For each value of ρ, $Z^{\rho\sigma\tau}$ is of the form of a multiple of the metric tensor plus a self-dual tensor, that is of the form

$$Wg^{\sigma\tau}+W^{\sigma\tau}.$$

However, if such a tensor is multiplied by

$$Wg_{\tau\lambda}-W_{\tau\lambda},$$

and summed on τ, we obtain

$$(W^2+\tfrac{1}{4}W^{\tau\mu}W_{\tau\mu})\delta_\lambda{}^\sigma.$$

This means that we may solve Eqs. (5.4) for $T_{\kappa,\lambda}$ as functions of T_ν and $T_{\rho,\lambda}$ $(\rho\neq\kappa)$. The discussion of the existence of solutions of the resulting system of equations reduces to a discussion of the integrability conditions.

To solve Eqs. (5.4) for $T_{\kappa,\lambda}$ for one value of κ, we must multiply Eqs. (5.4) by

$$W^\epsilon(g_{\tau\lambda}\delta_\epsilon{}^\kappa-g_{\alpha\tau}g_{\beta\lambda}\eta_{\epsilon\delta}{}^{\alpha\beta}g^{\delta\kappa}),$$

and sum on τ. We obtain for the right hand side,

$$\begin{aligned}W^\epsilon(g_{\tau\lambda}\delta_\epsilon{}^\kappa-g_{\alpha\tau}g_{\beta\lambda}\eta_{\epsilon\delta}{}^{\alpha\beta}g^{\delta\kappa})\bar{F}_\nu{}^\tau T^\nu \\ =H_{\kappa\lambda}=W^\kappa\bar{F}_{\lambda\nu}T^\nu+T^\kappa\bar{F}_{\lambda\nu}W^\nu+W^\nu T_\nu\bar{F}_{\kappa\lambda} \\ +W_\lambda\bar{F}_\nu{}^\kappa T^\nu-T_\lambda\bar{F}_\nu{}^\kappa W^\nu-\delta_\lambda{}^\kappa\bar{F}_{\sigma\nu}W^\sigma T^\nu.\end{aligned}\quad(5.6)$$

The left-hand side involves

$$\begin{aligned}W^\mu(g^{\sigma\tau}\delta_\mu{}^\rho+\eta_{\mu\nu}{}^{\sigma\tau}g^{\rho\nu})(g_{\tau\lambda}\delta_\epsilon{}^\kappa-g_{\alpha\tau}g_{\beta\lambda}\eta_{\epsilon\delta}{}^{\alpha\beta}g^{\delta\kappa})W^\epsilon \\ =a^2\delta_\lambda{}^\sigma g^{\kappa\rho}+\delta_\lambda{}^\rho(2W^\sigma W^\kappa-g^{\sigma\kappa}a^2)-\delta_\lambda{}^\kappa(2W^\sigma W^\rho-g^{\sigma\rho}a^2) \\ +2(g^{\sigma\kappa}W^\rho W_\lambda-g^{\sigma\rho}W^\kappa W_\lambda).\end{aligned}\quad(5.7)$$

The differential Eq. (5.4) may then be written as

$$\begin{aligned}T_{\rho,\sigma}[a^2\delta_\lambda{}^\sigma g^{\kappa\rho}+\delta_\lambda{}^\rho(2W^\sigma W^\kappa-g^{\sigma\kappa}a^2)-\delta_\lambda{}^\kappa(2W^\sigma W^\rho \\ -g^{\sigma\rho}a^2)+2(g^{\sigma\kappa}W^\rho W_\lambda-g^{\sigma\rho}W^\kappa W_\lambda)]=H_{\kappa\lambda}.\end{aligned}\quad(5.8)$$

Multiplying this by $g^{\kappa\lambda}$ and summing, we obtain

$$T_{\rho,\sigma}(g^{\rho\sigma}a^2-2W^\rho W^\sigma)=-2\bar{F}_{\rho\sigma}W^\rho T^\sigma.\quad(5.9)$$

Thus (5.8) becomes

$$\begin{aligned}a^2(T_{\kappa,\lambda}-T_{\lambda,\kappa})+2(T_{\lambda,\sigma}W^\sigma W_\kappa+T_{\sigma,\kappa}W^\sigma W_\lambda \\ -T_{,\rho}{}^\rho W_\kappa W_\lambda)=G_{\kappa\lambda},\end{aligned}\quad(5.10)$$

where

$$\begin{aligned}G_{\kappa\lambda}=W_\kappa\bar{F}_{\lambda\nu}T^\nu+T_\kappa\bar{F}_{\lambda\nu}W^\nu+W^\nu T_\nu\bar{F}_{\kappa\lambda}+W_\lambda\bar{F}_{\kappa\nu}T_\nu \\ -T_\lambda\bar{F}_{\kappa\nu}W^\nu+g_{\kappa\lambda}\bar{F}_{\sigma\nu}W^\sigma T^\nu.\end{aligned}\quad(5.11)$$

Setting $\kappa=4$ in Eqs. (5.10), and assuming that the coordinate system is a galilean one in which $V^\sigma=\delta_4{}^\sigma$, we obtain the four equations

$$\left.\begin{aligned}T_{4,i}&=-T_{i,4}-2(T_{i,j}W^j-T_{,j}{}^jW_i)+\frac{G_{4i}}{W_4}, \\ T_{4,4}&=2\frac{T_{i,j}W^iW^j}{W_4}-2(1-W_4)T_{,j}{}^j+\frac{G_{44}-2G_{4i}W^i}{2(W_4)^2},\end{aligned}\right\}\quad(5.12)$$

where G_{4i} and G_{44} are obtained from (5.11). The existence of solutions of these equations depends on the nature of the functions χ_σ and their derivatives.

