Theory of the Slowing Down of Neutrons by Elastic Collision with Atomic Nuclei*

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I. INTRODUCTION AND FUNDAMENTAL EQUATIONS

THE nuclear fission program has stimulated the solution of a great variety of neutron diffusion problems. Many of these problems have been solved on the assumption that the diffusion of neutrons takes place without loss of energy. Or, if the neutrons do lose energy, it is supposed that the losses occur in large (fixed) amounts—as in inelastic scattering where the scattering nuclei are left in excited states. As long as the neutrons can be regarded as belonging to a small number of energy groups, the methods appropriate to the treatment of neutron diffusion without energy loss can easily be generalized. However, if the energy loss is essentially "continuous"—as in the slowing down of neutrons by elastic collision with atomic nuclei (which take up variable amounts of recoil kinetic energy of translation)—the group treatment becomes artificial and unsatisfactory for most purposes.[1] In the latter case, a direct approach through the rigorous transport equation—subject to the conditions discussed below—is indicated. By means

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[1] Ten years ago, Westcott (W4) attempted an approximate treatment of the slowing-down process on the basis of three groups of neutrons—fast, slow, and intermediate.

of this direct approach, the outstanding problems connected with the slowing down of neutrons by elastic collision with nuclei have been investigated. While some work still remains to be done, a great deal has been accomplished and it seems useful to give a comprehensive review of the theory at the present time. In order that the results to be presented will be applicable to all cases where the slowing down of neutrons plays a role—the moderation of cosmic-ray neutrons in the atmosphere, the shielding of high voltage accelerators against neutron background, etc., in addition to pile design—it will be assumed that no multiplication takes place in the slowing-down medium.

As mentioned above, the possibility of obtaining fairly rigorous results on the slowing down of neutrons by elastic collision with nuclei depends on certain conditions being fulfilled. If these conditions are not fulfilled, the theory must be suitably (and usually approximately) corrected. The conditions in question are: (1) inelastic scattering[2] is assumed absent, (2) the elastic scattering is assumed spherically symmetric in the center of the mass system, (3) the effects of chemical binding are neglected.

Condition (1) implies that a neutron energy less than that of the first excited level of the nucleus with which it collides is contemplated. The energy of the first excited level is (on the average) smaller, the heavier the nucleus. Light "alpha-particle" nuclei like C^{12} and O^{16} have their first excited levels between 4 and 6 Mev, whereas heavy nuclei have their first excited levels nearer 100 kev.

Condition (2) implies that the energy of the neutron is low enough so that deviations from s-wave scattering need not be considered. At high energies, i.e., when the DeBroglie wavelength of the neutron becomes of the order of nuclear dimensions, p-wave scattering starts contributing appreciably. This requires an energy of several Mev for light nuclei and several hundred kev for heavy nuclei, or more explicitly, $E\approx 10/M^{\frac{2}{3}}$ Mev, where E is the neutron energy and M is the mass of the scattering nucleus measured in units of the neutron mass. Thus, the deviations from s-scattering become significant at energies similar to those necessary for the onset of inelastic scattering and are usually unimportant compared with the latter effect. Deviations from s-scattering may occur at a lower energy than that indicated by the above criterion because of resonances in the scattering cross section; thus, there are p-resonances in the scattering of neutrons by helium at 1.05 Mev and 1.35 Mev. However, such resonances occur rather infrequently and when they do occur, the character of the resonance is not generally known. At any rate, the practical consequences are not very great.

Condition (3) implies that the energy of the neutron is large compared with the vibration frequency associated with the chemical bond. One would expect that the picture of a neutron suffering an elastic collision with a free nucleus would break down when the *energy transfer* becomes of the order of the vibration quanta, i.e., several volts in heavy substances. However, it can be shown that while the collision function changes when the nucleus is bound, the average energy loss stays the same until the *energy* of the neutron becomes comparable with the quantum of the chemical bond. In other words, the formulae which will be derived will remain valid down to energies of the order of the vibration quanta—below one electron volt—provided the capture is small and slowly varying within this energy range, so that only the average energy loss counts (A2, A3).

