loses all trace of its initial state as time progresses. Such a gradual loss of "memory" can be achieved only by the operation of a dissipative force like dynamical friction which will gradually damp out any given initial velocity. Thus, if we assume for the sake of simplicity, that η is independent of $|\mathbf{u}|$, then the average velocity at later times will tend to zero like

$$\bar{\mathbf{u}} = \mathbf{u}_0 e^{-\eta t}; \tag{38}$$

but this is not to imply that the mean square velocity also tends to zero. Indeed, the restoration of a Maxwellian distribution of velocities from an arbitrary initial state requires that

$$\bar{\mathbf{u}} \rightarrow 0 \text{ while } \langle |\mathbf{u}|^2 \rangle_{\text{Av}} \rightarrow \text{a constant as } t \rightarrow \infty. \tag{39}$$

To achieve the first of these conditions we need dynamical friction and to achieve the second we need random fluctuations as expressed by a diffusion coefficient. The recognition of these facts is, of course, Einstein's achievement.

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A Special Method for Solving the Dirac Equations

A. H. TAUB
Mathematics Department, University of Illinois, Urbana, Illinois

Exact solutions to the Dirac equations for an electron in an external field of an arbitrary plane wave are obtained by transforming plane wave solutions for a free electron by variable Lorentz matrices. These matrices are those which occur in the discussion of the classical orbits. An example shows that this method, which applies classically, may fail for the Dirac equations when the external field is a constant one.

1. INTRODUCTION

IN a previous paper[1] it was shown that solutions of the classical relativistic equations of motion for a charged particle in the external field of a plane electromagnetic wave may be obtained in terms of Lorentz matrices determined by the antisymmetric tensor describing the external field. In this paper it is shown that when plane wave solutions of the Dirac equations for a free particle are transformed by these Lorentz matrices then the exact solutions to the Dirac equations for an electron in the external field described above are obtained. These solutions have been discussed by Volkow[2] and Singupta.[3]

We first obtain the necessary and sufficient conditions that a variable set of Lorentz matrices must satisfy in order that they be able to transform plane wave solutions of the Dirac equation for a free particle into solutions of these equations when an external field is present. It is then shown that these conditions can be satisfied in case the external field is that of a plane wave.

In the binary spinor formalism[4] the Dirac equations are

$$\begin{aligned} g^{\sigma}\left(\frac{\hbar}{i}\frac{\partial}{\partial x^{\sigma}} - \frac{e}{c}\Phi_{\sigma}\right)\psi &= -imc\bar{\varphi}, \\ g^{\sigma}\left(\frac{\hbar}{i}\frac{\partial}{\partial x^{\sigma}} + \frac{e}{c}\Phi_{\sigma}\right)\varphi &= -imc\bar{\psi}, \end{aligned} \tag{1.1}$$

where m and e are the mass and charge of the particle, $\hbar$ is Planck's constant divided by 2π, c is the velocity of light, Φ_{σ} is the four-vector potential describing the external field and ψ, φ and g^{σ} are spinors. The first two are single index spinors and the g^{σ} are two index ones satisfying the matrix equation

$$\tfrac{1}{2}(\bar{g}^{\sigma}g^{\tau} + \bar{g}^{\tau}g^{\sigma}) = -g^{\sigma\tau}\cdot 1, \tag{1.2}$$

where 1 is the 2×2 identity matrix and in a galilean frame,

$$g_{\sigma\tau} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = g^{\sigma\tau}. \tag{1.3}$$

An explicit set of matrices satisfying (1.2) are given in T.E., p. 938.

