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The Angular Distribution of Scattering and Reaction Cross Sections

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The general formula for the angular distribution in collisions between pairs of particles is simplified by performing explicitly all sums over magnetic quantum numbers. The resulting expressions involve coefficients introduced by Racah for the study of complex atomic spectra. The cross sections are expressed as series in Legendre polynomials, each coefficient in the series being manifestly real.

The general theory is then specialized for the case of nuclear reactions and scattering associated with one isolated resonance level of the compound nucleus. Formulas are derived for the various differential reaction cross sections and for scattering with and without change of channel spin. The interference terms between resonance and potential scattering are written explicitly, both for neutral and for charged particles.

1. INTRODUCTION

THE general expression for the differential scattering or reaction cross sections for an unpolarized beam in terms of the scattering matrix has been given in the literature.[1] However, the practical evaluation of this expression runs into difficulties as soon as some of the particles involved have intrinsic spins. In that case, we must average over the spin directions of the incident particles and sum over the spin directions of the outgoing particles. The resulting sums over Clebsch-Gordan (vector addition) coefficients are quite tedious to evaluate directly. However, all sums over magnetic quantum numbers are essentially geometrical in character and can therefore be performed without any detailed knowledge of the particular collision process (of the elements of the scattering matrix). We have obtained explicit expressions for the scattering and reaction cross sections free of all sums over magnetic quantum numbers. These explicit forms have the additional advantage that they express the cross sections directly as sums of Legendre polynomials, with coefficients that are manifestly real numbers.

In Sec. 2 of this paper we illustrate the method by application to a very simple problem: the elastic scattering of a single spinless particle by a center of force. Sections 3 and 4 are devoted to the derivation of the general result, applicable to all scattering and reaction processes of particles of arbitrary spins. In Secs. 5 through 7, we apply these general expressions to the special case of reactions and elastic scattering from a single isolated resonance level of the compound nucleus. The Racah coefficients, and the associated coefficients used in this paper (see Eq. 4.3), are described in an accompanying paper,† which contains, in addition, a summary of the relevant properties of these coefficients.

2. ELASTIC SCATTERING OF A SPINLESS PARTICLE BY A CENTER OF FORCE

The expression for the differential cross section $d\sigma$ for scattering into the solid angle element $d\Omega$ at an angle θ to the incident beam is well known:[2]

$$d\sigma = |f(\theta)|^2 d\Omega, \tag{2.1}$$

where the scattering amplitude $f(\theta)$ is a complex function depending upon a set of real parameters δ called phaseshifts:

$$f(\theta) = i(\pi)^{\frac{1}{2}} \lambda\!\!\!^{-} \sum_{l=0}^{\infty} (2l+1)^{\frac{1}{2}} (1-e^{2i\delta_l}) Y_{l,0}(\theta), \tag{2.2}$$

where $\lambda\!\!\!^{-} = \lambda/2\pi = k^{-1}$ is the deBroglie wavelength (divided by 2π) of the incident particles and $Y_{l,0}(\theta)$ is the normalized spherical harmonic defined as in Condon and Shortley.[3]

While (2.1) and (2.2) give an explicit expression for the scattering cross section, it is tedious to compare directly with experiment because the absolute square of an infinite sum must be evaluated for each scattering angle θ. It would be more convenient to have a formula

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[1] E. P. Wigner and L. Eisenbud, Phys. Rev. 72, 29 (1947), see Eqs. (33), (35), and (42); L. Eisenbud, J. Franklin Inst. 251, 231 (1951). See also: L. Diesendruck, thesis, The Johns Hopkins University (1950); D. R. Inglis, Phys. Rev. 74, 21 (1948); 76, 1319 (1949); E. Gerjuoy, Phys. Rev. 58, 503 (1940); C. L. Critchfield and E. Teller, Phys. Rev. 60, 10 (1941); E. Eisner, Phys. Rev. 65, 85 (1944); F. Bloch, Phys. Rev. 58, 829 (1940); G. Breit and B. T. Darling, Phys. Rev. 71, 402 (1947); W. Hauser and H. Feshbach, Phys. Rev. 87, 366 (1952); L. Wolfenstein, Phys. Rev. 82, 690 (1951).

† Biedenharn, Blatt, and Rose, Revs. Modern Phys. 24, 248 (1952); references to this paper will be designated by BBR.

[2] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, England, 1933), Chapter II.

[3] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Cambridge, England, 1935).

which contains only real terms with simple angular dependences. To get such a formula, we write $|f(\theta)|^2 = f^*(\theta)f(\theta)$ and use the fact that a product of two spherical harmonics can be expressed (like any other function of θ and ϕ) as a linear superposition of spherical harmonics:

$$Y_{lm}^*(\theta,\phi)\, Y_{l'm'}(\theta,\phi) = (-)^m \sum_{L=|l-l'|}^{l+l'} \sum_{M=-L}^{L} \left[\frac{(2l+1)(2l'+1)}{4\pi(2L+1)}\right]^{\frac{1}{2}} \times (ll'00|ll'L0)\,(ll',-m,m'|ll'LM)\, Y_{LM}(\theta,\phi), \quad (2.3)$$

where $(ll'mm'|ll'LM)$ are the Clebsch-Gordan (vector-addition) coefficients defined as in Condon and Shortley.[3] The sum over M is actually unnecessary since the only nonzero term is the one with $M=m'-m$. It is sometimes useful to retain the sum over M for formal reasons. Using (2.3) we get the following expression for the cross section:

$$d\sigma = \lambda\!\!\!^{-2} \sum_{L=0}^{\infty} B_L P_L(\cos\theta) d\Omega, \quad (2.4)$$

where $P_L(\cos\theta)$ is the usual Legendre polynomial, and

$$B_L = \sum_{l=0}^{\infty} \sum_{l'=|l-L|}^{l+L} (2l+1)(2l'+1)[(ll'00|ll'L0)]^2 \times \sin\delta_l \sin\delta_{l'} \cos(\delta_l - \delta_{l'}). \quad (2.5)$$

Expression (2.4) is directly comparable to experiment. The coefficients B_L can be determined by straightforward analysis of the data. All quantities in (2.4) and (2.5) are manifestly real. A simple explicit formula for the Clebsch-Gordan coefficients $(ll'00|ll'L0)$ has been given by Racah.[4] (This formula is reproduced in BBR, Eq. 5.) We remark here that this Clebsch-Gordan coefficient vanishes unless $l+l'-L$ is an even number. This is a consequence of the conservation of parity. This selection rule reduces the number of terms in (2.5) by about a factor of two.

For practical computations, (2.5) should be contracted so that each separate term appears only once in the sum (in the form stated so far, the terms $l=a$, $l'=b$ and $l=b$, $l'=a$ give identical contributions). The result is

$$B_L = \sum_{l=0}^{\infty} (2l+1)^2[(ll00|llL0)]^2 \sin^2\delta_l + 2\sum_{l=0}^{\infty} \sum_{\substack{l'=l+1 \\ (l+l'-L=\text{even})}}^{l+L} (2l+1)(2l'+1) \times [(ll'00|ll'L0)]^2 \sin\delta_l \sin\delta_{l'} \cos(\delta_l-\delta_{l'}). \quad (2.6)$$

We conclude this section by pointing out that the usual formula for the total cross section is contained in (2.4), namely, it is given by the term with $L=0$:

$$\sigma = 4\pi\lambda\!\!\!^{-2} B_0. \quad (2.7)$$

The usual formula follows directly from (2.6) and the relation

$$(ll'00|ll'00) = \delta_{ll'}(-)^l(2l+1)^{-\frac{1}{2}}. \quad (2.8)$$

[4] G. Racah, Phys. Rev. **61**, 186 (1942); **62**, 438 (1942).

3. NOTATION AND GENERAL EXPRESSION FOR THE CROSS SECTION

We now drop the assumption of pure elastic scattering as well as the assumption of spinless particles. We consider the reaction

$$a+X=Y+b, \quad (3.1)$$

in which particle a collides with nucleus X. After the collision, particle b emerges at an angle θ to the direction of the beam, the recoil nucleus Y being projected in the opposite direction. All quantities are measured in the center-of-gravity system. We use the language of nuclear reactions for the sake of convenience only. *The formulas derived below are applicable to any collision process in which two particles collide and two particles emerge.*

The system before the collision is described by three numbers: the channel index α, the channel spin s, and the orbital angular momentum (in the center-of-gravity system) l. The channel index α defines the type of the incoming particle (neutron, proton, α-particle, etc.) and the state of the struck nucleus (usually the ground state). The channel spin s[5] is the total spin angular momentum in the channel; it is formed by vector addition of the intrinsic spin i of the incoming particle and the spin I of the struck nucleus. For neutron-proton scattering, $i=I=\frac{1}{2}$, and $s=1$ for triplet state scattering, $s=0$ for singlet state scattering. The observed scattering cross section is the weighted average of the cross sections for pure triplet and pure singlet scattering. The expressions derived below are for "pure" collisions in which both the initial channel spin s and the final channel spin s' are known; (e.g., $s=s'=1$ corresponds to pure triplet scattering in the neutron-proton case). These expressions must therefore still be averaged over the possible incident channel spins s and summed over the possible out-going channel spins s', both with the proper statistical weights, in order to obtain quantities comparable to experiment. These further sums are not geometrical in character but depend upon the dynamics of the collision process; e.g., the phase shifts for neutron-proton scattering in the singlet and triplet states are not related, in general, but must be found separately from the force laws for these two spin states.[6] Thus, the geometrical (group theoretical)

[5] This quantity is called j_s by Wigner and Eisenbud; their "s" includes both of our α and s.

[6] One gets tremendous simplifications, of course, if one assumes that the dynamics of the collision is independent of the intrinsic spins of the incoming particle and struck nucleus. This assumption is, however, quite unjustified in general: in neutron-proton

methods used here do not lead to simplifications for the average over s and sum over s'.

The state of the system after the collision is described by the channel coordinate α' (which includes specification of the outgoing particle and of the quantum state of the residual nucleus), the channel spin s' (which is formed by vector addition of the intrinsic spin i' of the outgoing particle and of the spin I' of the residual nucleus in whatever quantum state it is left), and the outgoing orbital angular momentum l'.

The angular momenta s and l combine to form the total angular momentum J of the system; in resonance reactions, this J is the angular momentum of the compound nucleus; J is preserved during the collision, so that s' and l' combine to give the same J. Similarly, the parity is conserved. We shall not write the index π for the parity explicitly until later on.

