Some Properties of the Racah and Associated Coefficients*

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I. INTRODUCTION

THE composition of angular momenta, one of the basic elements of quantum mechanics, is accomplished by means of the vector addition coefficients —known also as the Clebsch-Gordan or Wigner coefficients. This, in principle, constitutes a complete solution to problems with coupled angular momenta; but, in practice, say in the evaluation of matrix elements of composite systems, one is quite often led to involved summations of products of several vector addition coefficients that are carried out, if at all, with difficulty. A quite typical situation might involve the matrix elements, for a composite system, of operators acting on only one or more of the subsystems. An elementary example of this is the evaluation of the matrix elements $(lsjm|a\mathbf{L}+b\mathbf{S}|l's'jm')$ for the magnetic moment of a particle with spin. A direct evaluation—which, of course, in this example could be avoided—involves a cumbersome sum of the product of three vector addition coefficients. This direct procedure, in effect, computes the matrix elements in a "decoupled" scheme, and then seeks to relate them to the desired matrix elements in the "coupled" scheme. The complicated sum of vector addition coefficients automatically effects this transformation between coupling schemes, keeping proper account of the conservation of angular momentum.

Racah, in his work on complex atomic spectra, discussed in detail the properties of these transformations, and defined the coefficient $W(abcd;ef)$—since known as the Racah coefficient—as the transformation between the coupling schemes ($\mathbf{a}+\mathbf{b}=\mathbf{e}$; $\mathbf{e}+\mathbf{d}=\mathbf{c}$) and ($\mathbf{b}+\mathbf{d}=\mathbf{f}$; $\mathbf{a}+\mathbf{f}=\mathbf{c}$). We discuss this in more detail in Sec. III.

Since Racah's pioneer work, the W coefficients have been applied to a wide variety of problems, the angular correlation of successive radiations and the angular distribution of scattering and reaction cross sections being conspicuous examples. From the utilitarian point of view (effecting difficult summations of vector addition coefficients), the usefulness of the Racah coefficients for these angular correlation problems is apparent. The relationship of the "recoupling" approach to these problems is less obvious, but more fundamental, as has been clearly pointed out by Fano (reference 12).‖

The coefficients that enter most naturally into the angular distribution of scattering and reaction cross sections are not the Racah coefficients themselves but a combination designated in reference 6 as the "Z coefficients" (defined by Eq. (23) below). For the specific application of the W and Z coefficients we shall, however, refer to the original literature. A bibliography for this purpose is included below. Besides giving algebraic tables of the W coefficients, we shall summarize the relevant algebraic properties of the coefficients.

The algebraic tables of the W coefficients are sufficiently complicated, especially for large values of the variables, that a numerical tabulation would be of great value. We have prepared such tables (reference 19) not only for the W but also the Z coefficients. However, space does not permit their inclusion here. Copies of these tables can be obtained from the Oak Ridge National Laboratory. The tabulation consists of 54 numerical tables of $W(l_1J_1l_2J_2;sL)$ for $s=\frac{1}{2}$ through 3 in steps of $\frac{1}{2}$ and $L=0\cdots 8$ in integer steps. The remaining parameters have the range $l_i=0$ through 4 or 5 (integer steps) and $J_i=0$ through 9/2 ($\frac{1}{2}$ integer steps) with the necessary restriction $J_i-s=$integer [see discussion following Eq. (12)]. The Z coefficients are tabulated in 54 tables for the same range of parameters.

II. VECTOR ADDITION COEFFICIENTS

We shall not repeat here the definition of the vector addition coefficients, since this is extensively treated in references 1 and 2. Tables of these coefficients are given in reference 2. However, some of the symmetry relations for the vector addition coefficients are less well known, and we repeat the rules given independently by Racah (reference 3) and Eisenbud.¶

$$\begin{aligned}(ab\alpha\beta|abc\gamma)&=(ba-\beta-\alpha|bac-\gamma)\\&=(-1)^{a+b-c}(ba\beta\alpha|bac\gamma)\\&=(-1)^{a+b-c}(ab-\alpha-\beta|abc-\gamma)\\&=(-1)^{a-\alpha}\left[\frac{2c+1}{2b+1}\right]^{\frac{1}{2}}\cdot(ac\alpha-\gamma|acb-\beta)\\&=(-1)^{b+\beta}\left[\frac{2c+1}{2a+1}\right]^{\frac{1}{2}}\cdot(cb-\gamma\beta|cba-\alpha).\end{aligned}\tag{1}$$

Other symmetry rules result from a combination of these basic symmetries.

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‖ Numbered references are given at the end of this article.

¶ L. Eisenbud, Ph.D. thesis, Princeton University (1948).

Racah also gives an explicit formula for the often-occurring coefficient $(ab00|abc0)$. This formula is most easily written in terms of the "triangle" coefficient $\Delta(abc)$ defined by

$$\Delta(abc)=\left[\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}\right]^{\frac{1}{2}}. \quad (2)$$

