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Multiple Meson Production in Nucleon-Nucleon Collisions

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I. INTRODUCTION

IN recent years, a great deal of work has appeared, both theoretical and experimental, bearing on the nature of the events which characterize the collisions of relativistic nuclear particles with each other. The interest in this area arises partly from the fact that only in such collisions can one explore the fields (if this terminology is at all applicable) very near to the nucleons. Under the best conditions, in a nucleon-nucleon collision, a laboratory energy of around 2×10^{11} ev is needed to explore distances of $\frac{1}{10}$ the proton Compton wavelength, and, in actuality, even more energy is required. Thus, in so far as one believes that the meson fields exist at such short distances, the study of the production of mesons at high energies will yield information applicable to the study of these fields. Indeed, since one is dealing with phenomena in so completely unknown an area, it would be remarkable if we could achieve a modicum of understanding on the basis of our conventional theoretical structure.

We will not discuss the experimental information,[1] nor its interpretation. This survey will be devoted to theoretical work which seeks to describe the presumably multiple production of mesons in a single nucleon-nucleon collision. We shall not concern ourselves with the question of the importance of plurality at low and moderate cosmic-ray energies. There is little doubt that plural events (in the broad sense that includes interactions by the produced mesons) play a role at moderate energies in complex nuclei. However, the multiple events seem also to occur, and can be fairly unambiguously separated at sufficiently high energies. We shall be interested in the latter.

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[1] See Chapter VIII of the book on high energy processes, to be published shortly by R. E. Marshak.

We shall also not consider questions related to the problem of meson identification. We shall tacitly suppose that we are speaking of the normal π-mesons, but little of what we say will be changed by another meson identification. If, on the other hand, there is a long chain of strongly coupled meson types, we will have to reconsider the entire situation.

The theoretical work is quite varied (as one might expect in an area in which one is not restricted by the existence of a sound and sensible theory), and we shall attempt some sort of classification according to content and concept (hypotheses would be much too precise a word). In the first place, we shall speak of "fundamental" and "kinematic" work. By fundamental, we shall mean work which has as its aim the derivation (in the loosest sense of the word), from the combined equations of motion of the nucleons and mesons, of the probability of transition from the initial state to a given final state—thus, the probability, under given conditions, of emitting a number of mesons of given momenta and charge. By kinematic work we will mean the pursuit of the problem from this point, taking into account the form of the nucleon-nucleon potential† to the prediction of those properties of such events as are experimentally observed, namely, multiplicities, angular distributions, etc. The role of the potential is a separate chapter, and will be discussed in Sec. IIIB.

† A word is needed about the use of the idea of potential in these problems. Clearly, if one believes that the nuclear forces are largely due to the same meson fields as are radiated in a collision (there is no good evidence on this point) then a consistent combined treatment of force and field is needed. This has been partially possible only in one unnatural case (Glauber), so that one is forced, in so far as he considers this point, to a sort of Born approximation in the potential, based on the very short collision time at high energies. To go past this would introduce the self-energy problem, which remains unsolved, even in the renormalization sense, in meson theory. This is a presently unsatisfactory state of affairs.

The fundamental treatments we shall further divide into the "pre-existing field" and "excited field" types. This is, again, a loose classification of convenience, and refers to the concept one has of the strength of the couplings involved (All treatments which yield multiple emissions presuppose "strong coupling", but not in the usual precise meaning of the term). The pre-existing field treatments suppose that the characteristics of the radiated meson field are primarily determined by the interaction of the field with the free nucleons long before the collision, and that the major function of the collision is to shake free these pre-existing fields and to provide enough energy and momentum to make this possible. On the other hand, the excited field treatments minimize this mechanism and suppose that the collision has the primary effect of transferring a great deal of energy and momentum into the volume immediately surrounding the point of collision, and that the character of the subsequent radiation is largely determined by the strong fields and interactions in the "hot spot". This is, in a crude sense, somewhat analogous to the compound nucleus idea in the theory of low energy nuclear reactions.

The only reason, of course, that such diametrically opposed viewpoints are possible is, as we have said, that there is no theoretically sound means of treating the problem.

II. THE FUNDAMENTAL PROBLEM—ROLE OF THE NONLINEARITY

A. Excited Field Theories

1. Heisenberg's Turbulence Theory

Heisenberg was the first to mention the possibility of multiple events, and his early work[2] fell more nearly into the pre-existing field category. His later ideas,[3] however, are in the excited field area, and we shall confine ourselves to them. Heisenberg's viewpoint is that a large number of mesons, trying to get out of the area of the collision, interact with each other so strongly that they come to a sort of dynamic equilibrium before they can escape. The form of the equilibrium spectrum can clearly not be calculated in this highly nonlinear situation, and one can only guess at it. If one supposes that the energy spectrum in wave-number space is

$$F(k)dk = Adk/k^{\alpha}, \tag{II.1}$$

where A and α are constants, then the total number n is, roughly,

$$n = \int F(k)dk/k = \frac{A}{\alpha}[(\mu c^2)^{-\alpha} - \epsilon^{-\alpha}], \tag{II.2}$$

where we have taken for the lower limit the rest energy of the meson, and for the upper limit the total energy in the event. The total energy is

$$\epsilon = \int F(k)dk = \frac{A}{\alpha - 1}[(\mu c^2)^{1-\alpha} - \epsilon^{1-\alpha}], \tag{II.3}$$

so that

$$n = \epsilon \cdot \frac{\alpha - 1}{\alpha} \cdot \frac{(\mu c^2)^{-\alpha} - \epsilon^{-\alpha}}{(\mu c^2)^{1-\alpha} - \epsilon^{1-\alpha}}. \tag{II.4}$$