Equations (1.1) are numerically invariant under a proper Lorentz transformation of the independent variables x^{σ}, namely,

$$x^{\sigma} \rightarrow x^{\sigma *} = L_{\tau}{}^{\sigma} x^{\tau},$$

where the $L_{\tau}{}^{\sigma}$ are constants, provided the spinors ψ and φ have the transformation law

$$\begin{aligned} \psi &\rightarrow \psi^*(x) = \Gamma\psi(L^{-1}x), \\ \varphi &\rightarrow \varphi^*(x) = \Gamma\varphi(L^{-1}x), \end{aligned} \tag{1.4}$$

where Γ is the spin-image of the Lorentz matrix L. Thus Γ is determined in terms of L by the equations:

$$\bar{\Gamma}^{-1} g^{\sigma} \Gamma = L_{\tau}{}^{\sigma} g^{\tau}, \tag{1.5}$$

and satisfied the condition

$$\det\Gamma = 1. \tag{1.6}$$

[1] A. H. Taub, Phys. Rev. **73**, 786 (1948).
[2] D. M. Volkow, Zeits. f. Physik **94**, 25 (1935).
[3] N. D. Singupta, Bull. Calcutta Math. Soc. **39**, 147 (1947).
[4] The notation used here is that of an earlier paper: "Tensor equations equivalent to the Dirac equations," Ann. of Math. **40**, 937 (1939). This will be referred to as T.E.

We shall assume that the elements of the Lorentz matrix L (and those of its spin image Γ) occurring in Eqs. (1.4) are functions of coordinates and determine the conditions L must satisfy in order that φ^* and ψ^* in this equation be solutions of (1.1) if ψ and φ are plane wave solutions of (1.1) with $\Phi_\sigma=0$. We shall also allow a transformation of gauge. Thus the problem is that of finding conditions on L (and hence Γ) in order that

$$\psi(x)=\Gamma\psi_1(p)\exp\left(\frac{i}{\hbar}(p_\sigma l_\tau{}^\sigma x^\tau+S(x))\right),$$
$$\varphi(x)=\Gamma\varphi_1(p)\exp\left(-\frac{i}{\hbar}(p_\sigma l_\tau{}^\sigma x^\tau+S(x))\right), \tag{1.7}$$

be solutions of (1.1), where

$$\psi_1(p)=p_\sigma\bar{g}^\sigma\bar{\psi}_0-imc\varphi_0,$$
$$\varphi_1(p)=p_\sigma\bar{g}^\sigma\bar{\varphi}_0+imc\psi_0, \tag{1.8}$$

ψ_0 and φ_0 are arbitrary constant spinors, p_σ are four constants satisfying

$$p_\sigma p^\sigma=m^2c^2; \tag{1.9}$$

$l_\tau{}^\sigma$ are the elements of the inverse matrix of $L_\tau{}^\sigma$, that is,

$$l_\tau{}^\sigma L_\rho{}^\tau=\delta_\rho{}^\sigma; \tag{1.10}$$

$l_\tau{}^\sigma$ are functions of the coordinates, and $S(x)$ is an arbitrary function of x^σ.

2. THE NECESSARY AND SUFFICIENT CONDITIONS

Substituting Eqs. (1.7) into (1.1) we obtain

$$g^\sigma\frac{\hbar}{i}\Gamma_{,\sigma}\psi_1+g^\sigma\Gamma p_\rho l_\sigma{}^\rho\psi_1+g^\sigma A_\sigma\Gamma\psi_1=-imc\Gamma\bar{\varphi}_1,$$
$$g^\sigma\frac{\hbar}{i}\Gamma_{,\sigma}\varphi_1-g^\sigma\Gamma p_\rho l_\sigma{}^\rho\varphi_1-g^\sigma A_\sigma\Gamma\varphi_1=-imc\bar{\Gamma}\bar{\psi}_1, \tag{2.1}$$

where

$$A_\sigma=p_\rho l_{\tau,\sigma}{}^\rho x^\tau+S_{,\sigma}-(e/c)\Phi_\sigma, \tag{2.2}$$

where the comma denotes the derivative, i.e.,

$$f_{,\sigma}=\partial f/\partial x^\sigma.$$

Multiplying Eqs. (2.1) by $\bar{\Gamma}^{-1}$ on the left and using (1.5) and the fact that ψ_1 and φ_1 satisfy the algebraic equations

$$g^\sigma p_\sigma\psi_1=-imc\bar{\varphi}_1,\quad g^\sigma p_\sigma\varphi_1=imc\bar{\psi}_1,$$

we obtain

$$g^\sigma(\hbar/i)\Gamma_{,\sigma}\psi_1+g^\sigma A_\sigma\Gamma\psi_1=0,$$
$$g^\sigma(\hbar/i)\Gamma_{,\sigma}\varphi_1-g^\sigma A_\sigma\Gamma\varphi_1=0.$$