$\Delta(abc)$ is clearly unchanged by any permutation of a, b, c. In general, we shall need the triangle coefficients only for values of a, b, c which satisfy the triangular inequalities

$$a+b\geqslant c, \quad b+c\geqslant a, \quad c+a\geqslant b. \quad (3)$$

Racah then defined the quantity g by

$$2g=a+b+c \quad (4)$$

and obtained

$$(ab00|abc0)=(-1)^{g+c}(2c+1)^{\frac{1}{2}}\Delta(abc)\frac{g!}{(g-a)!(g-b)!(g-c)!} \quad (a+b+c=\text{even})$$

$$(ab00|abc0)=0 \quad (a+b+c=\text{odd}). \quad (5)$$

III. RACAH COEFFICIENTS

The physical significance of the W coefficients, as well as their properties, can be most easily obtained by a procedure due to Racah. Consider orthonormal wave functions, $\Psi(jm)$, of sharp angular momenta, that is to say, $\Psi(jm)$ is an eigenfunction of the rotation operator with eigenvalues $j(j+1)$ for J^2 and m for J_z. If we have two such wave functions, in different spaces, then a product wave function with sharp total angular momentum, is obtained in the usual way by application of the vector addition coefficients:**

$$\Phi[j_1j_2;J_{12}M_{12}]=\sum_{m_1}(j_1j_2m_1M_{12}-m_1|j_1j_2J_{12}M_{12})\Psi(j_1m_1)\Psi(j_2M_{12}-m_1). \quad (6)$$

The unitary property of the vector addition coefficients guarantees that $\Phi[j_1j_2;J_{12}M_{12}]$ is itself orthonormal in J_{12} and M_{12}. For the addition of three angular momentum vectors we consider a third wave function $\Psi(j_3m_3)$. A composite function with sharp total angular momentum can be obtained by combining $\Psi(j_3m_3)$ with $\Phi[j_1j_2;J_{12}M_{12}]$ using again the vector addition coefficients:

$$\begin{aligned}\Phi[j_1j_2(J_{12})j_3;JM]&=\sum_{m_3}(J_{12}j_3m_3M-m_3|J_{12}j_3JM)\Psi(j_3m_3)\times\Phi[j_1j_2;J_{12}M-m_3]\\&=\sum_{m_1,m_3}(j_1j_2m_1M-m_3-m_1|j_1j_2J_{12}M-m_3)\\&\quad\cdot(J_{12}j_3M-m_3m_3|J_{12}j_3JM)\\&\quad\cdot\Psi(j_1m_1)\Psi(j_2M-m_3-m_1)\Psi(j_3m_3). \quad (7)\end{aligned}$$

** The vector addition $(ab\alpha\beta|abc\gamma)$ vanishes unless $\alpha+\beta=\gamma$. In the following, we shall explicitly satisfy this requirement, eliminating a formal sum over γ.

The wave functions thus formed are again orthonormal in J, M, and also in J_{12}. It is clear, though, that such wave functions are not unique, since we could just as well have combined first j_2 and j_3 to give J_{23} and then j_1 and J_{23} to give J. That is,

$$\begin{aligned}\Phi[j_1,j_2j_3(J_{23});JM]&=\sum_{m_2,m_1}(j_2j_3m_2M-m_1-m_2|j_2j_3J_{23}M-m_1)\\&\quad\cdot(j_1J_{23}m_1M-m_1|j_1J_{23}JM)\\&\quad\cdot\Psi(j_1m_1)\Psi(j_2m_2)\Psi(j_3M-m_1-m_2). \quad (8)\end{aligned}$$

To identify the composite wave functions we must indicate the coupling scheme, as in Eqs. (7) and (8). Composite wave functions for one coupling scheme are, however, linearly related to the wave functions for another scheme. Thus, we may write

$$\Phi[j_1,j_2j_3(J_{23});JM]=\sum_{J_{12}}\langle j_1,j_2j_3(J_{23})J|j_1j_2(J_{12})j_3J\rangle\cdot\Phi[j_1j_2(J_{12})j_3;JM]. \quad (9)$$

By using Eqs. (7) and (8), and the orthonormality of the $\Psi(j_im_i)$, one can obtain a relation for the transformation coefficient in terms of the vector addition coefficients:

$$\begin{aligned}\langle j_1,j_2j_3(J_{23})J|j_1j_2(J_{12})j_3J\rangle&=\sum_{m,m_2}(j_1j_2M-mm_2|j_1j_2J_{12}M-m+m_2)\\&\quad\cdot(j_2j_3m_2m-m_2|j_2j_3J_{23}m)\\&\quad\cdot(j_1J_{23}M-mm|j_1J_{23}JM)\\&\quad\cdot(J_{12}j_3M-m+m_2\,m-m_2|J_{12}j_3JM). \quad (10)\end{aligned}$$

It is this relation that Racah used to define the W coefficients originally. In his notation we have

$$(2e+1)^{\frac{1}{2}}(2f+1)^{\frac{1}{2}}W(abcd;ef)\equiv\langle ab(e)dc|a,bd(f)c\rangle. \quad (11)$$

Racah was able to perform the sum indicated on the right hand side of (10). His result is

$$W(abcd;ef)=\Delta(abe)\Delta(cde)\Delta(acf)\Delta(bdf)w(abcd;ef)$$

$$w(abcd;ef)=\sum_z\frac{(-1)^{z+a+b+c+d}\cdot(z+1)!}{(z-a-b-e)!(z-c-d-e)!\cdot(z-a-c-f)!(z-b-d-f)!}\cdot\frac{1}{(a+b+c+d-z)!(a+d+e+f-z)!(b+c+e+f-z)!}. \quad (12)$$

As the interpretation in terms of recoupling angular momenta indicates, the Racah function is defined for integral or half-integral values of the quantities a, b, c, d, e, f, with the limitation that each of the four triads

$$(a, b, e) \quad (c, d, e) \quad (a, c, f) \quad (b, d, f) \tag{13}$$

has an integral sum. The sum in (12) goes over integral values of z such that none of the factorials in the denominator has a negative argument.

