

Classical Electrodynamics in Terms of Direct Interparticle Action¹

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“... the energy tensor can be regarded only as a provisional means of representing matter. In reality, matter consists of electrically charged particles. . . .”³

INTRODUCTION AND SUMMARY

MANY of our present hopes to understand the behavior of matter and energy rely upon the notion of field. Consequently it may be appropriate to re-examine critically the origin and use of this century-old concept. This idea developed in the study of classical electromagnetism at a time when it was considered appropriate to treat electric charge as a continuous substance. It is not obvious that general acceptance in the early 1800's of the principle of the atomicity of electric charge would have led to the field concept in its present form. Is it after all essential in classical field theory to require that a particle act upon itself? Of quantum theories of fields and their possibilities we hardly know enough to demand on quantum grounds that such a *direct* self-interaction should exist. Quantum theory defines those possibilities of measurement which are consistent with the principle of complementarity, but the measuring devices themselves after all necessarily make use of classical concepts to specify the quantity measured.⁴ For this reason it is appropriate to begin a re-analysis of the field concept by returning to classical electrodynamics. We therefore propose here to go back to the great basic problem of classical physics—the motion of a system of charged particles under the influence of electromagnetic forces—and to inquire what description of the interactions and motions is possible which is at the same time (1) well defined (2) economical in postulates and (3) in agreement with experience.

We conclude that these requirements are satisfied by the theory of action at a distance of Schwarzschild,⁵ Tetrode,⁶ and Fokker.⁷ In this description of nature no direct use is made of the notion of field. Each particle moves in compliance with the principle of stationary

action,⁸

$$J = -\sum_a m_a c \int (-da_\mu da^\mu)^{\frac{1}{2}} + \sum_{a < b} (e_a e_b / c) \times \int \int \delta(ab_\mu ab^\mu) (da_\nu db^\nu) = \text{extremum.} \quad (1)$$

All of mechanics and electrodynamics is contained in this single variational principle.

However unfamiliar this direct interparticle treatment compared to the electrodynamics of Maxwell and Lorentz, it deals with the same problems, talks about the same charges, considers the interaction of the same current elements, obtains the same capacities, predicts the same inductances and yields the same physical conclusions. Consequently action at a distance must have a close connection with field theory. But never does it consider the action of a charge on itself. The theory of direct interparticle action is equivalent, not

⁸ Here the letters $a, b \dots$ denote the respective particles. Particle a has in c.g.s. units a mass of m_a grams, a charge of e_a franklins (e.s.u.), and has at a given instant the coordinates

$$\left. \begin{aligned} a^1 &= a_1 \\ a^2 &= a_2 \\ a^3 &= a_3 \end{aligned} \right\} \text{the three space coordinates, measured in cm.}$$

$a^4 = -a_4$, a quantity which has also the dimensions of a length, and which represents the product of the time coordinate by the velocity of light, c ($ct = \text{“cotime”}$).

(Note: In comparing formulas here with those in the literature, note that not all authors use the same convention about signs of covariant and contravariant components.)

The expression ab^m is an abbreviation for the vector, $a^m - b^m$. Greek indices indicate places where a summation is understood to be carried out over the four values of a given label. The argument $ab_\mu ab^\mu$ of the delta-function thus vanishes when and only when the locations of the two particles in space-time can be connected by a light ray. Here the delta-function $\delta(x)$ is the usual symbolic operator defined by the conditions $\delta(x) = 0$ when $x \neq 0$ and $\int_{-\infty}^{+\infty} \delta(x) dx = 1$. In the evaluation of the action, J , from (1), the world lines of the several particles are considered to be known for all time; i.e., the coordinates a^m are taken to be given functions of a single parameter, a , which increases monotonically along the world line of the first particle; likewise for b, c , etc. An arbitrary assumed motion of the particles is not in general in accord with the variation principle: a small change of the first order, $\delta a^m(a), \delta b^m(b), \dots$ in the world lines of the particles (this change here being limited for simplicity to any finite interval of time, and the length of this time interval later being increased without limit) produces in general a non-zero variation of the first order, δJ , in J itself. Only if all such first order variations away from the originally assumed motion produce no first order change in J is that originally assumed motion considered to satisfy the variational principle. It is such motions which are in this article concluded to be in agreement with experience.

¹ Part II of a critique of classical field theory of which another part here referred to as III appeared in Rev. Mod. Phys. 17, 157 (1945). For related discussion see also R. P. Feynman, Phys. Rev. 74, 1430 (1948).

² Now at Cornell University, Ithaca, N. Y.

³ A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1945), second edition, p. 82.

⁴ See in this connection Niels Bohr, *Atomic Theory and the Description of Nature* (Cambridge University Press, 1934) and chapter by Bohr in *Einstein*, of the *Living Philosophers Series* (Northwestern University, scheduled for 1949).

⁵ K. Schwarzschild, Göttinger Nachrichten, 128, 132 (1903).

⁶ H. Tetrode, Zeits. f. Physik 10, 317 (1922).

⁷ A. D. Fokker, Zeits. f. Physik 58, 386 (1929); Physica 9, 33 (1929) and 12, 145 (1932).

to the usual field theory, but to a modified or *adjunct field theory*, in which

(1) the motion of a given particle is determined by the sum of the fields produced by—or *adjunct to*—every particle *other* than the given particle.

(2) the field adjunct to a given particle is uniquely determined by the motion of that particle, and is given by half the retarded plus half the advanced solution of the field equations of Maxwell for the point charge in question.

This description of nature differs from that given by the usual field theory in three respects:

(1) There is no such concept as “the” field, an independent entity with degrees of freedom of its own.

(2) There is no action of an elementary charge upon itself and consequently no problem of an infinity in the energy of the electromagnetic field.

(3) The symmetry between past and future in the prescription for the fields is not a mere logical possibility, as in the usual theory, but a postulational requirement.

There is no circumstance of classical electrodynamics which compels us to accept the three excluded features of the usual field theory. Indeed, as regards the question of the action of a particle upon itself, there never was a consistent theory, but only the hope of a theory. It is therefore appropriate now and hereafter to formulate classical electrodynamics in terms of the adjunct field theory or the theory of direct interparticle action. The agreement of these two descriptions of nature with each other and with experience assures us that we arrive in this way at the *natural and self-consistent generalization of Newtonian mechanics to the four-dimensional space of Lorentz and Einstein*.