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Forms of Relativistic Dynamics

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For the purposes of atomic theory it is necessary to combine the restricted principle of relativity with the Hamiltonian formulation of dynamics. This combination leads to the appearance of ten fundamental quantities for each dynamical system, namely the total energy, the total momentum and the 6-vector which has three components equal to the total angular momentum. The usual form of dynamics expresses everything in terms of dynamical variables at one instant of time, which results in specially simple expressions for six or these ten, namely the components of momentum and of angular momentum. There are other forms for relativistic dynamics in which others of the ten are specially simple, corresponding to various sub-groups of the inhomogeneous Lorentz group. These forms are investigated and applied to a system of particles in interaction and to the electromagnetic field.

1. INTRODUCTION

EINSTEIN'S great achievement, the principle of relativity, imposes conditions which all physical laws have to satisfy. It profoundly influences the whole of physical science, from cosmology, which deals with the very large, to the study of the atom, which deals with the very small. General relativity requires that physical laws, expressed in terms of a system of curvilinear coordinates in space-time, shall be invariant under all transformations of the coordinates. It brings gravitational fields automatically into physical theory and describes correctly the influence of these fields on physical phenomena.

Gravitational fields are specially important when one is dealing with large-scale phenomena, as in cosmology, but are quite negligible at the other extreme, the study of the atom. In the atomic world the departure of space-time from flatness is so excessively small that there would be no point in taking it into account at the present time, when many large effects are still unexplained. Thus one naturally works with the simplest kind of coordinate system, for which the tensor $g^{\mu\nu}$ that defines the metric has the components

$$g^{00}=-g^{11}=-g^{22}=-g^{33}=1$$
$$g^{\mu\nu}=0 \quad \text{for} \quad \mu\neq\nu.$$

Einstein's restricted principle of relativity is now of paramount importance, requiring that physical laws shall be invariant under transformations from one such coordinate system to another. A transformation of this kind is called an inhomogeneous Lorentz transformation. The coordinates u_μ transform linearly according to the equations

$$\left.\begin{aligned} u_\mu{}^* &= \alpha_\mu + \beta_\mu{}^\nu u_\nu \\ \text{with} \qquad \beta_\mu{}^\nu \beta^{\mu\rho} &= g^{\nu\rho}, \end{aligned}\right\} \quad (1)$$

the α's and β's being constants.

A transformation of the type (1) may involve a reflection of the coordinate system in the three spacial dimensions and it may involve a time reflection, the direction du_0 in space-time changing from the future to the past. I do not believe there is any need for physical laws to be invariant under these reflections, although all the exact laws of nature so far known do have this invariance. The restricted principle of relativity arose from the requirement that the laws of nature should be independent of the position and velocity of the observer, and any change the observer may make in his position and velocity, taking his coordinate system with him, will lead to a transformation (1) of a kind that can be built up from infinitesimal transformations and cannot involve a reflection. Thus it appears that restricted relativity will be satisfied by the requirement that physical laws shall be invariant under infinitesimal transformations of the coordinate system of the type (1). Such an infinitesimal transformation is given by

$$\left.\begin{aligned} u_\mu{}^* &= u_\mu + a_\mu + b_\mu{}^\nu u_\nu \\ \text{with} \qquad b_{\mu\nu} &= -b_{\nu\mu}, \end{aligned}\right\} \quad (2)$$

the a's and b's being infinitesimal constants.

A second general requirement for dynamical theory has been brought to light through the discovery of quantum mechanics by Heisenberg and Schrödinger, namely the requirement that the equations of motion shall be expressible in the Hamiltonian form. This is necessary for a transition to the quantum theory to be possible. In atomic theory one thus has two over-riding requirements. The problem of fitting them together forms the subject of the present paper.

The existing theories of the interaction of elementary particles and fields are all unsatisfactory in one way or another. The imperfections may well arise from the use of wrong dynamical systems to represent atomic phenomena, i.e., wrong Hamiltonians and wrong interaction energies. *It thus becomes a matter of great importance to set up new dynamical systems and see if they will better describe the atomic world.* In setting up such a new dynamical system one is faced at the outset by the two requirements of special relativity and of Hamiltonian equations of motion. The present paper is intended to make a beginning on this work by providing the simplest methods for satisfying the two requirements simultaneously.

2. THE TEN FUNDAMENTAL QUANTITIES

The theory of a dynamical system is built up in terms of a number of algebraic quantities, called dynamical variables, each of which is defined with respect to a system of coordinates in space-time. The usual dynamical variables are the coordinates and momenta of particles at particular times and field quantities at particular points in space-time, but other kinds of quantities are permissible, as will appear later.

In order that the dynamical theory may be expressible in the Hamiltonian form, it is necessary that any two dynamical variables, ξ and η, shall have a P.b. (Poisson bracket) $[\xi, \eta]$, subject to the following laws,

$$\left.\begin{aligned} [\xi, \eta] &= -[\eta, \xi] \\ [\xi, \eta+\zeta] &= [\xi, \eta]+[\xi, \zeta] \\ [\xi, \eta\zeta] &= [\xi, \eta]\zeta+\eta[\xi, \zeta] \\ [[\xi, \eta], \zeta]+[[\eta, \zeta], \xi]+[[\zeta, \xi], \eta] &= 0. \end{aligned}\right\} \quad (3)$$

A number or physical constant may be counted as a special case of a dynamical variable, and has the property that its P.b. with anything vanishes.

Dynamical variables change when the system of coordinates with respect to which they are defined changes, and must do so in such a way that P.b. relations between them remain invariant. This requires that with an infinitesimal change in the coordinate system (2) each dynamical variable ξ shall change according to the law

$$\xi^* = \xi + [\xi, F], \quad (4)$$

where F is some infinitesimal dynamical variable independent of ξ, depending only on the dynamical system involved and the change in the coordinate system. We are thus led to associate one F with each infinitesimal transformation of coordinates.

