Topics in Scattering Theory

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INTRODUCTION

SOME questions in scattering theory will be discussed herein. They arose in the treatment of experimental material on proton-proton and proton-neutron scattering. Part of the reason for presenting the material is the mistaken impression produced in the literature that the variational treatment of the problem differs in an essential way from the older methods and that it offers special advantages. The variational method has advantages in cases for which the solution can be obtained only approximately or if the exact solution is very involved. It also has advantages in supplying approximate analytic forms for functions which must otherwise be evaluated by numerical methods. For the two-body central field problem it will be seen that the variational method as used by Blatt and Jackson following some work of Schwinger's may be considered as not being superior to the direct solution in the technique of calculation except for being a valuable guide in indicating the existence of identities. These identities can be established very simply, however, by direct methods. The variational method, as carried out in the literature, will be seen to be more involved than the direct solution. Some of this material has been previously pointed out by Peierls and Preston,[1] by Bethe,[2] by Chew and Goldberger,[3] by Hatcher, Arfken, and Breit,[4] and somewhat more systematically by Breit and Hatcher.[5] The treatment of this phase is based on some unpublished work done in collaboration with Mr. M. C. Yovits. It will be seen in this connection that the function f of Breit, Condon, and Present can be expanded in powers of the energy by a general and simple procedure and that simple formulas for the general coefficient can be given. The expansion can be made about any energy, and the range of applicability of the method can be extended, therefore, beyond the energy at which the series about the energy $E=0$ stops converging. Related expansions can be given for the effect of linear parameters entering the potential energy. Both types of expansions have a bearing on those in a paper by Breit, Thaxton, and Eisenbud.

The f function has been first introduced for $L=0$ by suitably regrouping quantities in the equation connecting the phase shift with the energy. Since the primary object was to establish the behavior of the phase shift for low energies, the coulomb function power series in the energy and distance have been isolated from terms which contained the energy logarithmically. The result was to obtain a relation between energy, the logarithmic derivative of the wave function with respect to distance at the boundary of the potential well and the phase shift. The function f which contains only the energy and phase shift is expressed in this relation as a quotient of power series in the energy with coefficients which are functions of the radius of the nuclear potential well and of the logarithmic derivative of the wave function with respect to distance at its boundary. This function f has the convenient property of being energy independent if the range of force is made zero. For small ranges of forces f has a small positive slope. The same approach is presented in the present paper for $L>0$. The first attempt to generalize f to $L>0$ is found in a paper by Landau and Smorodinsky.[6] Their discussion is not complete because no use is made of the logarithmic derivative of the wave function or of any equivalent information. The analogous treatment given by them for $L=0$ also has the same feature; but in this case it agrees with the previously established results of Breit, Condon, and Present provided the range of force is made zero. It will be seen below that for $L>0$ the generalized f function has in general a nonvanishing energy derivative at zero energy which may become zero in special cases. As the range of force becomes zero, this energy derivative will be seen to approach $-\infty$. The method of the comparison potential has to be applied in this case without the convenience of there being a potential having zero range and a constant f value. This circumstance is not a serious drawback because the relations for calculating coefficients of the energy in the power series for f apply also if the comparison potential does not have zero range.

The calculation of phase shifts has a bearing on the calculation of energy states. In fact, the relatively insignificant modification of the phase shift problem which is produced by enclosing the system in a large perfectly reflecting sphere enables one to deal with discrete states. The first section of the present paper is concerned with showing how the relations for changes in energy are connected with those for changes in phase shifts. It is shown that the Ferretti-Hulthén exact relation between the change in phase shift and the initial and final wave functions is closely related to the relation between the change in energy, the perturbing potential, and the initial and final wave functions.

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[1] R. E. Peierls and M. A. Preston, Phys. Rev. **72**, 250 (1947).
[2] H. A. Bethe, Phys. Rev. **76**, 38 (1949).
[3] G. F. Chew and M. L. Goldberger, Phys. Rev. **75**, 1637 (1949).
[4] Hatcher, Arfken, and Breit, Phys. Rev. **75**, 1389 (1949).
[5] G. Breit and R. D. Hatcher, Phys. Rev. **78**, 110 (1950).
[6] L. Landau and J. Smorodinsky, J. Phys. (U.S.S.R.) **8**, 154 (1944).

The generalization of f to $L>0$ can be used for the treatment of nuclear reactions.

The main part of the paper is concerned with considerations concerning the expansion of f in powers of the energy and in powers of parameters entering the assumed expressions for the potential energy as well as with the generalization to $L>0$. At the end, a brief discussion is given of the usefulness of f in obtaining fits to experimental data. It is brought out that the essential uncertainties cannot be remedied by the employment of any kind of plot and that the effect of shape of potential energy wells remain as an eventual limitation on the accuracy of conclusions concerning the depth and range of the potential wells. The advantages and disadvantages of the f plots are also considered.

In the last section of the paper the generalization of the method of the f function to the many-body case is considered employing the viewpoint of first section as a starting point.

I. RELATION OF PHASE SHIFTS TO ENERGY

Notation

The notation followed herein is, on the whole, the same as that used by Yost, Wheeler, and Breit, by Breit, Condon, and Present, and by Breit. A few minor changes appeared advisable. The function $F_L(kr)$ is defined in the usual manner as being a solution of the differential equation for r times the radial function with the requirements of an asymptotic form having unit amplitude at $r=\infty$ and of approaching zero at $r=0$ from the side of positive numbers. The subscript L is dropped occasionally, however, when this omission causes no ambiguity.

The function $\mathfrak{F}=F\cos K+G\sin K$ does not include the factor $\exp(iK)$. This function is understood to be continued into the nuclear interior as a solution of the appropriate radial equation with the actual rather than coulombian potential.

Let there be a system for which the hamiltonian function is H_0 and let ψ_0 be the wave function of a stationary state corresponding to energy E_0. Then one has

$$H_0\psi_0=E_0\psi_0. \tag{1}$$

If the hamiltonian and wave function are changed to

$$H=H_0+H', \quad \psi=\psi_0+\psi', \tag{1.1}$$

the energy is changed to the value E and one has

$$H\psi=(H_0+H')\psi=E\psi. \tag{2}$$

It follows from Eqs. (1) and (2) that

$$(\psi_0, H\psi)=E(\psi_0, \psi),$$

$$(\psi_0, H_0\psi)=(H_0\psi_0, \psi)=E_0(\psi_0, \psi),$$

and hence by subtraction

$$(\psi_0, H'\psi)=E'(\psi_0, \psi), \quad (E'=E-E_0). \tag{2.1}$$

The change in the energy, E', is thus expressible exactly as

$$E'=(\psi_0, H'\psi)/(\psi_0, \psi), \tag{2.2}$$

which may also be expressed as

$$E'=(\psi, H'\psi_0)/(\psi, \psi_0), \tag{2.3}$$

since H' is hermitian. The two forms of H' show the symmetry of the situation between the unperturbed state ψ_0 and the perturbed state ψ. If one sets $\psi\cong\psi_0$ in Eq. (2), there results the first-order perturbation theory formula,

$$E'\cong(\psi_0, H'\psi_0)/(\psi_0, \psi_0); \tag{2.2'}$$

and from Eq. (2.3) one similarly has on replacing ψ_0 by its approximate value ψ the approximation

$$E'\sim(\psi, H'\psi)/(\psi, \psi). \tag{2.3'}$$

If both functions ψ, ψ_0 are available, one can obtain E by means of either Eq. (2.2) or Eq. (2.3); but in practical problems one is concerned with finding ψ and E when ψ_0 and E_0 are known.

The Rayleigh-Schroedinger perturbation method and the Ritz variational method give Eqs. (2.2'), (2.3') in first approximation, and there is no difference between any two of these methods at this stage. On the other hand, Eq. (2.3) suggests that if there is available an expansion of ψ,

$$\psi=\sum_n\lambda^n\psi_n, \tag{2.4}$$

then one can obtain E by solving the equation which results from the substitution of Eq. (2.4) into Eq. (2.2). The parameter λ may be taken for example as a factor multiplying H' which is set equal to unity at the end of the calculation. Such a determination of E is possible only if an expansion of the type written in Eq. (2.4) is available. For the discrete spectrum of an eigenvalue problem the knowledge of the expansion (2.4) is equivalent to the knowledge of E in any consideration. Equation (2.2) differs from the Rayleigh-Schroedinger procedure in representing the energy change as a quotient of two power series; in this respect it is similar to the Ritz method.

It is well known that the discussion of problems in the continuum can be reduced to that of a discrete spectrum. The artifice consists usually in the introduction of a periodic boundary condition or of other similar modifications of the original problem. The relation of E' to E which has just been discussed becomes translated into a relation for the phase shift. A simple example is that of a spherically symmetric potential energy for which the wave function may be considered as a sum of terms, each term being representable as the product of a radial factor $R(r)=F(kr)/(kr)$ which depends only on the distance r and an angular factor which depends only on polar angles. The wave number is here denoted by $k/(2\pi)$. The equation satisfied by F is then

$$d^2F/dr^2+(2\mu/\hbar^2)[E-V_e]F=0, \tag{3}$$

with

$$V_e=V+L(L+1)\hbar^2/(2\mu r^2), \tag{3.1}$$

where $V(r)$ is the potential energy, L is the angular

momentum in units $\hbar$, and μ is the reduced mass, while

$$k^2=(2\mu/\hbar^2)E. \tag{3.2}$$

Here E is the energy of relative motion rather than the total energy. It is supposed that $V(r)$ vanishes sufficiently rapidly at $r=\infty$ to make it possible to neglect it beyond a certain $r=b$. This statement implies that all results concerning phase shifts are obtainable by carrying out the limiting process $b\to\infty$ and taking the limit for the phase shift. For $V=0$ the regular solutions of Eq. (3) have the asymptotic form,

$$\sin\varphi, \quad \varphi=kr-L\pi/2,$$

which become modified by V so as to become

$$F\sim\sin(\varphi+\delta). \tag{3.3}$$

For a very large value of $r=R$ the functions F will now be required to vanish. The original problem is modified by doing so. The energy spectrum becomes discrete. The spacing between successive energies of the spectrum will be called ΔE, the corresponding difference in k is

$$\Delta k\cong\pi/R.$$

The solutions F, F^0 will now be considered corresponding, respectively, to energies E, E^0 for the potential energies $V=V^0+V'$ and V^0, respectively. The number of nodes between $r=0$ and $r=R$ is here supposed to be the same for F and F^0. The function F may be considered as arising from F^0 as the result of an adiabatic change of V from V^0 to V^0+V'. According to Eq. (2.2), one has

$$E-E^0=\left[\int_0^\infty FF^0V'dr\right]\Big/\left[\int_0^\infty FF^0dr\right]. \tag{4}$$