We now introduce the probability amplitude $S_{\alpha's'l';\alpha sl}{}^J$ for a collision with total angular momentum J from channel αsl into channel $\alpha's'l'$. This quantity is often referred to as an element of the scattering matrix.[7] For reactions with only one possible channel (i.e., pure elastic scattering) S is related to the phase shift δ through $S=\exp(2i\delta)$. In general, for a reaction with N open channels S is an N-by-N matrix which must be unitary and symmetric. An example is neutron-proton scattering with tensor forces in the state with $J=1$ and even parity, for which two channels are open: the ${}^3S_1$ state and the ${}^3D_1$ state. In terms of our formalism, $\alpha=\alpha'$, $s=s'=1$, and l as well as l' can take two values, 0 and 2; thus, S is a 2-by-2 matrix, whose matrix elements are the probability amplitudes for ${}^3S_1$-to-${}^3S_1$, ${}^3D_1$-to-${}^3D_1$, ${}^3S_1$-to-${}^3D_1$, and ${}^3D_1$-to-${}^3S_1$ collisions, respectively; the last two matrix elements are equal, by reciprocity. All matrix elements are functions of the energy E of the collision.

Since the general expression for the cross section in terms of the scattering matrix is not given in standard treatments of quantum mechanics, we include the main steps of the derivation here. The first step consists in *defining the scattering matrix* itself. To do this, we restrict ourselves for the moment to one definite value of the total angular momentum J and its z component $J_z=M$. At any one energy E, the channel wave number k_α and the relative speed v_α in each channel α are given by the energetics of the reaction. We consider only those channels α which are "open" at the energy in question, i.e., for which k_α is real. Let $\Phi_{\alpha s}$ be the product of the wave function of the nucleus X and the particle a, both in the quantum states appropriate to the specification α, s. We define the spin- and angle-dependence of a wave function with total angular momentum J, z component thereof $J_z=M$, orbital angular momentum l, and spin angular momentum s, by[8]

$$\mathcal{Y}_{Jls}{}^M=\sum_{m_l=-l}^{l}\sum_{m_s=-s}^{s}(lsm_lm_s|lsJM)\,Y_{l,m_l}(\theta,\phi)\,\chi_{s,m_s}. \tag{3.2}$$

This expression is a function of the angular coordinates θ, ϕ as well as of the spin coordinates which appear in the spin function χ_{s,m_s}.

In terms of these definitions, the most general wave function in channel α, s with total angular momentum quantum numbers J, M consists of the superposition of an ingoing and outgoing spherical wave, each with spin-angle-dependence (3.2). At sufficiently large distances, we can write

$$\Psi_{\alpha s}(JM)=\frac{1}{r_\alpha(v_\alpha)^{\frac{1}{2}}}\mathcal{Y}_{Jls}{}^M\,\Phi_{\alpha s}\times\{A_{\alpha sl}{}^{JM}\exp[-i(k_\alpha r_\alpha-\tfrac{1}{2}l\pi)]-B_{\alpha sl}{}^{JM}\exp[+i(k_\alpha r_\alpha-\tfrac{1}{2}l\pi)]\}. \tag{3.3}$$

The coefficients $A_{\alpha sl}{}^{JM}$ and $B_{\alpha sl}{}^{JM}$ are not independent of each other. Rather, if the amplitudes of the ingoing waves are known, the amplitudes of the outgoing waves are determined uniquely by the wave equation. The relation between them defines the *scattering matrix*:

$$B_{\alpha's'l'}{}^{JM}=\sum_\alpha\sum_s\sum_l S_{\alpha's'l';\alpha sl}{}^J A_{\alpha sl}{}^{JM}. \tag{3.4}$$

Notice that the factor $\sqrt{v_\alpha}$ in (3.3) is different in different channels α. Because of it, the coefficients A and B correspond to amplitudes of probability flux, rather than amplitudes of probability density. With these definitions, S is a unitary matrix (to conserve the normalization of the wave function in time) as well as symmetric (reciprocity theorem). We also observe that the coefficients S in (3.4) are independent of M. This is necessary because different values of M can be obtained by simple rotation of the coordinate system, an operation which cannot affect the dynamics of the collision.

The next step consists in the *decomposition of a plane wave.* The incident wave in channel α, s is given by $\exp(ik_\alpha z_\alpha)\chi_{s,m_s}\Phi_{\alpha s}$. We may assume one definite spin orientation m_s even though all spin orientations are contained in the usual beam, because different values of m_s lead to *incoherent* contributions. This can be seen

scattering the basic force is spin-dependent, while in resonance reactions the system may go through a resonance in one spin state s, and be nonresonant at the same energy in another spin state.

[7] Our S^J is practically the same as the u^J of Wigner and Eisenbud; the precise relation is

$$S_{\alpha's'l';\alpha sl}{}^J=i^{l+l'}u_{sl;s'l'}{}^J,$$

where their "s" stands for our α and s combined. The factor $i^{l+l'}$ occurs because their standard wave functions go asymptotically (for large r) like $\exp(\pm ikr)$ whereas we use standard forms which behave asymptotically like $\exp[\pm i(kr-\frac{1}{2}l\pi)]$. Our choice has the advantage that the scattering mstrix for no events at all is given by $S=\delta_{\alpha'\alpha}\delta_{s's}\delta_{l'l}$.

[8] The Clebsch-Gordan coefficients $(lsm_lm_s|lsJM)$ vanish unless $m_l+m_s=M$. Hence one of the two sums in (3.2) is purely formal. It is a considerable convenience in the later work, however, to carry along these formal sums, and we shall do so from now on.

by observing that the spin direction m_s can be measured (e.g., by a Stern-Gerlach experiment) without interfering with the plane wave character of the beam. On the other hand, contributions of terms with different values of l are *coherent* and cannot be separated from each other without destroying the plane wave nature of the beam. We now write

$$\exp(ikz)\chi_{s,m_s}=(4\pi)^{\frac{1}{2}}\sum_{l=0}^{\infty} i^l(2l+1)^{\frac{1}{2}}\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} J_{l+\frac{1}{2}}(kr)Y_{l,0}(\theta)\chi_{s,m_s} \quad (3.5)$$

and

$$Y_{l,0}(\theta)\chi_{s,m_s}=\sum_{J=|l-s|}^{l+s}\sum_{M=-J}^{J}(ls0m_s|lsJM)\mathcal{Y}_{Jls}{}^{M}. \quad (3.6)$$

Using the asymptotic forms of the Bessel functions $J_{l+\frac{1}{2}}$ for large values of kr ($kr\gg l$) we get the following expression for the plane wave in channel α, s with spin direction given by m_s:

$$\begin{aligned}\exp(ik_\alpha z_\alpha)\chi_{s,m_s}\Phi_{\alpha s} &\cong \frac{i(\pi)^{\frac{1}{2}}}{k_\alpha r_\alpha}\Phi_{\alpha s}\sum_{J=0}^{\infty}\sum_{M=-J}^{J}\sum_{l=|J-s|}^{J+s} \\ &\times(ls0m_s|lsJM)i^l(2l+1)^{\frac{1}{2}}\mathcal{Y}_{Jls}{}^{M} \\ &\times\{\exp[-i(k_\alpha r_\alpha-\tfrac{1}{2}l\pi)] \\ &\quad-\exp[+i(k_\alpha r_\alpha-\tfrac{1}{2}l\pi)]\}. \quad (3.7)\end{aligned}$$

Comparing (3.7) with the standard form (3.3), we see that the amplitudes of the ingoing spherical waves are given by

$$\begin{aligned}A_{\alpha sl}{}^{JM}&=i\lambda\!\!\!^{-}_\alpha(\pi v_\alpha)^{\frac{1}{2}}(ls0m_s|lsJM)i^l(2l+1)^{\frac{1}{2}} \quad \text{in channel } \alpha,\ s, \\ A_{\alpha sl}{}^{JM}&=0 \quad \text{in other channels.}\end{aligned} \quad (3.8)$$

The amplitudes of the outgoing spherical waves are then determined by (3.4), i.e.,

$$B_{\alpha's'l'}{}^{JM}=i\lambda\!\!\!^{-}_\alpha(\pi v_\alpha)^{\frac{1}{2}}\sum_{l=|J-s|}^{J+s}\times(ls0m_s|lsJM)i^l(2l+1)^{\frac{1}{2}}S_{\alpha's'l';\alpha sl}{}^{J}. \quad (3.9)$$

Notice that there is no sum over α or s in (3.9). This corresponds to the fact that there are no ingoing spherical waves in any channels other than the entrance channel α, s.

We now investigate the asymptotic form of the wave function Ψ of the system at energy E in the region of configuration space corresponding to channel α', s'. For α', s' not equal to α, s the wave function corresponds to outgoing waves with amplitudes proportional to the corresponding matrix elements of the scattering matrix. However, the behavior is more complicated in the entrance channel α, s itself. There we have both ingoing and outgoing spherical waves. For the purpose of investigating reaction and scattering cross sections, we are interested only in that part of the wave function Ψ which is due to the occurrence of the reaction; i.e., we write

$$\Psi=\Psi_{\text{inc}}+\Psi_{\text{reac}}, \quad (3.10)$$

where Ψ_{inc} is given by (3.7). We now combine (3.7), (3.9), and (3.3) to obtain the *asymptotic form (for large r) of Ψ_{reac} in channel α', s'*:

$$\begin{aligned}\Psi_{\text{reac}}(\alpha's')&=i\lambda\!\!\!^{-}_\alpha\left(\frac{\pi v_\alpha}{v_{\alpha'}}\right)^{\frac{1}{2}}\frac{1}{r_{\alpha'}}\Phi_{\alpha's'} \\ &\times\sum_{J=0}^{\infty}\sum_{M=-J}^{J}\sum_{l=|J-s|}^{J+s}\sum_{l'=|J-s'|}^{J+s'} \\ &\times(ls0m_s|lsJM)i^l(2l+1)^{\frac{1}{2}} \\ &\times\exp[+i(k_{\alpha'}r_{\alpha'}-\tfrac{1}{2}l'\pi)] \\ &\times(\delta_{\alpha'\alpha}\delta_{s's}\delta_{l'l}-S_{\alpha's'l';\alpha sl}{}^{J})\mathcal{Y}_{Jl's'}{}^{M}. \quad (3.11)\end{aligned}$$

This expression would be sufficient if the detector in channel α', s' could be made to select the particular outgoing spherical wave with total angular momentum J, z component thereof M, and orbital angular momentum l'. Actually the common detectors select particles traveling in a given direction θ, ϕ. For the purposes of this discussion, we shall assume, however, that the detector is able to select particles corresponding to outgoing channel spin s' and spin direction $m_{s'}$. (Sums over $m_{s'}$ and over s' will be performed later). We therefore use (3.2) to decompose the spin-angle-function $\mathcal{Y}_{Jl's'}{}^{M}$ in (3.11), and we write $\Psi_{\text{reac}}(\alpha's')$ in the form

$$\Psi_{\text{reac}}(\alpha's')=i\lambda\!\!\!^{-}_\alpha\left(\frac{v_\alpha}{v_{\alpha'}}\right)^{\frac{1}{2}}\frac{\exp(ik_{\alpha'}r_{\alpha'})}{r_{\alpha'}}\Phi_{\alpha's'}\times\sum_{m_{s'}=-s'}^{s'}q_{\alpha's'm_{s'};\alpha sm_s}(\theta,\phi)\ \chi_{s',m_{s'}}. \quad (3.12)$$

The quantity q defined by (3.2), (3.11), and (3.12) can properly be called the *reaction amplitude* for the reaction $\alpha sm_s\rightarrow\alpha's'm_{s'}$. q is given explicitly by formula (3.14) below.

