Influence on the Cosmic-Ray Spectrum of Five Heavenly Bodies*

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I. INTRODUCTION

THIS paper investigates the trapping of externally incident charged particles in the magnetic field of the sun through deflection by the magnetic field of the earth, and the consequent form of the cosmic-ray spectrum to be expected at the earth, on the hypothesis of a constant solar field and relatively small or zero magnetic moments for the other heavenly bodies which also intercept these trapped particles—Mars, Venus, and the moon.

The remarkable drop of the cosmic-ray intensity for particles of energy below a few Bev[1] —by the work first of Cosyns and later of Millikan and his collaborators[2] clearly evidenced to be a property of the radiation *before* it strikes the earth—was interpreted by Janossy[3] in terms of the magnetic field of the sun and its action in deflecting away from the earth particles of lesser energy. No other reasonable explanation for the observed cut-off has ever been put forward. Moreover, Janossy's idea is supported by the agreement in order of magnitude between the required value of the magnetic moment of the sun and the value obtained by Hale and his collaborators[4] from measurements of the magnetic splitting of lines in the solar spectrum.

There is, nevertheless, one consequence of the solar cut-off hypothesis which has denied this theory complete acceptance. Particles of magnetic rigidity[5] between the cut-off limit and a rigidity $(1+2^{\frac{1}{2}})^2=5.83$ times that limit will come from outer space into the earth's field only from a certain cone of directions ("cone of allowed directions;" Table I). The particles will then be further deflected in the short scale field of the earth itself. However complicated the resulting pattern of allowed directions may be at the earth's surface, it is evident that this pattern has a fixed relation to the earth-sun line. As the earth rotates, the cosmic-ray intensity at any fixed geographical location will, consequently, be expected to vary in time, as already noted by Janossy. Moreover, the diurnal variation as calculated on this picture by Rossi[6] is very substantial at mountain elevations at geomag-

* The A.B. senior thesis of E. O. Kane, *Theory of the Allowed Cone of Cosmic Radiation* (Princeton, June 1948), contains a preliminary account of the considerations on absorption by the sun and radiative deceleration in the sun's field which come into the present discussion. The further considerations on the effect of Mars were reported by John A. Wheeler at the Pasadena Conference, while the detailed results given below were for the most part obtained by T. J. B. Shanley after the time of that meeting.

[1] Bev used here and below as abbreviation for 10^9 ev.

[2] M. G. E. Cosyns, Nature **137**, 616 (1936), reported definitely that the latitude effect begins at about 49° and is independent of altitude between 7 and 18 cm Hg. See also Bennett, Dunham, Bramhall, and Allen, Phys. Rev. **42**, 447 (1932); R. A. Millikan, New York Times, Dec. 31, 1933; and A. H. Compton, Phys. Rev. **43**, 387 (1933) for earlier indications that altitude does not influence the position of the knee in the curve of cosmic-ray intensity as a function of latitude. For a summary of the more detailed investigations of the Pasadena group, see R. A. Millikan, H. V. Neher, and W. H. Pickering, Phys. Rev. **63**, 234 (1943). See especially Carmichael and Dymond, Proc. Roy. Soc. **A171**, 321 (1939) for evidence that the intensity does not increase with altitude up to 88°N, even at very high altitudes.

[3] L. Janossy, Zeits. f. Physik **104**, 430 (1937). The solar cut-off hypothesis was further developed by M. S. Vallarta, Nature **139**, 839 (1937) and P. S. Epstein, Phys. Rev. **53**, 862 (1938).

[4] Hale, Seares, Van Maanen, and Ellerman, Astrophys. J. **47**, 206 (1918). Thiessen, Observatory **36**, 230 (1946). See later discussions for redeterminations of the sun's field.

[5] The earth and the sun are magnetic spectrometers, and therefore fix neither the energy nor the charge nor the mass nor the momentum of the incoming particles, but only their magnetic rigidity. This quantity has the same dimensions as a magnetic or electric potential, and may therefore be expressed either in gauss cm (for example, 20×10^6 gauss cm) or in volts (not electron volts; in the example, $300\times(20\times10^6)=6\times10^9$ volts=6 Bv). Two particles, however they may differ in mass or charge, will behave exactly alike for the purposes either of the Stormer theory or of the present analysis, provided only that they have the same magnetic rigidity. This quantity is, of course, so useful because it is a constant of the motion for a particle moving in a static magnetic field: (magnetic rigidity in gauss cm) = (velocity of light in cm/sec.)(momentum in g cm/sec.)/(charge of particle in electrostatic units) = (cp/e).

[6] These calculations were reported by B. Rossi at the 1947 Conference on Astrophysical Implications of the Cosmic Radiation sponsored by the New York Academy of Sciences and New York University. We are indebted to Professor Rossi for sending us a fuller report of these calculations.

TABLE I. Cut-off of cosmic-ray spectrum by sun's field expected on basis of simple theory, overlooking contribution of particles trapped in sun's field. A minimum magnetic rigidity is required to come in as far as the earth's orbit. A positive particle which just gets in so far will be traveling in the same direction as the motion of the earth in its orbit ("earth's directrix").* For a particle of higher rigidity the angle with respect to the earth's directrix may have any value between 0 and the figure given in the table (90°, for example, for a 3.4-Bv particle if sun's moment is 10^{34} gauss cm^3).**

Value assumed for sun's magnetism, expressed in several equivalent forms	Magnetic rigidity of entering particle	Permissible angle with respect to earth's directrix for positive particle on arrival at earth's orbit from outer space
Sun's moment	<0.97 Bv	Can't arrive
$=0.42\times10^{34}$ gauss cm^3	1.15 Bv	0° to 60.4°
$=1.26\times10^{27}$ Bv cm^2	1.42 Bv	0° to 90°
$=2.61\times10^{5}$ Bv (sun radius)^2	2.04 Bv	0° to 123.8°
$=5.67$ Bv (sun-earth distance)^2	2.78 Bv	0° to 144.9°
Sun's polar field, 25 gauss	3.63 Bv	0° to 159.9°
Sun's equatorial field, 12.5 gauss	>5.67 Bv	All directions
Sun's moment	<2.32 Bv	Can't arrive
$=1.0\times10^{34}$ gauss cm^3	2.74 Bv	0° to 60.4°
$=3.0\times10^{27}$ Bv cm^2	3.38 Bv	0° to 90°
$=6.21\times10^{5}$ Bv (sun radius)^2	4.87 Bv	0° to 123.8°
$=13.52$ Bv (sun-earth distance)^2	6.62 Bv	0° to 144.9°
Sun's polar field, 59.5 gauss	8.65 Bv	0° to 159.9°
Sun's equatorial field, 29.8 gauss	>13.52 Bv	All directions

* We neglect here any deviation between the direction of the sun's magnetic moment and the normal to the plane of the earth's orbit. Such a deviation appeared to be indicated by the early work of Hale and his collaborators, but the variations in solar magnetic field shown by more recent determinations presumably render doubtful the magnitude found at Mt. Wilson for this effect, and possibly put in question even the existence of any well defined deviation.

** The angular opening of the allowed cone would appear to depend upon the azimuth of the particle's velocity with respect to the earth's directrix, if we were to apply the Lemaitre-Vallarta theory at face value to the present problem (see M. S. Vallarta, Theory of the Allowed Cone of Cosmic Radiation, Section 3, Toronto University Applied Math. Series (1935–43) for this azimuthal dependence). This theory would only be relevant, however, if the sun were surrounded by a sphere of solid matter which extended out to the radius of the earth's orbit and prevented the return of particles from smaller distances. The absence of such absorption very much simplifies the theory, as described below.

netic latitudes in the range 40° to 50°. Yet no evidence for such a large variation has been found.

The absence of the expected large diurnal variation in cosmic-ray intensity presents to the solar cut-off hypothesis a fundamental difficulty, a way out of which has only recently been suggested by Alfvén.[7] He recalls the distinction usually made between unbounded and bounded orbits for a particle moving in the magnetic field of the sun (Fig. 1). However, he points out that it is incorrect to assume that there will be no particles moving in the bounded orbits. To be sure, Alfvén notes, particles placed on such trajectories will disappear in a time of the order of a century, through collision with the earth, if in no other way. But there will be continual new entries into such orbits by particles which came in from infinity on unbounded orbits, passed through the outer reaches of the earth's magnetic field, and were thereby deviated into bounded or "trapped" trajectories (Fig. 2). The intensity of the cosmic radiation in forbidden directions in the sun's field (Table I), instead of being zero, is therefore determined by the relative magnitudes of the scattering cross section of the earth and its absorption cross section. Rough estimates of these two quantities by Alfvén lead him to conclude that the intensity in forbidden directions may be appreciable in comparison with that in allowed directions. Thus there appears a distinct possibility of

[7] H. Alfvén, Phys. Rev. **72**, 88 (1947). We are indebted to Dr. Alfvén for sending us a copy of this communication in advance of publication.

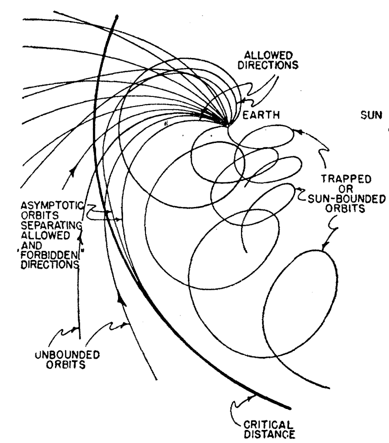

FIG. 1. Qualitative sketch of orbits of positive particles of ~3-Bv rigidity moving in field of sun, here assumed to have a magnetic moment of 10^{34} gauss cm^3 = 13.52 Bv (earth-sun distance)^2. The deviation of the orbits by the field of the earth itself is on too small a scale to be shown in this diagram. Full intensity will be observed near the earth for 3-Bv particles coming from orbits unbounded at infinity. It might appear that no particles arrive from other directions for the corresponding orbits are confined to a limited region of space, with no obvious source to replenish particles lost by collision with the earth. Such a source is, however, provided by the field of the earth itself; it deviates into trapped orbits some of the particles which come from infinity in unbounded orbits. To experience such a deviation a particle must pass close enough to the earth to experience a significant magnetic field, but not so close that it strikes any solid matter. Particles which satisfy these conditions may subsequently circulate in trapped orbits ~5000 years before they eventually hit the earth or some other heavenly body. Those among these particles which at this time strike the earth come from directions in the sun's field which are normally called "forbidden." That the intensity coming from allowed and forbidden directions cannot greatly differ must be concluded purely empirically from the absence of a strong diurnal variation in the cosmic-ray intensity.

explaining the absence of any substantial diurnal variation in cosmic-ray intensity.

We take up here the idea of Alfvén and investigate it quantitatively, for the following reasons:

(a) Recent determinations of the magnetic field of the sun by observations of the magnetic splitting of lines in the solar spectrum[8] show apparent variations from time to time. These variations are sufficiently great to throw some doubt on the significance for the long range field of the sun of any measurements which refer only to the layer of the sun where light absorption takes place. A quantitative investigation of the theory of trapping should permit an *independent determination of the magnetic moment of the sun*, based on the position of the cut-off of the cosmic-ray spectrum.