On the other hand, $E-E_0$ can be related to the phase shifts δ, δ^0 which correspond to F and F^0. The result of consideration will be to obtained the relation,

$$\sin(\delta-\delta^0)=-(k/E)\int_0^b V'F^0Fdr, \tag{5}$$

which first appears in the papers of Hulthén and Ferretti.[7] This relation is readily derived from the original differential equations. The object of the following discussion is to provide an explicit connection of Eq. (5) to Eq. (4). The slight difference in the relationship of F to F^0 in these two formulas must first be mentioned. In Eq. (4) the functions F, F^0 correspond to different potentials as well as to slightly different energies, while in Eq. (5) the energy is exactly the same for the two functions. On account of the equality of phase of F and F^0 at $r=R$, one obtains by usual means

$$[(FdF^0/dr)-F^0dF/dr]_{r=b}=(k_0^2-k^2)\int_b^R FF^0dr, \tag{5.1}$$

where b is a distance such that $V'=0$ whenever $r>b$. Increasing R/b to ∞ subordinates the effect of the energy difference $E-E_0$ on the left side of Eq. (5.1) in comparison with the effect of V' at a fixed energy. In fact, for a fixed value of b the effect of $E-E_0$ can be made as small as desired, while the effect of V' approaches a finite limit. Since for a vanishing value of $E-E_0$ the left side of (5.1) is the invariant wronskian, it may be evaluated by making b so large that the asymptotic forms of F and F^0 apply. There results

$$\int_b^R FF^0dr=[k/(k_0^2-k^2)]\sin(\delta-\delta^0). \tag{5.2}$$

On the left side, one may replace the lower limit of integration by 0 because $b\ll R$. Thus, in the limit of $b/R=0$, one has

$$\int_0^R FF^0dr\cong[1/2(k_0-k)]\sin(\delta-\delta^0), \tag{5.2'}$$

which approaches ∞. Introducing this value in the denominator of Eq. (4) one has

$$\int_0^\infty FF^0V'dr=-(dE/2dk)\sin(\delta-\delta^0). \tag{5.3}$$

Since

$$dE/E=2dk/k, \tag{5.4}$$

there results

$$\int_0^\infty FF^0V'dr=-(E/k)\sin(\delta-\delta^0), \tag{5.5}$$

which is Eq. (5). It is seen, therefore, that the general relation for energies of stationary states implies also the Ferretti-Hulthén form for the phase shift. The sine of the phase shift difference is seen to enter in the normalization integral as in Eq. (5.2).

While both of these relations show the symmetry of the perturbed and unperturbed states and indicate very clearly the relationship of first-order perturbation theory results to the exact values, they have not proved to be of great practical value because the knowledge of the perturbed wave function is usually not available.

The connection between Eq. (5) and Eq. (2.1) is seen to be based on the possibility of the simple approximation made in Eq. (5.2'). It will be noted that Eq. (2) and its consequence Eq. (2.1) are applicable also in the many-body problem, suggesting that Eq. (5) can be generalized so as to apply to the collision of two nuclei rather than two nucleons. In this case, quantization of the whole system restricting the dis-

[7] B. Ferretti, Nuovo cimento **1**, 30 (1943); L. Hulthén, Arkiv Mat. Astron. Fysik **29A**, 14 (1943). Other related material by Hulthén is referred to in a subsequent footnote. This relation has been employed for checking purposes by W. H. Ramsey, Proc. Camb. Phil. Soc. **44**, 87 (1947).

tance between the centers of mass of the colliding aggregates of particles to lie in the interval $0<r<R$ gives results which are analogous to Eqs. (5.2), (5.2′). There results then a generalization of the formula of Hulthén to the many-body problem.

II. THE PROPERTIES OF THE f FUNCTION FOR $L=0$

In the discussion of the scattering of protons by protons and of neutrons by protons, it has proved to be useful to deal with a function of the phase shift which will be denoted here by f in the notation of Breit, Condon, and Present.[8] This function has been invented because some of the preliminary experiments of Tuve, Heydenburg, and Hafstad[9] were in disagreement with the expected decrease of the logarithmic derivative of the wave function with energy and indicated in a sense a negative range of force. It was important, therefore, to arrange calculations in such a way as to see clearly the influence of the assumed range of force. Since the logarithmic derivative of the function usually denoted by F can be expanded as a power series in the energy at the boundary of the nuclear potential energy well, the expression for this logarithmic derivative in terms of the phase shift was arranged in terms of power series in the energy, which also contained the range of force as a parameter. This relation is

$$\left[\frac{rd\mathfrak{F}_i}{\mathfrak{F}_i dr}\right]_{r=b}=\left\{\left(\frac{r}{a}\right)\frac{X_0+[f+2\ln(2r/a)]\Phi_0^*}{\Psi_0+(r/a)[f+2\ln(2r/a)]\Phi_0}\right\}_{r=b}, \quad (6)$$

where Φ_0, Φ_0^*, Ψ_0, X_0 are power series in the energy which arise in the evaluation of coulomb functions and their derivatives. The quantity a is the Bohr collision length

$$a=\hbar^2/\mu e^2 \quad (6.1)$$

and

$$f=(C_0^2/\eta)\cot K_0-2\ln\eta+q_0/\eta, \quad (6.2)$$

where

$$q_0/2\eta=2\gamma-1+\text{R.P.}[\Gamma'(i\eta)/\Gamma(i\eta)],$$
$$\gamma=0.5772\cdots=\text{Euler's constant}, \quad (6.3)$$

so that

$$\frac{q_0}{2\eta}=\gamma-\frac{1}{1+\eta^2}+\sum_2^\infty\frac{\eta^2}{s(s^2+\eta^2)} \quad (6.3')$$

and

$$C_0^2/\eta=2\pi/(e^{2\pi\eta}-1), \quad (6.4)$$

while

$$\eta=e^2/\hbar v, \quad 1/\eta^2=40.01E_{\text{Mev}}. \quad (6.5)$$

The series for Φ_0, Φ_0^*, X_0, Ψ_0 may be found in the original papers. For purposes of orientation a few terms are listed below:

$$\begin{aligned}X_0&=2-(4+\eta^{-2})(r/a)-4(r/a)^2+\cdots,\\ \Phi_0&=1+(r/a)+[(1/3)-(1/6\eta^2)](r/a)^2+\cdots,\\ \Phi_0^*&=1+2(r/a)+[1-(1/2\eta^2)](r/a)^2+\cdots,\\ \Psi_0&=1-[3+(1/2\eta^2)](r/a)^2+\cdots.\end{aligned} \quad (6.6)$$

The arrangement of terms in Eq. (6) is such as to remove the logarithmic term in energy and distance from everything but the combination $f+2\ln(2r/a)$. Since the solution of Eq. (6) for f is obviously a power series in the energy and since for a potential of zero range the limiting form of Eq. (6) expresses f entirely in terms of constants, one has

$$f=\text{const} \quad (b=0). \quad (6.7)$$

In this case Eq. (7.9) of Breit, Condon, and Present is applicable; and it becomes, on clearing fractions,

$$f=f',$$

where f' is the value of f evaluated at a standard energy for which K_0 has a preassigned value. The discussion just referred to applies to potential energy curves of any shape. It gives[10]

$$\left[f+2\ln\left(\frac{r}{2-a}\right)-\frac{X_0-(a/r)Y\Psi_0}{Y\Phi_0-\Phi_0^*}\right]_{r=b}=0, \quad (7)$$

where

$$Y=rd\mathfrak{F}/\mathfrak{F}dr. \quad (7')$$

As long as Y may be expanded in powers of the energy, this relation gives for f an expression which is the ratio of two power series. The quantities X_0, Ψ_0, Φ_0, Φ_0^* have their origin in hypergeometric functions; and the series converge for all energies.

The convenience of dealing with the function f has been brought out later again by Landau and Smorodinsky.[6] The lack of agreement with zero range of force which is discussed by them was to be expected from Eq. (7.5) of BCP; and a linear variation of f was, in fact, to be expected as a consequence of the entrance of $1/\eta^2$ in X_0, Φ_0 together with r/a. The subject has been reopened by Schwinger in lectures at Harvard, a hectographed form of which has been circulated. This treatment is said to form a justification for expecting a linear variation of f with energy, and a natural delight has been taken by some in the employment of the variational method. It is nevertheless unlikely that one can produce anything but a complication by solving a problem by approximate methods if the problem admits of a simple exact solution. It is to be expected, however, that the existence of a variational equation can serve as a valuable guide in indicating the existence of relations (identities) which can produce essential simplifications in otherwise complicated formulas. The main logical point involved here is that, if a function of

[8] Breit, Condon, and Present, Phys. Rev. **50**, 825 (1936).
[9] Tuve, Heydenberg and Hafstad, Phys. Rev. **50**, 806 (1936).
[10] G. Breit and W. G. Bouricius, Phys. Rev. **75**, 1029 (1948).

the phase shift is "stationary" for the correct wave function, then an error in the wave function produces an error in the phase shift of a higher order. Therefore, if one has an expansion of the wave function in powers of a parameter, then one may be sure of the existence of identities involving the functions which multiply the different powers, for otherwise the calculation of the wave function would have to be carried out to the same order as that of the phase shift. The way in which this situation works out will be illustrated presently in some applications, but even without a concrete working out of the scheme one sees that such simplifications depend on the *existence* of a variational equation *rather than* on its *form*.