We are interested in the differential cross section for the process $\alpha sm_s\rightarrow\alpha's'm_{s'}$ corresponding to known spin directions m_s and $m_{s'}$ in the incident and outgoing channels, respectively, for a collision in which the final particle emerges within the solid angle element $d\Omega$ in the direction θ, ϕ with respect to the incident beam (all angles, etc., are measured in the center-of-gravity system). This cross section can be expressed in terms of the reaction amplitude $q_{\alpha's'm_{s'};\alpha sm_s}(\theta,\phi)$ for this particular collision, as follows:

$$d\sigma_{\alpha's'm_{s'};\alpha sm_s}=\lambda\!\!\!^{-}_\alpha{}^2|q_{\alpha's'm_{s'};\alpha sm_s}(\theta,\phi)|^2d\Omega, \quad (3.13)$$

where $\lambda_\alpha = k_\alpha^{-1}$ is the de Broglie wavelength (divided by 2π) of the relative motion in the incident channel α, and where the reaction amplitude q for the collision is related to the scattering matrix S as follows:[9]

$$q_{\alpha's'm_s';\alpha s m_s}(\theta,\phi)=\sum_{J=0}^{\infty}\sum_{M=-J}^{J}\sum_{l=|J-s|}^{J+s}\sum_{l'=|J-s'|}^{J+s'}\sum_{\mu'=-l'}^{l'} \times i^{l-l'}\pi^{\frac{1}{2}}(2l+1)^{\frac{1}{2}}(ls0m_s|lsJM)(l's'\mu'm_s'|l's'JM) \times(\delta_{\alpha'\alpha}\delta_{s's}\delta_{l'l}-S_{\alpha's'l';\alpha sl}{}^{J})\,Y_{l'\mu'}(\theta,\phi). \quad (3.14)$$

Since the vector addition coefficients vanish unless $m_l+m_s=M$, the sums over M and μ' in (3.14) are purely formal, the only contributing term being the one with $M=m_s$ and $\mu'=M-m_s'=m_s-m_s'$. Nevertheless, it is advantageous to keep these sums formally, for later work. The collision amplitude q, and hence also the cross section $d\sigma$, depend upon the angle ϕ, since the spin directions m_s and m_s' are specified. The cross section for the $\alpha s\rightarrow\alpha's'$ collision with an unpolarized beam is obtained by averaging over the incident spin directions m_s and summing over the final spin directions m_s':

$$d\sigma_{\alpha's';\alpha s}=(2s+1)^{-1}\sum_{m_s=-s}^{s}\sum_{m_s'=-s'}^{s'}d\sigma_{\alpha's'm_s';\alpha s m_s}. \quad (3.15)$$

We shall show that this expression can be simplified greatly by using sum rules for the Clebsch-Gordan coefficients. Finally, the differential cross section for the $\alpha\rightarrow\alpha'$ collision, without regard for the channel spins s or s', is obtained by averaging over the possible values of s and summing over the possible values of s':

$$d\sigma_{\alpha';\alpha}=\sum_{s=|I-i|}^{I+i}\sum_{s'=|I'-i'|}^{I'+i'}\frac{2s+1}{(2I+1)(2i+1)}d\sigma_{\alpha's';\alpha s}, \quad (3.16)$$

where I and i are the spins of the struck nucleus and incident particle, I' and i' those of the residual nucleus and outgoing particle, respectively, and the fraction represents the statistical weight of the channel spin s. As was mentioned before, the sum (3.16) cannot be simplified by geometrical considerations. Thus, we shall restrict ourselves to the differential cross section (3.15) in what follows; it being understood that the sums (3.16) must be performed on the final expression before comparison can be made with experiment.

4. REDUCTION OF THE DIFFERENTIAL CROSS SECTION

It is apparent from symmetry considerations that the sum (3.15) is essentially simpler than the individual terms which enter into it. For example, the averaged cross section (3.15) is independent of the angle ϕ, whereas the individual terms depend upon ϕ. The restrictions upon the differential cross section (3.15) have been discussed by many authors,[10] the paper of Yang being prominent for its derivation of these rules directly from symmetry considerations. While the symmetry considerations show that (3.15) can be simplified considerably, the actual evaluation of this expression has not been carried out so far, except for special cases.[11]

We can write (3.15) explicitly by performing the operations indicated in (3.13) and (3.14), as follows:

$$d\sigma_{\alpha's';\alpha s}=\lambda_\alpha^2(2s+1)^{-1}\sum_{J_1}\sum_{l_1}\sum_{l_1'}\sum_{J_2}\sum_{l_2}\sum_{l_2'} \times i^{-l_1+l_1'+l_2-l_2'}(\delta_{\alpha'\alpha}\delta_{s's}\delta_{l_1'l_1}-S_{\alpha's'l_1';\alpha sl_1}{}^{J_1})^* \times(\delta_{\alpha'\alpha}\delta_{s's}\delta_{l_2'l_2}-S_{\alpha's'l_2';\alpha sl_2}{}^{J_2}) \times K(J_1l_1'l_1;J_2l_2'l_2;s's;\theta)\,d\Omega, \quad (4.1)$$

where K is a purely geometrical quantity which is independent of the nature of the channels α, α' and of the dynamical aspects of the collision process (independent of the scattering matrix); K depends only upon the indicated angular momentum quantum numbers and upon the angle θ. In accordance with the general theorems, K is independent of the angle ϕ; it is defined as follows:

$$K(J_1l_1'l_1;J_2l_2'l_2;s's;\theta) =(2l_1+1)^{\frac{1}{2}}(2l_2+1)^{\frac{1}{2}}\pi\sum_{m_s}\sum_{m_s'}\sum_{M_1}\sum_{M_2}\sum_{\mu_1'}\sum_{\mu_2'} \times(l_1s0m_s|l_1sJ_1M_1)(l_2s0m_s|l_2sJ_2M_2) \times(l_1's'\mu_1'm_s'|l_1's'J_1M_1)(l_2's'\mu_2'm_s'|l_2's'J_2M_2) \times Y_{l_1'\mu_1'}(\theta\phi)^*\,Y_{l_2'\mu_2'}(\theta\phi). \quad (4.2)$$

K can be reduced completely to an expression involving no sums over magnetic quantum numbers. The necessary formalism has been developed by Racah[4] in connection with the theory of complex atomic spectra.

Introducing (2.3) into (4.2) gives a sum over seven magnetic quantum numbers and over L. These various sums can be reduced by a sum rule of Racah (BBR, Eq. (19)) which expresses sums of products of three vector addition coefficients in terms of "Racah coefficients" W which depend upon 6 angular momentum quantum numbers (none of which, however, has the interpretation of a magnetic quantum number). After substitution of (2.3) into (4.2), the sums over m_s', μ_1', and μ_2' combine into a Racah sum. After performing this sum, the sums over m_s, M_1, and M_2 again combine into a Racah sum. The result of this second Racah sum

[9] This follows directly from a comparison of (3.11) and (3.12), using (3.2).

[10] R. D. Myers, Phys. Rev. **54**, 361 (1938); E. Eisner and R. G. Sachs, Phys. Rev. **72**, 680 (1947); L. Wolfenstein and R. G. Sachs, Phys. Rev. **73**, 528 (1948); C. N. Yang, Phys. Rev. **74**, 764 (1948). The theorems were originally stated by E. Teller.

[11] The paper of Myers (reference 10) comes nearest to the present work, but it is restricted from the outset to resonance reactions and suffers from the fact that no explicit expressions are given for some of the coefficients (similar to the W of Racah) which enter the final expression.

is proportional to the vector addition coefficient $(l_1l_200|l_1l_2LM)$. This coefficient vanishes unless $M=0$, thereby eliminating the need for a summation over M. This is of course the well-known result that the cross section for an unpolarized beam is independent of the azimuthal angle ϕ, so that the only $Y_{LM}(\theta\phi)$ which can appear in the final expression for the cross section are the ones with $M=0$.

The final result can be written most simply in terms of quantities $Z(l_1J_1l_2J_2, sL)$ defined as follows:[12]

$$Z(l_1J_1l_2J_2, sL) = i^{L-l_1+l_2}(2l_1+1)^{\frac{1}{2}}(2l_2+1)^{\frac{1}{2}}(2J_1+1)^{\frac{1}{2}}(2J_2+1)^{\frac{1}{2}} W(l_1J_1l_2J_2, sL)\,(l_1l_200|l_1l_2L0), \quad (4.3)$$

where W is the Racah coefficient defined in reference (4). The definition of W is reprinted in BBR, Eq. (12), and the properties of W are discussed there. The factor $i^{L-l_1+l_2}$ is either $+1$ or -1, never imaginary.

In order to write the cross section as a sum of terms all of which are manifestly real numbers, we also need the identity

$$Z(l_1J_1l_2J_2, sL) = (-)^L Z(l_2J_2l_1J_1, sL). \quad (4.4)$$

The differential cross section can then be written as follows:

$$d\sigma_{\alpha's';\alpha s} = \frac{\lambda\!\!\!^{-}{}_\alpha^2}{2s+1}\sum_{L=0}^{\infty} B_L(\alpha's';\alpha s)\,P_L(\cos\theta)\,d\Omega, \quad (4.5)$$

where

$$B_L(\alpha's';\alpha s) = \frac{(-)^{s'-s}}{4}\sum_{J_1}\sum_{J_2}\sum_{l_1}\sum_{l_2}\sum_{l_1'}\sum_{l_2'} \times Z(l_1J_1l_2J_2, sL)\,Z(l_1'J_1l_2'J_2, s'L) \times \text{R.P.}[(\delta_{\alpha'\alpha}\delta_{s's}\delta_{l_1'l_1} - S_{\alpha's'l_1';\alpha sl_1}{}^{J_1})^* \times(\delta_{\alpha'\alpha}\delta_{s's}\delta_{l_2'l_2} - S_{\alpha's'l_2';\alpha sl_2}{}^{J_2})]. \quad (4.6)$$

Here R.P. stands for the real part of the expression in brackets. All sums in (4.6) are unrestricted and go from 0 to ∞; however, in practice only one of these sums is really infinite (say the sum over J_1). The other five sums are finite because of selection rules for non-vanishing Z coefficients (see BBR, Sec. IV). It should be noted that $B_L(\alpha s;\alpha's') = B_L(\alpha's';\alpha s)$ in accordance with the reciprocity theorem.

