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The Approximate Solution of One-Dimensional Wave Equations*

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CONTENTS

INTRODUCTION

THE rigorous and general solution of linear partial differential equations with constant coefficients can be obtained without difficulty, in the form of a Fourier integral.[1,2] However, the very generality of the Fourier integral makes it difficult to interpret in any detail, unless it can be evaluated in terms of elementary functions. The need for a simple and systematic method of approximate evaluation must have been felt very early. The Fresnel method of zones[3] is such an approximation, although the geometric derivation usually associated with it obscures its generality. The abstract reformulation necessary for its wider application was accomplished by Lord Kelvin[4] and has become known as the method of stationary phase. In this form, the close mathematical relation between Fresnel's ideas and those of Debye's method of steepest descent[5] becomes apparent.

The latter has been extensively considered by mathematicians. As a general method for approximating Fourier integrals, it has only one defect: the spectrum function must be susceptible of analytic extension into the complex plane. Formally, this is a very weak restriction; practically, it is a forbidding one, but it may be essential to consider the matter more fully than has been done here.

The Kelvin-Fresnel methods do not operate in the complex plane, and hence are more readily applied to many problems. An excellent summary of the work in this field prior to 1914 has been given by T. H. Havelock.[6] Since that time, wave theory has been very actively studied, under the impetus provided by atomic theory on the one hand, and by radio and sound on the other. Recently, there also has been a marked interest in geophysical wave problems.

One important relationship has become clear

* This work was supported by Contracts N6-ori-111 and NObs-2074 with the Office of Naval Research and the Bureau of Ships, Navy Department.

[1] A. G. Webster, *Partial Differential Equations of Mathematical Physics* (Julius Springer Verlag, Berlin, 1933), p. 44.

[2] H. Bateman, *Partial Differential Equations* (Dover Publications, New York, 1944), Chapter III.

[3] A. Fresnel, *Oeuvres* (Paris, 1866), tome I.

[4] Lord Kelvin, Proc. Roy. Soc. **42**, 80 (1887).

[5] P. Debye, Math. Ann. **67**, 535 (1908).

[6] T. H. Havelock, *The Propagation of Waves in Dispersive Media* (Cambridge University Press, Teddington, England, 1914).

in the years since Havelock's summary was prepared: the method of stationary phase is also the mathematical foundation for the Hamilton-Jacobi ray theory. The exposition of this relationship will therefore be the principal subject of the following pages.

This review is based largely on notes accumulated over a period of years. Some attempt has been made to give the original sources, but no systematic search of the literature has been made. After the major part of the manuscript had been prepared, an unexpected problem was encountered in applying the method of stationary phase to a dissipative medium. Sections 11, 12, and 13 are devoted to this problem, without, however, reaching definite conclusions.

1. EXAMPLES OF PHYSICAL WAVE EQUATIONS

The various kinds of physical phenomena known as waves are all described by partial differential equations. The simplest of these are the linear equations with constant coefficients, and it is only these that will be discussed here. A further simplification results from the exclusion of equations having more than two independent variables—the time, t, and one space coordinate, x. This restriction is not basic, however.

Waves on a stretched string, sound waves in a tube of uniform cross section, water waves in a shallow uniform canal, and many other phenomena, are governed by an equation of the form

$$(\partial^2\psi/\partial x^2)-(1/c_0^2)(\partial^2\psi/\partial t^2)=0, \qquad (1.1)$$

where c_0 is a constant, while ψ is the dependent variable.

The propagation of electric currents along a uniform wire is governed (1) by the more general equation

$$\frac{\partial^2\psi}{\partial x^2}-\frac{1}{c_0^2}\frac{\partial^2\psi}{\partial t^2}-\frac{2a}{c_0}\frac{\partial\psi}{\partial t}+(k_0^2-a^2)\psi=0, \qquad (1.2)$$

where c_0, a, and k_0 are constants defined as follows:

$$c_0^2=1/LC,$$
$$2a/c_0=RC+LS,$$
$$a^2-k_0^2=RS,$$

C = capacity per unit length, L = inductance per unit length, R = resistance per unit length, S = leakance per unit length. This equation, or special cases of it, also appears in other connections. Diffusion is not usually considered as a wave phenomenon for reasons that will appear below, but is governed by

$$D(\partial^2\psi/\partial x^2)=\partial\psi/\partial t, \qquad (1.3)$$

which is a particular case of Eq. (1.2).

The de Broglie equation for electron waves[7] is

$$\frac{\partial^2\psi}{\partial x^2}-\frac{1}{c_0^2}\frac{\partial^2\psi}{\partial t^2}-\frac{c_0^2}{\alpha^2}\psi=0,$$
$$\alpha=\hbar/m. \qquad (1.4)$$

This equation also governs the motion of long gravitational waves[8] on a rotating disk; c_0^2/α^2 is the Coriolis parameter, and $c_0^2=gh$, where h is the water depth, and g the acceleration of gravity.

Schrödinger's equation is

$$\partial\psi/\partial t=\tfrac{1}{2}i\alpha(\partial^2\psi/\partial x^2), \qquad (1.5)$$

and has a strong formal relationship[9] to that for flexural waves on a rod:

$$\partial^2\psi/\partial t^2=-\tfrac{1}{4}\alpha^2(\partial^4\psi/\partial x^4), \qquad (1.6)$$

although α has a different meaning in the two cases.

All of these equations have special solutions of the form

$$\psi(x,t)=\exp i(kx-nt), \qquad (1.7)$$

for, on substituting this function into any one of them, the result reduces to

$$\psi W(k,n)=0,$$

where W is a different function in each case; and this equation will be satisfied for all x and t if the (real or complex) constants k and n are roots[10] of the characteristic equation.

$$W(k,n)=0. \qquad (1.8)$$

[7] P. A. M. Dirac, *Quantum Mechanics* (The Clarendon Press, Oxford, England, 1935), p. 253.
[8] H. U. Sverdrup, "Dynamics of tides on the North Siberian shelf," Geofys. Publikas. **4** (1927).
[9] E. U. Condon and P. M. Morse, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1929), p. 26.
[10] H. Bateman, *Partial Differential Equations of Mathematical Physics* (Dover Publications, New York, 1944), p. 101.

Symbolically, the partial differential equations may all be written

$$W\left(\frac{1}{i}\frac{\partial}{\partial x}, -\frac{1}{i}\frac{\partial}{\partial t}\right)\psi=0. \tag{1.9}$$

In what follows, no particular assumption will be made concerning the function W, although the above examples will be used for illustration.

In these examples W is always a polynomial. In some generalized forms of wave motion, such as surface waves in moderately deep water, W may be a transcendental function of its arguments:

$$W=n^2-gk\tanh kh, \tag{1.10}$$

where g is the gravitational acceleration, and h the water depth.[11] Ripples of short wave-length are also influenced by the surface tension, T, and, approximately,

$$W=n^2-|k|(g+Tk^2/\rho), \tag{1.11}$$

where ρ is the density of the fluid.[11]

2. THE HAMILTONIAN FUNCTIONS AND THE LOGARITHMIC DECREMENTS

Equation (1.7) may be solved for n as a function of k. Thus, Eq. (1.1) leads to the characteristic equation

$$-k^2+n^2/c_0^2=0,$$

whence $n=\pm c_0 k$. Equation (1.2) leads to

$$-k^2+n^2/c_0^2+2ian/c_0-a^2+k_0^2=0,$$

or

$$n=c_0(-ia\pm[k^2-k_0^2]^{\frac{1}{2}}).$$

Equation (1.3) yields $n=iDk^2$. Equation (1.4) leads to $n=\pm[k^2+k_0^2]^{\frac{1}{2}}$ where $k_0=c_0/\alpha$, while Eq. (1.5) results in the single-valued function $n=\frac{1}{2}\alpha k^2$ and Eq. (1.6) results in $n=\pm\frac{1}{2}\alpha k^2$.

These examples serve to illustrate two facts: for a given real value of k, there will, in general, be several values of n, and these generally will be complex.

For this reason, it is convenient to replace n by $n-ia$ in all the above formulae, and to write the characteristic equation in the form

$$W(k, n-ia)=0, \tag{2.1}$$

where k, n, and a are all real numbers. Then, in general, there will be several single-valued solutions of Eq. (2.1).

$$\begin{aligned} n&=H_1(k), \quad a=D_1(k);\\ n&=H_2(k), \quad a=D_2(k);\\ &\cdots\cdots, \quad \cdots\cdots. \end{aligned} \tag{2.2}$$

The real functions H_1, H_2, etc., are called the Hamiltonian functions of the wave equation, while the functions D_1, D_2, etc., are called its logarithmic decrements.

The Eq. (1.6) then becomes

$$\psi=\exp\{i[kx-H(k)t]-D(k)t\}, \tag{2.3}$$

where H and K are any pair of associated functions.

