Review of a Generalized Electrodynamics

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This paper presents a review of five papers in which a generalized electrodynamics has been developed. The purpose of the review is to present the results obtained so far, leaving out duplications, false starts, and detailed calculations. The emphasis is on the sequence of ideas, the difficulties encountered, and the methods of procedure.

1. INTRODUCTION

In a series of papers published in the last five years[1–5] a field theory has been developed which, in spite of the simplicity of its fundamental assumptions, leads to results free of infinities usually associated with a point source. In view of a promising recent extension of the theory to meson fields[6] it seems worth while to review the results obtained so far, leaving out duplications, false starts, and sometimes confusing notation inherent in the process of active development by several workers.

In generalizing the equations of electrodynamics the idea was to leave the usual assumptions of the Maxwell-Lorentz theory as nearly unaltered as possible. Thus we continue to assume that the field equations and the equations of motion of the particles are derivable from a variation principle

$$\delta\int \bar{L}dt=0, \tag{1.1}$$

in which $\bar{L}$ is the sum of the Lagrangians of the field, of the particles, and of their interactions.

The Lagrangians of the particles and of the interactions are left unchanged. This at once results in preserving the Lorentz equations of motion for particles. Further, we assume, as usual, that the field equations are to be linear in the field quantities.

The Lagrangian of the field, however, is generalized by permitting dependence upon first derivatives of the field quantities **E** and **H**. This is the only new assumption used.

[1] B. Podolsky, Phys. Rev. **62**, 68 (1942).
[2] B. Podolsky and C. Kikuchi, Phys. Rev. **65**, 228 (1944).
[3] B. Podolsky and C. Kikuchi, Phys. Rev. **67**, 184 (1945).
[4] D. J. Montgomery, Phys. Rev. **69**, 117 (1946).
[5] Alex E. S. Green, Phys. Rev. **72**, 628 (1947).
[6] Alex E. S. Green, Phys. Rev. **73**, 26 (1948).

2. CLASSICAL EQUATIONS

Using $x_4=ict$ and $-ds^2=dx_\alpha{}^2$, with the usual summation convention, we need not distinguish between covariant and contravarient tensors. Letting

$$F_{\alpha\beta}=-F_{\beta\alpha}=(\partial A_\beta/\partial x_\alpha)-(\partial A_\alpha/\partial x_\beta) \tag{2.1}$$

with

$$A_\alpha=(\mathbf{A},\,i\varphi), \tag{2.2}$$

where A is the vector potential and φ the scalar potential, we have: $F_{12}=H_3$, $F_{23}=H_1$, $\cdots F_{14}=-iE_1$, $\cdots$etc.; then, as usual,

$$\mathbf{E}=-\nabla\varphi-(1/c)(\partial\mathbf{A}/\partial t)\quad\text{and}\quad\mathbf{H}=\nabla\times\mathbf{A}. \tag{2.3}$$

One set of the field equations is then

$$\partial F_{\alpha\beta}/\partial x_\gamma+\partial F_{\beta\gamma}/\partial x_\alpha+\partial F_{\gamma\alpha}/\partial x_\beta=0, \tag{2.4}$$

or

$$\nabla\times\mathbf{E}+(1/c)\partial\mathbf{H}/\partial t=0\quad\text{and}\quad\nabla\cdot\mathbf{H}=0, \tag{2.5}$$

which follow at once from Eq. (2.1). These are, of course, one pair of the Maxwell-Lorentz equations.

Equations of motion of a particle are, in the usual way,

$$(d^2x_\alpha/ds^2)=(e/mc^2)F_{\alpha\beta}(dx_\beta/ds), \tag{2.6}$$

or

$$\frac{d}{dt}\left[\frac{m\mathbf{v}}{(1-v^2/c^2)^{\frac{1}{2}}}\right]=e\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{H}\right). \tag{2.7}$$

The Lagrangian of the field is:

$$\bar{L}_f=\int L_f dV$$

where, in Heaviside's units,

$$L_f=-\frac{1}{2}\left[\frac{1}{2}F_{\alpha\beta}{}^2+a^2\left(\frac{\partial F_{\alpha\beta}}{\partial x_\beta}\right)^2\right] \tag{2.8}$$

$$=\tfrac{1}{2}(\mathbf{E}^2-\mathbf{H}^2)+\frac{1}{2}a^2\left[(\nabla\cdot\mathbf{E})^2-\left(\nabla\times\mathbf{H}-\frac{\partial\mathbf{E}/\partial t}{c}\right)^2\right] \tag{2.9}$$

and a is some constant of nature having the dimension of length. The resulting field equations are:

$$\left(1-a^2\frac{\partial^2}{\partial x_\alpha{}^2}\right)\frac{\partial F_{\beta\gamma}}{\partial x_\gamma}=j_\beta, \quad (2.10)$$

or

$$(1-a^2\Box)\nabla\cdot\mathbf{E}=\rho \quad (2.11)$$

and

$$(1-a^2\Box)\left(\nabla\times\mathbf{H}-\frac{\partial\mathbf{E}/\partial t}{c}\right)=\rho\mathbf{v}/c, \quad (2.12)$$

where the four-vector j_α is defined by

$$j_\alpha=(\rho\mathbf{v}/c,\ i\rho), \quad (2.13)$$

with

$$\rho(\mathbf{r})=\sum_s e_s\delta(\mathbf{r}-\mathbf{r}_s). \quad (2.14)$$

The summation here is over all the particles; the sth particle having the charge e_s and the position $\mathbf{r}_s=\mathbf{r}_s(t)$.

Equations (2.11) and (2.12) are a generalization of the second pair of Maxwell-Lorentz equations. In the classical theory, in which $\partial F_{\alpha\beta}/\partial x_\beta=j_\alpha$, one substitutes from (2.1) obtaining:

$$(\partial^2A_\beta/\partial x_\alpha\partial x_\beta)-(\partial^2A_\alpha/\partial x_\beta{}^2)=j_\alpha;$$

then, if

$$(\partial/\partial x_\alpha)(\partial A_\beta/\partial x_\beta)=0,$$

one obtains the wave equation

$$\partial^2A_\alpha/\partial x_\beta{}^2=-j_\alpha.$$

The usual Lorentz condition $\partial A_\beta/\partial x_\beta=0$ is thus a sufficient but not a necessary condition for the wave equation to hold; the necessary condition being merely $\partial A_\beta/\partial x_\beta=\text{constant}$.

Similarly, in our theory the same procedure leads to the necessary condition

$$(1-a^2\Box)(\partial A_\beta/\partial x_\beta)=\text{constant}, \quad (2.15)$$

but we choose the more restrictive Lorentz condition

$$\partial A_\alpha/\partial x_\alpha=0. \quad (2.16)$$

The remaining arbitrariness in A_α is then given by the usual gage transformation

$$A_\alpha'=A_\alpha+\partial X/\partial x_\alpha \quad (2.17)$$

with X being any solution of the wave equation

$$\Box X=0. \quad (2.18)$$

The gage transformation (2.17) will, of course, leave $F_{\alpha\beta}$ unaltered. We then obtain:

$$(1-a^2\Box)\Box A_\alpha=-j_\alpha, \quad (2.19)$$

a set of fourth-order partial differential equations.

In the static case the vector potential can be set equal to zero, and the scalar potential due to a point charge turns out to be, in electrostatic units,

$$\varphi=(e/r)(1-e^{-r/a}), \quad (2.20)$$

which approaches a finite value e/a as r approaches zero. This result was also obtained in a different way by Landé and Thomas.[7]

3. CANONICAL EQUATIONS FOR THE FIELD

Before quantizing the field equations by the method of Heisenberg and Pauli[8] we must be able to write the field equations in Hamiltonian canonical form.