Heisenberg finds fair agreement with some of the experiments, for $\alpha = 1$, in which case (4) becomes

$$n = \frac{\epsilon - \mu c^2}{\mu c^2 \ln(\epsilon/\mu c^2)} \approx \frac{\epsilon}{\mu c^2 \ln(\epsilon/\mu c^2)}, \tag{II.4$'$}$$

but the results are not too sensitive to the choice of α. The selection of a power law (II.1) for the spectrum is motivated by the statistical theory of hydrodynamic turbulence, which is also a nonlinear problem, and in which there seems to be some evidence that a law such as (II.1) with $\alpha = 5/3$ applies over part of the spectrum.

One may also note that, in this theory, since the strongly interacting turbulent meson fluid is supposed to have plenty of time to come to equilibrium, it will have forgotten from which directions the incident nucleons came. Consequently, the meson emissions will be spherically symmetrically distributed in the center of mass system.‡ This is a feature that seems to be ruled out by the event recently observed by Schein,[4] which demonstrates the existence of some memory. That this memory may be a simple consequence of the conservation law for angular momentum has been suggested by Fermi, below. Also the multiplicity observed in the Schein event is much lower than that predicted by Eq. (II.4′).

2. Fermi's Theory—Phase Space

Fermi has developed a theory[5] which is not dissimilar in concept from Heisenberg's, but is more specific in the major details. Fermi assumes the following:

(a) The "hot spot" has the shape of an oblate spheroid, contracted in the forward direction by the Lorentz factor $\gamma = [1 - v^2/c^2]^{-\frac{1}{2}}$, where v is the speed of each nucleon in the center of mass system.

(b) Mesons and neutron and proton pairs come to thermal equilibrium in this volume, with the "temperature" determined by the amount of energy available and the one free parameter in the dimensions of the spheroid.

(c) The states involved in the equilibrium are the ordinary free particle states of the mesons and nucleons, *not* the states of free particles confined to the spheroid.

[2] W. Heisenberg, Z. Physik **101**, 533 (1936); **113**, 61 (1939).

[3] W. Heisenberg, Z. Physik **126**, 569 (1949).

‡ I am informed that very recent work of Heisenberg differs somewhat from the description given here, and in particular predicts an anisotropic angular distribution.

[4] Lord, Fainberg, and Schein, Phys. Rev. **80**, 970 (1950).

[5] E. Fermi, Prog. Theoret. Phys. **5**, 570); Phys. Rev. **81**, 683 (1951).

In this way, dropping constant factors, one writes that the energy per unit volume is proportional to the fourth power of the temperature,

$$E/V \sim T^4,$$

and that the temperature is the energy per particle,

$$T \sim E/N,$$

so that

$$N \sim VT^3 \sim E^{\frac{3}{4}} V^{\frac{1}{4}}.$$

With the subsidiary assumption that $V \sim 1/E$, this is

$$N \sim E^{\frac{1}{2}},$$

where E is the available energy in the center-of-mass system. By suitably choosing the value of the parameter in the volume of the spheroid, one can obtain reasonable values of the multiplicity, etc.

This procedure again leads to spherically symmetric emissions, but Fermi has shown that a more literal interpretation of the model, coupled with an invocation of the law of conservation of angular momentum, will lead to an asymmetric distribution in good agreement with the Schein event, namely, a double cone with about a 30° half-width.

Criticism has been directed at the Fermi theory, on the following bases. In the first place, assumption (c) seems to be in conflict with the supposition that, if the particles are to be confined to the volume involved, they have uncertainties in momentum given by $\hbar/a$, where a is the appropriate dimension of the spheroid. Another way of expressing this is to say that the states confined to the spheroid start with a lowest state which has already much more energy than any of the states considered by Fermi to be most important. This objection is rendered invalid if one supposes that the wavelengths in the spheroid are, by virtue of large interparticle interactions, much shorter than those of the corresponding free particles. In this case, however, one can hardly speak of free particle states, and is led to something like the Heisenberg treatment above. There one does not presuppose any information about the spectrum of the normal modes. Fermi's view is somewhat different. He feels that confinement of particles with small energies to the volume is not impossible, since, in effect, the wave function for the situation has a "tail" in this region. He feels that the magnitude of the wave function will not play an essential role, as long as it is not precisely zero, and that the statistical considerations will be decisive.§

There is also a problem with regard to the attainment of equilibrium in the spheroid. If we assume that effects in the spheroid travel more slowly than the speed of light, the time required to reach equilibrium is of the order R_0/c. The collision time, however, is of order $R_0/\gamma c$, so that the spheroid of energy must remain in place for a time γ times the collision time, and must, therefore, be a relatively stable configuration. In the language of the old quantum theory, this seems to imply that the mesons will bounce back and forth between the walls of the spheroid γ times before escaping, or that the transmittivity of the wall is $1/\gamma$, corresponding to a large surface tension.