In view of (1.5) these may be rewritten as

$$L_\mu{}^\sigma g^\mu(A_\sigma+(\hbar/i)\Gamma^{-1}\Gamma_{,\sigma})\psi_1=0,$$
$$L_\mu{}^\sigma g^\mu(A_\sigma-(\hbar/i)\Gamma^{-1}\Gamma_{,\sigma})\varphi_1=0.$$

These equations must hold for any ψ_1 and φ_1 given by (1.7), where ψ_0 and φ_0 are arbitrary. Hence we must have

$$L_\mu{}^\sigma g^\mu A_\sigma=0,\quad L_\mu{}^\sigma g^\mu\Gamma^{-1}\Gamma_{,\sigma}=0.$$

Since $L_\mu{}^\sigma$ is a Lorentz matrix and hence non-singular and since the g^μ are linearly independent, the necessary and sufficient conditions on $L_\tau{}^\sigma$ are

$$p_\rho l_{\tau,\sigma}{}^\rho x^\tau+S_{,\sigma}-(e/c)\Phi_\sigma=0 \tag{2.3}$$

and

$$g^\sigma\Gamma_{,\sigma}=0. \tag{2.4}$$

The remainder of this section will be devoted to a rewording of these equations. We begin with (2.3). Defining a function f by the equation

$$S=\frac{e}{c}(x^\tau\Phi_\tau+x^\tau f_{,\tau}-f), \tag{2.5}$$

and substituting this for S into (2.3) we obtain

$$x^\tau\left(p_\rho l_{\tau,\sigma}{}^\rho+\frac{e}{c}\Phi_{\tau,\sigma}+\frac{e}{c}f_{,\sigma\tau}\right)=0.$$

Since this equation must hold for arbitrary x^τ we must have

$$p_\rho l_\tau{}^\rho=-\frac{e}{c}(\Phi_\tau+f_\tau{}'),$$

where f' is a scalar function which may differ from f by a linear function of x. However, S will remain unaffected if a linear function of x is added to f in (2.5). The velocity vector associated with p_σ is defined by

$$V^\sigma=(1/mc)p^\sigma,\quad V^\sigma V_\sigma=1,$$

and the above equation may be written as

$$L_\tau{}^\sigma V^\tau=\chi^\sigma, \tag{2.6}$$

where

$$\chi_\sigma=-e/mc^2(\Phi_\sigma+f_{,\sigma}{}'). \tag{2.7}$$

Thus Eq. (2.3) is equivalent to (2.6) which means that the Lorentz matrices required must carry the unit vector V^σ into the vector χ^σ related to the external field vector potential by (2.7). Of course f' must be such that χ^σ is a unit vector.

The condition expressed by (2.4) is best dealt with after it has been written in tensor form. It follows from T.E. §1 that Γ may be expressed as

$$\Gamma=z1+\tfrac{1}{4}Z_{\sigma\tau}S^{\sigma\tau}. \tag{2.8}$$

Where z is a complex scalar, $Z_{\sigma\tau}$ is a self-dual anti-symmetric tensor, and the matrices $S^{\sigma\tau}$ are defined in terms of the g^σ by

$$S_{\sigma\tau}=\tfrac{1}{2}(\bar{g}_\tau g_\sigma-\bar{g}_\sigma g_\tau). \tag{2.9}$$

The coefficients of (2.8) are given by

$$z=\tfrac{1}{2}\,\mathrm{trace}\Gamma,\quad Z_{\sigma\tau}=-\tfrac{1}{2}\,\mathrm{trace}(\Gamma S_{\sigma\tau}).$$

The condition $\det\Gamma=1$ becomes

$$z^2+\tfrac{1}{4}Z_{\sigma\tau}Z^{\sigma\tau}=1. \qquad (2.10)$$

$L_\tau{}^\sigma$ may be expressed in terms of z and $Z^{\sigma\tau}$ as

$$L_\tau{}^\sigma=(\bar{z}\delta_\rho{}^\sigma+\bar{Z}_\rho{}^\sigma)(z\delta_\tau{}^\rho=Z_\tau{}^\rho)$$
$$=(z\delta_\rho{}^\sigma+Z_\rho{}^\sigma)(\bar{z}\delta_\tau{}^\rho+\bar{Z}_\tau{}^\rho), \qquad (2.11)$$

where the bar denotes the complex conjugate, as follows from Eq. (1.5) and the identities given in T.E.