It will be noted that the diffusion equation is characterized by a Hamiltonian that vanishes identically. This is perhaps the first indication of the fundamental role of the Hamiltonian in determining the character of the solutions of partial differential equations. Figure 1 shows

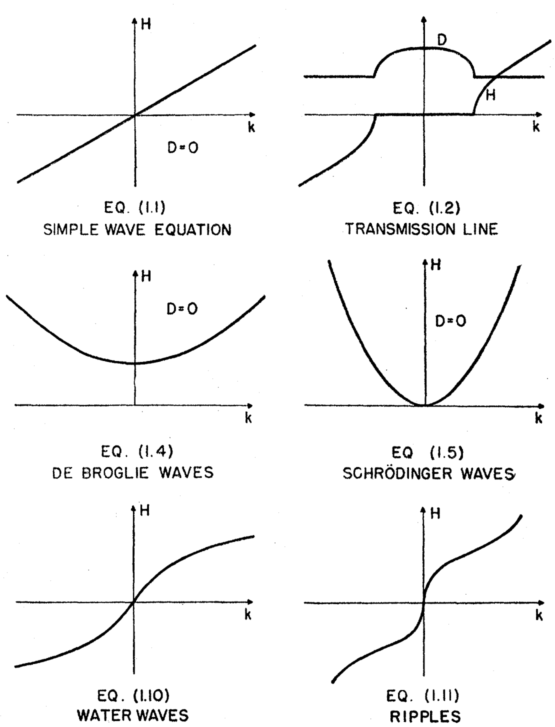

FIG. 1. Graphs of the Hamiltonian function and the related logarithmic decrement for some kinds of wave motion. In general, there will be several pairs of such functions, corresponding to different roots of the equation $W(k, n-ia)=0$.

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[11] L. M. Milne-Thomson, *Theoretical Hydrodynamics* (MacMillan and Company, Ltd., London, 1938), p. 376.

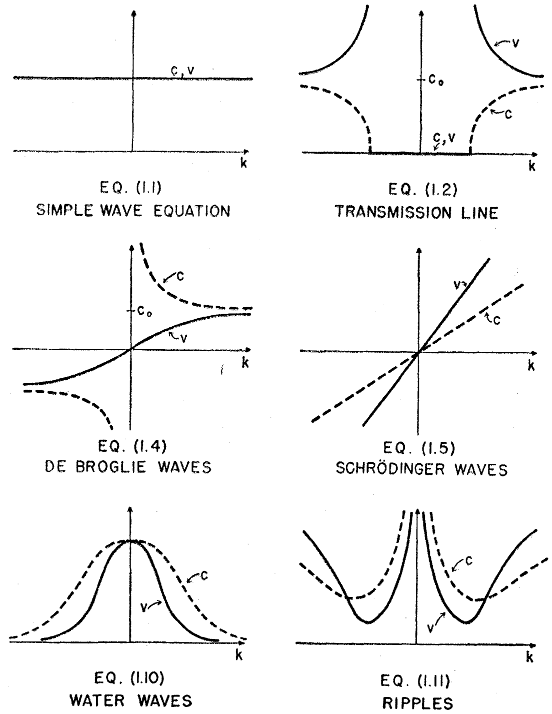

FIG. 2. Graphs of the phase and group velocities for some kinds of wave motion. In general, there will be one pair of these graphs for each Hamiltonian.

one set of H and D functions for some of the wave equations given above.

The real part of the expression on the right of Eq. (2.3) is

$$\exp\{-D(k)t\}\cos\{kx-H(k)t\}.$$

The roots of this quantity are given by

$$kx-H(k)t=(m+\tfrac{1}{2})\pi,\quad m=0,\ \pm 1,\ \cdots,$$

or

$$x-c(k)t=\text{constant},$$

where

$$c(k)=H(k)/k. \tag{2.4}$$

The roots, therefore, move with the velocity, $c(k)$, which is called the phase velocity of the waves. Figure 2 shows graphs of $c(k)$ for some of the wave equations given above.

3. THE FOURIER SOLUTION OF THE INITIAL VALUE PROBLEM

Consider the function

$$\psi=\frac{1}{2\pi}\sum\int_{-\infty}^{\infty}F(k)\times\exp\{i[kx-H(k)t]-D(k)t\}dk, \tag{3.1}$$

where the sum is to be extended over all of the Hamiltonian functions. The integrands are clearly solutions of the wave equation. If, moreover, the functions F_1, F_2, etc., which appear in the integrands are such that it is permissible to differentiate under the integral sign a sufficient number of times, ψ itself will satisfy the wave Eq. (1.9).

In the following sections, the multiplicity of Hamiltonians will be ignored as a non-essential complication. For the same reason, the logarithmic decrement will be set equal to zero; both of these complicating features of the Eq. (3.1) can readily be reintroduced at any stage. The further development of the theory will therefore be based on the single integral

$$\psi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)\times\exp\{i[kx-H(k)t]\}dk. \tag{3.2}$$

If the time, t, is set equal to zero, Eq. (3.2) becomes

$$\psi(x,0)\equiv\psi_0(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)\exp(ikx)dk. \tag{3.3}$$

An application of Fourier's theorem to this equation results in

$$F(k)=\int_{-\infty}^{\infty}\psi_0(x)\exp(-ikx)dx. \tag{3.4}$$

Thus, F is determined by the initial value of ψ, and conversely.**

In general, $F(k)$ will be a complex number, and may be written

$$F(k)=\phi(k)\exp[i\theta(k)], \tag{3.5}$$

where ϕ and θ are real. The function ϕ is called the (amplitude) spectrum of the wave, and θ its eikonal.[12,13] The further abbreviation

$$\Theta(k,x,t)=\theta(k)+kx-H(k)t \tag{3.6}$$

** In deriving Eq. (3.4), it has been explicitely assumed that there is only one Hamiltonian. If there are several, it becomes necessary to complicate the derivation by considering a certain number of the quantities $\partial\psi/\partial t$, $\partial^2\psi/\partial t^2$, $\cdots$; the functions F_1, $F_2\cdots$ are then determined by the initial values of these time derivatives in addition to the initial value of ψ itself.

[12] M. Born, *Optik* (Julius Springer Verlag, Berlin, 1933), Chapter II.

[13] C. Manneback, *Travaux de l'Assemblée Generale de l'Union Radio Scientifique Internationale* (London, 1934, published Brussels, 1935).

is convenient, and reduces Eq. (3.2) to the form

$$\psi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(k)\exp[i\Theta(k,x,t)]dk. \quad (3.7)$$

It will be noted that if $\psi_0(x)=\psi_0{}^*(x)$ (where the asterisk indicates the complex conjugate) Eq. (3.4) shows that F is real, and therefore that the eikonal vanishes identically. Solutions characterized by an eikonal which is constant, or at most a linear function of k, are of special importance in many applications; they are called centered waves. Waves whose initial form is characterized by the above type of symmetry are common in practical problems. It should be noted that centered waves do not necessarily originate from an instantaneous point source, although it will be shown that they have some properties in common with this fictional type.

4. THE METHOD OF STATIONARY PHASE, AND THE EIKONAL

The function Θ is called the phase of the integrand of Eq. (3.7). It should be noted that x and t are constant during the integration, while k is the variable. As k (and Θ) varies, the exponential factor will oscillate, with the period 2π in Θ. If, in any interval on the k axis, Θ increases by many times 2π, while ϕ remains sensibly constant, the contribution of this interval to the value of ψ will be disproportionately small compared to its length. In fact, it can be shown that under certain conditions, the value of the integral is determined almost entirely by the values of the integrand near the stationary points of the phase Θ. (See Section 15.)

These are the values of k that satisfy the

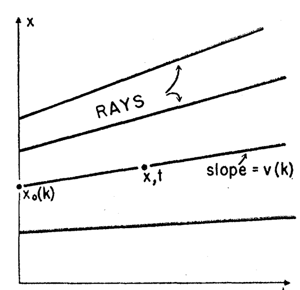

FIG. 3. The rays in the $x-t$ diagram. This diagram is the basis for the graphical method of finding the function $\kappa(x,t)$, described in the text.

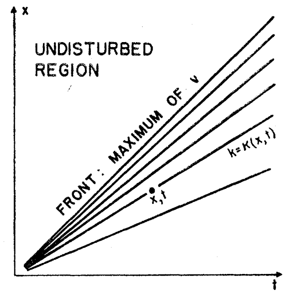

FIG. 4. The $x-t$ diagram for a centered wave, showing the front associated with a maximum of the group velocity.

equation

$$(\partial/\partial k)\Theta(k,x,t)=0. \quad (4.1)$$

In general, there will be several such values of k:

$$k=\kappa_1(x,t),\quad k=\kappa_2(x,t),\quad \cdots, \quad (4.2)$$

but for the present, it will be supposed that there is only one: $k=\kappa(x,t)$. Near this point, the approximate value of Θ is

$$\Theta(k,x,t)=S(x,t)-\tfrac{1}{2}R(x,t)(k-\kappa)^2, \quad (4.3)$$

where

$$S(x,t)=\Theta[\kappa(x,t),x,t],$$
$$R(x,t)=-\partial^2\Theta(\kappa,x,t)/\partial\kappa^2. \quad (4.4)$$

To a certain approximation, it is permissible to set

$$\phi(k)=\phi(\kappa).$$

Hence, Eq. (3.7) becomes, approximately

$$\psi(x,t)=\phi(\kappa)\exp[iS(x,t)]\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[-\tfrac{1}{2}iR(k-\kappa)^2]dk. \quad (4.5)$$

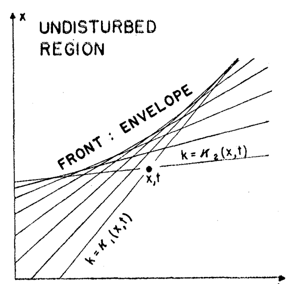

FIG. 5. The $x-t$ diagram for a general wave, showing the front and the two roots, κ_1 and κ_2, of the equation $\partial\Theta/\partial\kappa=0$, which are associated with an envelope of the family of rays.

