Av}$ is the flow of the charged particles.

If we let $\phi=c$, we get

$$\frac{\partial}{\partial t}(n\langle c\rangle_{Av})+\frac{\partial}{\partial r}\cdot(n\langle cc\rangle_{Av})$$

$$-\frac{en}{m}(E+\langle c\rangle_{Av}\times H)=-Nn\langle Jc\rangle_{Av}, \quad (3.14)$$

where cc in the second term denotes such a tensor as two vectors a and b determine

$$ab=\begin{bmatrix}a_x b_x & a_x b_y & a_x b_z\\a_y b_x & a_y b_y & a_y b_z\\a_z b_x & a_z b_y & a_z b_z\end{bmatrix}.$$

The differentiation of the second term is to be interpreted as

$$\left(\frac{\partial}{\partial r}\cdot ab\right)_x=\frac{\partial a_x b_x}{\partial x}+\frac{\partial a_y b_x}{\partial y}+\frac{\partial a_z b_x}{\partial z}.$$

$n\langle cc\rangle_{Av}$ is, so to speak, the partial pressure of the charged particles divided by their mass, m.

4. Quasi-Homogeneous Phenomena

In dealing with Boltzmann's equation, let us confine ourselves within the "quasi-homogeneous phenomena," i.e., let us assume that the spatial variations of such quantities as E, H, $\langle c\rangle_{Av}$, and $\langle cc\rangle_{Av}$ are small.

The spatial variation of $\langle c\rangle_{Av}$ being small, we have in place of (3.13), the equation of continuity

$$\frac{\partial n}{\partial t}+\langle c\rangle_{Av}\cdot\frac{\partial n}{\partial r}=0. \quad (4.1)$$

Substituting (4.1) into (3.14) and taking account of the smallness of $\partial\langle cc\rangle_{Av}/\partial r$, we get

$$N\langle Jc\rangle_{Av}+\frac{\partial\langle c\rangle_{Av}}{\partial t}+(\langle cc\rangle_{Av}-\langle c\rangle_{Av}\langle c\rangle_{Av})\cdot\frac{1}{n}\frac{\partial n}{\partial r}$$

$$-\frac{e}{m}(E+\langle c\rangle_{Av}\times H)=0. \quad (4.2)$$

Now let us consider the tensor

$$m(\langle cc\rangle_{Av}-\langle c\rangle_{Av}\langle c\rangle_{Av})=m\langle(c-\langle c\rangle_{Av})(c-\langle c\rangle_{Av})\rangle_{Av}.$$

If the velocity-distribution of charged particle is spherically symmetric around the drift velocity $\langle c\rangle_{Av}$, the tensor is diagonal and the diagonal elements are all the same,

$$m\langle(u-\bar{u})^2\rangle_{Av}=m\langle(v-\bar{v})^2\rangle_{Av}=m\langle(w-\bar{w})^2\rangle_{Av}.$$

By equating these elements to kT_e, the temperature T_e of the charged particle is defined. Generally let us define the temperature of the charged particle T_e as a tensor by the relation

$$m\langle(c-\langle c\rangle_{Av})(c-\langle c\rangle_{Av})\rangle_{Av}=kT_e. \quad (4.3)$$

Then (4.2) becomes

$$N\langle Jc\rangle_{Av}+\frac{\partial\langle c\rangle_{Av}}{\partial t}-\frac{e}{m}\langle c\rangle_{Av}\times H=\frac{e}{m}\left(E-\frac{kT_e}{e}\cdot\frac{\partial\ln n}{\partial r}\right). \quad (4.4)$$

The first term on the right side is the effect of the electric field on the drift of the charged particle, and the second term indicates the effect of the diffusion. The relative decrease in the number-density of the charged particle $-\partial(\ln n)/\partial r$ multiplied by kT_e/e is equivalent to the electric field. We have, therefore, the relation between the mobility and the diffusion-coefficient,

$$\text{diffusion-coefficient}=(kT_e/e)\cdot\text{mobility}. \quad (4.5)$$

In the special case when the tensor T_e degenerates to the scalar T_e,

$$\text{diffusion-coefficient}=(kT_e/e)\cdot\text{mobility}. \quad (4.6)$$

Later we shall see that the velocity-distribution of electrons in an electric field is spherically symmetric around the drift velocity, and we can consider the "electron temperature" as a scalar. Regarding ions in such a weak electric field that we can neglect the square of the field strength, the temperature of ions is equal to the gas temperature T. If we regard ions in a strong field, however, T_e does not generally become scalar. (In the next section we shall call $\frac{1}{3}$ of the diagonal sum of the tensor T_e the temperature of ions.)

The temperature of the charged particle is nearly the same whether the field is homogeneous or quasi-homogeneous; consequently, the effect of diffusion on the temperature can be neglected. Under this condition let us first treat the homogeneous phenomena, in which Boltzmann's equation for the velocity-distribution function of charged particles becomes

$$\frac{\partial f}{\partial t}+\frac{e}{m}(E+c\times H)\cdot\frac{\partial f}{\partial c}=\frac{\partial_e f}{\partial t}. \quad (4.7)$$

The auxiliary theorem regarding any function, ϕ, of the

velocity, **c**, is given by

$$N\langle J\phi\rangle_{Av}+\frac{\partial\bar{\phi}}{\partial t}-\frac{e}{m}\left\langle(\mathbf{E}+\mathbf{c}\times\mathbf{H})\cdot\frac{\partial\phi}{\partial\mathbf{c}}\right\rangle_{Av}=0, \quad (4.8)$$

which, in case $\phi=\mathbf{c}$, reduces to

$$N\langle J\mathbf{c}\rangle_{Av}+\frac{\partial\langle\mathbf{c}\rangle_{Av}}{\partial t}-\frac{e}{m}(\mathbf{E}+\langle\mathbf{c}\rangle_{Av}\times\mathbf{H})=0. \quad (4.9)$$

5. Maxwellian Model of Molecules

The coefficient of elastic scattering, $I(g,\theta)$, between gas molecule and electron or ion is rather complicated. We can, however, much simplify the mathematical treatment by assuming a model for this fundamental quantity—namely, by assuming that $I(g,\theta)$ is inversely proportional to the relative speed g. This type of model was introduced by Maxwell, and it may be reasonable to call it by his name.

When the interaction potential between an ion and a gas molecule is inversely proportional to the fourth power of the distance, (attracting or repulsing) the foregoing assumption holds accurately. In fact, a neutral molecule and an ion attract mutually with an inverse-fourth-power potential beyond a certain distance. In case of ions, therefore, this model for the elastic encounter has some ground. As for electrons, we cannot authorize the assumption in such a manner. However, since the elastic collision cross section of molecules decrease almost monotonously with increasing speed of electrons (excepting Ramsauer effect for slow electron), the assumption aforementioned is not far from the reality.

If $gI(g,\theta)$ does not depend on g,

$$J\mathbf{c}=\lambda\mathbf{c}; \quad (5.1)$$

i.e., the velocity $\mathbf{c}$ of the charged particle is an eigenfunction of the operator J. The corresponding eigenvalue is

$$\lambda=2\pi\frac{m_1}{m_1+m}\int_0^\pi(1-\cos\theta)gI(g,\theta)\sin\theta d\theta,^2 \quad (5.2)$$

which is a constant related to the encountering pair.

Inserting (5.1) into (4.9), we have

$$N\lambda\langle\mathbf{c}\rangle_{Av}+\frac{d\langle\mathbf{c}\rangle_{Av}}{dt}-\frac{e}{m}(\mathbf{E}+\langle\mathbf{c}\rangle_{Av}\times\mathbf{H})=0. \quad (5.3)$$

Let us investigate some special cases.

(i) When only a static electric field exists (i.e., when $\mathbf{H}=0$ and $\partial/\partial t=0$), the drift-velocity is given by

$$\langle\mathbf{c}\rangle_{Av}=K\mathbf{E}, \quad K=e/mN\lambda. \quad (5.4)$$

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[2] T. Kihara, *Imperfect Gases* (originally written in Japanese, translated into English by U. S. Air Force, and to be published as Air Force Technical Reports), Eq. (21.1).

K is called the mobility, which in general depends on the gas temperature or the field strength. In case of Maxwellian model, however, the mobility does not depend on temperature or field strength, and it is a constant except that it is inversely proportional to the number-density N of the gas molecules.