Just as in (2.5), the sum needs to be contracted for practical computation. The generalization of (2.6) is the contracted general formula[13]

$$\begin{aligned} B_L(\alpha's';\alpha s) &= \frac{(-)^{s'-s}}{4}\sum_{J=0}^{\infty}\sum_{l=|J-s|}^{J+s}\sum_{l'=|J-s'|}^{J+s'} Z(lJlJ, sL)\,Z(l'Jl'J, s'L)\,|\delta_{\alpha'\alpha}\delta_{s's}\delta_{l'l} - S_{\alpha's'l';\alpha sl}{}^J|^2 \\ &+ \frac{(-)^{s'-s}}{2}\sum_{J_1=0}^{\infty}\sum_{l_1=|J_1-s|}^{J_1+s}\sum_{l_1'=|J_1-s'|}^{J_1+s'}\Bigg\{\sum_{J_2=J_1+1}^{\infty}\sum_{l_2=|J_2-s|}^{J_2+s}\sum_{l_2'=|J_2-s'|}^{J_2+s'} Z(l_1J_1l_2J_2, sL)\,Z(l_1'J_1l_2'J_2, s'L)\,\text{R.P.}[\;] \\ &+ \sum_{l_2=l_1+1}^{J_1+s}\sum_{l_2'=|J_1-s'|}^{J_1+s'} Z(l_1J_1l_2J_1, sL)\,Z(l_1'J_1l_2'J_1, s'L)\,\text{R.P.}[J_2=J_1] \\ &+ \sum_{l_2'=l_1'+1}^{J_1+s'} Z(l_1J_1l_1J_1, sL)\,Z(l_1'J_1l_2'J_1, s'L)\,\text{R.P.}[J_2=J_1,\, l_2=l_1]\Bigg\}, \end{aligned} \quad (4.7)$$

where "R.P. [ ]" stands for the corresponding expression in (4.6). R. P. $[J_2=J_1]$ means the real part of the square bracket in (4.6) with J_2 set equal to J_1. While (4.7) looks much less symmetrical than (4.6), it is better adapted for actual computation since each term appears only once. In addition to the restrictions indicated on the various sums, the following restrictions reduce the number of actual terms:

$$l_1+l_2-L=\text{even}, \quad l_1'+l_2'-L=\text{even}, \quad (4.8)$$

$$(l_1+l_1') \text{ and } (l_2+l_2') = \begin{bmatrix}\text{even}\\ \text{odd}\end{bmatrix} \text{ if channels } \alpha \text{ and } \alpha' \text{ have } \begin{bmatrix}\text{equal}\\ \text{opposite}\end{bmatrix} \text{ parities.} \quad (4.9)$$

The parity of channel α is the product of the parity of the incident particle a and the parity of target nucleus X. The intrinsic parity of neutrons, protons, deuterons, tritons, and alpha-particles is $+1$, so the parity of channel α usually is equal to the parity of the target nucleus, and the parity of channel α' usually is equal to the parity of the residual nucleus Y in whatever state it is left.

[12] This definition differs slightly from the one used in our earlier Letter to the Editor of the Phys. Rev. (**82**, 123 (1951)).

[13] In our earlier Letter to the Editor [Phys. Rev. **82**, 123 (1951)] the restrictions on the sums over J_2, l_2, and l_2' were stated incorrectly resulting in the omission of some contributing terms (e.g., with $J_2>J_1$ but $l_2<l_1$). The expression of Hauser and Feshbach (reference 1) corresponds to the omission of all but the first line of Eq. (4.7).

We emphasize that nowhere in the derivation did we assume anything about the mechanism responsible for the reaction. Thus (4.5) and (4.6) apply to all possible collisions in which two particles enter and two particles emerge; they can be used, for example, for the exchange scattering of mesons by single nucleons, for neutron-proton scattering, and for nuclear reactions in which the compound nucleus treatment is not correct. The rather complicated appearance of these formulas is a direct consequence of their generality. In any special case of practical importance, considerable simplifications can be made. We shall give one example in this paper (reactions and scattering due to a single resonance level of the compound nucleus) and another example in a different paper (neutron-proton scattering with spin-orbit coupling).‡ The rules of reference (10) about the limitations upon the complexity of the angular distributions can be shown to follow from (4.6) by the use of Racah's selection rules for nonvanishing W coefficients. There is of course no sense in re-doing Yang's very nice derivation by such a sledge-hammer method; the advantage of (4.6) lies in the fact that it gives an *explicit* formula for the differential cross section.

For collisions of spinless particles (4.6) simplifies greatly owing to the fact that

$$Z(l_1J_1l_2J_2,\,0L)=\delta_{l_1J_1}\delta_{l_2J_2}(-i)^{L-l_1+l_2}\times[(2l_1+1)(2l_2+1)]^{\frac{1}{2}}(l_1l_200|l_1l_2L0). \quad (4.10)$$

In particular, the differential cross section for pure elastic scattering of spinless particles is obtained by substituting (4.10) into (4.6) and using the relation

$$S_{\alpha sl';\alpha sl}{}^J=\delta_{l'l}\exp(2i\delta_l),$$

valid for $s=0$. The result is identical with (2.5).

Finally, the total cross section (integrated over all angles) for the $\alpha s\rightarrow\alpha's'$ reaction is also contained in (4.6). We use the relation

$$Z(l_1J_1l_2J_2,\,s0)=\delta_{l_1l_2}\delta_{J_1J_2}(-)^{J_1-s}(2J_1+1)^{\frac{1}{2}} \quad (4.11)$$

to get the well-known result

$$\sigma_0(\alpha's';\alpha s)=\frac{4\pi\lambda\!\!\!^{-}{}_\alpha{}^2}{2s+1}B_0(\alpha's';\alpha s)=\frac{\pi\lambda\!\!\!^{-}{}_\alpha{}^2}{2s+1}\sum_{J=0}^{\infty}\sum_{l=|J-s|}^{J+s}\sum_{l'=|J-s'|}^{J+s'}(2J+1)|\delta_{\alpha'\alpha}\delta_{s's}\delta_{l'l}-S_{\alpha's'l';\alpha sl}{}^J|^2. \quad (4.12)$$

So far we have written all expressions as if the elements of the scattering matrix were independent of each other. For each value of J and parity, there is a certain number, say N, of different combinations α, s, l. Each scattering matrix element is a complex number; hence the formulas involve $2N^2$ real parameters for each value of J and Π. Actually these parameters are not independent of each other. The scattering matrix S must be unitary and symmetric. This implies that S can be written in the form

$$S^{J\Pi}=\exp(2iQ^{J\Pi}), \quad (4.13)$$

where $Q^{J\Pi}$ is a Hermitean (S=unitary) and symmetric (S=symmetric) N-by-N matrix. Q is therefore real and symmetric, i.e., it corresponds to a rotation in a real N-dimensional vector space. We can always write Q in the form

$$Q^{J\Pi}=U_{J\Pi}{}^{-1}\Delta_{J\Pi}U_{J\Pi}, \quad (4.14)$$

where Δ is a real diagonal N-by-N matrix, and U is an orthogonal (unitary and real) N-by-N matrix. The diagonal elements of Δ are the "eigen-phaseshifts." The scattering matrix (4.13) then becomes[14]

$$S^{J\Pi}=(U_{J\Pi})^{-1}\exp(2i\Delta_{J\Pi})\,U_{J\Pi}. \quad (4.15)$$

There are N eigen-phaseshifts necessary to specify $\Delta_{J\Pi}$, and $\frac{1}{2}N(N-1)$ independent real parameters necessary to specify rotation $U_{J\Pi}$ in an N-dimensional space. Thus, there are altogether $\frac{1}{2}N(N+1)$ real independent parameters necessary to specify the scattering matrix $S^{J\Pi}$ for a definite J and parity, rather than the $2N^2$ real but dependent parameters contained in the formulas written so far.§

The parametrization (4.15) is advantageous for the cases $N=1$, 2, and perhaps $N=3$, where it is fairly straightforward to write the orthogonal transformation U explicitly in terms of the $\frac{1}{2}N(N-1)$ real independent parameters (Euler angles for $N=3$) on which it depends. For larger values of N, it is preferable to have a more direct way of constructing U from a matrix into which the $\frac{1}{2}N(N-1)$ parameters defining the rotation enter directly. This is supplied by the Cayley transform for orthogonal transformations:

$$U_{J\Pi}=(B_{J\Pi}-1)/(B_{J\Pi}+1), \quad (4.16)$$

where B is a real, antisymmetric (i.e., a skew-Hermitean) N-by-N matrix. The representation (4.16) is possible whenever U has all its eigenvalues different from $+1$. In terms of this B matrix, then, we obtain an explicit parametrization of the S matrix through

$$S^{J\Pi}=\frac{B_{J\Pi}+1}{B_{J\Pi}-1}\exp(2i\Delta_{J\Pi})\frac{B_{J\Pi}-1}{B_{J\Pi}+1}. \quad (4.17)$$

‡ J. M. Blatt and L. C. Biedenharn, Phys. Rev. **86**, 399 (1952).

[14] We recall that if $f(x)$ is any function of x which allows a power series expansion, then $f(U^{-1}AU)=U^{-1}f(A)U$.

§ E. Wigner, Proc. Nat'l Acad. Sci. **32**, 302 (1946).

We illustrate this procedure by writing down the most general S matrix for a two-channel reaction, i.e., we assume $N=2$. This applies, for example, to neutron-proton scattering with tensor forces in triplet states with total angular momentum J and parity $\Pi=-(-1)^J$.

Omitting the indices J and Π, we can write

$$\Delta=\begin{pmatrix}\delta_1 & 0\\ 0 & \delta_2\end{pmatrix},\quad B=\begin{pmatrix}0 & \cot(\frac{1}{2}\epsilon)\\ -\cot(\frac{1}{2}\epsilon) & 0\end{pmatrix}. \tag{4.18}$$

Substitution into (4.16) and (4.17) then yields

$$U=\begin{pmatrix}\cos\epsilon & \sin\epsilon\\ -\sin\epsilon & \cos\epsilon\end{pmatrix},$$

$$S=\begin{pmatrix}\cos^2\epsilon\ e^{2i\delta_1}+\sin^2\epsilon\ e^{2i\delta_2} & \frac{1}{2}\sin(2\epsilon)(e^{2i\delta_1}-e^{2i\delta_2})\\ \frac{1}{2}\sin(2\epsilon)(e^{2i\delta_1}-e^{2i\delta_2}) & \sin^2\epsilon\ e^{2i\delta_1}+\cos^2\epsilon\ e^{2i\delta_2}\end{pmatrix}. \tag{4.19}$$

This is the expression for the most general S matrix of a two-channel reaction, in terms of the three real parameters δ_1, δ_2 (the two eigen-phaseshifts), and ϵ.