According to its definition, W satisfies various selection rules, all of which can be summarized by saying that each of the four triads (13) must form a possible triangle, i.e., must satisfy the condition that any side of a triangle is smaller than or equal to the sum of the other two sides. If one or more of those four triangles degenerates into a straight line, the summation in (12) reduces to one term (see Eq. 29).

The Racah coefficients are highly symmetrical functions of the parameters a, b, c, d, e, f. The basic symmetry relations are

$$\begin{aligned} W(abcd;ef) &= W(badc;ef) = W(cdab;ef) = W(acbd;fe) \\ &= (-1)^{e+f-a-d}W(ebcf;ad) \\ &= (-1)^{e+f-b-c}W(aefd;bc). \end{aligned} \tag{14}$$

Additional symmetry relations follow from the ones stated here, so that there are altogether 24 different permutations of a, b, c, d, e, f [which correspond to all possible permutations between the four triads (13)], for which the corresponding W's differ at most by a sign.

One can readily obtain the properties of the W coefficients from the above "recoupling" technique. Since the composite wave functions $\Phi[\cdots;JM]$ are orthonormal, we know immediate from Eq. (9) that the transformation coefficients are unitary. In terms of the W functions the unitary property is expressed by

$$\sum_e (2e+1)(2f+1)W(abcd;ef)W(abcd;eg) = \delta_{fg}. \tag{15}$$

Another sum rule given by Racah, which can be obtained from this procedure, is

$$W(agfb;cd) = \sum_e (2e+1)(-1)^{a+b-e} \cdot W(abcd;ef)\cdot W(bacd;eg). \tag{16}$$

An extension of the recoupling procedure to four wave functions yields yet another sum rule for the Racah functions:††

$$W(a\alpha b\beta;c\gamma)W(a'\alpha b'\beta;c'\gamma) = \sum_\lambda (2\lambda+1) \cdot W(a'\lambda\alpha c;ac')W(b\lambda\beta c';b'c)W(a'\lambda\gamma b;ab'). \tag{17}$$

Racah‡‡ has shown that, except for a phase, Eqs. (14), (15), (16), and (17) define the W functions completely. Hence, no further independent relations can exist.

The Racah coefficients are useful for the study of angular distributions since their application effects the summations over the magnetic quantum numbers. If we substitute Eqs. (7), (8), into (9), and use the orthonormality of the $\Psi(j_i m_i)$ we find the relation

$$\begin{aligned} (ab\alpha\beta|abe\alpha+\beta)(ed\alpha+\beta\delta|edc\alpha+\beta+\delta) &= \sum_f (2e+1)^{\frac{1}{2}}(2f+1)^{\frac{1}{2}}(bd\beta\delta|bdf\beta+\delta) \\ &\quad \cdot(af\alpha\beta+\delta|afc\alpha+\beta+\delta)\cdot W(abcd;ef). \end{aligned} \tag{18}$$

The usefulness of (18) may not be immediately apparent. In a fairly typical problem where one is faced with summing a product of several vector addition coefficients over a single magnetic quantum number, however, successive application of (18) will allow "recoupling" of the angular momenta involved in the vector addition coefficients until the magnetic quantum number sum can be carried out.

Using the unitary property of the vector addition coefficients Eq. (18) can be given a form that proves very useful for summations involving products of three vector addition coefficients:

$$\begin{aligned} \sum_\beta (ab\alpha\beta|abe\alpha+\beta)\cdot(ed\alpha+\beta\gamma-\alpha-\beta|edc\gamma) &\cdot(bd\beta\gamma-\alpha-\beta|bdf\gamma-\alpha) = (2e+1)^{\frac{1}{2}}(2f+1)^{\frac{1}{2}} \\ &\cdot(af\alpha\gamma-\alpha|afc\gamma)\cdot W(abcd;ef). \end{aligned} \tag{19}$$

The result given in Eq. (19) is more particularized than that in (18) above, since in the former we are restricted to a single Racah transformation.

For calculational purposes, a recursion formula for the W coefficients is desirable. Equation (17), it will be readily observed is, in fact, a generalized recursion relation. By specializing this formula various recursion relations can be derived. For example, take $c'=\frac{1}{2}$. The W's involving c' then have a simple algebraic form (see Sec. V) and we get as one case

$$\begin{aligned} W(a\alpha+\tfrac{1}{2}b\beta+\tfrac{1}{2};c+\tfrac{1}{2}\gamma) &= (2c+1)\cdot\left[\frac{(\alpha+\beta+\gamma+2)(\alpha+\beta-\gamma+1)}{(\alpha+c+a+2)(\alpha+c+1-a)(\beta+b+c+2)(\beta+c+1-b)}\right]^{\frac{1}{2}}\cdot W(a\alpha b\beta;c\gamma) \\ &+\left[\frac{(a+\alpha+1-c)(a+c-\alpha)(b+\beta+1-c)(b+c-\beta)}{(\alpha+a+c+2)(\alpha+c+1-a)(\beta+b+c+2)(\beta+c+1-b)}\right]^{\frac{1}{2}}\cdot W(a\alpha+\tfrac{1}{2}b\beta+\tfrac{1}{2};c-\tfrac{1}{2}\gamma). \end{aligned} \tag{20}$$

†† L. C. Biedenharn, to appear in J. Math. Phys. (M.I.T.).
‡‡ Unpublished manuscript.