If the above conditions are accepted, the time-dependent transport equation for a single element[3] can be written in the form:

$$\frac{\partial N}{\partial t}(\mathbf{r},\,\mathbf{\Omega},\,u,\,t)+\mathbf{v}\cdot\mathrm{grad}N(\mathbf{r},\,\mathbf{\Omega},\,u,\,t)$$
$$=-\frac{vN(\mathbf{r},\,\mathbf{\Omega},\,u,\,t)}{l(u)}+\int_0^u du'\int d\mathbf{\Omega}'\frac{v'N(\mathbf{r},\,\mathbf{\Omega}',\,u',\,t)}{l_s(u')}\times f(\mu_0,\,u-u')+S(\mathbf{r},\,u,\,t). \quad (1)$$

In Eq. (1), $N(\mathbf{r},\,\mathbf{\Omega},\,u,\,t)d\mathbf{r}d\mathbf{\Omega}du$ is the number of neutrons between $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$, $\mathbf{\Omega}$ and $\mathbf{\Omega}+d\mathbf{\Omega}$, u and $u+du$ at time t, where $\mathbf{r}$ represents the three space coordinates, $\mathbf{\Omega}$ is a unit vector

[2] Pure capture, which completely removes the neutron, will be permitted (cf. below).

[3] The transport equation for a mixture is an easy generalization and will be treated below in connection with certain special problems.

in the direction of the neutron velocity, and[4] $u \equiv \log(E_0/E)$ with E_0 some initial energy (which is unique if the source is mono-energetic), and E the energy of interest. The first two terms on the left-hand side of Eq. (1) represent the time rate of change of the neutron distribution function moving with the neutron stream in the direction $\mathbf{\Omega}$. The first term on the right-hand side of Eq. (1) represents the neutrons removed from the beam by scattering and capture, i.e., $vN(\mathbf{r}, \mathbf{\Omega}, u, t)/l(u)$ represents the number of scattering and capture collisions per unit time at $\mathbf{r}$ and t which occur to neutrons with parameters $\mathbf{\Omega}$ and u, where $l(u)$, the total mean free path, is defined by:

$$\frac{1}{l(u)} = \frac{1}{l_s(u)} + \frac{1}{l_c(u)},$$

with l_s and l_c the scattering and capture mean free paths, respectively. The second term on the right-hand side of (1) represents the neutrons scattered into the beam: $v'N(\mathbf{r}, \mathbf{\Omega}', u', t)/l_s(u')$ is the number of collisions per second at $\mathbf{r}$ and t, which occur to neutrons with parameters $\mathbf{\Omega}'$ and u', whereas $f(\mu_0, u-u')$ (the function $f(\mu_0, u-u')$ is normalized so that $\int d\mathbf{\Omega} \int du' f(\mu_0, u-u') = 1$ for all $\mathbf{\Omega}$ and u) is the relative probability of a neutron having the parameters $\mathbf{\Omega}$, u after a scattering collision before which their values were $\mathbf{\Omega}'$ and u'. The scattering function $f(\mu_0, u-u')$ depends only on $\mu_0 = \mathbf{\Omega} \cdot \mathbf{\Omega}'$ and the difference between u and u', i.e., the ratio of the final to the initial energy.[5] The derivation of the explicit form of the function of $f(\mu_0, u-u')$ is given directly below. The last term on the right-hand side of Eq. (1) is the source term (assumed isotropic) representing the neutrons emitted per unit time at $\mathbf{r}$ and t, and with energy corresponding to u.

The form of $f(\mu_0, u)$ follows from the assumption of spherically symmetric scattering in the center of mass system (s-scattering), and the laws of conservation of energy and momentum. If a neutron, moving with velocity v_0, collides

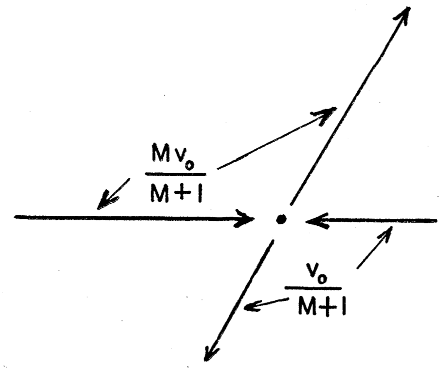

FIG. 1a. Center of mass system.

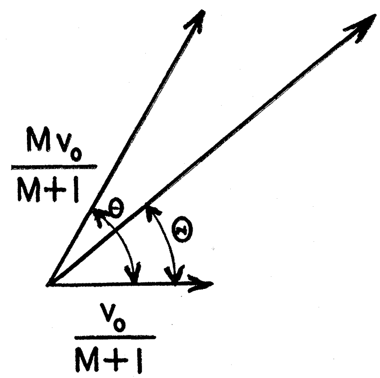

FIG. 1b. Relation between θ and Θ.

