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On the Decomposition of Tensors by Contraction

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The decomposition of tensors into irreducible representations of the orthogonal groups is calculated for three and four dimensions. The connection is shown with the problem of the allowed values of ordinary and isotopic spin for a given symmetry of the spacial eigenfunction of a nuclear system.

IT is a known fact[1] that a tensor which is irreducible in the affine space, i.e., which cannot be decomposed by imposing further symmetry conditions, may be decomposed in the metric space by the contraction of some indices. Particular examples of this decomposition are given by Schouten,[2] but the problem of determining the decomposition of a tensor of given symmetry has not yet been solved in general.[3]

It is the purpose of this paper to solve this problem for three and four dimensions, i.e., to determine how the irreducible representations of the linear groups $GL(3)$ and $GL(4)$ break up as representations of the proper orthogonal groups $O^+(3)$ and $O^+(4)$.

The four-dimensional case is of particular interest for a better understanding of the different attempts to construct a relativistic theory of particles with integer spin. Another application of the four-dimensional case, which will be treated at the end of this paper, is connected with the symmetry of nuclear systems.

It is known that an irreducible representation of $GL(n)$ is characterized by a diagram of Young,[4] i.e., by a set of n numbers

$$f_1 \geqslant f_2 \geqslant \cdots \geqslant f_n; \tag{1}$$

this representation will be denoted by $\langle P(f_1 f_2 \cdots f_n)\rangle$ or by $\langle P(f_i)\rangle$. It is also known that[5] if we restrict $GL(n)$ to its unimodular subgroup $SL(n)$, the representations $\langle P(f_i)\rangle$ and $\langle P(f_i+e)\rangle$ become equivalent,

$$\langle P(f_i+e)\rangle \doteq \langle P(f_i)\rangle \quad \text{in } SL(n), \tag{2}$$

and that[6] $\langle P(f_i)\rangle$ and $\langle P(f_i')\rangle$ are contragredient to each other if

$$f_i' = e - f_{n+1-i}; \tag{3}$$

if we restrict $SL(n)$ to its orthogonal subgroup $O^+(n)$, also contragredient representations become equivalent, and break up in the same parts:

$$\langle P(f_i)\rangle \doteq \langle P(f_i')\rangle \quad \text{in } O^+(n). \tag{4}$$

[1] H. Weyl, *The Classical Groups* (Princeton University Press, Princeton 1939), Chapter VB.

[2] J. A. Schouten, *Der Ricci-Kalkül* (Verlag Julius Springer, Berlin 1924), pp. 266, 288.

[3] Reference 1, p. 158. *Editor's Note:* It has been pointed out to us by Professor Hermann Weyl that a fairly explicit solution for general n is contained in theorem (7.9.C), p. 229 of Weyl's book (reference 1). It would still need further computation to derive therefrom Racah's explicit theorem.

[4] Reference 1, p. 120.

[5] Reference 1, p. 132.

[6] Reference 1, p. 135.

THE THREE-DIMENSIONAL CASE

The irreducible representations of $O^+(3)$ are the representations $D(L)$, whose substratum are the $2L+1$ spherical harmonics of degree L; according to (2) the number of times that a representation $D(L)$ appears in the decomposition of $\langle P(f_1 f_2 f_3)\rangle$ depends only on the differences between the f_i, and we shall write

$$\langle P(J+f,K+f,f)\rangle = \textstyle\sum_L N_L(J,K) D(L); \tag{5}$$

it follows from (4) that

$$N_L(J,K) = N_L(J,J-K). \tag{6}$$

In order to calculate the $N_L(J,K)$ we shall build a recursion formula based on the comparison of the decomposition laws of the products of the representations of $GL(3)$ and of $O^+(3)$, with the representation $\langle P(100)\rangle = D(1)$. The decomposition law of $D(L)\times D(1)$ is well known:

$$D(0)\times D(1) = D(1) \tag{7a}$$

$$D(L)\times D(1) = D(L-1)+D(L)+D(L+1) \quad (L\geqslant 1). \tag{7b}$$

The decomposition law of $\langle P(f_1 f_2 f_3)\rangle \times \langle P(100)\rangle$ is[7]

$$\langle P(f_1 f_2 f_3)\rangle \times \langle P(100)\rangle = \langle P(f_1+1, f_2 f_3)\rangle + \langle P(f_1 f_2+1, f_3)\rangle + \langle P(f_1 f_2 f_3+1)\rangle; \tag{8}$$

this law holds, however, only for $f_1 > f_2 > f_3$, since for $f_1 = f_2$ the second term is missing, and for $f_2 = f_3$ the third term is missing.