Suppose the Lagrangian density L_f is a function of the potentials $A_\alpha=(\mathbf{A},\ i\varphi)$ as well as their first and second derivatives:[9]

$$L_f=L_f(A_\alpha,\ A_{\alpha,\beta},\ A_{\alpha,\beta\gamma}),$$

where A_α are functions of the space-time coordinates x_α, and

$$A_{\alpha,\beta}=\partial A_\alpha/\partial x_\beta,\quad A_{\alpha,\beta\gamma}=\partial^2A_\alpha/\partial x_\beta\partial x_\gamma.$$

The variational equation,

$$\delta W_f=\delta\int\int L_f dVdt=0,\quad dV=dx_1dx_2dx_3$$

or

$$ic\delta W_f=\delta\int L_f d\Omega=0,\quad d\Omega=dVdx_4, \quad (3.1)$$

in the absence of particles leads to the field equation,

$$\frac{\partial L_f}{\partial A_\alpha}-\frac{\partial}{\partial x_\beta}\frac{\partial L_f}{\partial A_{\alpha,\beta}}+\frac{\partial^2}{\partial x_\beta\partial x_\gamma}\frac{\partial L_f}{\partial A_{\alpha,\beta\gamma}}=0, \quad (3.2)$$

provided A_α and $A_{\alpha,\beta}$ are specified and are unvaried over the boundaries of the four-dimensional manifold Ω over which the integration is performed.

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[7] A. Landé and H. Thomas, Phys. Rev. **60**, 514 (1941).

[8] W. Heisenberg and W. Pauli, Zeits. f. Physik **56**, 1 (1929).

[9] Greek indices will range from 1 to 4, while Latin subscripts range from 1 to 3. Repeated indices are summed.

We introduce as the new generalized coordinates[2]

$$q_\alpha = A_\alpha \quad \text{and} \quad Q_\alpha = \partial A_\alpha/\partial t = \dot{q}_\alpha, \tag{3.3}$$

and define the momenta conjugate to q_α and Q_α by

$$p_\alpha = (\partial L_f/\partial \dot{q}_\alpha) - \partial/\partial t(\partial L_f/\partial \ddot{q}_\alpha) - \partial/\partial x_j(\partial L_f/\partial \dot{q}_{\alpha,j}) \tag{3.4}$$

and

$$P_\alpha = \partial L/\partial \ddot{q}_\alpha, \tag{3.5}$$

respectively. In performing differentiation with respect to $\dot{q}_{\alpha,j}$ it is necessary to realize that both $icA_{\alpha,j4}$ and $icA_{\alpha,4j}$ are its equivalent forms, so that Eq. (3.4) can be written more fully thus:

$$p_\alpha = \frac{1}{ic}\left[\frac{\partial L_f}{\partial A_{\alpha,4}} - \frac{\partial}{\partial x_4}\frac{\partial L_f}{\partial A_{\alpha,44}} - \frac{\partial}{\partial x_j}\left(\frac{\partial L_f}{\partial A_{\alpha,4j}} + \frac{\partial L_f}{\partial A_{\alpha,j4}}\right)\right].$$

The Hamiltonian density is now defined by

$$H_f = -L_f + p_\alpha \dot{q}_\alpha + P_\alpha \dot{Q}_\alpha. \tag{3.6}$$

The time derivatives of the coordinates, $\dot{q}_\alpha$ and $\dot{Q}_\alpha$, can, in general, be eliminated by using Eqs. (3.3) and (3.5). The result is

$$H_f = H_f(q_\alpha, p_\alpha, q_{\alpha,i}, q_{\alpha,ij}, Q_\alpha, P_\alpha, Q_{\alpha,i}). \tag{3.7}$$

If now we define the Hamiltonian

$$\bar{H}_f = \int H_f dv,$$

then it can be shown[2] that

$$\dot{p}_\alpha = -\delta\bar{H}_f/\delta q_\alpha, \quad \dot{P}_\alpha = -\delta\bar{H}_f/\delta Q_\alpha, \tag{3.8}$$

$$\dot{q}_\alpha = \delta\bar{H}_f/\delta p_\alpha, \quad \dot{Q}_\alpha = \delta\bar{H}_f/\delta P_\alpha. \tag{3.9}$$

We note that

$$d\bar{H}_f/dt = \int \{(\delta\bar{H}_f/\delta q_\alpha)\dot{q}_\alpha + (\delta\bar{H}_f/\delta p_\alpha)\dot{p}_\alpha + (\delta\bar{H}_f/\delta Q_\alpha)\dot{Q}_\alpha + (\delta\bar{H}_f/\delta P_\alpha)\dot{P}_\alpha\} dV = 0. \tag{3.10}$$

Equations (3.8) and (3.9) are the desired canonical equations of the field, and definitions (3.4) and (3.5) are thereby justified. Equation (3.10) is, of course, the law of conservation of energy of the field, in the absence of particles.

The total momentum of the field, $\mathbf{P}_\alpha$ can be calculated in the usual way[2] and we obtain

$$\mathbf{P}_\alpha = \int t_{\alpha\beta} dS_\beta, \tag{3.11}$$

where the energy-momentum tensor $t_{\alpha\beta}$ is given by

$$ict_{\alpha\beta} = L_f \delta_{\alpha\beta} - A_{\mu,\alpha} p_{\mu\beta} - A_{\mu,\alpha\lambda} P_{\mu\lambda\beta} \tag{3.12}$$

with

$$p_{\alpha\beta} = \partial L_f/\partial A_{\alpha,\beta} - \partial/\partial x_\mu(\partial L_f/\partial A_{\alpha,\beta\mu}) \tag{3.13}$$

and

$$P_{\alpha\beta\gamma} = \partial L_f/\partial A_{\alpha,\beta\gamma}. \tag{3.14}$$

The tensor $t_{\alpha\beta}$ is not symmetric, but satisfies the equation of conservation of energy and momentum, namely:

$$t_{\alpha\beta,\beta} = 0. \tag{3.15}$$

However, while the theory makes $\mathbf{P}_\alpha$ definite, $t_{\alpha\beta}$ can be changed by addition of any $t_{\alpha\beta}'$ which gives a vanishing contribution to the integral in (3.11) and satisfies the equation

$$t_{\alpha\beta,\beta}' = 0.$$

This fact is made use of in "symmetrizing" $t_{\alpha\beta}$; that is, replacing it by $T_{\alpha\beta} = T_{\beta\alpha} = t_{\alpha\beta} + t_{\alpha\beta}'$ with suitably chosen $t_{\alpha\beta}'$.

Applying the theory of this section to the function L_f given in Eq. (2.8), we obtain:

$$ict_{\alpha\beta} = L_f\delta_{\alpha\beta} - A_{\mu,\alpha}(1 - a^2\Box)F_{\mu\beta} + a^2 F_{\mu\beta,\alpha} F_{\mu\nu,\nu},$$

or upon symmetrization,

$$\begin{aligned} icT_{\mu\nu} = F_{\mu\alpha}F_{\nu\alpha} &- (1/4)F_{\alpha\beta}F_{\alpha\beta}\delta_{\mu\nu} \\ &+ (a^2/2)[F_{\alpha\beta}\Box F_{\alpha\beta} + (\partial F_{\alpha\beta}/\partial x_\beta) \\ &\times(\partial F_{\alpha\gamma}/\partial x_\gamma)]\delta_{\mu\nu} \\ &- a^2[F_{\mu\alpha}\Box F_{\nu\alpha} + F_{\nu\alpha}\Box F_{\mu\alpha} \\ &+ (\partial F_{\mu\alpha}/\partial x_\alpha)(\partial F_{\nu\beta}/\partial x_\beta)]. \end{aligned} \tag{3.16}$$

In electrostatics this gives for the energy

$$E = \tfrac{1}{2}\int \{\mathbf{E}^2 - a^2[(\nabla\cdot\mathbf{E})^2 + 2\mathbf{E}\cdot\nabla^2\mathbf{E}]\} dV. \tag{3.17}$$

Making use of $\nabla\times\mathbf{E} = 0$, and assuming that $\mathbf{E}\nabla\cdot\mathbf{E}$ vanishes at infinity faster than $1/r^2$, one finds that Eq. (3.17) can easily be put in the form

$$E=\tfrac{1}{2}\int\{\mathbf{E}^2+a^2(\nabla\cdot\mathbf{E})^2\}dV, \qquad (3.18)$$

which is obviously positive. For the fields of a point charge given by Eq. (2.20) this energy turns out to be $e^2/2a$, in electrostatic units.