The integral in Eq. (4.5) is the complete[14] Fresnel integral:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[-\tfrac{1}{2}iRu^2]du=(2\pi iR)^{-\frac{1}{2}}. \quad (4.6)$$

Hence, the approximate value of ψ is

$$\psi(x,t)=[2\pi iR]^{-\frac{1}{2}}\phi(\kappa)\exp[iS]. \quad (4.7)$$

The calculations necessary to estimate the error in this approximation are outlined in Section 15.

If there is more than one root of Eq. (4.1)—i.e., more than one point of stationary phase—the single term on the right of Eq. (4.7) must be replaced by a sum of terms of the same structure. Other methods of treating the case of two stationary points are discussed in Section 7.

If the logarithmic decrement is not zero, Eq. (4.7) is only slightly more complicated:

$$\psi(x,t)=[2\pi iR]^{-\frac{1}{2}}\phi(\kappa)\exp[-D(\kappa)t+iS]. \quad (4.8)$$

5. GROUP VELOCITY AND THE HAMILTON-JACOBI FUNCTION

Using Eq. (3.6), one finds that the condition for stationary phase (Eq. (4.1)) becomes

$$x=x_0(k)+v(k)t, \quad (5.1)$$

where

$$x_0=-d\theta/dk \quad (5.2)$$

and

$$v=dH/dk. \quad (5.3)$$

Equation (5.1) suggests that something starts at $x_0(k)$ at time $t=0$, and travels to x with the velocity $v(k)$. Obviously, this "something" is characterized by the wave number k; it is called, for reasons that will not become clearer below, the "group of waves" having that wave number. The initial coordinate, x_0, is called the origin of the group, and v the group velocity. Equations (5.2) and (5.3) relate these two quantities to the eikonal and the Hamiltonian, respectively. Figure 2 shows graphs of phase and group velocity for the various examples.

The Eq. (5.1) can be made the basis of a graphic solution for $\kappa(x, t)$, as is indicated in Fig. 3. It represents a family of straight lines—the rays—in the x, t plane. Each of these rays is characterized by a particular value of k, and intersects the x axis at $x_0(k)$. Its slope is $v(k)$. To determine $\kappa(x, t)$, one has only to note the value of k for the ray that passes through the point x, t of the diagram.

In the case of a centered wave, x_0 is independent of k—say equal to zero—and the rays all emanate from the origin of the x, t diagram, as shown in Fig. 4. In this case, κ is obtained by solving the equation

$$v(\kappa)=x/t$$

and κ is thus a function of the ratio x/t only.

In those cases where v has a maximum and the wave is centered, there will be a region into which rays penetrate. The ray of greatest speed is called the advancing front. In the same way, the case of a minimum of group velocity results in a receding front.

It is also possible that $v(k)$ has several extrema, in which case even centered waves will have a rather complicated ray diagram and there will be several fronts, some advancing and others receding. An interesting case of this sort has been discussed in detail by C. L. Pekeris.[15] In the following, the discussion will be much simplified by assuming that v has only one extremum and that there is only one front.

Figure 5 also shows a region of the x, t plane that is not reached by any rays. Since there are no real roots of Eq. (5.1) for values of x, t in this region, there will be no points of stationary phase, and the integral of Eq. (3.7) is zero to the present approximation. (See Section 15.) This approximate result is not valid too near the boundary of the undisturbed region, as will be shown in Sections 7 and 8.

It may also happen that even if v has no extremum, several rays pass through a single point of the diagram, as shown in Fig. 5. In this case, Eq. (5.1) will have several roots, $k=\kappa_1(x, t), \cdots$, and Eq. (4.7) will have several terms, as already noted. As is evident from an inspection of Fig. 5, multiple roots of Eq. (5.1) are accompanied by an envelope of the family of rays, and an undisturbed region beyond the envelope. Such fronts

[14] E. Jahnke and F. Emde, *Funktionentafeln mit Formeln und Kurven* (B. G. Teubner, Leipzig and Berlin, 1938; Dover Publications, New York, 1943), p. 35.

[15] C. L. Pekeris, "Theory of propagation of explosive sound in shallow water," Report of the Columbia University Division of War Research, No. 6.1-sr1131-1891, January 1945. (PB 31063).

may be either advancing or receding, but do not have a constant velocity; they will also be discussed further in Section 7.

Turning now to the function

$$S(x, t) = \Theta(\kappa, x, t) \\ = \theta(\kappa) + \kappa x - H(\kappa)t, \qquad (5.4)$$

which is known as the Hamilton-Jacobi function: for, on differentiating Eq. (5.4), one obtains

$$\partial S/\partial x = \kappa + [x - x_0(\kappa) - v(\kappa)t]\partial\kappa/\partial x,$$
$$\partial S/\partial t = -H(\kappa) + [x - x_0(\kappa) - v(\kappa)t]\partial\kappa/\partial t.$$

Because κ is a root of Eq. (5.1), it follows that

$$\partial S/\partial x = \kappa, \quad \partial S/\partial t = -H(\kappa), \qquad (5.5)$$

whence elimination of κ results in the Hamilton-Jacobi[16] equation

$$(\partial S/\partial t) + H(\partial S/\partial x) = 0. \qquad (5.6)$$

This is also variously known as the geometric wave equation, and as the "equation of the characteristics"[10] (not to be confused with the characteristic equation, Eq. (1.8)).

The Hamilton-Jacobi function enters into Eq. (4.7) in the factor

$$\exp(iS).$$

Equation (5.5) thus shows that, near x, t, the disturbance is approximately a sinusoidal wave of frequency $\nu = H(\kappa)$, and wave number κ. The individual wave crest thus moves with the phase velocity

$$(\partial x/\partial t)_S = \nu/\kappa = H(\kappa)/\kappa = c(\kappa) \qquad (5.7)$$

as it passes through the point x, t. The crests are not constantly associated with the same wave number; rather, the wave number is associated with the ray as has already been seen.

Since c is thus expressed as a function of x and t, the space-time locus of an individual wave crest can be obtained by solving the differential equation:

$$dx/dt = c[\kappa(x, t)]. \qquad (5.8)$$

It follows from the above that the solution of this equation will be a curve in the x, t plane, whose equation is $S(x, t) = \text{const}$. In general, this curve cuts across many rays. This result has a simple physical interpretation: the individual crest does not move with a constant velocity, nor does the separation between successive crests remain constant during their motion. This separation between successive crests is appropriately called the wave-length, λ, and varies from place to place and from time to time. For any given pair of crests, it is given approximately (but only approximately) by the equation

$$\lambda = 2\pi/\kappa(x, t), \qquad (5.9)$$

and thus varies as they move across the rays. This approximation is valid only when the value of κ does not change appreciably from one crest to the next.

The amplitude of the wave crests is (at least partially) determined by the spectrum amplitude $\phi(\kappa)$. If the attention be focused not on the motion of the individual wave crests, but on the motion of the point where the amplitude ϕ, or what is the same thing, the wave number κ, has a given value, another velocity will be obtained. This is

$$(\partial x/\partial t)_\phi = -(\partial\phi/\partial t)/(\partial\phi/\partial x), \\ = -(\partial\kappa/\partial t)/(\partial\kappa/\partial x).$$

But

$$\frac{\partial\kappa}{\partial t} = \frac{\partial^2 S}{\partial x\partial t} = -\frac{\partial}{\partial x}H(\kappa) = -v(\kappa)\frac{\partial\kappa}{\partial x};$$

hence,

$$(\partial x/\partial t)_\phi = v(\kappa). \qquad (5.10)$$

The solution of this equation is a straight line in the x, t diagram, as is obvious from Figs. 2 to 4: κ is constant along any ray, and therefore also ϕ. Moreover, the slope of the ray passing through x, t is $v(\kappa)$.

These considerations lead to the following description of the disturbance: the individual crests move with a variable velocity and amplitude. Both are determined by the ray which the crest is momentarily crossing; the wave-length and one factor, ϕ, in the expression for the amplitude are the same at all points of a given ray. The motion of the individual crests is along the curves $S = \text{const}$. The slope of any member of this family, at the point where it crosses the ray characterized by the wave number κ, is $c(\kappa)$. There are thus two families of curves in the x, t diagram: the rays, which are straight lines, and the crest paths, all of which intersect a given ray at the same angle.

[16] N. Born, *Atommechanik* (Julius Springer Verlag, Leipzig, 1925).

If several rays pass through a given point of the x, t diagram, there will be several trains of waves passing through it simultaneously, each moving according to its own appropriate laws. The interference of these various trains may result in complicated patterns. This will be considered in more detail in Section 7.

Another consequence of Eq. (5.5) is that

$$\partial\kappa/\partial t=\partial^2 S/\partial x\partial t=-(\partial/\partial x)H(\kappa)=-\partial\nu/\partial x$$

or

$$(\partial\kappa/\partial t)+(\partial\nu/\partial x)=0, \tag{5.11}$$

which is an equation derived by Rossby[17] and Munk[18] from elementary considerations concerning the motion of wave crests.

Finally, it should be noted that $S(x, t)$ satisfies the Hamilton-Jacobi partial differential equation, no matter what form the eikonal $\theta(k)$ may be given, provided only that κ is a root of Eq. (5.1). Thus, a by-product of these considerations is a graphical method of solving partial differential equations of the first-order and constant coefficients.

6. THE RESOLUTION OF THE WAVES INTO A SPECTRUM

Equation (4.7) shows that the amplitude of the waves at x and t is determined directly by the spectrum ϕ and by the function R, but only indirectly by the initial value of the wave function ψ. Obviously, this cannot be true for all values of the time. At instants immediately following $t=0$, the function $\psi(x, t)$ must differ only slightly from its initial value $\psi_0(x)$. This immediately suggests that the approximate value of ψ yielded by the method of stationary phase will be accurate only for large values of t.