There are considerable simplifications which take place in the cross section formula (4.6) when the $S_{\alpha's'l';\alpha sl}{}^{J\Pi}$ are parametrized according to (4.15). We start by deriving a simple formula for the total (transmission) cross section in a channel α. The cross section in question is given in terms of (3.5) and (4.12) by

$$\sigma_0(\alpha)=\sum_s\sum_{\alpha'}\sum_{s'}\frac{2s+1}{(2I+1)(2i+1)}\sigma_0(\alpha's';\alpha s)$$
$$=\frac{\pi\lambda\!\!\!^{-}{}_\alpha^2}{(2I+1)(2i+1)}\sum_{J=0}^{\infty}\sum_{\Pi=\pm1}(2J+1)\sum_s\sum_l\sum_{\alpha'}\sum_{s'}\sum_{l'}|\delta_{\alpha'\alpha}\delta_{s's}\delta_{l'l}-S_{\alpha's'l';\alpha sl}{}^{J\Pi}|^2.$$

The sums over α', s', and l' combine to give the diagonal (αsl, αsl) element of the matrix $(1-S^+)(1-S)$. Since S is unitary, this matrix is equal to $2-S-S^+$. Thus the total cross section becomes

$$\sigma_0(\alpha)=\frac{\pi\lambda\!\!\!^{-}{}_\alpha^2}{(2I+1)(2i+1)}\sum_{J=0}^{\infty}\sum_{\Pi=\pm1}(2J+1)\sum_s\sum_l[2-S^{J\Pi}-(S^{J\Pi})^+]_{\alpha sl;\alpha sl}. \tag{4.20}$$

This formula is the direct analog of the usual relation between the total cross section and the imaginary part of the amplitude for elastic scattering in the forward direction.

In the general case of many different channels α, (4.20) involves not only the eigen-phaseshifts contained in the diagonal matrix Δ but also the transformation coefficients contained in the orthogonal matrix U. However, if elastic scattering ($\alpha'=\alpha$) is the only energetically possible reaction, (4.20) allows a further, drastic simplification. For then we can add a formal sum over α on the right side of (4.20) without changing the result. We then get the trace of the matrix $2-S-S^+$. By use of the parametrization (4.15) or (4.17), this can be written as

$$\mathrm{trace}[(1-S^+)(1-S)]=\mathrm{trace}[(1-e^{-2i\Delta})(1-e^{2i\Delta})]$$
$$=4\ \mathrm{trace}\ (\sin^2\Delta).$$

Thus the total cross section is given by

$$\sigma_0(\alpha;\alpha)=\frac{4\pi\lambda\!\!\!^{-}{}_\alpha^2}{(2I+1)(2i+1)}\sum_{J=0}^{\infty}\sum_{\Pi=\pm1}$$
$$\times(2J+1)\ \mathrm{trace}\ (\sin^2\Delta_{J\Pi}). \tag{4.21}$$

This is the natural analog of the usual formula for the total scattering cross section in elastic collisions of spinless particles. The trace of the matrix $\sin^2\Delta$ is equal to the sum of the $\sin^2(\delta_i)$ of the various eigen-phaseshifts δ_i since Δ is a diagonal matrix. Three remarks should be made concerning (4.21): (1) This formula is invalid if reactions other than elastic scattering are energetically permitted. An analogous formula can be derived for that case, but it is of no practical interest. (2) In the special case of neutron-proton scattering, it is usually assumed that there are no singlet-to-triplet-state transitions during the scattering; that is, s is taken to be a constant of the motion. In that case it is not necessary to sum over s and s' in order to get a simple formula analogous to (4.21). Rather, the singlet scattering and the triplet scattering separately are given by expressions similar to (4.21). (3) While the total scattering cross section can be given in terms of a small number of parameters [N eigen-phaseshifts for each J and Π instead of the full $\frac{1}{2}N(N+1)$ parameters necessary to define $S^{J\Pi}$ completely], this is no longer true for the angular distribution.

We now derive a general formula for the angular distribution valid when the scattering matrix is parametrized according to (4.15). It is well known that the eigenvectors $u_1, u_2, \cdots, u_k, \cdots, u_N$ of the scattering matrix associated with the eigen-phaseshifts $\delta_1, \delta_2, \cdots, \delta_k, \cdots, \delta_N$ can be read off from the unitary matrix U in (4.15). That is, the first row of U contains the components of u_1, the second row of U contains the components of u_2, etc.[15] We now introduce the projection operators $P(u_k)$ whose matrix elements are defined by

$$(P(u_k))_{\alpha's'l',\alpha sl}=(u_k)_{\alpha sl}(u_k)_{\alpha's'l'} \tag{4.22}$$

and rewrite the scattering matrix in the spectral form

$$S=\sum_{k=1}^{N}\exp(2i\delta_k)P(u_k). \tag{4.23}$$

The matrix which enters the cross section formulas is not S itself, but $1-S$, which becomes

$$1-S=-2i\sum_{k=1}^{N}\exp(i\delta_k)\sin(\delta_k)P(u_k). \tag{4.24}$$

Formula (4.6) can then be rewritten as follows:

$$B_L(\alpha's';\alpha s)=(-1)^{s'-s}\sum_{J_1}\sum_{\Pi_1}\sum_{J_2}\sum_{\Pi_2}\sum_{k_1=1}^{N_1}\sum_{k_2=1}^{N_2}\sin(\delta_{J_1\Pi_1k_1})\sin(\delta_{J_2\Pi_2k_2})\cos(\delta_{J_1\Pi_1k_1}-\delta_{J_2\Pi_2k_2})T, \tag{4.25}$$

where T is the following sum:

$$T=\sum_{l_1}\sum_{l_2}\sum_{l_1'}\sum_{l_2'}Z(l_1J_1l_2J_2,sL)Z(l_1'J_1l_2'J_2,s'L)(P(u_{J_1\Pi_1k_1}))_{\alpha's'l_1';\alpha sl_1}(P(u_{J_2\Pi_2k_2}))_{\alpha's'l_2';\alpha sl_2}.$$

This can be rewritten by the use of (4.22) in the product form

$$T=\Big[\sum_{l_1}\sum_{l_2}Z(l_1J_1l_2J_2,sL)(u_{J_1\Pi_1k_1})_{\alpha sl_1}(u_{J_2\Pi_2k_2})_{\alpha sl_2}\Big]\times\Big[\sum_{l_1'}\sum_{l_2'}Z(l_1'J_1l_2'J_2,s'L)(u_{J_1\Pi_1k_1})_{\alpha's'l_1'}(u_{J_2\Pi_2k_2})_{\alpha's'l_2'}\Big]. \tag{4.26}$$

Expressions (4.25) and (4.26) are the direct analog of (2.5) for spinless particles. While the notation appears rather formidable, the actual evaluation of the sums (4.26) is quite simple. An example appears in a separate paper on neutron-proton scattering in the presence of spin-orbit coupling. We remark that for the special case of pure elastic scattering (i.e., scattering without change of either the energy or the channel spin, symbolized by $\alpha'=\alpha$ and $s'=s$), the two square brackets in (4.26) become identical, so that the coefficient T is the square of an expression linear in the Z coefficients.

5. EVENTS CONNECTED WITH A SINGLE RESONANCE LEVEL OF THE COMPOUND NUCLEUS; (A) REACTIONS

The general formulas can be simplified greatly if special assumptions are made about the nature of the scattering matrix. A very useful special case is obtained by assuming that the reaction proceeds via a definite resonance level of the compound nucleus, with angular momentum J_0 and parity Π_0. The expression for the scattering matrix in that case has been given by Wigner and Eisenbud,[1] their Eq. (56). In order to get simple expressions, we shall make two additional restrictive assumptions: (1) The channel radius R [equal to the a_s of reference (1)] is independent of the channel spin s, although it may vary for different channels α. (2) The constant matrix $\mathcal{R}_\infty$ in Eq. (46) of reference 1 can be neglected compared to the resonant term. It is advantageous to rewrite Eq. (56) of reference 1 in an equivalent form. We define the partial widths $\Gamma_{\alpha sl}$ in the same way as Wigner and Eisenbud [the $\Gamma_{\lambda sl}$ in their Eq. (55)], but instead of their $\alpha_{\lambda sl}$ we introduce a real quantity $g_{\alpha sl}$ defined by

$$g_{\alpha sl}=\pm(\Gamma_{\alpha sl})^{\frac{1}{2}}. \tag{5.1}$$

The ambiguity in sign is typical of the dispersion formulas; this uncertainty does not appear in the formula for the total cross section, but we shall see that a study of the angular distribution of scattering and reaction products allows in principle a determination, not only of the magnitude of the partial widths $\Gamma_{\alpha sl}$, but also of the relative signs of the parameters $g_{\alpha sl}$ which enter into the dispersion formula. We mention that both $g_{\alpha sl}$ and $\Gamma_{\alpha sl}$ are functions of the channel energy through the usual penetration factor.

We also introduce the phaseshifts for the "potential" (hard sphere) scattering ξ_l, through

$$\exp(2i\xi_l)=\frac{G_l(R)-iF_l(R)}{G_l(R)+iF_l(R)} \quad \text{(for neutral particles)}, \tag{5.2}$$

$$\exp(2i\xi_l)=\exp(2i\sigma_l)\frac{G_l(R)-iF_l(R)}{G_l(R)+iF_l(R)} \quad \text{(for charged particles)}, \tag{5.3}$$

where $F_l(r)$ and $G_l(r)$ are the conventional regular and irregular solutions of the radial wave equation outside the nuclear surface, R is the channel radius (i.e., the sum of the radii of the target nucleus and incident

[15] In general, the rows of U would contain the complex conjugates of these components; however, U is a real matrix.

particle), and σ_l is the phaseshift for Coulomb scattering from an impenetrable sphere of radius R; the formula for σ_l is

$$\exp(2i\sigma_l)=\frac{(l+i\eta)(l-1+i\eta)\cdots(1+i\eta)}{(l-i\eta)(l-1-i\eta)\cdots(1-i\eta)}\frac{(i\eta)!}{(-i\eta)!},\quad (5.4)$$

where $\eta=Z_aZ_Xe^2/\hbar v$ and $z!$ is the usual factorial function. The potential (hard-sphere) scattering cross section is given by Eqs. (2.4) and (2.5) with δ_l replaced by ξ_l.