It is easy to see why no unified presentation of classical electrodynamics along these lines has yet been given, though the elements for such a description are all present in isolated form in the literature. The development of electromagnetic theory came before the era of relativity. Most minds were not prepared for the requirement that interactions should be propagated with a certain characteristic speed, still less for the possibility of both advanced and retarded interactions. Newtonian instantaneous action at a distance with its century and a half of successes seemed the natural

framework about which to construct a description of electromagnetism. Attempt after attempt failed.⁹ And unfortunately uncompleted was the work of Gauss, who wrote to Weber on the 19th of March, 1845: “I would doubtless have published my researches long since were it not that at the time I gave them up I had failed to find what I regarded as the keystone, *Nil actum reputans si quid superesset agendum*: namely, the derivation of the additional forces—to be added to the interaction of electrical charges at rest, when they are both in motion—from an action which is propagated not instantaneously but in time as is the case with light.”¹⁰ These failures and the final success via the apparently quite different concept of field were taken by physicists generally as convincing arguments against electromagnetic action at a distance.

Field theory taught gradually and over seven decades difficult lessons about constancy of light velocity, about relativity of space and time, about advanced and retarded forces, and in the end made possible by this circuitous route the theory of direct interparticle interaction which Gauss had hoped to achieve in one leap. On this route and historically important was Liénard¹¹ and Wiechert’s¹² derivation from the equations of Maxwell of an expression for the elementary field generated by a point charge in an arbitrary state of motion. With this expression as starting point Schwarzschild arrived at a law of force between two point charges which made no reference to field quantities. Developed without benefit of the concept of relativity, and expressed in the inconvenient notation of the prerelativistic period, his equations of motion made no appeal to the physicists of the time. After the advent of relativity Schwarzschild’s results were rederived independently by Tetrode and Fokker. These results are most conveniently summarized in Fokker’s principle of stationary action of Eq. (1).

To investigate the consistency of the Schwarzschild-Tetrode-Fokker theory of direct interparticle interaction and its relation to field theory, we have first to

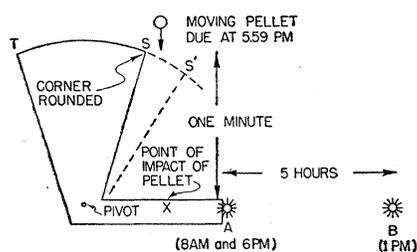


FIG. 1. The paradox of advanced effects. Does the pellet strike X at 6 p.m.? If so, the advanced field from A sets B in motion at 1 p.m., and B moves A at 8 a.m. Thereby the shutter TS is set in motion and the path of the pellet is blocked, so it cannot strike X at 6 p.m. If it does not strike X at 6 p.m., then its path is not blocked at 5.59 p.m. via this chain of actions, and therefore the pellet ought to strike X.

⁹ For a stimulating and instructive if not always objective account of early researches on field theory and action at a distance see A. O’Rahilly, *Electromagnetics* (Longmans, Green and Company, New York (1938)). See also J. J. Thomson, Report of the British Assn. for the Adv. of Science for 1885, p. 97; J. C. Maxwell, *Electricity and Magnetism* (Oxford University Press, London, 1892), third edition, Chapter 23); R. Reif and A. Sommerfeld, *Encyclopädie der Math. Wiss.* 5, Part 2, Section 12 (1902). A recent very brief account has been given by H. J. Groenewold, report on Puntladingen en stralingsveld, *Ned. Nat. Ver.*, Amsterdam (May 1947). M. Schönberg regards field and direct action not as two equivalent representations of the same force, but as two different parts of the total force: *Phys. Rev.* 74, 738 (1948); *Sum. Bras. Math.* 1, Nos. 5 and 6 (1946); J. L. Lopes and M. Schönberg, *Phys. Rev.* 67, 122 (1945).

¹⁰ C. F. Gauss, *Werke* 5, 629 (1867).

¹¹ A. Liénard, *L’Éclairage Électrique* 16, pp. 5, 53, 106 (1898).

¹² E. Wiechert, *Archives Néerland* (2) 5, 549 (1900); *Ann. d. Physik* 4, 676 (1901). Compare these derivations in prerelativistic notation with that given for example by W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1944), second edition, p. 19, or A. Sommerfeld, *Ann. d. Physik* 33, 668 (1910).

examine in the next section the paradox of advanced interactions. In the following section is recalled the derivation of the equations of motion from the variation principle. Next these equations of motion are shown to satisfy the principle of action and reaction as generalized to the non-instantaneous forces of a relativistic theory of action at a distance. In a subsequent section the corresponding formulation of the laws of conservation of energy and momentum is given. Finally the connection is established between these conservation laws and the field-theoretic description of a stress-energy tensor defined throughout space and time.

THE PARADOX OF ADVANCED ACTIONS

The greatest conceptual difficulty presented by the theory of direct interparticle interaction is the circumstance that it associates with the retarded action of *a* on *b*, for example, an advanced action of *b* on *a*. A description employing retarded forces alone would violate the law of action and reaction or, in mathematical terms, could not be derived from a single principle of stationary action.

Advanced actions appear to conflict both with experience and with elementary notions of causality. Experience refers not to the simple case of two charges, however, but to a universe containing a very large number of particles. In the limiting case of a universe in which all electromagnetic disturbances are ultimately absorbed it may be shown¹ that the advanced fields combine in such a way as to make it appear—except for the phenomenon of radiative reaction—that each particle generates only the usual and well-verified retarded field. It is only necessary to make the natural postulate that we live in such a completely absorbing universe to escape the apparent contradiction between advanced potentials and observation.

In a universe consisting of a limited number of charged particles advanced effects occur explicitly. It is no objection if the character of physics under such idealized conditions conflicts with our experience. It is only required that the description should be logically self-consistent. In particular in analyzing the behavior of an idealized universe containing only a few particles we cannot introduce the human element as we know it into the systems under study. To do so would be to assume tacitly the possibility of a clean cut separation between the effects of past and future. This possibility is denied in a description of nature in which both advanced and retarded effects occur explicitly.

The apparent conflict with causality begins with the thought: If the present motion of *a* is affected by the future motion of *b*, then the observation of *a* attributes a certain inevitability to the motion of *b*. Is not this conclusion in direct conflict with our recognized ability to influence the future motion of *b*?

All essential elements of the general paradox appear in the following idealized example: Charged particles *a* and *b* are located in otherwise charge-free space at a

distance of 5 light-hours. A clockwork mechanism is set to accelerate *a* at 6 p.m. Thereby *b* will be affected, not only at 11 p.m. via retarded effects, but also at 1 p.m. via advanced forces. This afternoon motion will cause *a* to suffer a premonitory movement at 8 a.m. Seeing this motion in the morning, we conclude the clockwork will go off in the evening. We return to the scene a few seconds before 6 p.m. and block the clockwork from acting on *a*. But then why did *a* move in the morning?