(b) An investigation of the equilibrium of particles between bounded and unbounded orbits should provide the basis for a quantitative discussion of the distribution of the *cosmic-ray spectrum* with respect to magnetic rigidity, and for the prediction of the magnitude of *diurnal variations* in intensity.

Features of the present treatment which require mention are the following:

(a) The difference between the intensity in allowed and forbidden directions is found on the whole not to be great. Consequently, it is permissible to determine the magnitude of this difference—and to develop the whole theory—on the basis of certain approximations, without which the calculations would be more difficult by several orders of magnitude.

(b) The sun's *general* field is assumed not to change substantially over a period of the order of magnitude of the time—say 5000 years—during which a particle circulates in a trapped orbit. Even a very substantial fluctuation in solar magnetic moment about a suitable average value might conceivably by some kind of principle of adiabatic invariance leave the situation very much as we have treated it here on the basis of a constant moment. It would, of course, be important to investigate this point carefully. The present work may serve as the basis for such a more detailed study. On account of this question of possible variations in the sun's moment it is important in using any of the present results to recognize that they are valid only on the basis of the assumptions which we have explicitly stated.

(c) Mars, Venus, the moon, and the sun compete with the earth for trapped particles.

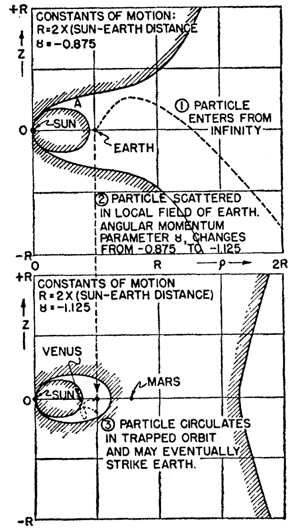

FIG. 2. Trapping in the sun's field of a positive particle which has a characteristic radius, R, equal to twice the radius of the earth's orbit—in other words, a particle of 3.38-Bv rigidity, if the magnetic moment of the sun is 10^{34} gauss $cm^3 = 13.52$ Bv (sun-earth distance)^2. The particular particle under discussion (upper diagram) comes from infinity with just the right total angular momentum (angular momentum parameter, γ, equal to -0.875) to arrive at the earth's orbit moving at an angle of 60° with respect to the direction of motion of the earth in its orbit ($\cos x = 0.50$). The laws of conservation of energy and angular momentum permit this particle to move anywhere in the unshaded region. On arrival near the earth, the particle by chance experiences a large net deflection in the local magnetic field. It is fortunate enough not to hit any matter and departs from the earth in a direction which—for an example—makes an angle of 120° with respect to the earth's directrix ($\cos x = -0.50$). The scattering process changes the angular momentum of the particle with respect to the sun. For the new value of the angular momentum parameter, $\gamma = -1.125$, the particle is constrained to remain in the unshaded toroid-like volume element which centers at the sun and contains the earth. The particle circulates in this region until lost by impact on the earth, or by rescattering, or by collision with another heavenly body. If it hits the earth, it is moving at the time of impact in a direction which would normally be called forbidden, for a particle with $R = 2\times$ (sun-earth distance) and $\gamma = -1.125$ cannot get in to the earth from infinity. Note in the present case that the trapped particles can strike neither Mars nor Venus nor any other planet.

[8] Recent indications from magnetic splitting of spectral lines for changes in the sun's moment. Thiessen, personal communication to the author, June 1948; H. Von Kluber, Zeits. f. Astrophys. **24** (No's. 1–2): 1 and 21, 1947; H. W. Babcock, Oral report at meeting of American Astronomical Society, Pasadena, June 29, 1948.

TABLE II. Values adopted in present work. For the magnetic moment of the earth, 8.1×10^{25} gauss $cm^3=2.43\times10^{19}$-Bv $cm^2=59.8$ Bv (earth radius)$^2=1.10\times10^{-7}$ Bv (sun-earth distance)^2. Values adopted for dimensions tabulated below.

Object	Distance, r, from sun	Radius, a	Geometrical cross section, πa^2
Sun	—	6.95×10^{10} cm	1.52×10^{22} cm^2*
Earth	1.49×10^{13} cm	6.38×10^{8}	$12.8\ \times10^{17}$
Moon	1.49×10^{13}	1.74×10^{8}	0.95×10^{17}
Mars	2.28×10^{13}	3.43×10^{8}	3.70×10^{17}
Venus	1.08×10^{13}	6.31×10^{8}	$12.5\ \times10^{17}$

* Only an exceedingly small fraction of the sun's geometrical cross section has any part in the absorption of trapped cosmic-ray particles, as will be seen below.

Some of these competitive effects, which were not mentioned by Alfvén, are quite important for the quantitative determination of the intensity in trapped orbits, and are taken into account here.

(d) We assume that the magnetic moments of Mars, Venus, and the moon are sufficiently small that the scattering cross sections of these objects can be neglected in comparison with their absorption cross sections (see more detailed discussion below).

(e) Clouds of particles emerging from the sun will carry local magnetic fields. These fields may produce deflections over and above those considered here. Deflections from this source cannot be extremely frequent. If they were, they would make it possible for particles of very low energy to reach the earth via scattering into trapped orbits. Such an effect would be inconsistent with the observed low energy cut-off of the cosmic-ray intensity. The possible effects of ion clouds nevertheless deserve further investigation.

Outline of Method of Analysis

The analysis of the trapped particles is carried out in the following way. The shape of those regions in space where these particles may move is first determined (Fig. 2). The motion of a particle in such a region is represented by a trajectory which weaves about and ultimately covers the trapped region to an effectively smooth density. A density function is therefore defined for the trapped particles. Some come close to the sun, where their increased acceleration causes them to radiate. This radiation is calculated. So likewise is the expected rate of loss of particles by collision with interplanetary matter. Next is considered the rate of loss by impacts on the earth and on the moon, for each of which objects an absorption cross section is defined, of the order of magnitude of 10^{17} cm^2. Trapped particles which find the earth accessible move in some cases in zones which include Mars or Venus (Table II). In these cases we define for each of these particles likewise an effective absorption cross section. In addition, all the zones of trapping intersect the sun in rings of small angular diameter near the sun's north and south poles. The loss of particles by collisions with this surface is also expressed in the form of an effective cross section (Fig. 3). From the total cross section for all absorption processes, and from the size of the zone of trapping, we estimate a mean time in such a zone of the order of magnitude of 5000 years.

The scattering of particles in the magnetic field of the earth is next considered. The magnetic field experienced by a particle which goes by in nearly rectilinear motion at the distance r will be of the order (charge$\times$magnetic moment)$/r^3$, and the time of action will be of the order r/velocity. Thus the deflection, θ, in the case of distant encounters will be proportional to $1/r^2$; or the cross section for deflections greater than θ will be proportional to $1/\theta$. Consequently, the cross section, $d\sigma$, for a deflection into an element of solid angle $d\Omega$ at the angle θ will be $d\sigma\sim(\text{const.}/\theta^3)d\Omega$. By a more detailed analysis of this kind, together with certain simplifying assumptions, we obtain an approximate formula for the differential scattering cross section at all angles.

We next formulate the general integral equation which determines how the intensity, I, of particles of a given energy depends upon direction of arrival in the neighborhood of the earth (I=number of particles per cm^2 and per sec. and per unit solid angle). To solve this difficult equation, we translate it into an equivalent variational problem. We then represent I as a function of angle by a suitably chosen analytic expression with two adjustable constants. These constants are determined to give the variational integral a stationary value. The value of I as a function of angle as so determined is shown in Fig. 4. The dotted segment of each intensity curve represents a lower limit to the intensity

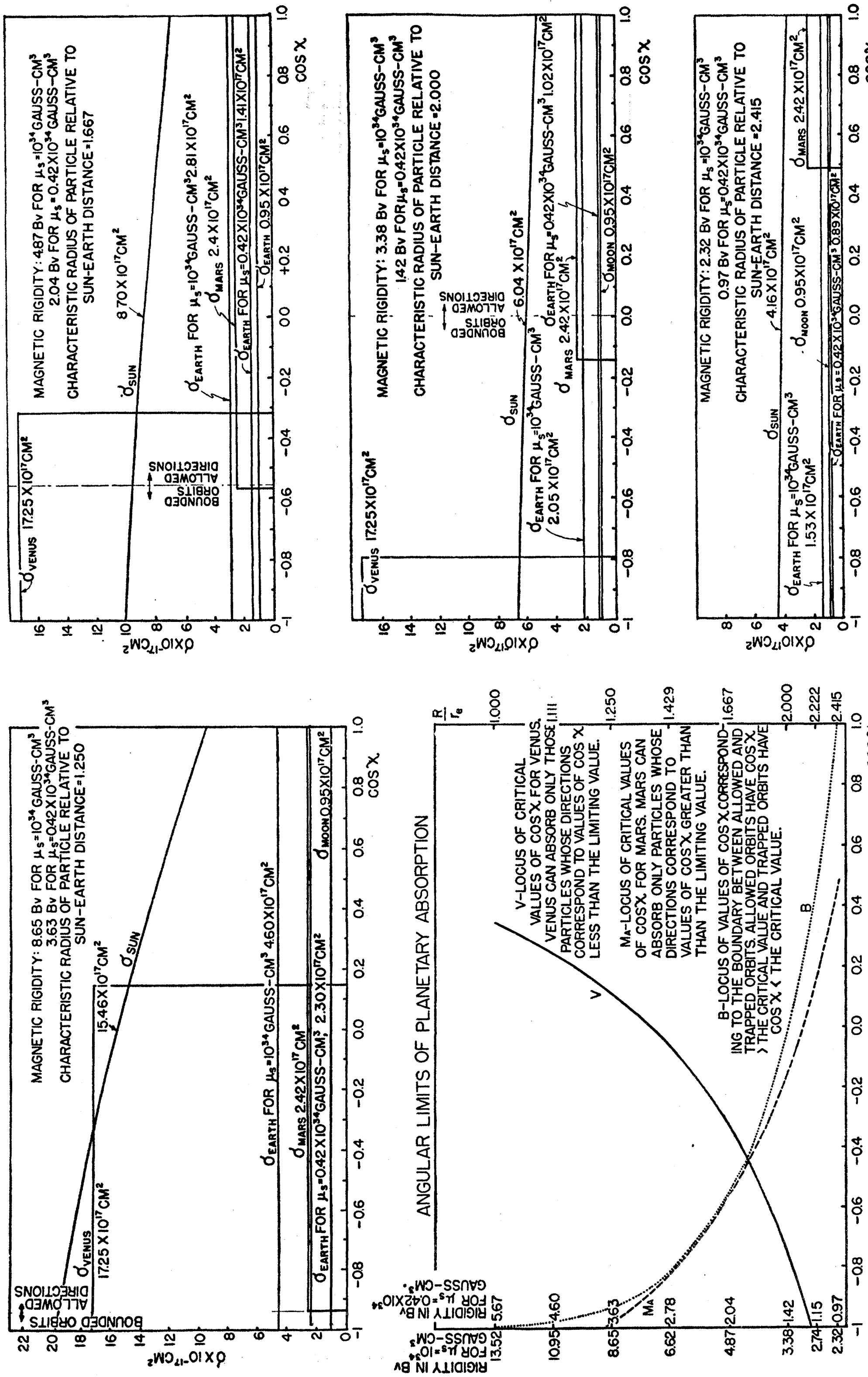

FIG. 3. Absorption cross sections presented by five heavenly bodies as function of direction of arrival of cosmic-ray particles at earth's orbit.