Let us apply two infinitesimal transformations of coordinates in succession. Suppose the first one changes the dynamical variable ξ to ξ^* according to

$$\xi^* = \xi + [\xi, F_1],$$

and the second one changes ξ^* to $\xi^\dagger$ according to

$$\xi^\dagger = \xi^* + [\xi^*, F_2{}^*] = \xi^* + [\xi, F_2]^*.$$

The two transformations together change ξ to $\xi^\dagger$

according to

$$\xi^{\dagger}=\xi+[\xi, F_1]+[\xi, F_2]+[[\xi, F_2], F_1],$$

to the accuracy of the order F_1F_2 (with neglect of terms of order F_1^2 or F_2^2). If these two transformations are applied in the reverse order, they change ξ to $\xi^{\dagger\dagger}$ according to

$$\xi^{\dagger\dagger}=\xi+[\xi, F_2]+[\xi, F_1]+[[\xi, F_1], F_2].$$

Thus

$$\begin{aligned}\xi^{\dagger\dagger}&=\xi^{\dagger}+[[\xi, F_1], F_2]-[[\xi, F_2], F_1]\\&=\xi^{\dagger}+[\xi, [F_1, F_2]],\end{aligned}$$

with the help of the first and last of Eqs. (3). This gives the change in a dynamical variable associated with that change of the coordinate system which is the commutator of the two previous changes. It is of the standard form

$$\xi^{\dagger\dagger}=\xi^{\dagger}+[\xi^{\dagger}, F],$$

with an F that is the P. b. of the F's associated with the two previous changes of coordinates. Thus the commutation relations between the various infinitesimal changes of coordinates correspond to the P.b. relations between the associated F's.

The F associated with the transformation (2) must depend linearly on the infinitesimal numbers a_μ, $b_{\mu\nu}$ that fix this transformation. Thus we can put

$$\left.\begin{aligned}F&=-P^{\mu}a_{\mu}+\tfrac{1}{2}M^{\mu\nu}b_{\mu\nu}\\M^{\mu\nu}&=-M^{\nu\mu},\end{aligned}\right\}\quad(5)$$

where P^μ, $M^{\mu\nu}$ are finite dynamical variables, independent of the transformation of coordinates.

The ten quantities P_μ, $M_{\mu\nu}$ are characteristic for the dynamical system. They will be called the ten *fundamental quantities*. They determine how all dynamical variables are affected by a change in the coordinate system of the kind that occurs in special relativity. Each of them is associated with a type of infinitesimal transformation of the inhomogeneous Lorentz group. Seven of them have simple physical interpretations, namely, P_0 is the total energy of the system, P_r ($r=1$, 2, 3) is the total momentum, and M_{rs} is the total angular momentum about the origin. The remaining three M_{r0} do not correspond to any such well-known physical quantities, but are equally important in the general dynamical scheme.

From the commutation relations between particular infinitesimal transformations of the coordinate system we get at once the P.b. relations between the ten fundamental quantities,

$$\left.\begin{aligned}&[P_\mu, P_\nu]=0\\&[M_{\mu\nu}, P_\rho]=-g_{\mu\rho}P_\nu+g_{\nu\rho}P_\mu\\&[M_{\mu\nu}, M_{\rho\sigma}]=-g_{\mu\rho}M_{\nu\sigma}+g_{\nu\rho}M_{\mu\sigma}-g_{\mu\sigma}M_{\rho\nu}+g_{\nu\sigma}M_{\rho\mu}.\end{aligned}\right\}\quad(6)$$

To construct a theory of a dynamical system one must obtain expressions for the ten fundamental quantities that satisfy these P.b. relations. *The problem of finding a new dynamical system reduces to the problem of finding a new solution of these equations.*

An elementary solution is provided by the following scheme. Take the four coordinates q_μ of a point in space-time as dynamical coordinates and let their conjugate momenta be p_μ, so that

$$\begin{gathered}[q_\mu, q_\nu]=0, \quad [p_\mu, p_\nu]=0\\ [p_\mu, q_\nu]=g_{\mu\nu}.\end{gathered}$$

The q's will transform under an infinitesimal transformation of the coordinate system in the same way as the u's in (2). This leads to

$$P_\mu=p_\mu, \quad M_{\mu\nu}=q_\mu p_\nu-q_\nu p_\mu, \quad (7)$$

and provides a solution of the P.b. relations (6). The solution (7) does not seem to be of any practical importance, but it may be used as a basis for obtaining other solutions that are of practical importance, as the next three sections will show.

The foregoing discussion of the requirements for a relativistic dynamical theory may be generalized somewhat. We may work with dynamical variables that are connected by one or more relations for all states of motion that occur physically. Such relations are called *subsidiary equations*. They will be written

$$A\approx 0 \quad (8)$$

to distinguish them from dynamical equations. They are less strong than dynamical equations, because with a dynamical equation one can take the P.b. of both sides with any dynamical variable and get another equation, while with a subsidiary equation one cannot do this in general. The lesser assumption is made, however, that from two subsidiary equations $A\approx 0, B\approx 0$ one can infer a third

$$[A, B]\approx 0. \quad (9)$$

A subsidiary equation must remain a subsidiary equation under any change of coordinate system. This enables one to infer from (8)

$$[P_\mu, A]\approx 0, \quad [M_{\mu\nu}, A]\approx 0. \quad (10)$$

A dynamical variable is of physical importance only if its P.b. with any subsidiary equation gives another subsidiary equation, i.e., its P.b. with A in (8) must vanish in the subsidiary sense. Such a dynamical variable will be called a *physical variable*. The P.b. of two physical variables is a physical variable. Equations (10) show that the ten fundamental quantities are physical variables.