To formulate the paradox acceptably, we have to eliminate human intervention. We therefore introduce

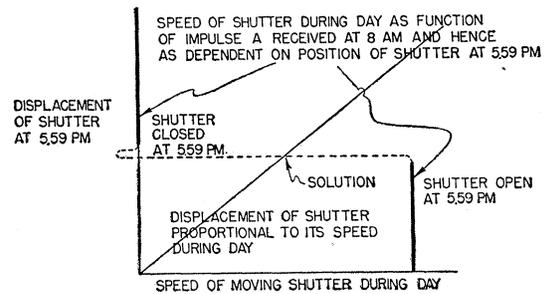


FIG. 2. Analysis and resolution of the paradox of advanced effects. The action of the shutter on the pellet—the interaction of past and future—is continuous (dashed line in diagram) and the curves of action and reaction cross. See text for physical description of solution.

a mechanism which saves charge *a* from a blow at 6 p.m. only if this particle performs the expected movement at 8 a.m. (Fig. 1). Our dilemma now is this: Is *a* hit in the evening or is it not? If it is, then it suffered a premonitory displacement at 8 a.m. which cut off the blow, so *a* is *not* struck at 6 p.m.! If it is not bumped at 6 p.m. there is no morning movement to cut off the blow and so in the evening *a* is jolted!

To resolve, we divide the problem into two parts: effect of past of *a* upon its future, and of future upon past. The two corresponding curves in Fig. 2 do not cross. We have no solution, because the action of the shutter on the pellet, of the future on the past, has been assumed discontinuous in character.

The paradox, and the case it presents against advanced potentials, evidently depends on the postulate that discontinuous forces can exist in nature. From a physical point of view we are led to make just the contrary assumption, that the influence of the future upon the past depends in a continuous manner upon the future configuration.

Our general assumption about continuity is explicitly verified in the present case. The action of shutter on pellet is not discontinuous. The pellet will strike the point *S* a glancing blow if the shutter lies only part way across its path (dashed curve in Fig. 2).

Of the problem of influence of future upon past, and past upon future, we now have in Fig. 2 a self-consistent solution: Charge *a* by late afternoon has moved a

very slight distance athwart the path of the pellet. Thus one second before 6 p.m. it receives a glancing blow in the counter-clockwise sense and at 6 p.m. a stronger acceleration in the clockwise direction. The accelerations received by a at these two moments are by electromagnetic interaction transmitted in reduced measure to b at 1 p.m. and back from b in yet greater attenuation to a . Thus this particle receives one second before 8 a.m. a certain counter-clockwise impulse and at 8 a.m. an opposite impulse. The net rotational momentum imparted to the lever is clockwise. It carries the point S in the course of 10 hours the necessary distance across the path of the pellet. The chain of action and reaction is completed. The paradox is resolved.

Generalizing, we conclude advanced and retarded interactions give a description of nature logically as acceptable and physically as completely deterministic as the Newtonian scheme of mechanics. In both forms of dynamics the distinction between cause and effect is pointless. With deterministic equations to describe the event, one can say: the stone hits the ground because it was dropped from a height; equally well: the stone fell from a height because it was going to hit the ground.

The distinction between Newtonian and relativistic mechanics is one of detail—instantaneous interactions *versus* forces unconfined to a single plane in space time. The interrelations between the world lines are more complicated than those of Newtonian mechanics, but just as definite. There a well-defined division of past and present was possible; here these divisions of time are inextricably mixed.

EQUATIONS OF MOTION

Advanced and retarded forces being accepted on equal footing in the description of nature, we now reproduce the derivation from Fokker's action principle of equations of motion which contain them both. Let the world line of a typical particle a be altered from $a^m(a)$ to $a^m(a) + \delta a^m(a)$. Let the abbreviation be introduced,

$$A_m^{(b)}(x) = e_b \int \delta(x b_\mu x b^\mu) db_m(b) \quad (2)$$

(vector potential of particle b at point x). Also denote by $a^{m'}(a)$ the derivative $da^m(a)/da$. Then the change in action produced by the alteration of the world line of a is

$$\begin{aligned} \delta J = m_a c \int \{ a_\mu' (\delta a^\mu)' / (-a_\nu' a^{\nu'})^{\frac{1}{2}} \} da \\ + \delta \sum_{b \neq a} (e_a/c) \int A_\nu^{(b)}(a) a^{\nu'}(a) da, \end{aligned}$$

or, by partial integration, and dropping terms at the

limits where the variations δa^m vanish,

$$\begin{aligned} \delta J = \int da \sum_m \delta a^m(a) \{ -m_a c (d/da) [a_m' / (-a_\nu' a^{\nu'})^{\frac{1}{2}}] \\ + (e_a/c) \sum_{b \neq a} [(\partial A_\rho^{(b)}/\partial a^m) - (\partial A_m^{(b)}/\partial a^\rho)] a^{\rho'} \}. \quad (3) \end{aligned}$$

The condition that δJ be zero to the first order for arbitrary δa^m is the vanishing of the curly bracket in (3) for all four values of m , whence result the four components of the equation of motion for particle a . Instead of expressing the motion in terms of the arbitrary parameter a , introduce a new parameter, $\alpha = \alpha(a)$, the "proper cotime," defined in terms of a up to an unimportant additive constant by the equation $d\alpha/da = (-a_\nu' a^{\nu'})^{\frac{1}{2}}$ and denote by dots derivatives with respect to the proper cotime. Introduce also the abbreviation

$$F_{mn}^{(b)}(x) = \partial A_n^{(b)}(x) / \partial x^m - \partial A_m^{(b)}(x) / \partial x^n \quad (4)$$

(field at point x due to b).¹³ Then the four-vector equation of motion takes a form,

$$m_a c^2 \ddot{a}_m = e_a \sum_{b \neq a} F_{m\rho}^{(b)}(a) \dot{a}^\rho, \quad (5)$$

identical with that of Lorentz, with the following exceptions: self-actions are explicitly excluded; no fields act except those adjunct to the other particles; each such adjunct field is uniquely determined by the prescription of Eqs. (2) and (4).