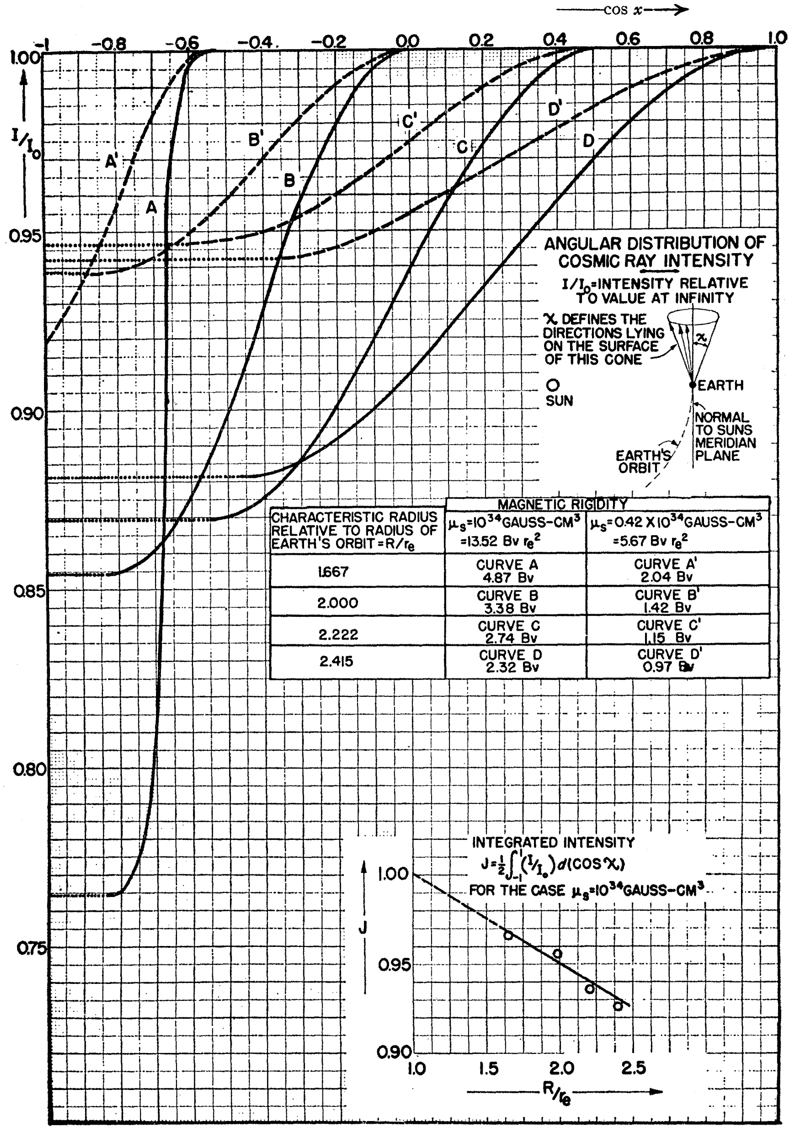

FIG. 4. Intensity of primaries arriving at earth's orbit from trajectories trapped in sun's field. Calculations made for sun's moment 10^{34} gauss cm^3 (smooth curves) and 0.42×10^{34} gauss cm^3 (dashed curves). If earth were a pure absorber (no magnetic moment), intensity would drop sharply to 0 instead of falling off smoothly as shown. Lower diagram shows omnidirectional flux at earth's orbit relative to value it would have in absence of sun's field.

in the given range of angles. There our analytic expression incorrectly predicts a rise in intensity because it consists of only the first few terms of an infinite series (see discussion after Eq. (66)).

Discussion of Results

From the results shown in Fig. 4 we make the following observations:

(a) The higher the energy of the cosmic-ray primaries the more rapidly their intensity falls off in the so-called forbidden region of directions, but the smaller is the angular range of forbidden directions. Absorption becomes stronger relative to scattering as the energy increases.

(b) The less the assumed value of the sun's magnetic field, the less will the intensity fall off in the forbidden region.

(c) The integrated intensity J increases with increasing magnetic rigidity. We have no points on our integrated intensity curve for $R/r_e<1.667$ (magnetic rigidity>4.87 Bev for sun's moment of 10^{34} gauss-cm^3, or >2.04 Bev for sun's moment of 0.42×10^{34} gauss-cm^3) because the evaluation of I/I_0 becomes exceedingly laborious for greater rigidities. But we know that $J=1$ for $R/r_e=1$ (all directions allowed), so it is safe to assume that J increases continually as R/r_e increases.

Conclusions

(a) Starting with Alfvén's original suggestion, it is possible to develop a quantitative equilibrium theory for the trapping of cosmic-ray particles in the magnetic field of the sun, where in addition to the effect of scattering in the magnetic field of the earth considered by him there is also taken into account the direct absorption of the rays by five heavenly bodies—Mars, Venus, the earth, the sun, and the moon.

(b) Radiative deceleration of trapped protons is completely negligible, but the same effect for trapped electrons is calculated to be quite important. Neither this particular effect nor electrons in general are considered in the present calculations, however, because of existing experimental evidence against more than 1 percent of such particles in the primary radiation.

(c) Absorption of trapped protons by disperse interstellar matter appears from uncertain astrophysical evidence to be unimportant and is therefore neglected. It may be possible to check this conclusion from the more detailed experimental study of the cosmic radiation itself.

(d) If Mars has a magnetic moment as large as that of the earth, particles of a magnetic rigidity otherwise insufficient to reach the earth directly will be passed inward through the intermediation of Mars, and the cut-off of the cosmic-ray spectrum will be substantially lowered, a point probably susceptible to experimental check. In the present calculations it has however appeared most reasonable to consider the moment of Mars to be negligible.

(e) The distribution in direction has been calculated for the cosmic-ray protons of any given energy which arrive in the neighborhood of the earth (Fig. 4). The expected departure from uniformity of intensity with respect to direction is small but finite. Consequently, a diurnal variation is to be expected in the cosmic-ray intensity at high altitudes and intermediate latitudes. This diurnal variation will be small.

(f) The nature of the calculated cut-off of the cosmic-ray spectrum at low magnetic rigidities is in general accord with observation if the sun is assigned a magnetic moment about 10^{34} gauss $\text{cm}^3=13.52$ Bv (sun-earth distance)^2.

MORE DETAILED ANALYSIS

Review of Relevant Parts of Stoermer-Lemaitre-Vallarta Theory

The equations of motion of a charged particle in a static divergence-free magnetic field of axial symmetry contain three coordinates but ordinarily admit only two first integrals; the motion itself is ordinarily quasi-ergodic.[9] Specifically, the two independent constants of the motion are (a) the kinetic energy (or momentum; or velocity), unaffected by a force which always acts perpendicular to the direction of motion; and (b) the component parallel to the axis of symmetry of the *total* angular momentum, in which we include both the kinetic and potential angular momenta. These quantities are most easily defined by employing cylindrical coordinates ρ, θ, z, to describe the position of the particle in space, and s to measure arc length along the trajectory. Then, in terms of the momentum, p, of the particle, the kinetic or ordinary angular momentum about the z axis is

$$\rho\cdot(\rho d\varphi/ds)\cdot p, \tag{1}$$

where the quantity in parenthesis is the cosine of the angle between p and the direction of increasing φ. The increase in this z component of the kinetic angular momentum in a given time is equal to the integral of the applied angular impulse, and by Lorentz' law of force is

$$\int \rho(\text{force})_\varphi dt=\int \rho(e/c)(\mathbf{v}\times\mathbf{H})_\varphi dt$$
$$=-(e/c)\int(\rho H_z d\rho-\rho H_\rho dz). \tag{2}$$

But the value of the last integral is independent of the path followed by the particle, because of the fact that the divergence of H vanishes:

$$0=\text{div}\mathbf{H}=\rho^{-1}[(\partial/\partial z)(\rho H_z)+(\partial/\partial\rho)(\rho H_\rho)]. \tag{3}$$

Consequently, the term following the minus sign on the right-hand side of (2), apart from a constant of integration, is a function only of the

[9] M. S. Vallarta, *Theory of the Allowed Cone of Cosmic Radiation* (Toronto University Applied Math. Series, 1935–43), Section 3.

position of the particle. The increase in the kinetic angular momentum is thus associated with an equal and opposite decrease in another function, to which we therefore assign the name of potential angular momentum. The sum of the two is evidently a constant of the motion:

$$p\rho^2 d\varphi/ds + (e/c)\int_{\infty}^{\rho,z} (\rho H_z d\rho - \rho H_\rho dz) = M_z = \text{constant}. \quad (4)$$

The integral in (4) represents ρ times the φ-component of the usual vector potential. In the case of a magnetic dipole of strength μ (gauss cm^3) directed along the *negative* z axis,[10] where, for example, in the equatorial plane the magnetic field in the z direction has the value $H_z = +\mu/r^3$, we have

$$p\rho^2 d\varphi/ds - (e/c)(\mu\rho^2/r^3) = M_z, \quad (5)$$

where $r^2 = \rho^2 + z^2$.

We recall that for any given value of the magnetic moment μ (gauss cm^3 or Bv cm^2) and for a positive particle of any given magnetic rigidity, (cp/e) (gauss cm or Bv), there exists one circular path ($\rho d\varphi/ds$ = minus 1) in which the given particle can circulate about the given dipole. The circle has the *characteristic radius* R, where

$$R^2 = \mu/(cp/e). \quad (6)$$

The radius R, as noted by Stoermer, provides a unit of length convenient to describe also the movement about the same dipole of a particle of the same rigidity, in any other orbit. The angular momentum of such an arbitrary motion may usefully be expressed in terms of the characteristic angular momentum, pR,

$$M_z = pR \cdot 2\gamma,$$

where γ is the so-called angular momentum parameter. For the special case of the circular orbit, $\gamma = -1$.

In terms of the characteristic radius R and the angular momentum parameter γ the law of conservation (5) takes the form $\rho d\varphi/ds$ (= cosine of the angle x between the momentum of the particle and the east, if we are dealing with motion in the field of the earth—or the directrix of the earth's motion, if we are dealing with motion in the field of the sun)

$$= \rho R^2/r^3 + 2\gamma R/\rho. \quad (7)$$

This is Stoermer's fundamental equation for the allowed cone of the cosmic radiation.