Returning to the general case, and using Eqs. (3.4), (3.5), and (2.8), we find

$$p_4=\frac{a^2}{ic}F_{j4,\,j4} \qquad (3.19)$$

and

$$p_j=\frac{1}{ic}(1-a^2\Box)F_{j4}; \qquad (3.20)$$

having made use of the symmetry properties of $F_{\alpha\beta}$, namely:

$$F_{\alpha\beta}=-F_{\beta\alpha} \quad \text{and} \quad F_{\alpha\beta,\,\gamma}+F_{\beta\gamma,\,\alpha}+F_{\gamma\alpha,\,\beta}=0.$$

Similarly,

$$P_4=0, \quad P_j=-(a^2/c^2)F_{j\beta,\,\beta}. \qquad (3.21)$$

We note that p_4 no longer vanishes, as it does in the usual electrodynamics, but because of the vanishing P_4, we shall encounter the usual difficulty in quantization.

4. QUANTIZATION PRELIMINARIES

To avoid the quantization difficulty just mentioned we use the device previously used by Fermi[10] and by Fock and Podolsky.[11] We use, instead of Eq. (2.8), the Lagrangian density

$$L_f=-\tfrac{1}{2}(A_{\alpha,\,\beta}A_{\alpha,\,\beta}+a^2A_{\alpha,\,\beta\beta}A_{\alpha,\,\gamma\gamma}). \qquad (4.1)$$

To see the meaning of the change, we can put this in the form

$$L_f=-\tfrac{1}{2}(\tfrac{1}{2}F_{\alpha\beta}{}^2+a^2F_{\alpha\beta,\,\beta}{}^2) -\tfrac{1}{2}A_{\alpha,\,\alpha}(1-a^2\Box)A_{\beta,\,\beta}+V_{\beta,\,\beta}, \qquad (4.2)$$

where

$$V_\beta=\tfrac{1}{2}(A_{\alpha,\,\alpha}A_\beta-A_\alpha A_{\beta,\,\alpha}) +(a^2/2)A_{\alpha,\,\alpha}(A_{\gamma,\,\gamma\beta}-2A_{\beta,\,\gamma\gamma}).$$

The first term in (4.2) is the old L_f, the last term is a four divergence and therefore does not affect the resulting field equations, and the middle term vanishes when the auxiliary condition (2.16) is taken into account. Thus with condition (2.16) this L_f is equivalent to the old. The term $V_{\beta,\,\beta}$ is chosen to simplify L_f as much as possible so that it removes the third derivative contained in the middle term.

The Lagrangian density (4.1) has two advantages. In the first place the field equation obtained by its use is immediately Eq. (2.19). In the second place it leads to

$$c^2p_\alpha=(1-a^2\Box)\dot{A}_\alpha \qquad (4.3)$$

and

$$c^2P_\alpha=a^2\Box A_\alpha, \qquad (4.4)$$

None of these vanish identically. The Hamiltonian density is now

$$H_f=-L_f-A_{\alpha,\,4}(1-a^2\Box)A_{\alpha,\,4} -a^2A_{\alpha,\,44}\Box A_\alpha. \qquad (4.5)$$

As another preliminary to quantization it is convenient to introduce a generalization of the Fourier development for field quantities. When no particles are present, any field quantity $F(\mathbf{r},t)$ satisfies the generalized wave equation

$$(1-a^2\Box)\Box F=0, \qquad (4.6)$$

which can be seen from Eq. (2.19), since all field quantities are obtainable by linear operations on A_α. A general solution of Eq. (4.6) is

$$F=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int\{F(\mathbf{k})\exp(ik_\alpha x_\alpha) +F^*(\mathbf{k})\exp(-ik_\alpha x_\alpha) +\bar{F}(\mathbf{k})\exp(i\bar{k}_\alpha x_\alpha) +\bar{F}^*(\mathbf{k})\exp(-i\bar{k}_\alpha x_\alpha)\}\mathbf{dk}, \qquad (4.7)$$

where

$$k_\alpha=(\mathbf{k},\,ik), \quad \bar{k}_\alpha=(\mathbf{k},\,i\bar{k}),$$

$$\bar{k}=(1+a^2k^2)^{\frac{1}{2}}/a, \quad k=|\mathbf{k}|,$$

and

$$\mathbf{dk}=dk_xdk_ydk_z.$$

This is a very convenient form, as it enables us to use generalized Fourier amplitudes $F(\mathbf{k})$, without having to introduce fictitious "boxes with periodic boundary conditions" or any other artificial devices.

The sum of the first two terms in Eq. (4.7)

[10] E. Fermi, Rev. Mod. Phys. **4**, 87 (1932).

[11] V. Fock and B. Podolsky, Physik. Zeits. Sowjetunion **1**, 801 (1932).

satisfies the usual wave equation and may be called the *ordinary* field; the other two terms then give the *extraordinary* field.

In classical theory, where all Fourier amplitudes commute, $F^*(\mathbf{k})$ is the complex conjugate of $F(\mathbf{k})$ in case $F(\mathbf{r}, t)$ is real and the negative of the complex conjugate of it when $F(\mathbf{r}, t)$ is imaginary. Thus, for example, since $A_4 = i\varphi$ is imaginary, $\mathbf{A}_4^*(\mathbf{k})$ is the negative of the complex conjugate of $A_4(\mathbf{k}) = i\varphi(\mathbf{k})$; thus

$$A_4^*(\mathbf{k}) = i\varphi^*(\mathbf{k}). \tag{4.8}$$

Quantum mechanically, $F(\mathbf{k})$ and $F^*(\mathbf{k})$ are generally non-commuting quantities.

Let F_1 and F_2 be the ordinary and extraordinary parts of $F = F(\mathbf{r}, t)$, respectively. Then one can easily verify the following relations:

$$a^2\Box F = F_2, \quad (1-a^2\Box)F = F_1; \tag{4.9}$$

$$\Box F_1 = 0, \quad (1-a^2\Box)F_2 = 0. \tag{4.10}$$

When working with $F(\mathbf{k})$, $F^*(\mathbf{k})$, etc., we shall speak of working in $\mathbf{k}$ space; $\mathbf{k}$ is, of course, the usual wave vector. The Hamiltonian of the field in $\mathbf{k}$ space can be computed, using Eq. (4.5), and turns out to be

$$\begin{aligned}\bar{H}_f &= \int H_f dV \\ &= 2\int \{k^2 A_\alpha^*(\mathbf{k}) A_\alpha(\mathbf{k}) \\ &\qquad - \bar{k}^2 \bar{A}_\alpha^*(\mathbf{k}) \bar{A}_\alpha(\mathbf{k})\} d\mathbf{k}. \end{aligned} \tag{4.11}$$

We could now pass from the classical to the quantum equations by using the usual method of Heisenberg and Pauli.[2] Accordingly, we would have

$$\begin{aligned}[p_\alpha(\mathbf{r}, x_4), q_\beta(\mathbf{r}', x_4)] &= -\hbar i \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'), \\ [P_\alpha(\mathbf{r}, x_4), Q_\beta(\mathbf{r}', x_4)] &= -\hbar i \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}');\end{aligned} \tag{4.12}$$

with all other pairs commuting, where

$$[A, B] \equiv AB - BA.$$

Here, however, we run into the second usual difficulty of quantum electrodynamics. It turns out that the Lorentz auxiliary condition, $\partial A_\alpha/\partial x_\alpha = 0$, is inconsistent with the commutation rules (4.6). This well-known difficulty is usually resolved with the help of a suggestion made by Fermi.[10] Instead of the condition $\partial A_\alpha/\partial x_\alpha = 0$ on A_α, one uses the condition

$$(\partial A_\alpha/\partial x_\alpha)\psi = 0 \tag{4.13}$$

on ψ, assuming that this equation must hold for all states ψ that actually occur in nature.