The slightly critical remarks concerning the application of variational techniques to the solution of one-dimensional problems are not meant to include such obviously useful matters as the construction of convenient analytic approximations[11] to wave functions for one-dimensional problems or the application of the variational phase shift methods to the solution of problems involving more than one dimension. That some special results obtained for the two-body problem by the variational method can be obtained by other means has been pointed out by H. A. Bethe,[2] by Peierls and Preston,[1] by Chew and Goldberger,[3] and by the writer.[5] The presentation of the more general situation which is given below is based on work done in collaboration with Mr. M. C. Yovits which is being prepared for publication.

The considerations are concerned with properties of the function frequently called u, which may be defined by

$$u=C_0[G_0+F_0\cot K_0]=C_0\mathfrak{F}_0/\sin K_0. \tag{8}$$

It satisfies the differential equation

$$(d^2u/dr^2)+[\kappa+\lambda_1\varphi_1(r)+\cdots]u=0, \tag{8.1}$$

where

$$\kappa=2\mu E/\hbar^2=k^2, \tag{8.2}$$

while λ_1, λ_2 are parameters multiplying various parts of expressions tried for the potential energy. Here E represents again the energy of relative motion. One has in general[8]

$$\cot K_0=[1/(F_0^2\Delta)]-G_0/F_0, \tag{8.3}$$

$$\Delta=(dF_0/F_0kdr)-d\mathfrak{F}_0/\mathfrak{F}_0kdr, \tag{8.3'}$$

the evaluation of all quantities being made for any $r\geqq b$. Equation (8.3) is a direct consequence of requirements of continuity of the wave function and its derivative. At a fixed energy one can obtain the same K_0 by employing a large variety of potentials. For all of them Δ is the same. The rate of change of Δ with κ may be different, however, for two potentials chosen at random; and comparison of Eq. (8.3) with a similar relation for another potential gives

$$(\partial/\partial\kappa)[\cot K_0-\cot K_0']=(1/F_0^2)(\partial/\partial\kappa)[(1/\Delta)-1/\Delta'], \tag{8.4}$$

the relation being applicable only at the energy for which

$$K_0=K_0'. \tag{8.5}$$

The right side of Eq. (8.4) simplifies, on account of $\Delta=\Delta'$, to the form,

$$F_0^{-2}\Delta^{-2}(\partial/\partial\kappa)[(\partial\mathfrak{F}/k\mathfrak{F}dr)-\partial\mathfrak{F}'/k\mathfrak{F}'dr]=[1/(k\Delta^2F_0^2u^2)]\int_0^b(u'^2-u^2)dr. \tag{8.6}$$

On the other hand, one obtains

$$1/\Delta=F_0^2[(G_0/F_0)+\cot K_0]=uF_0/C_0,$$

and hence, by means of Eq. (8.6),

$$(\partial/\partial\kappa)[\cot K_0-\cot K_0']=(1/kC_0^2)\int_0^b(u'^2-u^2)dr, \tag{8.7}$$

which may[5] also be written as

$$(\partial/\partial\kappa)(f-f')=a\int_0^b(u'^2-u^2)dr. \tag{9}$$

This equation may be applied at any energy. Its employment is convenient not only as a means of determining the slope of the f curve, but also as a way of obtaining the coefficients of the power series expansion around any κ. It has been first made known in substantially the form of Eq. (9) for $E=0$ by Schwinger and has been used by Bethe, by Peierls and Preston, by Chew and Goldberger, and by Hatcher, Arfken, and Breit also at $E=0$. The discussion in the paper by Breit and Hatcher[5] makes explicit use of the applicability of the equation at any energy; and a method of working out coefficients of f in its power series expansion in E has been described in this paper, making use of the applicability of Eq. (9) at any energy. This method has proved to be somewhat laborious, however; and the relations in it are less transparent than those sketched subsequently.

The procedure to be described is somewhat related to some of the calculations of Jackson and Blatt,[12] especially in the explicit employment power series expansion of u in the energy. As a consequence of not employing the variational procedure except as a guide, the general term in the expansion can be written down in a compact form and the expansion can be seen to be

[11] L. Hulthén, Dixième Congrès des Mathématiciens Scandinaves, Copenhagen, 1946, p. 201. L. Hulthén, Kgl. Fysiograf. Sällskap. Lund Förh. **14**, 8 (1944); **14**, 21 (1944); **15**, 22 (1945). L. Hulthén, Arkiv Mat. Astron. Fysik **35a**, 1 (1948). C. E. Fröberg, Arkiv Mat. Astron. Fysik **34a**, 1 (1947); **35a**, 1 (1948); **36a**, 1 (1948). I. Hansson, Kgl. Fysiograf. Sällskab. Lund Förh. **18**, 1 (1948).

[12] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950).

applicable around any value of the energy. This fact is somewhat trivial from the mathematical point of view because the subdivision of $\kappa+\lambda_1\varphi_1+\lambda_2\varphi_2+\cdots$ into parts κ, $\lambda_1\varphi_1$, $\cdots$ is arbitrary and a change in κ by a constant can be compensated by a change of the other terms in this expression. The differential equation for u will be written therefore as

$$(d^2u/dr^2)+[\kappa-\kappa_0+\cdots]u=0, \tag{9.1}$$

so that

$$\kappa-\kappa_0=k^2-k_0^2; \tag{9.2}$$

and it will be supposed that it is possible to expand

$$u=\sum_0^\infty(\kappa-\kappa_0)^n u_n. \tag{9.3}$$

Denoting the operator on the left side of Eq. (9.1) by $\mathfrak{L}$, one has a chain of equations

$$\mathfrak{L}u_j+u_{j-1}=0, \tag{9.4}$$

where it is understood that $u_{-1}=0$. On the other hand, substitution of Eq. (9.3) into Eq. (9) yields

$$a\sum_n(\kappa-\kappa_0)^n\sum_{i+j=n}\int_0^b(u_i'u_j'-u_iu_j)dr = \sum_n n(\kappa-\kappa_0)^{n-1}[f^{(n)}-f'^{(n)}],$$

where it supposed that the power series expansions of f and f' are

$$f=\sum_n(\kappa-\kappa_0)^n f^{(n)}, \quad f'=\sum_n(\kappa-\kappa_0)^n f'^{(n)}. \tag{9.5}$$

When coefficients of equal powers of $\kappa-\kappa_0$ are identified, there results

$$n[f^{(n)}-f'^{(n)}]=a\sum_{i+j=n-1}\int_0^b(u_i'u_j'-u_iu_j)dr. \tag{9.6}$$

This relation enables one to determine the coefficients $f^{(n)}$ in terms of the known $f'^{(n)}$ and the functions u_i, u_i'. On the right side of this relation there occur inconveniently high values of j and i. The variational principle indicates, however, that if the function u is known to within the term $(\kappa-\kappa_0)^{n-1}u_{n-1}$, that is, if the error in the function is of the order $(\kappa-\kappa_0)^n$, then the error in f is no greater than of order $(\kappa-\kappa_0)^{2n}$, so that $f^{(2n-1)}$ is just determinable. The knowledge of u_0 alone thus suffices for the determination of $f^{(1)}$, that of u_1 for $f^{(3)}$, that of u_2 for $f^{(5)}$, etc. These facts have been well realized by Schwinger,[13] by Blatt,[14] and by Jackson and Blatt[12] and have also been made use of to some degree by direct substitution into the variational form for f. In Eq. (9.6) one has to have u_0 in order to determine $f^{(1)}$, u_1 to determine $f^{(2)}$, u_2 to determine $f^{(3)}$, etc. From $f^{(3)}$ on, Eq. (9.6) appears to require the knowledge of many more u_j than the existence of the variational principle indicates as necessary. There must exist, therefore, identities between the integrals on the right side of Eq. (9.6). This is, in fact, the case as may be seen from the following consideration. The chain of differential equation (9.4) gives the identity

$$\int_0^b(u_{t+1}\mathfrak{L}u_s+u_{t+1}u_{s-1})dr=0, \tag{9.7}$$

which becomes on partial integration

$$[u_{t+1}(du_s/dr)-u_s du_{t+1}/dr]_0^b + \int_0^b[u_s\mathfrak{L}u_{t+1}+u_{t+1}u_{s-1}]dr=0, \tag{9.7'}$$

and since $\mathfrak{L}u_{t+1}+u_t=0$, one has

$$[u_{t+1}(du_s/dr)-u_s du_{t+1}/dr]_0^b + \int_0^b[-u_su_t+u_{t+1}u_{s-1}]dr=0. \tag{9.8}$$

There is a similar equation for the functions of the comparison potential. Since Eq. (9.6) has been derived by differentiating Eq. (9), the comparison potential must be varied with energy so as to give the same K_0 and hence the same $u_j(b)$. The values of the square bracket outside the integral sign in Eq. (9.8) are therefore the same for the primed and unprimed u_j. All the unprimed u_j vanish at $r=0$, and the same is the case for a nonsingular comparison potential of finite range. For a comparison potential of zero range the function u_0' is finite and nonvanishing at $r=0$, while $u_1'=u_2'=\cdots=0$ at $r=0$. The function u_0' is not needed, however, for $u_{s-1}'=u_0'$ under the integral call for u_1' in the square bracket. The values of the square bracket are, therefore, the same for the two potentials. There results the identity

$$(s-1,\ t+1)=(s,\ t), \tag{10}$$

where

$$(s,\ t)=\int_0^b(u_s'u_t'-u_su_t)dr. \tag{10.1}$$

Returning to Eq. (9.6), one sees that the sum on the right consists of a number of equal terms. Each term occurs as many times as there are ways of partitioning the integer $n-1$ into two integers, counting zero as an integer, and considering $i+j$ as distinct from $j+i$ if $i\neq j$. This number of partitions is the number of terms in the sequence $0, 1, \cdots, n-1$ and is therefore n. One has thus

$$f^{(n)}-f'^{(n)}=a(s,\ s), \qquad (n=2s+1), \tag{10.2}$$

$$f^{(n)}-f'^{(n)}=a(s-1,\ s), \quad (n=2s). \tag{10.3}$$

These coefficients in the power series expansion are seen to be of very simple form when expressed in terms of the u_s, u_s'.