Consider positive particles of 3.38-Bv rigidity moving in the field of the sun, assumed for the present to have a moment of 10^{34} gauss cm^3 = 13.52 Bv (earth-sun distance)^2. For such particles the characteristic distance R of Eq. (6) is twice the radius of the earth's orbit.

Consider those among these particles which can arrive at the world's path ($\rho = r = R/2$) moving at the angle $x = 60°$ to the direction of the earth's motion. The angular momentum parameter for such particles is given by the equation

$$0.50 = \cos x = \rho R^2/r^3 + 2\gamma R/\rho = 4 + 4\gamma, \quad (8)$$

whence $\gamma = -0.875$. Conversely, the value of the angular momentum parameter γ determines those points in the meridian or (ρ, z) plane at which the particle might in principle arrive at some time in the course of its motion: all those points (ρ, z) which give to the right-hand side of (7)—and therefore give to $\cos x$—a value between -1 and $+1$. These points form an open domain which includes the earth (upper portion of Fig. 2).

Particles of the class we are considering, with γ in the neighborhood of -0.875, $R = 2\times$(sun-earth distance), arrive earth's orbit at angles $x \doteqdot 60°$ with respect to the earth's directrix and at all—or practically all—azimuths, and with the same intensity, I_0 (particles per cm^2 per sec. per unit solid angle) which characterized particles of the same rigidity at infinity. This uniformity of the intensity in all allowed directions follows from the well-known application of Liouville's theorem to the movement of cosmic-ray particles. A false impression of the number of forbidden azimuths would follow, however, from an uncritical application to the sun's field of the usual Stoermer-Lemaitre-Vallarta theory of motion in the earth's field. To take over that theory unchanged would be legitimate if we were on the surface of the sun, or if that body were

[10] The magnetic moments of both the sun and the earth are directed approximately opposite to the angular momenta of rotation and revolution of the earth, which we take to determine the direction of the positive z axis. Thus the direction of increasing φ agrees with the direction of both terrestrial movements.

surrounded by an opaque sphere with radius the same as that of the earth's orbit. Then many cosmic-ray trajectories would be cut off before they got to the point of observation. But no such great blocking effect actually occurs. Trace backwards a trajectory with *any* azimuth and with an angle of inclination, $x=60°$, with respect to the earth's directrix (upper portion in Fig. 2). The trajectory may, for example, lead back into the region A in the allowed zone, executing many loops and turns. But there is only an exceedingly small probability that the trajectory will wind its way sufficiently far down into the horn of region A to intersect the sun's surface and be blocked. Practically all trajectories will turn around before having reached in so close. Followed back, they will lead to infinity. They supply an almost full measure of channels to supply the earth from all azimuths with particles of the given class.

We conclude that an observer located at the earth's orbit (but for simplicity free of the local perturbing effects of the earth's field) will see the full intensity of 3.38-Bv particles in the cone of directions making an angle of 60° with respect to the earth's directrix.

These particles advance abreast in armies. Only a part of them actually hit the earth. Many more are deflected in the local field of the earth. This alteration in direction, without change of velocity, gives these particles new angular momenta about the sun. Consider the particles which depart from the earth at the angle $x=120°$, having in this way acquired an angular momentum parameter $\gamma=-1.125$. The allowed zone for these particles, determinable by the condition $-1\leqslant \cos x\leqslant 1$, or by

$$-1\leqslant(\rho R^2/r^3-2\times 1.125R/\rho)\leqslant 1, \qquad (9)$$

is a bounded region (lower portion of Fig. 2). The particles are trapped.

If there were no matter in the bounded zone, and if no radiative deceleration occurred, then the trajectory of the typical particle would in the course of time come indefinitely close to every point of the allowed volume. Only for a class of orbits of measure zero are certain periodicity conditions fulfilled which prevent in such cases the uniform coverage of all of the allowed zone. Such orbits have no influence on intensity questions. With this understanding we can say that every condition of motion allowed by the conservation laws occurs with equal probability. Specifically, particles of characteristic radius $R=2\times$(sun-earth distance), which have a rigidity of 3.38 Bv if the moment of the sun is 13.52 Bv (sun-earth distance)^2—or more particularly, those among these particles which have an angular momentum parameter, γ, between -1.125 and $(-1.125+d\gamma)$—will eventually become distributed with equal density in all portions of the meridian plane. Moreover, at any one point (ρ, z) of the meridan plane, where the particle's velocity necessarily makes an angle between x and $(x+dx)$ with the direction of increasing φ, and where

$$\cos x=\rho R^2/r^3-2\times 1.125R/\rho, \qquad (10)$$

$$d(\cos x)=2d\gamma(R/\rho), \qquad (11)$$

all values of the azimuth of the velocity vector are equally probable.

We can summarize and generalize the foregoing discussion of the intensity question as follows.

(a) To particles of a given rigidity (or more accurately, in a certain narrow band of rigidities) we ascribe at large distances from the sun the standard intensity of I_0 particles per cm^2 and per sec. and per unit solid angle. (b) We denote by R the characteristic radius of these particles in the sun's field (Eq. (6)).

We consider for the time being particles of such a rigidity, or of such a characteristic radius, that coming from infinity they can arrive at the earth's orbit, $r=r_e$, from a certain range of directions, between $x=0$ and $x=x_0$, where $x_0<\pi$. (c) Over this range of directions particles arrive in the neighborhood of the earth with the intensity I_0 per cm^2 and per sec. and per unit solid angle. (d) For directions between x_0 and π the intensity near the earth has the smaller value I, dependent in general upon x, but not dependent upon azimuth with respect to the earth's directrix. The value of I would be zero but for the circumstance that the stock of trapped particles is replenished fast enough to keep pace with losses. (e) The trapped particles which have angular momentum parameters between γ and $\gamma+d\gamma$ and which lie in the ring-like region of space in the allowed zone in the limits ρ to $\rho+d\rho$,

$z+dz$, near the earth, amount in number to the quantity

$$(I/v)2\pi\cdot d(\cos x)\cdot 2\pi\rho_e d\rho dz \tag{12}$$

$$=(I/v)\cdot 2\pi\cdot 2d\gamma(R/\rho_e)\cdot 2\pi\rho_e d\rho dz. \tag{13}$$

Here v is the velocity of the particles and I/v the number per unit volume and per unit solid angle. (f) At any other point (ρ, z) in the bounded zone the number of particles between γ and $d\gamma$ per unit ranges of ρ and z has a value identical with that at the earth.

Thus the total number of particles of the given class in the bounded region is

$$4\pi d\gamma(I/v)2\pi R\int_{\text{bounded zone}} d\rho dz. \tag{14}$$

The variables ρ and z may be expressed in terms of new variables, α and x, defined by the equations

$$\cos x=(\rho R^2/r^3)+(2\gamma R/\rho), \tag{15}$$
$$\cos^3\alpha=(\cos x)(\rho^2/R^2)-(2\gamma\rho/R).$$

Then the expression for the number of trapped particles reduces to the form

$$4\pi d\gamma(I/v)\cdot 2\pi R\cdot R^2\int_0^{\pi/2}[-2\gamma-(\gamma^2+\cos^3\alpha)^{\frac{1}{2}} -(\gamma^2-\cos\alpha^3)^{\frac{1}{2}}](4d\alpha/\cos^2\alpha)$$
$$=4\pi d\gamma\cdot(I/v)\cdot 2\pi R^3 g(\gamma). \tag{16}$$

Here $g(\gamma)$ is a transcendental integral which has been evaluated numerically. The solid angle $2\pi d(\cos x)$ spanned by the given class of particles at any permitted distance ρ from the axis of revolution of the earth is given from (11) by the expression $4\pi d\gamma(R/\rho)$. This solid angle increases or decreases as the particles approach or recede from the sun, in just such a way as to compensate the change in the volume $2\pi\rho$ associated with a unit range of ρ and z. (g) Thus the number of particles per unit volume and per unit solid angle at the general point ρ, z in the trapped zone has either the value zero, or the same value I/v which applies at the earth itself. However, every particle which at the earth had the direction cosine, $\cos x_e$, has at the new point the direction cosine $\cos x$

$$=\rho R^2/r^3+2\gamma R/\rho, \tag{17}$$

where γ is determined by the equation

$$\cos x_e=R^2/r_e^2+2\gamma R/r_e. \tag{18}$$

Negligible Effect of Radiative Deceleration on the Trapped Particles

The horn-shaped region where trapped particles move becomes narrower and narrower for a particle which approaches close to the sun. Moreover, the only region of the sun which the particle can approach is the neighborhood of the south or north pole. If the field here is 50 gauss, a particle of rigidity 3 Bv or 10^7 gauss cm will, if it moves perpendicular to the lines of force, describe a circle with a radius, r, of 2 km. The energy loss in one revolution at constant velocity is

$$-\Delta E=(4\pi/3)(e^2/r)(v/c)^3(E/Mc^2)^4, \tag{19}$$

where v is the velocity of the particle, E its energy, M its mass, and e its charge. If the particle is an electron, the loss of energy in one revolution is $\sim$3000 ev. For a proton the loss is $\sim 2\times 10^{-10}$ ev. In either case the change in energy is negligible. Moreover, after a few such nearly circular turns the particle will spiral out into the wide reaches of the horn where the rate of radiation is smaller by many orders of magnitude. Thus the losses by radiation will be negligible in comparison to the losses by such accidents as direct collision with the surface of the sun.

To confirm this conclusion about negligible rate of radiation by a more detailed analysis, we shall calculate how long a time is required for the particles in the bounded zone to lose ten percent of their energy by radiation. The number of these particles with angular momentum parameters in the interval $d\gamma$ and with meridian plane coordinates in the interval $d\rho dz$ is given by (13). The rate of radiation by a particle at the point ρ, z is

$$-(dE/dt)=(2e^2/3c^3)(vE/Mc^2)^4 \times(\text{radius of curvature})^{-2}$$
$$=(2e^4/3M^4c^5)(\mathbf{p}\times\mathbf{H})^2, \tag{20}$$

where $\mathbf{H}$, the magnetic field, lies in the meridian plane, while the momentum, $\mathbf{p}$, on the other hand, has a component in the meridian plane which points in one direction with as much probability as in another. Thus, averaging over azimuths of $\mathbf{p}$, we have

$$(\mathbf{p}\times\mathbf{H})_{\mathrm{Av}}{}^2 = H^2p^2 - (\mathbf{p}\cdot\mathbf{H})_{\mathrm{Av}}{}^2 = H^2p^2(1-\tfrac{1}{2}\sin^2 x). \quad (21)$$

Summing over all the trapped particles in the bounded zone, we have for the rate of radiation of energy

$$4\pi d\gamma(I/v)2\pi R\int\int d\rho dz(2e^4/3M^4c^5) \times H^2p^2(1-\tfrac{1}{2}\sin^2 x). \quad (22)$$

Integral (22) receives contributions of significant magnitude only for regions near the sun. There our calculations will not be quite right, because the absorption by the sun itself will decrease the intensity I to approximately half the value which obtains some distance from the sun. This computation of the radiative losses is therefore conservative.