The physical variables are the only ones that are really important. One could eliminate the non-physical variables from the theory altogether and one could then make the subsidiary equations into dynamical equations. However, the elimination may be awkward and may spoil some symmetry feature in the scheme of equations, so it is desirable to retain the possibility of subsidiary equations in the general theory.

3. THE INSTANT FORM

The ten fundamental quantities for dynamical systems that occur in practice are usually such that some of them are specially simple and the others are complicated. The complicated ones will be called the Hamiltonians. They play jointly the rôle of the single Hamiltonian in non-relativistic dynamics. Since the P.b. of two simple quantities is a simple quantity, the simple ones of the ten fundamental quantities must be those associated with some sub-group of the inhomogeneous Lorentz group.

In the usual form of dynamics one works with dynamical variables referring to physical conditions at some instant of time, e.g., the coordinates and momenta of particles at that instant. An instant in the four-dimensional relativistic picture is a flat three-dimensional surface containing only directions which lie outside the light-cone. The simplest instant referred to the u coordinate system is given by the equation

$$u_0=0. \tag{11}$$

The effect of working with dynamical variables referring to physical conditions at this instant will be to make specially simple those of the fundamental quantities associated with transformations of coordinates that leave the instant invariant, namely P_1, P_2, P_3, M_{23}, M_{31}, M_{12}. The remaining ones, P_0, M_{10}, M_{20}, M_{30}, will be complicated in general and will be the Hamiltonians. We get in this way a form of dynamics which is associated with the sub-group of the inhomogeneous Lorentz group that leaves the instant invariant, and which may appropriately be called the *instant form.*

Let us take as an example a single particle by itself. The ten fundamental quantities in this case are well known, but they will be worked out again here to illustrate a method that can be used also with the other forms of dynamics.

We take as dynamical coordinates the three coordinates of the particle at the instant (11). Calling these coordinates q_r, we can base our work on the scheme (7), with the additional equation

$$q_0=0. \tag{12}$$

With this equation p_0 no longer has a meaning. We must therefore modify the expressions for the ten fundamental quantities given by (7) so as to eliminate p_0 from them, without invalidating the P.b. relations (6).

Let us change the expressions for the ten fundamental quantities by multiples of $p^\sigma p_\sigma-m^2$, where m is a constant, i.e., let us put

$$\left.\begin{aligned} P_\mu&=p_\mu+\lambda_\mu(p^\sigma p_\sigma-m^2)\\ M_{\mu\nu}&=q_\mu p_\nu-q_\nu p_\mu+\lambda_{\mu\nu}(p^\sigma p_\sigma-m^2),\end{aligned}\right\} \tag{13}$$

where

$$\lambda_{\mu\nu}=-\lambda_{\nu\mu}$$

and the coefficients λ are functions of the q's and p's that do not become infinitely great when one puts $p^\sigma p_\sigma=m^2$ with $p_0>0$. Since $p^\sigma p_\sigma-m^2$ has zero P.b. with all the expressions (7), the modified expressions (13) must still satisfy the P.b. relations (6), apart from multiples of $p^\sigma p_\sigma-m^2$, with any choice of the λ's. If we now choose the λ's so as to make the P_μ, $M_{\mu\nu}$ given by (13) independent of p_0, the P.b. relations (6) must be satisfied apart from terms that are independent of p_0 as well as being multiples of $p^\sigma p_\sigma-m^2$. Such terms must vanish, so we get in this way a solution of our problem.

The λ's have the values

$$\left.\begin{aligned} \lambda_r&=0, \quad \lambda_0=-\{p_0+(p_sp_s+m^2)^{\frac{1}{2}}\}^{-1},\\ \lambda_{sr}&=0, \quad \lambda_{r0}=-q_r\{p_0+(p_sp_s+m^2)^{\frac{1}{2}}\}^{-1},\end{aligned}\right\} \tag{14}$$

and Eqs. (13) become

$$P_r=p_r, \qquad M_{rs}=q_rp_s-q_sp_r, \tag{15}$$

$$P_0=(p_sp_s+m^2)^{\frac{1}{2}}, \quad M_{r0}=q_r(p_sp_s+m^2)^{\frac{1}{2}}, \tag{16}$$

with the help of (12). Equations (15) and (16) give all the ten fundamental quantities for a particle with rest-mass m. Those given by (15) are the simple ones: those given by (16) are the Hamiltonians.

For a dynamical system composed of several particles, P_r and M_{rs} will be just the sum of their values for the particles separately,

$$P_r=\sum p_r, \quad M_{rs}=\sum(q_rp_s-q_sp_r). \tag{17}$$

The Hamiltonians P_r, M_{r0} will be the sum of their values for the particles separately plus interaction terms,

$$\left.\begin{aligned} P_0&=\sum(p_sp_s+m^2)^{\frac{1}{2}}+V,\\ M_{r0}&=\sum q_r(p_sp_s+m^2)^{\frac{1}{2}}+V_r.\end{aligned}\right\} \tag{18}$$

The V's here must be chosen to make P_0, M_{r0} satisfy all the P.b. relations (6) in which they appear.

Some of these relations are linear in the V's and are easily fulfilled. The P.b. relations for $[M_{rs}, P_0]$ and $[M_{rs}, M_{t0}]$ are fulfilled provided V is a three-dimensional scalar (in the space u_1, u_2, u_3) and V_r a three-dimensional vector. The P.b. relation for $[P_r, P_0]$ will be fulfilled provided V is independent of the position of the origin in the three-dimensional space u_1, u_2, u_3. The P.b. relation for $[M_{r0}, P_s]$ will be fulfilled provided

$$V_r=q_rV+V_r', \tag{19}$$

where the q_r are the coordinates of any one of the particles and the V_r' are independent of the position of the origin in three-dimensional space.

The remaining conditions for the V's are quadratic, involving $[V, V_r]$ or $[V_r, V_s]$. These conditions are not easily fulfilled and provide the real difficulty in the problem of constructing a theory of a relativistic dynamical system in the instant form.