with a nucleus of mass M (measured in units of the neutron mass) at rest, then in the center of mass system the initial velocity of the neutron is $Mv_0/(M+1)$ (cf. Fig. 1a) and that of the nucleus $v_0/(M+1)$. After the collision, the momenta of the neutron and nucleus must again (in the center of mass system) be equal in magnitude and oppositely directed. Furthermore, since energy is conserved in the collision, it is clear that the velocity of the neutron (in the center of mass system) is the same as before, i.e., $Mv_0/(M+1)$. If the deflection of the neutron from its initial direction is denoted by θ, its deflection in the laboratory system by Θ, and its final velocity in the laboratory system is denoted by v, then (cf. Fig. 1b):

$$\frac{Mv_0}{M+1}\cos\theta + \frac{v_0}{M+1} = v\cos\Theta, \tag{2}$$

$$\left(\frac{Mv_0}{M+1}\right)^2 + \left(\frac{v_0}{M+1}\right)^2 - \frac{2Mv_0^2}{M+1}\cos\theta = v^2. \tag{3}$$

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[4] It is convenient to use logarithmic energy units because the asymptotic neutron density (for large u) is constant on this scale (cf. below).

[5] The dependence on the difference $(u-u')$ is the result of the special character of the scattering (s-scattering) while the dependence on μ_0 holds generally for isotropic scattering media.

From (3):

$$\cos\theta = 1 - \frac{(M+1)^2}{2M}\left[1-\left(\frac{v}{v_0}\right)^2\right], \quad (4)$$

or since $u = \log(E_0/E)$:

$$\cos\theta = 1 - \frac{(M+1)^2}{2M}(1-e^{-u}). \quad (4a)$$

The assumption of scattering which is spherically symmetric in the center of mass system implies that the differential cross section is proportional to $d(\cos\theta)$; but:

$$d(\cos\theta) = -\frac{(M+1)^2}{2M}e^{-u}du. \quad (5)$$

Furthermore, substituting (4a) into (2) leads to the relation:

$$\cos\Theta = -\frac{(M+1)}{2}e^{-u/2} - \frac{(M-1)}{2}e^{u/2}. \quad (6)$$

Equation (6) shows that the maximum possible logarithmic energy loss is $q_M \equiv \log(M+1/M-1)^2$ corresponding to[6] $\Theta = \pi$. Combining (5) and (6) leads to the desired result for the relative probability, $f(\mu_0, u)$ of a collision changing the neutron velocity from v_0 to v through a deflection (in the laboratory system) of $\mu_0 = \cos\Theta$, namely:

$$f(\mu_0, u) = \frac{(M+1)^2}{8\pi M}e^{-u} \times \delta\left\{\mu_0 - \left[\frac{(M+1)}{2}e^{-u/2} - \frac{(M-1)}{2}e^{u/2}\right]\right\}. \quad (7)$$

In Eq. (7), $(M+1)^2/8\pi M$ is the normalization constant chosen so that $\int d\mathbf{\Omega}\int du f(\mu_0, u) = 1$ and $\delta(x-a)$ is the Dirac δ-function defined by: $\delta(x-a) = 0$ when $x \neq a$ and $\int \delta(x-a)F(x)dx = F(a)$. It is understood that $u \leqslant q_M$; for $u > q_M$, $f(\mu_0, u)$ is identically zero.

Two quantities of interest later may be derived immediately from (7), i.e. the average logarithmic energy loss in one collision, ξ, and the average cosine of the angle of deflection (in the laboratory system) in one collision, $\langle\cos\Theta\rangle_{\text{Av}}$. We find:

$$\xi = \int_0^{q_M} u du \int d\mathbf{\Omega}_0 f(\mu_0, u) = 1 - \frac{(M+1)^2}{4M}q_M e^{-q_M}, \quad (8)$$

$$\langle\cos\Theta\rangle_{\text{Av}} = \int_0^{q_M} du \int d\mathbf{\Omega}_0 \mu_0 f(\mu_0, u) = 2/3M. \quad (9)$$

No loss of generality is incurred by assuming that $S(\mathbf{r}, u, t) = Q(\mathbf{r})\delta(u)\delta(t)$ where $Q(\mathbf{r})$ is a function of $\mathbf{r}$ alone and the $\delta(x)$ is the Dirac δ-function. A solution of the integro-differential Eq. (1) corresponding to an arbitrary distribution of sources (in u and t) is obtained by superposing solutions of (1) with a δ-distribution of sources (in u and t). Using the above equation for $S(\mathbf{r}, u, t)$, $h(u) = l(u)/l_s(u)$ and the abbreviation $\psi(\mathbf{r}, \mathbf{\Omega}, u, t)$($\psi(\mathbf{r}, \mathbf{\Omega}, u, t)d\mathbf{r}d\mathbf{\Omega}du$ is the total number of collisions per unit time between $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$, etc.) for $[v/l(u)]N(\mathbf{r}, \mathbf{\Omega}, u, t)$, we find Eq. (1) becomes:

$$\frac{l(u)}{v}\frac{\partial\psi}{\partial t} + l(u)\mathbf{\Omega}\cdot\text{grad}\psi + \psi(\mathbf{r}, \mathbf{\Omega}, u, t) = \int_0^u du' \int d\mathbf{\Omega}'\psi(\mathbf{r}, \mathbf{\Omega}', u', t)f(\mu_0, u-u') \times h(u') + Q(\mathbf{r})\delta(u)\delta(t). \quad (10)$$

If we are not interested in the spatial distribution of neutrons, we may integrate (10) over $d\mathbf{\Omega}$ and over all space; we then obtain the following integro-differential equation:

$$\frac{l(u)}{v}\frac{\partial\Psi_0}{\partial t} + \Psi_0(u, t) = \int_0^u du'\Psi_0(u', t)h(u') \times f_0(u-u') + Q\delta(u)\delta(t). \quad (11)$$

In Eq. (11), $\Psi_0(u, t) = \int d\mathbf{r}\int d\mathbf{\Omega}\psi(\mathbf{r}, \mathbf{\Omega}, u, t)$ is the average number of collisions a neutron experiences per unit time per unit logarithmic energy interval, $Q = \int d\mathbf{r}Q(\mathbf{r})$ is the total source strength, and $f_0(u-u') = \int d\mathbf{\Omega}_0 f(\mu_0, u-u')$ is the relative probability that a neutron will be scattered into the logarithmic energy interval du

[6] This remark does not hold for hydrogen ($M=1$) where the maximum possible logarithmic energy loss is infinity corresponding $\Theta = \pi/2$.

from the logarithmic energy interval du'. If we use (7), we find:

$$f_0(u)=\frac{(M+1)^2}{4M}e^{-u} \quad \text{for } u\leq q_M,$$
$$=0, \qquad \text{for } u>q_M. \tag{12}$$

The rest of this article consists essentially of a discussion of solutions of Eqs. (10) and (11) under different assumptions regarding $l(u)$ and $h(u)$. Equation (11) is simpler and is treated in Part II. We present rigorous solutions for the energy distribution of slowed down neutrons in the stationary non-capturing case. Fairly accurate—although not rigorous—solutions for the time-dependent and capturing cases are also given. In Part III, Eq. (10) is treated and information is obtained about the more complicated problem of the spatial distribution of slowed down neutrons. The problem of the spatial moments of the neutron density is considered first. In principle, knowledge of all the moments of a distribution function yields the distribution function itself. Inability to write down rigorous expressions for all the spatial moments leads to the development of a method of successive approximations for obtaining the neutron density directly. The well-known "age" equation is the first approximation. Higher approximations, i.e., improvements on the "age" theory, are also presented. Finally, the asymptotic neutron density is discussed. Part IV contains an appendix on the predictions of the theory with regard to the second spatial moment of the neutron density in C, O, H, D_2O, H_2O.

II. ENERGY DISTRIBUTION OF SLOWED-DOWN NEUTRONS

In this part, we discuss solutions of Eq. (11)—the spatially independent equation. We assume that a mono-energetic source of neutrons of energy E_0 is emitted first continuously (stationary case), and then at time $t=0$. In the stationary case we inquire into the distribution of neutrons at all energies E. In the time-dependent case, we also ask for the distribution of neutrons in time.

A. Stationary Case

If stationarity is assumed, Eq. (11) becomes

$$\Psi_0(u)=\int_0^u du'\Psi_0(u')h(u')f_0(u-u')+\delta(u), \tag{13}$$

where $f_0(u)$ is defined by Eq. (12). The total number of neutrons produced per unit time is assumed to be unity. For hydrogen ($M=1$), Eq. (13) takes on an especially simple form, namely:

$$\Psi_0(u)=\int_0^u du'\Psi_0(u')h(u')e^{-(u-u')}+\delta(u). \tag{14}$$

The solution of (14) can be obtained by converting it into a differential equation; we get, if we omit the neutrons experiencing no collisions at all and set $g(u)=1-h(u)$ (F2):**

$$\Psi_0(u)=h(0)\exp\left[-\int_0^u g(u')du'\right]. \tag{15}$$

The physical significance of (15) becomes clearer if we rewrite it as:

$$\Psi_0(u)=\frac{l(0)}{l_s(0)}\exp\left[-\int_0^u \frac{l(u')}{l_c(u')}du'\right]. \tag{15a}$$

For zero-capture, i.e., $l(u)\equiv l_s(u)$, (15a) reduces to unity for all u. In other words, in the absence of capture, the average number of collisions in hydrogen per unit time and per unit logarithmic energy loss is a constant, equal to unity for unit source strength; with capture present, the same quantity decreases exponentially in accordance with (15a).