Introducing (5) in (8), taking into account (7), and equating the coefficients of the various $D(L)$, we have

$$N_0(J+1,K)+N_0(J,K+1) + N_0(J-1,K-1) = N_1(J,K) \tag{9a}$$

$$N_L(J+1,K)+N_L(J,K+1)+N_L(J-1,K-1) = N_{L-1}(J,K)+N_L(J,K)+N_{L+1}(J,K) \quad (L\geqslant 1); \tag{9b}$$

owing to the limitations of (8), these equations hold only for $J>K>0$, but it is possible to consider them valid for $J\geqslant K\geqslant 0$, if we put conventionally

$$N_L(J,J+1) = N_L(J,-1) = 0. \tag{10}$$

If all values of $N_L(J',K')$ are known for $J'\leqslant J$, Eqs. (9) determine the $N_L(J+1,K)$ for $K\leqslant J$, and the $N_L(J+1,J+1)$ are then given by (6). All $N_L(J,K)$

[7] We are unable to give a direct reference, since, owing to war conditions, we have no free access to the University grounds and to its library; we may only refer to reference 1, Chapter VII (22).

may therefore be deduced from the $N_L(0,0)$, and these are

$$N_L(0,0)=\delta_{L0}, \tag{11}$$

since

$$\langle P(000)\rangle = D(0).$$

If we are able to find a function $N_L(J,K)$ which satisfies the system (6), (9a), (9b), (10), and (11), we may be sure that this function will solve our problem, since we saw that this system has only one solution.

It is evident that (9b) is satisfied by

$$N_L(J,K)=f(J-L)+g(K-L)+g'(J-K-L), \tag{12}$$

where f, g, and g' are functions whatsoever of their arguments; we shall seek to specialize these functions in order to satisfy also the other equations of the system.

It follows from (6) that $g=g'$, and from (10) that

$$f(J-L)+g(J+1-L)+g(-1-L)=0; \tag{13}$$

varying J and L by the same amount, we obtain that g is a constant for negative values of its argument; since f and g contain an arbitrary additional constant, we shall put

$$g(n)=0 \quad \text{for } n<0$$

and deduce from (13) that

$$g(n)=-f(n-1).$$

Assembling our results, we have

$$N_L(J,K)=f(J-L)-f(K-L-1)-f(J-K-L-1) \tag{14}$$

with

$$f(n)=0 \quad \text{for } n<-1. \tag{15a}$$

Equation (11) is satisfied by (14) and (15a) if $L>1$; it is satisfied also for $L=1$ and $L=0$ if

$$f(-1)=0 \tag{15b}$$

and

$$f(0)=1. \tag{15c}$$

Introducing (14) in (9a), we have

$$f(J+1)+f(J)-f(K)-f(K-1)-f(J-K)-f(J-K-1)=0; \tag{16}$$

putting $K=0$ and taking into account (15), we get

$$f(J+1)=f(J-1)+1 \tag{17}$$

and therefore

$$f(n)=[P(n+2)/2], \tag{18}$$

where $[x]$ means the greatest integer contained in x, and

$$P(y)=(y+|y|)/2=\begin{cases} y & \text{for } y\geqslant 0 \\ 0 & \text{for } y\leqslant 0. \end{cases} \tag{19}$$

It is easy to verify that (18) satisfies (16) also for $K>0$, and therefore

$$N_L(J,K)=[P(J-L+2)/2]-[P(K-L+1)/2]-[P(J-K-L+1)/2]. \tag{20}$$

THE FOUR-DIMENSIONAL CASE

It is known that the group $O^+(4)$ is not a simple group, but is homomorph with the direct product of two unimodular groups in two dimensions; its irreducible representations are the $(2R+1)(2S+1)$-dimensional representations

$$\theta(R,S)=D(R)\times D(S), \tag{21}$$

and we shall write

$$\langle P(f_1f_2f_3f_4)\rangle = \sum_{RS} N_{RS}(f_1f_2f_3f_4)\theta(R,S). \tag{22}$$

The four-dimensional extension of (9) does not allow a direct calculation of N_{RS} as in the three-dimensional case, and it appears more convenient to proceed by successive steps.