It will be observed that Eq. (20) is indeterminate if either $a=\alpha+c+1$ or $b=\beta+c+1$ or both. Although a recursion formula can be written for these cases, it is unnecessary to do so, since the desired W coefficient has a simple explicit form [see Eq. (29)].

Useful algebraic formulas for the W coefficients result from giving numerical values to one variable (say the e) in Eq. (12). Two triangle conditions then restrict the remaining variables:

$$\begin{aligned}&(1)\quad |a-e|\leqslant b\leqslant a+e,\\ &(2)\quad |c-e|\leqslant d\leqslant c+e.\end{aligned}\tag{21}$$

The formulas can thus be conveniently tabulated in a square array of $(2e+1)^2$ entries, similar to the tabulation of the vector addition coefficients. We give such tables for $e=\frac{1}{2}$, 1, $\frac{3}{2}$, 2 in Sec. V. If algebraic formulas for higher values of e are desired, the recursion formulas can be used to generate them.

In the correlation of successive radiations involving pure multipoles the Racah coefficients that enter have two repeated indices, in the form $W(a\nu cd;ad)$. Only integer values of ν can occur. A recursion relation for these W's takes a particularly simple and useful form:

$$\begin{aligned}W(a\nu+1\,cd;ad)&=\frac{2\nu+1}{\nu+1}\cdot\frac{2a(a+1)+2d(d+1)-2c(c+1)-\nu(\nu+1)}{[(2a+2+\nu)(2a-\nu)(2d+2+\nu)(2d-\nu)]^{\frac{1}{2}}}\\ &\quad\cdot W(a\nu cd;ad)-\frac{\nu}{\nu+1}\\ &\quad\cdot\left[\frac{(2a+\nu+1)(2a+1-\nu)(2d+\nu+1)(2d+1-\nu)}{(2a+\nu+2)(2a-\nu)(2d+2+\nu)(2d-\nu)}\right]^{\frac{1}{2}}\\ &\quad\cdot W(a\nu-1\,cd;ad).\end{aligned}\tag{22}$$

Equation (22) is by far the simplest way to obtain algebraic forms for the $W(a\nu cd;ad)$. A short tabulation of these W's is given in Sec. V.

IV. *Z* COEFFICIENTS

Coefficients which are more appropriate for the angular distribution problem, as mentioned earlier, are not the Racah coefficients themselves, but the combination

$$\begin{aligned}Z(abcd;ef)=i^{f-a+c}[(2a+1)(2b+1)(2c+1)(2d+1)]^{\frac{1}{2}}\\ \cdot W(abcd;ef)(ac00|acf0).\end{aligned}\tag{23}$$

Since there exists the simple explicit formula (7) for the relevant Clebsch-Gordan coefficient, the computation of Z is a simple matter once the tabulation of the W coefficients has been carried out.

The Z coefficients obey all the selection rules for the Racah coefficients [see (12) and the discussion there] as well as the selection rule

$$Z=0\text{ unless }a+c+f=\text{even},\tag{24}$$

which follows from (5). This restriction has the consequence that the phase factor i^{f-a+c} in (23) is always real and equal to ±1. Of the various symmetry relations for the Racah coefficients, only one is needed for the angular distribution problem, namely,

$$Z(l_1J_1l_2J_2;sL)=(-1)^LZ(l_2J_2l_1J_1;sL).\tag{25}$$

A sum rule for Z coefficients follows from Racah's sum rule (15). This becomes for the Z coefficients

$$\sum_b Z(abcd;ef)Z(abc'd;ef)=\delta_{cc'}(2a+1)(2d+1)[(ac00|acf0)]^2.\tag{26}$$

Finally, we give here the values of Z when either e or f vanish. $e=0$ corresponds to vanishing channel spin; $f=0$ is related to the total cross section.

$$\begin{aligned}Z(abcd;0f)=\delta_{ab}\delta_{cd}(-1)^{2f}i^{f-a+c}\\ \cdot[(2a+1)(2c+1)]^{\frac{1}{2}}(ac00|acf0),\end{aligned}\tag{27}$$

$$Z(abcd;e0)=\delta_{ac}\delta_{bd}(-1)^{b-e}(2b+1)^{\frac{1}{2}}.\tag{28}$$

In the application to angular distributions in nuclear reactions $f=L$ is integral, so that the factor $(-1)^{2f}$ in (27) is always equal to $+1$.

Tables I, II, III, and IV for the W coefficients combined with Eqs. (5) and (23) suffice to determine the Z coefficients explicitly.

V. ALGEBRAIC FORMULAS FOR THE *W* COEFFICIENTS

The summation in Eq. (12) reduces to a single term if any one of the triangles formed from the triads (abe) (cde) (acf) (bdf) reduces to a line. In all such cases the symmetry conditions (16) allow the coefficient in question to be permuted to the form

$$W(abcd;a+bf)=\left[\frac{2a!2b!(a+b+c+d+1)!(a+b+c-d)!(a+b+d-c)!(c+f-a)!(d+f-b)!}{(2a+2b+1)!(c+d-a-b)!(a+c-f)!(a+f-c)!(a+c+f+1)!(b+d-f)!(b+f-d)!(b+f+d+1)!}\right]^{\frac{1}{2}}.\tag{29}$$

For any one variable equal to zero we have a special case of Eq. (29), and the result is [after using symmetry conditions (14)]

$$W(abcd;0f)=(-1)^{b+c-f}(2b+1)^{-\frac{1}{2}}(2c+1)^{-\frac{1}{2}}\delta_{ab}\delta_{cd}.\tag{30}$$

TABLE I. $W(l_1J_1l_2J_2;\frac{1}{2},L)$.