Substituting (2.8) into (2.4), using Eq. (1.28) of T.E. and using the linear independence of the g^σ we obtain the equations

$$(z\delta_\tau{}^\sigma+Z_\tau{}^\sigma)_{,\sigma}=0, \qquad (2.12)$$

as the equivalent tensor form of (2.4), where z and $Z_{\sigma\tau}$ are related by (2.10).

It follows from Eqs. (2.11), (2.12) and (2.6) that we must have

$$\chi_{,\sigma}{}^{\sigma}=0. \qquad (2.13)$$

To prove this we take the divergence of (2.11) which reduces to

$$L_{\tau,\sigma}{}^\sigma=(\bar{z}\delta_\rho{}^\sigma+\bar{Z}_\rho{}^\sigma)(z\delta_\tau{}^\rho+Z_\tau{}^\rho)_{,\sigma}$$
$$=\bar{Z}^{\sigma\rho}(zg_{\rho\tau}+Z_{\rho\tau})_{,\sigma}.$$

However, since $Z^{\sigma\tau}$ is a self-dual tensor Eqs. (2.12) may be written as

$$Z_{\sigma\tau,\rho}+Z_{\tau\rho,\sigma}+Z_{\rho\sigma,\tau}=z_{,\nu}g^{\nu\lambda}(g)^{\frac{1}{2}}\epsilon_{\lambda\rho\sigma\tau}. \qquad (2.14)$$

Substituting from this equation for $Z_{\rho\tau,\sigma}$ we obtain

$$L_{\tau,\sigma}{}^\sigma=\tfrac{1}{2}\bar{Z}^{\sigma\rho}Z_{\rho\sigma,\tau}=\tfrac{1}{2}Z^{\sigma\rho}\bar{Z}_{\rho\sigma,\tau}.$$

The last equation is a consequence of the reality of $L_\tau{}^\sigma$. But for any self-dual tensor

$$\bar{Z}^{\sigma\rho}Z_{\rho\sigma}=0,$$

and hence

$$(\bar{Z}^{\sigma\rho}Z_{\sigma\rho})_{,\tau}=0.$$

Therefore we must have

$$L_{\tau,\sigma}{}^\sigma=0,$$

as a consequence of (2.12). Thus (2.13) is proved.

3. THE PLANE WAVE CASE

If the method described above is to be used to obtain exact solutions to the Dirac equations for a given Φ_σ, we must determine χ_σ and then find z and $Z_{\sigma\tau}$ such that Eqs. (2.10), (2.12) and (2.6) are satisfied for every time-like vector V^σ. It is sometimes impossible to find χ_σ. To see this we must recall that χ_σ is essentially Φ_σ plus the gradient of a scalar. This scalar must satisfy the wave equation if the divergence of Φ_σ vanishes and must be such that its sum with Φ_σ gives a time-like vector of constant length. In case Φ_σ describes a constant electric field, say $\Phi_\sigma=Ex^1\delta_\sigma{}^4$, it is impossible to fulfill these conditions. Thus the method described is a special one and gives exact solutions in very few cases. However, it can be used to obtain approximate solutions for in some problems it may be possible to obtain a χ_σ which satisfies the required conditions to a sufficient degree of approximation. If this is the case, the discussion in the subsequent sections may be used to determine the matrices $L_\tau{}^\sigma$ in terms of the approximate χ^σ.