Now we come to the well known proof that each adjunct field satisfies Maxwell's equations when for charge and current are introduced the appropriate expressions for the given particles. We employ Dirac's identity¹⁴

$$\begin{aligned} (\partial^2 / \partial x_\mu \partial x^\mu) \delta(x b_\nu x b^\nu) = -4\pi \delta(x_1 - b_1) \delta(x_2 - b_2) \\ \times \delta(x_3 - b_3) \delta(x_4 - b_4), \quad (6) \end{aligned}$$

multiply both sides by $db_m(\beta) = \dot{b}_m(\beta) d\beta$, integrate with respect to β from $-\infty$ to $+\infty$, and conclude that $A_m^{(b)}(x)$ satisfies the equation

$$(\partial^2 / \partial x_\mu \partial x^\mu) A_m^{(b)}(x) = -4\pi j_m^{(b)}(x). \quad (7)$$

Here

$$\begin{aligned} j_m^{(b)}(x) = e_b \int_{-\infty}^{+\infty} \delta(x_1 - b_1) \delta(x_2 - b_2) \\ \times \delta(x_3 - b_3) \delta(x_4 - b_4) \dot{b}_m(\beta) d\beta. \quad (8) \end{aligned}$$

is an abbreviation for the density-current four-vector at point x due to particle b , an obviously singular quantity, obeying certain evident conservation rela-

¹³ The electric field E_x is $F_{14} = -F_{41}$ and the magnetic field H_x is $F_{23} = -F_{32}$: the vector potential A_x is $A^1 = A_1$ and the scalar potential is $A^4 = -A_4$. Likewise in Eq. (8) $j^4 = -j_4$ represents the charge density in franklins (e.s.u.) / cm³ and $j^1 = j_1$ gives (1/c) times (x-component of the charge flux in franklins/cm² sec.).

¹⁴ P. A. M. Dirac, Proc. Roy. Soc. London. A167, 148 (1938).

tions. The vector potential (2), in addition to satisfying the inhomogeneous wave equation (6), has a four-dimensional divergence which vanishes:

$$\begin{aligned} (\partial/\partial x_\mu)A_\mu^{(b)}(x) &= e_b \int_{-\infty}^{+\infty} \delta'(xb_\nu xb^\nu) 2xb^\mu \dot{b}_\mu d\beta \\ &= 2e_b \delta(xb_\nu xb^\nu) \Big|_{-\infty}^{+\infty} = 0. \end{aligned} \quad (9)$$

We differentiate this zero divergence with respect to x^m and subtract from it (7), obtaining the field equations

$$\partial F_{m\mu}^{(b)}(x)/\partial x_\mu = 4\pi j_m, \quad (10)$$

equivalent to the usual relations $\text{div}E = 4\pi\rho$ and $\text{curl}H = \dot{E}/c + 4\pi J_x/C$. The other pair of Maxwell's equations follow identically from the definition (4) of the F 's in terms of the A 's.

The fields (2) are distinguished from all other solutions of Maxwell's equations by being half the sum of the advanced and retarded Liénard-Wiechert potentials of particle b :

$$\begin{aligned} A_m^{(b)}(x) &= e_b \int \frac{db_m/d\beta}{d(xb_\mu xb^\mu)/d\beta} \delta(xb_\nu xb^\nu) d(xb_\rho xb^\rho) \\ &= (1/2)R_m^{(b)}(x) + (1/2)S_m^{(b)}(x). \end{aligned} \quad (11)$$

Here, for example, R represents the retarded potential

$$R_m^{(b)}(x) = e_b \dot{b}^m / \dot{b}^\mu b x_\mu, \quad (12)$$

evaluated at that point on the world line of b which intersects the light cone drawn from the point of observation into the past:

$$xb_\mu xb^\mu = 0; \quad x^4 > b^4, \quad (13)$$

and S similarly represents the advanced potential.

By way of illustration of these results in familiar cases consider first the case of a point charge, b , at rest at the origin. Then retarded and advanced fields are identical, all components of the four-potential vanish except the last, $\dot{b}^4 = d(\text{cotime})/d(\text{proper cotime}) = 1$, $b x_4 = b_4 - x_4 = x^4 - b^4 = \text{elapsed cotime} = \text{distance to point of observation} = r$, and the scalar potential has the familiar value e_b/r . Next, in the case of a slowly moving point charge, it similarly follows that $A^m = e_b(\dot{b}^m/2r)_{\text{ret}} + e_b(\dot{b}^m/2r)_{\text{adv}}$. If this point charge is at the same time being accelerated, then the derived electric field has at large distances the value $E = -e_b(\dot{b}_\perp/2r)_{\text{ret}} - e_b(\dot{b}_\perp/2r)_{\text{adv}}$, where \dot{b}_\perp is the component of the three-vector \dot{b} perpendicular to the line r . This result refers only to the field of the particle in question. In the idealized case of a universe containing charged particles sufficient in number to absorb all electromagnetic disturbances, the advanced fields of the particles of the absorber will combine with the given field to produce the full retarded field of experience, $-e_b(\dot{b}_\perp/r)_{\text{ret}}$, as shown in III. As a final example consider a fixed linear

conductor past any point of which flow per second i/e particles of charge e . The interval of cotime between the k th and the $(k+1)$ st particle is ce/i . The coordinates of the k th particle are

$$\begin{aligned} k^m(\gamma) &= s^m(\gamma) \quad (m=1, 2, 3) \\ k^4(\gamma) &= s^4(\gamma) + kce/i \quad (k=-\infty, \dots, -1, 0, 1, \dots) \end{aligned} \quad (14)$$

where $s^m(\gamma)$ is the parametric representation of the curve of the wire. The four-potential at a point of observation an appreciable distance from the wire is obtained by summing over all the particles or equivalently, because of the close spacing of the charges, by integrating over k :

$$\begin{aligned} A^m(x) &= e \iint \delta[r^2(\gamma) - (ct - s^4(\gamma) - kce/i)^2] \\ &\quad \times dk(ds^m(\gamma)/d\gamma)d\gamma \\ &= \begin{cases} i \int ds^m(\gamma)/cr(\gamma) & \text{for } m=1, 2, 3 \\ e \int dk/r & \text{for } m=4. \end{cases} \end{aligned} \quad (15)$$

Here $r(\gamma)$ is the magnitude of the vector $x(\gamma)$, $y(\gamma)$, $z(\gamma)$ which runs from the point γ of the curve to the point of observation. The scalar potential of Eq. (15) will normally be compensated wholly or in part by contributions from opposite charges at rest and need not be considered here. From the vector potential follows an expression for the magnetic field

$$H = \text{curl}A = i \int (ds \times r)/cr^3, \quad (16)$$

identical with that due to Ampère.

To go further in deriving well known results would be pointless. Adequate textbooks exist. They treat well defined problems of electromagnetism, where there is no compelling reason to consider a particle to act on itself. Thus all their analyses are immediately translatable into terms of the present modified or *adjunct field theory*. However, this point of view is mathematically identical with that of action at a distance. Consequently the theory of direct interparticle action, far from attempting to replace field theory, joins with field theory to provide the science of electromagnetism with additional techniques of mathematical analysis and to facilitate deeper physical insight. The rest of this article may illustrate how the two points of view join hands to elucidate in four-dimensional mechanics the principle of action and reaction and the laws of conservation of momentum and energy.