In our case, however, even Eq. (4.13) cannot be used. The difficulty arises due to the fact that this equation must hold at all points of space time. Thus, if $\partial A_\beta'/\partial x_\beta'$ is the operator in Eq. (4.13) at another point x_α', we must also have

$$(\partial A_\beta'/\partial x_\beta')\psi = 0, \tag{4.14}$$

and therefore

$$[\partial A_\alpha/\partial x_\alpha, \partial A_\beta'/\partial x_\beta']\psi = 0. \tag{4.15}$$

If Eqs. (4.13) and (4.14) are consistent, Eq. (4.15) must be a consequence of them, thus not imposing new restrictions on ψ. This, however, does not turn out to be the case, because of the commutation properties of the extraordinary part of $A_{\alpha,\alpha}$ required by Eqs. (4.12).

Therefore, we find it necessary to make certain classically admissible changes before performing quantization. These consist in carrying out of a gage transformation,[12]

$$A_\alpha = \varphi_\alpha + a\partial B/\partial x_\alpha, \tag{4.16}$$

thus introducing new potentials φ_α. We wish the ordinary part of φ_α to be the same as that of A_α. This means that the ordinary part of B must be zero, which by Eq. (4.9) means that

$$(1-a^2\Box)B = 0. \tag{4.17}$$

By (4.16), the Lorentz condition on A_α becomes

$$A_{\alpha,\alpha} = \varphi_{\alpha,\alpha} + a\Box B = 0,$$

or, taking account of Eq. (4.17),

$$a\varphi_{\alpha,\alpha} + B = 0. \tag{4.18}$$

From Eqs. (4.17) and (4.18) it follows that φ_α satisfies the equation

$$(1-a^2\Box)\varphi_{\alpha,\alpha} = 0, \tag{4.19}$$

which replaces the Lorentz condition on A_α. This has the form of Eq. (2.15), so that φ_α will satisfy the same generalized wave equation

$$(1-a^2\Box)\Box\varphi_\alpha = 0, \tag{4.20}$$

[12] This is analogous to the method of E. C. G. Stückelberg, Helv. Phys. Acta 11, 299 (1938).

as do all the other field quantities. This equation can also be derived directly from the corresponding equation for A_α, together with Eqs. (4.16) and (4.17).

In terms of the Fourier amplitudes Eqs. (4.16) and (4.18), respectively, can now be written as follows:

$$\begin{aligned} A_\alpha(\mathbf{k}) &= \varphi_\alpha(\mathbf{k}), \\ \bar{A}_\alpha(\mathbf{k}) &= \bar{\varphi}_\alpha(\mathbf{k}) + ia\bar{k}_\alpha\bar{B}(\mathbf{k}), \\ A_\alpha^*(\mathbf{k}) &= \varphi_\alpha^*(\mathbf{k}), \\ \bar{A}_\alpha^*(\mathbf{k}) &= \bar{\varphi}_\alpha^*(\mathbf{k}) - ia\bar{k}_\alpha\bar{B}^*(\mathbf{k}); \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} &k_\alpha\varphi_\alpha(\mathbf{k}) = 0, \quad ia\bar{k}_\alpha\bar{\varphi}_\alpha(\mathbf{k}) + \bar{B}(\mathbf{k}) = 0, \\ &k_\alpha\varphi_\alpha^*(\mathbf{k}) = 0, \quad -ia\bar{k}_\alpha\bar{\varphi}_\alpha^*(\mathbf{k}) + \bar{B}^*(\mathbf{k}) = 0. \end{aligned} \tag{4.22}$$

Substituting from Eq. (4.21) into Eq. (4.11) and simplifying with the help of Eq. (4.22), we obtain:

$$\begin{aligned} \bar{H}_f = &\int k^2\{\varphi_\alpha^*(\mathbf{k})\varphi_\alpha(\mathbf{k}) + \varphi_\alpha(\mathbf{k})\varphi_\alpha^*(\mathbf{k})\}d\mathbf{k} \\ &- \int \bar{k}^2\{\bar{\varphi}_\alpha^*(\mathbf{k})\bar{\varphi}_\alpha(\mathbf{k}) + \bar{\varphi}_\alpha(\mathbf{k})\varphi_\alpha^*(\mathbf{k}) \\ &\qquad + \bar{B}^*(\mathbf{k})\bar{B}(\mathbf{k}) + \bar{B}(\mathbf{k})\bar{B}^*(\mathbf{k})\}d\mathbf{k}, \end{aligned} \tag{4.23}$$

where we have rearranged factors and made use of the fact that $a^2\bar{k}_\alpha{}^2 = -1$. We take this expression to be the quantum-mechanical Hamiltonian of the field.[13] Equations (4.18) can be regarded as our new auxiliary condition, or Eqs. (4.22) in **k**-space. Quantum mechanically we must again take them as conditions on ψ, namely:

$$\begin{aligned} &k_\alpha\varphi_\alpha(\mathbf{k})\psi = 0, \quad (ia\bar{k}_\alpha\bar{\varphi}_\alpha(\mathbf{k}) + \bar{B}(\mathbf{k}))\psi = 0, \\ &k_\alpha\varphi_\alpha^*(\mathbf{k})\psi = 0, \quad (ia\bar{k}_\alpha\bar{\varphi}_\alpha^*(\mathbf{k}) - \bar{B}^*(\mathbf{k}))\psi = 0. \end{aligned} \tag{4.24}$$

The commutation rules for the Fourier amplitudes can be derived either by means of Eqs. (4.12), or making use of the fact that for consistent application of quantum mechanical ideas it is only necessary to have the relation

$$[\bar{H}_f, F(\mathbf{r}, t)] = -i\hbar\dot{F}(\mathbf{r}, t), \tag{4.25}$$

for every F. The calculation involved is very lengthy. It is therefore simpler to assume commutation rules by analogy with the usual quantum electrodynamics, and then verify that Eq. (4.25) is satisfied. We thus obtain the following:

$$\begin{aligned} [\varphi_\alpha^*(\mathbf{k}), \varphi_\beta(\mathbf{k}')] &= -\frac{c\hbar}{2k}\delta_{\alpha\beta}\delta(\mathbf{k}-\mathbf{k}'), \\ [\bar{\varphi}_\alpha^*(\mathbf{k}), \bar{\varphi}_\beta(\mathbf{k}')] &= \frac{c\hbar}{2\bar{k}}\delta_{\alpha\beta}\delta(\mathbf{k}-\mathbf{k}'), \\ [\bar{B}^*(\mathbf{k}), \bar{B}(\mathbf{k}')] &= \frac{c\hbar}{2\bar{k}}\delta(\mathbf{k}-\mathbf{k}'). \end{aligned} \tag{4.26}$$

All other pairs commute.