(ii) When the electric and magnetic fields are static (i.e., when $\partial/\partial t=0$),

$$\langle\mathbf{c}\rangle_{Av}-K(\mathbf{E}+\langle\mathbf{c}\rangle_{Av}\times\mathbf{H})=0 \quad (5.5)$$

from which we see that the vector $\langle\mathbf{c}\rangle_{Av}-K\mathbf{E}$ is perpendicular to both $\langle\mathbf{c}\rangle_{Av}$ and $\mathbf{H}$. As shown in Fig. 3, draw from the origin, O, the vector $K\mathbf{E}=OP$, then the end Q of the drift velocity $\langle\mathbf{c}\rangle_{Av}=OQ$ lies on the sphere with diameter OP. Since $\langle\mathbf{c}\rangle_{Av}-K\mathbf{E}$ is perpendicular to $\mathbf{H}$, Q lies also on the plane on which P lies and to which $\mathbf{H}$ is perpendicular. In the special case when $\mathbf{H}$ is perpendicular to $\mathbf{E}$, Q lies on the circle whose diameter is OP and whose plane is perpendicular to $\mathbf{H}$. Generally we have from (5.5)

$$\langle\mathbf{c}\rangle_{Av}=(1+K^2H^2)^{-1}[K\mathbf{E}+K^2(\mathbf{E}\times\mathbf{H})+K^3(\mathbf{E}\cdot\mathbf{H})\mathbf{H}],$$
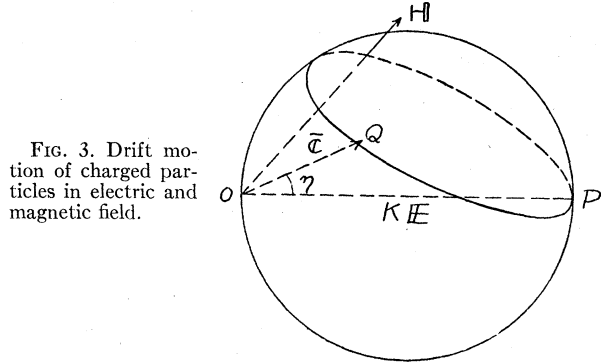$$(5.6)$$

FIG. 3. Drift motion of charged particles in electric and magnetic field.

where $H=|\mathbf{H}|$ is expressed in electrostatic unit. From this equation we have

$$|\langle\mathbf{c}\rangle_{Av}|=KE\cos\eta,$$
$$\cos\eta=[E^2+K^2(\mathbf{E}\cdot\mathbf{H})^2]/[E^2+K^2E^2H^2]. \quad (5.7)$$

η equals angle POQ, i.e., the angle between the electric field $\mathbf{E}$ and the drift velocity $\langle\mathbf{c}\rangle_{Av}$. The absolute value of drift velocity does not change if one imposes a magnetic field and at the same time makes the density of the gas $(\cos\eta)^{-1}$ times as dense as before.

(iii) When only a periodic field

$$\mathbf{E}=\mathbf{E}_0\exp(j\omega t)$$

exists,

$$\langle\mathbf{c}\rangle_{Av}+\frac{1}{N\lambda}\frac{\partial\langle\mathbf{c}\rangle_{Av}}{\partial t}-K\mathbf{E}_0\exp(j\omega t)=0,$$

from which we have as the periodic solution of the drift velocity

$$\langle\mathbf{c}\rangle_{Av}=[1+(j\omega/N\lambda)]^{-1}K\mathbf{E}_0\exp(j\omega t), \quad (5.8)$$

which is the same as the relation between ac voltage and current.

Equation (5.1) indicates that the velocity, **c**, of charged particles is one of the eigenfunctions of the operator J, and the corresponding eigenvalue is λ. Next to **c**, $(mc^2/2kT) - (\frac{3}{2})$ is an important eigenfunction—namely,

$$J\left(\frac{mc^2}{2kT} - \frac{3}{2}\right) = 2\lambda \frac{m}{m_1+m}\left(\frac{mc^2}{2kT} - \frac{3}{2}\right). \quad (5.9)$$

Adopting this eigenfunction as ϕ into the auxiliary theorem (4.8), we have

$$2N\lambda \frac{m}{m_1+m}\left(\frac{m\bar{c}^2}{2kT} - \frac{3}{2}\right) + \frac{d}{dt}\left(\frac{m\bar{c}^2}{2kT} - \frac{3}{2}\right) - \frac{e}{kT}\langle\mathbf{c}\rangle_{Av}\cdot\mathbf{E} = 0.$$

On the other hand, Eq. (5.3) gives

$$e\langle\mathbf{c}\rangle_{Av}\cdot\mathbf{E} = m\langle\mathbf{c}\rangle_{Av}\cdot\left(N\lambda\langle\mathbf{c}\rangle_{Av} + \frac{d\langle\mathbf{c}\rangle_{Av}}{dt}\right).$$

From these two equations we have, after some calculation,

$$2N\lambda\frac{m}{m_1+m}[\tfrac{1}{2}m(c^2 - \langle\mathbf{c}\rangle_{Av}\cdot\langle\mathbf{c}\rangle_{Av}) - \tfrac{3}{2}kT]$$

$$+\frac{d}{dt}[\tfrac{1}{2}m(c^2 - \langle\mathbf{c}\rangle_{Av}\cdot\langle\mathbf{c}\rangle_{Av}) - \tfrac{3}{2}kT] = \frac{m_1 m}{m_1+m}N\lambda\langle\mathbf{c}\rangle_{Av}\cdot\langle\mathbf{c}\rangle_{Av}.$$

If we introduce the temperature of the charged particle T_e by the relation,

$$\tfrac{1}{2}m(c^2 - \langle\mathbf{c}\rangle_{Av}\cdot\langle\mathbf{c}\rangle_{Av}) = \tfrac{1}{2}m\langle|\mathbf{c}-\langle\mathbf{c}\rangle_{Av}|^2\rangle_{Av} = \tfrac{3}{2}kT_e, \quad (5.10)$$

the preceding equation becomes

$$3N\lambda\frac{m}{m_1+m}(kT_e - kT) + \frac{3}{2}\frac{d}{dt}(kT_e - kT)$$

$$= \frac{m_1 m}{m_1+m}N\lambda|\langle\mathbf{c}\rangle_{Av}|^2. \quad (5.11)$$

A characteristic feature of this result is the difference of the temperature of the charged particle, T_e, and the gas temperature, T, is proportional to the square of the drift velocity $|\langle\mathbf{c}\rangle_{Av}|^2$. And, since $|\langle\mathbf{c}\rangle_{Av}|$ is proportional to the field strength E, $T_e - T$ is proportional to E^2.

Especially in static or quasi-static cases, the simple relation,

$$kT_e - kT = \tfrac{1}{3}m_1|\langle\mathbf{c}\rangle_{Av}|^2, \quad (5.12)$$

holds even if a magnetic field exists. Taking into account the effect of the magnetic field on $|\langle\mathbf{c}\rangle_{Av}|$, we find that, as regards the temperature of the charged particle, an imposing of the magnetic field is equivalent to magnification of the gas-density by $(\cos\eta)^{-1}$ (see Eq. (5.7)).

In the periodic case $\langle\mathbf{c}\rangle_{Av} = \mathbf{a}\cos\omega t$, we have

$$kT_e - kT = \tfrac{1}{6}m_1 a^2[1 + \cos\delta\cos(2\omega t - \delta)], \quad (5.13)$$

where

$$\delta = \tan^{-1}[(m_1+m)\omega/mN\lambda]. \quad (5.14)$$

For large frequency or small mass of the charged particle, i.e., for

$$(m_1+m)\omega/mN\lambda \gg 1, \quad (5.15)$$

the time variation of $T_e - T$ is small and

$$kT_e - kT \doteqdot \tfrac{1}{6}m_1 a^2 = \tfrac{1}{3}m_1 \text{ mean } |\langle\mathbf{c}\rangle_{Av}|^2 \quad (5.16)$$

where "mean" denotes the time-mean.

In cases of extremely high frequencies where the electrons make many oscillations per collision, namely when

$$0 < N\lambda/\omega \ll 1,$$

we have from (5.8)

$$\langle\mathbf{c}\rangle_{Av} = \frac{1}{j\omega}\frac{e}{m}\mathbf{E}_0\exp(j\omega t)$$

and from (5.16)

$$kT_e - kT = \tfrac{1}{6}m_1(eE_0/\omega m)^2.$$

These relations are independent of the gas density N (so long as $N \gg n$) and cross section λ,—a fact which may be easily understood.

APPENDIX TO PART I. VELOCITY-DISTRIBUTION OF HEAVY IONS IN A LIGHT GAS

The aim of this paper is chiefly to develope the theory on the basis of a special model of molecules, namely Maxwellian. Here, however, let us insert a discussion of the motion of *heavy* ions without restricting to special models. The necessary assumptions are:

1. The mass of ions is large enough in comparison to that of gas molecules, i.e., $m \gg m_1$.

2. The drift-velocity of ions is much smaller than the thermal velocity of gas molecules, i.e., $\langle c^2\rangle_{Av} \ll kT/m_1$.

3. Quasi-homogeneity.

Under the above assumptions 1 and 2, the ion-velocity **c** is an eigenfunction of the operator J

$$J\mathbf{c} = \lambda\mathbf{c}, \quad (A.1)$$

the eigenvalue λ being[3]

$$\lambda = \frac{16}{3}\frac{m_1}{m_1+m}\Omega^{(1)}(1).$$

Here, generally,

$$\Omega^{(l)}(r) = (\pi)^{\frac{1}{2}}\int_0^\infty \phi^{(l)}V^{2r+2}\exp(-V^2)dV, \quad r = l, l+1, \cdots$$

$$\left[V = \left(\frac{m_1 m}{m_1+m}\frac{1}{2kT}\right)^{\frac{1}{2}}g\right]$$

[3] T. Kihara, see reference 2, Eq. (20.8).

is the statistical mean of the general collision cross section

$$\phi^{(l)} = \int_0^\pi (1 - \cos l\theta) g I(g, \theta) \sin\theta d\theta. \quad l = 1, 2, 3, \cdots.$$

It depends on the temperature unless the interaction is Maxwellian.