Physically, the reason for this is to be found in the different velocity $v(k)$ with which the various Fourier components are propagated. When it is necessary to consider many components as being present at a given time and place, the extremum of the phase function Θ will be very shallow and the above approximation will not be good. Only when the differing velocity of the Fourier components has dispersed the disturbance into a well-resolved spectrum will the above approximation be an accurate one.

The mathematics of this physical argument is best approached by a study of the function R. Using Eqs. (3.6), (4.4), (5.2), and (5.3), it follows that

$$R=-\partial^2\Theta/\partial\kappa^2,$$
$$=(dx_0/d\kappa)+t(dv/d\kappa), \tag{6.1}$$
$$=(\partial x/\partial\kappa)_t, \tag{6.2}$$
$$=1/(\partial\kappa/\partial x)_t. \tag{6.3}$$

Of these three alternative forms, Eq. (6.2) is the most enlightening. Referring to Fig. 4, it is easily seen that $\partial x/\partial\kappa$ will be large where the rays are widely spaced (for given increments of k or κ) and small where they are poorly resolved. Since the amplitude is inversely proportional to $R^{\frac{1}{2}}$, it follows that the wave disturbance is "spread thin" and has a small amplitude at those places where the space occupied by a given spectral interval is large. The function R may be appropriately called the resolution of the wave spectrum.

Equation (6.1) may also be written

$$R=R_0+\alpha t, \tag{6.4}$$

where

$$\alpha=dv(\kappa)/d\kappa$$
$$=d^2H(\kappa)/d\kappa^2 \tag{6.5}$$

will be recognized as the coefficient of dispersion of the medium. Since α and R_0 are functions of κ only, they will both be constant along a given ray. Thus, the resolution of a given part of the spectrum increases linearly with time from an initial value, at the rate α. For sufficiently large values of t, therefore, $|R|$ must always be great; for small values of t, its magnitude will depend on R_0. These considerations are also obvious from a consideration of the rays in the x, t diagram.

7. WAVE FRONTS

Since α and R_0 may have opposite signs, there may come a time when $R=0$ for a given ray. Reference to Fig. 5 and the interpretation $R=\partial x/\partial\kappa$, shows that this will occur at the envelope of the family of rays, if there is such an envelope. Equation (4.7) is clearly not a good

[17] C. G. Rossby, *On the Propagation of Frequencies and Energy in Certain Types of Oceanic and Atmospheric Waves*, J. Meteorol. 2, No. 4, 187 (1945).

[18] W. H. Munk, *Increase in the Period of Waves Traveling Over Large Distances, with Applications to Tsunamis, Swell and Seismic Surface Waves*, Trans. Am. Geophys. Union, April 1947.

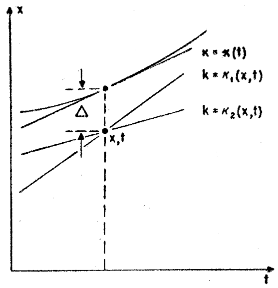

FIG. 6. Notation for the extrema and the point of inflection of Θ, which are associated with an envelope (cf. Figs. 5 and 7).

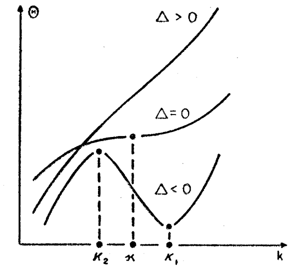

FIG. 7. Schematic graphs of Θ as a function of k for various values of Δ.

approximation to ψ at the wave front, since it yields the absurd value $\psi=\infty$ when $R=0$. It thus becomes necessary to develop a more suitable approximation for this region.

The front is a curve in the x, t plane which separates it into two regions: one is the disturbed region, in which at least two rays pass through each point, the other is the undisturbed region, in which there are no rays. This is the typical description, but more elaborate cases are possible. Only this typical case will be treated here; for the moment it will also be supposed that, at the wave front, κ is not infinite.

The details of the situation are illustrated in Figs. 6 and 7. Let Δ be the distance from the momentary position of the front to the point x, t, counted positive when x is in the undisturbed region. Figure 7 shows schematic graphs of the phase function Θ for $\Delta>0$, $\Delta=0$, and $\Delta<0$. When $\Delta>0$, no ray passes through x, t, and correspondingly, the graph of Θ has no stationary point. When $\Delta<0$, two rays pass through x, t, and Θ, correspondingly, has one maximum and one minimum at $k=\kappa_1$ and $k=\kappa_2$. For $\Delta=0$, Θ has a minimax for $k=K$.

In the previous sections, Θ has been approximated by a quadratic in k, in the neighborhood of its stationary point. Figure 7 makes it obvious that for small values of Δ it will be better to approximate it as a cubic in k.

Let K be the value of k for which Θ has a point of inflection, so that K is a root of

$$-(\partial^2\Theta/\partial k^2)=R_0(k)+\alpha(k)t=0. \qquad (7.1)$$

From this equation, or from Fig. 6, it is clear that K is a function of t only. The cubic approximation to Θ is then

$$\begin{aligned}\Theta(k,x,t)&=\Theta(K,x,t)+(k-K)(\partial\Theta/\partial K)\\&\quad+\tfrac{1}{3}!(k-K)^3(\partial^3\Theta/\partial K^3),\\&=\theta(K)+Kx-H(K)t\\&\quad+(k-K)[x-x_0(K)-v(K)t]\\&\quad-\tfrac{1}{3}!(k-K)[R_0'(K)+\alpha'(K)t]\\&=S_0(x,t)+(k-K)\Delta(x,t)\\&\quad+\tfrac{1}{3}(k-K)^3\delta^3(t),\end{aligned} \qquad (7.2)$$

where S_0, Δ, and δ are obvious abbreviations. It will be noted that $\Delta(x,t)$ is the distance of x from the momentary position of the wave front as already defined. S_0 is formally similar to the Hamilton-Jacobi function, but does not satisfy the same partial differential equation. The real quantity

$$\delta(t)=-\{\tfrac{1}{2}[R_0'(K)+\alpha'(K)t]\}^{\frac{1}{3}} \qquad (7.3)$$

is a scale factor which, it will be seen, determines the steepness of the front.

Arguments similar to those which lead to Eq. (4.5) now lead to the approximate expression

$$\psi(x,t)=\phi(K)\exp[iS_0]\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{i[(k-K)\Delta+\tfrac{1}{3}(k-K)^3\delta^3]\}dk. \qquad (7.4)$$

The integral which appears in this equation is essentially Airy's integral:

$$\mathrm{Ai}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[i(xu+\tfrac{1}{3}u^3)]du, \qquad (7.5)$$

so that Eq. (7.4) becomes

$$\psi(x,t)=(\phi/\delta)\mathrm{Ai}(\Delta/\delta)\exp(iS_0). \qquad (7.6)$$

The Airy function can be evaluated[19] in terms of Bessel functions of order $\frac{1}{3}$. Another definition of the function has been discussed by Kramers[20] in connection with the "WKB method" without apparently noting the identity of the function with Airy's. For sufficiently large positive values of ξ, the approximations

$$\mathrm{Ai}(\xi)=\tfrac{1}{2}\pi^{-\frac{1}{2}}\frac{\exp(-\frac{2}{3}\xi^{\frac{3}{2}})}{\xi^{\frac{1}{4}}}, \tag{7.7}$$

$$\mathrm{Ai}(-\xi)=\tfrac{1}{2}\pi^{-\frac{1}{2}}\frac{\cos(\frac{2}{3}\xi^{\frac{3}{2}}-\frac{1}{4}\pi)}{\xi^{\frac{1}{4}}}, \tag{7.8}$$

are valid. Figure 8 has been prepared using these equations to supplement a recently published table[21] of this function for small absolute values of the argument; ordinates for values of $x<-2.4$ may be considerably in error because of the approximate nature of Eq. (7.8). Eight-place tables have been prepared by the BAAS and are being published as this is written.[22]

8. WAVE FRONTS OF CONSTANT VELOCITY, FIRST KIND

It is clear from Fig. 5 that the envelope of the family of rays will be, in general, a curve, so that the velocity of the wave front will not be constant. There is one limiting case, however, in which the envelope becomes a straight line and the wave front moves with constant velocity. This arises when the group velocity has an extremum for some finite value of k—say k_0—and the wave is centered. In the case of water waves governed by Eq. (1.10), the velocity is a maximum for $k_0=0$. Ripples governed by Eq. (1.11) have a minimum group velocity for k_0 given by Eq. (8.9) below.