Operators occurring in Eq. (4.24) commute among themselves so that these four conditions are consistent with each other. For example,

$$\begin{aligned} &[ia\bar{k}_\alpha\bar{\varphi}_\alpha(\mathbf{k}) + \bar{B}(\mathbf{k}), ia\bar{k}_\beta'\bar{\varphi}_\beta^*(\mathbf{k}') - \bar{B}^*(\mathbf{k}')] \\ &= -a^2\bar{k}_\alpha\bar{k}_\beta'[\bar{\varphi}_\alpha(\mathbf{k}), \bar{\varphi}_\beta^*(\mathbf{k}')] - [\bar{B}(\mathbf{k}), \bar{B}^*(\mathbf{k}')] \\ &= \frac{c\hbar a^2}{2\bar{k}}\bar{k}_\alpha\bar{k}_\beta'\delta_{\alpha\beta}\delta(\mathbf{k}-\mathbf{k}') + \frac{c\hbar}{2\bar{k}}\delta(\mathbf{k}-\mathbf{k}') \\ &= \frac{c\hbar}{2\bar{k}}(a^2\bar{k}_\alpha{}^2 + 1)\delta(\mathbf{k}-\mathbf{k}') = 0, \end{aligned}$$

since

$$a^2\bar{k}_\alpha{}^2 = -1.$$

These operators however, do not commute with the Hamiltonian[14] $\bar{H}_f$, but the equations

$$[\bar{H}_f, k_\alpha\varphi_\alpha(\mathbf{k})]\psi = 0, \quad [\bar{H}_f, ia\bar{k}_\beta\bar{\varphi}_\beta(\mathbf{k}) + \bar{B}(\mathbf{k})]\psi = 0,$$

etc., are merely Eqs. (4.24) again, and thus do not imply new conditions on ψ. Thus, using Eqs. (4.26),

$$\begin{aligned} &[\bar{H}_f, ia\bar{k}_\beta\bar{\varphi}_\beta(\mathbf{k}) + \bar{B}(\mathbf{k})]\psi \\ &= -2\int \bar{k}'^2[\bar{\varphi}_\alpha^*(\mathbf{k}'), ia\bar{k}_\beta\bar{\varphi}_\beta(\mathbf{k})]\bar{\varphi}_\alpha(\mathbf{k}')d\mathbf{k}'\psi \\ &\quad -2\int \bar{k}'^2[\bar{B}^*(\mathbf{k}'), \bar{B}(\mathbf{k})]\bar{B}(\mathbf{k}')d\mathbf{k}'\psi \\ &= -c\hbar\int \bar{k}'\{ia\bar{k}_\beta\bar{\varphi}_\beta(\mathbf{k}') + \bar{B}(\mathbf{k}')\}\delta(\mathbf{k}-\mathbf{k}')d\mathbf{k}'\psi \\ &= -c\hbar\bar{k}\{ia\bar{k}_\beta\bar{\varphi}_\beta(\mathbf{k}) + \bar{B}(\mathbf{k})\}\psi = 0 \end{aligned}$$

by Eq. (4.24).

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[13] In reference 2, the terms in B were inadvertently omitted.

[14] In reference 2, it was erroneously stated that they do commute with the Hamiltonian.

In what follows we shall have no occasion to refer to equations preceding Eq. (4.23). It will therefore introduce no confusion to put

$$\varphi_\alpha = (A_x, A_y, A_z, i\varphi) = (\mathbf{A}, i\varphi); \qquad (4.27)$$

thus using for the components of φ_α the symbols we previously used for the components of A_α.

5. FIELD IN THE PRESENCE OF PARTICLES

When particles are present, we introduce a separate time for each particle. This procedure, suggested by Dirac,[15] preserves relativistic invariance of equations. We assume that for each particle we have the Dirac wave equation,

$$(R_s - i\hbar\partial/\partial t_s)\Psi = 0, \qquad (5.1)$$

where $s = 1, 2, \cdots n$, n = number of particles,

$$R_s = c\boldsymbol{\alpha}_s \cdot (\mathbf{p}_s - \epsilon_s \mathbf{A}(\mathbf{r}_s, t_s)/c) + m_s c^2 \beta_s + \epsilon_s \varphi(\mathbf{r}_s, t_s), \qquad (5.2)$$

and Ψ is a function of all $\mathbf{r}_s$ and t_s as well as the variables $\mathbf{A}(\mathbf{k})$, $\varphi(\mathbf{k})$, $\mathbf{A}^*(\mathbf{k})$, $\varphi^*(\mathbf{k})$, $\bar{\mathbf{A}}(\mathbf{k})$, $\bar{\mathbf{A}}^*(\mathbf{k})$, $\bar{\varphi}(\mathbf{k})$, $\bar{\varphi}^*(\mathbf{k})$ describing the field.

On the other hand, the Heisenberg-Pauli equation for the system *particles+field* is

$$(\bar{H}_f + \textstyle\sum_s R_s)\psi = i\hbar\partial\psi/\partial t, \qquad (5.3)$$

where t is the common time of the system. $\bar{H}_f$ can be eliminated from this equation, when we wish to consider merely the behavior of particles, by a transformation due to Rosenfeld,[16] namely:

$$\psi' = \exp(i\bar{H}_f t/\hbar)\psi, \qquad (5.4)$$

which gives

$$\textstyle\sum_s R_s \psi' = i\hbar\partial\psi'/\partial t; \qquad (5.5)$$

the quantities $\mathbf{A}(\mathbf{r}_s, t)$ and $\varphi(\mathbf{r}_s, t)$ occurring in R_s are the potentials at points $\mathbf{r}_s$. They satisfy, except for auxiliary conditions, the equations for the field without the particles.

It can be shown[17] that, provided Eqs. (5.1) are consistent with each other, one can obtain Eq. (5.5) as their consequence, by adding them together, and putting all $t_s = t$. For, if we put

$$\Psi_{t_s = t} = \psi',$$

then

$$\partial\psi'/\partial t = \textstyle\sum_s (\partial\Psi/\partial t_s)_{t_s = t} \qquad (5.6)$$

and Eq. (5.5) follows immediately. In order that Eq. (5.1) be consistent with each other it is necessary and sufficient that all R_s commute among themselves. Dirac's investigation showed that such is the case when for every pair of values of s, say u and v,

$$(t_u - t_v) < (\mathbf{r}_u - \mathbf{r}_v)^2.$$

This condition is certainly satisfied when $t_u = t_v = t$, the only case of physical interest.

The presence of particles requires another modification of our auxiliary conditions. Equations (4.24), with ψ replaced by Ψ are inconsistent with Eq. (5.1). As was done by Dirac, Fock, and Podolsky,[17] the situation is remedied by adding certain terms to the operators in Eqs. (4.24) to secure the desired consistence. The additional terms are completely defined by this requirement. If we introduce

$$Q(\mathbf{k}) = \frac{\mathbf{k}\cdot\mathbf{A}(\mathbf{k})}{k}, \quad \bar{Q}(\mathbf{k}) = \frac{a\mathbf{k}\cdot\bar{\mathbf{A}}(\mathbf{k}) - i\bar{B}(\mathbf{k})}{a\bar{k}}, \qquad (5.7)$$

and

$$\begin{aligned} f(\mathbf{r}_s, t_s) &= (2\pi)^{-\frac{3}{2}} \sum_{s=1}^{n} \epsilon_s \exp(i\varphi_s), \\ &\qquad \varphi_s \equiv ckt_s - \mathbf{k}\cdot\mathbf{r}_s, \\ \bar{f}(\mathbf{r}_s, t_s) &= (2\pi)^{-\frac{3}{2}} \sum_{s=1}^{n} \epsilon_s \exp(i\bar{\varphi}_s), \\ &\qquad \bar{\varphi} \equiv c\bar{k}t - \mathbf{k}\cdot\mathbf{r}_s; \end{aligned} \qquad (5.8)$$

the modified auxiliary conditions become

$$\begin{aligned} [Q(\mathbf{k}) - \varphi(\mathbf{k}) + f/2k^2]\Psi &= 0, \\ [\bar{Q}(\mathbf{k}) - \bar{\varphi}(\mathbf{k}) - \bar{f}/2\bar{k}^2]\Psi &= 0, \\ [Q^*(\mathbf{k}) - \varphi^*(\mathbf{k}) + f^*/2k^2]\Psi &= 0, \\ [\bar{Q}^*(\mathbf{k}) - \bar{\varphi}^*(\mathbf{k}) - \bar{f}^*/2\bar{k}^2]\Psi &= 0, \end{aligned} \qquad (5.9)$$

which differ from Eqs. (4.24) only by terms in f and $\bar{f}$. The operators in brackets commute among themselves and with the operators $R_s - i\hbar\partial/\partial t$ of Eq. (5.1),

If $\mathbf{E}$ and $\mathbf{H}$ are defined by equations

$$\mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}/\partial t}{c}, \quad \mathbf{H} = \nabla\times\mathbf{A} \qquad (5.10)$$