4. THE POINT FORM

One can build up a dynamical theory in terms of dynamical variables that refer to physical conditions on some three-dimensional surface other than an instant. The surface must satisfy the condition that the world-line of every particle must meet it, otherwise the particle could not be described by variables on the surface, and preferably the world-line should meet it only once, for the sake of uniqueness.

To get a simple form of theory one should take the surface to be such that it is left invariant by some sub-group of the group of inhomogeneous Lorentz transformations. A possible sub-group is the group of rotations about some point, say the origin $u_\mu=0$. The surface may then be taken to be a branch of a hyperboloid

$$u^\rho u_\rho=\kappa^2, \quad u_0>0, \tag{20}$$

with κ a constant. The fundamental quantities associated with the infinitesimal transformations of the sub-group, namely the $M_{\mu\nu}$, will then be specially simple, while the others, namely the P_μ, will be complicated in general and will be the Hamiltonians. A new form of dynamics is thus obtained, which may be called the *point form*, as it is characterized by being associated with the sub-group that leaves a point invariant.

To illustrate the new form, let us take again the example of a single particle. The dynamical coordinates must determine the point where the world-line of the particle meets the hyperboloid (20). Let the four coordinates of this point in the u system of coordinates be q_μ. Only three of these are independent, but instead of eliminating one of them, it is more convenient to work with all four and introduce the subsidiary equation

$$q^\rho q_\rho \approx \kappa^2. \tag{21}$$

It is then necessary that the ten fundamental quantities, and indeed all physical variables, shall have zero P.b. with $q^\rho q_\rho$. The condition for this is that they should involve the p's only through the combinations $q_\mu p_\nu - q_\nu p_\mu$.

The ten fundamental quantities may be obtained by a method parallel to that of the preceding section, with the subsidiary equation (21) taking the place of Eq. (12). We again assume Eqs. (13), and now choose the λ's so as to make their right-hand sides have zero P.b. with $q^\rho q_\rho$. The resulting expressions for the ten fundamental quantities will again satisfy the P.b. relations (6), as may be inferred by a similar argument to the one given in the preceding section.

We find at once

$$\lambda_{\mu\nu}=0.$$

To obtain λ_μ, instead of arranging directly for the P_μ to have zero P.b. with $q^\rho q_\rho$, it is easier to make $q_\mu P_\nu - q_\nu P_\mu$ and $P^\mu P_\mu$ have zero P.b. with $q^\rho q_\rho$. Now

$$q_\mu P_\nu - q_\nu P_\mu = q_\mu p_\nu - q_\nu p_\mu + (q_\mu\lambda_\nu - q_\nu\lambda_\mu)(p^\sigma p_\sigma - m^2),$$

so we must have

$$q_\mu\lambda_\nu - q_\nu\lambda_\mu = 0,$$

and hence

$$\lambda_\mu = q_\mu B,$$

where B is some dynamical variable independent of μ. Further

$$\begin{aligned} P^\mu P_\mu &= \{p^\mu + q^\mu B(p^\sigma p_\sigma - m^2)\}\{p_\mu + q_\mu B(p^\rho p_\rho - m^2)\} \\ &= m^2 + \{1 + 2p^\mu q_\mu B + q^\mu q_\mu B^2(p^\sigma p_\sigma - m^2)\}(p^\rho p_\rho - m^2). \end{aligned}$$

In order that $P^\mu P_\mu$ shall have zero P.b. with $q^\rho q_\rho$, we must take

$$1 + 2p^\mu q_\mu B + q^\mu q_\mu B^2(p^\sigma p_\sigma - m^2) = 0,$$

so that

$$B(p^\sigma p_\sigma - m^2) = (q^\rho q_\rho)^{-1}\{[(p^\nu q_\nu)^2 - q^\lambda q_\lambda(p^\sigma p_\sigma - m^2)]^{\frac{1}{2}} - p^\nu q_\nu\}$$

The right-hand side here tends to zero as $p^\sigma p_\sigma - m^2 \to 0$, so it is a multiple of $p^\sigma p_\sigma - m^2$, as it ought to be. We now get finally

$$\left.\begin{aligned} P_\mu &= p_\mu + q_\mu \kappa^{-2}\{[(p^\nu q_\nu)^2 - \kappa^2(p^\sigma p_\sigma - m^2)]^{\frac{1}{2}} - p^\nu q_\nu\} \\ M_{\mu\nu} &= q_\mu p_\nu - q_\nu p_\mu, \end{aligned}\right\} \tag{22}$$

in which the expression for P_μ has been simplified with the help of (21).

It is permissible to take $\kappa=0$ and so to have a light-cone instead of a hyperboloid. The expression for B then becomes much simpler and gives

$$P_\mu = p_\mu - \tfrac{1}{2} q_\mu (p^\nu q_\nu)^{-1}(p^\sigma p_\sigma - m^2). \tag{23}$$

instead of the first of Eqs. (22).

For a dynamical system composed of several particles, the $M_{\mu\nu}$ will be just the sum of their values for the particles separately,

$$M_{\mu\nu} = \sum (q_\mu p_\nu - q_\nu p_\mu). \tag{24}$$

The Hamiltonians P_μ will be the sum of their values for the particles separately plus interaction terms,

$$P_\mu = \sum \{p_\mu + q_\mu B(p^\sigma p_\sigma - m^2)\} + V_\mu. \tag{25}$$

The V_μ must be chosen so as to make the P_μ satisfy the correct P.b. relations. The relations for $[M_{\mu\nu}, P_\rho]$ are satisfied provided the V_μ are the components of a 4-vector. The remaining relations, which require the P_μ to have zero P.b. with one another, lead to quadratic conditions for the V_μ. These cause the real difficulty in the problem of constructing a theory of a relativistic dynamical system in the point form.