We will now show that the method applies exactly if Φ_σ describes a plane wave. That is, we assume that Φ_σ is a function only of a variable $n_\tau x^\tau$ where n_τ is a null vector orthogonal to Φ_σ. Thus we assume

$$\Phi_\sigma=\Phi_\sigma(n_\tau x^\tau),\quad n^\sigma\Phi_\sigma=0,\quad n^\sigma n_\sigma=0. \qquad (3.1)$$

Now consider the spin-transformation

$$\Gamma=1+\frac{e}{2mc^2}n_\sigma\Phi_\tau S^{\sigma\tau}. \qquad (3.2)$$

The Lorentz transformation corresponding to this is a parabolic one given by

$$L_\tau{}^\sigma=\delta_\tau{}^\sigma+\frac{e}{mc^2(n^\rho V_\rho)}(n^\sigma\Phi_\tau-\Phi^\sigma n_\tau)$$
$$-\frac{e(\Phi_\lambda\Phi^\lambda)}{2mc^2(n^\rho V_\rho)}n^\sigma n_\tau. \qquad (3.3)$$

It may readily be verified that for Γ given by (3.2) we have

$$z=1\quad\text{and}\quad Z_{\sigma\tau}=\frac{e}{2mc^2}\eta_{\mu\nu}{}^{\sigma\tau}n^\mu\Phi^\nu, \qquad (3.4)$$

where

$$\eta_{\mu\nu}{}^{\sigma\tau}=\delta_\mu{}^\sigma\delta_\nu{}^\tau-\delta_\nu{}^\sigma\delta_\mu{}^\tau+\frac{1}{(g)^{\frac{1}{2}}}\epsilon^{\sigma\tau\xi\eta}g_{\xi\mu}g_{\eta\nu}, \qquad (3.5)$$

and these quantities satisfy (2.10) and (2.12).

If we compute χ^τ from (2.6) using (3.3)

$$\chi_\mu=-\frac{e}{mc^2}(\Phi_\mu+f_{,\mu}),$$

where

$$f_{,\mu}=-\frac{mc^2}{e}V_\mu-\frac{n_\mu}{n^\rho V_\rho}\left(\Phi_\sigma V^\sigma-\frac{e}{2mc^2}\Phi_\sigma\Phi^\sigma\right). \qquad (3.6)$$

Since the term in parentheses in the last equation is a function of $n_\sigma x^\sigma$, $f_{,\mu}$ is the gradient of a scalar. Thus in this case we may obtain exact solutions of the Dirac equation by transforming plane wave solutions. The solutions are

$$\psi(x)=\Gamma\exp[-(ie/\hbar c)f]\psi_1(p),$$
$$\varphi(x)=\Gamma\exp[(ie/\hbar c)f]\varphi_1(p). \qquad (3.7)$$

Where f is defined by (3.6), $\psi_1(p)$ and $\varphi_1(p)$ are defined by (1.8) and Γ is given by (3.2). The solutions (3.7) were derived by Volkow[2] and Singupta[3] by another method.

4. THE EQUATIONS (2.6)

These conditions determine $L_\tau{}^\sigma$ as functions of V^σ and χ^σ and three functions of coordinates as follows from the fact that for a given set of values for the coordinates these four equations are linear equations for the $L_\tau{}^\sigma$. However, these are only three independent equations in (2.6) since $L_\tau{}^\sigma$ must be a Lorentz matrix and hence it is a consequence of (2.6) that

$$\chi^\sigma\chi_\sigma = V^\sigma V_\sigma = 1. \quad (4.1)$$

We next express $L_\tau{}^\sigma$ as a function of V^σ, x^σ and three arbitrary functions. The Lorentz transformation

$$M_\rho{}^\sigma = \delta_\rho{}^\sigma - (1+\alpha)^{-1}(V^\sigma V_\rho + \chi^\sigma\chi_\rho + V^\sigma V_\rho - (1+2\alpha)V^\sigma x_\rho), \quad (4.2)$$

where

$$\alpha = V^\sigma\chi_\sigma$$

has the property that

$$M_\rho{}^\sigma\chi^\rho = V^\sigma.$$

Hence

$$N_\tau{}^\sigma V^\tau = M_\rho{}^\sigma L_\tau{}^\rho V^\tau = V^\sigma. \quad (4.3)$$