§ I would like to thank Professor Fermi for an illuminating conversation on this subject.

B. Pre-existing Field Theories

1. Lewis, Oppenheimer, and Wouthuysen (LOW)

LOW[6] have made an attempt to treat the fundamental problem according to the pre-existing field picture. The idea here is that the emission mechanism is very similar to that in the case of ordinary bremsstrahlung, where the radiated electromagnetic field can be thought of as the difference between the stable field appropriate to the incident electron, and that appropriate to the final scattered electron. Thus, since the electromagnetic coupling is to the velocity of the electron, through the term $-e/c\,\mathbf{v}.\mathbf{A}$, the change of velocity in scattering is responsible for the radiation. In the case of a pseudoscalar meson field, the coupling is to the spin of the nucleon, and, if the field is charged or symmetrical type neutral, also to the isotopic spin. Thus, if either of these changes in a collision (change of isotopic spin measuring charge exchange), mesons will be radiated. This problem can then be treated by a method exactly analogous to that developed by Bloch and Nordsieck for the treatment of the electromagnetic radiation, provided that one can treat the spin and isotopic spin as classical vectors. This corresponds to an extreme form of the "strong-coupling" theories, hence to zero isobar energy.

These authors then suppose that there is no essential interference, in the impact of two nucleons, between the interaction process between them, which leads to mutual exchange of momentum, energy, charge, and angular momentum, and the meson emission process. The nucleons are supposed to undergo the latter at their leisure.

The main results of this treatment are given in Sec. III, and need not be repeated here. It suffices to say that, in the theory there considered, the probability of emitting a given group of mesons is proportional to their invariant volume in phase space, and their total number is limited only by the over-all conservation laws.

The major difficulty with this method is that it does not properly take into account the effect of radiation damping, but depends instead upon an argument, based upon the Bloch-Nordsieck calculations, that the damping will affect only the total cross section, and not the relative cross sections calculated. It is hoped that this argument will have some range of applicability.‖

[6] Lewis, Oppenheimer, and Wouthuysen, Phys. Rev. **73**, 127 (1948).

‖ A detailed comparison of the Fermi and LOW theories has been carried out by H. Fukuda, in a paper now in press. I am indebted to Dr. Fukuda for a copy of his paper, and for an interesting conversation concerning it.

2. *Fukuda and Takeda*

Fukuda and Takeda[7] have carried out essentially the same calculation as LOW, but using the new, fully covariant perturbation-theoretic formalism, and taking into account properly the effect of virtual pair production of nucleons. It is supposed that the nucleons obey the Dirac theory; the potential between them is treated in the manner mentioned at the beginning of Sec. IIIB and is subject to the difficulties there described. This procedure than leads to the same results as in LOW.

3. *Umezawa et al.*

Umezawa[8] and collaborators have carried out a similar calculation, but have worked in the Heisenberg representation. They assert that this makes their results valid in strong-coupling as well as weak-coupling theory. The Heisenberg representation has an advantage of perspicuity, and leads to essentially the same results as the other treatments.

4. *Clementel and Dallaporta*

Clementel and Dallaporta[9] have worked with a semiclassical method, which, again, leads to essentially the same results, and have carried out a detailed analysis of the relation to cosmic-ray phenomena.

5. *Glauber*

Glauber[10] has been able, through a canonical transformation, to treat the multiple emission problem exactly, in the limit in which the mass of the nucleon is much less than the mass of the mesons. This is an entirely unreal limit, but the results are of considerable methodological interest. There is, somewhat surprisingly, no difference in results from the previous treatments, except in the effect of the potential. Glauber was able to show that, in the very extreme case of point coupling between the nucleons, the effect of interference between the scattering and the emissions is to increase the multiplicity by a factor $2^{\frac{1}{3}}$, so that it is entirely likely that one needn't be concerned about this effect.

C. Discussion

The pre-existing theories have some fundamental ambiguities that need to be mentioned. The central idea of all such theories is that a free nucleon is surrounded by an intense meson field, whose Fourier components describe many mesons of a wide range of energy. In a collision, it is supposed that a part of this field is shaken loose, and that this part increases with increasing energy. Consequently, at higher and higher energies, one draws mesons from the field nearer and nearer the nucleons (the more energetic mesons are, naturally, nearer the nucleons, in the sense of the uncertainty principle). Thus, in an event in which a dozen mesons are produced, one can estimate that distances less than the nucleon Compton wavelength are involved. In this sense, one can regard the existence of such multiple events as indirect evidence that the meson fields exist roughly as described, to such short distances from the nucleon. But if this is the case, one should expect the nuclear forces to exhibit this fact by showing a rapid increase of intensity with decreasing distance, in at least the range between a nucleon Compton wavelength and a meson Compton wavelength. The scattering experiments do not seem to show such an increase, so that one is forced to the somewhat ambivalent view that the fields are there to be shaken off, but not to manifest themselves as a force field. This is in conflict with the *raison d'être* of a field theory, and must be regarded as a serious problem. The analog of this problem does not exist in the excited field theories, which may be regarded as an argument in their favor.