5. THE FRONT FORM

Consider the three-dimensional surface in space-time formed by a plane wave front advancing with the velocity of light. Such a surface will be called a *front* for brevity. An example of a front is given by the equation

$$u_0 - u_3 = 0. \tag{26}$$

We may set up a dynamical theory in which the dynamical variables refer to physical conditions on a front. This will make specially simple those of the fundamental quantities associated with infinitesimal transformations of coordinates that leave the front invariant, and will give a third form of dynamics, which may be called the *front form.*

If A_μ is any 4-vector, put

$$A_0+A_3=A_+, \quad A_0-A_3=A_-.$$

We get a convenient notation by using the $+$ and $-$ suffixes freely as tensor suffixes, together with 1 and 2. They may be raised with the help of

$$g^{++}=g^{--}=0, \quad g^{+-}=\tfrac{1}{2},$$
$$g^{i+}=g^{i-}=0, \quad \text{for } i=1, 2,$$

as one can verify by noting that these g values lead to the correct value for $g^{\mu\nu}A_\mu A_\nu$ when μ and ν are summed over 1, 2, $+$, $-$.

The equation of the front (26) becomes in this notation

$$u_-=0.$$

The fundamental quantities P_1, P_2, P_-, M_{12}, M_{+-}, M_{1-}, M_{2-} are associated with transformations of coordinates that leave this front invariant and will be specially simple. The remaining ones P_+, M_{1+}, M_{2+} will be complicated in general and will be the Hamiltonians.

Let us again work out the example of a single particle. The dynamical coordinates are now q_1, q_2, q_+. We again assume Eqs. (13), and add to them the further equation $q_-=0$. We must now choose the λ's so as to make the right-hand sides of (13) independent of p_+. The resulting expressions for the ten fundamental quantities will then again satisfy the required P.b. relations.

We find

$$\lambda_+=-1/p_-, \quad \lambda_{i+}=-q_i/p_-,$$

the other λ's vanishing. Thus

$$\left.\begin{aligned} &P_i=p_i, \quad P_-=p_-, \\ &M_{12}=q_1p_2-q_2p_1, \quad M_{i-}=q_ip_-, \quad M_{+-}=q_+p_-, \end{aligned}\right\}(27)$$

$$\left.\begin{aligned} &P_+=(p_1^2+p_2^2+m^2)/p_-, \\ &M_{i+}=q_i(p_1^2+p_2^2+m^2)/p_--q_+p_i. \end{aligned}\right\}(28)$$

Equations (27) give the simple fundamental quantities. Equations (28) give the Hamiltonians.

For a dynamical system composed of several particles, P_i, P_-, M_{12}, M_{+-}, M_{i-} will be just the sum of their values for the particles separately. The Hamiltonians P_+, M_{i+} will be the sum of their values for the particles separately plus interaction terms,

$$\left.\begin{aligned} &P_+=\sum(p_1^2+p_2^2+m^2)/p_-+V \\ &M_{i+}=\sum\{q_i(p_1^2+p_2^2+m^2)/p_--q_+p_i\}+V_i. \end{aligned}\right\}(29)$$

The V's must satisfy certain conditions to make the Hamiltonians satisfy the correct P.b. relations.

As before, some of these conditions are linear and some are quadratic. The linear conditions for V require that it shall be invariant under all transformations of the three coordinates u_1, u_2, u_+ of the front except those for which du_+ gets multiplied by a factor, and for the latter transformations V must get multiplied by the same factor. The linear conditions for V_i require it to be of the form

$$V_i=q_iV+V_i', \tag{30}$$

where q_i are the coordinates 1, 2 of any one of the particles, and V_i' has the same properties as V with regard to all transformations of the three coordinates of the front except rotations associated with M_{12}, and under these rotations it behaves like a two-dimensional vector. The quadratic conditions for the V's are not easily fulfilled and give rise to the real difficulty in the construction of a theory of a relativistic dynamical system in the front form.

6. THE ELECTROMAGNETIC FIELD

To set up the dynamical theory of fields on the lines discussed in the three preceding sections, one may take as dynamical variables the three-fold infinity of field quantities at all points on the instant, hyperboloid, or front, and use them in place of the discrete set of variables of particle theory. The ten fundamental quantities P_μ, $M_{\mu\nu}$ are to be constructed out of them, satisfying the same P.b. relations as before.

For a field which allows waves moving with the velocity of light, a difficulty arises with the point form of theory, because one may have a wave packet that does not meet the hyperboloid (20) at all. Thus physical conditions on the hyperboloid cannot completely describe the state of the field. One must introduce some extra dynamical variables besides the field quantities on the hyperboloid. A similar difficulty arises, in a less serious way, with the front form of theory. Waves moving with the velocity of light in exactly the direction of the front cannot be described by physical conditions on the front, and some extra variables must be introduced for dealing with them.

An alternative method of setting up the dynamical theory of fields is obtained by working with dynamical variables that describe the Fourier components of the field. This method has various advantages. It disposes of the above difficulty of extra variables, and it usually lends itself more directly to physical interpretation. It leads to expressions for the ten fundamental quantities that can be used with all three forms. For a field by itself, there is then no difference between the three forms. A difference occurs, of course, if the field is in interaction with something. The dynamical variables of the field are then to be understood as the Fourier components that the field would have, if the interaction were suddenly cut off at the instant, hyperboloid or front, after the cutting off.