ACTION AND REACTION

Laws of conservation of angular momentum, energy and linear momentum are well known to exist in any

theory for which the equations of motion are derivable from an action principle which is invariant with respect to rotation, translation, or displacement of the time coordinate.¹⁵ Thus Fokker¹⁶ has derived an energy-momentum conservation principle for an idealized situation in which there are only two particles, of which a acts on b via purely retarded forces. The present treatment is the natural generalization of Fokker's analysis to the case of a theory which is symmetric between every pair of particles and which is based on the action principle (1). It will be sufficient to prove the conservation law for a single pair of particles in order to see the corresponding result for a system of particles.

For the typical particle a let the four-vector of energy and comomentum be denoted by

$$mcv(1-v^2/c^2)^{-\frac{1}{2}} = \begin{cases} G^1 = G_1 \\ G^2 = G_2 \\ G^3 = G_3 \end{cases} \begin{array}{l} \text{three space components of} \\ \text{the kinetic comomentum} \\ \text{(velocity of light times} \\ \text{kinetic momentum: ex-} \\ \text{pressible in energy units).} \end{array}$$

$$mc^2(1-v^2/c^2)^{-\frac{1}{2}} = G^4 = -G_4, \text{ kinetic energy plus rest-mass energy.}$$

Then the change in kinetic comomentum and energy in the interval of proper cotime, $d\alpha$, on account of the action of particle b follows directly from the equations of motion (5) and the expressions (4) and (2) for the force coefficients:

$$dG_m^{(a)}(\alpha) = m_a c^2 \ddot{a}_m d\alpha = e_a e_b d a^\mu \times \left\{ (\partial/\partial a^\mu) \int \dot{b}_\mu - (\partial/\partial a^\mu) \int \dot{b}_m \right\} \delta(ab, ab^r) d\beta. \quad (17)$$

We carry out the differentiations with respect to the coordinates a and add to the result the following zero quantity

$$e_a e_b d a_m \int_{-\infty}^{+\infty} (d/d\beta) \delta(ab, ab^r) d\beta, \quad (18)$$

thus finding for the impulse

$$dG_m^{(a)}(\alpha) = 2e_a e_b \int_{\beta=-\infty}^{\beta=+\infty} \delta'(ab, ab^r) \times (ab_m d a^\mu d b_\mu - db_m d a^\mu a b_\mu - da_m d b^\mu a b_\mu). \quad (19)$$

In this expression the integrand is changed in sign but unaltered in value by an interchange of the roles of particles a and b .

To the result just obtained we give the following obvious interpretation:

(1) The right hand side of (19), after removal of the integral sign, represents in terms of the symbolic delta-function the

¹⁵ E. Noether, Göttinger Nachrichten, Math. Phys. Klasse. 235 (1918); E. Bessel-Hagen, Math. Ann. 84, 258 (1921).

¹⁶ A. D. Fokker, Zeits. f. Physik 58, 386 (1929).

transfer of impulse or energy to a during the stretch of cotime $d\alpha$ from effects which originate at b in the cotime interval $d\beta$.

(2) There is no energy or impulse transfer except when the stretch $d\beta$ of the world line of b is intersected by either the forward or backward light cone drawn from a : i.e., b acts on a through both retarded and advanced forces.

(3) The impulse communicated to a over the portion $d\alpha$ of its world line via retarded forces, for example, from the stretch $d\beta$ of the world line of b is equal in magnitude and opposite in sign to the impulse transfer from a to b via advanced forces over the same world line intervals (*equality of action and reaction*).

The relativistic generalization of the Newtonian principle of action and reaction as just stated is obviously not identical with the non-relativistic formulation. In no Lorentz frame of reference are action and reaction simultaneous. For the instant at which a experiences a force from b there is not one corresponding time at which b gets a back reaction, but two instants.¹⁷ Thus for a given point on the world line of a we can make two statements about the transfer of energy (or impulse) from b . Each statement refers to a single one of the two parts of the total transfer. It is evidently reasonable that the law of action and reaction should have this Jacob's ladder character in 4-dimensional space-time.

ENERGY AND MOMENTUM OF INTERACTION

Considering two isolated particles a and b , we immediately conclude from the law of action and reaction as just stated the constancy in time of the *total energy and comomentum four-vector*

$$G_m(\alpha, \beta) = m_a c^2 \dot{a}_m(\alpha) + m_b c^2 \dot{b}_m(\beta) + 2e_a e_b \left\{ - \int_{-\infty}^{\alpha} \int_{\beta}^{\infty} + \int_{\alpha}^{\infty} \int_{-\infty}^{\beta} \right\} \delta'(ab, ab^r) (ab_m d a^\mu d b_\mu - db_m d a^\mu a b_\mu - da_m d b^\mu a b_\mu) = (\text{constant})_m. \quad (20)$$

In the case of more particles we have a corresponding expression with a kinetic term for each individual particle and an interaction term for each pair of charges. Thus G_m becomes a function of as many parameters $\alpha, \beta, \gamma, \dots$ as there are particles. To prove constancy with respect to a given parameter, such as α , we have only to differentiate (20) and insert for $m_a c^2 \ddot{a}_m(\alpha)$ the quotient $dG_m^{(a)}(\alpha)/d\alpha$ obtained from (19).

Evidently we have in (20) what may be called a many-time formulation of the conservation laws, derived of course from the equations of motion, but from which conversely the equations of motion are derivable with equal ease.

The interpretation of the double integral in (20) as an interaction energy is obvious in the case of two stationary charges separated by a distance R . Thus by integration we find for G^4 the familiar result $m_a c^2 + m_b c^2 + e_a e_b / R$.

¹⁷ L. Page, Am. J. Phys. 13, 141 (1945), has reviewed the complications which come from comparing action and reaction at the same time.

In the case of individual moving charges it is sometimes convenient to add to the idea of kinetic comomentum and energy $G_m^{(a)}$ the notion of potential comomentum and energy

$$U_m^{(a)} = e_a \sum_{b \neq a} A_m^{(b)}(a(\alpha)), \quad (21)$$

and total comomentum and energy,

$$P_m^{(a)} = G_m^{(a)} + U_m^{(a)}. \quad (22)$$

In terms of these expressions, the four-vector of energy and comomentum of the whole system takes the form

$$G_m(\alpha, \beta, \dots) = \sum_a P_m^{(a)}(\alpha) + \sum_{a < b} 2e_a e_b \times \left\{ \int_{\alpha}^{\infty} \int_{-\infty}^{\beta} - \int_{-\infty}^{\alpha} \int_{\beta}^{\infty} \right\} \delta'(ab, ab') ab_m da^{\mu} db_{\mu}. \quad (23)$$

The summation of the potential energies so to speak counts twice the interaction between each pair of particles. The double integrals in (23) correct for this oversight.