They, on the other hand, have in common the difficulty that they seek to describe an essentially nonlinear situation in which, even during the course of the collision, the mesons interact with each other many times. Since there is no adequate method for treating such a problem, one is led to more or less intuitive pictures of the events, which are not derivable from a meson theory. This may be an advantage.

Note added in proof: A large star has recently been observed in India, by Lal, Pal, Peters, and Swami, which has a number of interesting features. It is initiated by a magnesium nucleus of about 2×10^{14} ev, and contains over 200 charged particles, most of which must therefore be mesons. The development of both the soft and hard components can be followed in some detail, and the following facts, among others, emerge from the analysis: (1) The ratio of neutral to charged mesons seems to be near unity, which is hard to understand on any existing theory. (2) In the secondary stars, produced by the mesons emerging from the first event, the amount of nuclear disturbance seems to be less than that associated with a nucleon-induced star of the same number of prongs. (3) In the same secondary stars, the mesons seem to be produced with an angular distribution consisting of a single cone pointing backwards in the center of mass system. While by no means conclusive, this is naturally to be expected from a pre-existing field theory, in which the effect of the collision is to shake off part of the field of the nucleon or nucleons involved. I would like to thank Professor Oppenheimer for calling my attention to this star.

III. KINEMATICS OF PRE-EXISTING THEORIES

A. Role of the Conservation Laws

If we consider that the only interaction between the two nucleons involved in the collision occurs through

[7] H. Fukuda and G. Takeda, Prog. Theoret. Phys. 5, 957 (1950).
[8] Umezawa, Takahashi, and Kamefuchi, Phys. Rev. 85, 505 (1952).
[9] E. Clementel and N. Dallaporta, Nuovo cimento 5, 235 (1948); 5, 298 (1948).
[10] R. Glauber, Phys. Rev. 84, 395 (1951).

the potential (which is discussed below), then we can consider each nucleon individually, the effect of the other being represented only by the transfer of spin, momentum, energy, and charge. In so doing, we neglect interference effects between the nucleons, which is justified by the short wavelength involved. Glauber has shown that, even in the unrealistic extreme case of point interaction between the nucleons, the effect of interference is to raise the multiplicity by a factor $2^{\frac{1}{3}}=1.26$. Then the differential cross section for the emission by *one* of the nucleons of mesons of momentum $p_1, \cdots, p_n$, ending in the state $\mathbf{P}_f$, is proportional to

$$\frac{A^N}{N!}d\mathbf{P}_f\prod_{\nu=1}^{N}\frac{d\mathbf{p}_\nu}{\epsilon_\nu}, \tag{III.1}$$

where A is given in LOW.

A word should be added about the isobar question. The expression (III.1) is derived from a model in which the isobar energy is assumed to be zero, so that the nucleon in the final state may have any charge it likes, and the mesons are uninhibited in their choice of charge. One can, however, easily take into account deviations from this supposed extreme behavior, qualitatively, at least. There are two other views (at least) that one can have toward the charge behavior of the nucleons. One can believe, with Fukuda and Takeda, that the isobar energy is so high that, at each intermediate step in the process, the nucleon is either a proton or a neutron. Then, at each stage, the nucleon is restricted in the charge type of the mesons it can emit; if a proton, it can emit either a positive or neutral meson; if a neutron it can emit either a negative or neutral meson. Or one can believe that something intermediate occurs. that the isobar energy is high enough to insure that the final nucleons will not be highly charged, but that in the intermediate states anything can happen. Such a model has also been discussed.[11]

We consider the case of large isobar energy more carefully. In a symmetric theory the charged mesons are coupled to the nucleon just twice as strongly as the neutral mesons, so that, at each emission, the probability is twice as large that the charged alternative will be chosen. Thus, there will be twice as many charged mesons produced as there are neutral, and the naively expected even division among positive, negative, and neutral mesons, will obtain.

How, then, are the charges distributed in the low isobar energy case, since then the restriction mentioned above is lifted, and we might offhand expect a larger fraction of charged mesons, given by $2+2/2+2+1=\frac{4}{5}$. That this is not so, and that the fraction remains $\frac{2}{3}$ (as given in LOW) is connected with the fact that we must reinterpret the isotopic spin vector in the strong coupling case. τ_+, the part of the isotopic spin that is involved in the production of positive mesons, can be represented by $\frac{1}{2}(\tau_\xi+i\tau_\eta)$, and has the same matrix element for emission of a positive meson as does τ_ζ for a neutral one. But the coupling is twice as strong for the charged meson, so that the $\frac{2}{3}$ ratio holds when we also take into account the negatives. When we go over to the strong coupling theory, however, we must deal with $|\Delta\tau_+|^2$, which is equal to $\frac{1}{4}[|\Delta\tau_\xi|^2+|\Delta\tau_\eta|^2]$, and therefore, if there is no preference among the $\xi\eta\zeta$ directions, $|\Delta\tau_+|^2$ will be just one-half of $|\Delta\tau_\zeta|^2$, on the average, so that, again, one will have the even distribution among positives, negatives, and neutrals. The expected value of $|\Delta\tau_\zeta|^2$ is discussed in LOW, and the corresponding symmetrical case, in perturbation theory, is given by Fukuda and Takeda.