Let us take as an example the electromagnetic field, first without any interaction. We may work with the four potentials $A_\lambda(u)$ satisfying the subsidiary equation

$$\partial A_\lambda/\partial u_\lambda \approx 0. \tag{31}$$

Their Fourier resolution is

$$A_\lambda(u)=\int\{A_{k\lambda}\exp(ik^\mu u_\mu) + A^\dagger_{k\lambda}\exp(-ik^\mu u_\mu)\}k_0^{-1}d^3k, \tag{32}$$

with

$$k_0=(k_1^2+k_2^2+k_3^2)^{\frac{1}{2}},\quad d^3k=dk_1dk_2dk_3.$$

The factor k_0^{-1} inserted in (32) leads to simpler transformation laws for the Fourier coefficients $A_{k\lambda}$, since the differential element $k_0^{-1}d^3k$ is Lorentz invariant. We now take the $A_{k\lambda}$, $A^\dagger_{k\lambda}$ as dynamical variables.

Under the transformation of coordinates (2) the potential $A_\lambda(u)$ at a particular point u changes to a potential at the point with the same u-values in the new coordinate system, i.e., the point with the coordinates $u_\mu - a_\mu - b_\mu{}^\nu u_\nu$ in the original coordinate system. This causes a change in $A_\lambda(u)$ of amount

$$-(a_\mu+b_\mu{}^\nu u_\nu)\partial A_\lambda/\partial u_\mu.$$

There is a further change, of amount $b_\lambda{}^\nu A_\nu$, owing to the change in the direction of the axes. Thus, from (4) and (5)

$$\begin{aligned}[A_\lambda(u),\ -P^\mu a_\mu+\tfrac{1}{2}M^{\mu\nu}b_{\mu\nu}] &= A_\lambda(u)^*-A_\lambda(u)\\ &= -(a_\mu+b_\mu{}^\nu u_\nu)\partial A_\lambda/\partial u_\mu+b_\lambda{}^\nu A_\nu,\end{aligned}$$

and hence

$$\left.\begin{aligned}&[A_\lambda(u),\ P^\mu]=\partial A_\lambda/\partial u_\mu,\\ &[A_\lambda(u),\ M_{\mu\nu}]=u_\mu\partial A_\lambda/\partial u^\nu-u_\nu\partial A_\lambda/\partial u^\mu\\ &\qquad\qquad +g_{\lambda\mu}A_\nu-g_{\lambda\nu}A_\mu.\end{aligned}\right\} \tag{33}$$

Taking Fourier components according to (32), we get

$$\left.\begin{aligned}&[A_{k\lambda},\ P_\mu]=ik_\mu A_{k\lambda},\\ &[A_{k\lambda},\ M_{\mu\nu}]=(k_\mu\partial/\partial k^\nu-k_\nu\partial/\partial k^\mu)A_{k\lambda}\\ &\qquad\qquad +g_{\lambda\mu}A_{k\nu}-g_{\lambda\nu}A_{k\mu},\end{aligned}\right\} \tag{34}$$

in which $A_{k\lambda}$ may be considered as a function of four independent k's for the purpose of applying the differential operator $k_\mu\partial/\partial k^\nu - k_\nu\partial/\partial k^\mu$ to it.

The Maxwell theory gives for the energy and momentum of the electromagnetic field

$$P_\mu=-4\pi^2\int k_\mu A_k{}^\lambda A^\dagger_{k\lambda}k_0^{-1}d^3k, \tag{35}$$

the $-$ sign being needed to make the transverse components contribute a positive energy. In order that this may agree with the first of Eqs. (34), we must have the P.b. relations

$$\left.\begin{aligned}&[A_{k\lambda},\ A_{k'\mu}]=0\\ &[A_{k\lambda},\ A^\dagger_{k'\mu}]=-ig_{\lambda\mu}/4\pi^2\\ &\qquad\cdot k_0\delta(k_1-k_1')\delta(k_2-k_2')\delta(k_3-k_3').\end{aligned}\right\} \tag{36}$$

The second of Eqs. (34) then leads to

$$M_{\mu\nu}=4\pi^2 i\int\{A^\dagger_{k\lambda}(k_\mu\partial/\partial k^\nu-k_\nu\partial/\partial k^\mu)A_k{}^\lambda + A^\dagger_{k\mu}A_{k\nu}-A^\dagger_{k\nu}A_{k\mu}\}k_0^{-1}d^3k. \tag{37}$$

Equations (35) and (37) give the ten fundamental quantities.

For the electromagnetic field in interaction with charged particles, the ten fundamental quantities will be the sum of their values for the field alone, given by (35) and (37), and their values for the particles, given in one of the three preceding sections, with interaction terms involving the field variables $A_{k\lambda}$, $A^\dagger_{k\lambda}$ as well as the particle variables. One usually assumes that there is no direct interaction between the particles, only interaction between each particle and the field. The ten fundamental quantities then take the form

$$\left.\begin{aligned}P_\mu&=P_\mu{}^F+\textstyle\sum_a P_\mu{}^a\\ M_{\mu\nu}&=M_{\mu\nu}{}^F+\textstyle\sum_a M_{\mu\nu}{}^a,\end{aligned}\right\} \tag{38}$$

where $P_\mu{}^F$, $M_{\mu\nu}{}^F$ are the contributions of the field alone, given by (35) and (37), and $P_\mu{}^a$, $M_{\mu\nu}{}^a$ are the contributions of the a-th particle, consisting of terms for the particle by itself and interaction terms. For point charges, the interaction terms will involve the field variables only through the $A_\lambda(q)$ and their derivatives at the point q where the world-line of the particle meets the instant, hyperboloid or front. The expressions for $P_\mu{}^a$, $M_{\mu\nu}{}^a$ may easily be worked out for this case by a generalization of the method of the three preceding sections, as follows.