[13] J. Schwinger, Phys. Rev. **72**, 724 (1947).
[14] J. M. Blatt, Phys. Rev. **74**, 92 (1948).

In a similar way one can expand f in terms of the parameters $\lambda_1, \lambda_2, \cdots$ which enter Eq. (8.1). In order to simplify the notation the subscripts i will be omitted in the λ_i and φ_i. Differentiating Eq. (8.3) with respect to a λ and employing Eq. (8), one finds

$$(\partial/\partial\lambda)[k\cot K_0]=-\int_0^b (\mathfrak{F}^2/\sin^2 K)\varphi dr, \quad (11)$$

which is equivalent to

$$\partial f/\partial\lambda=-a\int_0^b \varphi u^2 dr. \quad (11.1)$$

The functions u, f will now be expanded as

$$u=u_0+\lambda u_{1\lambda}+\lambda^2 u_{2\lambda}+\cdots, \quad (11.2)$$

$$f=f_0+\lambda f_{1\lambda}+\lambda^2 f_{2\lambda}+\cdots. \quad (11.3)$$

There are the relations

$$\mathfrak{L}u_{(j+1)\lambda}+\varphi u_{j\lambda}=0, \quad (11.4)$$

which are similar to Eq. (9.4). One has, similarly to Eq. (9.8),

$$[u_{t\lambda}(du_{s\lambda}/dr)-u_{s\lambda}du_{t\lambda}/dr]_0^b + \int_0^b [-u_{(t-1)\lambda}u_{s\lambda}+u_{t\lambda}u_{(s-1)\lambda}]\varphi dr=0. \quad (11.5)$$

The square brakets outside the integral are zero. This fact may be seen as follows. According to Eq. (8), one may obtain u at $r=b$ in terms of F_0 and G_0 and K_0. Expressing K_0 in terms of f, one has

$$u(r)=A(r)+B(r)f \quad (r\geqq b), \quad (11.6)$$

where $A(r)$, $B(r)$ are functions whose precise form does not matter for the present discussion. The only essential fact is that K_0 does not enter $A(r)$ and $B(r)$. From Eq. (11.6) it follows that

$$u_{t\lambda}(du_{s\lambda}/dr)-u_{s\lambda}du_{t\lambda}/dr=0 \quad (r=b). \quad (11.7)$$

An exception to this argument is found for t or s having the value zero. This case does not arise, however, in the needed applications of Eq. (11.5), since it corresponds to $s-1$ or $t-1$ having the value -1. At $r=0$ one has $u=0$ according to the boundary condition and hence all $u_{s\lambda}(0)=0$. There results from Eq. (11.5)

$$\int_0^b u_{(t-1)\lambda}u_{s\lambda}\varphi dr=\int_0^b u_{t\lambda}u_{(s-1)\lambda}\varphi dr. \quad (12)$$

Introducing

$$[s, t]_\lambda=\int_0^b u_{s\lambda}u_{t\lambda}\varphi dr, \quad (12.1)$$

one has on account of Eq. (11.1)

$$\int_0^\infty \varphi u^2 dr=\sum_0^\infty \lambda^n \sum_{s+t=n}[s, t]_\lambda = -(1/a)\sum n\lambda^{n-1}f_{n\lambda}, \quad (12.2)$$

so that

$$nf_{n\lambda}=-a\sum_{s+t=n-1}[s, t]_\lambda. \quad (12.3)$$

According to Eq. (12), there are just n equal terms on the right side of Eq. (12.3) and hence

$$f_{n\lambda}=-a[s, s]_\lambda, \quad (n=2s+1), \quad (12.4)$$

$$f_{n\lambda}=-a[s-1, s]_\lambda, \quad (n=2s). \quad (12.5)$$

The coefficients of powers of λ in Eq. (11.3) are thus obtainable by Eqs. (12.4), (12.5) with the abbreviation of Eq. (12.1). The solution presupposes that one has available the $u_{s\lambda}$, but the number of these that is required is about half the number of the $f_{n\lambda}$.

The expansions of Eq. (9.5) may be rewritten in a form in which less care has to be used regarding the units in which the parameter κ is expressed:

$$\frac{[f-f^{(0)}-(f'-f'^{(0)})]}{[a(\kappa-\kappa_0)]}=\sum_0^\infty \int_0^b [v_s'^2-v_s^2]dr + \sum_0^\infty \int_0^b [v_s'v_{s+1}'-v_sv_{s+1}]dr, \quad (12.6)$$

where

$$u=\sum_0^\infty v_n, \quad v_0=u_0, \quad u'=\sum_0^\infty v_n', \quad v_0'=u_0', \quad (12.7)$$

it being understood that the successive terms in the expansions of u in Eq. (12.7) are the same as in Eq. (9.3). For a zero range potential, $f'-f'^{(0)}$ drops out and the formula gives $f-f^{(0)}$ directly. In this form the successive terms in the expansion of f contain integrals which are expressed directly in terms of the successive terms in the expansion of u. The denominator on the left side of Eq. (12.6) is

$$a(\kappa-\kappa_0)=(2E/e^2)[1-(\kappa_0/\kappa)], \quad (12.8)$$

E = energy of relative motion.

If one wishes to express r in units a in the integrals of Eq. (12.6), the factor $2E/e^2$ becomes replaced by $1/\eta^2$. Similar convenience regarding units can be obtained in the expansions with respect to a general linear parameter by introducing $\lambda^n u_{n\lambda}$ in one combination. They were kept separate as a matter of convenience in the presentation.

The power series described here can be expanded to cases of simultaneous entrance of several parameters without bringing in essentially new elements. They are somewhat more readily adaptable to systematic treatment than the method of expanding the logarithmic derivative of a function in powers of parameters. This method has been used by Breit, Thaxton, and Eisen-

bud.[15] It could be used to construct the functions u_κ, $u_{s\lambda}$ of the present paper in terms of iterated integrals. Circumspection has to be used, however, when one goes through a node of the wave function as described by Bloch, Hull, Broyles, Bouricius, Freeman, and Breit.[16] The difference between the method used by BTE and that employed here is primarily one of arrangement of calculation, aside from expressing the results in terms of the f function rather than the phase shift.

III. GENERALIZATION TO $L>0$

The following formulas of Yost, Wheeler, and Breit and of subsequent papers[17,8,15,10,16] are useful for the present purpose:

$$F_L=C_L\rho^{L+1}\Phi_L,\quad G_L=D_L\rho^{-L}\Theta_L,\quad (2L+1)C_LD_L=1, \tag{13}$$

$$C_L^2=\frac{2^{2L}}{[(2L+1)!]^2}[L^2+\eta^2][(L-1)^2+\eta^2]\cdots[1^2+\eta^2]\left[\frac{2\pi\eta}{(e^{2\pi\eta}-1)}\right], \tag{13.1}$$

$$\Theta_L=\Psi_L+\rho^{2L+1}(p_L\ln 2\rho+q_L)\Phi_L, \tag{13.2}$$

$$\frac{p_L}{2\eta}=\frac{2^{2L}}{(2L)!(2L+1)!}[L^2+\eta^2]\times[(L-1)^2+\eta^2]\cdots[1^2+\eta^2], \tag{13.3}$$

$$q_L=p_L\left[\sum_1^L\frac{s}{s^2+\eta^2}-\sum_1^{2L+1}\frac{1}{s}+2\gamma+\text{R.P.}\frac{\Gamma'(i\eta)}{\Gamma(i\eta)}\right]+\check{q}_L, \tag{13.4}$$

$$\check{q}_L=\text{I.P.}(-)^{L+1}\frac{2^L}{(2L)!}\left\{\frac{2^{-L}}{2L+1}+\frac{2^{1-L}(i\eta-L)}{1!(2L)}+\cdots+\frac{2^{s+1-L}(i\eta-L)(i\eta-L+1)\cdots(i\eta-L+s)}{(s+1)!(2L-s)}+\cdots+\frac{2^L(i\eta-L)\cdots(i\eta+L-1)}{(2L)!1}\right\}. \tag{13.5}$$

In addition to these formulas giving the coulomb functions, use will be made also of power series representations of the derivatives

$$dF_L/d\rho=C_L\rho^L\Phi_L^*,$$
$$dG_L/d\rho=D_L\rho^{-L-1}[\Psi_L^*+\rho^{2L+1}(p_L\ln 2\rho+q_L)\Phi_L^*+\rho^{2L+1}p_L\Phi_L] \tag{13.6}$$

and of the wronskian relation on the Φ_L, Φ_L^*, Ψ_L, Ψ_L^*:

$$(2L+1)[(G_LdF_L/d\rho)-F_LdG_L/d\rho]=\Phi_L^*\Psi_L-\Psi_L^*\Phi_L-\rho^{2L+1}p_L\Phi_L^2=2L+1. \tag{13.7}$$