Integrating (III.1) over the meson space, we obtain approximately, for mesons of total energy E and total momentum $\mathbf{P}$ (see reference 11),

$$B^N d\mathbf{P}_f dE d\mathbf{P}(E^2-\mathbf{P}^2)^N/N!^3. \tag{III.2}$$

This result is obtained by using the Pauli trick of representing

$$\delta(E-\sum_1^N\epsilon_\nu)\quad\text{and}\quad\delta(\mathbf{P}-\sum_1^N\mathbf{p}_\nu)$$

by their Fourier transforms, so that one has to do integrals of the type

$$\int\exp[i(\lambda\epsilon+\mathbf{u}\cdot\mathbf{p})]d\mathbf{p}/\epsilon \tag{III.3}$$

for each meson, followed by one final integral in λ and $\mathbf{u}$, which leads to (III.2). Actually (III.3) is related to a Hankel function of the first kind, and (III.2) is obtained by replacing the Hankel function by its value near the origin of λ and $\mathbf{u}$. This is equivalent to neglect of the meson mass, and can be seen to be justified as follows: The integral that leads from the N factors (III.3) to (III.2) is an integral over λ and $\mathbf{u}$ that can be easily seen to have a saddle at $\lambda_s=2iNE/E^2-P^2)$, $\mathbf{u}_s=-\mathbf{P}/E\lambda_s$, so that $\lambda_s^2-\mathbf{u}_s^2=-4N^2/(E^2-P^2)$; but $E^2-P^2\propto\bar{N}^3$, where $\bar{N}$ is the mean multiplicity, so that $\lambda_s^2-\mathbf{u}_s^2\propto 1/\bar{N}$. The ratio of the second to the first term in the expansion of (III.3) is of the order $(\lambda^2-\mu^2)\ln(\lambda^2-\mu^2)$, therefore $1/N\ln N$. This factor appears N times, so that the net result of the next term is to multiply (III.2) by $(1+\beta/N\ln N)^N\exp(\beta\ln N)=N^\beta$, where β is a number of order unity. For large N this factor is swamped, by the $N!^3$ and the B^N, so that, since we are not concerned with the absolute value of (III.2), our approximation is adequate.

If the original energy and momentum of the nucleon are $(E_0, \mathbf{P}_0)$, and the energy and momentum transferred to it in the collision are $(\delta, \boldsymbol{\Delta})$, then the conservation laws of energy and momentum are

$$E_0+\delta=E+E_f,\quad \mathbf{P}_0+\boldsymbol{\Delta}=\mathbf{P}+\mathbf{P}_f. \tag{III.4}$$

So that we can replace $dEd\mathbf{P}$ in (III.2) by $d\boldsymbol{\Delta}d\delta$, multiply by $\mathcal{P}(\delta, \boldsymbol{\Delta})$, the probability of a momentum-

[11] H. W. Lewis, Proc. Berkeley Statistical Symposium (University of California Press, Berkeley, 1951).

energy transfer $(\mathbf{\Delta}, \delta)$, and obtain for the unnormalized probability for the emission of N mesons of total energy and momentum $(E, \mathbf{P})$, the nucleon going into the final state $\mathbf{P}_f$, with the relations (III.4),

$$\mathcal{P}(\delta, \mathbf{\Delta})d\delta d\mathbf{\Delta} d\mathbf{P}_f \cdot B^N(E^2-\mathbf{P}^2)^N/N!^3. \quad \text{(III.5)}$$

We can also find the energy-momentum distribution of the emitted mesons by substituting the values of λ_s and $\mathbf{u}_s$ into the integrand of (III.3), so that the meson distribution, for given $(E, \mathbf{P})$, is

$$\exp\left\{-\frac{2N}{E^2-\mathbf{P}^2}[E\epsilon-\mathbf{P}\cdot\mathbf{p}]\right\}\frac{d\mathbf{p}}{\epsilon}, \quad \text{(III.6)}$$

which consists, as it should, entirely of relativistic invariants. It is spherically symmetric in the coordinate system in which $\mathbf{P}=0$.

For the final nucleons the situation is analogous, since the factor involved, corresponding to (III.3), is

$$\exp[i(\lambda E_f+\mathbf{u}\cdot\mathbf{P}_f)]d\mathbf{P}_f, \quad \text{(III.7)}$$

which differs from (III.3) only in the absence of the energy in the denominator. Thus the final nucleon distribution is, corresponding to (III.6),

$$\exp\left\{-\frac{2N}{E^2-\mathbf{P}^2}[EE_f-\mathbf{P}\cdot\mathbf{P}_f]\right\}d\mathbf{P}_f. \quad \text{(III.8)}$$

This doesn't relate in any way, of course, to the possibility of creation of additional nucleons in the collision, which is a higher order effect, in our approximation.