For $M>1$, a simple solution of Eq. (13) does not exist, since the lower limit of the integral is no longer 0 for all u (as for $M=1$) but

$$u-q_M(q_M=\log(M+1/M-1)^2)$$

for $u>q_M$. This peculiar lower limit prevents the reduction of Eq. (13) to a differential equation, although a solution can still be found, in principle, by repeated integration from one collision interval to the next [$(0, q_M)$ to $(q_M, 2q_M)$ to $(2q_M, 3q_M)$, etc.]. The latter procedure is cumbersome and, in practice, has only limited interest

** References to the bibliography are given in parentheses.

since capture generally sets in for energies considerably lower than the initial energy, i.e., $u \gg q_M$. Hence, it suffices to examine how the asymptotic solution of the non-capture problem is modified by the presence of capture. The non-capture case will therefore be treated first. We shall then consider the novel modifications introduced by capture, both when the capture is slowly varying in one collision interval (i.e., q_M), and when it is rapidly varying (resonance capture). Finally, the case of "$1/v$"—capture (v is the velocity) will be discussed separately because of its intrinsic interest and the usefulness of the solution for the time-dependent problem without capture (cf. Section B).

1. No Capture

If capture is absent, Eq. (13) becomes:

$$\Psi_0(u)=\int_0^u du'\Psi_0(u')f_0(u-u')+\delta(u). \qquad (16)$$

The integral Eq. (16) can easily be solved by the method of Laplace transform (A1). If we write:

$$\Phi_0(\eta)\equiv \mathfrak{L}\Psi_0(u)=\int_0^\infty due^{-\eta u}\Psi_0(u),$$

$$G(\eta)\equiv \mathfrak{L}f_0(u)=\int_0^\infty due^{-\eta u}f_0(u),$$

then taking the Laplace transform of both sides of (16) yields:

$$\Phi_0(\eta)=\Phi_0(\eta)G_0(\eta)+1, \qquad (17)$$

where

$$G_0(\eta)=\frac{\alpha}{\eta+1}[1-\exp[-q_M(\eta+1)]], \quad (\alpha=(M+1)^2/4M). \qquad (17a)$$

The first term on the right-hand side of (17) follows from the convolution theorem[7] for Laplace transform; Eq. (17) leads immediately to the following solution for $\Psi_0(u)$:

$$\Psi_0(u)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d\eta e^{\eta u}}{\left\{1-\dfrac{\alpha}{\eta+1}(1-\exp[-q_M(\eta+1)])\right\}}, \qquad (18)$$

where the integration is taken over a line to the right of all the poles of the integrand. The poles of the integrand are at:

$$(\eta+1)/\alpha=1-\exp[-q_M(\eta+1)]. \qquad (19)$$

If we separate $\delta(u)$ (this is the direct source term) from $\Psi_0(u)$, we can write the solution in the form:

$$\Psi_0(u)=\delta(u)+\sum_{\substack{\text{all poles}\\(j=0,1,\cdots)}}\frac{\{1-\exp[-q_M(\eta_j+1)]\}e^{\eta_j u}}{\{(1/\alpha)-q_M\exp[-q_M(\eta_j+1)]\}}, \qquad (20)$$

where η_j is the jth root of Eq. (19).

The only pole for which η_j has a non-negative real part is at $\eta_0=0$; all other poles ($j>0$) lie to the left of the imaginary axis. Consequently, for large u, the asymptotic solution for $\Psi_0(u)$ is:

$$\Psi_0(u)_{\text{asym}}=1/\xi, \qquad (21)$$

where ξ is defined by Eq. (11a). It is precisely because the asymptotic behavior of $\Psi_0(u)$ is a constant that we chose u as a variable rather than the energy itself.