It is desirable to note the relations

$$\Phi_L^*=\rho^{-L}(d/d\rho)(\rho^{L+1}\Phi_L),\qquad \Psi_L^*=\rho^{L+1}(d/d\rho)(\rho^{-L}\Psi_L),$$

and the facts that the series for Φ_L, Ψ_L start with 1 and that the series for Φ_L^* starts with $L+1$, while the series for Ψ_L^* starts with $-L$. Both Ψ_L and Ψ_L^* are defined so as to have a vanishing coefficient of ρ^{2L+1} in their contribution to Θ_L, Θ_L^*. The symbol $\mathfrak{F}_L$ will be used in the sense of a real function with phase shift so that

$$\mathfrak{F}_L=F_L\cos K_L+G_L\sin K_L;\quad (r>b). \tag{14}$$

The logarithmic derivative has a deficiency

$$\Delta_L=(\partial F_L/F_L\partial\rho)-\partial\mathfrak{F}_L/\mathfrak{F}_L\partial\rho,$$

in terms of which one obtains from the wronskian of F_L, $\mathfrak{F}_L$ the relation

$$\cot K_L=[1/(F_L^2\Delta_L)]-G_L/F_L. \tag{14.1}$$

By means of this formula one obtains

$$-\frac{r\partial\mathfrak{F}_L}{\mathfrak{F}_L\partial r}=\frac{\rho/F_L^2}{(G_L/F_L)+\cot K_L}-\frac{\Phi_L^*}{\Phi_L}, \tag{14.2}$$

and by means of Eq. (13.6) this relation can be rearranged to read

$$\frac{r\partial\mathfrak{F}_L}{\mathfrak{F}_L\partial r}=\frac{(\Psi_L^*/\Phi_L)+y^{2L+1}p_L\eta^{-2L-1}+(y^{2L+1}\Phi_L^*/\Phi_L)[f_L+p_L\eta^{-2L-1}\ln 2y]}{(\Psi_L/\Phi_L)+^{2L+1}[f_L+p_L\eta^{-2L-1}\ln 2y]}, \tag{14.3}$$

where

$$y=r/a, \tag{14.4}$$

and

$$f_L=[(2L+1)C_L^2\cot K_L-p_L\ln\eta+q_L]/\eta^{2L+1}. \tag{14.5}$$

A formula related to the above expression for f_L appears in a paper by Landau and Smorodinsky.[6] There are some differences, however; and since the paper just referred to does not arrive at correct results even for $L=0$, it is difficult to discuss these differences. Equation (14.3) shows that f_L is expressible in terms of $r\partial\mathfrak{F}_L/\mathfrak{F}_L\partial r$ and the quantities Ψ_L, Φ_L, Ψ_L^*, Φ_L^*, which

[15] Breit, Thaxton, and Eisenbud, Phys. Rev. **55**, 1018 (1939); the unnumbered equation $y(0)=1$ on p. 1046 of this paper was meant to be $xy(0)=1$.

[16] Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, Phys. Rev. **80**, 553 (1950).

[17] Yost, Wheeler and Breit, Phys. Rev. **49**, 174 (1936); G. Breit and M. H. Hull, Jr., Phys. Rev. **80**, 392 (1950); **80**, 561 (1950).

may be arranged as power series in the energy. Disregarding questions of convergence, one may express f_L, therefore, as a power series in $1/\eta^2$ by expanding

$$f_L+p_L\eta^{-2L-1}\ln 2y=y^{-2L-1}\frac{\Psi_L^*+y^{2L+1}p_L\eta^{-2L-1}\Phi_L-\Psi_L(r\partial\mathfrak{F}_L/\mathfrak{F}_L\partial r)}{\Phi_L(r\partial\mathfrak{F}_L/\mathfrak{F}_L\partial r)-\Phi_L^*}. \tag{14.6}$$

It will be noted that, according to Eq. (13.3), the quantity $p_L\eta^{-2L-1}$ is a polynomial in $1/\eta^2$ and that the quantities Φ_L, Φ_L^*, Ψ_L, Ψ_L^* have been introduced by means of Eqs. (13) through Eq. (13.6) in such a way that they are power series in $1/\eta^2$. For $E=0$ the quantities p_L/η^{2L+1}, Φ_L, Φ_L^*, Ψ_L, Ψ_L^*, and $r\partial\mathfrak{F}_L/\mathfrak{F}_L\partial r$ approach finite limits. For a finite y, therefore, the right side of Eq. (14.6) is finite. While the convergence of the series expressing these quantities as power series in ρ is a standard result of the theory of hypergeometric functions, the convergence of the series rearranged as power series in $1/\eta^2$ does not appear to be covered by standard treatments. In addition, the quantity $r\partial\mathfrak{F}_L/\mathfrak{F}_L\partial r$ cannot be expressed as a power series converging at all energies because $r\partial\mathfrak{F}_L/\mathfrak{F}_L\partial r$ becomes infinite when $\mathfrak{F}_L=0$. When this is the case, the right side of Eq. (14.6) approaches $-y^{-2L-1}\Psi_L/\Phi_L$; and this quantity has poles at the roots of Φ_L. At most, therefore, one can expect a finite radius of convergence for the right side of Eq. (14.6); and simple examples verify this expectation. The logarithmic terms in η enter only in f_L. For large η, i.e., low energies, the asymptotic expansion of $\Gamma'(i\eta)/\Gamma(i\eta)$ which enters q_L cancels the term $-p_L\ln\eta$ in accordance with Eq. (13.4) and Stirling's series:

$$\text{R.P. }[\Gamma'(i\eta)/\Gamma(i\eta)]=\ln\eta+(\eta^{-2}/12)+(\eta^{-4}/120)+\cdots. \tag{14.7}$$

The removal of $\ln\eta$ occurs asymptotically only.

A generalization of Eq. (9) to $L>0$ is also possible. For this purpose it is convenient to introduce the function

$$u_L=C_L\mathfrak{F}_L/\sin K_L, \tag{15}$$

which will be used in conjunction with a function u_L' similarly defined for a comparison potential which is adjusted so as to reproduce K_L at some specified energy. Quantities referring to the comparison potential will be primed below. It follows from Eq. (14.1) that

$$[\partial/\partial(k^2)][\cot K_L-\cot K_L']=F_L^{-2}[\partial/\partial(k^2)][(1/\Delta_L)-1/\Delta_L'] \tag{15.1}$$

and at the comparison energy

$$[\partial/\partial(k^2)][(1/\Delta_L)-1/\Delta_L']=\Delta_L^{-2}[\partial/\partial(k^2)][(\partial\mathfrak{F}_L/\mathfrak{F}_L\partial\rho)-\partial\mathfrak{F}'/\mathfrak{F}'\partial\rho]. \tag{15.2}$$

One also obtains, in the standard manner, from the differential equation satisfied by u_L the formula

$$u_L^2[\partial/\partial(k^2)](\partial\mathfrak{F}_L/\mathfrak{F}_L\partial r)\,|_r^b+\int_r^b u_L^2dr=0. \tag{15.3}$$

This formula cannot be applied to the treatment of a potential having zero range because for $L>0$ the integral becomes infinite if the lower limit is made to approach zero. For a nonsingular potential, which, therefore, has a non-zero range, u_L is regular at $r=0$. For two such potentials, one has for the corresponding functions, adjusted so as to have $u_L(b)=u_L'(b)$:

$$[\partial/\partial(k^2)][(\partial\mathfrak{F}_L/\mathfrak{F}_L\partial r)-\partial\mathfrak{F}_L'/\mathfrak{F}_L'\partial r]=u_L^{-2}(b)\int_0^b(u_L'^2-u_L^2)dr; \tag{15.4}$$

and hence, combining Eq. (15.4) with Eqs. (15), (15.1), (15.2), one obtains

$$[\partial/\partial(k^2)](f_L-f_L')=(2L+1)a\eta^{-2L}\int_0^b(u_L'^2-u_L^2)dr. \tag{15.5}$$

This relation is analogous to Eq. (9), which applies to $L=0$. For $L>0$ one cannot use Eq. (15.5) with a comparison potential having zero range because u_L' behaves as r^{-2L}. It may be possible to replace $\int_0^b u_L'^2dr$ by

$$-u_L'^2(r)[\partial/\partial(k^2)][\partial\mathfrak{F}_L'(r)/\mathfrak{F}_L'(r)\partial r]+\int_r^b u_L'^2dr \tag{15.6}$$

in this case and then go to the limit of $r=0$ with a suitable consideration of the behavior of $\partial/\partial(k^2)$ occurring in Eq. (15.6). The writer does not know of the existence of an investigation of this point. It appears unlikely that one can make use of a zero range potential, because one could apply Eq. (15.4) in this case with the understanding that the potential has a finite range which is being made very small and that u' starts out regularly at $r=0$ within the potential well. The contribution to $\int_0^b u_L'^2dr$ arising from the interior of the potential well is then positive, and it cannot cancel the infinite contribution arising from values of r outside the potential well of the comparison potential. The right side of Eq. (15.5) becomes plus infinity, which means that $\partial f_L'/\partial E$ is minus infinity.

The difference between $L=0$ and $L>0$ may also be seen from Eq. (14.3). The coefficients of ρ^n in Ψ_L^* vanish for $n=L$ and $n=2L+1$. If $L=0$, the first nonvanishing power of ρ in Ψ_L^* is ρ^2 and the numerator of right side of Eq. (14.3) vanishes for $r\to 0$. The effect of a finite f is to produce a finite effect on $\partial\mathfrak{F}_0/\mathfrak{F}_0\partial r$ just outside the very narrow potential well before one passes to the limit of zero range. But for $L>0$ the dominant

term in the numerator is Ψ_L^*/Φ_L, and a finite f_L has a vanishing effect on $\partial\mathfrak{F}_L/\mathfrak{F}_L\partial r$ at the boundary of the narrow potential well. The value of f_L is thus very sensitive to $\partial\mathfrak{F}_L/\mathfrak{F}_L\partial r$, as is also apparent from Eq. (14.6). Here one sees that for $L=0$ the term $-\Psi_0(r\partial\mathfrak{F}_0/\mathfrak{F}_0\partial r)$ is dominant in the numerator and that energy rate of change of this term is then proportional to $ru_0^{-2}\int_0^r u_0^2dr$; i.e., to r^2. The effect on f is then proportional to r. For $L>0$ there is a direct energy dependence of Ψ_L^*. Besides the effect of the potential well is to produce an energy rate of change of f_L caused by $-\Psi_L(r\partial\mathfrak{F}_L/\mathfrak{F}_L\partial r)$ proportional to $r^2/r^{2L+1}=r^{1-2L}$.