For a centered wave

$$\Theta=kx-H(k)t, \tag{8.1}$$

and, therefore,

$$R=\alpha(k)t, \tag{8.2}$$

so that $R=0$ only at $t=0$, or at some value of k for which the dispersion $\alpha=dv/dk=0$. The former case is to be considered in Section 10, so that only the latter will be treated here. Letting $c_0=c(k_0)$, $v_0=v(k_0)$, etc., it is seen that

$$S_0=k_0(x-c_0t), \tag{8.3}$$

$$\Delta=x-v_0t, \tag{8.4}$$

$$\delta=-\{\tfrac{1}{2}\alpha_0' t\}^{\frac{1}{3}}. \tag{8.5}$$

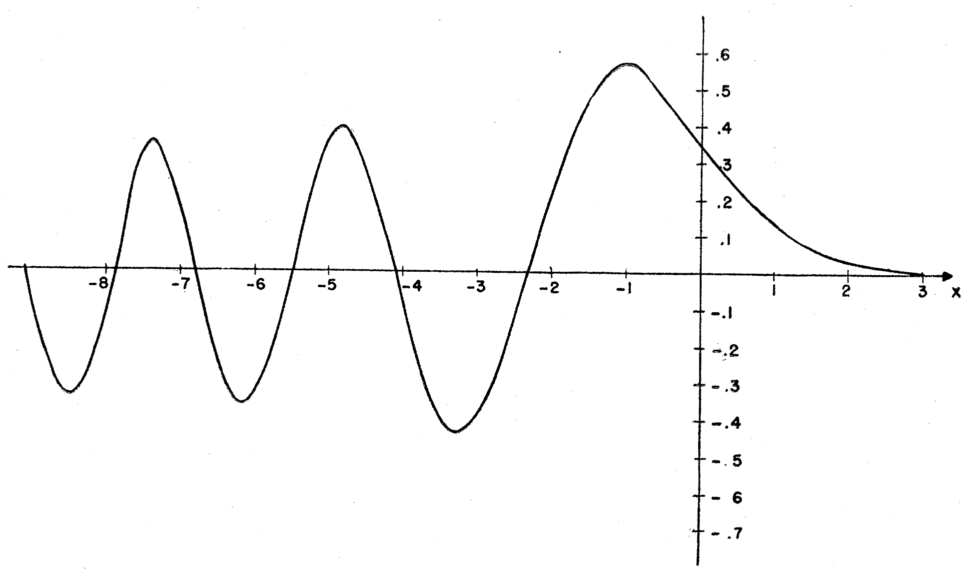

FIG. 8. Graph of the Airy function, Ai(x), defined by Eq. (7.5).

[19] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1944), pp. 188, 201, 659.
[20] H. A. Kramers, Zeits. f. Physik **39**, 828 (1926).
[21] P. M. Woodward, A. M. Woodward, R. Hensman, H. Davies, and N. Gamble, Phil. Mag. **37**, 236 (1946).
[22] *Mathematical Tables*, Volume B (to be published for the British Association for the Advancement of Science at the University Press, Cambridge).

Hence the disturbance in the neighborhood of the wave front will be given by

$$\psi=(\phi_0/\delta)\mathrm{Ai}[(x-v_0t)/\delta]\cdot\exp[ik_0(x-c_0t)]. \quad (8.6)$$

The significant point to be noted is that the scale factor δ increases as the cube root of the time.

This is of special significance in case $k_0=0$: for then the "wave-length" of the disturbance will be determined by the oscillations of the Airy integral, and will be determined empirically by measuring the interval between its maxima. The first two of these intervals are

$$L_1=3.8\delta,$$
$$L_2=2.6\delta.$$

Thus the "wave-length" near the front will increase as the cube root of t. In the case of long surface waves, Eq. (1.9) yields

$$\alpha_0'=-g^{\frac{1}{2}}h^{5/2}=-v_0h^2,$$

so that

$$\delta=\{\tfrac{1}{2}h^2v_0t\}^{\frac{1}{3}}>0, \quad \text{for } t>0. \quad (8.7)$$

The situation near this front is very peculiar. It may be described as a carrier wave which is amplitude-modulated. However, contrary to the customary case, the carrier has a longer (infinitely longer!) wave-length than the modulation. Consequently, the empirically determined "wave-length" bears no relation whatever to the spectrum of the disturbance. Such wave fronts are of considerable importance as they constitute the seismic sea waves, commonly called "tidal waves." Observations of these are in qualitative,[18] and possibly quantitative, agreement with the present calculations.

The case of ripples governed by Eq. (1.11) is somewhat more normal. For these, the minimum of the group velocity is at

$$k_0=0.393(g\rho/T)^{\frac{1}{2}}. \quad (8.8)$$

The corresponding velocities are

$$v_0=1.086(gT/\rho)^{\frac{1}{4}}, \quad (8.9)$$

$$c_0=1.713(gT/\rho)^{\frac{1}{4}}, \quad (8.10)$$

while $\delta<0$ (receding front), where

$$-k_0\delta=0.768(gt/v_0)^{\frac{1}{3}}. \quad (8.11)$$

When t is much greater than v_0/g, the wave-length L_1 will be greater than $2\pi/k_0$, and the modulation will have a greater wave-length than the carrier. However, for small values of t the situation is similar to that of the gravitational waves. The question of the extent to which these formulae are applicable for small t will be considered in Section 10.

9. WAVE FRONTS OF CONSTANT VELOCITY, SECOND KIND

While the ray diagram of Fig. 4 for centered waves applies to the Eqs. (1.2) and (1.4), the previous analytical procedures do not. This is because the limiting value of the group velocity is reached for $k=\infty$, and it is not possible to use Taylor's series in this neighborhood. Instead of a cubic approximation to Θ, the expansion

$$H(k)=c_0k+b/k+\cdots \quad (9.1)$$

is valid in this neighborhood. In the case of a centered wave, $\theta=0$, so that approximately

$$\psi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(k)\times\exp[ik(x-c_0t)-ibt/k]dk \quad (9.2)$$

when $|x-c_0t|$ is sufficiently small.

The value of this integral depends strongly on the behavior of ϕ for large values of k. Only the special case

$$\phi(k)\rightarrow(ik)^{-n-1} \quad (9.3)$$

will be considered. Then the approximation

$$\psi(x,t)=\frac{1}{2\pi}\int(ik)^{-n-1}\times\exp[ik(x-c_0t)-ibt/k]dk \quad (9.4)$$

will be valid. However, because $(ik)^{-n-1}$ becomes infinite for $k=0$, the path of integration must be deformed into the complex k plane, so as to pass above the origin.

The integral of Eq. (9.4) can then be evaluated in terms of Bessel functions. The substitution

$$u=k/bt,$$
$$\xi=bt(x-c_0t)$$

reduces Eq. (9.4) to

$$\psi=(bt)^{-n}f(\xi), \quad (9.5)$$

where

$$f(\xi)=\frac{1}{2\pi}\int(iu)^{-n-1}\exp[i(u\xi-1/u)]du. \quad (9.6)$$

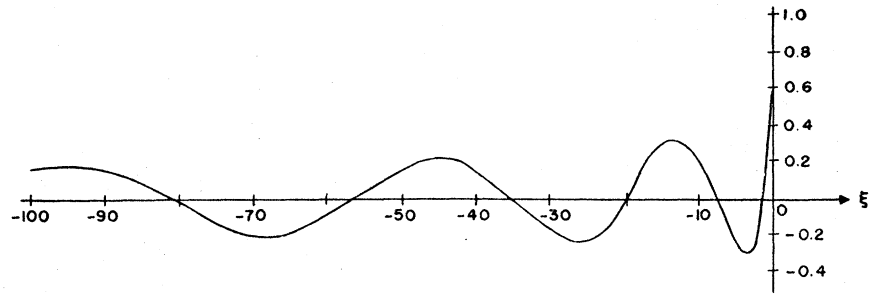

FIG. 9. Graph of the function $J_0(2\xi^{\frac{1}{2}})$.

The rigorous formula

$$f(\xi)=0, \quad \text{where } \xi>0 \tag{9.7}$$

can be proven readily. For negative values of the argument it can be shown that

$$f(-\xi)=\xi^{n/2}J_n[2(\xi)^{\frac{1}{2}}], \tag{9.8}$$

where J_n is Bessel's function of order n. Graphs of these functions for $n=0$ and 1 are shown in Figs. 9 and 10.

10. UNRESOLVED WAVES

There are circumstances under which none of the methods described above will lead to a suitable approximation. The precise delineation of these conditions will be omitted, but by now their general nature should be fairly clear. In each of the three approximate methods, it has been assumed that $\phi(k)$ may be treated as a constant, while $\Theta(k, x, t)$ is approximated by a polynomial in k. In other words, it has been assumed that the principal variations in the integrand are caused by Θ, those caused by ϕ being either non-existent or else canceling out in the end result, so that they may be ignored from the beginning.

In the present section, it will be assumed that $\phi(k)$ has a sharp maximum at $k=k_0$. In this case, it is fairly certain that its variation will dominate over that of Θ.