Suppose there is only one particle, for simplicity. We must replace Eqs. (13) by

$$\left.\begin{aligned}P_\mu&=P_\mu{}^F+p_\mu+\lambda_\mu(\pi^\sigma\pi_\sigma-m^2)\\ M_{\mu\nu}&=M_{\mu\nu}{}^F+q_\mu p_\nu-q_\nu p_\mu+\lambda_{\mu\nu}(\pi^\sigma\pi_\sigma-m^2),\end{aligned}\right\} \tag{39}$$

where

$$\pi_\sigma=p_\sigma-eA_\sigma(q),$$

and $P_\mu{}^F$, $M_{\mu\nu}{}^F$ are the right-hand sides of (35) and (37). From (33),

$$\begin{aligned}&[A_\lambda(q),\ P_\mu{}^F+p_\mu]=0\\ &[A_\lambda(q),\ M_{\mu\nu}{}^F+q_\mu p_\nu-q_\nu p_\mu]=g_{\lambda\mu}A_\nu(q)-g_{\lambda\nu}A_\mu(q),\end{aligned}$$

and hence

$$\begin{aligned}&[\pi_\lambda,\ P_\mu{}^F+p_\mu]=0\\ &[\pi_\lambda,\ M_{\mu\nu}{}^F+q_\mu p_\nu-q_\nu p_\mu]=g_{\lambda\mu}\pi_\nu-g_{\lambda\nu}\pi_\mu.\end{aligned}$$

It follows that $\pi^\sigma\pi_\sigma$ has zero P.b. with each of the quantities $P_\mu{}^F+p_\mu$, $M_{\mu\nu}{}^F+q_\mu p_\nu-q_\nu p_\mu$. One can now infer, by the same argument as in the case of no field, that if the λ's in (39) are chosen to make P_μ, $M_{\mu\nu}$ have zero P.b. with q_0, $q^\rho q_\rho$ or q_-, the P.b. relations (6) will all be satisfied. Such a choice of λ's, in conjunction with one of the equations $q_0=0$, $q^\rho q_\rho\approx\kappa^2$, $q_-=0$, will provide the ten fundamental quantities for a charged particle in interaction with the field in the instant, point and front forms, respectively. The subsidiary Eq. (31) must be modified when a charge is present.

The point form will be worked out as an illustration. In this case we have at once $\lambda_{\mu\nu}=0$. We can get λ_μ conveniently by arranging that $q_\mu(P_\nu-P_\nu{}^F)-q_\nu(P_\mu-P_\mu{}^F)$ and $\{P^\mu-P^{\mu F}-eA^\mu(q)\}\{P_\mu-P_\mu{}^F-eA_\mu(q)\}$ shall have zero P.b. with $q^\rho q_\rho$. The first condition gives $\lambda_\mu=q_\mu B$. The second then gives

$$1+2\pi^\mu q_\mu B+q^\mu q_\mu B^2(\pi^\sigma\pi_\sigma-m^2)=0.$$

Thus we get finally

$$\left.\begin{aligned} P_\mu&=P_\mu{}^F+p_\mu+q_\mu\kappa^{-2}\{[(\pi^\nu q_\nu)^2\\ &\qquad\qquad-\kappa^2(\pi^\sigma\pi_\sigma-m^2)]^{\frac{1}{2}}-\pi^\nu q_\nu\}\\ M_{\mu\nu}&=M_{\mu\nu}{}^F+q_\mu p_\nu-q_\nu p_\mu.\end{aligned}\right\}\tag{40}$$

The above theory of point charges is subject to the usual difficulty that infinities will arise in the solution of the equations of motion, on account of the infinite electromagnetic energy of a point charge. The present treatment has the advantage over the usual treatment of the electromagnetic equations that it offers simpler opportunities for departure from the point-charge model for elementary particles.

8. DISCUSSION

Three forms have been given in which relativistic dynamical theory may be put. For particles with no interaction, any one of the three is possible. For particles with interaction, it may be that all three are still possible, or it may be that only one is possible, depending on the kind of interaction. If one wants to set up a new kind of interaction between particles in order to improve atomic theory, the way to proceed would be to take one of the three forms and try to find the interaction terms V, or to find directly the Hamiltonians, satisfying the required P.b. relations. The question arises, which is the best form to take for this purpose.

The instant form has the advantage of being the one people are most familiar with, but I do not believe it is intrinsically any better for this reason. The four Hamiltonians P_0, M_{r0} form a rather clumsy combination.

The point form has the advantage that it makes a clean separation between those of the fundamental quantities that are simple and those that are the Hamiltonians. The former are the components of a 6-vector, the latter are the components of a 4-vector. Thus the four Hamiltonians can easily be treated as a single entity. All the equations with this form can be expressed neatly and concisely in four-dimensional tensor notation.

The front form has the advantage that it requires only three Hamiltonians, instead of the four of the other forms. This makes it mathematically the most interesting form, and makes any problem of finding Hamiltonians substantially easier. The front form has the further advantage that there is no square root in the Hamiltonians (28), which means that one can avoid negative energies for particles by suitably choosing the values of the dynamical variables in the front, without having to make a special convention about the sign of a square root. It may then be easier to eliminate negative energies from the quantum theory. This advantage also occurs with the point form with $\kappa=0$, there being no square root in (23).

There is no conclusive argument in favor of one or other of the forms. Even if it could be decided that one of them is the most convenient, this would not necessarily be the one chosen by nature, in the event that only one of them is possible for atomic systems. Thus all three forms should be studied further.

The conditions discussed in this paper for a relativistic dynamical system are necessary but not sufficient. Some further condition is needed to ensure that the interaction between two physical objects becomes small when the objects become far apart. It is not clear how this condition can be formulated mathematically. Present-day atomic theories involve the assumption of localizability, which is sufficient but is very likely too stringent. The assumption requires that the theory shall be built up in terms of dynamical variables that are each localized at some point in space-time, two variables localized at two points lying outside each other's light-cones being assumed to have zero P.b. A less drastic assumption may be adequate, e.g., that there is a fundamental length λ such that the P.b. of two dynamical variables must vanish if they are localized at two points whose separation is space-like and greater than λ, but need not vanish if it is less than λ.

I hope to come back elsewhere to the transition to the quantum theory.