The error introduced by using (21) may be estimated by finding the poles $\eta_{1,2}$ with the largest negative real part. Since the poles depend on M, two cases have been distinguished: $M=2$, and $M\gg 1$. The results are (M2):

$$\begin{aligned} M=2:&\quad \eta_{1,2}=-1.55\pm 3.37i,\\ M\gg 1:&\quad \eta_{1,2}=-0.52M\pm 1.87Mi. \end{aligned} \qquad (22)$$

Substitution into (20) shows that the dominant

[7] The convolution theorem for Laplace transforms states that $\mathfrak{L}\int_0^u du'F_1(u-u')F_2(u')=G_1(\eta)G_2(\eta)$, where $G_1(\eta)=\mathfrak{L}F_1(u)$, and $G_2(\eta)=\mathfrak{L}F_2(u)$. This theorem and all other cited theorems on Laplace transform are proved in G. Doetsch, *Laplace Transformation* (Dover Publishers).

terms, which have been neglected in writing down (21), are:

$$(M=2)e^{-1.55u}, \quad (M\gg 1)\cdot Me^{-0.52Mu}.$$

For the special case $M=1$ (hydrogen) it can easily be shown that the only pole is at $\eta_0=0$; hence the rigorous solution is $\Psi_0(u)\equiv 1$, obtained by using the fact that $q_{M=1}=\infty$ and $\alpha=1$. This is in agreement with the previous derivation (cf. Eq. (15)).

Placzek has solved Eq. (16) by an alternative method, and has obtained curves for

$$\left[\frac{\Psi_0(u)-\Psi_0(u)_{\text{asym}}}{\Psi_0(u)_{\text{asym}}}\right]$$

as a function of (u/q_M) for different values of M (P1). His results are shown in Figs. 2 a, b, c; it is seen that the fluctuations of $\Psi_0(u)$ about its asymptotic value die out fairly completely when $u>3q_M$. Since q_M is the larger, the lighter the nucleus (e.g., $q_{M=2}=2.2$ as compared with $q_{M=12}=0.159$), the deviations of $\Psi_0(u)$ from its asymptotic value will thus extend over a larger energy region for the light nuclei. Hence, for a given width in initial energy of the neutrons, the deviations will be observed more easily in light nuclei.

Placzek (P1) has also examined the solution of Eq. (16) for a mixture of elements. He finds, for a mixture of m elements with all the mean free paths constant or varying in the same way with energy, the following asymptotic behavior for $\Psi_0(u)$:

$$\Psi_0(u)_{\text{asym}}=1\Big/\left\{1-\sum_{s=1}^{m}c_s\alpha_s q_{M_s}\exp[-q_{M_s}]\right\}, \quad (23)$$

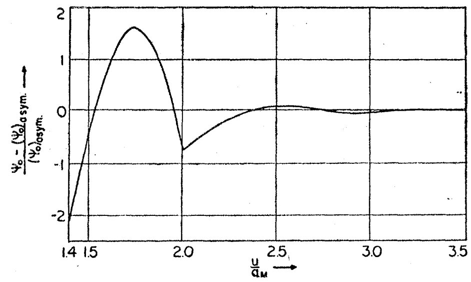

FIG. 2a. $[\psi_0(u)-\psi_0(u)_{\text{asym}}]/\psi_0(u)_{\text{asym}}$ as a function of u/q_M (from (P1)) for $M=2$.

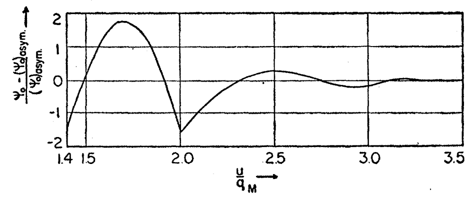

FIG. 2b. $[\psi_0(u)-\psi_0(u)_{\text{asym}}]/\psi_0(u)_{\text{asym}}$ as a function of u/q_M (from (P1)) for $M=12$.

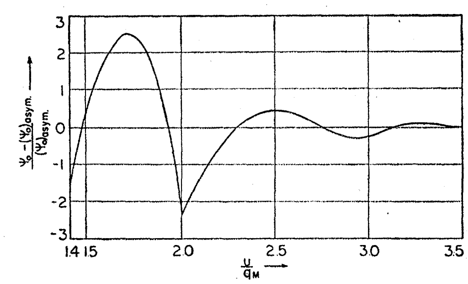

FIG. 2c. $[\psi_0(u)-\psi_0(u)_{\text{asym}}]/\psi_0(u)_{\text{asym}}$ as a function of u/q_M (from (P1)) for $M=\infty$.

where $c_s=l(u)/l_s(u)$, with $l(u)$ the total mean free path, and $l_s(u)$ the mean free path for scattering at a nucleus of type s. Equation (23) is valid for variable c_s provided the variation is small in a region of the extension of one collision interval.