It is convenient to return to Eq. (3.2),

$$\psi(x, t)=\tfrac{1}{2}\pi\int_{-\infty}^{\infty} F(k)\exp[ikx-iH(k)t]dk,$$

or the following calculation: since $\phi=|F|$ has a maximum at $k=k_0$, the Hamiltonian may profitably be expanded by Taylor's theorem near this point:

$$\begin{aligned}H(k)&=H(k_0)+H'(k_0)(k-k_0)\\&\quad+\tfrac{1}{2}H''(k_0)(k-k_0)^2+\cdots,\\&=c_0k_0+v_0(k-k_0)+\tfrac{1}{2}\alpha_0(k-k_0)^2+\cdots,\end{aligned} \tag{10.1}$$

where c_0, v_0, and α_0 have meanings that is obvious from the previous discussions. For the first, it is convenient to neglect the quadratic term; doing this and substituting in Eq. (3.2),

$$\psi(x, t)=\exp[ik_0t(v_0-c_0)]\cdot\tfrac{1}{2}\pi\int_{-\infty}^{\infty} F(k)\exp[ik(x-v_0t)]dk. \tag{10.2}$$

Comparing this with Eq. (3.3),

$$\psi_0(x)=\tfrac{1}{2}\pi\int_{-\infty}^{\infty} F(k)\exp(ikx)dk,$$

it is seen that

$$\psi(x, t)=\psi_0(x-v_0t)\exp[ik_0t(v_0-c_0)]. \tag{10.3}$$

This remarkably simple result can be put in a more familiar form, if we write

$$\psi_0(x)=\Psi(x)\exp ik_0x, \tag{10.4}$$

where, by definition, Ψ is the modulation, and $\exp ik_0x$ is the carrier wave. It can be shown that (when the maximum of ϕ is sufficiently sharp) Ψ

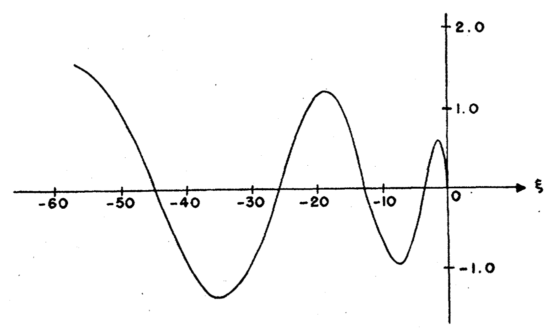

FIG. 10. Graph of the function $\xi^{\frac{1}{2}}J_1(2\xi^{\frac{1}{2}})$.

is a slowly varying function of x in comparison to $\exp ik_0x$. Then Eq. (10.3) becomes

$$\psi(x, t) = \Psi(x - v_0t) \exp[ik_0(x - c_0t)]. \quad (10.5)$$

Thus, under these conditions, we recover the well-known result that the carrier is propagated with the phase velocity, while the modulation is propagated with the group velocity and without distortion.

Equation (10.5) obviously results because the dispersion, α_0, has been neglected in this calculation. If the quadratic terms are included in Eq. (10.1), Eq. (3.2) becomes ($\xi = x - v_0t$):

$$\begin{aligned}\psi(x, t) &= \exp[ik_0(x - c_0t)] \cdot \tfrac{1}{2}\pi \int_{-\infty}^{\infty} F(k) \\ &\quad \times \exp\{i(k - k_0)\xi \\ &\quad - (i\alpha_0/2)t(k - k_0)^2\} dk \\ &= \Psi(\xi, t) \exp[ik_0(x - c_0t)]. \quad (10.6)\end{aligned}$$

The function $\Psi(\xi, t)$ remains to be studied more carefully; clearly

$$\Psi(\xi, 0) = \psi_0(\xi), \quad (10.7)$$

but its changes with time must be investigated.

A change of variable reduces Ψ to the form

$$\begin{aligned}\Psi(\xi, t) &= \tfrac{1}{2}\pi \int_{-\infty}^{\infty} F(k + k_0) \\ &\quad \times \exp[ik\xi - (i\alpha_0/2)tk^2] dk, \quad (10.8)\end{aligned}$$

and the general principles of Section 3 then show that it satisfies a wave equation in ξ and t, with the Hamiltonian $\frac{1}{2}\alpha_0k^2$. This uniquely determines the differential equation

$$\partial\Psi/\partial t = (i\alpha_0/2)(\partial^2\Psi/\partial\xi^2), \quad (10.9)$$

which is, except for the replacement of x by ξ and α by α_0, identical with Schrödinger's equation (Eq. (1.5)).

Equations (10.7) and (10.9) serve to determine Ψ completely, but the present assumptions concerning F make it necessary to find a new method of calculating it approximately. In case Ψ is an analytic function of t, the Taylor expansion,

$$\begin{aligned}\Psi(\xi, t) &= \Psi(\xi, 0) + t(\partial\Psi/\partial t)_{t=0} \\ &\quad + \tfrac{1}{2}t^2(\partial^2\Psi/\partial t^2)_{t=0} + \cdots,\end{aligned}$$

may be used. But, because of Eq. (10.9),

$$(\partial^n\Psi/\partial t^n)_{t=0} = (i\alpha_0/2)^n(\partial^{2n}\psi_0/\partial\xi^{2n}).$$

Hence, the expansion

$$\begin{aligned}\Psi(\xi, t) &= \psi_0(\xi) + (i\alpha_0t/2)(d^2\psi_0/d\xi^2) \\ &\quad + \tfrac{1}{2}(i\alpha_0t/2)^2(d^4\psi_0/d\xi^4) + \cdots \quad (10.10)\end{aligned}$$

is obtained. For sufficiently small values of α_0t, it often converges very rapidly; for large values of α_0t, the approximations on which it is based fail, and it is useless to attempt to transform it into a more rapidly convergent series. Fortunately, it is clear that for very large values of α_0t the variation of Θ will again dominate over that of ϕ, and one of the previous approximations can be used.

Consequently, Eq. (10.10) should be considered merely as indicating how the process of distortion begins. The different phase velocities of the Fourier components do not have appreciable effect until the ratio

$$\alpha_0t(d^2\psi_0/d\xi^2)/\psi_0$$

becomes appreciable. When it is comparable to unity, the original modulation will have been distorted, probably beyond recognition. When this ratio is much greater than unity, the divergence of the rays in the x, t diagram will have produced a complete resolution of the wave into its periodic components.

The considerations of this section have neglected the possible effects of dissipation. These can be included approximately by multiplying Eq. (10.6) by the factor $\exp[-D(k_0)t]$.

11. THE BOUNDARY VALUE PROBLEM

In the general wave equation

$$W\left(\frac{1}{i}\frac{\partial}{\partial x}, -\frac{1}{i}\frac{\partial}{\partial t}\right)\psi = 0, \quad (1.8)$$

the variables x and $-t$ enter in a symmetric manner. This symmetry is lost in Fourier's solution of the initial value problem:

$$\psi(x, t) = \tfrac{1}{2}\pi \int_{-\infty}^{\infty} F(k) \exp[i(kx - H(k)t)] dk, \quad (3.2)$$

$$F(k) = \int_{-\infty}^{\infty} \psi_0(x) \exp(-ikx) dx. \quad (3.3)$$

This problem can be formulated as follows:

Initial Value Problem: Given that at the initial instant $t = 0$, $\psi = \psi_0(x)$, and that it satisfies Eq. (1.8) for all later times, find $\psi(x, t)$ for all $t > 0$.

Because of the symmetry of Eq. (1.8), there is a second problem, which is of fundamental importance in transmission line theory. It reads:

Boundary Value Problem: Given that at the boundary point $x=0$, $\psi=\psi_1(t)$, and that it satisfies Eq. (1.8) for all points to the right of the boundary, find $\psi(x, t)$ for all $x>0$.

Having solved the initial value problem, the boundary value problem presents no new mathematical difficulties. Its approximate solutions can be obtained from the foregoing pages by replacing x by $-t$ and conversely.

However, there are certain consequences of this procedure that do not appear to have been noticed,*** so that it is profitable to examine the first stages of the boundary value problem in some detail. The symmetry of x and $-t$ has not yet been lost in Eq. (1.6): the special solutions

$$\psi=\exp(ikx-int), \tag{1.6}$$

$$W(k, n)=0, \tag{1.7}$$

are equally useful in the boundary value problem.

The next step in the initial value problem was the solution of this last equation for n as a function of k. In the boundary value problem, this is reversed, and k is expressed as a function of n. If there are no dissipative terms, it follows that

$$k=G(n) \tag{11.1}$$

and that G will have the same central importance as the Hamiltonian function had.

When there is dissipation, this step would be preceded by replacing

$$W(k, n-ia)=0 \tag{2.1}$$

by the equation

$$W(k+ib, n)=0, \tag{11.2}$$

and solving for the real quantities k and b as functions of the real variable n:

$$k=G(n), \quad b=A(n).$$

The function $A(n)$ is the attenuation and is analogous to the logarithmic decrement $D(k)$ of the initial value problem.

The case $A(n)=0$, $D(k)=0$ is of such fundamental importance that physicists have studied it almost to the exclusion of the more general one.

*** They were called to the writer's attention by Dr. R. W. Raitt, after the preceding sections had been written.

This case may be illustrated by

$$W(k, n)=-k^2+n^2/c_0^2$$

or

$$H(k)=c_0k, \quad G(n)=n/c_0.$$

Then G is the inverse function to H, and G and H have the same graph in the k, n plane. In this (and in more general cases of the same sort)

$$c=H(k)/k=n/G(n),$$

and

$$v=dH/dk=1/(dG/dn),$$

always provided that $W(k, n)=0$.

Physically, these last two results mean that the same phase and group velocities govern the propagation of the waves, regardless of whether one solves the initial or the boundary value problems. This numerical equality of two quantities with mathematically different definitions is the justification for ascribing a physical reality to them and calling them "it" rather than "they." Moreover, it follows that the family of rays is the same for the two problems. In the one, the rays are specified by their intercept on the x axis, in the other, by their intercept on the t axis, but the ray passing through a given point has the same slope.

All of this is altered when there are dissipative terms. In the k, n diagram, the equations $n=H(k)$ and $k=G(n)$ have different graphs; in the x, t diagram there are two families of rays. This casts serious doubt on the validity of the previous results in this case, for a given solution of the wave equation may be uniquely specified either by its initial or by its boundary values. The method of stationary phase will yield different approximations in the two cases; in general, at least one of these will be poor.