It will be seen presently that u_L/η^L approaches a finite limit as $E\to 0$; and it will therefore be apparent that Eq. (15.5) gives, in general, a finite difference in the slopes of the f_L *versus* E curves at $E=0$. One may use Eq. (15.5), therefore, for the calculation of such differences. The expansions of f which have been carried out for $L=0$ are applicable in the case $L>0$ as well. It will be noted that the comparison potential has been left arbitrary in these expansions and that no specialization to potentials of zero range has been made. It follows from Eqs. (13), (14), (15) that

$$u_L/\eta^L=\{\Theta_L/[(2L+1)(r/a)^L]\}+C_L^2\eta^{-2L-1}(r/a)^{L+1}\Phi_L\cot K_L. \quad (16)$$

In this formula the quantity C_L^2/η^{2L+1} approaches zero as $\eta\to\infty$ and $E\to 0$, but $\cot K_L$ approaches ∞. Substitution of the value of $\cot K_L$ in terms of f_L gives

$$u_L/\eta^L=\{\Theta_L/[(2L+1)(r/a)^L]\}+(r/a)^{L+1}\Phi_L[f_L+(p_L\ln\eta-q_L)/\eta^{2L+1}]/(2L+1). \quad (16.1)$$

Since $(p_L\ln\eta-q_L)/\eta^{2L+1}$ and f_L approach finite limits as $E\to 0$, this form of u_L/η^L may be used at any energy. For sufficiently low energies one may use the asymptotic forms of Φ_L and Θ_L in terms of bessel functions of argument ix with

$$x=(8r/a)^{\frac{1}{2}}=(8\rho\eta)^{\frac{1}{2}}. \quad (16.2)$$

These expansions give

$$\Phi_L=\Xi_L+\frac{1}{\eta^2}\left\{\left[\frac{(x/2)^6}{12(\nu+1)}-\frac{\nu-1}{24}\left(\frac{x}{2}\right)^4\right]\frac{\Xi_{L+1}}{(\nu+1)(\nu+2)}-\frac{(x/2)^4}{12(\nu+1)}\Xi_L\right\}$$
$$+\frac{1}{24\eta^4}\left\{\left[\frac{(x/2)^8}{5(\nu+1)}+\frac{(\nu+2)(\nu+3)(7-5\nu)}{120(\nu+1)}\left(\frac{x}{2}\right)^6-\frac{(\nu+2)(\nu+3)(7-5\nu)(\nu-1)}{240}\left(\frac{x}{2}\right)^4\right]\frac{\Xi_{L+1}}{(\nu+1)(\nu+2)}$$
$$+\left[-\frac{7-5\nu}{60(\nu+1)}\left(\frac{x}{2}\right)^6+\frac{(\nu-1)(\nu+3)(7-5\nu)}{240(\nu+1)}\left(\frac{x}{2}\right)^4\right]\Xi_L\right\}+\cdots, \quad (16.3)$$

where $\nu=2L+1$,

$$\Xi_L=(2L+1)!(x/2)^{-2L-1}I_{2L+1}(x), \quad (16.4)$$

and $I_\nu(x)$ is the usual function of the first kind of order ν of imaginary argument ix, for the bessel equation. The function Θ_L may be obtained[18] from the formula for Φ_L, as

$$[\Theta_L]_{\text{symb}}=-4(2L+1)(C_L^2/C_0^2)\eta^{-2L}\times(r/a)^{2L+1}[\Phi_L]_{I\to K} \quad (16.5)$$

with the understanding that the last factor in Eq. (16.5) is obtained by replacing the $I_\nu(x)$ by the corresponding $K_\nu(x)$. The $K_\nu(x)$ are bessel functions of imaginary argument of the second kind defined in the manner of Whittaker and Watson[19] rather than that of Watson.[20] The advantage of this definition is that the recurrence formulas for the K_ν have the form of those for the I_ν and that the simple substitution indicated in Eq. (16.5) is possible. One finds in this manner

$$u_L/\eta^L=[(r/a)^{L+1}/(2L+1)]\times\{\Phi_L[f_L+(p_L\ln\eta-q_L)/\eta^{2L+1}]-2(p_L/\eta^{2L+1})[\Phi_L]_{I\to K}\}. \quad (16.6)$$

All of the forms of u_L/η^L from Eq. (16) on are applicable only for values of r greater than the largest r at which the potential is non-coulombian.

It is seen that the method of the f function is applicable to $L>0$ and that the expansions of f in powers of the energy and of parameters entering linearly in the expression for the potential energy are also applicable. The method can thus be applied to the calculation of phase shifts. Since the value of $\mathfrak{F}_L$ at the nuclear boundary is related to K_L by Eq. (15), it appears desirable to consider also the relationship of $\mathfrak{F}_L$ to f_L. One finds by a straightforward calculation that

$$|\mathfrak{F}_L|^2=\frac{[(2L+1)/\eta][(2L+1)C_L^2/\eta^{2L+1}](u_L/\eta^L)^2}{[f_L+(p_L\ln\eta-q_L)/\eta^{2L+1}]^2+[(2L+1)C_L^2/\eta^{2L+1}]^2}, \quad (17)$$

[18] Sixth reference of footnote 17.
[19] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, London, 1920), third edition, Chapter XVII.
[20] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, London, 1922), first edition Chapter III.

where the quantities entering the formula have been grouped in such a way as to give a simple behavior at low energies which is as follows:

$$(2L+1)C_L^2/\eta^{2L+1}\sim 2^{2L+1}(2L+1)\pi e^{-2\pi\eta}/[(2L+1)!]^2,$$

$$f_L+(p_L\ln\eta-q_L)/\eta^{2L+1}\sim\text{const},$$

$$u_L/\eta^L\sim\text{const}.$$

The function $|\mathfrak{F}_L|^2$ thus behaves as $ve^{-2\pi\eta}$ at low energies. Aisde from factors varying slowly with the energy, this dependence is independent of L, in agreement with the fact that at low energies the collision cross section for a reaction is expected[21] to vary as

$$v^{-2}e^{-2\pi\eta}$$

independently of the value of L. It is seen from Eq. (17) that the factor $\exp(-2\pi\eta)$ enters not only the numerator but also one of the two terms in the denominator in the formula for $|\mathfrak{F}_L|^2$. The value of $|\mathfrak{F}_L|^2$ may be expected, therefore, to increase somewhat more slowly with the energy than one would expect from the numerator alone.

IV. COMPARISON OF THE METHOD OF THE f FUNCTION WITH OTHER METHODS

It has been pointed out by Schwinger in some lectures at Harvard that a plot of experimental values of f can be valuable in determining the range and depth parameters of potential wells which are commonly used as models for nucleon interactions in nuclear physics. This method has been made use of by Blatt and Jackson[12] for the presentation of experimental material and for the determination of nuclear interaction parameters from scattering experiments. In the earlier work only part of the simplification produced by f function has been made use of. This consisted in plotting $(\eta/C_0^2)\tan K_0$, a quantity which is the reciprocal of the phase shift dependent part of f, against energy as has been done by Breit, Thaxton, and Eisenbud. These plots showed that the differences between the expected behavior of the phase shift for different assumed potentials of different shapes were minor. The families of curves for potentials of different shapes are the same to a good approximation. The advantage of plotting $(\eta/C_0^2)\tan K_0$ rather than K_0 is that this feature is more obvious and also that the theoretically expected differences show up more clearly at low energies. The same advantages are present in a more pronounced manner in the plots of f against E.

In the earlier publications[22,15] the determination of nuclear parameters was presented by direct comparison of experimental phase shift with the theoretical curves and by comparing scattering yield for different models with experiment. The latter method is the most laborious of all. It has the advantage, however, of dealing as directly as possible with the experimental material and of bringing out the possible influence of systematic sources of errors on the interpretation. This procedure is capable of suggesting the importance of systematic errors which could be overlooked. The method of fitting data to a nuclear potential well by means of f function plots has advantages of relative ease and simplicity as well as of ease of presentation in publication. Its simplicity is perhaps more apparent than real. The following considerations matter.

(a) In the region of 300–400 kev the experimental errors appear very large on the f plots. Visual judgment concerning the goodness of one or another straight-line fit becomes difficult for this reason. In this respect the situation would be changed if the accuracy of observations in the 200- to 400-kev energy region were improved by a large factor. It is believed that a visual fit of a straight line to data is impaired if the statistical errors are much larger for one set of data than for another. One's eye compares the deviations of the straight line from the statistical limits of error with each other. It gives little information if some statistical errors exceed a certain limit in comparison with others and discounts information from such observations almost altogether. In this respect there are individual differences between people; and it is not desired, therefore, to stress this point too strongly. The fact is, however, that a (K_0, E) plot shows up the errors more nearly in proportion to the actual errors of observation; and an inspection of a plot of observation against theory does not produce a bias for considering some of the measurements as valueless and does not make one discard completely information contained in them.

(b) The visual fit is made less definite by the presence of a curvature in the theoretically expected curves. This curvature differs for different shapes of nuclear potential wells. One's eye has the task of having to determine the best fit to a set of points by an imaginary ruler with a curvature which is allowed to vary within certain limits. This is more difficult than trying for the best straight-line fit. Since the data on proton-proton scattering at energies of from about 800 kev to 4 Mev are more valuable for the determination of nuclear well parameters than data at lower energies, a visual straight-line fit has to do mainly with drawing a secant to the desired curve. Some indefiniteness is introduced by this circumstance. It is not quite so small as indicated by a casual inspection of fits to experiment. This circumstance may be seen by means of a simple argument brought out by Hatcher, Arfken, and Breit,[4] some of which appears to be worth repeating in a slightly different form. If two potentials such as the Yukawa and square well types are fitted to experimental material so as to agree with each other at two energies E_I, E_{II}, then the expansion coefficients are readily found to satisfy the relations

$$f_Y^{(0)}-f_S^{(0)}=[f_Y^{(2)}-f_S^{(2)}]E_IE_{II},$$

$$f_Y^{(1)}-f_S^{(1)}=-[f_Y^{(2)}-f_S^{(2)}](E_I+E_{II}).$$

[21] Ostrofsky, Breit, and Johnson, Phys. Rev. **49**, 22 (1936).
[22] Hoisington, Share, and Breit, Phys. Rev. **56**, 884 (1939).