We also introduce the notation E_0 for the observed resonance energy, i.e., in terms of the notation of reference 1,

$$E_0=E_\lambda+\Delta_\lambda. \quad (5.5)$$

In principle, E_0 is not a constant but depends on the channel energy through the quantity Δ_λ. In practice the energy variation of Δ_λ can usually be neglected, however, so that E_0 can be considered a constant.

In terms of the notation introduced so far, and making the approximations indicated above, Wigner and Eisenbud's formula for the scattering matrix can be rewritten in the form[16]

$$S_{\alpha's'l';\alpha sl}{}^{J_0\Pi_0}=\exp[i(\xi_{\alpha l}-\eta_\alpha \ln 2k_\alpha r_\alpha)]\ \exp[i(\xi_{\alpha' l'}-\eta_{\alpha'}\ln 2k_{\alpha'}r_{\alpha'})]\times\left[\delta_{\alpha'\alpha}\delta_{s's}\delta_{l'l}+i\frac{g_{\alpha sl}g_{\alpha's'l'}}{E_0-E-\frac{1}{2}i\Gamma}\right]\quad (J=J_0,\ \Pi=\Pi_0). \quad (5.6)$$

In this formula, r_α is a screening radius for the Coulomb field in channel α, and $r_{\alpha'}$ is a similar quantity for channel α'. These screening radii drop out of the final formulas. Γ is the total width of the level J_0, Π_0, E_0:

$$\Gamma=\sum_\alpha\sum_s\sum_l \Gamma_{\alpha sl}. \quad (5.7)$$

This quantity depends on the energy. Formula (5.6) is appropriate for the particular value of $J\ (=J_0)$ and $\Pi\ (=\Pi_0)$ for which the resonance occurs. We assume that resonances of different J and parity are sufficiently far away in energy that we can neglect their influence.

Thus, for all other values of J and Π, we get

$$S_{\alpha's'l';\alpha sl}{}^{J\Pi}=\delta_{\alpha'\alpha}\delta_{s's}\delta_{l'l}\exp[2i(\xi_{\alpha l}-\eta_\alpha \ln 2k_\alpha r_\alpha)]\quad (J,\ \Pi\neq J_0,\ \Pi_0). \quad (5.8)$$

In this section we shall give an expression for the differential *reaction* cross section, that is, we assume either $\alpha'\neq\alpha$ or $s'\neq s$ or both.[17] In that case the scattering matrix elements vanish unless $J=J_0$ and $\Pi=\Pi_0$, and simple substitution of (5.6) into (4.6) gives [we introduce the notation $R_L(\alpha's';\alpha s)$ for later use]

$$B_L(\alpha's';\alpha s)=R_L(\alpha's';\alpha s),$$

where

$$R_L(\alpha's';\alpha s)\equiv\frac{(-)^{s'-s}}{4[(E-E_0)^2+(\frac{1}{2}\Gamma)^2]}\sum_{l_1=|J_0-s|}^{J_0+s}\ \sum_{l_2=|J_0-s|}^{J_0+s}\ \sum_{l_1'=|J_0-s'|}^{J_0+s'}\ \sum_{l_2'=|J_0-s'|}^{J_0+s'}\times Z(l_1J_0l_2J_0,sL)Z(l_1'J_0l_2'J_0,s'L)g_{\alpha sl_1}g_{\alpha sl_2}g_{\alpha's'l_1'}g_{\alpha's'l_2'}\cos[\xi_{\alpha l_1}-\xi_{\alpha l_2}+\xi_{\alpha' l_1'}-\xi_{\alpha' l_2'}]\quad (\text{for } \alpha,\ s\neq\alpha',\ s'). \quad (5.9)$$

In addition to the restrictions on the ranges of l_1, etc., indicated on the sums, all these orbital angular momentum quantum numbers must satisfy the parity conservation laws; if we denote the channel parity of channel α by Π_α, that of channel α' by $\Pi_{\alpha'}$, and of the compound nucleus by Π_0, the restrictions are

$$(-)^{l_1}=(-)^{l_2}=\Pi_\alpha\Pi_0\quad (-)^{l_1'}=(-)^{l_2'}=\Pi_{\alpha'}\Pi_0. \quad (5.10)$$

Each term in (5.9) occurs twice. Hence for practical computations the sum should be contracted along the lines of (4.7). We need not write down the result here.

The total cross section is given by (4.12) which becomes

$$\sigma_0(\alpha's';\alpha s)=\pi\lambda\!\!\!^{-}{}_\alpha^2\frac{2J_0+1}{2s+1}\sum_l\sum_{l'}\frac{\Gamma_{\alpha sl}\Gamma_{\alpha's'l'}}{(E-E_0)^2+(\frac{1}{2}\Gamma)^2}\quad (\text{for } \alpha,\ s\neq\alpha',\ s'), \quad (5.11)$$

where the ranges of summation over l and l' are as in (5.9) and (5.10). We see that the total cross section does not allow a determination of the signs of the quantities $g_{\alpha sl}$, but that a measurement of the differential cross section for the $\alpha s\rightarrow\alpha's'$ reaction allows such a determination, at least in principle.

According to the selection rules for nonvanishing Z coefficients (see BBR, Sec. IV), B_L vanishes for values

[16] We have not been able to get simple results by reducing (5.6) to the standard form (4.15). Simplifications do occur, however, near the peak of the resonance, provided that the resonance scattering dominates over the potential scattering.

[17] From the theoretical point of view, scattering without change of energy but with a change in the channel spin should be considered inelastic scattering; it can be treated by the same expressions which are applicable to reactions (for which $\alpha'\neq\alpha$). In practice, of course, the spin-flip events are included in the measured cross section for elastic scattering. We shall return to this point in Sec. VI.

of L which violate the condition

$$L \leqslant 2J_0. \tag{5.12a}$$

Furthermore, according to (5.10), the sum l_1+l_2 is always an even number, so that B_L vanishes for odd values of L,

$$L = \text{even}. \tag{5.12b}$$

Sometimes the penetration coefficients are such that there is a maximum value of l of the incident particles for which the widths are appreciable; or else there may be a maximum effective value of l' for the outgoing particles. If we assume that $g_{\alpha s l}=0$ for $l>l_{\max}$ and $g_{\alpha' s' l'}=0$ for $l'>l_{\max}'$, we get the additional restrictions

$$L \leqslant 2l_{\max} \quad L \leqslant 2l_{\max}'. \tag{5.12c}$$

These well-known restrictions on the complexity of the angular distribution[10] therefore follow also from our explicit expressions for the reaction cross sections, as of course they must. We mention that the quantities $g_{\alpha s l}$ involve the *square roots* of the penetration coefficients. Thus, care must be exercised in estimating the value of $l_{\max}$.

The formulas given so far apply to reactions in which the incident particles collide with a definite channel spin s and the outgoing particles emerge with a definite channel spin s'. In practice, the channel spins are not measured, and *the observed cross section for the $\alpha \rightarrow \alpha'$ reaction* is given by

$$d\sigma_{\alpha',\alpha} = \frac{\lambda\!\!\!^{-}{}_{\alpha}^2}{(2i+1)(2I+1)} \sum_{L=0}^{L_{\max}} B_L(\alpha';\alpha)\, P_L(\cos\theta)\, d\Omega, \tag{5.13}$$

where $B_L(\alpha';\alpha)$ is given in terms of (5.9) by

$$B_L(\alpha';\alpha) \equiv R_L(\alpha';\alpha) \equiv \sum_{s=|I-i|}^{I+i} \sum_{s'=|I'-i'|}^{I'+i'} R_L(\alpha's';\alpha s). \tag{5.14}$$

We emphasize that (5.13) and (5.14) apply only to reactions, i.e., to the case $\alpha' \neq \alpha$, whereas (5.9) applies also to the case $\alpha'=\alpha$ provided that $s' \neq s$. We have introduced the notations $R_L(\alpha's';\alpha s)$ and $R_L(\alpha';\alpha)$ for later use, since these sums also enter into the cross section for elastic scattering. The range of summation over L in (5.13) is determined by the restrictive conditions (5.12).

6. EVENTS CONNECTED WITH A SINGLE RESONANCE LEVEL OF THE COMPOUND NUCLEUS; (B) ELASTIC SCATTERING OF NEUTRAL PARTICLES

We now use expressions (4.6), (5.6), and (5.8) for the case $\alpha=\alpha'$, $s=s'$ which corresponds to true elastic scattering, i.e., scattering without change of either the energy or the channel spin. We observe that we would get pure hard-sphere (potential) scattering by using (5.8) for all values of J and Π, including $J=J_0$ and $\Pi=\Pi_0$. It is therefore advantageous to add and subtract the cross section for potential scattering. The actual cross section then consists of three parts: the resonance nuclear part, the potential scattering, and the interference term between these two.[18] We write the cross section in the form (4.5), with $\alpha'=\alpha$ and $s'=s$, but we break up the coefficient $B_L(\alpha s;\alpha s)$ into these three contributions:

$$B_L(\alpha s;\alpha s) = R_L(\alpha s;\alpha s) + (2s+1)H_L(\alpha;\alpha) + I_L(\alpha s;\alpha s), \tag{6.1}$$

where R_L stands for the resonance contribution, H_L for the hard sphere contribution, and I_L for the interference term. The calculation of these terms is straightforward but tedious. Use must be made of the sum rule [BBR, Eq. (26)] for the Z coefficients. As indicated by the notation, the resonance contribution R_L is equal to (5.9) with α' set equal to α and s' set equal to s. The hard-sphere scattering is by assumption independent of the channel spin s [the factor $2s+1$ merely serves to cancel the corresponding factor in (4.5)] and is given by (2.5) with the hard-sphere scattering phase shifts ξ_l in place of δ_l, i.e.,

$$H_L(\alpha;\alpha) = \sum_{l=0}^{\infty} \sum_{l'=|l-L|}^{l+L} (2l+1)(2l'+1)[(ll'00|ll'L0)]^2 \sin\xi_l \sin\xi_{l'} \cos(\xi_l - \xi_{l'}). \tag{6.2}$$

In order to write the interference term in a simple form, we introduce an angle parameter β to measure the deviation of the energy from resonance. We define β through

$$\tan\beta = (E-E_0)/\tfrac{1}{2}\Gamma. \tag{6.3}$$

The interference term in the cross section is then given by

$$I_L(\alpha s;\alpha s) = -(2J_0+1) \sum_{l=|J_0-s|}^{J_0+s} \sum_{l'=|l-L|}^{l+L} (2l'+1)[(ll'00|ll'L0)]^2 \frac{\Gamma_{\alpha s l}}{[(E-E_0)^2+(\frac{1}{2}\Gamma)^2]^{\frac{1}{2}}} \sin\xi_{l'} \sin(\beta + 2\xi_l - \xi_{l'}). \tag{6.4}$$

[18] The last two parts do not make any contribution to scattering without change of energy but with change of channel spin.