From either Eq. (20) or Eq. (23) for the energy of the system it is clear (see Fig. 3) that the electromagnetic energy of a finite number of particles is definable from a knowledge of only a finite stretch of their world lines. It is also evident that particles which come together in otherwise charge free space, interact, and then separate in a regular way, will in the end experience no net loss of energy to outer space. Both features of the four-vector G_m are reasonable in the mathematical description of a physically closed system.

RELATION OF INTERACTION ENERGY TO FIELD ENERGY

In field theory it is customary to attempt to define throughout space a symmetrical stress energy tensor¹⁸ $T_{mn}(x)$ with the following properties:

- (1) The divergence $\partial T_{mv}/\partial x_v$ vanishes at every place where there is no particle.
- (2) At the location of a typical charge a this divergence becomes singular in such a way that its integral over a small volume element containing the charge gives the value of the electromagnetic force acting on that charge:

$$- \hat{a}_i \int \int \int_{\text{neighborhood of } a} (\partial T_{mv}/\partial x_v) dx^1 dx^2 dx^3 = m_a c^2 \hat{a}_m \quad (24)$$

when the integration extends over a region of constant time which contains a . When the integration proceeds over an arbitrary space-like region or "surface," σ , such that no pair of points in

¹⁸ Typical components are

- T_{11} , force in positive x -direction across unit area in yz plane exerted upon medium on negative side of plane by medium on positive side (equal in the Maxwell theory to $(8\pi)^{-1} \times (H_x^2 - H_y^2 - H_z^2 + E_x^2 - E_y^2 - E_z^2)$).
- T_{14} , velocity of light times energy flux in x direction per cm^2 of yz plane and per sec. (Maxwell value $(4\pi)^{-1}(E_y H_z - E_z H_y)$).
- T_{44} , negative of the energy density (usual expression $-(8\pi)^{-1} \times (E^2 + H^2)$).

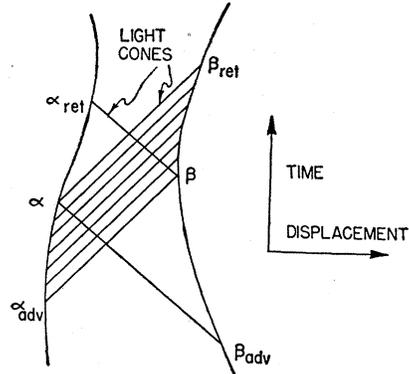


Fig. 3. Interactions considered in formulating the law of conservation of momentum and energy. Note that the stretches of world line from α_{adv} to α_{ret} and from β_{adv} to β_{ret} completely determine the value of the energy-momentum four vector $G_m(\alpha, \beta)$. It is also natural to specify these two world-line segments as initial conditions in dealing with the two-particle problem.

the surface can be connected by a light ray, then the corresponding statement is

$$m_a c^2 \hat{a}_m + \hat{a}_\mu \int \int \int_{\text{neighborhood of } a} (\partial T_{mv}/\partial x_v) d\sigma^\mu = 0. \quad (25)$$

Here, if the surface is defined by a parametric representation in terms of three quantities u, v, w , then

$$d\sigma^4 = [\partial(x^1, x^2, x^3)/\partial(u, v, w)] du dv dw$$

with corresponding expressions for the other three components of $d\sigma^m$.

(3) For every space-like surface σ there is defined a four-vector of energy and comomentum

$$G_m(\sigma) = \sum_a m_a c^2 \hat{a}_m(\alpha) + \int \int \int T_{m\alpha} d\sigma^\alpha, \quad (26)$$

which is conserved in the sense that its value is completely independent of the choice of σ . Thus consider a change $\delta\sigma$ in the surface σ —i.e., an alteration from $x^m(u, v, w)$ to $x^m + \delta x^m(u, v, w)$ —and the associated alterations $d\alpha, d\beta, \dots$ in the points where the respective world lines intersect this surface. Then the change in G_m is expressible via the theorem of Gauss in terms of an integral over the volume, ω , comprised between the two surfaces:

$$\delta G_m = \sum_a m_a c^2 \hat{a}_m d\alpha + \int \int \int (\partial T_{m\alpha}/\partial x_\alpha) d\omega. \quad (27)$$

But the integrand vanishes everywhere except in the immediate neighborhood of the typical particle, a , and there—writing $d\omega = da_\mu d\sigma^\mu$, and using (25)—we conclude that the contribution from the integral just cancels out the first term in δG_m .

Is there any choice of the tensor T_{mn} in the adjunct field theory which will yield for the energy-comomentum vector $G_m(\sigma)$ of (26) a value identical with the corresponding vector $G_m(\alpha, \beta, \dots)$ of the theory of direct interparticle action? The appropriate tensor may be constructed when one recalls that the field of a given particle is to produce changes only in the motions of the other particles, and that the principle of action and reaction connects the retarded effects exerted for example by a on b via the retarded field $(1/2)R_{mn}^{(a)}$ with the advanced effects exerted by b on a via the advanced field $(1/2)S_{mn}^{(b)}$:

$$T_{mn}(x) = \sum_{a \neq b} (R^{(a)}(x) \& S^{(b)}(x))_{mn}. \quad (28)$$

Here R and S denote the retarded and advanced Liénard-Wiechert fields, so that $F_{mn}^{(a)} = (1/2)R_{mn}^{(a)} + (1/2)S_{mn}^{(a)}$. For a convenient abbreviation we have adopted the notation

$$(R \& S)_{mn} = (R_m^\mu S_{\mu n} + S_m^\mu R_{\mu n} + \frac{1}{2}g_{mn}R^{\mu\nu}S_{\mu\nu})/8\pi \quad (29)$$

with $g_{mn} = 0$ for $m \neq n$ and $g_{11} = g_{22} = g_{33} = 1 = -g_{44}$.

That the tensor T_{mn} of (28) does lead to the energy-momentum four-vector (20) of the theory of action at a distance is proven in the appendix. Here we shall only establish that the stress energy tensor satisfies the conditions (1) and (2) (and hence (3)). Thus, we

TABLE I. Correspondence of principal alternative expressions for interaction energy in adjoint field theory and in theory of direct interparticle interaction.