[15] P. A. M. Dirac, Proc. Roy. Soc. **136**, 453 (1932).
[16] L. Rosenfeld, Zeits. f. Physik **76**, 729 (1932).
[17] P. A. M. Dirac, V. A. Fock, and B. Podolsky, Physik. Zeits. Sowjetunion **2**, 473 (1932), or P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, 1935), Chapter XIII.

then, of course,

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{H}/\partial t}{c}=0 \quad\text{and}\quad \nabla\cdot\mathbf{H}=0; \tag{5.11}$$

this pair of Maxwell-Lorentz equations remaining valid operator equations. It can now be shown, with the help of Eqs. (5.9), that the analogs of the other pair of Maxwell-Lorentz equations are[2]

$$[(1-a^2\Box)\nabla\cdot\mathbf{E}]\psi=(\textstyle\sum_s \epsilon_s\delta(\mathbf{r}-\mathbf{r}_s))\psi \tag{5.12}$$

and

$$\left[(1-a^2\Box)\left(\nabla\times\mathbf{H}-\frac{\partial\mathbf{E}/\partial t}{c}\right)\right]\psi =(\textstyle\sum_s \epsilon_s\alpha_s\delta(\mathbf{r}-\mathbf{r}_s))\psi. \tag{5.13}$$

These equations show the effect of particles on the field, and are natural generalizations of the usual equations.

6. INTERACTION BETWEEN PARTICLES

Interaction between particles arises in this theory as a result of interaction between the field and particles. The field is affected by particles only through the auxiliary conditions (5.9), while particles are affected by the field through the entry of potentials in Eq. (5.1). Thus, if the auxiliary conditions were solved for Ψ, and the result be substituted into Eq. (5.1), we would obtain equations containing interactions between particles, and in these equations one would no longer need consider the effect of particles on the field. The way to perform this elimination of auxiliary conditions was shown by Fock.[18]

Fock makes a natural assumption that Ψ is a *functional* of the field variables $\varphi_\alpha(\mathbf{k})$. Then from the commutation rules for the Fourier amplitudes he can infer the way in which these amplitudes, as operators, operate on the functional. Thus, for the amplitudes of the scalar potential we have

$$[\varphi^*(\mathbf{k}),\ \varphi(\mathbf{k}')]=\frac{c\hbar}{2k}\delta(\mathbf{k}-\mathbf{k}'),$$

and one can consider operation with $\varphi(\mathbf{k})$ as a multiplication by $\varphi(\mathbf{k})$, while

$$\varphi^*(\mathbf{k})\sim\frac{c\hbar}{2k}\frac{\delta}{\delta\varphi(\mathbf{k})}. \tag{6.1}$$

This is analogous to interpreting p_x as $-hi\partial/\partial x$. Making use of this idea, we observe that since

$$[\bar{\varphi}^*(\mathbf{k}),\ \bar{\varphi}(\mathbf{k}')]=-\frac{c\hbar}{2\bar{k}}\delta(\mathbf{k}-\mathbf{k}'),$$

we can put

$$\bar{\varphi}^*(\mathbf{k})\sim-\frac{c\hbar}{2\bar{k}}\frac{\delta}{\delta\bar{k}\ \delta\bar{\varphi}(\mathbf{k})}. \tag{6.2}$$

Further, since the commutation rules between Q's of Eq. (5.7) and their conjugates, obtained by the use of Eqs. (4.26), are

$$\begin{aligned}[Q(\mathbf{k}),\ Q^*(\mathbf{k}')]&=\frac{c\hbar}{2k}\delta(\mathbf{k}-\mathbf{k}'),\\ [\bar{Q}(\mathbf{k}),\ \bar{Q}^*(\mathbf{k}')]&=\frac{c\hbar}{2\bar{k}}\delta(\mathbf{k}-\mathbf{k}');\end{aligned} \tag{6.3}$$

we may assume that

$$Q(\mathbf{k})\sim\frac{c\hbar}{2k}\frac{\delta}{\delta Q^*(\mathbf{k})},\quad \bar{Q}(\mathbf{k})\sim-\frac{c\hbar}{2\bar{k}}\frac{\delta}{\delta\bar{Q}^*(\mathbf{k})}. \tag{6.4}$$

Further, we observe that $Q(\mathbf{k})$ is merely the longitudinal part of $\mathbf{A}(\mathbf{k})$; the transverse part may be called $\mathbf{D}(\mathbf{k})$, and

$$\mathbf{D}(\mathbf{k})=\mathbf{A}(\mathbf{k})-\mathbf{k}Q(\mathbf{k})/k. \tag{6.5}$$

Analogously, we can define $\overline{\mathbf{D}}(\mathbf{k})$ as

$$\overline{\mathbf{D}}(\mathbf{k})=\overline{\mathbf{A}}(\mathbf{k})-\mathbf{k}Q(\mathbf{k})/k, \tag{6.6}$$

which is not the transverse part of $\overline{\mathbf{A}}(\mathbf{k})$, but merely an analog of Eq. (6.5).

With the help of Eqs. (6.1), (6.2), and (6.4) we can now solve Eqs. (5.9). It rurns out that Ψ must have the form[3]

$$\Psi=\exp(\chi-\bar{\chi})\Omega, \tag{6.7}$$

where Ω is a functional of $\mathbf{D}(\mathbf{k})$, $\mathbf{D}^*(\mathbf{k})$, $\overline{\mathbf{D}}(\mathbf{k})$, and $\overline{\mathbf{D}}^*(\mathbf{k})$ only (it is, of course, a function of

[18] V. A. Fock, Physik. Zeits. Sowjetunion 6, 449 (1934).

coordinates and times of particles), while

$$\chi=\frac{2}{c\hbar}\int Q^*(\mathbf{k})\varphi(\mathbf{k})k\mathbf{dk}+\frac{1}{c\hbar}\int\varphi(\mathbf{k})\frac{f^*}{k}\mathbf{dk}-\frac{1}{c\hbar}\int Q^*(\mathbf{k})\frac{f}{k}\mathbf{dk}+\chi',$$
$$\bar{\chi}=\frac{2}{c\hbar}\int \bar{Q}^*(\mathbf{k})\bar{\varphi}(\mathbf{k})\bar{k}\mathbf{dk}-\frac{1}{c\hbar}\int\bar{\varphi}(\mathbf{k})\frac{\bar{f}^*}{\bar{k}}\mathbf{dk}+\frac{1}{c\hbar}\int \bar{Q}^*(\mathbf{k})\frac{\bar{f}}{\bar{k}}\mathbf{dk}+\bar{\chi}', \quad (6.8)$$

χ' and $\bar{\chi}'$ being arbitrary functions of the space-time coordinates of the particles.

As we wish to obtain wave equations for Ω corresponding to Eqs. (5.1) for Ψ, we must replace each operator F in (5.1) and (5.2) by

$$\exp(-\chi+\bar{\chi})F\exp(\chi-\bar{\chi}). \quad (6.9)$$

This amounts to substitution of Ψ from Eq. (6.7) into Eqs. (5.1) and multiplication on the left by $\exp(-\chi+\bar{\chi})$. After the transformation (6.9) is carried out, terms containing $Q(\mathbf{k})$, $\bar{Q}(\mathbf{k})$, $\varphi^*(\mathbf{k})$, or $\bar{\varphi}^*(\mathbf{k})$ can be omitted, since

$$Q(\mathbf{k})\Omega=\bar{Q}(\mathbf{k})\Omega=\varphi^*(\mathbf{k})\Omega=\bar{\varphi}^*(\mathbf{k})\Omega=0.$$

Finally, the quantities χ' and $\bar{\chi}'$ are determined by the requirement that the transformed operator R_s should be Hermitian. This gives[3]

$$\chi'=-\frac{1}{4c\hbar}\int\frac{f^*f}{k^3}\mathbf{dk},\quad \bar{\chi}'=-\frac{1}{4c\hbar}\int\frac{\bar{f}^*\bar{f}}{\bar{k}^3}\mathbf{dk}. \quad (6.10)$$