2. *Capture*

Equation (21) gives us the asymptotic behavior of $\Psi_0(u)$ in the case of no capture. When capture sets in for some $u=u_0\gg q_M$, the solution (21) is modified. To determine the character of the new solution, we rewrite Eq. (15) for $u>q_M$:

$$\Psi_0(u)=\alpha\int_{u-q_M}^{u}du'\Psi_0(u')h(u')e^{-(u-u')}, \quad (24)$$

where $\Psi_0(u_0)=1/\xi$, $h(u_0)=1$. If we multiply both sides of Eq. (24) by ξ, and introduce the notation $S(u)=\xi\Psi_0(u)$, we get:

$$S(u)=\alpha\int_{u-q_M}^{u}du'S(u')h(u')e^{-(u-u')}, \quad (24a)$$

where $S(u_0)=1$. The quantity $S(u)$ represents the so-called slowing-down function, i.e., the number of neutrons reaching the energy corresponding to u per unit time. Equation (24a) can be written in another form, namely (P1):

$$S(u)=P(u)+\alpha\frac{e^{-q_M}}{\xi}\int_{u-q_M}^{u}du'[h(u')S(u')-S(u)], \tag{24b}$$

where

$$P(u)=1-\frac{1}{\xi}\int_{u_0}^{u}du'g(u')S(u')$$

is the probability per unit time, that the neutron is not captured in the logarithmic energy interval (u_0, u). The presence of the second term on the right-hand side of Eq. (24b) distinguishes hydrogen from all the other nuclei (since $q_{M=1}=\infty$), and is responsible for the physical difference between S and P.[8] If the capture is slowly varying within one collision interval, an approximate solution of (24a) is (P1):

$$S(u)=\exp\left[-\int_{u_0}^{u}\nu_0(u')du'\right], \tag{25}$$

where $\nu_0(u)$ is the positive root of:

$$(1-\nu_0)+\alpha h\exp[-q_M(1-\nu_0)]-\alpha h=0. \tag{25a}$$

For large M and $g(u)\ll 1$, Eq. (25) reduces to:

$$S(u)=\exp\left[-\frac{1}{\xi}\int_{u_0}^{u}g(u')du'\right]. \tag{25b}$$

Equation (25) has been generalized to mixtures for $h(u)$ equal to a constant slightly less than unity (weak capture). The result for a mixture of m elements is (W2):

$$\nu_0=(1-h)\Big/1-\sum_{s=1}^{m}c_s\alpha_s q_{M_s}\exp[-q_{M_s}], \tag{25c}$$

where $\Psi_0(u_0)$ is now given by the right-hand side of (23).

If the capture is rapidly varying within one collision interval (resonance capture), it is necessary to solve (24a) rigorously. This is feasible if the rapidly varying capture extends over several collision intervals. We seek a solution of (24) which satisfies the boundary condition $S(u_0)=1$, and which reduces to $S(u)\equiv 1$ for $h(u)\equiv 1$ for all $u\geqslant u_0$. This solution turns out to be (P1):

$$S(u)=\left\{e^{\chi(u)}\left[1-\alpha e^{-q_M}\int_{u_0}^{u}e^{-\chi(u')}du'\right]\right\}, \tag{26}$$

where

$$\chi(u)=\alpha\int_{u_0}^{u}h(u')du'-(u-u_0).$$

The solution (26) is valid in the interval (u_0, u_0+q_M); the solution in the following intervals can be found from (26) by successive integrations. After the capturing region has been passed, $S(u)$ will fluctuate for a while and finally tend to an asymptotic value. This asymptotic value can be determined without a knowledge of $S(u)$ in the fluctuating region. In particular, if $\bar{u}_0(>u_0)$ is the value

[8] It is the connection between S and P which leads to the introduction of the quantity S; in the time-dependent case we shall return to Ψ_0.

of u beyond which the capture ceases, it follows from (24b) that the asymptotic value of $S(u)$ is given by:

$$S(u)_{\text{asym}}=\left\{1-\frac{1}{\xi}\int_{u_0}^{u}du'S(u')g(u')\right\}. \tag{27}$$

If the resonance capture takes place within a small portion of one collision interval, i.e., the width of resonance is small compared to q_M, then Eq. (27) leads to the following simple result:[9]

$$S(u)_{\text{asym}}=\left[1-\frac{1}{\xi}\int_{\text{res}}\frac{l(u')}{l_c(u')}du'\right]. \tag{28}$$

Equation (28) follows from the fact that it is self-consistent to replace $S(u')$ under the integral sign in (27) by its "no-capture" value, i.e., unity.