	Canonical form	Frenkel form
Basic type of field coupling envisaged	Those partial fields which are reciprocally responsible for equality of action and reaction	Total (time-symmetric) field adjunct to each of the coupled particles
Typical term in stress-energy tensor	$R^{(a)} \& S^{(b)}$	$F^{(a)} \& F^{(b)}$
Expression for interaction energy:	Eq. (20)	Eq. (20) plus expression (40)
Depends upon:	Finite stretch of the two world lines	Shape of the two world lines from $t = -\infty$ to $+\infty$

evaluate the divergence of the typical term in the tensor of Eq. (28), finding

$$\begin{aligned} \partial T_{m\mu}/\partial x_\mu = \sum_{a \neq b} \{ & (S^{(b)\rho\eta}/16\pi)(\partial R_{m\rho}^{(a)}/\partial x^\eta \\ & + \partial R_{\rho\eta}^{(a)}/\partial x^m + \partial R_{\eta m}^{(a)}/\partial x^\rho) \\ & + (S_m^{(b)\rho}/8\pi)(\partial R_{\rho\sigma}^{(a)}/\partial x_\sigma) \\ & + \text{similar term with } S^{(b)} \text{ and} \\ & R^{(a)} \text{ interchanged} \}. \quad (30) \end{aligned}$$

Here the first three cyclically related terms cancel, as seen for example from the antisymmetrical representation of the fields via potentials; and the divergence of R gives the same charge and current distribution (8) which appeared in the time-symmetric case. Using this circumstance, and combining terms, we have

$$\begin{aligned} \partial T_{m\alpha}/\partial x_\alpha = \sum_{b \neq a} F_{m\mu}^{(b)}(x) j^{(a)\mu}(x) \\ = \sum_{b \neq a} F_{m\mu}^{(b)}(x) e_a \int \delta(x^1 - a^1) \delta(x^2 - a^2) \\ \times \delta(x^3 - a^3) \delta(x^4 - a^4) \dot{a}^\mu(\alpha) d\alpha \\ = \sum_a \int \delta(x^1 - a^1) \delta(x^2 - a^2) \delta(x^3 - a^3) \\ \times \delta(x^4 - a^4) m_a c^2 \ddot{a}_m d\alpha, \quad (31) \end{aligned}$$

in complete satisfaction of requirements (1) and (2).

As alternative choice for the stress energy tensor which also has the properties (1), (2) and (3) is that proposed by Frenkel,¹⁹ who was among the first to stress the notion of fields as always adjunct to specific particles:

$$T_{mn}^*(x) = \sum_{a \neq b} (F^{(a)}(x) \& F^{(b)}(x))_{mn}. \quad (32)$$

Thus the difference between Frenkel's tensor and the canonical tensor (28) is a quantity

$$T_{mn}^* - T_{mn} = \sum_{a \neq b} (\frac{1}{2}R^{(a)} - \frac{1}{2}S^{(a)}) \& (\frac{1}{2}R^{(b)} - \frac{1}{2}S^{(b)}) \quad (33)$$

which has everywhere a zero divergence.

The possibility of more than one expression for the stress-energy tensor with the same divergence is well known in the usual single-field formulation of electrodynamics,²⁰ and is not surprising here. However, the expressions for field energy also turn out to differ (Table I).

The energy-momentum four-vector G_m defined by (26) and (28), and the alternative four-vector G_m^* defined by (26) and (32), are both ordinarily finite for a system of point charges. In illustration, note that near a typical particle a the corresponding field varies as $1/r^2$, the field of any other particle b is finite, the volume element is proportional to $4\pi r^2 dr$ and the integral of (26) converges, yielding for example in the interaction energy $e_a e_b / r_{ab}$ for two stationary point charges separated by the distance r_{ab} . The density of field energy, while finite, is not positive definite, even for two particles of the same charge. Also the flow of energy and momentum may have finite values at a point in space where the total field, $F^{(a)} + F^{(b)} + \dots$, actually vanishes. This result, unexpected from the point of view of the usual field theory, nevertheless presents no logical difficulties.

ENERGY OF RADIATION

The canonical and the Frenkel tensors, which give the same interaction energy in the case of two charges which are at rest, give different results for the case of a

TABLE II. Energy flux at distance r from accelerated charge for adjoint field theory in completely absorbing universe.

Time of observation relative to moment of acceleration	Form of stress-energy tensor		
	Canonical	Frenkel	Maxwell
r/c seconds earlier	no flux	$-E^2/8\pi$ towards the source	no flux
r/c seconds later	$E^2/4\pi$ outward	$E^2/8\pi$ outward	$E^2/4\pi$ outward
at other times	no flux	no flux	no flux

¹⁹ J. Frenkel, Zeits. f. Physik **32**, 518 (1925). See also J. L. Synge, Trans. Roy. Soc. Canada **34**, 1 (1940) and Proc. Roy. Soc. London **A177**, 118 (1940) as well as the discussion of Synge's treatment in III.

²⁰ See in particular M. H. L. Pryce, Proc. Roy. Soc. London. **A168**, 398 (1938).

single accelerated particle in a completely absorbing universe. There we have in the neighborhood of the radiating source $F^{(a)} = (1/2)R^{(a)} + (1/2)S^{(a)}$ and $F^{(b)} + F^{(c)} + \dots = (\text{sum of advanced fields of absorber particles})^{21} = (1/2)R^{(a)} - (1/2)S^{(a)}$. For the parts of these fields which are proportional to the acceleration of the charge, and which vary at large distance as $1/r$, we have for $R^{(a)}$ and $S^{(a)}$ respectively a zero value except for an instant r/c seconds after or before the moment of acceleration. The corresponding energy flux (Table II) satisfies in both the Frenkel and the canonical formulations the law of conservation of energy, but agrees only in the canonical case with customary ideas of energy localization. From the standpoint of pure electrodynamics it is not possible to choose between the two tensors. The difference is of course significant for the general theory of relativity, where energy has associated with it a gravitational mass. So far we have not attempted to discriminate between the two possibilities by way of this higher standard.

CONCLUSION

We conclude that the theory of direct interparticle action, and the equivalent adjunct field theory, provide a physically reasonable and experimentally satisfactory account of the classical mechanical behavior of a system of point charges in electromagnetic interaction with one another, free of the ambiguities associated with the idea of a particle acting upon itself.