The boundaries of the trapping region over which we integrate are found by solving (7) for ρ:

$$\rho = (\cos^2 x r^6/4R^4 - 2\gamma r^3/R)^{\frac{1}{2}} + \cos x r^3/2R^2. \quad (23)$$

As $\cos x$ varies from -1 to $+1$, ρ varies over the range r^3/R^2. For the case of a 3.38-Bv particle near a sun of polar field strength 50 gauss this distance is just the 4 km expected from our previous calculation. Moreover, compared to the radius of the sun, $a_{\mathrm{sun}} = 695{,}000$ km, the value of ρ itself is so small, $50{,}300\pm 2$ km, that we can in evaluating the radiation integral (22) make the following simplifications:

$$d\rho dz \to d\rho dr \to r^3dr/R^2, \quad (\sin^2 x)_{\mathrm{Av}} \to \tfrac{2}{3}, \quad H\to 2\mu/r^3 \to 2(cp/e)R^2/r^3. \quad (24)$$

We find for the rate of loss of energy

$$4\pi d\gamma(I/v)2\pi R(8e^2c/9)(R^2/a_{\mathrm{sun}}{}^2)(p/Mc)^4. \quad (25)$$

The amount of energy present is, on the other hand, in the approximation $E \doteqdot cp$,

$$(cp)4\pi d\gamma(I/v)2\pi R\cdot R^2\cdot(\text{a dimensionless factor of order unity}).$$

Consequently, the time required for radiation to reduce the energy of the trapped particles by 10 percent is

$$T\sim 0.1[a_{\mathrm{sun}}{}^2/c(e^2/Mc^2)](Mc/p)^3. \quad (26)$$

The critical time for electrons of 5-Bv rigidity, is of the order of 2000 years, and for

TABLE III. Values of $g(\gamma)$ in Eq. (16).

γ	large and negative	-2.0	-1.5	-1.2	-1.1	-1.0	-1.0
$g(\gamma)$	$3\pi/16(-\gamma)^3$	0.0762	0.179	0.393	0.510	0.885	not defined, zone not bounded

protons of the same rigidity is roughly 10^{16} years. The time spent in the trapped state, on the other hand, is comparable to 5000 years only (see below). Consequently, for protons the energy lost via radiation in the available time is negligible. For electrons, on the other hand, these losses are not at all negligible. If electrons constitute a significant fraction of the incoming radiation, then one should indeed expect on this account a lowering of intensity near the earth from those directions in the sun's field from which arrival of particles if forbidden by the simple Stoermer theory. In view of evidence[11] that electrons amount to one percent or less of the incident intensity, we disregard electrons in the present analysis and therefore conclude that we may neglect the radiative effect.

Losses by Collision with Dust Particles

Alfvén has already noted the possibility of an absorption of the trapped particles by dust and interplanetary matter. Our calculation of the magnitude of this effect differs somewhat from his. (a) We estimate below for the order of magnitude of the time of trapping 5000 years, instead of his 200 years. (b) We adopt for the density of interplanetary matter the recent figure 5×10^{-21} g/cm^3 given by van de Hulst.[12] Thus we find for the average amount of matter traversed (making no allowance for the quite probable decrease of the density in regions of the trapped zone well above and below the plane of the earth's orbit) the figure

$$(5\times 10^3\text{ yr.})\times(10^{18}\text{ cm/year}) \times(5\times 10^{-21}\text{ g/cm}^3) = 25\text{ g/cm}^2. \quad (27)$$

In contrast to this figure the mean free path of protons for interaction with matter is of the

[11] R. L. Hulsizer and B. Rossi, Phys. Rev. **73**, 1402 (1948).

[12] H. C. van de Hulst, Astrophys. J. **105**, 471 (1947). Contrast his figure with the earlier value of P. van Rhyn cited in his paper or Baumbach's value of $\sim 10^{-19}$ g/cm^3 for the corona (S. Baumbach, Astro. Nachr. **236**, 121 (1937)).

order of 100 g/cm^2.[13] Thus the calculations taken at face value would indicate a loss of the order of 25 percent of the trapped particles via interaction with matter. However, we have here made no allowance for the quite probable decrease of the density of interplanetary matter in regions of the trapped zone well above and below the plane of the earth's orbit.[14] We shall therefore assume the present absorption effect to be negligible, as did Alfvén. Nevertheless, we have to recognize the existence of a large uncertainty in all existing figures for the density of interplanetary matter. Instead of trying to use this uncertain data to estimate the decrement of the cosmic-ray intensity in the so-called forbidden regions, it would probably be more reasonable in future considerations of this kind to use cosmic-ray data to draw conclusions about dust and other matter.

Losses in Collisions with the Moon

It appears reasonable to consider the magnetic moment of the moon so small that we can treat this object as absorbing cosmic rays like an opaque sphere of cross section π(moon radius)^2 $=\pi a_{\mathrm{moon}}^2=0.95\times 10^{17}$ cm^2. (a) The moon has only 0.012 the mass of the earth. (b) Relative to the earth its iron content is presumably even smaller, as indicated by the characteristic difference in density of the two planets (5.52 *vs.* 3.36). (c) Its rotation is slower by the factor 28. (d) Even for our standard of comparison, the earth, the magnetic scattering effect is weak enough so that a significant fraction of this planet is accessible to the cosmic radiation.[15]

Losses in Collisions with the Earth

Consider particles of a given rigidity approaching the earth from random directions. If every portion of the surface, $4\pi a_{\mathrm{earth}}^2$, could be reached from the full solid angle of the sky, 2π, then we could say that the earth behaves like an opaque, non-magnetic object, and we should attribute to it an absorption cross section equal to the geometrical cross section, πa_{earth}^2. This is the case for particles of rigidity 60 Bv and greater. For particles of lower rigidity the allowed cone fills at any given latitude, λ, a solid angle, $\Omega(\lambda)$, which is, in general, less than 2π. In this case the earth absorbs from the stream of cosmic-ray particles like a disk of cross section

$$(1/4)\int_{-\pi/2}^{\pi/2}[\Omega(\lambda)/2\pi]a_{\mathrm{earth}}^2 2\pi\cos\lambda d\lambda \equiv \pi a_{\mathrm{earth}}^2 F, \quad (28)$$

where F is a dimensionless factor less than unity. This *accessibility factor* was determined as follows: (a) We integrated the curves given by Johnson[16] to find

$$\Omega(\lambda)=2\pi\int_0^{\pi/2} f\sin\zeta d\zeta$$

(in terms of his notation). (b) We then integrated Ω over the surface of the earth for several values of the rigidity to obtain the accessibility factor, F, shown in Fig. 5.

Collisions with Mars and Venus

The lower portion of Fig. 2 shows that particles with a characteristic radius, R, in the sun's field of twice the sun-earth distance, and with an angular momentum parameter $\gamma=-1.125$, move in a cage, the borders of which are not far from including Mars or Venus. For what values of R and γ does the cage include both the earth and Mars? For Mars to be included, the inclination x at that planet must lie between 0 and π, or

$$-1\leqslant R^2/r_{\mathrm{Mars}}^2+2\gamma R/r_{\mathrm{Mars}}\leqslant 1, \quad (29)$$

whence we conclude that the angular momentum parameter must lie between the limits

$$-r_{\mathrm{Mars}}/2R-R/2r_{\mathrm{Mars}} \leqslant\gamma\begin{cases}\leqslant r_{\mathrm{Mars}}/2R-R/2r_{\mathrm{Mars}} & \text{(30a)}\\ \leqslant -1. & \text{(30b)}\end{cases}$$

The inequality (30b) is added here as the condition for a bounded zone to exist at all (compare upper and lower portions of Fig. 2). The upper limit on γ set by (30a) is the relevant one when the characteristic radius, R, of the particle in the sun's field exceeds 2.4142 times the radius of Mars' orbit (3.69 times r_{earth}; rigidity 13.52 Bv/(3.69)^2=0.99 Bv *or less* for sun moment$=10^{34}$

[13] See, for example, B. Rossi, Rev. Mod. Phys. **20**, 537 (1948).

[14] L. Spitzer, Jr., Astrophys. J. **95**, 329 (1942).

[15] See M. S. Vallarta, Nature **161**, 646 (1948), and Dauvillier, C.R. Acad. Sci., Paris, **225**, 839 (1947), for other arguments for a negligible magnetic moment for the moon.

[16] T. H. Johnson, Rev. Mod. Phys. **10**, 221 (1938).

gauss cm³). When the magnetic rigidity is greater or the characteristic radius is less, then the limit (30b) applies. This is the case in which we shall be interested.

Positive particles in the range of angular momenta set by (30b) arrive at the earth with inclinations to the earth's directrix over the whole range given by the limits

$$(\cos x)_{\min} = R^2/r_e^2 - r_{\text{Mars}}/r_e - R^2/r_e r_{\text{Mars}}, \quad (31)$$

and

$$(\cos x)_{\max} = R^2/r_e^2 - 2R/r_e, \quad (32)$$

insofar as these limits lie inside the geometrically possible limits $-1 < \cos x < 1$. The various possibilities are summarized in Table IV.

Consider those trapped particles of characteristic radius R and angular momentum parameter γ to $\gamma + d\gamma$ which find accessible both Mars and the earth. The number of these particles in the bounded zone in the range of coordinates $d\rho dz$ is

$$4\pi d\gamma (I/v) 2\pi R d\rho dz, \quad (33)$$

where I depends upon R and γ but is independent of ρ or z. We divide by $2\pi\rho d\rho dz$ to obtain the particle density, multiply by v to obtain the flux, and multiply again by the effective cross section of the appropriate planet to obtain the number of particles lost per second. Thus, the rate of loss due to the earth and moon together is

$$4 d\gamma I(R/r_e)(\pi a_e^2 F + \pi a_{\text{moon}}^2). \quad (34)$$

The contribution of Mars we similarly take to be

$$4\pi d\gamma I(R/r_{\text{Mars}}) \pi a_{\text{Mars}}^2. \quad (35)$$

Thus Mars, when it can absorb at all earth-accessible trapped particles, has the same effect as a fictitious planet located on the earth's orbit with a cross section

$$(r_e/r_{\text{Mars}}) \pi a_{\text{Mars}}^2 = 2.42 \times 10^{17}\ \text{cm}^2. \quad (36)$$

Figure 3 shows in comparison with the cross sections of the earth and the moon the so defined effective cross section of Mars, for particles of several rigidities, as a function of the inclination of the particle's trajectory near the earth's orbit (or equivalently, as a function of the angular momentum).