	$l_1=J_1+\frac{1}{2}$	$l_1=J_1-\frac{1}{2}$
$l_2=J_2+\frac{1}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{(J_1+J_2+L+2)(J_1+J_2-L+1)}{(2J_1+1)(2J_1+2)(2J_2+1)(2J_2+2)}\right]^{\frac{1}{2}}$	$(-1)^{J_1+J_2-L}\left[\frac{(L-J_1+J_2+1)(L+J_1-J_2)}{(2J_1)(2J_1+1)(2J_2+1)(2J_2+2)}\right]^{\frac{1}{2}}$
$l_2=J_2-\frac{1}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1-J_2+1)(L-J_1+J_2)}{(2J_1+1)(2J_1+2)(2J_2)(2J_2+1)}\right]^{\frac{1}{2}}$	$(-1)^{J_1+J_2-L-1}\left[\frac{(J_1+J_2+L+1)(J_1+J_2-L)}{2J_1(2J_1+1)(2J_2)(2J_2+1)}\right]^{\frac{1}{2}}$

TABLE II. $W(l_1J_1l_2J_2;1,L)$.

	$l_2=J_2+1$
$l_1=J_1+1$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+3)(L+J_1+J_2+2)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{4(2J_1+3)(J_1+1)(2J_1+1)(2J_2+3)(J_2+1)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_1=J_1$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2+1)(L+J_1-J_2)}{4J_1(2J_1+1)(J_1+1)(2J_2+1)(J_2+1)(2J_2+3)}\right]^{\frac{1}{2}}$
$l_1=J_1-1$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1-J_2)(L+J_1-J_2-1)(L-J_1+J_2+2)(L-J_1+J_2+1)}{4(2J_1+1)(2J_1-1)(J_1)(J_2+1)(2J_2+1)(2J_2+3)}\right]^{\frac{1}{2}}$
	$l_2=J_2$
$l_1=J_1+1$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+2)(L+J_1-J_2+1)(J_1+J_2-L+1)(L-J_1+J_2)}{4(2J_1+1)(J_1+1)(2J_1+3)(J_2)(J_2+1)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_1=J_1$	$(-1)^{J_1+J_2-L-1}\left[\frac{J_1(J_1+1)+J_2(J_2+1)-L(L+1)}{[4J_1(J_1+1)(2J_1+1)(J_2)(J_2+1)(2J_2+1)]^{\frac{1}{2}}}\right]$
$l_1=J_1-1$	$(-1)^{J_1+J_2-L-1}\left[\frac{(L+J_1+J_2+1)(-L+J_1+J_2)(L+J_1-J_2)(L-J_1+J_2+1)}{4(2J_1+1)(J_1)(2J_1-1)(J_2)(2J_2+1)(J_2+1)}\right]^{\frac{1}{2}}$
	$l_2=J_2-1$
$l_1=J_1+1$	$(-1)^{J_1+J_2-L}\left[\frac{(L-J_1+J_2)(L-J_1+J_2-1)(L+J_1-J_2+2)(L+J_1-J_2+1)}{4(2J_1+1)(J_1+1)(2J_1+3)(2J_2-1)(J_2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_1=J_1$	$(-1)^{J_1+J_2-L-1}\left[\frac{(L+J_1+J_2+1)(L+J_1-J_2+1)(L+J_2-J_1)(J_1+J_2-L)}{4J_1(2J_1+1)(J_1+1)(J_2)(2J_2+1)(2J_2-1)}\right]^{\frac{1}{2}}$
$l_1=J_1-1$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+1)(L+J_1+J_2)(-L+J_1+J_2)(-L+J_1+J_2-1)}{4(2J_1+1)(J_1)(2J_1-1)(2J_2+1)(J_2)(2J_2-1)}\right]^{\frac{1}{2}}$

The Racah Coefficient $W(a\nu cd;ad)$

In order to remove irrational normalizing factors, define

$$W(a\nu cd;ad)\equiv\left[\frac{(2a-\nu)!(2d-\nu)!}{(2a+\nu+1)!(2d+\nu+1)!}\right]^{\frac{1}{2}}Y_\nu(acd). \quad (31)$$

In addition, define the variable x by

$$x\equiv c(c+1)-a(a+1)-d(d+1), \quad (32)$$

and introduce the convention that $\bar{a}=a(a+1)$, etc.

The Y_ν are then rational polynomials of the νth order in x with coefficients involving $\bar{a}$, $\bar{d}$ rationally. The lowest polynomials are

$$Y_0=1,$$
$$Y_1=-2x,$$
$$Y_2=6x^2+6x-8\bar{a}\bar{d},$$
$$Y_3=-20x^3-80x^2+16x[3\bar{a}\bar{d}-\bar{a}-\bar{d}-3]+80\bar{a}\bar{d},$$
$$Y_4=70x^4+700x^3+x^2[1560-240\bar{a}\bar{d}+200\bar{a}+200\bar{d}]+x[720+480\bar{a}+480\bar{d}-1360\bar{a}\cdot\bar{d}]+48\bar{a}\bar{d}[2\bar{a}\bar{d}-4\bar{a}-4\bar{d}-27].$$

The Y_ν for higher values of ν may be generated from the recursion relation

$$Y_{\nu+1}=\left(\frac{2\nu+1}{\nu+1}\right)Y_1Y_\nu-(2\nu+1)Y_\nu-\left(\frac{\nu}{\nu+1}\right)(4\bar{a}+1-\nu^2)(4\bar{d}+1-\nu^2)Y_{\nu-1}. \quad (33)$$

TABLE III. $W(l_1J_1l_2J_2;\,\frac{3}{2},\,L)$.