It is interesting to observe that the sign ambiguity associated with the quantities $g_{\alpha sl}$ does not appear in this interference term.

The contribution R_L vanishes as soon as L violates one or more of the conditions (5.12). The quantities H_L and I_L never become exactly zero. However, in the case of scattering of neutral particles, the phase shifts ξ_l for the potential scattering approach zero as l becomes large, and hence H_L as well as I_L become small for large values of L, so that the sum (4.5) converges well. This is not true for the elastic scattering of charged particles, and this case will be treated separately in the next section.

Equations (6.1) through (6.4) give the cross section for scattering without change of either the energy or the channel spin s. Experimentally, this quantity is not measured. The experimental "elastic scattering" cross section includes events in which the spins are flipped. These events are described correctly by (5.9) with $\alpha'=\alpha$ but $s'\neq s$. The experimentally observed elastic scattering cross section is given by (3.16) with $\alpha'=\alpha$. We introduce the partial widths $\Gamma_{\alpha l}$ for emission into channel α with orbital angular momentum l, summed over all possible channel spins s, i.e.,

$$\Gamma_{\alpha l}\equiv\sum_{s=|I-i|}^{I+i}\Gamma_{\alpha sl}. \tag{6.5}$$

We then get the following formula for the *experimentally observed differential cross section for elastic scattering in the neighborhood of a resonance level of angular momentum* J_0 *and parity* Π_0:

$$d\sigma_{\alpha,\alpha}=\frac{\lambda\!\!\!^{-}_\alpha{}^2}{(2I+1)(2i+1)}\sum_{L=0}^{L_{\max}}R_L(\alpha;\alpha)\,P_L(\cos\theta)\,d\Omega+\lambda\!\!\!^{-}_\alpha{}^2\sum_{L=0}^{\infty}H_L(\alpha;\alpha)\,P_L(\cos\theta)\,d\Omega$$
$$-\lambda\!\!\!^{-}_\alpha{}^2\frac{2J_0+1}{(2I+1)(2i+1)}\sum_{L=0}^{\infty}\sum_{l=l_{\min}}^{J_0+I+i}\sum_{l'=|l-L|}^{l+L}(2l'+1)[(ll'00|ll'L0)]^2$$
$$\times\frac{\Gamma_{\alpha l}}{[(E-E_0)^2+(\frac{1}{2}\Gamma)^2]^{\frac{1}{2}}}\sin\xi_{l'}\sin(\beta+2\xi_l-\xi_{l'})\,P_L(\cos\theta)\,d\Omega. \tag{6.6}$$

In this formula the first term represents resonance nuclear scattering; the coefficient $R_L(\alpha;\alpha)$ is given by (5.14) with $\alpha'=\alpha$; $L_{\max}$ is determined by the restrictions (5.12). The second term is the hard-sphere (potential) scattering; the coefficient $H_L(\alpha;\alpha)$ is given by (6.2). The last term contains the interference between nuclear and potential scattering. The range of summation over l is such that the three angular momenta l, I, and i can combine by the vector addition rules to give a total angular momentum J_0. That is, the maximum value of l is as indicated on the sum, while the minimum value of l is given by

$$\begin{aligned} l_{\min}&=J_0-(I+i) && \text{if } J_0>I+i,\\ l_{\min}&=0 && \text{if } |I-i|\leqslant J_0\leqslant I+i,\\ l_{\min}&=|I-i|-J_0 && \text{if } J_0<|I-i|.\end{aligned} \tag{6.7}$$

Finally we give the total cross section, which is 4π times the coefficient of $P_0(\cos\theta)$ in (6.6):

$$\sigma_0(\alpha;\alpha)=\pi\lambda\!\!\!^{-}_\alpha{}^2\frac{2J_0+1}{(2I+1)(2i+1)}\sum_l\sum_{l'}\frac{\Gamma_{\alpha l}\Gamma_{\alpha l'}}{(E-E_0)^2+(\frac{1}{2}\Gamma)^2}+4\pi\lambda\!\!\!^{-}_\alpha{}^2\sum_{l=0}^{\infty}(2l+1)\sin^2\xi_l$$
$$-4\pi\lambda\!\!\!^{-}_\alpha{}^2\frac{2J_0+1}{(2I+1)(2i+1)}\sum_{l=l_{\min}}^{J_0+I+i}\frac{\Gamma_{\alpha l}\sin\xi_l\sin(\beta+\xi_l)}{[(E-E_0)^2+(\frac{1}{2}\Gamma)^2]^{\frac{1}{2}}}. \tag{6.8}$$

As before, the first term gives the contribution of pure resonance scattering, the second is the pure hard-sphere (potential) scattering, and the last is the interference between the two. If ξ_l is negative and $|\xi_l|$ is less than $\frac{1}{2}\pi$, the corresponding interference term is negative below resonance and at resonance and then becomes positive; the range of energy over which this term is appreciable is much wider than the usual half-width, because its magnitude decreases far away from resonance like $|E-E_0|^{-1}$ rather than like $(E-E_0)^{-2}$. We also observe that the interference term in both (6.6) and (6.8) can be considered the sum of terms with different l, each of which is equal to the interference term obtained from a resonance of given l and partial width for emission into channel α equal to $\Gamma_{\alpha l}$ if all spins I, I', i, i' are set equal to zero; the only effect of the spins consists in the multiplication by the usual statistical factor, and in the need for summing over the many possible values of l consistent with J_0 and Π_0. One sometimes encounters the statement that this is the only effect of the spins for the total scattering cross section. We see that this is not entirely accurate because the treatment neglecting spins does not lead to any scattering with change of orbital angular momentum l, whereas such scattering does occur when spins are present. However, this scattering contributes to the resonance term [the first term in (6.6) and (6.8)] but not to the interference between resonance and potential scattering.

7. EVENTS CONNECTED WITH A SINGLE RESONANCE LEVEL OF THE COMPOUND NUCLEUS; (C) ELASTIC SCATTERING OF CHARGED PARTICLES

The expressions derived in the preceding section are still correct for the elastic scattering of charged particles, but they are not useful in practice because the sum over L converges very slowly. Thus we must treat the Coulomb scattering explicitly in order to get usable expressions. The potential scattering is now scattering by a charged hard sphere of radius R, which can be written as scattering by a point change plus correction terms to take into account the finite nuclear radius. We shall first derive the expressions for the scattering of spinless charged particles by a nucleus of zero spin ($I=i=0$). The cross section is given by (2.1) where $f(\theta)$ consists of three parts:

$$f(\theta)=f_C(\theta)+f_{CH}(\theta)+f_{Rl}(\theta), \tag{7.1}$$

where $f_C(\theta)$ is the scattering amplitude for pure Rutherford scattering, $f_{CH}(\theta)$ is the difference between the scattering amplitude from a charged hard sphere and $f_C(\theta)$, and $f_{Rl}(\theta)$ is the nuclear resonance scattering amplitude, which we assume to be associated with a resonance of angular momentum l (since $i=I=0$ by assumption, this l is equal to the total angular momentum of the compound nucleus), of partial width $\Gamma_{\alpha l}$, total width Γ, and resonance energy E_0.

The expressions for the scattering matrix, (5.6) and (5.8), contain factors $\exp[-i\eta\ln(2kr)]$, where r is a screening radius for the electrostatic field of the nucleus within the atom. In practice the effect of screening on the angular distribution of elastically scattered particles is negligible at all angles except for very small-angle forward scattering (where the screening effect prevents the cross section from becoming infinite). We can therefore ignore those factors. It is useful to factor out a common phase factor $\exp(2i\sigma_0)$ from all terms of (7.1). We introduce the notation

$$z\equiv Z_a Z_X e^2/2Mv^2, \tag{7.2}$$

where Z_a and Z_X are the atomic numbers of incident particle and target nucleus, respectively, and M is the reduced mass for the relative motion in the center-of-gravity system. We also define phase shifts ϕ_l and ψ_l through [$F_l(R)$ and $G_l(R)$ are the usual regular and irregular Coulomb wave functions, evaluated at the nuclear radius R]

$$\phi_l\equiv\xi_l-\sigma_l \qquad \exp(2i\phi_l)=\frac{G_l(R)-iF_l(R)}{G_l(R)+iF_l(R)}, \tag{7.3}$$

$$\psi_l\equiv\sigma_l-\sigma_0 \qquad \exp(2i\psi_l)=\frac{(l+i\eta)(l-1+i\eta)\cdots(1+i\eta)}{(l-i\eta)(l-1-i\eta)\cdots(1-i\eta)}. \tag{7.4}$$

Thus we have

$$\xi_l=\psi_l+\phi_l+\sigma_0. \tag{7.4a}$$

The S wave Coulomb phase shift σ_0 cancels out in the argument of the cosine in expression (5.9). This is typical of all subsequent formulas: The value of σ_0 is never needed; the final results can all be expressed in terms of differences of phase shifts, such as ϕ_l and ψ_l. ϕ_l can be interpreted as the additional phase shift by which scattering from a charged hard sphere differs from scattering by a point charge. This additional phase shift approaches zero for large values l. In terms of this notation, and factoring out a common $\exp(2i\sigma_0)$, we get

$$f_C(\theta)\ =-z\operatorname{cosec}^2(\tfrac{1}{2}\theta)\exp[-2i\eta\ln\sin(\tfrac{1}{2}\theta)], \tag{7.5}$$

$$f_{CH}(\theta)=i\lambda\pi^{\frac{1}{2}}\sum_{l'=0}^{\infty}(2l'+1)^{\frac{1}{2}}\exp(2i\psi_{l'})\,[1-\exp(2i\phi_{l'})]\,Y_{l',0}(\theta), \tag{7.6}$$

$$f_{Rl}(\theta)\ =i\lambda\pi^{\frac{1}{2}}(2l+1)^{\frac{1}{2}}\exp[2i(\psi_l+\phi_l)]\,\frac{\Gamma_{\alpha l}\exp(i\beta)}{[(E-E_0)^2+(\frac{1}{2}\Gamma)^2]^{\frac{1}{2}}}\,Y_{l,0}(\theta). \tag{7.7}$$