Just as for Mars, we define for Venus the effective cross section

$$(r_e/r_{\text{Venus}}) \pi a_{\text{Venus}}^2 = 17.25 \times 10^{17}\ \text{cm}^2 \quad (37)$$

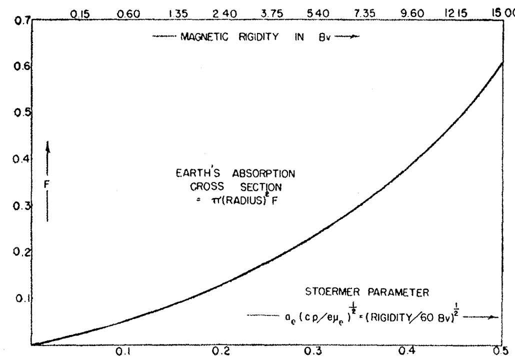

FIG. 5. Accessibility factor, F.

which applies to trapped particles which can reach both Venus and the earth (Table V and Fig. 3).

It is seen from Fig. 3 that Venus, while endowed with an effective cross section much larger than that of Mars or the earth, is able to bring that cross section into action only for a small fraction of the particles of interest to us here.

In the present analysis we neglect the magnetic moments of Mars and Venus. To do so is not obviously legitimate. The diameter and density of Venus agree within 6 percent with the corresponding quantities for the earth. There appears to be no reliable information about the rate of rotation of Venus. However, the rotational period of Mars agrees within 10 percent with the period of the earth, and there is no evident reason why the same should not be true for Venus. Consequently, it would not be surprising if Venus had a magnetic moment of the same order as that of the earth. Such a moment would have the following effects: (a) The absorption cross section of Venus would be reduced by a factor F similar to that presented graphically as

TABLE IV. Trapped particles which find both Mars and the earth accessible. Here r_e = radius of earth's orbit, R = "characteristic radius" of particle = (sun's moment in Bv cm²/particle rigidity in Bv)$^{\frac{1}{2}}$.

R/r_e	Rigidity in Bv for sun moment of 10^{34} gauss cm³	Limiting inclinations near earth of those trapped particles which find Mars accessible
3.69 to 2.70	0.99 to 1.85	Can't reach earth
2.70 to 2.414	1.85 to 2.32	$\left[(\cos x)_{\min} =\right.$ $(\cos x)_{\max} = 1$
2.414 to 1.236	2.32 to 8.85	$\left.0.346(R/r_e)^2 - 1.53\right]$ $\left[(\cos x)_{\max} =\right.$
1.236 to 1	8.85 to 13.52	$(\cos x)_{\min} = -1$ $\left.(R/r_e)^2 - 2R/r_e\right]$
Less than 1	Greater than 13.52 Bv	Can't be trapped if they reach earth

TABLE V. Trapped particles which find both Venus and the earth accessible.

R/r_e	Rigidity in Bv for sun moment of 10^{34} gauss cm^3	Limiting inclinations near earth of those trapped particles which find Venus accessible
Above 2.13	Below 2.98	Can't reach earth.
2.13 to 1.75	2.98 to 4.41	$(\cos x)_{\min}=-1$; $(\cos x)_{\max}=0.725-0.38R^2/r_e^2$.
1.75	4.42	$(\cos x)_{\min}=-1$; $(\cos x)_{\max}=-0.438$.
1.75 to 1	4.42 to 13.52	$(\cos x)_{\min}=-1$; $(\cos x)_{\max}=R^2/r_e^2-2R/r_e$.
Below 1	Above 13.52	Can't be trapped if they reach earth.

a function of rigidity in Fig. 5. (b) The losses from the zone of trapping due to Venus, while decreased by this reduction in absorption, would seem at first sight to be on the whole increased via the presence of a large scattering cross section. In this process Venus would imitate the action which is illustrated for the earth itself in Fig. 2, deflecting the already trapped particles into a new bounded zone, ordinarily even closer to the sun. (c) But the particles thus deprived of access to the earth, after many circuits of the new cage, may be rescattered by Venus into an earth-containing zone. Consequently, it is quite conceivable that assignment of a magnetic moment to Venus will actually decrease the net absorptive effect of that planet. (d) The absorptive effect of Venus is already small when we take into account the narrowness of the zones of action depicted in Fig. 3. Consequently, we shall neglect any reduction in this effect which may come about via a possible moment of Venus.

In contrast to Venus, Mars will be expected to have a moment much smaller than that of the earth: (a) Its average density is 3.95, compared to the earth's 5.53, indicating a lower proportion of iron. (b) Its mass is only 11 percent of the earth's mass, indicating even for the same degree of magnetization a 9-times smaller moment. Thus it is not unreasonable to neglect the magnetic moment of Mars in the present calculations.

If Mars does, nevertheless, possess a substantial moment, the consequences will be interesting. To be specific in discussing these consequences, let us fix on a value for the moment of the sun equal to 10^{34} gauss $cm^3=13.52$ Bv (sun-earth distance)^2. Then the lowest rigidity for which particles from outer space have access to the earth is 13.52 Bv$/(2.4142)^2=2.32$ Bv, which will then represent the lower limit to the cosmic-ray spectrum. To Mars, on the other hand, particles can arrive from outer space with rigidities down to $(r_e/r_{\mathrm{Mars}})^2 2.32$ Bv$=0.99$ Bev. If that planet had a scattering cross section significant in comparison with its absorption cross section, it could trap these particles after the manner of the earth's trapping effect. Some of the thus caught particles would subsequently reach the earth—specifically, those among these particles which have rigidities greater than 1.85 Bv (Table IV). Consequently, the presence of a substantial magnetic moment on Mars would alter the cut-off of the cosmic-ray spectrum from 2.32 Bv to 1.85 Bv, probably an experimentally detectable effect. We look apart from such an effect in the present paper, but hope at a later time to analyze the possibility of determining in this way the magnetic moment of Mars.

The Other Planets and the Interplanetary Transfer Process

If Mars could, in principle, pass particles of low rigidity on to the earth, cannot planets still further out pass on particles of still lower rigidity to Mars for eventual delivery to the earth? Is there any limit to the length of the chain of planets which can be formed in this way?

It is a remarkable feature of the interplanetary transfer process that it can take place only if the radii of successive orbits differ by a factor less than 2.4142. It is an equally remarkable feature of the solar system that the planetary orbital radii differ from one another by a factor only a little less than 2 (Bode's law). But it is most remarkable of all that the chain which would thus be possible is, in fact, completely broken by the absence of a planet between Mars and Jupiter. The radii of these two planets differ by the factor 3.42. Instead of the planet which at one time presumably revolved between them, we now have only the multitudinous asteroidal fragments. Magnetic scattering by such fragments can be completely neglected. For spheres of equal degrees of magnetization the magnetic moment and scattering cross section decrease as R^3, while the absorption cross section decreases only as R^2. Thus the absorption dominates for fragments of small size.

Granted that the asteroids constitute a broken link in an otherwise possible interplanetary transfer process, there can be no effect on the cosmic-ray intensity near the earth of these objects and more remote planets. Neither group of bodies intercept any trapped particles to which the earth is accessible.

As for planets closer to the sun than Venus, there is only Mercury to consider. It can have no effect at the earth if the intervening link—Venus itself—is a complete absorber; and even if Venus does have a significant scattering cross section, our results, as noted above will be relatively little affected. For these reasons we limit our attention to Mars, Venus, the earth, the moon, and the sun.

Absorption by the Sun

A particle which approaches the sun follows a spiral path with radius of the order of magnitude of a kilometer about a line of force which leads in close to the north or south pole. The location of the particle as projected onto the line of force moves in towards the magnetic dipole, comes to a minimum distance, and then moves away again without any repeated in-and-out movements (so long as the radius of the loops is small in comparison with distance from the sun). Consequently, the sun directly decreases the number of trapped particles only insofar as it cuts down to zero the flux of outward moving particles in the horn of the bounded zone close to the sun. But in calculating the number of impacts on the sun we need only consider the flux of inward moving particles.

This inward flux can in a very good approximation be considered as unaffected by the sun, and therefore (see Eq. (33) and following discussion) amounts to

$$4\pi d\gamma I(R/a_{\mathrm{sun}})(d\varphi/2\pi) \tag{38}$$

particles per cm^2 and per sec. Here φ is the azimuthal angle of the velocity vector about an eastbound line of constant latitude on the surface of the sun. The particles in question move with respect to this line at an inclination x which varies uniformly from 0 to π as we move over the surface of the sun from one boundary of the trapping region to the other. According to (23), we can represent the ρ-coordinate of the typical point on the surface to a good approximation by the expression

$$\rho=(-2\gamma a_{\mathrm{sun}}{}^3/R)^{\frac{1}{2}}+\cos x\, a_{\mathrm{sun}}{}^3/2R^2. \tag{39}$$

The direction cosine of the velocity vector with respect to the inward drawn normal is $\sin x \sin\varphi$, with φ running from 0 to π only (zero outward flux). The number of trapped particles of the class (R; γ to $\gamma+d\gamma$) lost per second by collision with the sun is thus

$$\int \text{flux}\cdot\text{direction cosine}\cdot d(\text{surface})$$

$$\begin{aligned}&=\int_{-1}^{1}\int_{0}^{\pi}\{4\pi d\gamma I(R/a_{\mathrm{sun}})(d\varphi/2\pi)\}\\ &\qquad\times\sin x\sin\varphi\{a_{\mathrm{sun}}{}^3 d(\cos x)/2R^2\}\\ &\qquad\qquad\times 2\pi(-2\gamma a_{\mathrm{sun}}{}^3/R)^{\frac{1}{2}}, \qquad (40)\\ &=4\pi d\gamma I(\pi a_{\mathrm{sun}}{}^2/2)(-2\gamma a_{\mathrm{sun}}{}^3/R^3)^{\frac{1}{2}}.\end{aligned}$$

This rate of loss is the same which would occur if there existed at the earth's orbit a fictitious cross section obtained by dividing (40) through by the expression $4\pi d\gamma I(R/r_e)$ (see Eq. (35)). Thus the equivalent effective cross section of the sun is

$$(\pi a_{\mathrm{sun}}{}^2/2)(r_e/R)(-2\gamma a_{\mathrm{sun}}{}^3/R^3)^{\frac{1}{2}}$$

or

$$\begin{aligned}&(\pi a_{\mathrm{sun}}{}^2/2)(a_{\mathrm{sun}}{}^3 r_e/R^4)^{\frac{1}{2}}(1-\cos x\, r_e{}^2/R^2)^{\frac{1}{2}}\\ &\quad=24.2\times10^{17}\ \mathrm{cm}^2(r_e{}^2/R^2)(1-\cos x\, r_e{}^2/R^2)^{\frac{1}{2}}\\ &\quad=(\pi a_{\mathrm{sun}}{}^{7/2} r_e{}^{\frac{1}{2}}/2)\{(cp/e)/\mu_{\mathrm{sun}}\}\\ &\qquad\qquad\times\{1-(cp/e)\cos x\, r_e{}^2/\mu_{\mathrm{sun}}\}^{\frac{1}{2}}, \qquad (40a)\end{aligned}$$

where we have expressed the angular momentum parameter in terms of the inclination, x, of the particles at the time they near the *earth's* orbit.