B. Role of the Potential

We want now to consider the form of the function $\mathcal{P}(\delta, \mathbf{\Delta})$ in (III.5), and its effect on the multiplicity and energy and angular distributions of the mesons and nucleons. One's first guess would be choose as the potential something like a Lorentz-contracted Yukawa well, so that

$$\mathcal{P}_y(\delta, \mathbf{\Delta}) \propto (\kappa^2+\mathbf{\Delta}^2-\delta^2)^{-2}, \quad \text{(III.9)}$$

and then to integrate (III.5) over δ and $\mathbf{\Delta}$. Essentially this was done by Fukuda and Takeda; LOW, on the other hand, have supposed that $\delta=0$, since they were working in the old three-dimensional formalism. In either case, we find that the factor $(E^2-\mathbf{P}^2)^N$ dominates the picture, and, in fact, almost the entire contribution to the integral comes from $\delta\approx 0$, $\mathbf{\Delta}\approx -\mathbf{P}_0$, and the resulting distribution of mesons is spherically symmetric in the center-of-mass system. This is the result found by all workers in the pre-existing theories, and corresponds to $\mathbf{\Delta}^2-\delta^2\approx \mathbf{P}_0^2\approx(\gamma\mu)^2\cdot(M/\mu)^2$, which means an impact parameter of the order of the nucleon Compton wavelength. Such collisions are, however, expected to be rare, and their predominance here stems from the large volume in phase space allotted to them. LOW have mentioned briefly the problem of determining what happens in the more distant collisions, and we want to discuss this in somewhat more detail here.

It is clear, in the first place, that no modifications of (III.9) can alter the basic situation. $(E^2-\mathbf{P}^2)^N$ will dominate any potential we can think of (within reason), so that we cannot carry out any such formal procedure. We will, therefore, simply discuss the question of what are the likely values of δ and $\mathbf{\Delta}$, and will forego the integration, since it is there that we get into trouble.

What, then, are the likely values of δ and $\mathbf{\Delta}$? This question can only be answered in so far as we are willing to adopt some physical picture of the origin of the forces between the nucleons. If, as seems very likely, they originate in the exchange of a meson of some kind, then the characteristic length involved in the law of force will be the Compton wavelength of the meson involved. Call this $1/\kappa$. Thus, there will be a field of force surrounding each nucleon, with momenta and energies of order κ in the rest system of that nucleon. A collision then consists in the capture by one nucleon, while passing by, of one of the mesons in the field of the other. If v is the speed of each nucleon in the center-of-mass system, $\gamma=(1-v^2)^{-\frac{1}{2}}$, and δ_0, $\mathbf{\Delta}_0$ are the energy and momentum of the "virtual" meson in the rest system of the parent nucleon, then

$$\delta=\gamma(\delta_0+\mathbf{v}\cdot\mathbf{\Delta}_0)\approx\gamma\kappa, \quad \mathbf{\Delta}=\gamma(\mathbf{\Delta}_0+\mathbf{v}\delta_0)\approx\gamma\kappa\mathbf{v}, \quad \text{(III.10)}$$

since we have supposed that all particles are very fast. We have taken into account the fact that $\mathbf{\Delta}_0$ will have no directional preference (since there is no preferred axis in the rest system of the particle), and have set terms linear in $\mathbf{\Delta}_0$ equal to zero. We note also that $\delta^2-\mathbf{\Delta}^2=\delta_0^2-\mathbf{\Delta}_0^2\approx\kappa^2$, so that we are, in fact, in the region of large contributions of (III.9).

We are now in a position to see one of the most surprising results of this treatment. We need only remember that the expression we are dealing with for the total cross section is the product of two factors of (III.5) (with, of course, only one set of the terms involving δ and $\mathbf{\Delta}$), one for each nucleon. But δ and $\mathbf{\Delta}$ have equal and opposite values for the two nucleons, since the exchanged meson delivers the whole amount of energy and momentum with which it started out. For the momentum part this is all right, since the two nucleons are initially traveling in opposite directions, so that each can be slowed down by the impact, corresponding to the same sign of the physical effect. For the energy δ, however, one must lose energy, and one must gain. That this causes trouble can be seen formally, since we have, neglecting for the moment, the final nucleon energies,

$$\begin{aligned}[(E_0+\delta)^2-(\mathbf{P}_0+\mathbf{\Delta})^2]^{N_1}[(E_0-\delta)^2-(-\mathbf{P}_0-\mathbf{\Delta})^2]^{N_2} \\ \approx[M^2+2(E_0\delta-\mathbf{P}_0\cdot\mathbf{\Delta})]^{N_1} \\ \times[M^2-2(E_0\delta+\mathbf{P}_0\cdot\mathbf{\Delta})]^{N_2}. \quad \text{(III.11)}\end{aligned}$$

We have assumed here that the second nucleon is the parent of the exchanged meson, so that, from