12. FORMAL THEORY OF MULTIPLE HAMILTONIANS†

The difficulty just discussed is partly inherent in the differential equations, but results mainly from the use of the approximate method of stationary phase, as will be shown in the next section. The formal aspects of the problem are related to the multiplicity of Hamiltonians encountered in solving a general partial differ-

† This section contains unpublished material supplied by Professor Marcel Riesz of the University of Lund; it is a pleasure to acknowledge his kind permission to use it in this review.

ential equation. This multiplicity was noted explicitly by Dirac,[23] and led to the prediction of the positron. In terms of the ray diagram, it leads to several families of rays in the x, t diagram, each given by Eqs. (5.1) to (5.3) with their appropriate Hamiltonians. Physically, it leads to the appearance of several superposed trains of waves, although the converse is not true. A single Hamiltonian can also give rise to several trains if several rays pass through a single point (cf. Fig. 5).

The mathematical theory of these matters is radically different in the cases of conservative and dissipative media. The former will be considered first. Then the characteristic equation, Eq. (1.8), may be factored into

$$W(k, n) = P \prod_{\alpha=1}^{N} [H_\alpha(k) - n], \qquad (12.1)$$

where the Hamiltonians H_α are distinct real functions of k. The quantity P may be a function of k and/or n, but, by definition, has no roots other than trivial ones. Often W will be a polynomial, of degree N in n and degree K in k; then P will be a constant. Only this case will be discussed here.

There will then be N sets of equations similar to Eqs. (5.1) to (5.3), and any one of them may be written

$$\frac{dx}{dt} = \frac{\partial H_\beta(k)}{\partial k}, \quad \frac{dk}{dt} = \frac{\partial H_\beta(k)}{\partial x} = 0. \qquad (12.2)$$

Each of these sets gives rise to a family of rays, so that there are N of these.

Now consider the canonical equations

$$\begin{aligned} dx/d\tau &= \partial W/\partial k, & dk/d\tau &= -\partial W/\partial x = 0, \\ dn/d\tau &= \partial W/\partial t = 0, & dt/d\tau &= -\partial W/\partial n, \end{aligned} \qquad (12.3)$$

whose interpretation will shortly appear. The function W will be called the comprehensive Hamiltonian. They have the integral $W(k, n) =$ constant, and this constant may have any real value whatever; comparing this with Eq. (12.1) it is seen that only those solutions for which it is zero (the null lines of the comprehensive Hamiltonian) will be of interest. It will be shown that the family of null lines consists of all N families of rays, and no others.

The first and last of the Eqs. (12.3) may be written

$$\begin{aligned} \frac{dx}{d\tau} &= \sum_{\alpha=1}^{N} P_\alpha \frac{\partial H_\alpha}{\partial k}, \\ \frac{dt}{d\tau} &= \sum_{\alpha=1}^{N} P_\alpha, \end{aligned} \qquad (12.4)$$

where

$$P_\alpha(kn) = W(k, n)/[H_\alpha(k, n) - n]. \qquad (12.5)$$

When Eq. (12.1) is satisfied, one and only one of the factors $H_\alpha - n$ will vanish; if this is the one for $\alpha = \beta$, all of the P_α except P_β will vanish, and Eqs. (12.4) reduce to

$$\begin{aligned} dx/d\tau &= P_\beta(\partial H_\beta/\partial k), \\ dt/d\tau &= P_\beta; \end{aligned} \qquad (12.6)$$

eliminating the parameter τ, these reduce exactly to Eq. (12.2).

Thus far, the discussion has been directed toward the ray theory appropriate to the initial value problem. Turning now to the boundary value problem in a conservative medium, Eq. (11.1) implies that Eq. (1.8) can also be factored into

$$W(k, n) = Q \sum_{\alpha=1}^{K} [G_\alpha(n) - k], \qquad (12.7)$$

where K is not necessarily equal to N, and Q will again be constant. Proceeding as before, the quantities

$$Q_\beta = W(k, n)/[G_\beta(n) - k] \qquad (12.8)$$

are defined. Because of Eq. (12.1), all but one of them vanish, and hence Eqs. (12.3) reduce to

$$\begin{aligned} dx/d\tau &= Q_\beta(k, n), \\ dt/d\tau &= Q_\beta(k, n)(\partial G_\beta/\partial n), \end{aligned} \qquad (12.9)$$

or, finally to

$$dt/dx = \partial G_\beta/\partial n, \qquad (12.10)$$

in keeping with the alternative formula for the group velocity.

It has thus been shown that the multiplicity of ray families introduced in connection with the Fourier solutions are all determined by a single set of Hamiltonian equations, Eqs. (12.1) and (12.3). But this is the case only for a conservative

[23] P. A. M. Dirac, *Quantum Mechanics* (Clarendon Press, Oxford, 1935), second edition, p. 252.

medium; for a dissipative medium, $W(k, n)$ assumes complex values for real values of k, n, and cannot be resolved into real factors.†† Formally, Eqs. (12.1), (12.3), and (12.10) may still be set up, but their solutions will lead to complex values of x and t. It is not clear that their solutions have any relevance to the Fourier solution of the wave equation.

Rather, this solution leads to the separation of the complex Hamiltonian into H_α and D_α or into G_α and A_α as discussed in Sections 2 and 11. While this is decidedly relevant to the Fourier solution, it makes Eqs. (12.2), (12.3), and (12.10) entirely distinct and unrelated. The curves (in the $x-t$ diagram) determined by each set of equations are different, and those determined by Eq. (12.3) have complex values of the coordinates.

13. ARBITRARINESS OF THE METHOD OF STATIONARY PHASE

Apart from the formal difficulties of treating dissipative media in the same manner as conservative, there are certain difficulties common to the two cases, which are inherent in the method of stationary phase. The mathematical discussion can best be based on the integral

$$I=\int_a^b \exp[w(z)]dz, \tag{13.1}$$

where w is a function of the complex variable $z=x+iy$, analytic at all points of a region, Γ,

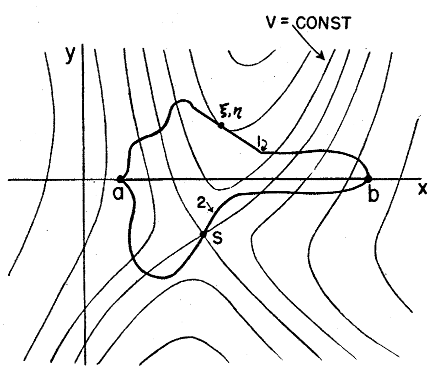

FIG. 11. Contours for the methods of stationary phase and steepest descent.

†† Something similar may occur even for a non-dissipative medium. It may happen that one or more of the functions H or G becomes imaginary or complex for certain real values of its variable.

that includes the segment ab of the x axis. Setting $w(z)=u(x, y)+iv(x, y)$ where u and v are real, the function w may be graphically represented by plotting the contours $v=\text{const.}$ as shown by the light curves on Fig. 11. The contours $u=\text{const.}$ intersect the plotted curves orthogonally.

In Eq. (13.1), the path of integration may be along the real axis, but the value of I is the same for any path that runs from a to b and remains entirely within the region Γ. The method of stationary phase can be applied to any one of these paths, and will ordinarily result in a different *approximate* expression for I in each case, despite the fact that the *exact* value of I is independent of the path.

The reason why the difficulty did not obtrude itself in the case of waves in a non-dissipative medium now becomes clear: then the initial value and boundary value problems lead to integrals that can be transformed into each other by a real change of variable. In the dissipative case, the necessary change of variable becomes complex, and must be accompanied by a deformation of the path of integration.

It is of interest to investigate these various approximate values of I more closely. Let a general path of integration be specified by giving x and y as functions of s, the distance along the curve. Then the angle, $\alpha(s)$, between the tangent to the curve and the x axis is given by

$$dx/ds=\cos\alpha,\quad dy/ds=\sin\alpha,\\ dz/ds=\exp(i\alpha), \tag{13.2}$$

and hence

$$I=\int_{s_a}^{s_b} \exp[u+i(v+\alpha)]ds. \tag{13.3}$$

The phase of the integrand is $v+\alpha$, and its stationary point (if any) is determined by the equation

$$\partial v/\partial x \cos\alpha+\partial v/\partial y \sin\alpha+d\alpha/ds=0. \tag{13.4}$$

The geometric interpretation of this equation is simplest if, at the stationary point, the path is a straight line: then it states that the path is tangent to that v-contour which passes through the stationary point. This is shown by the path 1 in Fig. 11; the result could also have been obtained by inspection.

Let $x=\xi$, $y=\eta$, $s=s_0$ be the stationary point in this path; then the expression of the phase near this point will be

$$v=v(\xi,\eta)+\alpha_0-(s-s_0)^2/2r^2+\cdots, \quad (13.5)$$

where, as a consequence of the Cauchy-Riemann equations,

$$r(\xi,\eta)=[(\partial^2u/\partial\xi\partial\eta)\cos 2\alpha_0 - (\partial^2v/\partial\xi\partial\eta)\sin 2\alpha_0]^{-\frac{1}{2}}. \quad (13.6)$$

Hence, the method of stationary phase yields the approximation

$$I=(2\pi)^{\frac{1}{2}}r(\xi,\eta) \times\exp[w(\xi+i\eta)+i(\alpha_0-\pi/4)]. \quad (13.7)$$

The disconcerting fact is that the point ξ, η may be chosen arbitrarily in Γ, and the path of integration then deformed so that ξ, η becomes the point of stationary phase. Thus, one is left with a double infinity of approximate values of I, each of which usually differs from every other. Under such conditions an investigation is indicated to determine which of the many approximate values is most accurate. Such an investigation is beyond the scope of this review, but even if it were carried out, it would still leave the physicist with an awkward but interesting problem.