The cross section for elastic scattering is then given by (2.1), i.e.,

$$d\sigma=|f_C(\theta)+f_{CH}(\theta)+f_{Rl}(\theta)|^2d\Omega=[|f_C|^2+2\text{ R.P. }(f_C{}^*f_{CH})+|f_{CH}|^2]\,d\Omega$$
$$+[2\text{ R.P. }(f_C{}^*f_R)+2\text{ R.P. }(f_{CH}{}^*f_R)]\,d\Omega+|f_{Rl}|^2\,d\Omega. \tag{7.8}$$

The separation above is into the usual three parts: potential scattering, interference between potential and resonance scattering, and pure resonance scattering. By performing the indicated operations and using relation

(2.3) in various places, we get

$$|f_C|^2+2\text{ R.P. }(f_C{}^*f_{CH})+|f_{CH}|^2=z^2\operatorname{cosec}^4(\tfrac{1}{2}\theta)$$
$$-2\lambda z\sum_{l=0}^{\infty}(2l+1)\sin\phi_l\cos[2\eta\ln\sin(\tfrac{1}{2}\theta)+2\psi_l+\phi_l]\operatorname{cosec}^2(\tfrac{1}{2}\theta)\,P_l(\cos\theta)$$
$$+\lambda^2\sum_{L=0}^{\infty}\sum_{l=0}^{\infty}\sum_{l'=|l-L|}^{l+L}(2l+1)(2l'+1)[(ll'00|ll'L0)]^2\sin\phi_l\sin\phi_{l'}\cos[(2\psi_l+\phi_l)-(2\psi_{l'}+\phi_{l'})]\,P_L(\cos\theta). \quad (7.9)$$

This expression reduces to the differential cross section for potential scattering of neutral particles if we set $z=\psi_l=0$ and $\phi_l=\xi_l$. The interference term between resonance and potential scattering is equal to

$$2\text{ R.P. }(f_C{}^*f_{Rl})+2\text{ R.P. }(f_{CH}{}^*f_{Rl})$$
$$=\lambda z(2l+1)\frac{\Gamma_{\alpha l}}{[(E-E_0)^2+(\tfrac{1}{2}\Gamma)^2]^{\frac{1}{2}}}\sin[2\eta\ln\sin(\tfrac{1}{2}\theta)+2\psi_l+2\phi_l+\beta]\operatorname{cosec}^2(\tfrac{1}{2}\theta)\,P_l(\cos\theta)$$
$$-\lambda^2\sum_{L=0}^{\infty}\sum_{l'=|l-L|}^{l+L}(2l+1)(2l'+1)[(ll'00|ll'L0)]^2\frac{\Gamma_{\alpha l}}{[(E-E_0)^2+(\tfrac{1}{2}\Gamma)^2]^{\frac{1}{2}}}$$
$$\times\sin\phi_{l'}\sin(\beta+2\psi_l+2\phi_l-2\psi_{l'}-\phi_{l'})\,P_L(\cos\theta). \quad (7.10)$$

Finally, the pure resonance scattering cross section in the absence of spins is given by

$$|f_{Rl}|^2=\pi\lambda^2(2l+1)\frac{(\Gamma_{\alpha l})^2}{(E-E_0)^2+(\tfrac{1}{2}\Gamma)^2}|Y_{l,0}(\theta)|^2. \quad (7.11)$$

In this simple case it is not advantageous to use (2.3).

Formulas (7.8), (7.9), (7.10), and (7.11) give the elastic scattering cross section for the special case $i=I=0$. They can be used directly to analyze the elastic scattering of α-particles by even-even nuclei. We now show that the more general case $i\neq 0$, $I\neq 0$ can be obtained from these expressions and (6.6) by simple inspection. The first line of (6.6) is the pure resonance scattering. This expression can be used as it stands even in the presence of Coulomb fields, since the sums over L and the various l are finite and no question of convergence arises. Thus, the first line of (6.6) replaces (7.11). The second line of (6.6) is the pure potential scattering and is identical with (7.9); the only difference is that (7.9) involves rapidly convergent sums, whereas the sum in (6.6) converges very slowly. Finally, consider the last line of (6.6), the interference between resonant and potential scattering. We remarked at the end of Sec. 6 that this term can be considered the sum of terms with different values of l, each of which is equal to the interference term (7.10) for a resonance of given l and $i=I=0$, except for the statistical factor in front. The latter change means replacement of $(2l+1)$ by $(2J_0+1)/(2I+1)(2i+1)$. This can be seen formally also by considering (7.10) with $z=\psi_l=0$ and $\phi_l=\xi_l$ (corresponding to no Coulomb field), which differs from the interference term in (6.6) only through the statistical factor and the sum over permissible values of l.

Putting together all this information, we arrive at the following formula for *the elastic scattering of charged particles near a resonance level of the compound nucleus:*

$$d\sigma_{\alpha\alpha}=\frac{\lambda_\alpha{}^2}{(2I+1)(2i+1)}\sum_{L=0}^{L_{\max}}R_L(\alpha,\alpha)\,P_L(\cos\theta)\,d\Omega+z^2\operatorname{cosec}^4(\tfrac{1}{2}\theta)\,d\Omega$$
$$-2\lambda_\alpha z\sum_{l=0}^{\infty}(2l+1)\sin\phi_l\cos[2\eta\ln\sin(\tfrac{1}{2}\theta)+2\psi_l+\phi_l]\operatorname{cosec}^2(\tfrac{1}{2}\theta)\,P_l(\cos\theta)\,d\Omega$$
$$+\lambda_\alpha{}^2\sum_{L=0}^{\infty}\sum_{l=0}^{\infty}\sum_{l'=|l-L|}^{l+L}(2l+1)(2l'+1)[(ll'00|ll'L0)]^2\sin\phi_l\sin\phi_{l'}\cos[(2\psi_l+\phi_l)-(2\psi_{l'}+\phi_{l'})]\,P_L(\cos\theta)\,d\Omega$$
$$+\frac{\lambda_\alpha z(2J_0+1)}{(2I+1)(2i+1)}\sum_{l=l_{\min}}^{J_0+I+i}\frac{\Gamma_{\alpha l}}{[(E-E_0)^2+(\tfrac{1}{2}\Gamma)^2]^{\frac{1}{2}}}\operatorname{cosec}^2(\tfrac{1}{2}\theta)\sin[2\eta\ln\sin(\tfrac{1}{2}\theta)+2\psi_l+2\phi_l+\beta]\,P_l(\cos\theta)\,d\Omega$$
$$-\frac{\lambda_\alpha{}^2(2J_0+1)}{(2I+1)(2i+1)}\sum_{L=0}^{\infty}\sum_{l=l_{\min}}^{J_0+I+i}\sum_{l'=|l-L|}^{l+L}(2l'+1)[(ll'00|ll'L0)]^2\sin\phi_{l'}$$
$$\times\frac{\Gamma_{\alpha l}}{[(E-E_0)^2+(\tfrac{1}{2}\Gamma)^2]^{\frac{1}{2}}}\sin(\beta+2\psi_l+2\phi_l-2\psi_{l'}-\phi_{l'})\,P_L(\cos\theta)\,d\Omega. \quad (7.12)$$

We repeat the interpretation of the various terms in this formula: the first term is pure resonance scattering, with $R_L(\alpha, \alpha)$ defined by (5.14). The next term is pure Rutherford (point charge) scattering, with z defined by (7.2). The following two terms represent the correction due to the finite size of the nucleus; the phaseshifts ψ_l and ϕ_l are defined by (7.3) and (7.4), respectively. The next to last term is the interference between resonance scattering and pure Rutherford scattering, while the last term is the finite nuclear size correction to this interference term.

There is one case of special interest here, in which the rather complicated expression (7.12) reduces to more manageable proportions. This is elastic scattering of protons on even-even ($I=0$) target nuclei. In that case $i=\frac{1}{2}$, $I=0$, and there is only one possible channel spin, namely, $s=\frac{1}{2}$. Furthermore, for a resonance of given $J=J_0$ and parity Π_0, there is only one value $l=l_0$ of the orbital angular momentum of the protons; that is, l_0 is defined uniquely by the conditions that (1) l_0 is one of $J_0+\frac{1}{2}$, $J_0-\frac{1}{2}$; and (2) $(-1)^{l_0}=\Pi_0\Pi_X$, where Π_X is the parity of the target nucleus. In that case the quantities $g_{\alpha sl}$ do not enter; indeed we need to know only two widths, the width Γ_α for elastic scattering, and the total width Γ. *The cross section for elastic scattering of charged particles with* $s=\frac{1}{2}$ (protons, tritons, or He^3 nuclei) *by nuclei with* $I=0$ (even-even nuclei) is given by

$$d\sigma_{\alpha\alpha}=\frac{\lambda_\alpha^2\Gamma_\alpha^2}{8[(E-E_0)^2+(\frac{1}{2}\Gamma)^2]}\sum_{L=0}^{2J_0-1}[Z(l_0J_0l_0J_0,\tfrac{1}{2}L)]^2P_L(\cos\theta)\,d\Omega+(d\sigma)_{\text{pot}}$$
$$+\tfrac{1}{2}\lambda_\alpha z(2J_0+1)\frac{\Gamma_\alpha}{[(E-E_0)^2+(\frac{1}{2}\Gamma)^2]^{\frac{1}{2}}}\operatorname{cosec}^2(\tfrac{1}{2}\theta)\sin[2\eta\ln\sin(\tfrac{1}{2}\theta)+2\psi_{l_0}+2\phi_{l_0}+\beta]\,P_{l_0}(\cos\theta)\,d\Omega$$
$$-\tfrac{1}{2}\lambda_\alpha^2(2J_0+1)\frac{\Gamma_\alpha}{[(E-E_0)^2+(\frac{1}{2}\Gamma)^2]^{\frac{1}{2}}}\sum_{L=0}^{\infty}\sum_{l'=|l_0-L|}^{l_0+L}(2l'+1)$$
$$\times[(l_0l'00|l_0l'L0)]^2\sin\phi_{l'}\sin(\beta+2\psi_{l_0}+2\phi_{l_0}-2\psi_{l'}-\phi_{l'})\,P_L(\cos\theta)d\Omega. \quad (7.13)$$

In the first term, the resonance scattering, the sum over L contains only even values of L. It is interesting to observe that this resonance scattering has a definite angular distribution which depends on l_0 and J_0 but not on the widths or the energy of the resonance. The next term, $(d\sigma)_{\text{pot}}$, is the differential cross section for the potential (charged hard-sphere) scattering, and is given by the second, third, and fourth terms of (7.12). Experimentally, one usually measures the scattering cross section at some definite angle, as a function of energy E. The cross section for the potential scattering can be subtracted directly by interpolating a smooth curve between the data above and below resonance and then taking the difference between that smooth curve and the observed cross section in the resonance. The last two terms in (7.13) represent the interference between resonance scattering and Rutherford (point charge) scattering and the finite nuclear size correction to this interference term, respectively.

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