In the case of static electric field alone ($\partial/\partial t = 0$, $\partial/\partial r = 0$, $H = 0$), we get from (4.2),

$$e E/m = N\lambda \langle c \rangle_{Av}, \quad (A.2)$$

N being the number-density of the gas molecules. We thus get the result that the mobility of heavy ions is independent of the field strength, though it depends on the temperature—a fact which agrees with observation.[4]

The relation (A.2), inserted into Boltzmann's equation (2.2), yields

$$N\lambda \langle c \rangle_{Av} \frac{\partial f}{\partial c} = \frac{\partial_e f}{\partial t}, \quad (A.3)$$

the solution of which is given by[5]

$$f = n \left(\frac{m}{2\pi k T} \right)^{\frac{3}{2}} \exp\left[-\frac{m}{2kT} (c - \langle c \rangle_{Av})^2 \right], \quad (A.4)$$

where T, the temperature of ions, is the same as the gas temperature.

Let us show in the following that this Maxwell distribution satisfies the most general Boltzmann equation (2.2), so far as $\langle c \rangle_{Av}$ satisfies a certain differential equation.

When n depends on time t and position r and $\langle c \rangle_{Av}$ depends on t ($\partial \langle c \rangle_{Av}/\partial r = 0$ is assumed under quasi-homogeneous condition) we get from (A.4)

$$\frac{\partial f}{\partial t} = -\frac{\partial \langle c \rangle_{Av}}{\partial t} \cdot \frac{\partial f}{\partial c} + \frac{f}{n} \frac{\partial n}{\partial t}, \quad c \cdot \frac{\partial f}{\partial r} = \frac{f}{n} c \cdot \frac{\partial n}{\partial r}$$

which yield, together with the equation of continuity (4.1)

$$\frac{\partial f}{\partial t} + c \cdot \frac{\partial f}{\partial r} = -\frac{\partial \langle c \rangle_{Av} \cdot}{\partial t} \frac{\partial f}{\partial c} + \frac{f}{n} (c - \langle c \rangle_{Av}) \cdot \frac{\partial n}{\partial r}$$

$$= -\left(\frac{\partial \langle c \rangle_{Av}}{\partial t} + \frac{kT}{mn} \frac{\partial n}{\partial r} \right) \cdot \frac{\partial f}{\partial c}.$$

On the other hand, we have if we assume (A.4),

$$c \times H \cdot \frac{\partial f}{\partial c} = \langle c \rangle_{Av} \times H \cdot \frac{\partial f}{\partial c}.$$

The Boltzmann equation (2.2), therefore, takes the form

$$\left[-\frac{\partial \langle c \rangle_{Av}}{\partial t} - \frac{kT}{mn} \frac{\partial n}{\partial r} + \frac{e}{m} (E + \langle c \rangle_{Av} \times H) \right] \cdot \frac{\partial f}{\partial c} = \frac{\partial_e f}{\partial t}. \quad (A.5)$$

Since from (4.2) holds

$$N\lambda \langle c \rangle_{Av} + \frac{\partial \langle c \rangle_{Av}}{\partial t} + \frac{kT}{mn} \frac{\partial n}{\partial r} - \frac{e}{m} (E + \langle c \rangle_{Av} \times H) = 0. \quad (A.6)$$

Equation (A.5) is just the same as Eq. (A.3), from whose solution we have started. We thus obtain the theorem; the velocity-distribution of heavy ions in a light gas is Maxwellian function (A.4), the center of which is given by Eq. (A.6).

Equation (A.6) differs from (5.3) by the fact that λ here depends on the temperature and that the effect of diffusion is taken into consideration without any mathematical difficulties. (We considered in 5 homogeneous cases, since the effect of diffusion would make the problem complex.)

II. VELOCITY-DISTRIBUTION OF ELECTRONS

In this part we investigate the velocity-distribution of electrons adopting a simple and suitable model of molecules. The contents of this part lay the foundation of our molecular theory of electrical discharges in gases.

6. Velocity-Distribution of Electrons in the Region of Elastic Collisions

Ions and electrons are both charged particles. These two types of particles in gases, however, have rather different quality. First, the mass of the electron, m, is much smaller than that of gas molecules, m_1. Secondly, inelastic collisions are important between the electron and the gas molecule while they can be neglected between ions. In this section we investigate, as preparation, the velocity distribution of electrons in case only elastic collisions occur—i.e., in case the mean energy of electrons is not sufficient to excite or ionize the gas molecule.

The fundamental quantity determining the motion of electrons in a gas is the effective collision cross section of the gas molecule for electrons. In this respect we adopt the Maxwellian model which was mentioned in 5; namely, we assume that the scattering coefficient, $I(g, \theta)$, for angle θ is inversely proportional to the relative speed g. Then, what we have investigated in 5 can be applied to the case of electrons, if we interprete m as the mass of electrons and take into account the relation $m \ll m_1$. The electron velocity c is one of the eigenfunctions of the operator J and the corresponding eigenvalue is

$$\lambda = 2\pi \int_0^\pi (1 - \cos\theta) g I(g, \theta) \sin\theta d\theta. \quad (6.1)$$

[4] A. M. Tyndall, *The Mobility of Positive Ions in Gases* (Cambridge Physical Tracts, 1938), Fig. 31.
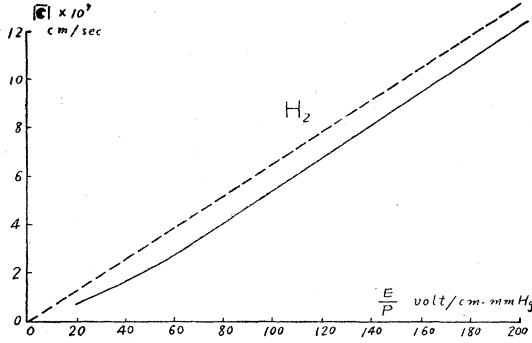[5] T. Kihara, see reference 2, §20.

FIG. 4. Drift velocity of electrons in hydrogen.

The drift velocity $\langle c \rangle_{Av}$ can be determined by the equation

$$N\lambda\langle c \rangle_{Av} + \frac{d\langle c \rangle_{Av}}{dt} - \frac{e}{m}(E + \langle c \rangle_{Av} \times H) = 0. \qquad (6.2)$$

Especially in the case of static field

$$\langle c \rangle_{Av} = KE, \quad K = e/mN\lambda,$$

i.e., the drift velocity of the electrons is proportional to E/N or E/P. Measuring E in volt/cm, P in mm of Hg, $\langle c \rangle_{Av}$ in cm/sec, and λ in cm³/sec, we have the numerical relation at 0°C

$$\langle c \rangle_{Av} = 0.050E/\lambda P$$

(at 0°C, $N/P = 3.55 \times 10^{16}/\text{cm}^3$ mm Hg). In fact, the drift velocity of the electrons is roughly proportional to E/P. In Fig. 4 the full line is the observed curve[6] in hydrogen, and the dotted line is the calculated one choosing $\lambda = 7.7 \times 10^{-8}$ cm³/sec, which may be regarded as one of the molecular constants of hydrogen.

Generally, when we eliminate E from Boltzmann's equation (4.7) making use of (6.2), we have

$$N\lambda\langle c \rangle_{Av} \cdot \frac{\partial f}{\partial c} + \frac{e}{m}(c - \langle c \rangle_{Av}) \times H \cdot \frac{\partial f}{\partial c}$$

$$+ \left(\frac{\partial f}{\partial c} \cdot \frac{d\langle c \rangle_{Av}}{dt} + \frac{\partial f}{\partial t} \right) = \frac{\partial_e f}{\partial t}. \qquad (6.3)$$

From (5.10) and (5.12) follows the relation

$$|\langle c \rangle_{Av}|^2 \ll \langle |c - \langle c \rangle_{Av}|^2 \rangle_{Av}, \quad \text{or} \quad |\langle c \rangle_{Av}|^2 \ll \bar{c}^2,$$

indicating that the drift motion of electrons is much smaller than the thermal (i.e., random) motion. This we can understand easily considering that the transfer of energy between electron and gas molecule is very small because of a great difference between their masses. After all, the velocity-distribution of electrons, f, is a function of the scalar c to the first approximation. Let[7]

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[6] L. B. Loeb, *Fundamental Processes of Electrical Discharge in Gases* (John Wiley & Sons, Inc., New York, 1939), Table 29.

[7] This definition of $f^{(0)}$ is different from that in 3.

it be $f^{(0)}$. To the second approximation f is a function of scalar $|c - \langle c \rangle_{Av}|$. It is our purpose to derive the form of the function.