	$l_1=J_1+3/2$
$l_2=J_2+\frac{3}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+4)(L+J_1+J_2+3)(L+J_1+J_2+2)(-L+J_1+J_2+3)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2=J_2+\frac{1}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{3(L+J_1+J_2+3)(L+J_1+J_2+2)(L+J_1-J_2+1)(L-J_1+J_2)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)}\right]^{\frac{1}{2}}$
$l_2=J_2-\frac{1}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{3(L+J_1+J_2+2)(L+J_1-J_2+2)(L+J_1-J_2+1)(L-J_1+J_2)(L-J_1+J_2-1)(J_1+J_2-L+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)}\right]^{\frac{1}{2}}$
$l_2=J_2-\frac{3}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{(L-J_1+J_2)(L-J_1+J_2-1)(L-J_1+J_2-2)(L+J_1-J_2+3)(L+J_1-J_2+2)(L+J_1-J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}$
	$l_1=J_1+\frac{1}{2}$
$l_2=J_2+\frac{3}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{3(L+J_1+J_2+3)(L+J_1+J_2+2)(-L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2+1)(L+J_1-J_2)}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2=J_2+\frac{1}{2}$	$(-1)^{J_1+J_2-L-1}\frac{[(L+J_1+J_2+2)(-L+J_1+J_2+1)]^{\frac{1}{2}}[(L+J_1+J_2+3)(-L+J_1+J_2)-2(L-J_1+J_2)(L+J_1-J_2)]}{[(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)]^{\frac{1}{2}}}$
$l_2=J_2-\frac{1}{2}$	$(-1)^{J_1+J_2-L-1}\frac{[(L+J_1-J_2+1)(L-J_1+J_2)]^{\frac{1}{2}}[2(L+J_1+J_2+2)(-L+J_1+J_2)-(L-J_1+J_2-1)(L+J_1-J_2)]}{[(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)]^{\frac{1}{2}}}$
$l_2=J_2-\frac{3}{2}$	$(-1)^{J_1+J_2-L-1}\left[\frac{3(L+J_1+J_2+1)(L-J_1+J_2)(L-J_1+J_2-1)(L+J_1-J_2+2)(L+J_1-J_2+1)(-L+J_1+J_2)}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}$
	$l_1=J_1-\frac{1}{2}$
$l_2=J_2+\frac{3}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{3(L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2+2)(L-J_1+J_2+1)(L+J_1-J_2)(L+J_1-J_2-1)}{(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2=J_2+\frac{1}{2}$	$(-1)^{J_1+J_2-L}\frac{[(L-J_1+J_2+1)(L+J_1-J_2)]^{\frac{1}{2}}[(L-J_1+J_2)(L+J_1-J_2-1)-2(L+J_1+J_2+2)(-L+J_1+J_2)]}{[(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)]^{\frac{1}{2}}}$
$l_2=J_2-\frac{1}{2}$	$(-1)^{J_1+J_2-L}\frac{[(L+J_1+J_2+1)(-L+J_1+J_2)]^{\frac{1}{2}}[(L+J_1+J_2+2)(-L+J_1+J_2-1)-2(L+J_1-J_2)(L-J_1+J_2)]}{[(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)]^{\frac{1}{2}}}$
$l_2=J_2-\frac{3}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{3(L+J_1+J_2+1)(L+J_1+J_2)(L-J_1+J_2)(L+J_1-J_2+1)(-L+J_1+J_2)(-L+J_1+J_2-1)}{(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}$
	$l_1=J_1-3/2$
$l_2=J_2+\frac{3}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1-J_2)(L+J_1-J_2-1)(L+J_1-J_2-2)(L-J_1+J_2+3)(L-J_1+J_2+2)(L-J_1+J_2+1)}{(2J_1+1)(2J_1)(2J_1-1)(2J_1-2)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2=J_2+\frac{1}{2}$	$(-1)^{J_1+J_2-L-1}\left[\frac{3(L+J_1+J_2+1)(-L+J_1+J_2)(L+J_1-J_2)(L+J_1-J_2-1)(L-J_1+J_2+2)(L-J_1+J_2+1)}{(2J_1+1)(2J_1)(2J_1-1)(2J_1-2)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)}\right]^{\frac{1}{2}}$
$l_2=J_2-\frac{1}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{3(L+J_1+J_2+1)(L+J_1+J_2)(-L+J_1+J_2)(-L+J_1+J_2-1)(L+J_1-J_2)(L-J_1+J_2+1)}{(2J_1+1)(2J_1)(2J_1-1)(2J_1-2)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)}\right]^{\frac{1}{2}}$
$l_2=J_2-\frac{3}{2}$	$(-1)^{J_1+J_2-L-1}\left[\frac{(L+J_1+J_2+1)(L+J_1+J_2)(L+J_1+J_2-1)(-L+J_1+J_2)(-L+J_1+J_2-1)(-L+J_1+J_2-2)}{(2J_1+1)(2J_1)(2J_1-1)(2J_1-2)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}$

Algebraic formulas that result from taking the variable e in Eq. (12) to have the values $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2 are given in Tables I through IV below. Table I gives the Racah coefficient $W(l_1J_1l_2J_2;\,\frac{1}{2}L)$; Table II, $W(l_1J_1l_2J_2;\,1L)$; Table III, $W(l_1J_1l_2J_2;\,\frac{3}{2}L)$; and Table IV, $W(l_1J_1l_2J_2;\,2L)$.