Thus the matrix $N_\tau{}^\sigma$ is a Lorentz transformation which leaves the vector V^σ invariant. The most general form for $N_\tau{}^\sigma$ is

$$N_\tau{}^\sigma = \cos\theta\delta_\rho{}^\sigma - \frac{i\sin\theta}{(g)^{\frac{1}{2}}}\epsilon^{\sigma\tau\mu\nu}Y_\mu V_\nu + (1-\cos\theta)(V^\sigma V_\rho - Y^\sigma Y_\rho), \quad (4.4)$$

where θ is arbitrary and Y^σ is an arbitrary space-like vector satisfying

$$Y^\sigma V_\sigma = 0, \quad Y^\sigma Y_\sigma = -1. \quad (4.5)$$

Therefore the matrix L is given by the matrix equation

$$L = M^{-1}N,$$

where N is given by (4.3) and M^{-1} is the inverse of the Lorentz matrix given by (4.2). M^{-1} is obtained from (4.2) by interchanging V^σ and x^σ.

The spin image of M^{-1} is

$$\mu = a1 - \frac{1}{2a}V_\mu\chi_\nu S^{\mu\nu}, \quad (4.6)$$

and of N is

$$\text{N} = \cos\theta/2 + i\sin\theta/2V_\mu Y_\nu S^{\mu\nu}. \quad (4.7)$$

Hence the spin image of L is

$$\Gamma = \mu\nu = (a\cos\theta/2 + i((1/2a)\sin\theta/2\chi_\nu Y^\nu) + (1/2a)(V_\sigma + \chi_\sigma)(V_\tau\cos\theta/2 + i\sin\theta/2Y_\tau)S^{\sigma\tau}. \quad (4.8)$$

It may be readily verified that Eqs. (4.8) reduce to (3.2) if we take θ and Y_ν such that

$$a\cos\theta/2 = 1, \quad \chi_\nu Y^\nu = 0, \quad (4.9)$$

and further write

$$Y^\tau = \frac{1}{[2(\alpha-1)]^{\frac{1}{2}}}\frac{e}{mc^2(n^\rho V_\rho)}\epsilon^{\tau\sigma\mu\nu}\Phi_\sigma V^\mu n_\nu. \quad (4.10)$$

In general we may write

$$z = c + iW^\nu\eta_\nu = W^\mu T_\mu, \quad (4.11)$$

$$Z^{\sigma\tau} = \eta_{\mu\nu}{}^{\sigma\tau}(cV^\nu + i\eta^\nu)W^\mu = \eta_{\mu\nu}{}^{\sigma\tau}W^\mu T^\nu, \quad (4.12)$$

where

$$W^\mu = \frac{\chi^\mu + V^\mu}{2}, \qquad \eta^\mu = a\sin\theta/2Y^\mu,$$
$$T^\mu = \frac{1}{a^2}(cV^\mu + i\eta^\mu), \qquad c = a\cos\theta/2. \quad (4.13)$$

The algebraic conditions (2.10) are satisfied by (4.11) and (4.12). We have as a consequence of (4.13) that

$$\eta^\nu\eta_\nu = c^2 - a^2, \quad \eta^\nu V_\nu = 0,$$
$$V^\nu W_\nu = a^2 = W^\mu W_\mu = (1/T^\mu T_\mu). \quad (4.14)$$

The three real quantities to be determined, namely θ and the two independent components of Y^σ, must be chosen so that the eight real first order partial differential Eqs. (2.12) are satisfied. These are differential equations for θ and Y^ν in which the vectors χ_σ and V_σ enter, and which must be satisfied for all V_σ. However, since these are tensor equations we may always choose the coordinate system so that V_σ has the value $\delta_\sigma{}^4$ if we so desire. Solutions for other values of V_σ may then be obtained by making a Lorentz transformation.