To this approximation, the second term of (6.3) is obviously zero. Since

$$\frac{\partial f}{\partial t} = \frac{\partial f^{(0)}}{\partial t} + \frac{\partial f}{\partial \langle c \rangle_{Av}} \cdot \frac{d\langle c \rangle_{Av}}{dt} = \frac{\partial f^{(0)}}{\partial t} - \frac{\partial f}{\partial c} \cdot \frac{d\langle c \rangle_{Av}}{dt},$$

the Eq. (6.3) can be transformed into the following form,

$$N\lambda\langle c \rangle_{Av} \cdot \frac{\partial f}{\partial c} + \frac{\partial f^{(0)}}{\partial t} = \frac{\partial_e f}{\partial t}. \qquad (6.4)$$

After expanding the function f of the scalar $|c - \langle c \rangle_{Av}|$, we have

$$f = f^{(0)} + \langle c \rangle_{Av} \cdot c f^{(1)}, \quad f^{(1)} = -\frac{1}{c} \frac{\partial f^{(0)}}{\partial c}. \qquad (6.5)$$

This expression being inserted into (6.4), the first term on the left side becomes

$$N\lambda |\langle c \rangle_{Av}|^2 \left(f^{(1)} + \frac{1}{3}c\frac{\partial f^{(1)}}{\partial c} \right) + N\lambda\langle c \rangle_{Av} \cdot \frac{c}{c}\frac{\partial f^{(0)}}{\partial c},$$

and the right side of (6.4) becomes

$$\frac{\partial_e f^{(0)}}{\partial t} + \frac{\partial_e (\langle c \rangle_{Av} \cdot c f^{(1)})}{\partial t}.$$

Now, since the electron speed suffers no considerable alteration by collisions and molecules, we have

$$\frac{\partial_e (\langle c \rangle_{Av} \cdot c f^{(1)})}{\partial t} = -N f^{(1)} \langle c \rangle_{Av} \cdot Jc$$

$$= -N\lambda\langle c \rangle_{Av} \cdot c f^{(1)}$$

$$= N\lambda\langle c \rangle_{Av} \cdot \frac{c}{c} \cdot \frac{\partial f^{(0)}}{\partial t}.$$

Equation (6.4), therefore, becomes finally

$$N\lambda |\langle c \rangle_{Av}|^2 \left(f^{(1)} + \frac{1}{3}c\frac{\partial f^{(1)}}{\partial c} \right) + \frac{\partial f^{(0)}}{\partial t} = \frac{\partial_e f^{(0)}}{\partial t}, \qquad (6.6)$$

the right side of which has been expressed as

$$\frac{\partial_e f^{(0)}}{\partial t} = N\lambda\frac{1}{c^2}\frac{\partial}{\partial c}\left(\frac{kT}{m_1}c^2\frac{\partial f^{(0)}}{\partial c} + \frac{m}{m_1}c^3 f^{(0)} \right) \qquad (6.7)$$

after complicated calculations.[8]

Now let us expect that $f^{(0)}$ may be Maxwellian

$$f^{(0)} = n\left(\frac{m}{2\pi kT_e} \right)^{\frac{3}{2}} \exp\left(-\frac{mc^2}{2kT_e} \right), \quad f^{(1)} = \frac{m}{kT_e}f^{(0)}, \qquad (6.8)$$

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[8] Chapman-Cowling, see reference 1, p. 349.

then the terms of (6.6) are calculated as follows:

$$\frac{\partial_e f^{(0)}}{\partial t} = N\lambda \frac{m}{kT_e} \frac{kT_e - kT}{m_1}\left(3 - \frac{mc^2}{kT_e}\right)f^{(0)},$$

$$\frac{\partial f^{\circ}}{\partial t} = \frac{\partial f^{(0)}}{\partial T_e}\frac{dT_e}{dt} = \frac{1}{kT_e}\left(\frac{mc^2}{2kT_e} - \frac{3}{2}\right)f^{(0)}\frac{dkT_e}{dt},$$

$$N\lambda|\langle \mathbf{c}\rangle_{Av}|^2\left(f^{(1)} + \frac{1}{3}c\frac{\partial f^{(1)}}{\partial c}\right)$$

$$= N\lambda|\langle \mathbf{c}\rangle_{Av}|^2\frac{m}{kT_e}\left(1 - \frac{1}{3}\frac{mc^2}{kT_e}\right)f^{(0)}.$$

Hence, when the electron temperature, T_e, satisfies the equation

$$3N\lambda\frac{m}{m_1}(kT_e - kT) + \frac{3}{2}\frac{dkT_e}{dt} = mN\lambda|\langle \mathbf{c}\rangle_{Av}|^2, \quad (6.9)$$

the assumed expression of (6.8) really satisfies the equation of Boltzmann. Thus we get the theorem. The velocity of electrons is described by a Maxwellian distribution whose center is the drift-velocity $\langle \mathbf{c}\rangle_{Av}$ determined by (6.2) and whose temperature satisfies (6.9).

Equation (6.9) is the same as we have from (5.11) if we neglect m/m_1 for 1. In case of such a high frequency field that the angular frequency ω satisfies $m_1\omega/mN\lambda \gg 1$, the electron temperature T_e suffers no time-variation, and

$$kT_e - kT = \tfrac{1}{3}m_1 \text{ mean } |\langle \mathbf{c}\rangle_{Av}|^2$$

as already mentioned in 5.

7. Velocity-Distribution in Case Excitations Occur

Since the mass of the electron is much smaller than that of molecules, transfer of energy between electron and molecule at the time of an elastic collision is small. At high electron temperatures, however, electrons can excite gas molecules, and be deprived of the excitation energy. Therefore, the effect of exciting collisions on the energy distribution of electrons cannot be neglected even if their frequency is so small that the effect on the drift velocity can be neglected.

In the case exciting collisions occur, we should add to the right side of Boltzmann's equation (6.3) the time variation of f due to exciting collisions, $(\partial f/\partial t)_{\text{excite}}$, which can be replaced by its spherically symmetric approximation $(\partial f^{(0)}/\partial t)_{\text{excite}}$.

If the number of electrons per unit volume and per unit time, which undergo inelastic collisions and whose speeds decrease from values above c to values below c because of the collisions, are denoted by $S(c)$ we have

$$\left(\frac{\partial f^{(0)}}{\partial t}\right)_{\text{excite}} = \frac{1}{4\pi c^2}\frac{\partial S(c)}{\partial c}. \quad (7.1)$$

By making use of Eq. (6.7)

$$\frac{\partial_e f^{(0)}}{\partial t} + \left(\frac{\partial f^{(0)}}{\partial t}\right)_{\text{excite}}$$

$$= N\lambda\frac{1}{c^2}\frac{\partial}{\partial c}\left[\frac{kT}{m_1}c^2\frac{\partial f^{(0)}}{\partial c} + \frac{m}{m_1}c^3 f^{(0)} + \frac{S(c)}{4\pi N\lambda}\right], \quad (7.2)$$

by which the right side of (6.6) should be replaced.

Now, let us denote by $Q(c_0, c)$ the effective cross section for such a process that the speeds of electrons decrease from c_0 to values below c because of inelastic collisions. $Q(c_0, c)$ is a monotonously increasing function of $c(\leqq c_0)$ and $Q(c_0, c_0)$ is the total cross section of inelastic collisions for electrons with initial speed c_0. Regarding this quantity we assume a characteristic model,—namely,

$$Q(c_0, c) = \rho c^3/c_0^2, \quad (7.3)$$

which we shall make the basis of our mathematical theory. ρ is a molecular model-constant with the dimension of area divided by velocity. According to this model the total cross section $Q(c_0, c_0) = \rho c_0$ is proportional to the speed of colliding electrons.

Here let us see the basis of the fundamental assumption (7.3). In general a molecule can be raised to several excited states. Denoting by ϵ_s the energy difference between the ground state and the sth excited state, the molecule can be put in the sth excited state by the collision of an electron whose kinetic energy is larger than ϵ_s. By this transition the electron loses the energy ϵ_s. The effective cross section for this process is usually as shown in Fig. 5. For these molecules $Q(c_0, c)$ increases stepwise regarding c as shown in Fig. 6. And the total exciting cross section $Q(c_0, c_0)$ increases nearly monotonously as shown in Fig. 7. Replacing these real natures by such smooth curves as shown in the figures, we obtain the foregoing molecular model. (The effective cross section to be excited to the sth state may be approximated by such a formula as

$$a_s(\tfrac{1}{2}mc_0^2 - \epsilon_s)^{\frac{1}{2}}/\tfrac{1}{2}mc_0^2, \quad \tfrac{1}{2}mc_0^2 > \epsilon_s.$$

If we suppose that these excited states distribute with an equal interval Δ as shown in Fig. 5 and that a_s is constant independent of s, we obtain (7.3) by taking the limit $\Delta \to 0$ keeping $a_s\Delta$ constant.)
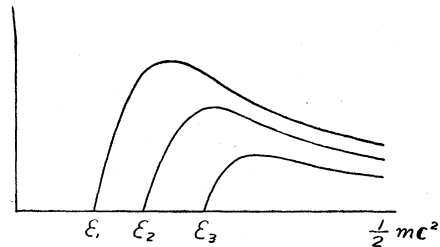

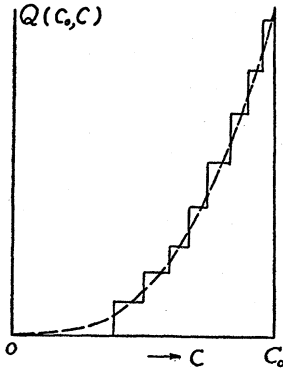DIG. 5. Effective cross section to be transferred to each excited state. The abscissa shows the electron energy.