TABLE IV. $W(l_1J_1l_2J_2;2,L)$.

	$l_1=J_1+2$
$l_2=J_2+2$	$(-1)^{L-J_1-J_2}\left[\frac{(L+J_1+J_2+5)(L+J_1+J_2+4)(L+J_1+J_2+3)(L+J_1+J_2+2)(-L+J_1+J_2+4)(-L+J_1+J_2+3)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{(2J_1+5)(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)\cdot(2J_2+5)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2=J_2+1$	$(-1)^{L-J_1-J_2}\left[\frac{4(L+J_1+J_2+4)(L+J_1+J_2+3)(L+J_1+J_2+2)(L+J_1-J_2+1)(-L+J_1+J_2+3)(-L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2)}{(2J_1+5)(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)\cdot(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)}\right]^{\frac{1}{2}}$
$l_2=J_2$	$(-1)^{L-J_1-J_2}\left[\frac{6(L+J_1+J_2+3)(L+J_1+J_2+2)(L+J_1-J_2+2)(L+J_1-J_2+1)(-L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2)(L-J_1+J_2-1)}{(2J_1+5)(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)\cdot(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)}\right]^{\frac{1}{2}}$
$l_2=J_2-1$	$(-1)^{L-J_1-J_2}\left[\frac{4(L+J_1+J_2+2)(L-J_1+J_2)(L-J_1+J_2-1)(L-J_1+J_2-2)(L+J_1-J_2+3)(L+J_1-J_2+2)(L+J_1-J_2+1)(-L+J_1+J_2+1)}{(2J_1+5)(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)\cdot(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}$
$l_2=J_2-2$	$(-1)^{L-J_1-J_2}\left[\frac{(L-J_1+J_2)(L-J_1+J_2-1)(L-J_1+J_2-2)(L-J_1+J_2-3)(L+J_1-J_2+4)(L+J_1-J_2+3)(L+J_1-J_2+2)(L+J_1-J_2+1)}{(2J_1+5)(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)\cdot(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)(2J_2-3)}\right]^{\frac{1}{2}}$
	$l_1=J_1+1$
$l_2=J_2+2$	$(-1)^{L-J_1-J_2}\left[\frac{4(L+J_1+J_2+4)(L+J_1+J_2+3)(L+J_1+J_2+2)(L-J_1+J_2+1)(L+J_1-J_2)(-L+J_1+J_2+3)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)\cdot(2J_2+5)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2=J_2+1$	$(-1)^{L-J_1-J_2-1}\left[\frac{(L+J_1+J_2+3)(L+J_1+J_2+2)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)\cdot(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)}\right]^{\frac{1}{2}}\cdot4\cdot[(J_1+1)(J_1-J_2)-L(L+1)+J_2(J_2+2)]$
$l_2=J_2$	$(-1)^{L-J_1-J_2-1}\left[\frac{6(L+J_1+J_2+2)(L-J_1+J_2)(L+J_1-J_2+1)(-L+J_1+J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)\cdot(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)}\right]^{\frac{1}{2}}\cdot2\cdot[J_1(J_1+2)+J_2(J_2+1)-L(L+1)]$
$l_2=J_2-1$	$(-1)^{L-J_1-J_2-1}\left[\frac{(L-J_1+J_2)(L-J_1+J_2-1)(L+J_1-J_2+2)(L+J_1-J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)\cdot(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}\cdot4\cdot[J_1(J_1+2)+J_2(J_2+1)+J_1J_2-L(L+1)]$
$l_2=J_2-2$	$(-1)^{L-J_1-J_2-1}\left[\frac{4(L+J_1+J_2+1)(-L+J_1+J_2)(L-J_1+J_2)(L-J_1+J_2-1)(L-J_1+J_2-2)(L+J_1-J_2+3)(L+J_1-J_2+2)(L+J_1-J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)\cdot(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)(2J_2-3)}\right]^{\frac{1}{2}}$
	$l_1=J_1$
$l_2=J_2+2$	$(-1)^{L-J_1-J_2}\left[\frac{6(L+J_1+J_2+3)(L+J_1+J_2+2)(L+2-J_1+J_2)(L+1-J_1+J_2)(L+J_1-J_2)(L+J_1-J_2-1)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)\cdot(2J_2+5)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2=J_2+1$	$(-1)^{L-J_1-J_2-1}\left[\frac{6(L+J_1+J_2+2)(L+J_1-J_2)(L-J_1+J_2+1)(-L+J_1+J_2+1)}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)\cdot(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)}\right]^{\frac{1}{2}}\cdot2\cdot[J_1(J_1+1)+J_2(J_2+2)-L(L+1)]$