In this way we obtain, instead of Eqs. (5.1), the set

$$(c\boldsymbol{\alpha}_s\cdot\mathbf{P}_s+m_sc^2\beta_s)\Omega=T_s\Omega; \quad (6.11)$$

where

$$\mathbf{P}_s=\mathbf{p}_s-(\epsilon_s/c)\mathbf{D}(\mathbf{r}_s,t_s)-(\epsilon_s/2c)\nabla_sU_s, \quad (6.12)$$

$$T_s=i\hbar\partial/\partial t_s-(\epsilon_s/2c)\partial U_s/\partial t_s-(\epsilon_s^2/8\pi a), \quad (6.13)$$

and

$$U_s\equiv\sum\nolimits'_u(\epsilon_u/8\pi^3)\cdot\int[(1/k^3)\sin(\varphi_s-\varphi_u)-(1/\bar{k}^3)\sin(\bar{\varphi}_s-\bar{\varphi}_u)]\mathbf{dk}, \quad (6.14)$$

the prime after the summation indicating that the term $u=s$ is to be omitted.

The vector $\mathbf{D}(\mathbf{r}_s, t_s)$ is the value of $\mathbf{D}(\mathbf{r}, t)$ at the point $\mathbf{r}=\mathbf{r}_s$ and $t=t_s$. $\mathbf{D}(\mathbf{r}, t)$ is obtained from $\mathbf{D}(\mathbf{k})$ and $\bar{\mathbf{D}}(\mathbf{k})$ by the formula (4.7).

To obtain a physical interpretation we must pass from the set (6.11) to an equation analogous to Eq. (5.5). This is done by adding together the equations for individual particles and setting all $t_s=t$, the common time. In fact, to consider several particles as interacting parts of one physical system is possible only when all of them are referred to a common time of the system. When this is done, the resulting equation is[4]

$$\{\textstyle\sum_s(\boldsymbol{\alpha}_s\cdot[c\mathbf{p}_s-\epsilon_s\mathbf{D}(\mathbf{r}_s,t)]+m_sc^2\beta_s+\epsilon_s^2/8\pi a)+\sum'_{s,u}\epsilon_s\epsilon_u[1-\exp(-|\mathbf{r}_s-\mathbf{r}_u|/a)]/8\pi|\mathbf{r}_s-\mathbf{r}_u|\}\Omega=i\hbar\partial\Omega/\partial t. \quad (6.15)$$

We can now recognize $\epsilon_s^2/8\pi a$ as the electrostatic self-energy of sth particle, and the double summation as the electrostatic interaction energy among the particles, which thus turn out to be the same as in non-quantum theory. The field enters this wave equation only through $\mathbf{D}(\mathbf{r}, t)$, which can be thought of as independent of the presence of particles. In fact, the properties of $\mathbf{D}$ are sufficiently defined by the following commutation rules, which can be easily derived from definitions and the Eqs. (4.26).

$$[D_i(\mathbf{k}),D_j^*(\mathbf{k}')]=\frac{-c\hbar}{2k}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\delta(\mathbf{k}-\mathbf{k}'),$$
$$[\bar{D}_i(\mathbf{k}),\bar{D}_j^*(\mathbf{k}')]=\frac{c\hbar}{2\bar{k}}\left(\delta_{ij}-\frac{k_ik_j}{\bar{k}^2}\right)\delta(\mathbf{k}-\mathbf{k}'). \quad (6.16)$$

These commutation rules are however inconvenient for interpretation of $D_j(\mathbf{k})$, etc., as functional operators. Montgomery[4] therefore introduces another set of operators $\mathbf{b}(\mathbf{k})$ and $\bar{\mathbf{b}}(\mathbf{k})$, whose components have more convenient commutation rules,

$$[b_j(\mathbf{k}),b_{j'}^*(\mathbf{k}')]=\delta_{jj'}\delta(\mathbf{k}-\mathbf{k}'),$$
$$[\bar{b}_j(\mathbf{k}),\bar{b}_{j'}^*(\mathbf{k}')]=-\delta_{jj'}\delta(\mathbf{k}-\mathbf{k}'), \quad (6.17)$$

with other combinations commuting. One can then put

$$\mathbf{D}(\mathbf{k})=(c\hbar/2k^3)^{\frac{1}{2}}\mathbf{k}\times\mathbf{b}(\mathbf{k}) \tag{6.18}$$

and

$$\bar{\mathbf{D}}(\mathbf{k})=(c\hbar/2\bar{k}^3)^{\frac{1}{2}}[\mathbf{k}\times\bar{\mathbf{b}}(\mathbf{k})+\bar{\mathbf{b}}(\mathbf{k})/a], \tag{6.19}$$

which, together with Eqs. (6.17), again give Eqs. (6.16).

Equation (6.15) can now be put into a more convenient form

$$(H_0-i\hbar\partial/\partial t)\Omega$$
$$=\int\{\mathbf{G}^*(\mathbf{k})\cdot\mathbf{b}(\mathbf{k})+\mathbf{G}(\mathbf{k})\cdot\mathbf{b}^*(\mathbf{k})$$
$$+\bar{\mathbf{G}}^*(\mathbf{k})\cdot\bar{\mathbf{b}}(\mathbf{k})+\bar{\mathbf{G}}(\mathbf{k})\cdot\bar{\mathbf{b}}(\mathbf{k})\}\mathbf{dk}\Omega, \tag{6.20}$$

where

$$H_0=\sum_s(\boldsymbol{\alpha}_s\cdot c\mathbf{p}_s+m_sc^2\beta_s+\epsilon_s^2/8\pi a)$$
$$+\sum_{s,u}{}'\epsilon_s\epsilon_u[1-\exp(-|\mathbf{r}_s-\mathbf{r}_u|/a)]/$$
$$8\pi|\mathbf{r}_s-\mathbf{r}_u|, \tag{6.21}$$

$$\mathbf{G}(\mathbf{k})=(1/2\pi k)^{\frac{3}{2}}(c\hbar/2)^{\frac{1}{2}}\sum_s\epsilon_s\boldsymbol{\alpha}_s$$
$$\times\mathbf{k}\exp[i(ckt-\mathbf{k}\cdot\mathbf{r}_s)] \tag{6.22}$$

and

$$\bar{\mathbf{G}}(\mathbf{k})=(1/2\pi\bar{k})^{\frac{3}{2}}(c\hbar/2)^{\frac{1}{2}}\sum_s\epsilon_s(\boldsymbol{\alpha}_s\times\mathbf{k}+\boldsymbol{\alpha}_s/a)$$
$$\times\exp[i(c\bar{k}t-\mathbf{k}\cdot\mathbf{r}_s)]. \tag{6.23}$$

Finally, the time dependence of G's can be eliminated, and we obtain[19]

$$(H_0-i\hbar\partial/\partial t)\Omega+\hbar c\int[k\mathbf{b}^*(\mathbf{k})\cdot\mathbf{b}(\mathbf{k})$$
$$-\bar{k}\bar{\mathbf{b}}^*(\mathbf{k})\cdot\bar{\mathbf{b}}(\mathbf{k})]\mathbf{dk}\Omega$$
$$=\int\{\mathbf{G}_0^*(\mathbf{k})\cdot\mathbf{b}(\mathbf{k})+\mathbf{G}_0(\mathbf{k})\cdot\mathbf{b}^*(\mathbf{k})$$
$$+\bar{\mathbf{G}}_0^*(\mathbf{k})\cdot\bar{\mathbf{b}}(\mathbf{k})+\bar{\mathbf{G}}_0(\mathbf{k})\cdot\mathbf{b}^*(\mathbf{k})\}\mathbf{dk}\Omega, \tag{6.24}$$

in which

$$\mathbf{G}_0(\mathbf{k})=\mathbf{G}(\mathbf{k})e^{-ickt},\ \text{etc.}$$

Equation (6.24) may be regarded as the fundamental wave equation of our theory for a system of particles in the presence of photons. The energy of the two kinds of photons appears as the integral operator on the left side. The effect of photons on the particles appears on the right side of the equation.