FIG. 6. Effective cross section for the process that the speeds of electrons decrease from c_0 to values below c because of inelastic collisions. Full line—real values. Dotted line—for the model (7.3).

Introducing this model of $Q(c_0, c)$, we have the afore-mentioned $S(c)$ as follows:

$$S(c) = 4\pi N \int_c^\infty f(c_0)Q(c_0, c)c_0^3 dc_0$$

$$= 4\pi N \rho c^3 \int_c^\infty f(c_0)c_0 dc_0,$$

where $f(c_0)$ is velocity-distribution function of electrons. Now again expecting the Maxwellian distribution

$$f = A \exp\left(-\frac{mc^2}{2kT_e}\right), \quad A = n\left(\frac{m}{2\pi kT_e}\right)^{\frac{3}{2}}, \quad (7.4)$$

we can show the self-consistency of this expectation. In this case

$$S(c) = 4\pi N \rho \frac{kT_e}{m} c^3 f, \quad (7.5)$$

and we see in fact that $f^{(0)}$ given by (7.4) satisfies (7.2), provided that the electron temperature satisfies the equation

$$3N\frac{\rho}{m}(kT_e)^2 + 3N\lambda\frac{m}{m_1}(kT_e - kT)$$

$$+ \frac{3}{2}\frac{dkT_e}{dt} = mN\lambda |\langle c \rangle_{Av}|^2. \quad (7.6)$$

We have, therefore, the theorem; if the cross section of the gas molecule for elastic collisions is inversely proportional to the electron speed, and if its effective cross section for excitation is expressed by (7.3), the electron velocity is described by the Maxwellian distribution whose center is the drift velocity $\langle c \rangle_{Av}$ given by (6.2) and whose temperature T_e is given by (7.6).

The electron temperature determined by (7.6) becomes nearly constant in case of static or sufficiently high frequency field. In these cases and when the electron temperature is so high that the excitations occur

sufficiently, i.e., when

$$kT_e/m \gg m\lambda/m_1\rho,$$

we have the simple relation,

$$(kT_e)^2 = \begin{cases} (m^2\lambda/3\rho)|\langle c \rangle_{Av}|^2 & \text{(static)} \\ (m^2\lambda/3\rho)\text{mean }|\langle c \rangle_{Av}|^2 & \text{(high frequency),} \end{cases} \quad (7.7)$$

which indicates that the electron temperature is proportional to the absolute value of the drift velocity and therefore to the field strength E.

In case of no magnetic field, we have for static field E,

$$kT_e = \frac{1}{(3\lambda\rho)^{\frac{1}{2}}}\frac{e}{N}E, \quad (7.8)$$

and for high frequency field $E_0 \cos\omega t$,

$$kT_e = \left(1 + \frac{\omega^2}{N^2\lambda^2}\right)^{-\frac{1}{2}}\frac{1}{(3\lambda\rho)^{\frac{1}{2}}}\frac{e}{N}\frac{E_0}{\sqrt{2}}. \quad (7.9)$$

When the frequency is so large that $\omega/N\lambda \gg 1$, holds the relation

$$kT_e = (eE_0/\omega\sqrt{2})(\lambda/3\rho)^{\frac{1}{2}},$$

which is independent of gas density but depends on the ratio of the elastic and the exciting cross sections.

8. Nature of Ionized Gases

We are dealing from the outset with gases in which the number-density of charged particles is much smaller than that of gas molecules, $n \ll N$. Even in these cases, if the density of charged particles grows to a certain amount, we cannot neglect their mutual interactions. These interactions are the long-range force of Coulomb which does not allow us to use the notion of collision cross section, and they are usually taken into account by considering the field of space charge.

An ionized gas in which positive and negative charges balance each other is called *plasma*. In this section we derive the well-known qualities of the plasma made of electrons and positive ions by means of our molecular model.

Since the mobility of ions is much smaller than that of electrons, current in plasma, on which an electric field is imposed, is mainly a result of electrons. Electric con-
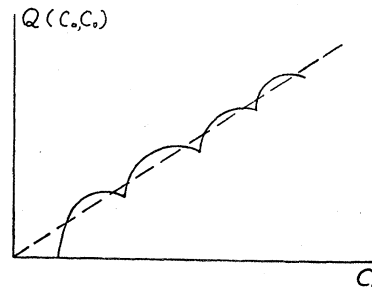

FIG. 7. Total cross section of excitation. The abscissa shows electron speed. Full line — real values. Dotted line—for the model (7.3).

ductivity, therefore, is equal to the product of the electron charge e, the number-density of electrons n, and the electronic mobility K,—i.e.,

$$enK.$$

For our molecular model the mobility is $K = e/mN\lambda$ independently of the field strength. We have, therefore, Ohm's law.

In a high frequency field, $\mathbf{E}$, the dielectric constant, ϵ, of plasma is defined by

$$\epsilon\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P}.$$

Here $\mathbf{P}$ is the electric polarization which, neglecting the motion of ions, is expressed by

$$\mathbf{P} = ne\mathbf{r},$$

$\mathbf{r}$ being displacement of electrons caused by the field. In case

$$\mathbf{E} = \mathbf{E}_0 \exp(j\omega t),$$

the displacement is

$$\mathbf{r} = \frac{1}{j\omega}\frac{e\mathbf{E}}{m}\frac{1}{N\lambda + j\omega}.$$

We have, therefore,

$$\epsilon = 1 + \frac{4\pi}{j\omega}\frac{ne^2}{m(N\lambda + j\omega)}$$

$$= 1 - 4\pi\frac{ne^2}{m(\omega^2 - j\omega N\lambda)}.$$

Let

$$\omega_0^2 = 4\pi ne^2/m; \tag{8.1}$$

then

$$\epsilon = 1 - \omega_0^2/(\omega^2 - j\omega N\lambda). \tag{8.2}$$

ω_0 is usually called the angular frequency of plasma-oscillation; it may not be reasonable, however, to associate with it a mechanical proper oscillation.

III. IONIZATION BY COLLISION

In this part we investigate the ionization by collision, most important among electron-generating processes, on the basis of a suitable molecular model. Since the processes of ionization and electron capture have, in general, no remarkable influence upon the velocity distribution, these processes can be treated as a kind of perturbation added to the processes mentioned in the preceding sections.

9. Ionization by Collision with Electron

The gas molecule can be deprived of its electrons by encountering with other electrons and change into a positive ion. This process is termed ionization by collision. In this case, a certain amount of work is necessary to deprive a molecule of an electron and change it into
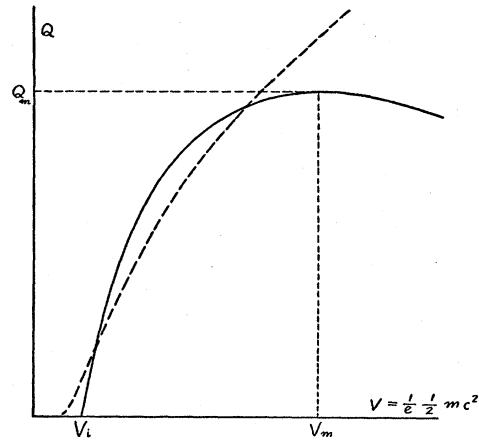

FIG. 8. Effective cross section for ionization. Full line—observed. Dotted line—for the model (9.1).

a positive monovalent ion. This amount of work is called the ionization energy or, being expressed in terms of electon-volts the ionization potential of the molecule. For a gas molecule to be ionized by collision with an electron, the electron must obviously have possessed kinetic energy larger than the ionization energy.

Denoting by Q the effective cross section of a molecule for the process of collision-ionization, we can plot the variation of Q vs electron energy $\frac{1}{2}mc^2$ or the accelerating voltage V of the electron as shown by the full line in Fig. 8. Q vanishes below the ionization voltage V_i, takes maximum Q_m for a voltage V_m several times greater than V_i, and tends to decrease for the larger electronic energy. For some important gas molecules, the values of V_i, V_m, and Q_m are given in Table I. In the phenomena of electrical discharge the mean energy of electrons is usually much smaller than the energy corresponding to the maximum of Q. Taking this fact into account, we adopt the following model of the effective cross section,

$$Q = \begin{cases} \sigma(c^2 - c_i^2)^{\frac{1}{2}}/c_ic^2 & (c > c_i) \\ 0 & (c < c_i), \end{cases} \tag{9.1}$$

which is a good representation of the actual quantity in the low energy region, Fig. 8. Here σ is a molecular constant with the dimension of area, and $mc_i^2/2$ is the ionization energy. c_i may be determined by equating $mc_i^2/2$ directly to the actual value eV_i. But it is more adequate to fix c_i and σ so that the model (1) may coincide with the actual form as closely as possible. The values[9] of σ and c_i determined in this manner are listed in Table II.