It is seen from (40a) that the equivalent effective absorption cross section presented by the sun goes up roughly in proportion to the rigidity of the particles under consideration, and is ordinarily significantly greater than the absorption cross section of the earth itself. The dependence upon rigidity and inclination can be seen in more detail from Fig. 3.

Time of Circulation of Trapped Particles

Imagine the earth suddenly to be demagnetized. Then the absorption effects of Mars, Venus, moon, and sun as just discussed, and the full geometrical cross section of the earth, would cause the number of trapped particles to decrease exponentially with time. The time, τ, for the intensity to fall to $1/e$ of its present value furnishes a crude measure of the average time of circulation of caged corpuscles. Denoting by σ the equivalent effective cross sections of the five heavenly bodies, we have from (16) and (III) the result

$$\tau = 2\pi R^2 r_e g(\gamma)/v(\sigma_e + \sigma_{\text{moon}} + \sigma_s + \sigma_m + \sigma_v). \quad (41)$$

Consider, for example, particles with a characteristic radius, R, equal to twice the radius r_e of the earth's orbit (rigidity 3.38 Bv if sun's moment is 10^{34} gauss $\text{cm}^3 = 13.52$ Bv (sun-earth distance)^2), with such a value of $\gamma(=-1.125)$ that when near the earth's orbit they move at an angle of $x = 120°$ with respect to the earth's directrix. The numerator of the expression for τ then has the value 4.1×10^{40} cm^3. In the denominator the values to be used for the equivalent cross sections *in this particular case* (see Fig. 3) are

$$\begin{aligned}
\sigma_{\text{earth}} &= 12.8 \times 10^{17}\ \text{cm}^2 \\
\sigma_{\text{moon}} &= 1.0 \times 10^{17} \\
\sigma_{\text{sun}} &= 6.4 \times 10^{17} \\
\sigma_{\text{Mars}} &= 2.4 \times 10^{17} \\
\sigma_{\text{Venus}} &= 0.0 \times 10^{17} \\
\hline
\sigma_{\text{total}} &= 22.6 \times 10^{17}\ \text{cm}^2.
\end{aligned} \quad (42)$$

With v set equal to the speed of light we find $\tau = 6.0 \times 10^{11}$ sec. $= 19{,}000$ years.

The definition of the time τ used here of course overlooks the loss of trapped particles via scattering at the earth itself. It is difficult to define quantitatively a time τ which takes into account this effect, because the earth's scattering cross section diverges for small angles. However, considering the scattering only for angles of appreciable magnitude, we conclude that σ_{total} may be increased effectively by a factor possibly as much as ten. Consequently, we take 5000 years as a reasonable order of magnitude estimate of the time of circulation of the trapped particles of interest to us here. This is the figure used in the considerations above on the influence of radiative deceleration and dust.

The Scattering Cross Section of the Earth

The qualitative discussion in the introduction of the scattering of charged particles by the earth's field already gave an order of magnitude estimate for the cross section, $d\sigma$, for a deviation of the direction vector into a solid angle element $d\Omega$, making an angle θ with the original direction:

$$d\sigma \sim \mu_e/(cp/e)(d\Omega/\theta^3). \quad (43)$$

A more nearly quantitative analysis must now be made. To give a completely precise treatment of the scattering would, of course, be much more complicated than the still uncompleted task of the Lemaitre-Vallarta theory to give a complete account of the allowed cone of the cosmic radiation. To avoid this prohibitive task, we shall treat the scattering in the approximation in which the deviation of the particle from a straight line path is at all times regarded as a relatively small quantity. Thus we shall have a thoroughly accurate treatment of the deflections of small angle, for which the differential cross section, $d\sigma/d\Omega$, becomes very large. A sum theorem for the integral cross section will then permit us to obtain a reasonable estimate of the number of the relatively improbable deflections of large angle.

When we consider a particle which experiences only small deviations from a straight line, we can calculate its deflection by integrating with respect to time over the unperturbed rectilinear motion the component of force perpendicular to the line of motion:

$$\theta = \int \frac{(\text{normal force})dt}{\text{momentum}}. \quad (44)$$

Expressing the force in terms of the magnetic field and the velocity, and writing $\mathbf{ds}$ for the element of distance $\mathbf{v}dt$, we have

$$\boldsymbol{\theta} = (cp/e)^{-1} \int \mathbf{ds} \times \mathbf{H}. \quad (45)$$

Here the vector $\boldsymbol{\theta}$ is evidently perpendicular to the line of motion, as it should be. This vector gives us not only the magnitude of the deflection, but also its azimuth.

Instead of evaluating (45) by direct integration for a general line of motion and for an arbitrary orientation of the earth's polar axis, we can obtain the same results much more readily by simple considerations of vector covariance.

The deflection $\boldsymbol{\theta}$ is a linear function of the magnetic field **H**. Moreover, **H** itself is a linear function of the magnetic moment vector, $\boldsymbol{\mu}$. Consequently, the vector $\boldsymbol{\theta}$ may be considered to be the sum of three parts, each one of which is, respectively, due to one of the three components of μ.

Let the magnetic moment $\boldsymbol{\mu}$ be analyzed into two components perpendicular to the line of motion of the particle and one component parallel to that line. Then this third component contributes nothing to $\boldsymbol{\theta}$: When the dipole moment of the earth is parallel to the line of motion of a particle, then for every element of path length **ds** above the equatorial plane on which the perpendicular component of **H** has one value, there is another element of path length an equal distance below the equatorial plane on which the perpendicular component of **H** has an equal and opposite value. Thus the integral (45) for the deflection reduces to zero. Consequently, only components of the magnetic moment perpendicular to the line of motion contribute to a deviation. More specifically, in a coordinate system of which the z axis is parallel to the line of motion, θ_x and θ_y must be linear functions of μ_x and μ_y.

The deflection vector $\theta=(\theta_x, \theta_y)$ depends not only upon the two-dimensional vector $\boldsymbol{\mu}=(\boldsymbol{\mu}_x, \boldsymbol{\mu}_y)$ but also upon the vector $\boldsymbol{\rho}=(X, Y)$ from the center of the dipole to the point where the line of motion pierces the X, Y plane.

The relation between $\boldsymbol{\theta}$ and the other two vectors $\boldsymbol{\mu}$ and $\boldsymbol{\rho}$ must be covariant with respect to coordinate transformations in the X, Y plane. Consequently, we can write $\boldsymbol{\theta}=a\boldsymbol{\mu}+b\mathbf{p}$. Here the coefficients a and b are invariant against rotations of the coordinate system in the (X, Y) plane. Moreover, a must not contain μ, while b must be linear in μ. The most general expression which satisfies these requirements has the form

$$\boldsymbol{\theta}=f\cdot(\rho)\boldsymbol{\mu}+f_2(\rho)(\boldsymbol{\rho}\cdot\boldsymbol{\mu})\boldsymbol{\rho}, \tag{46}$$

where $\rho=(x^2+y^2)^{\frac{1}{2}}$ is the so-called impact parameter of the scattering process and $f_1(\rho)$ and $f_2(\rho)$ are functions thus far arbitrary.

We determine the ***ratio*** of the unknown functions f_1 and f_2 by a simple physical argument. Consider a hollow cylindrical bundle of parallel wires each carrying the same amount of current and lined up parallel to the z axis, with the earth's dipole at the center of the bundle. The magnetic field vanishes inside the cylinder. Consequently, there is no force or torque exerted on the dipole. By the law of action and reaction it follows that the force exerted by the dipole on a typical wire must vanish on being averaged over wires of all azimuths. But this force is measured by the average value of the deflection $\boldsymbol{\theta}$ of Eq. (4), namely,

$$(\boldsymbol{\theta})_{\text{Av}}=[f_1(\rho)+\tfrac{1}{2}\rho^2 f_2(\rho)]\boldsymbol{\mu}. \tag{47}$$

We conclude that $f_2=-2f_1/\rho^2$. To calculate f_1 itself, we consider the case where the particle moves in the equatorial plane of the dipole and the integral (45) is easily computed: $H=\mu(\rho^2+z^2)^{-\frac{3}{2}}$; $\theta=2\mu e/cp\rho^2$. Comparing with (46), we find

$$\boldsymbol{\theta}=(2/\rho^2)(cp/e)^{-1}[-\boldsymbol{\mu}+2(\boldsymbol{\mu}\cdot\boldsymbol{\rho})\boldsymbol{\rho}/\rho^2]. \tag{48}$$

Writing

$$\begin{aligned}(\mu_x, \mu_y)&=\mu_\perp(\cos\alpha, \sin\alpha),\\ (\rho_x, \rho_y)&=\rho(\cos\beta, \sin\beta),\\ (\theta_x, \theta_y)&=\theta(\cos\gamma, \sin\gamma),\end{aligned}$$

we can examine separately the magnitude and azimuth of the deflection:

$$\theta=(2\mu_\perp/\rho^2)(cp/e)^{-1}, \tag{49}$$

$$\gamma=\alpha-2\beta. \tag{50}$$

We conclude from these results that (a) the magnitude of the deflection is independent of the azimuth of the impact parameter; (b) the azimuth of the deflection is altered by $2d\beta$ by a change $d\beta$ in the azimuth of the impact parameter; (c) all values of the azimuth of the deflection are covered twice as the azimuth of the impact parameter sweeps through 2π; and (d) all values of the azimuth of the deflection are equally probable, just as if we were dealing with deflections under the action of central forces.

The scattering cross section $d\sigma=2\pi\rho d\rho$ associated with deflections between θ and $\theta+d\theta$ follows directly from (49):

$$d\sigma=\pi d(\rho^2)=\pi d(2\mu_\perp/\theta)(cp/e)^{-1} \doteqdot \mu_\perp(cp/e)^{-1}\theta^{-3}d\Omega, \quad (51)$$

where $d\Omega$ represents the element of solid angle $d\Omega=2\pi\sin\theta d\theta$.

The expression just derived for the differential cross section for scattering is, of course, valid only for small angles. It may be regarded as merely the leading term in a power series for the cross section as a function of θ. It being most difficult completely to determine this function for all angles, we shall have to use an empirical expression for the cross section. On the choice of this expression we impose the following requirements:

(a) It must reduce for small θ to Eq. (51).

(b) It must have the same value for deflections of magnitude θ and $(2\pi-\theta)$.

(c) The integral $\int_\theta^\pi (d\sigma/d\Omega)2\pi\sin\theta d\theta$ for the probability of all deflections greater than θ must reduce for small values of θ to

$$\pi\rho^2=(2\pi\mu_\perp/\theta)(cp/e)^{-1}.$$

As simplest expression satisfying these requirements we adopt the formula

$$d\sigma/d\Omega=(\mu_\perp/8)(cp/e)^{-1}[(\sin\theta/2)^{-3}+2], \quad (52)$$

where the function $\sin(\theta/2)$ meets requirement (b), the power -3 satisfies (a), and the constant 2 is required to fit condition (c).