5. THE EQUATIONS (2.12)

It is convenient to work with the complex vector T^μ defined in terms of θ and Y^μ by Eqs. (4.13). We then have the four complex equations

$$(W^\mu T_\mu)_{,\sigma}g^{\sigma\tau} + \eta_{\mu\nu}{}^{\sigma\tau}(W^\mu T^\nu)_{,\sigma} = 0, \quad (5.1)$$

to determine the vector T^μ. However, there are only three independent components of T^μ as follows from (4.14). Equation (5.1) may be written as

$$W^\mu T_{\rho,\sigma}(\delta_\mu{}^\rho g^{\sigma\tau} + \eta_{\mu\nu}{}^{\sigma\tau}g^{\rho\nu}) = \bar{F}_\nu{}^\tau T^\nu, \quad (5.2)$$

where the bar denotes the complex conjugate and $F_\nu{}^\tau$ is the self-dual tensor describing the external field,

$$F_{\mu\nu} = \eta_{\mu\nu}{}^{\sigma\tau}W_{\sigma,\tau} = -\frac{e}{2mc^2}\eta_{\mu\nu}{}^{\sigma\tau}\Phi_{\sigma,\tau}. \quad (5.3)$$

In deriving (5.2), we made use of (2.13) and (3.5).

Equations (5.2) may be written as

$$T_{\rho,\sigma}Z^{\rho\sigma\tau} = \bar{F}_\nu{}^\tau T^\nu, \quad (5.4)$$

where

$$Z^{\rho\sigma\tau} = W^\rho g^{\sigma\tau} + \eta_{\mu\nu}{}^{\sigma\tau}W^\mu g^{\nu\rho}. \quad (5.5)$$

For each value of ρ, $Z^{\rho\sigma\tau}$ is of the form of a multiple of the metric tensor plus a self-dual tensor, that is of the form

$$Wg^{\sigma\tau}+W^{\sigma\tau}.$$

However, if such a tensor is multiplied by

$$Wg_{\tau\lambda}-W_{\tau\lambda},$$

and summed on τ, we obtain

$$(W^2+\tfrac{1}{4}W^{\tau\mu}W_{\tau\mu})\delta_\lambda{}^\sigma.$$

This means that we may solve Eqs. (5.4) for $T_{\kappa,\lambda}$ as functions of T_ν and $T_{\rho,\lambda}$ $(\rho\neq\kappa)$. The discussion of the existence of solutions of the resulting system of equations reduces to a discussion of the integrability conditions.

To solve Eqs. (5.4) for $T_{\kappa,\lambda}$ for one value of κ, we must multiply Eqs. (5.4) by

$$W^\epsilon(g_{\tau\lambda}\delta_\epsilon{}^\kappa-g_{\alpha\tau}g_{\beta\lambda}\eta_{\epsilon\delta}{}^{\alpha\beta}g^{\delta\kappa}),$$

and sum on τ. We obtain for the right hand side,

$$\begin{aligned}W^\epsilon(g_{\tau\lambda}\delta_\epsilon{}^\kappa-g_{\alpha\tau}g_{\beta\lambda}\eta_{\epsilon\delta}{}^{\alpha\beta}g^{\delta\kappa})\bar{F}_\nu{}^\tau T^\nu\\ =H_{\kappa\lambda}=W^\kappa\bar{F}_{\lambda\nu}T^\nu+T^\kappa\bar{F}_{\lambda\nu}W^\nu+W^\nu T_\nu\bar{F}_{\kappa\lambda}\\ +W_\lambda\bar{F}_\nu{}^\kappa T^\nu-T_\lambda\bar{F}_\nu{}^\kappa W^\nu-\delta_\lambda{}^\kappa\bar{F}_{\sigma\nu}W^\sigma T^\nu.\end{aligned}\quad(5.6)$$

The left-hand side involves

$$\begin{aligned}W^\mu(g^{\sigma\tau}\delta_\mu{}^\rho+\eta_{\mu\nu}{}^{\sigma\tau}g^{\rho\nu})(g_{\tau\lambda}\delta_\epsilon{}^\kappa-g_{\alpha\tau}g_{\beta\lambda}\eta_{\epsilon\delta}{}^{\alpha\beta}g^{\delta\kappa})W^\epsilon\\ =a^2\delta_\lambda{}^\sigma g^{\kappa\rho}+\delta_\lambda{}^\rho(2W^\sigma W^\kappa-g^{\sigma\kappa}a^2)-\delta_\lambda{}^\kappa(2W^\sigma W^\rho-g^{\sigma\rho}a^2)\\ +2(g^{\sigma\kappa}W^\rho W_\lambda-g^{\sigma\rho}W^\kappa W_\lambda).\end{aligned}\quad(5.7)$$