The intercepts $f^{(0)}$ and slopes $f^{(1)}$ are, therefore, different for the two fits. The quantity $f^{(1)}$ determines almost directly and to a good approximation the range parameter of a potential well, being approximately proportional to it. The effect of $f_Y^{(2)}-f_S^{(2)}$ on it is not negligible. Thus, for the Yukawa and square well potentials, employing Mev as the unit of energy, the approximate values are $f_Y^{(2)}=-0.0054$, $f_S^{(2)}=0.0061$. A fit at 0.8 and 3.2 Mev corresponds to $f_Y^{(1)}-f_Y^{(2)}$ of $\sim 0.00115 \times 4=0.046$, which corresponds to a 4.6 percent difference in error of judgment concerning the range parameter. It is thus seen that with an undecided preference for fitting data by a potential of one or another shape, there exists a non-negligible effect on the certainty of knowledge of the parameters which are primarily determined by the proton-proton scattering data. It can hardly be said that in this matter the approximately linear appearance of the (f, E) plots is a substantial advantage over other ways of plotting the data. The main problems of making a fit are not essentially simplified as long as the line which has to be used has an unknown curvature. The advantages are rather in having less interleaving of graphs and in there being somewhat fewer points required in the plotting of theoretical graphs.

(c) There appears to be a difference of opinion regarding the possibility of drawing conclusions concerning the shape of potential wells from experimental material. The fact that no obvious preference is visible in the (f, E) graphs is used by Jackson and Blatt as an argument for considering previous considerations[23] of this problem as being inapplicable. Regarding this matter, there appears to be some misunderstanding. It is clear that there can be no real difference between conclusions arrived at by different but mathematically equivalent methods. One of the main objects of the papers of Hoisington, Share, and Breit and of Breit, Broyles, and Hull was to point out that the effects of changes in the shape of potential energy curves were sufficiently large that their determination could be attempted, provided an accuracy of about one percent in the absolute scattering yield could be secured. The differences in shape which were meant here were mainly those having to do with small additions to the potential energy at one distance accompanied with compensating subtractions at other distances. Of the potentials commonly used, only the Yukawa potential has a theoretical justification; and even in this case one can have serious doubts concerning the applicability of theory to the calculation of the shape. The two papers on sensitivity to changes in the shape of the potential energy curves have been written, therefore, from the point of view of arbitrary changes obtained by addition and compensating subtraction of potential energy at different distances. This view admits freely one's ignorance concerning a theoretically expected shape. A possible direct usefulness is in connection with trying modifications of potentials produced by the action of some additional type of meson with a different mass or in trying out effects of the incompleteness of meson-theoretic views. Blatt and Jackson justly point out on page 101 of their paper on proton-proton scattering that some of the changes in potential energy considered by Breit, Broyles, and Hull are larger than the whole potential for the same distance for the Yukawa potential. There is no reason, however, why one should be interested only in potentials of types that have become customary largely through habit.

In view of the objections made by Jackson and Blatt to the discussion of Breit, Broyles, and Hull in terms of shielding of the nuclear potential by the coulomb barrier at 200 kev and their insistence on page 102 of their paper on the absence of any effect but the interference minimum at 400 kev, it appears necessary to disagree with them on account of the definitely positive slope of u at $r=e^2/mc^2$ at 200-kev bombarding energy. It is not true that the interference effect is the only one present. At this energy the function u increases by a factor ~ 1.5 from $r=e^2/mc^2$ to $r=3e^2/mc^2$. About 10 percent of this effect disappears if the coulomb field is removed in the calculations. A factor 1.5 in u corresponds to a factor 2 in sensitivity to potential lumps. At $r=4e^2/mc^2$ these effects are still more pronounced, and here the coulomb barrier is responsible for a factor ~ 1.5. On the other hand, it must be said that it would have been clearer not to put the matter in terms of the coulomb barrier, since effects of the barrier cannot be clearly differentiated from effects of shortening of wavelength which set in strongly at ~ 3 Mev for $r\sim 3e^2/mc^2$. For such energies the coulomb field can produce both an increase and a decrease of sensitivity at $r=e^2/mc^2$ as compared with the sensitivity between $rmc^2/e^2=3$ and 5. For this reason one could object to stating the situation in terms of the coulomb barrier, inasmuch as the differential effect at different energies can have either direction, depending on how the coulomb barrier effect combines with the shortening of wavelength effect. At 200 kev these complications are practically absent; and the statement by Breit, Broyles, and Hull was therefore made for this energy. The fact remains that there exist strong effects in addition to those caused by the interference minimum and that these effects are partly associated with the action of the coulomb barrier. It should also be mentioned that BBH have brought out the fact that at 400 kev the coulomb barrier effects are relatively unimportant; this statement must have been overlooked by Jackson and Blatt, since on page 102 of their article they take issue with a supposedly opposite point of view. The slight preference for a Yukawa-shaped well over the exponential well which has been expressed by Hoisington, Share, and Breit was meant for the two wells tried in that paper rather than for the best Yukawa well as compared with the best exponential well. In this respect the clarification of

[23] Hoisington, Share, and Breit, reference 23; Breit, Broyles, and Hull, Phys. Rev. **73**, 69 (1948).

an obscurely made statement of HSB which is found in the paper of Jackson and Blatt is very welcome, especially since HSB have not used the word "shape" in a consistent manner. The explanation of the difference in results for the particular Yukawa, and exponential wells which has been given by HSB is nevertheless applicable.

(d) If a fit to experiment is made in terms of the f function, the most objective method is that of weighted least squares first brought out at a public meeting by Jackson and Blatt[24] and simultaneously carried out by Hatcher, Arfken, and the writer.[4] Such a fit would be capable of determining the potential well parameters very fairly if one had a definite knowledge of the shape of the well. Since this is unavailable, the circumstances discussed in connection with the influence of $f_Y^{(2)}-f_S^{(2)}$ on $f_Y^{(0)}-f_S^{(0)}$ and on $f_Y^{(1)}-f_S^{(1)}$ in relation to the visual fits are relevant. A least-squares fit to data gives, therefore, different $f^{(0)}$, $f^{(1)}$ depending on the assumed shapes of potential energy curves. Still different values can result from a least squares determination of $f^{(2)}$. It is obviously impractical to try to determine too many coefficients $f^{(i)}$ by least squares, since by doing so with experimental material of necessarily limited accuracy one can spoil the accuracy of $f^{(0)}$ and $f^{(1)}$. The problems of fitting data are thus helped only partially by the employment of approximately linear plots. That no great help can be expected along such lines is clear from the fact that for any kind of plot one can use the curves for one potential system as a reference set of curves and that the adjustment of potential well parameters for potentials of different shapes to give the same phase shifts at two energies can be readily accomplished as is explained in the paper by Breit, Thaxton, and Eisenbud. The essential uncertainties are caused by the unknown magnitude of possible systematic experimental errors as well as the probable inadequacy of the potential energy point of view.

V. GENERALIZATION TO THE TWO-AGGREGATE MANY-BODY CASE

The connection of the method of the f function with the Ferretti-Hulthén equation may now be briefly discussed, together with generalizations to problems in which at least one of the colliding particles has a composite structure. A generalization of this type has been made by Verde[25] in the special case of neutrons scattered by deuterons. Following Verde's procedure, J. L. McHale[26] has obtained corresponding results for the scattering of deuterons by deuterons. These calculations are performed by an application of Green's theorem employing the unperturbed and the perturbed functions. These generalizations of Eq. (5) can be understood also by the method already employed for its presentation. The advantage of doing so is the direct relationship to the hamiltonian through Eq. (2.2) and a consequent directly assured symmetry of the answer. The difference in presentation is to some degree trivial, since Green's theorem considerations have to be used in establishing the hermitian character of the hamiltonian. It is satisfying, however, to have two problems stated in the same form. It has been brought out with reference to the one-body problem that the direct employment of Eq. (5) is prevented by the lack of knowledge of the wave function and that the infinitesimal form of this relation can in this case be used to advantage in making estimates of first-order phase shift changes, the situation being similar to that encountered in the calculation of first order energy changes. A consideration of Eq. (11) and its consequence Eq. (11.1) shows that they follow from the infinitesimal form of Eq. (5). The remaining work leading to the values of the coefficients of λ^n in the expansion of f depend only on the possibility of performing partial integrations as in Eqs. (11.6), (11.7) which result in Eqs. (12.4), (12.5). It is possible, therefore, to generalize the method of the f function to the many-body case in close analogy to the method already sketched. The generalization will be described for elastic scattering only.

The coordinates of the center of mass are separated first in the usual way. The process of separation can be arranged so as to leave one with internal coordinates of the two aggregates, and with the relative coordinates of the center of mass of one fragment with respect to the center of mass of the other. If the masses of individual particles are M_i and their cartesian coordinates (x_i, y_i, z_i), the employment of the variables $\xi_i=M_i^{\frac{1}{2}}(x_i, y_i, z_i)$ makes the transformation to relative and absolute coordinates of any pair appear as an orthogonal transformation, provided one employs the combinations $(M_1+M_2)^{\frac{1}{2}}(x_1+x_2)$, $M_1M_2/(M_1+M_2)^{\frac{1}{2}}(x_2-x_1)$ as the new variables in dealing with particles having masses M_1 and M_2. The coordinates thus introduced make the original kinetic energy operator of the whole collection expressible as $-\hbar^2\sum_i[(\partial^2/\partial\xi_i^2)+(\partial^2/\partial\eta_i^2)+\partial^2/\partial\zeta_i^2]$, and for the two fragments there results the usual expression containing the reduced mass. Since the jacobian of the orthogonal transformations is unity, the original volume element of the problem is equal to the product of the volume element of the internal coordinates of the two fragments and the volume element in the space of relative coordinates of the fragments with respect to each other.

On account of the requirements of antisymmetry of the particles and the symmetry of the hamiltonian with respect to rotations in spin and isotopic spin space, the wave equation can often be separated, giving rise to a set of coupled equations between functions containing space coordinates alone, each function corresponding to a special symmetry regarding permutations. These

[24] J. D. Jackson and J. M. Blatt, Phys. Rev. **75**, 1296(A) (1949). This paper was read at the New York meeting of the American Physical Society in January, 1949.

[25] M. Verde, Helv. Phys. Acta **22**, 339 (1949). This generalization is concerned only with a formula for $\sin(\delta-\delta^0)$.