Since a few electrons with exceptionally large energies usually take the main part of ionization, we may consider that the velocity distribution of electrons is not

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[9] These values are determined as follows: Making use of Table I we take $c_i = 0.8(2eV_i/m)^{\frac{1}{2}}$, and fix σ so that $Q = Q_m$ for $c = 0.9(2eV_m/m)^{\frac{1}{2}}$.

TABLE I. Molecular constants for ionization by collision.

	V_i volt	V_m volt	Q_m cm^2	Reference
He	24.5	112	0.355	a
Ne	21.7	174	0.849	a
A	15.7	88	3.66	a
Hg	10.4	85	5.46	b
H$_2$	15.6	68	1.02	c
N$_2$	15.5	102	2.90	c
O$_2$	12.5	118	2.93	d
CH$_4$	14.5	80	2.28	d

a P. T. Smith, Phys. Rev. 36, 1293 (1930).
b P. T. Smith, Phys. Rev. 37, 808 (1931).
c J. T. Tate and P. T. Smith, Phys. Rev. 39, 270 (1932).
d A. L. Hughes and E. Klein, Phys. Rev. 23, 450 (1924).

disturbed by the ionization process. When the velocity-distribution is Maxwellian as derived in the preceding Part II,—i.e., when

$$f = n \left(\frac{m}{2\pi k T_e} \right)^{\frac{1}{2}} \exp\left[-\frac{mc^2}{2kT_e} \right], \quad (9.2)$$

the relative rate of increase ν in the electron density n is given by

$$\nu = \frac{1}{n}\frac{dn}{dt} = N \int_{c_i}^{\infty} Q \frac{f}{n} 4\pi c^3 dc$$

$$= N \frac{3\sigma}{c_i} \frac{kT_e}{m} \exp\left[-\frac{mc_i^2}{2kT_e} \right]. \quad (9.3)$$

The coefficient of collision-ionization per unit time, ν, is not only proportional to the number-density of the gas molecules N but also a function of the electron temperature T_e.

Next, suppose the electrons move with a constant drift velocity $K\mathbf{E}$ under the influence of a uniform electrostatic field $\mathbf{E}$. Take the z axis in the direction of field and designate the relative increase of the current density per unit length, $(1/i)di/dz$, by α, then we have

$$\alpha = \frac{1}{KE}\frac{1}{n}\frac{dn}{dt} = \frac{\nu}{KE}, \quad K = \frac{e}{mN\lambda}. \quad (9.4)$$

α is called Townsend coefficient. When the molecules are ionized, the exciting collisions occur frequently enough in ordinary cases. Therefore, the electron temperature is given by (7.8), from which we have

$$\alpha = N\frac{\sigma}{c_i} \left(\frac{3\lambda}{\rho} \right)^{\frac{1}{2}} \exp\left[-\frac{N}{E}\frac{mc_i^2}{2e}(3\lambda\rho)^{\frac{1}{2}} \right]. \quad (9.5)$$

TABLE II. Four molecular constants.

	$c_i \times 10^{-8}$ cm/sec	$\sigma \times 10^{16}$ cm^2	$\lambda \times 10^{8}$ cm^3/sec	$\rho \times 10^{24}$ cm sec
He	2.35	0.20	2.0	0.14
He	2.22	0.31	4.4	0.20
A	1.89	1.74	7.8	1.35
H$_2$	1.89	0.59	5.5	0.81
N$_2$	1.87	1.23	16.5	1.80
CH$_4$	1.81	1.1	16.7	1.63

Under constant temperature the density of the gas molecule, N, is proportional to the pressure of the gas P. For instance, at $0°C$, measuring P in mm of Hg,

$$N/P = 3.55 \times 10^{16} 1/\text{cm}^3 \text{ mm of Hg.} \quad (9.6)$$

In this case the Townsend coefficient α is expressed in the form

$$\alpha = A_0 P \exp(-B_0 P/E), \quad (9.7)$$

where A_0 and B_0 are molecular constants independent of pressure and electric field (at $0°C$ or room temperature), namely,

$$A_0 = \frac{N}{P}\frac{\sigma}{c_i} \left(\frac{3\lambda}{\rho} \right)^{\frac{1}{2}}, \quad B_0 = \frac{N}{P}\frac{mc_i^2}{2e}(3\lambda\rho)^{\frac{1}{2}}. \quad (9.8)$$

According to (9.7), the variation of α/P vs E/P is shown by the dotted line in Fig. 9. The model we have assumed for the effective cross section for ionization is too large in the region of large electron energy. The α derived from this model, therefore, is presumed to be too large for large values of E/P. In fact, in the region
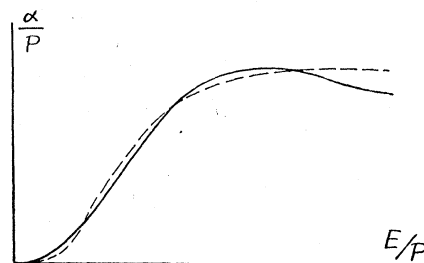

FIG. 9. Townsend coefficient (qualitative). Full line—observed. Dotted line—for the formula (9.7).

of exceedingly large E/P the decrease in α beyond the maximum has been observed as shown by the full line in Fig. 9.

Anyway, the Townsend coefficient α is sufficiently approximated by the formula (9.7) in the greater part of the region significant to discharge phenomena. Equation (9.7) is a semi-empirical formula which has been widely used for a long time.

The values of A_0 and B_0 at $0°C$ are given in Table III. These values substituted into (9.7) show good agreement with the experiment in the region of electric field $B_0/2 \lesssim E/P \lesssim 2B_0$.

With these values, together with the already fixed constants c_i and σ, we can determine the other two molecular constants λ and ρ. The results of this estimation are given in Table II.

The value of λ determined in 6 from the drift velocity of electrons in hydrogen is 7.7×10^{-8} cm^3/sec, which is in rather sufficient agreement with the value in the table, 5.5×10^{-8} cm^3/sec.

IV. DIELECTRIC BREAKDOWN

In the preceding parts we have laid the foundation for investigating discharge phenomena from the molecu-

lar point of view. In Part IV, therefore, we are concerned with the dielectric breakdown of the low pressure gas in a homogeneous electric field. In order to discern the characteristics of the phenomena, we may separately deal with three types of electric field distinguished by the frequency: namely, microwave of sufficiently large frequency, ordinary radiowave of several Mc, and static or quasi-static field.

10. Dielectric Breakdown by Microwave

Suppose a microwave field be imposed on a low pressure gas between two parallel plates which are at the distance L apart. We assume that the area of the plates is large enough in comparison with L to justify the one-dimensional treatment of this problem. Furthermore the wavelength of the electric field is assumed to be much longer than L. Take the origin of the coordinate at the center of the gap and let the z axis be perpendicular to the plates.

In the gas between the plates, the number of electrons is increased by collision-ionization and decreased by diffusion. If we disregard the other effects for the present, the rate of increase in the number-density is given by

$$\partial n/\partial t = \nu n + |D\partial^2 n/\partial z^2|, \qquad (10.1)$$

where D is the diffusion-coefficient and ν the relative rate of increase in n by collision-ionization. D is inversely, and ν is directly proportional to the number-density of gas molecules N. It may be reasonable to suppose that $n(z, t)$ vanishes on the plates $z = \pm L/2$.

Unless the density or the pressure P is exceedingly low, ν and D vary with E as shown in Fig. 10. That is to say, the collision ionization rarely occurs for a weak field; when the electric field reaches a certain strength, however, it supercedes the diffusion effect, resulting in the discontinuous increase in the electron density between the plates. This phenomenon is called dielectric breakdown. At the critical voltage—the breakdown voltage—$\partial n/\partial t = 0$, i.e.,

$$\nu n + D\frac{\partial^2 n}{\partial z^2} = 0, \quad (n = 0 \text{ at } z = \pm L/2). \quad (10.2)$$

The lowest eigenvalue is given by

$$(L/\pi)^2 \nu = D, \qquad (10.3)$$

for which we have the distribution

$$n = n_0 \cos(\pi z/L), \qquad (10.4)$$

where n_0 is the electron density at $z = 0$.

In the ideal case like (10.1), no electron density appears below the breakdown voltage. If ionizing agents like x-ray are irradiated from the outside, however, some electron density appears even below the breakdown voltage. Let G be the number of electrons created per unit volume and per unit time by such external

TABLE III. Coefficients in (9.7).

	A_0 1 cm mm Hg	B_0 volt/cm mm Hg	Reference
He	2.0	50	a
Ne	4.0	80	a
A	13.6	200	a
H_2	5.0	130	b
N_2	12.4	340	b
CH_4	12.0	300	c

a M. J. Druyvesteyn and F. M. Penning, Revs. Modern Phys. 12, 99 (1940), Table I.
b A. v. Engel and M. Steenbeck, *Elektrische Gasentladungen* I (1932), Table 15.
c M. E. Rose and W. E. Ramsey, Phys. Rev. 61, 199 (1942).

agents. Then the rate of increase in n is given by

$$\partial n/\partial t = G + \nu n + |D\partial^2 n/\partial z^2|. \qquad (10.5)$$

For the distribution of n in the stationary state we have

$$n = \frac{G}{\nu}\left(\frac{\cos(\nu/D)^{\frac{1}{2}} z}{\cos\Theta} - 1\right), \quad \Theta = \frac{L}{2}\left(\frac{\nu}{D}\right)^{\frac{1}{2}} < \frac{\pi}{2}. \quad (10.6)$$

The value of n at the center is estimated by

$$n_0 = \frac{G}{\nu}\left(\frac{1}{\cos\Theta} - 1\right). \qquad (10.7)$$

The relation of n_0 to the field strength E is shown on Fig. 11(a).