(III.10), $\mathbf{P}_0\cdot\mathbf{\Delta}\approx\gamma\kappa\mathbf{P}_0\cdot\mathbf{v}_2\approx-\gamma\kappa|\mathbf{P}_0|\approx-\gamma\kappa E_0\approx-E_0\delta$, since we have always assumed that all particles are fast. Consequently, the round bracket in the second factor is close to zero, so that there will be a tendency for N_2 to be very small compared to N_1, and we will see a highly asymmetric double cone in the center-of-mass system. This goes against one's physical intuition, but can be understood in the following manner. The exchanged meson comes from the virtual field of the second nucleon, which field is, of course, Lorentz-contracted in the direction of motion. Thus, thinking along the lines of the Weizsacker-Williams method, the meson will be very loosely bound, and will have many of the characteristics of a free particle. In particular, its energy, which would be $|\mathbf{\Delta}|[1+\mu^2/2|\mathbf{\Delta}|^2]$ if it were a free particle, is, instead, $|\mathbf{\Delta}|/v$, from (III.10) or

$$|\mathbf{\Delta}|[1+M^2/2\mathbf{P}_0^2],$$

which is almost the same. Thus, as far as energy and momentum are concerned, the nucleon almost thinks that it has emitted a free meson. To emit N_2 additional really free mesons, would then be almost the same ["almost" corresponds to the term in M^2 in (III.11)] as for a free particle of energy E_0 to emit N_2+1 free particles. The latter is well known to be forbidden by energy-momentum considerations, so one expects the former to be almost forbidden. The other nucleon has, of course, no such problem, since it absorbs the energy δ.

Thus, the asymmetric cone is really a consequence of the model we have adopted, and we must concern ourselves with the reasons for its apparent absence in nature. But first let us work out some of the quantitative features of the asymmetric cone.

1. The Asymmetric Cone

We have here simply to evaluate the consequences of (III.6) in light of the expressions for δ and $\mathbf{\Delta}$ given by (III.10). We have

$$E=E_0+\delta=\gamma(M+\kappa)$$
$$P=|\mathbf{P}_0-\mathbf{\Delta}|=\gamma v(M-\kappa)\approx\gamma(M-\kappa), \quad \text{(III.12)}$$

where we have, as always, set $c=1$. Substituting these values into (III.6), we have our basic distribution

$$\exp\left\{-\frac{N(M+\kappa)}{2\kappa E_0}\left[\epsilon-p\frac{M-\kappa}{M+\kappa}\cos\vartheta\right]\right\}\frac{d\mathbf{p}}{\epsilon}, \quad \text{(III.13)}$$

where ϑ is the angle between the initial direction of motion of the parent nucleon, and the direction of the emitted meson. It should be noted that we are specifically working in the center-of-mass system. Thus, the distribution in angle of mesons of momentum p is (after normalization)

$$f_p(\vartheta)=\frac{Ap}{2\sinh Ap}\exp[Ap\cos\vartheta], \quad \text{(III.14)}$$

where $A=N(M-\kappa)/2\kappa E_0$. For the angular distribution regardless of energy, we obtain, assuming that the mesons are fast,

$$f_{\text{total}}(\vartheta)=\frac{(E/p)^2-1}{2}\frac{\sin\vartheta d\vartheta}{(E/p-\cos\vartheta)^2}$$
$$=\frac{(M+\kappa/M-\kappa)^2-1}{2}\frac{\sin\vartheta d\vartheta}{[(M+\kappa/M-\kappa)-\cos\vartheta)]^2}, \quad \text{(III.15)}$$

again after normalization to unity. The corresponding mean cosines are

$$\langle\cos\vartheta\rangle_p=\coth Ap-(Ap)^{-1} \quad \text{(III.16)}$$

and

$$\langle\cos\vartheta\rangle_{\text{total}}=\frac{M+\kappa}{M-\kappa}-\frac{2M\kappa}{(M-\kappa)^2}\ln\left(\frac{M}{\kappa}\right), \quad \text{(III.17)}$$

where it is to be noted that the latter is independent of E_0. That is, the mean width of the cone, in the center-of-mass system, is independent of the primary energy. We also have, corresponding to these expressions,

$$\langle\sin^2\vartheta\rangle_p=\frac{2}{Ap}[\coth Ap-(Ap)^{-1}] \quad \text{(III.18)}$$

and

$$\langle\sin^2\vartheta\rangle_{\text{total}}=\frac{4M\kappa}{(M-\kappa)^2}\left[\frac{M+\kappa}{M-\kappa}\ln\left(\frac{M}{\kappa}\right)-2\right]. \quad \text{(III.19)}$$

Finally, if we call ϑ_1 the angle at which the number of mesons per unit solid angle is half that in the forward direction, and ϑ_2 the angle of the cone which includes half the mesons (in both cases, regardless of energy),

$$\cos\vartheta_1=(M+\kappa-2\kappa\sqrt{2}/(M-\kappa), \quad \text{(III.20)}$$

$$\cos\vartheta_2=(M-\kappa)/(M+\kappa). \quad \text{(III.21)}$$

If we were to guess that κ is equal to the mass of the π-meson, then $M/\kappa=1836/274=6.7$, and we would have $\vartheta_1=31°$, and $\vartheta_2=42°$, values which are in generally good agreement with the experiments.[4]