TABLE IV.—*Continued.*

$l_2=J_2$	$(-1)^{L-J_1-J_2}\left[\dfrac{1}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)\cdot(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)}\right]^{\frac{1}{2}}\cdot 6\cdot\left[A(A+1)-\frac{4}{3}J_1(J_1+1)J_2(J_2+1)\right]$
	where $A\equiv L(L+1)-J_1(J_1+1)-J_2(J_2+1)$
$l_2=J_2-1$	$(-1)^{L-J_1-J_2}\left[\dfrac{6(L+J_1+J_2+1)(L-J_1+J_2)(L+J_1-J_2+1)(-L+J_1+J_2)}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)\cdot(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}\cdot 2\cdot\left[J_1(J_1+1)-L(L+1)+J_2^2-1\right]$
$l_2=J_2-2$	$(-1)^{L-J_1-J_2}\left[\dfrac{6(L+J_1+J_2+1)(L+J_1+J_2)(L-J_1+J_2)(L-J_1+J_2-1)(-L+J_1+J_2)(-L+J_1+J_2-1)(L+J_1-J_2+2)(L+J_1-J_2+1)}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)\cdot(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)(2J_2-3)}\right]^{\frac{1}{2}}$
	$l_1=J_1-1$
$l_2=J_2+2$	$(-1)^{L-J_1-J_2}\left[\dfrac{4(L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2+3)(L-J_1+J_2+2)(L-J_1+J_2+1)(L+J_1-J_2)(L+J_1-J_2-1)(L+J_1-J_2-2)}{(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)(2J_1+2)\cdot(2J_2+1)(2J_2+2)(2J_2+3)(2J_2+4)(2J_2+5)}\right]^{\frac{1}{2}}$
$l_2=J_2+1$	$(-1)^{L-J_1-J_2-1}\left[\dfrac{(L-J_1+J_2+2)(L-J_1+J_2+1)(L+J_1-J_2)(L+J_1-J_2-1)}{(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)(2J_1+2)\cdot(2J_2)(2J_2+1)(2J_2+2)(2J_2+3)(2J_2+4)}\right]^{\frac{1}{2}}\cdot 4\cdot\left[(J_1-1)(J_1+J_2+2)-(L+J_2+2)(L-J_2-1)\right]$
$l_2=J_2$	$(-1)^{L-J_1-J_2}\left[\dfrac{6(L+J_1+J_2+1)(L+J_1-J_2)(-L+J_1+J_2)(L-J_1+J_2+1)}{(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)(2J_1+2)\cdot(2J_2-1)(2J_2)(2J_2+1)(2J_2+2)(2J_2+3)}\right]^{\frac{1}{2}}\cdot 2\cdot\left[J_1^2-1-(L+J_2+1)(L-J_2)\right]$
$l_2=J_2-1$	$(-1)^{L-J_1-J_2-1}\left[\dfrac{(L+J_1+J_2+1)(L+J_1+J_2)(-L+J_1+J_2)(-L+J_1+J_2-1)}{(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)(2J_1+2)\cdot(2J_2-2)(2J_2-1)(2J_2)(2J_2+1)(2J_2+2)}\right]^{\frac{1}{2}}\cdot 4\cdot\left[(J_1-1)(J_1-J_2+1)-(L+J_2)(L-J_2+1)\right]$
$l_2=J_2-2$	$(-1)^{L-J_1-J_2-1}\left[\dfrac{4(L+J_1+J_2+1)(L+J_1+J_2)(L+J_1+J_2-1)(L+J_1-J_2+1)(-L+J_1+J_2)(-L+J_1+J_2-1)(-L+J_1+J_2-2)(L-J_1+J_2)}{(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)(2J_1+2)\cdot(2J_2-3)(2J_2-2)(2J_2-1)(2J_2)(2J_2+1)}\right]^{\frac{1}{2}}$
	$l_1=J_1-2$
$l_2=J_2+2$	$(-1)^{L-J_1-J_2}\left[\dfrac{(L+4-J_1+J_2)(L+3-J_1+J_2)(L+2-J_1+J_2)(L+1-J_1+J_2)(L+J_1-J_2)(L+J_1-J_2-1)(L+J_1-J_2-2)(L+J_1-J_2-3)}{(2J_1-3)(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)\cdot(2J_2+5)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2=J_2+1$	$(-1)^{L-J_1-J_2-1}\left[\dfrac{4(L+1+J_1+J_2)(L+3-J_1+J_2)(L+2-J_1+J_2)(L+1-J_1+J_2)(L+J_1-J_2)(L+J_1-J_2-1)(L+J_1-J_2-2)(-L+J_1+J_2)}{(2J_1-3)(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)\cdot(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)}\right]^{\frac{1}{2}}$
$l_2=J_2$	$(-1)^{L-J_1-J_2}\left[\dfrac{6(L+1+J_1+J_2)(L+J_1+J_2)(L+2-J_1+J_2)(L+1-J_1+J_2)(L+J_1-J_2)(L+J_1-J_2-1)(-L+J_1+J_2)(-L+J_1+J_2-1)}{(2J_1-3)(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)\cdot(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)}\right]^{\frac{1}{2}}$
$l_2=J_2-1$	$(-1)^{L-J_1-J_2-1}\left[\dfrac{4(L+1+J_1+J_2)(L+J_1+J_2)(L+J_1+J_2-1)(L+1-J_1+J_2)(L+J_1-J_2)(-L+J_1+J_2)(-L+J_1+J_2-1)(-L+J_1+J_2-2)}{(2J_1-3)(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)\cdot(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}$
$l_2=J_2-2$	$(-1)^{L-J_1-J_2}\left[\dfrac{(L+1+J_1+J_2)(L+J_1+J_2)(L+J_1+J_2-1)(L+J_1+J_2-2)(-L+J_1+J_2)(-L+J_1+J_2-1)(-L+J_1+J_2-2)(-L+J_1+J_2-3)}{(2J_1-3)(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)\cdot(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)(2J_2-3)}\right]^{\frac{1}{2}}$

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