APPENDIX

To compute the integral of the field energy which appears in (26), we express each field as a superposition of elementary fields from each infinitesimal range of path $d\alpha$, and the tensor T_{mn} or T_{mn}^* as the superposition of parts due to stretches $d\alpha$ of the world line of a and $d\beta$ of b . We use the notation $T^\dagger d\alpha d\beta$, $F^\dagger d\alpha$ to indicate each such elementary contribution to T , F , etc. Thus the four-potential $\mathcal{R}^\dagger d\alpha$ arises from a charge which appears for an instant at $a(\alpha)$ and disappears at $a(\alpha + d\alpha)$.

The lack of conservation of the charge which generates the elementary potential causes the four-divergence of \mathcal{R}^\dagger to equal a non-zero scalar, r ,

$$\begin{aligned} \partial \mathcal{R}_\mu^{(a)\dagger} / \partial x_\mu &= (\partial / \partial x_\mu) 2e_a \dot{a}_\mu \begin{cases} \delta(ax_\nu a x^\nu) & \text{for } x^4 > a^4 \\ 0 & \text{for } x^4 < a^4 \end{cases} \\ &= r^{(a)}(x, \alpha) = 4e_a \dot{a}_\mu x a^\mu \begin{cases} \delta'(ax_\nu a x^\nu) & \text{for } x^4 > a^4 \\ 0 & \text{for } x^4 < a^4 \end{cases}, \end{aligned} \quad (34)$$

whose integral however satisfies the conservation condition $\int_{-\infty}^{+\infty} r^{(a)}(x, \alpha) d\alpha = 0$. This circumstance permits some latitude in the definition of the elementary field in terms of the potential. It will prove useful to adopt the definition

$$R_{mn}^{(a)\dagger} = \partial \mathcal{R}_n^{(a)} / \partial x^m - \partial \mathcal{R}_m^{(a)} / \partial x^n - (r^{(a)}/2) g_{mn}. \quad (35)$$

The elementary field is not antisymmetrical in the indices m and n , but the normal field $R_{mn}^{(a)} = \int R_{mn}^{(a)\dagger} d\alpha$ changes sign of course on this interchange of labels.

The elementary component of the stress-energy tensor is not symmetric in its two indices, but its divergence is found by direct

²¹ See part III for fuller discussion.

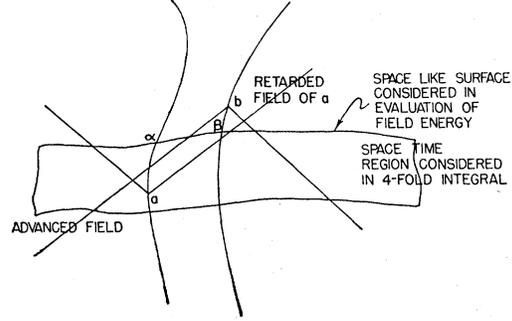


FIG. 4. Contribution to canonical expression for field energy which arises from coupling of retarded field of a and advanced field of b .

algebra to have the simple value

$$\begin{aligned} (\partial / \partial x_\rho) (R^{(a)\dagger} \& S^{(b)\dagger})_{m\rho} = (1/8\pi) (-\partial^2 R_\rho^{(a)\dagger} / \partial x^\sigma \partial x_\sigma) \\ \times (S_m^{(b)\dagger\rho} - (s^{(b)}/2) \delta_m^\rho) + (1/8\pi) (-\partial^2 S_\rho^{(b)\dagger} / \partial x_\sigma \partial x^\sigma) \\ \times (R_m^{(a)\dagger\rho} - (r^{(a)}/2) \delta_m^\rho), \end{aligned} \quad (36)$$

where the typical field d'Alembertian has the value

$$-\partial^2 R_m^{(a)\dagger} / \partial x^\sigma \partial x_\sigma = 4\pi e_a \delta(x^1 - a^1) \delta(x^2 - a^2) \delta(x^3 - a^3) \delta(x^4 - a^4) \dot{a}_m. \quad (37)$$

We integrate (36) over a four-dimensional region of the form shown in Fig. 4. Of the terms on the right the second vanishes throughout this region, and the first gives

$$\begin{aligned} (e_a/2) (S_{m\rho}^{(b)\dagger}(a) - (g_{m\rho}/2) s^{(b)}(a)) \dot{a}^\rho \\ = (e_a/2) (\partial S_\rho^{(b)\dagger} / \partial a^m - \partial S_m^{(b)\dagger} / \partial a^\rho - g_{m\rho} \partial S_\nu^{(b)\dagger} / \partial a_\nu) \dot{a}^\rho \\ = 2e_a e_b (ab_m \dot{a}^\mu \dot{b}_\mu - \dot{b}_m^* ab_\mu \dot{a}^\mu - \dot{a}_m ab_\mu \dot{b}^\mu) \delta'(ab_\sigma ab^\sigma) \end{aligned} \quad (38)$$

when $b^4 > a^4$, and zero otherwise. The four-integral on the left hand side may be expressed via the theorem of the Gauss in the form

$$-\int \int \int (R^{(a)\dagger} \& S^{(b)\dagger})_{m\rho} d\sigma^\rho. \quad (39)$$

Here the integral, which goes over the whole of the three-dimensional region or "surface" in the figure, contributes only over the upper region because of the vanishing elsewhere of at least one of the fields in question. The elementary contributions just computed we now sum over the world line of a from $-\infty$ to α and over the world line of b from β to ∞ , where α and β determine the points where the world lines of a and b intersect the space-like surface σ . We have then only to erase the daggers in (39). The converse expression, with $R^{(b)}$ & $S^{(a)}$, we obtain by interchanging the roles of b and a in (38) and in the limits of integration. In this way follows at once the identity of expression (20) for the energy in the theory of direct interparticle interaction and the canonical expression (26-28) of the adjunct field theory.

When instead the Frenkel expression (32) is used for the stress-energy tensor, then there results an increment in the energy-comomentum four-vector given by the expression

$$\begin{aligned} G_m^*(\alpha, \beta) - G_m(\alpha, \beta) &= e_a e_b \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (ab_m \dot{a}^\mu \dot{b}_\mu \\ &- \dot{b}_m ab_\mu \dot{a}^\mu - \dot{a}_m ab_\mu \dot{b}^\mu) \begin{cases} \delta'(ab_\nu ab^\nu) & \text{for } a^4 < b^4 \\ -\delta'(ab_\nu ab^\nu) & \text{for } a^4 > b^4 \end{cases} d\alpha d\beta \\ &= e_a e_b \int \int \begin{cases} +\delta' & \text{for } a^4 < b^4 \\ -\delta' & \text{for } b^4 > a^4 \end{cases} ab_m da_\mu db^\mu, \end{aligned} \quad (40)$$

a covariant which is independent of α and β and which has an interesting relation to the two world lines in question.