The energy distribution regardless of angle can now be found by integrating (III.13) over angles, which yields the distribution

$$\exp\left[-\frac{N(M+\kappa)\epsilon}{2\kappa E_0}\right]\cdot\sinh\left[\frac{N(M-\kappa)p}{2\kappa E_0}\right]\cdot d\epsilon \quad \text{(III.22)}$$

where, as usual, p can be replaced by ϵ, since the mesons are supposed fast. This yields, for the normalized distribution,

$$\frac{MN}{E_0(M-\kappa)}[\exp(-N\epsilon/E_0)$$
$$-\exp(-MN\epsilon/\kappa E_0)]\cdot d\epsilon. \quad \text{(III.23)}$$

We, therefore, find

$$\langle\epsilon\rangle = E_0(M+\kappa)/MN \qquad \text{(III.24)}$$

$$\langle\epsilon^2\rangle = 2E_0^2(M^2+\kappa M+\kappa^2)/M^2N^2. \qquad \text{(III.25)}$$

Finally, we can note that the mean multiplicity given by (III.5) is

$$\langle N\rangle = B^{\frac{1}{2}}(E^2-P^2)^{\frac{1}{4}} = (4\kappa BE_0^2/M)^{\frac{1}{4}}, \qquad \text{(III.26)}$$

which is reduced by a factor $(4\kappa/M)^{\frac{1}{4}}$ from the value given in LOW.

It should be mentioned that the numerical value of B given in LOW has changed, owing to better information about the value of $g^2/\hbar c$, and to the fact that we now know that we are talking about π-mesons (or heavier ones), whereas, in LOW, it was thought that μ-mesons were produced directly. The value of B given in LOW is (*cf.* Eq. 31 there)

$$B = g^2 q/\pi, \qquad \text{(III.27)}$$

where q is a number of order unity, and the meson mass enters in that energies are measured in units of the meson rest energy. Thus, taking the value $g^2 \sim \frac{1}{6}$, which is suggested by the meson production experiments, and setting $q\sim 1$, $B\sim 1/20$, so that, in terms of the primary energy in the laboratory system,

$$\langle N\rangle \approx (2BE_{\text{lab}})^{\frac{1}{4}} \approx 0.9 E_{\text{Bev}}^{\frac{1}{4}}, \qquad \text{(III.28)}$$

if we measure the laboratory energy of the primary in Bev. This is, it must be remembered, the multiplicity in a single cone—the corresponding expression for the double cone is slightly different.

2. The Symmetric Cone

We want now to consider the relative likelihood of formation of a double cone, within the limitations of the picture we have adopted. There is no escape from the fact that the single cone is the most immediate prediction, and we can only study the implications for the picture of the moderately well-established experimental fact that these collisions produce, in general, double cones of mesons. It is clear that our conclusion that one of the nucleons will decline to emit mesons is a simple consequence of the energy-momentum balance, so that the salvation must come from a different hypothesis about the probability of energy and momentum exchange between the nucleons.¶

One possibility which jumps to mind is the following: that if, in the collision, two scattering mesons rather than one are exchanged, the double cone will become kinematically possible. Then the volume of phase space available to the mesons will be much larger than in the single cone situation, so that one may take the point of view that the nucleons will prefer to exchange pairs of mesons, just because they know that this will make much more phase space available to them. This is unsatisfactory, largely because it means giving up the idea of independence of the collision and emission probabilities, but we will pursue it to find out what its consequences are for the various distributions.

Thus, we assume that nucleon A delivers a meson of energy and momentum $(\delta, \mathbf{\Delta})$ to nucleon B, and that nucleon B reciprocates by delivering $(\delta, -\mathbf{\Delta})$ to A. The consequence is that each nucleon loses momentum in magnitude equal to $2|\mathbf{\Delta}|$, and loses no energy in the initial collision. Thus, for each nucleon, $E=E_0$, and $|\mathbf{P}| = |\mathbf{P}_0| - 2|\mathbf{\Delta}|$. One can insert these values into (III.6), obtaining as our distribution [analogously to (III.13)],

$$\exp\left\{-\frac{NM^2}{2\kappa E_0(M-\kappa)}\left[\epsilon - p\frac{M-2\kappa}{M}\cos\vartheta\right]\right\}\frac{d\mathbf{p}}{\epsilon}. \qquad \text{(III.13}'\text{)}$$

This differs from (III.13) only through the appearance of $M^2/(M-\kappa)$ instead of $M+\kappa$ in one place, and $(M-2\kappa)/M$ instead of $(M-\kappa)/(M+\kappa)$ in another. Thus, none of the expressions (III.14) to (III.25) are substantially changed, and they are easily derived from (III.13′), so that there is no need to write them here. One should only note that the expressions for the multiplicity refer to the number of mesons in each cone, so that the total multiplicity is about doubled.

¶ Alternatively, one can believe that two cones have essentially different origins, and that, in the primary event, it is really the single cone that appears.