In view of (10.6) n increases with the voltage, and becomes infinitely large when $\Theta = \pi/2$, indicating the occurrence of dielectric breakdown. Actually a certain finite electron density is sufficient to cause the dielectric breakdown. This can be interpreted as described in the following in which account is taken of the influence of space charge.

When electrons are created by ionization, positive ions are brought forth at the same time. These ions, however, are diffused very slowly because of their large inertia, and remain within the gap as space charge. This positive space charge exerts Coulomb force on electrons and deters them from diffusing. Being affected by the space charge, therefore, the relation between the electron density and the field strength becomes as shown in Fig. 11(b). In this case the maximum field is the breakdown field which corresponds to a certain value of Θ below $\pi/2$. The breakdown field corresponding to $\Theta = \pi/2$ in (10.3) is the value at the limit when the external ionization effect vanishes.
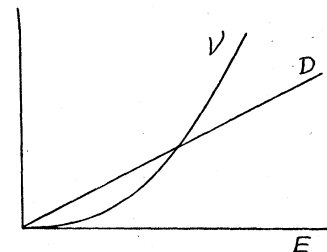

FIG. 10. Collision-ionization coefficient and diffusion coefficient vs electric field.
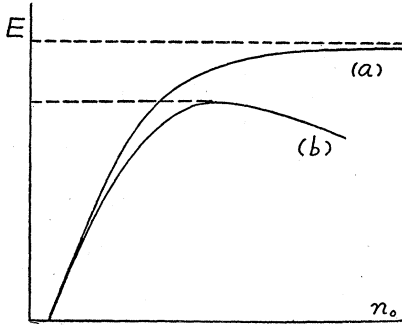
Fig. 11. Electron density *vs* electric field.

Since in our molecular model the diffusion coefficient D is given by

$$D = kT_e K/e = kT_e/mN\lambda$$

and the coefficient of collision ionization ν by

$$\nu = \frac{3N\sigma}{c_i}\frac{kT_e}{m}\exp\left[-\frac{mc_i^2}{2kT_e}\right],$$

the condition of breakdown (10.3) is written in the form

$$\exp\left[\frac{mc_i^2}{2kT_e}\right] = \frac{3\sigma\lambda}{c_i}\left(\frac{NL}{\pi}\right)^2. \qquad (10.8)$$

With the relation (7.9) between the electron temperature T_e and the amplitude of the electric field E_0, we know the functional form of E_0/N in terms of NL and ω/N. That is,

$$\frac{N}{E}\frac{mc_i^2}{2e}(3\lambda\rho)^{\frac{1}{2}}\left(1+\frac{\omega^2}{N^2\lambda^2}\right)^{\frac{1}{2}} = \ln\left[\frac{3\sigma\lambda}{c_i}\left(\frac{NL}{\pi}\right)^2\right].$$

$$(E = E_0/\sqrt{2}). \qquad (10.9)$$

It is often more convenient to express the breakdown field E in terms of PL, the product of the pressure and
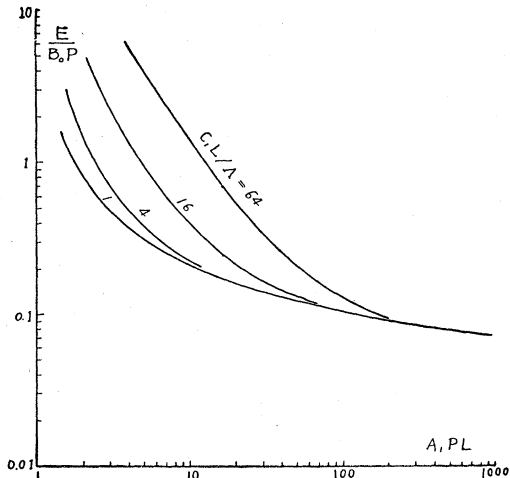

Fig. 12. Breakdown field for microwave (10.10).

the gap length, and L/Λ, the ratio of the gap length to the wavelength; we get from (10.9)

$$\frac{B_0 P}{E}\left[1+\left(\frac{C_1 L/\Lambda}{A_1 PL}\right)^2\right]^{\frac{1}{2}} = 2\ln(A_1 PL), \qquad (10.10)$$

where B_0 is the constant (9.8) involved in the Townsend coefficient α; A_1 and C_1 are given by

$$A_1 = (N/P\pi)(3\sigma\lambda/c_i)^{\frac{1}{2}}, \quad C_1 = (\omega\Lambda/\pi)(3\sigma/\lambda c_i)^{\frac{1}{2}}. \qquad (10.11)$$

The relation (10.10) is shown in Figs. 12 and 13. In the latter the ordinate shows the breakdown voltage.

By means of the molecular constants given by Table II in 9, we can calculate A_1 and C_1, the results being shown in Table IV (C_2 is a constant necessary in 11).

The full lines in Fig. 14 are the results observed by MacDonald-Brown[10] for the dielectric breakdown of H_2 in an electric field of 10-cm wavelength. If we take $B_0 = 172$ volt/cm mm Hg, $A_1 = 5.0$ 1/cm.mm Hg, and $C_1 = 377$ as the constants in Eq. (10.10) so that we may obtain a result as close as possible to the experiment, we
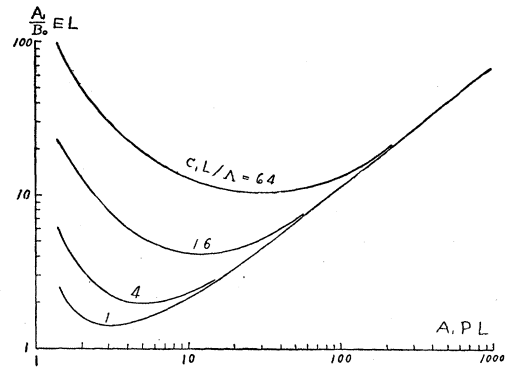

Fig. 13. Breakdown voltage for microwave.

obtain curves expressed by dotted lines in Fig. 14. There is no great discrepancy between these constants and A_1 and C_1 given in Table IV.

In the low pressure region, the theoretical and the experimental curves show remarkable discrepancy. The reason may be as follows: for the availability of ordinary notions in the kinetic theory of gases, especially the notion of diffusion, it is necessary that the mean free path of the particle is short enough in comparison with the gap length; this condition does not hold for extremely low pressures. In fact the mean free path of the electron with ionization energy is nearly equal to $c_i/N\lambda = 0.1/P$ cm/mm Hg in hydrogen gas.

11. Dielectric Breakdown for Radiofrequency Field

As in the preceding section, we treat the dielectric breakdown of low pressure gas by high frequency field between parallel plate electrodes. In this section we assume that the angular frequency ω satisfies the con-

[10] A. D. MacDonald and S. C. Brown, Phys. Rev. 76, 1634 (1949).

dition,

$$m/m_1 \ll \omega/N\lambda \ll 1. \quad (11.1)$$

For such a frequency, we must take into account the periodic mass-motion of electrons caused by the field. Hence, the equation to be satisfied by the electron density n is expressed in the form

$$\frac{\partial n}{\partial t} = \nu n + D\frac{\partial^2 n}{\partial z^2} - KE_0 \cos\omega t \frac{\partial n}{\partial z}, \quad (11.2)$$

where $KE_0 \cos\omega t$ is the drift velocity of electrons. Just as in the foregoing case, we have the boundary condition for $n(z, t)$

$$n(\pm L/2, t) = 0. \quad (11.3)$$

The breakdown field can be derived from the condition that (11.2) should have a solution periodic in time.

Since the drift velocity is usually larger than the diffusion velocity, the electrons in the periodic solution move to and fro as a whole as shown in Fig. 15. In the

TABLE IV. Coefficients in (10.10) and (11.8).