Two further notes are required. First, our later calculations will be little affected in accuracy and much simplified in execution by using an average value for the perpendicular component of the dipole moment, regardless of the direction of motion of the particle under consideration. Thus, particles which travel in the direction of the motion of the earth in its orbit will see the full value, $\mu_\perp=\mu$, while particles traveling perpendicular to that direction will see $\mu_\perp=0$ or $\mu_\perp=\mu$ in the two extreme azimuths, and $\mu_\perp=(2/\pi)\mu=0.636\mu$ on the average. Moreover, such an averaging with respect to azimuths is automatically guaranteed by ergodic considerations for different particles whose directions of motion make the same angle with respect to the direction of the earth's motion. Now the difference between 0.636μ and μ itself is percentagewise not very great, considering the other approximations which we are forced to make. Consequently, we shall here and in the following adopt for $\mu_\perp$ the value

$$(\mu_\perp)_{Av}=\mu(\sin x)_{Av} =\mu\left(\int\sin^2 x dx d\psi \Big/ \int \sin x dx d\psi\right) =(\pi/4)\mu=0.785\mu. \quad (53)$$

Second, going over from the consideration of a naked dipole as so far considered to a dipole sheathed in solid matter, we have to recognize that the scattering cross section will be reduced to whatever extent absorption occurs. The absorption cross section of the earth we have already represented in the form πa^2F, where a is the radius of the earth and F is a factor ranging from 1 for particles with a magnetic rigidity of 60 Bv down to 0.1 for 1-Bv particles.

Particles of rigidity 15 Bv and less, such as we are considering here, if they had found the earth a transparent object, would have traveled in tortuous orbits, ordinarily with several loops, and would have departed with nearly random directions. The loss in scattering on account of absorption is therefore most reasonably treated as a reduction in that term of the empirical expression (9) which represents isotropic scattering. Combining this reduction with our earlier averaging over values of $\mu_\perp$, we adopt as final simplified expression for the differential scattering cross section

$$d\sigma/d\Omega=[\pi\mu/32(cp/e)][(\sin\theta/2)^{-3}+2] -(1/4\pi)\pi a^2F. \quad (54)$$

Equilibrium between Scattering and Absorption

Of particles of a given rigidity—or equivalently of a given characteristic radius R in the sun's field—the flux, or number, I, per cm^2 and per sec. and per unit solid angle, is dependent upon the angle of inclination, x, at the earth's orbit, but independent of the azimuthal angle, φ, about the earth's directrix. For inclinations between $x=0$ and $x=x_0$, where

$$\cos x_0=R^2/r_e^2-2R/r_e, \quad (55)$$

the particles arrive from outer space. For these values of x the intensity I therefore has the standard value I_0 associated with particles of the given rigidity at great distances from the solar system. For inclinations between x_0 and π, on the other hand, the intensity has a lower value which is determined by the equilibrium between absorption and scattering.

When a steady state has been reached, and losses from a fixed solid angle element $d\Omega_1$, at $x=x_1$, balance gains into this solid angle from all other solid angle elements $d\Omega_2$, then the intensity $I(x)$ satisfies—for x between x_0 and π—the equation

$$\int_2 d\Omega_2 I(x_2)(d\sigma/d\Omega)_{21}d\Omega_1 = I(x_1)d\Omega_1\left\{\sigma(x_1)+\int_2 d\Omega_2(d\sigma/d\Omega)_{12}\right\}. \quad (56)$$

In this equation $\sigma(x)$ symbolizes the total absorption cross section of the five heavenly bodies as represented in Fig. 3, and $(d\sigma/d\Omega)_{12}$ is the differential cross section as estimated in the preceding section for scattering from the direction 1 to the direction 2 (equal to the cross section for scattering in the converse direction).

The integral equation (56) has only a formal significance because the scattering cross section diverges for small angles. The equation acquires a better defined meaning when the scattering terms are rearranged so that their difference appears on one side of the equation, and when the scattering cross section is cut off at a minimum angle, $\theta_{12}=\epsilon$, which is later allowed in the limit to go to zero.

The integral equation is transformed into a more convenient form by writing

$$I(x)=\sum_{L=0}^{\infty} C_L P_L(\cos x). \quad (57)$$

We recall from the theory of spherical harmonics the relation

$$P_L(\cos x_2)=\sum_{m=-L}^{L}\frac{(L-|m|)!}{(L+|m|)!}P_L^{(m)}(\cos\theta_{12}) \times P_L^{(m)}(\cos x_1)\exp(im\alpha), \quad (58)$$

where α is the dihedral angle between (a) the plane containing directions 1 and 2 and (b) the plane containing direction 1 and the earth's directrix (the axis of our system of spherical polar coordinates, x and φ). In the integral on the left-hand side of (56), the differential $d\Omega_2=\sin x_2 dx_2 d\varphi_2$ can be written in the equivalent form, $\sin\theta_{12}d\theta_{12}d\alpha$.

We perform first the integration with respect to α. Since it occurs only in the factor $\exp(im\alpha)$, every term vanishes except the term $m=0$. What we thus obtain we combine with the scattering term on the right-hand side of the equation, finding, finally, the result

$$\sum_L S_L C_L P_L(\cos x)+\sigma(x_1)\sum_L C_L P_L(\cos x_1)=0, \quad (59)$$

where S_L is an abbreviation for the "scattering coefficient."

$$S_L=\int_0^{\pi}\{1-P_L(\cos\theta)\}(d\sigma/d\Omega)2\pi\sin\theta d\theta. \quad (60)$$

By multiplying through by $P_L(\cos x)$ and integrating with respect to x, we can write this equation in the equivalent form

$$2S_L C_L/(2L+1)+\int P_L(\cos x)\sigma(x)I(x) \times\sin x dx=0, \quad (L=0,1,\cdots). \quad (61)$$

In either of these forms of the equilibrium equation it will be noted that it is no longer necessary to cut off the (physically meaningful) divergence of the scattering cross section at small angles. For such angles, θ, the quantity $1-P_L(\cos\theta)$ varies as $L(L+1)\theta^2/4$; $d\sigma/d\Omega$ varies as const.$/\theta^3$, and the element of solid angle as $2\pi\theta d\theta$, giving the integral the approximate behavior $\int_0^\theta d\theta$. It is clear from this discussion that we cannot treat the movement of direction vectors over the surface of the unit sphere as a diffusion process, as might have been justified if the scattering cross section had fallen off faster at larger angles. Deflections of all sizes are of comparable importance.

For the further analysis we compute the scattering coefficients, S_L, explicitly from expression (54) for $d\sigma/d\Omega$, finding

$$S_L=0 \quad \text{for} \quad L=0;$$

otherwise

$$S_L=(2L+1)(\pi^2/4)[\mu_e/(cp/e)]-\pi a_e^2F, \quad (62)$$

where $\mu e/(cp/e)$ represents the square of the characteristic radius of a particle of the given rigidity in the field of the earth's dipole moment, $\mu_e=60$ Bv (earth radius)^2.

Approximate Solution of Equilibrium Equation by Variational Method

Recalling the expression

$$I(x)=\sum_L C_LP_L(\cos x) \quad (63)$$

for the intensity at the inclination x, consider the single number J defined for every function $I(x)$ by the expression

$$J=\sum_L S_LC_L^2/(2L+1) + (1/2)\int_0^\pi I^2(x)\sigma(x)\sin x dx. \quad (64)$$

The variation, δJ, of J with respect to a small change δC_L in a coefficient in the Legendre series (63) for $I(x)$ is just the expression (61), which vanishes for the solution of the equilibrium equation. Consequently, J takes on a stationary value for this particular function $I(x)$. Conversely, by representing $I(x)$ as an empirical function with a certain number of adjustable parameters, and adjusting these parameters to give J a stationary value, we obtain the closest approximation to the exact solution which is attainable with a function of the given mathematical form.

This variational procedure is a special case of a more general procedure, in which we replace the first term in (64) by the quantity

$$(1/4)\int\int\{I(x_1)-I(x_2)\}^2(d\sigma/d\Omega)_{12}d\Omega_1d\Omega_2. \quad (65)$$

In the present case, we must take one precaution to guarantee the convergence of the series in the (64), which for large values of L goes approximately as

$$\text{const.}\sum_L C_L^2;$$

specifically, we shall require continuity not only for $I(x)$ itself, but also for its first derivative.

In the present calculations, we took as trial solution the expression

$$I(x)=I_0\{1-K_1(y-y_0)^2-K_2(y-y_0)^3\}, \quad (66)$$

with two adjustable constants K_1 and K_2, and with the abbreviation $y=\cos x$. Of course we apply this formula only from $x=x_0$ to $x=\pi$. At smaller inclinations we assign to I the constant value $I=I_0$. The integral for the second part of J is evaluated straightforwardly from (66) and the data of Fig. 3. For example, considering the contribution to the variational integral J from the absorption of Mars, we have only to (a) take one-half the equivalent effective cross section of that planet as defined earlier, and (b) multiply this number by the integral with respect to $\cos x$, between the two appropriate limiting angles, of the square of the trial function as just defined. All these calculations are done analytically, and lead to a second degree function of K_1 and K_2. Likewise,

$$C_L=\{(2L+1)/2\}\int_0^\pi I(x)P_L(\cos x)\sin x dx$$

becomes a linear function of K_1 and K_2, and thus J itself altogether a quadratic function. The condition that J be stationary with respect to variations in K_1 and K_2 thus leads to two linear equations for these two unknowns, which are then readily found.

The equations were solved only for values of $R/r_e\leqslant 1.667$. In these cases all the Legendre coefficients higher than the third lead to terms in K_1 and K_2 that are negligible compared to the terms of lower order. For larger values of R/r_e it is necessary to evaluate an increasingly large number of Legendre coefficients to guarantee adequate convergence.

Equation (66) represents the intensity as the sum of a quadratic and a cubic term in the deviation of $y=\cos x$ from the value $\cos x=y_0$ for the cut-off angle. The calculated intensity curve reaches a minimum for some negative value of y and then rises again as y further decreases to -1. This effect is a consequence of our use of only two constants in the variational function adopted to represent the intensity. The actual intensity curve must decrease continuously from

$y=y_0$ to $y=-1$. That our approximate curve will cross the actual curve, lying part of the time below it, part of the time above it, is guaranteed by the variational principle used in our computations. Consequently, by taking the approximate curves at the minimum, and arbitrarily cutting off the rise for values between the minimum and $y=-1$ (dotted lines in Fig. 4), we obtain a reasonable *lower limit* for the intensity in the "forbidden" cone of directions.

The equations were solved for two values of the sun's moment, $\mu_s=10^{34}$ gauss cm^3 (sun's polar field 59.5 gauss) and $\mu_s=0.42\times10^{34}$ gauss cm^3 (or 25 gauss at pole). The results of the calculations have already been shown in Fig. 4 and discussed in the first part of the paper.

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