[19] See reference 4. Eq. (2.19) of that article should have $(-i\hbar\partial/\partial t)$ in the left-hand member.

7. APPLICATIONS OF PERTURBATION METHOD

As was done by Fock,[18] the functional Ω may be expanded in series of simple functionals

$$\Omega=\sum_{r,s}\Omega_{rs}, \tag{7.1}$$

where Ω_{rs} is an eigenfunctional of the numbers of photons of the two kinds, corresponding to r ordinary and s tilde photons. It can be written in the form

$$\Omega_{rs}=\sum_{i_1\cdots i_r}\sum_{j_1\cdots j_s}\int\cdots\int\mathbf{dk}_1\mathbf{dk}_2\cdots$$
$$\times\mathbf{dk}_r\mathbf{dk}_1'\cdots\mathbf{dk}_s'\psi_{rs}(\mathbf{k}_1i_1\cdots\mathbf{k}_s'j_s)$$
$$\times b_{i_1}'(\mathbf{k}_1)\cdots b_{i_r}'(\mathbf{k}_r)$$
$$\times\bar{b}_{j_1}'(\mathbf{k}_1')\cdots\bar{b}_{j_s}'(\mathbf{k}_s'), \tag{7.2}$$

with i's and j's being 1, 2, 3; or in particular

$$\Omega_{00}=\psi_{00},$$
$$\Omega_{10}=\sum_i\int\mathbf{dk}\psi_{10}(\mathbf{k},i)b_i'(\mathbf{k}), \tag{7.3}$$
$$\Omega_{01}=\sum_j\int\mathbf{dk}\psi_{01}(\mathbf{k},j)\bar{b}_j'(\mathbf{k}).$$

This can be carried out as far as desired, but if one wishes to limit oneself in terms quadratic in particle charges, one may drop higher terms and put

$$\Omega=\Omega_{00}+\Omega_{10}+\Omega_{01}. \tag{7.4}$$

In all of the above equations ψ_{rs} are, of course, functions of t and the positions of the particles $\mathbf{r}_1, \mathbf{r}_2, \cdots\mathbf{r}_n$, but for brevity these variables have been suppressed.

The operators $\mathbf{b}(\mathbf{k})$, $\mathbf{b}^*(\mathbf{k})$, $\bar{\mathbf{b}}(\mathbf{k})$, and $\bar{\mathbf{b}}^*(\mathbf{k})$ may now be related to operations on Ω as follows:

$$b_i^*(\mathbf{k})\sim b_i'(\mathbf{k}),\quad \bar{b}_j^*(\mathbf{k})\sim\bar{b}_j'(\mathbf{k}),$$
$$b_i(\mathbf{k})\sim\delta/\delta b_i'(\mathbf{k}),\quad \bar{b}_j(\mathbf{k})\sim-\delta/\delta\bar{b}_j'(\mathbf{k}). \tag{7.5}$$

With these assumptions and neglect of terms in Ω corresponding to more than one photon, one then obtains[20]

$$(H_0-i\hbar\partial/\partial t)\psi_{00}$$
$$=\sum_j\int \mathbf{dk}[G_{0j}^*(\mathbf{k})\psi_{10}(\mathbf{k},j)-\bar{G}_{0j}^*(\mathbf{k})\psi_{01}(k,j)], \quad (7.6)$$

$$(H_0+\hbar ck-i\hbar\partial/\partial t)\psi_{10}(\mathbf{k},j)=G_{0j}(\mathbf{k})\psi_{00}, \quad (7.7)$$

$$(H_0+\hbar c\bar{k}-i\hbar\partial/\partial t)\psi_{01}(\mathbf{k},j)=\bar{G}_{0j}(\mathbf{k})\psi_{00}. \quad (7.8)$$

From this point two procedures have been used. Both assume an unperturbed value of ψ_{00}. This is substituted into Eqs. (7.7) and (7.8), which are then solved for ψ_{10} and ψ_{01}. If, with these quantities substituted into Eq. (7.6), the latter is written in the form

$$(H_0+U)\psi_{00}=i\hbar\partial\psi_{00}/\partial t; \quad (7.9)$$

then U may be regarded as the contribution to the energy of the particles due to the possibility of emission and absorption of photons.

Montgomery[4] assumes the unperturbed ψ_{00} to correspond to plane waves. Then, working in the momentum space, he obtains a generalization of Moller's formula.

More revealing is the result of the procedure of Green.[5] Assuming that $(H_0-i\hbar\partial/\partial t)\psi_{10}$ and $(H_0-i\hbar\partial/\partial t)\psi_{01}$ can be neglected compared with $\hbar ck\psi_{10}$ and $\hbar c\bar{k}\psi_{01}$, respectively, he at once obtains from Eqs. (7.7) and (7.8)

$$\psi_{10}=G_{0j}\psi_{00}/\hbar ck, \quad \psi_{01}=\bar{G}_{0j}\psi_{00}/\hbar c\bar{k}. \quad (7.10)$$

Thus,

$$U=-\int[\mathbf{G}_0^*(\mathbf{k})\cdot\mathbf{G}_0(\mathbf{k})/\hbar ck-\bar{\mathbf{G}}_0^*(\mathbf{k})\cdot\bar{\mathbf{G}}_0(\mathbf{k})/\hbar c\bar{k}]\mathbf{dk}, \quad (7.11)$$

$$=-\sum_{u,v}\frac{\epsilon_u\epsilon_v}{16\pi}\left\{\boldsymbol{\alpha}_u\cdot\boldsymbol{\alpha}_v\left[\frac{1-\exp(-R/a)}{R}\right]+\frac{\boldsymbol{\alpha}_u\cdot\mathbf{R}\boldsymbol{\alpha}_v\cdot\mathbf{R}}{R^2}\left[\frac{1-\exp(-R/a)}{R}-\frac{\exp(-R/a)}{a}\right]\right\}, \quad (7.12)$$

where $\mathbf{R}=\mathbf{r}_u-\mathbf{r}_v$, and $R=|\mathbf{R}|$.

There are no infinities in this result. The electromagnetic self-energy is obtained by taking terms for which $u=v$, and letting $R\to 0$. This gives $-3/4\epsilon_s^2/4\pi a$ for each particle. The electromagnetic interaction thus obtained is a generalization of Breit's formula, which can be obtained by letting $a\to 0$.

8. CONCLUDING REMARKS

It is interesting to note that although the tilde photon energy of Eq. (6.24),

$$-\hbar c\bar{k}\bar{\mathbf{b}}^*(\mathbf{k})\cdot\bar{\mathbf{b}}(\mathbf{k}),$$

is apparently negative, it gives a positive contribution $\hbar c\bar{k}$ to the energy in Eq. (7.8). This is due to the occurrence of the minus sign in the commutation rules of the tilde quantities, Eq. (6.17). If one should put

$$\bar{b}_j=\bar{b}_j', \quad \bar{b}_j^*=-\bar{b}_j'^*,$$

then the above energy expression becomes

$$+\hbar c\bar{k}\bar{\mathbf{b}}'^*(\mathbf{k})\cdot\bar{\mathbf{b}}'(\mathbf{k});$$

and, since the commutation rules for these $\bar{b}$'s are the same as for b's, the eigenvalues of this expression will be positive, just as for ordinary photons.

Recently, Green[6] has extended the theory to a meson type field, leaving out all considerations of isotopic spin and spin dependent interaction. Since no infinities occur in that case also, we are at present considering the case when spin dependence is explicitly taken into account.

[20] Reference 4. A slightly different choice in Eq. (7.5) is responsible for a sign in Eq. (7.8) being different from that of Montgomery's Eq. (2.27), but results are unaffected since there is a compensatory change of sign in Eq. (7.6).