[26] J. L. McHale, dissertation, Indiana University (1951).

equations have then the form,

$$\sum_k[(T-E)\delta_{ik}+V_{ik}]\psi_k=0, \tag{18}$$

as in the paper by Verde. The separation of the two fragments to an infinite distance by elastic scattering corresponds to the survival of one of these functions. The operator T arises from the kinetic energy; and according to the orthogonal transformation argument, it can be arranged to be symmetric in all variables. For the case of pure elastic scattering there may be in general several ψ_i which satisfy conservation of energy for large internuclear distances. These ψ_i will be denoted by ψ_s with s taking on the values s', s'', etc. The energy change $E-E^0$ which is produced by a change V_{ik}' in the V_{ik} assumes the form,

$$E-E^0=\left[\sum_{i,k}\int\psi_i^*V_{ik}'\psi_k^0 drd\tau\right]\Big/\left[\sum_i\int\psi_i^*N_{ij}\psi_j^0 r^2drd\tau\right] \tag{18.1}$$

where the volume element of the internal coordinates is $d\tau$ and the angular integrations as well as spin sums are supposed to have been performed. The ψ_s are supposed to have the asymptotic form,

$$\psi_s\sim\alpha_s\mathfrak{u}_s(\mathbf{r}_{int})\mathfrak{F}^{(s)}(kr)/r, \quad \int\mathfrak{u}_s^2(\mathbf{r}_{int})d\tau=1. \tag{18.2}$$

Here each of the $\mathfrak{F}^{(s)}$ is supposed to have a different phase shift δ_s in the general case. The asymptotic form of Eq. (18.2) is supposed to apply for a large value of the interfragment separation. The distance r should really be denoted by a different symbol for each s, and the internal coordinates must be different also. Since the same form of the equation results on relabeling the particles the different symbols have not been introduced, in order to complicate the notation. The system will be quantized in a big sphere of radius R employing the same value of R for each ψ_s. Introducing

$$\beta_s=(R/2)^{\frac{1}{2}}\alpha_s \tag{18.3}$$

and copying in other respects the two-particle treatment, one obtains

$$\sum_{i,k}\int\varphi_i^*V_{ik}'\varphi_k^0 r^2drd\tau$$

$$=-(E/k)\sum_{s',s''}\beta_{s'}\beta_{s''}\sin(\delta_{s'}-\delta_{s''}^0), \tag{18.4}$$

$$\sum_{s',s''}N_{s's''}\beta_{s'}\beta_{s''}\cos(\delta_{s'}-\delta_{s''})=1, \tag{18.5}$$

where the φ_i satisfy the same equation as the ψ_i but are normalized so that

$$\varphi_s\sim\beta_s\mathfrak{u}(\mathbf{r}_{int})\mathfrak{F}^{(s)}(kr)/r. \tag{18.6}$$

The infinitesimal form of Eq. (18.4) does not give the change in phase shifts in the general case written here. If, however, the changes in the phase shifts are equal, then Eqs. (18.4), (18.5) give

$$\delta-\delta^{(0)}\cong-(k/E)\sum_{i,k}\int\varphi_i^*V_{ik}'\varphi_k r^2drd\tau. \tag{18.7}$$

The equality of the phase change changes corresponds to to the usual situation of there being only one phase shift, so that the state for elastic scattering is obtainable by linearly combining some of the φ_i. In the subsequent discussion it is supposed that this is the case and that the remaining functions φ_i appear in the normalization integral with the coefficient 1. This can be accomplished by a suitable change of normalization. The possibility of obtaining the simple expansion (12.6) for the function f of the two-body problem depends on Eq. (11) which is a direct consequence of the H. M. Taylor formula for the first-order change in phase shift. In addition, it is essential to be able to carry through the partial integrations of Eq. (11.5) so as to lower the order of the derivatives of u. These steps can be generalized to the many-body, two-aggregate, elastic scattering problem.

The phase shift denoted by δ will be called K_L in the coulombian case. The function f_L will be understood to be given by Eq. (14.5). The phase shift K_L depends in general not only on L but also on the vector arrangement of L with the spin and other factors. This circumstance will be understood without taking explicit account of it in the notation. One obtains with this understanding

$$\partial f_L/\partial\lambda=(2L+1)(2/e^2)$$

$$\times\int\sum_{i,k}(u_i^*/\eta^L)(\partial V_{ik}/\partial\lambda)(u_k/\eta^L)r^2drd\tau, \tag{19}$$

where

$$u_i=C_L\varphi_i/\sin K_L. \tag{19.1}$$

Here λ' is a parameter entering the V_{ik}. The factor $2/e^2$ occurs in place of the factor $-a$ of Eq. (11.1) because the factor $-2\mu/\hbar^2$ has been absorbed in the $\varphi(r)$ of Eq. (8.1). The particular function u_s which corresponds to the channel[27] of elastic scattering is now given by Eqs. (16), (16.1) in the region of configuration space in which the aggregates have definitely separated and in which only the coulomb force acts between them. It should be remarked that Eq. (11) applies only to the special case $L=0$ and that its generalization to $L>0$ is obtained through the inclusion of the factor $(2L+1)/\eta^{2L}$ as in Eq. (15.5).

If the parameter enters the V_{ik} linearly, one obtains a chain of equations similar to Eqs. (11.4), *viz.*,

$$\sum_k\mathfrak{L}_{ik}(u_{L,s+1})_k+\lambda(-2\mu/\hbar^2)\sum_k V_{ik}^{(\lambda)}(u_{L,s})_k=0, \tag{19.2}$$

where

$$u_{L,k}=\sum_s\lambda^s(u_{L,s})_k \tag{19.3}$$

[27] G. Breit, Phys. Rev. **58**, 1068 (1940).

and $\lambda V_{ik}^{(\lambda)}$ is the part of V_{ik} containing λ. Here μ is the reduced mass of the two aggregates. It follows that

$$\int\sum(u_{L,t-1})_i^* V_{ik}^{(\lambda)}(u_{L,s})_k r^2 dr d\tau$$

$$=\int\sum(u_{L,t})_i^* V_{ik}^{(\lambda)}(u_{L,s-1})_k r^2 dr d\tau, \quad (19.4)$$

it being understood that the summations are over i and k. In order that Eq. (19.4) be a consequence of Eq. (19.2), it is necessary to require

$$\sum_{i,k}\int_{r<b}\{(u_{L,t})_i^*\mathfrak{L}_{ik}(u_{L,s})_k - (u_{L,s})_i^*\mathfrak{L}_{ik}(u_{L,t})_k\}r^2 dr d\tau=0, \quad (19.5)$$

with the understanding that the last integration is extended over all of the configuration space with the exception of the part of the scattering channel $r>b$, within which the aggregates are definitely free of each other. This step is similar to that of Eq. (11.5), where the expression in square brackets vanishes. An application of the Green's theorem transformation to the left side of Eq. (19.5) employing the orthogonal transformation described at the beginning of this section shows that it is zero for regular real functions and in some other cases. Under these conditions, one has for the expansion of f in terms of a parameter Eq. (11.3) with values of the coefficients given by Eqs. (12.4), (12.5), the symbol in square brackets being generalized to mean

$$[s,t]_\lambda=\int\sum_{i,k}(u_{L,s})_i^*[-(2\mu/\hbar^2)V_{ik}^{(\lambda)}](u_{L,t})_k r^2 dr d\tau. \quad (19.6)$$

It is helpful to observe that expansions in terms of energy are not essentially different from expansions in terms of parameters and that one can derive Eq. (9) from Eq. (11.1). To do so it suffices to consider first not an energy change but a change in the potential energy by the constant amount $(-\Delta E)$ everywhere except in the remote part $r>b$ of the elastic scattering channel. The join between the inner region and the remote part of the channel is made by rounding off the curve for ΔE. The change in the potential energy produced in this manner will be called ΔV. The change in phase shift produced, is, of course, still a change for the original energy E. One introduces next a comparison potential and adjusts it so that the two potentials give the same phase shift before ΔV is applied. The application of ΔV gives different changes in phase shifts for the two potentials. It is useful to consider the double difference between the four phase shifts, i.e., the difference in the phase shift differences for the two potentials which is produced by ΔV in the limit of a very smooth join and in the additional limit of a very small ΔV. In these two limits the joining region, i.e., the region within which the original ΔE is rounded off, produces the same effect on both phase shift differences, because the wave functions in the two cases approach either. The double difference is thus correctly given by the difference of two expressions like that in Eq. (11.1). If one now raises the energy to $E+\Delta E$ everywhere by a gradual distortion of the ΔV curve, there appears an additional change in phase which is the same for the two cases. Equation (9) is thus a direct consequence of Eq. (11.1). The whole argument can be transferred directly to the many-particle case by employing $(\Delta E)\delta_{ik}$ as the change in V_{ik}. One may apply Eq. (9), therefore, to the more general case by changing (s,t) to

$$(s,t)=\eta^{-2L}\int\sum_i[(u_{L,s'})_i^*(u_{L,t'})_i-(u_{L,s})_i^*(u_{L,t})_i]r^2 dr d\tau \quad (20)$$

in the expansion

$$(f_L-f_L')/(2L+1)=\sum_n(\kappa-\kappa_0)^n f_L^{(n)} \quad (20.1)$$

with $f_L^{(n)}$ being obtainable by means of Eqs. (10.2), (10.3). Specialization to the case of a vanishing coulomb field is obtained by observing that for $\eta=0$, $C_L^2=[2^L(L!)/(2L+1)!]^2=\text{const}$ and that $(\cot K)/\eta = ak\cot K$. The factor a is seen to disappear, since it is present on the right side of Eqs. (10.2), (10.3), (11.1); and one has then relations for $k\cot K$. After this manuscript was nearly completed, there arrived a second paper by Verde[28] which contains some of the results of the present section.

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[28] A. Troesch and M. Verde, Helv. Phys. Acta **24**, 39 (1951). In this paper formulas are given for the calculation of the linear term in the expansion of $k\cot\delta$ for the $d-n$ scattering problem, and approximate calculations are made for the expected scattering. The more general case discussed here does not seem to have been considered.