It has been shown above that the Hamilton-Jacobi theory of waves, inclusive of the concept of group velocity, is based on a single one of these approximations. It is unlikely that this one will always be the best. Can the Hamilton-Jacobi theory be extended so as to yield a better approximation to the solutions of the wave equation? It appears likely that this can be done, but if so, it will modify the concept of group velocity in a peculiar manner. It has been customary to consider it to be a property of the medium, and the same for all disturbances in that medium. In such an extended Hamilton-Jacobi theory, the group velocity will be strongly dependent on the nature of the disturbance.

14. THE METHOD OF STEEPEST DESCENT

There is yet another approximate value of I, which is obtained by choosing the path of integration so that it passes through the saddle point S, Fig. 11, if such a saddle point exists. The coordinates of this point are the roots of

$$\partial v/\partial x=0, \quad \partial v/\partial y=0. \quad (14.1)$$

If the path is straight at S, then Eq. (13.4) will be satisfied for all values of α. Moreover, because of the Cauchy-Riemann equations, one will also have

$$\partial u/\partial x=0, \quad \partial u/\partial y=0 \quad (14.2)$$

at S. Hence, in addition to Eqs. (13.5) and (13.6), one will also have

$$u=u(\xi,\eta)-(s-s_0)^2/2r_1^2+\cdots, \quad (14.3)$$

where

$$r_1=[-(\partial^2v/\partial\xi\partial\eta)\cos 2\alpha_0 - (\partial^2u/\partial\xi\partial\eta)\sin 2\alpha_0]^{-\frac{1}{2}}. \quad (14.4)$$

The value of α_0 can be chosen so as to simplify the remaining calculations. If it is chosen so that $1/r=0$, the value of r_1, becomes

$$r_1=r_0=\{[\partial^2v/\partial\xi\partial\eta]^2+[\partial^2u/\partial\xi\partial\eta]^2\}^{\frac{1}{4}}. \quad (14.5)$$

This is equivalent to requiring that α_0 be chosen so as to make $(1/r_1)$ a maximum. Hence this procedure is usually called the method of steepest descent[5] since it makes the maximum of u as sharp as possible.

The corresponding approximate value of I is

$$I=(2\pi)^{\frac{1}{2}}r_0(\xi\eta)\exp[w(\xi+i\eta)+i\alpha_0]. \quad (14.6)$$

The value of α_0 may also be chosen so that $1/r_1=0$; this value differs from the previous by $\pi/4$, and makes $r=r_0$, so that Eq. (12.7) becomes identical with Eq. (13.6). The method of steepest descent thus appears as a singular case of the method of stationary phase. It is likely that this singular case will often yield a better approximation than any of the others, but, it cannot be used as the basis for the Hamilton-Jacobi theory unless the saddle point happens to be on the real axis.

15. THE REMAINDER IN THE METHOD OF STATIONARY PHASE

The earliest work on this problem is again Fresnel's justification of his method of zones. Unfortunately, the widely current accounts of his argument leave much to be desired. They almost invariably involve the same reasoning by which it was "proven" that the hare cannot overtake the tortoise. There are more recent studies of the problem[24–26] which are more satisfactory.

In order to reduce the problem to its simplest mathematical terms, consider the integral

$$I(p)=\int_a^b \phi(k)\exp[ip\theta(k)]dk, \qquad (15.1)$$

where ϕ and θ are real functions of k, and p is a real parameter. In the foregoing applications the phase of the integrand was a linear function of the two parameters x and t, and it was seen that the approximations resulting from the method of stationary phase were valid only when these parameters were such as to make the phase a rapidly varying function of k. This last phrase is not very precise, nor is it easy to reword it precisely. Therefore, the parameter p has been introduced to simplify the problem: it is required to find an expansion for I in descending powers of p, and to investigate the remainder of this series after n terms, assuming that ϕ and θ are finite, continuous and adequately differentiable in the interval a, b.

The simplest case arises when θ has no extrema in a, b. Then a change of variable reduces the integral to

$$I(p)=\int_\alpha^\beta f(\theta)\exp(ip\theta)d\theta, \qquad (15.2)$$

where $\alpha=\theta(a)$, $\beta=\theta(b)$, and

$$f(\theta)=\phi(k)/\theta'(k). \qquad (15.3)$$

Because of our assumption, $f(\theta)$ will also be sufficiently regular for the following procedures, whereas it would become infinite if $\theta'(k)$ were to vanish somewhere in the interval of integration.

An integration by parts reduces Eq. (14.2) to

$$I(p)=(1/ip)[f(\beta)e^{i\beta}-f(\alpha)e^{i\alpha}]+i\mathfrak{R}_1/p, \qquad (15.4)$$

where

$$\mathfrak{R}_1=\int_\alpha^\beta f'(\theta)\exp[ip\theta]d\theta. \qquad (15.5)$$

[24] L. Brillouin, Ann. Ecole Normale 2, 33 (1916).
[25] Van der Corput, Compositio Mathematica 1, 15 (1934); 3, 328 (1936).
[26] J. Bijl, Dissertation, Groningen, 1937.

An iteration of this process leads to

$$\mathfrak{R}_1=(1/ip)[f'(\beta)e^{i\beta}-f'(\alpha)e^{i\alpha}]+i\mathfrak{R}_2/p,$$

where

$$\mathfrak{R}_2=\int_\alpha^\beta f''(\theta)\exp[ip\theta]d\theta.$$

Thus, the problem of expansion is solved, except for the trouble involved in differentiating Eq. (15.3) with respect to θ. In particular, it is seen that the series begins with a term proportional to $1/p$.

In order to discuss the remainders, it will be sufficient for the present purposes to deal only with $\mathfrak{R}_1$, since it is quite typical. The definition of an integral results in the inequality

$$|\mathfrak{R}_1|\leqslant\int_\alpha^\beta |f'(\theta)|d\theta, \qquad (15.6)$$

which shows that the extrema of the function f are again the central elements of the problem. If $f(\theta)$ has no extrema in the interval of integration, the value of the integral is simply $|f(\beta)-f(\alpha)|$. However, if f has one extremum, at $\theta=\theta_1$, the value of the integral increases to $|f(\beta)-f(\theta_1)|+|f(\theta_1)-f(\alpha)|$, and if it has m extrema, there will be $m+1$ such terms.

In fact, the integral in Eq. (14.6) is essentially a measure of the extent to which the function f varies or oscillates in the interval α to β. This concept is adumbrated by such phrases as "f is a slowly varying function," etc. In order to be able to use it conveniently, the notation

$$\int_\alpha^\beta |f'(\theta)|d\theta=Os[f(\theta),\alpha,\beta] \qquad (15.7)$$

will be introduced. Hence the remainder after n terms satisfies the inequality

$$|\mathfrak{R}_n|/p^n\leqslant(1/p^n)Os[f^{(n-1)}(\theta),\alpha,\beta]. \qquad (15.8)$$

The situation is materially altered if the phase has a stationary point in the interval of integration, for then the oscillation of $f(\theta)$ becomes infinite. Let $k=c$, $\theta(c)=\gamma$ be an extremum of $\theta(k)$ and the only stationary point in the interval

a, b; then Eq. (15.1) can be written

$$I(p)=\phi(c)\int_a^b \exp[ip\theta(k)]dk + \int_\gamma^\beta f(\theta)\exp[ip\theta]d\theta + \int_\alpha^\gamma f(\theta)\exp[ip\theta]d\theta, \quad (15.9)$$

where it is to be remembered that k is a two-valued function of θ, and where now

$$f(\theta)=[\phi(k)-\phi(c)]/\theta'(k). \quad (15.10)$$

The integrands of the second and third integrals are then sufficiently regular so that the preceding results can be applied to them at once: each has an expansion whose first term is proportional to $1/p$, and the remainder after n terms is subject to Eq. (15.8). It is, therefore, only necessary to discuss the integral

$$I_0(p)=\int_a^b \exp[ip\theta(k)]dk. \quad (15.11)$$

A new variable ξ, which is a single-valued function of k, is defined by the equation

$$\theta(k)=\theta(c)+\tfrac{1}{2}\theta''(c)\xi^2. \quad (15.12)$$

(If c were a minimax, $\theta''(c)$ would be zero, and this and the following equations would have to be altered in a manner analogous to the considerations of Section 7.) It is easily seen that, for $k=c$, $\xi=0$ and $d\xi/dk=1$, so that

$$I_0(p)=\exp[ip\theta(c)] \times\int_{\xi_a}^{\xi_b} \exp[(ip/2)\theta''(c)\xi^2]d\xi + \exp[ip\theta(c)]\int_{\xi_a}^{\xi_b}((d\xi/dk)-1) \times\exp[(ip/2)\theta''(c)\xi^2]d\xi. \quad (15.13)$$

The first integral is the incomplete Fresnel integral,[14] and requires no further discussion here. For large values of ξ_b and $-\xi_a$ it approaches the complete Fresnel integral in a known manner:

$$\int_{\xi_a}^{\xi_b} \exp[(ip/2)\theta''(c)\xi^2]d\xi \rightarrow [2\pi i/p\theta''(c)]^{\frac{1}{2}}. \quad (15.14)$$

Hence, the expansion of this integral will have its first term proportional to $1/p^{\frac{1}{2}}$ rather than to $1/p$, as has been the case with the others. The second integral is conveniently treated by changing to the variable

$$\eta=\tfrac{1}{2}\theta''(c)\xi^2, \quad (15.15)$$

which reduces it to two integrals of the same type as the last two in Eq. (15.9). Its expansion therefore begins with a term proportional to $1/p$.

In principle, the results of this section should make it possible to find the answers to the mathematical questions raised in Section 13. However, the analytical complications are so great that it is to be hoped that some more elegant treatment can be found.