	A_1 1/cm mm Hg	C_1	C_2
He	0.81	210	530
Ne	1.53	190	730
A	5.2	360	520
H_2	2.5	250	560
N_2	6.5	210	660
CH_4	6.2	200	750

analytical expression holds

$$n = \begin{cases} n_0 \cos\dfrac{\pi z'}{L - 2KE_0/\omega} & \text{for} \quad |z'| < \dfrac{L}{2} - \dfrac{KE_0}{\omega} \\ 0 & \text{for} \quad \dfrac{L}{2} - \dfrac{KE_0}{\omega} < |z'| < \dfrac{L}{2}, \end{cases}$$

where $z' \equiv z - (KE_0/\omega)\sin\omega t$. Then, to obtain the condition of breakdown we may replace L in (10.3) by L minus the amplitude of the mass-vibration of electrons,

$$L - \int_{-\pi/2}^{\pi/2} KE_0 \cos\omega t\, dt = L - \frac{2KE_0}{\omega}.$$

Hence, we have

$$\frac{1}{\pi^2}\left(L - \frac{2KE_0}{\omega}\right)^2 \frac{\nu}{D} = 1. \quad (11.4)$$

The explicit form for our molecular model is given in the form

$$\exp\left(\frac{mc_i^2}{2kT_e}\right) = \frac{3\sigma\lambda}{c_i}\left(\frac{NL}{\pi}\right)^2\left(1 - \frac{2KE_0}{\omega L}\right)^2$$
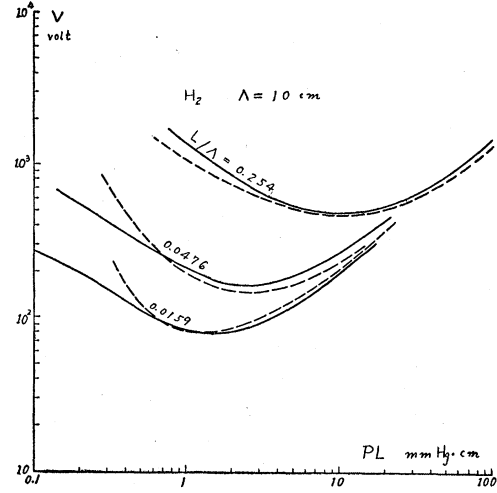
$$(K = e/mN\lambda), \quad (11.5)$$

FIG. 14. Dielectric breakdown of hydrogen for microwave. Wavelength $\Lambda = 10$ cm. V is the effective (rms) value of breakdown voltage.

corresponding to (10.8). The electron temperature T_e is given, from (7.9) and (11.1), by

$$kT_e = \frac{eE}{N(3\lambda\rho)^{\frac{1}{3}}} \quad (E = E_0/\sqrt{2}). \quad (11.6)$$

We get, therefore,

$$\exp\left[\frac{N}{E}\frac{mc_i^2}{2e}(3\lambda\rho)^{\frac{1}{3}}\right] = \frac{3\sigma\lambda}{c_i}\left(\frac{NL}{\pi}\right)^2\left(1 - \frac{2eE_0}{m\lambda NL}\right)^2$$

which indicates the functional form of E/N in terms of NL and ωL. With regard to the pressure P and the wavelength Λ we have

$$\exp\frac{B_0 P}{2E} = A_1 PL\left(1 - \frac{E/B_0 P}{C_2 L/\Lambda}\right), \quad (11.8)$$
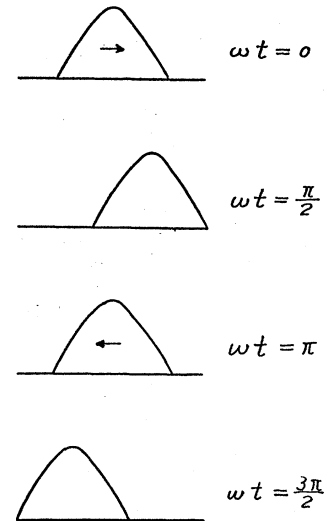
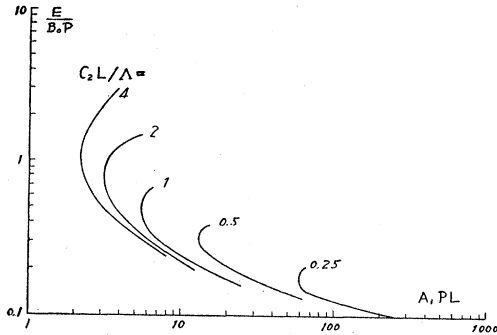FIG. 15. Mass-vibration of electrons in a radiofrequency field.

FIG. 16. Breakdown fields of radiowave frequency, (11.8).

where B_0 and A_1 are the constants in (10.10) and C_2 is given by

$$C_2 = \frac{\omega L}{\sqrt{2}} \frac{1}{c_i^2} \left(\frac{\lambda}{3\rho}\right)^{\frac{1}{2}}. \tag{11.9}$$

The numerical values can be calculated by means of the constants in Table II; the results are contained in Table IV. Figures 16 and 17 illustrate the relation (11.8).

The full lines in Fig. 18 show the experimental results of Githens[11] for the breakdown voltage $V = EL$ of hydrogen in an electric field of several Mc frequency. The dotted lines are derived from the formula (11.8) with the constants $B_0 = 105$ volt/cm mm Hg, $A_1 = 2.54$ 1/cm mm Hg, and $C_2 = 1800$. The constants have been fixed in such a way that (11.8) is in agreement with the experimental results as closely as possible. The comparison of these three values to B_0 in Table III, A_1 and C_2 in Table IV reveals that the difference in C_2 is large. It may be said, however, that we have roughly explained the discharge phenomena by means of the mechanism considered in this section.

The experimental curves for various frequencies show conspicuous discrepancy with theoretical curves in the region of small PL. It is because in the foregoing treatment we have not taken into account the effect of electron emission from the electrode (the γ-effect to be mentioned later) which takes an essential part in the low PL region. The peak value of the breakdown volt-
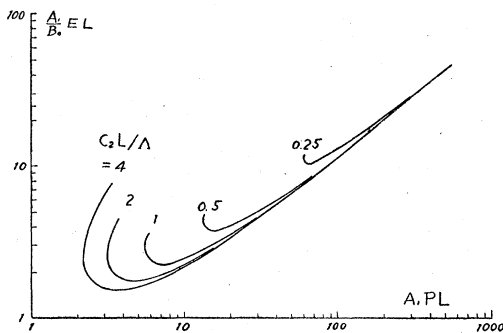

FIG. 17. Breakdown voltage for radiowave frequency.

[11] S. Githens, Phys. Rev. **57**, 822 (1940).

age in this region is known to be approximately equal to the static breakdown voltage.

12. Dielectric Breakdown for Static Field

For high frequency electric fields the breakdown takes place when collision-ionization supplies sufficient electrons to cancel the dissipation by diffusion. In case of electrostatic field, however, electrons created by ionization in the gas are carried away by the field itself. We must, therefore, consider a secondary electron-creating mechanism besides the collision-ionization in order to understand the dielectric breakdown. This role is usually played by the electron emission taking place when the positive ions created in the gas hit upon the cathode.

Assume that the voltage $V = EL$ is imposed on parallel plates electrodes with gap length L. Then one electron emitted from the cathode ionizes the gas molecule and increases in number to $\exp(\alpha L)$ until it reaches the anode. Since the number of the positive ions thus created is $\exp(\alpha L) - 1$, the breakdown takes place for the Town-
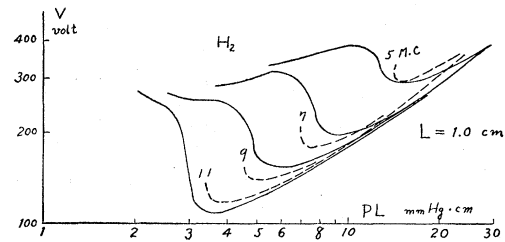

FIG. 18. Breakdown voltage of hydrogen for field of radiowave frequency. Gap $L = 1.05$ cm. V is the rms value of the voltage.

send coefficient given by

$$\gamma[\exp(\alpha L) - 1] = 1 \tag{12.1}$$

where we denote by γ the number of electrons liberated on the cathode by one positive ion; i.e., the breakdown condition is given by

$$\alpha L = \Gamma, \quad \text{where} \quad \Gamma = \ln(1+\gamma)/\gamma. \tag{12.2}$$

Such an electron-creating process is called γ-effect. The mechanism of this process, however, is rather complicated; for the electron emission from the cathode hit by ultraviolet photons and excited molecules in metastable states are likely to participate to a great extent besides the effect of the collision of positive ions. Since these effects are apparently involved in γ-effect, the estimation of γ from the molecular point of view is difficult.

In view of (12.2) we know that a change in γ has no considerable influence upon the breakdown voltage. Hence, we may be able to regard γ as a constant; in fact, the breakdown voltage calculated with α given by (9.7),

$$V = B_0 PL[\ln(A_0 PL/\Gamma)]^{-1}, \tag{12.3}$$

has been known to be in good agreement with experiment.

Figure 19 shows the relation (12.3). We note its remarkable characteristics in the fact that the breakdown voltage V has a minimum. By rewriting (12.2) as $V\alpha/E = \Gamma$, we can see that the existence of a minimum in V corresponds to the existence of a maximum in α/E.

ACKNOWLEDGMENT
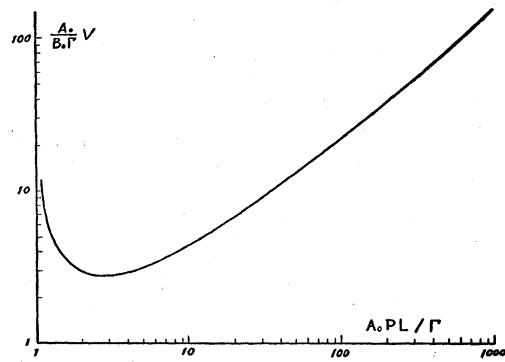
Fig. 19. Breakdown voltage for direct current.