The differential Eq. (5.4) may then be written as

$$\begin{aligned}T_{\rho,\sigma}[a^2\delta_\lambda{}^\sigma g^{\kappa\rho}+\delta_\lambda{}^\rho(2W^\sigma W^\kappa-g^{\sigma\kappa}a^2)-\delta_\lambda{}^\kappa(2W^\sigma W^\rho\\ -g^{\sigma\rho}a^2)+2(g^{\sigma\kappa}W^\rho W_\lambda-g^{\sigma\rho}W^\kappa W_\lambda)]=H_{\kappa\lambda}.\end{aligned}\quad(5.8)$$

Multiplying this by $g^{\kappa\lambda}$ and summing, we obtain

$$T_{\rho,\sigma}(g^{\rho\sigma}a^2-2W^\rho W^\sigma)=-2\bar{F}_{\rho\sigma}W^\rho T^\sigma.\quad(5.9)$$

Thus (5.8) becomes

$$\begin{aligned}a^2(T_{\kappa,\lambda}-T_{\lambda,\kappa})+2(T_{\lambda,\sigma}W^\sigma W_\kappa+T_{\sigma,\kappa}W^\sigma W_\lambda\\ -T_{,\rho}{}^\rho W_\kappa W_\lambda)=G_{\kappa\lambda},\end{aligned}\quad(5.10)$$

where

$$\begin{aligned}G_{\kappa\lambda}=W_\kappa\bar{F}_{\lambda\nu}T^\nu+T_\kappa\bar{F}_{\lambda\nu}W^\nu+W^\nu T_\nu\bar{F}_{\kappa\lambda}+W_\lambda\bar{F}_{\kappa\nu}T_\nu\\ -T_\lambda\bar{F}_{\kappa\nu}W^\nu+g_{\kappa\lambda}\bar{F}_{\sigma\nu}W^\sigma T^\nu.\end{aligned}\quad(5.11)$$

Setting $\kappa=4$ in Eqs. (5.10), and assuming that the coordinate system is a galilean one in which $V^\sigma=\delta_4{}^\sigma$, we obtain the four equations

$$\begin{aligned}T_{4,i}&=-T_{i,4}-2(T_{i,j}W^j-T_{,j}{}^jW_i)+\frac{G_{4i}}{W_4},\\ T_{4,4}&=2\frac{T_{i,j}W^iW^j}{W_4}-2(1-W_4)T_{,j}{}^j+\frac{G_{44}-2G_{4i}W^i}{2(W_4)^2},\end{aligned}\quad(5.12)$$

where G_{4i} and G_{44} are obtained from (5.11). The existence of solutions of these equations depends on the nature of the functions χ_σ and their derivatives.

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Forms of Relativistic Dynamics

P. A. M. DIRAC
St. John's College, Cambridge, England

For the purposes of atomic theory it is necessary to combine the restricted principle of relativity with the Hamiltonian formulation of dynamics. This combination leads to the appearance of ten fundamental quantities for each dynamical system, namely the total energy, the total momentum and the 6-vector which has three components equal to the total angular momentum. The usual form of dynamics expresses everything in terms of dynamical variables at one instant of time, which results in specially simple expressions for six or these ten, namely the components of momentum and of angular momentum. There are other forms for relativistic dynamics in which others of the ten are specially simple, corresponding to various sub-groups of the inhomogeneous Lorentz group. These forms are investigated and applied to a system of particles in interaction and to the electromagnetic field.

1. INTRODUCTION

EINSTEIN'S great achievement, the principle of relativity, imposes conditions which all physical laws have to satisfy. It profoundly influences the whole of physical science, from cosmology, which deals with the very large, to the study of the atom, which deals with the very small. General relativity requires that physical laws, expressed in terms of a system of curvilinear coordinates in space-time, shall be invariant under all transformations of the coordinates. It brings gravitational fields automatically into physical theory and describes correctly the influence of these fields on physical phenomena.

Gravitational fields are specially important when one is dealing with large-scale phenomena, as in cosmology, but are quite negligible at the other extreme, the study