The decomposition of $\langle P(f000)\rangle$ is easy to obtain: a symmetric tensor of rank f and vanishing trace is the substratum of the representation $\theta(f/2,f/2)$, and the trace of a symmetric tensor of rank f is a symmetric tensor of rank $f-2$; we have, therefore,

$$\langle P(f000)\rangle = \langle P(f-2,000)\rangle + \theta(f/2,f/2) = \sum_{R\leq f/2} \theta(R,R), \quad 2R\equiv f(\text{mod}2). \tag{23}$$

In order to obtain the decomposition of $\langle P(f_1f_200)\rangle$, we consider at first the external product $\langle P(f_1000)\rangle \times \langle P(f_2000)\rangle$ and its decomposition as representation of $GL(4)$,[7]

$$\langle P(f_1000)\rangle \times \langle P(f_2000)\rangle = \sum_0^{f_2}{}_\alpha \langle P(f_1+\alpha,f_2-\alpha,00), \tag{24}$$

from which we obtain

$$\langle P(f_1f_200)\rangle = \langle P(f_1000)\rangle \times \langle P(f_2000)\rangle - \langle P(f_1+1,000)\rangle \times \langle P(f_2-1,000)\rangle \tag{25}$$

with the convention

$$\langle P(-1000)\rangle = 0. \tag{26}$$

The decomposition of the product (24) as representation of $O^+(4)$,

$$\langle P(f_1000)\rangle \times \langle P(f_2000)\rangle = \sum_{RS} \omega_{RS}(f_1f_2)\theta(R,S), \tag{27}$$

may be obtained from (23) and from the well-known decomposition law of the product $D(R)\times D(R')$; the result is that $\omega_{RS}(f_1f_2)$ vanishes unless

$$R,S\leqslant (f_1+f_2)/2, \quad 2R\equiv 2S\equiv f_1+f_2 \pmod 2,$$

and that, if these conditions are satisfied, and $f_1\geqslant f_2$,

$$\omega_{RS}(f_1f_2)=\omega_{RS}(f_2f_1)=\varphi(f_2+2-|R-S|)-\varphi(f_2+1-R-S)+\varphi(R+S-f_1-1)-\tfrac{1}{2}\varphi(R+S-|R-S|-f_1+f_2+1), \tag{28}$$

where

$$\varphi(x) = [x^2/4] \quad \text{for } x \geq 0$$
$$\varphi(x) = 0 \quad \text{for } x \leq 0.$$

From (25) and (27) we obtain

$$N_{RS}(f_1f_200) = \omega_{RS}(f_1f_2) - \omega_{RS}(f_1+1, f_2-1). \quad (29)$$

The calculation for the general case is based on the following relation between representations of $O^+(4)$:

$$\begin{aligned}\langle P(f_1f_2f_3f_4)\rangle = &\langle P(f_1-f_3,000)\rangle \times \langle P(f_2-f_4,000)\rangle \\ &-\langle P(f_1-f_4+1,000)\rangle \times \langle P(f_2-f_3-1,000)\rangle \\ &-\langle P(f_1-f_2-1,000)\rangle \times \langle P(f_3-f_4-1,000)\rangle; \quad (30)\end{aligned}$$

if this relation holds for $f_1 \leq f_0$, it may be proved for $f_1 = f_0+1$, $f_2 \leq f_0$, by using (24) and the four-dimensional extension of (8), and for $f_2 = f_1$ by using (24) and (4); since it holds for $f_1 = 0$, it holds, therefore, in general. Introducing (27) into (30), we obtain

$$\begin{aligned}N_{RS}(f_1f_2f_3f_4) = &\omega_{RS}(f_1-f_3, f_2-f_4) \\ &-\omega_{RS}(f_1-f_4+1, f_2-f_3-1) \\ &-\omega_{RS}(f_1-f_2-1, f_3-f_4-1). \quad (31)\end{aligned}$$

SPIN, ISOTROPIC SPIN, AND SYMMETRY OF NUCLEAR SYSTEMS

In the classification of the nuclear states arises the question of the determination of the allowed values of the spin S and of the isotopic spin R of a nucleus, when the symmetry of the spacial eigenfunction is given.[8] Since the whole eigenfunction must be antisymmetrical, the function of spin and charge is the substratum of a representation $\langle P(f_1f_2f_3f_4)\rangle$ of $SL(4)$, and the symmetry of the spacial eigenfunction is characterized by a Young diagram with four columns whose lengths are f_1, f_2, f_3, f_4.

The separation of the spin coordinates from the charge coordinates is equivalent to the restriction of $SL(4)$ to the product of two $SL(2)$, and since the group $O^+(4)$ is homomorph with this product, the $N_{RS}(f_1f_2f_3f_4)$ calculated in the preceding section give us also the number of terms with ordinary spin S and isotopic spin R which are allowed for a nucleus with a spacial eigenfunction of given symmetry. (Wigner supermultiplet).

[8] E. Feenberg and M. Phillips, Phys. Rev. **51**, 597 (1937), Appendix 2.