3. "1/v" Capture

If the scattering mean free path is assumed constant, and the capture mean free path is proportional to the velocity ("1/v" law for the capture cross section), a rigorous solution of (24a) can be obtained (P1). Let us write $w=l_s/l_c=v_c/v$ where l_s is the scattering mean free path (assumed constant), l_c is the capture mean free path, and v_c is the velocity at which the mean free paths for scattering and capture are equal; then $h(u)=1/(1+w)$. If we rewrite Eq. (24a) in terms of w, we get $(S(u)\equiv\phi(w))$:

$$\phi(w)=\frac{2}{w^2(1-r^2)}\int_{rw}^{w}dw'\frac{w'\phi(w')}{(1+w')}, \tag{29}$$

where $r\equiv[(\alpha-1)/\alpha]^{\frac{1}{2}}=(M-1/M+1)$. Equation (29) implies that we restrict ourselves to values of $u>q_M$; the error introduced is small as long as $v_c\ll v_0$ (v_0 is the initial neutron velocity). The solution of (29) is most conveniently expressed in terms of an infinite series, namely:

$$\phi(w)=\sum_{j=0}^{\infty}\beta_j w^j, \tag{30}$$

where

$$\beta_0=1,\quad \beta_j=(-)^j\lambda_j\prod_{k=1}^{j}(1-\lambda_k)^{-1}, \tag{30a}$$

with

$$\lambda_j=\frac{2}{j+2}\,\frac{1-r^{j+2}}{1-r^2}. \tag{30b}$$

The expansion (30) is useful for small M; for hydrogen a rigorous solution is easily obtained, namely (B3):

$$\phi(w)=1/(1+w)^2. \tag{31}$$

For large M, an expansion of $\log[\phi(w)]$ is more convenient and one finds (P1):

$$\log[\phi(w)]=-\frac{2}{1-r}\left\{\frac{(1+r+r^2)w}{(1+2r)}-\frac{(1+3r+2r^3)}{2(1+2r)^2}w^2+\cdots\right\}; \tag{32}$$

or

$$\phi(w)=\exp[-(Mw-\tfrac{1}{3}Mvw^2)], \tag{32a}$$

where terms of order w^2 and Mw^3 have been neglected.

[9] A detailed discussion of the effect of resonance capture on the slowing down of neutrons in uranium piles will be found in the *Plutonium Project Record* of the Chicago Metallurgical Laboratory.

B. Time-Dependent Case

We now turn to the time-dependent Eq. (13), and assume that capture is absent and that the source strength is unity. We have:

$$\frac{l(u)}{v}\frac{\partial\Psi_0}{\partial t}+\Psi_0(u,t)=\int_0^u du'\Psi_0(u',t)e^{-(u-u')}+\delta(u)\delta(t). \tag{33}$$

Equation (33) is difficult to solve for a general element of mass M, if $l(u)$ is permitted to vary arbitrarily with energy. We therefore consider two special cases:

1. $M=1$ (hydrogen) and arbitrary variation of $l(u)$ with energy.
2. $M\neq 1$ and $l(u)\equiv$constant.

1. Hydrogen

For hydrogen, Eq. (33) assumes the simple form:

$$\frac{l(u)}{v}\frac{\partial\Psi_0}{\partial t}+\Psi_0(u,t)=\int_0^u du'\Psi_0(u',t)e^{-(u-u')}+\delta(u)\delta(t). \tag{34}$$

The solution of (34) is most readily obtained by taking the Laplace transform of both sides of Eq. (34) with respect to t; we get:

$$\left[1+\frac{sl(u)}{v}\right]\Phi_0(u,s)=\int_0^u du'\Phi_0(u',s)e^{-(u-u')}+\delta(u), \tag{35}$$

where

$$\Phi_0(u,s)=\int_0^\infty dte^{-st}\Psi_0(u,t).$$

It is convenient to separate out the δ-function part of the solution of (35); thus we write:

$$\Phi_0(u,s)=\frac{\delta(u)}{1+(sl_0/v_0)}+\chi(u,s). \tag{36}$$

In Eq. (36), the zero subscript on l and v indicates that the values of l and v at $u=0$ are to be chosen. Substituting (36) into (35) yields:

$$\left[1+\frac{sl(u)}{v}\right]\chi(u,s)=\int_0^u du'\chi(u',s)e^{-(u-u')}+\frac{e^{-u}}{1+(sl_0/v_0)}. \tag{37}$$

If we differentiate (37) with respect to u, we find:

$$\left[1+\frac{sl}{v}\right]\chi'+s\left[\left(\frac{l}{v}\right)'+\frac{l}{v}\right]\chi=0, \tag{38}$$

where the primes denote differentiation with respect to u (the parameter s is regarded as fixed). The solution of (38), subject to the boundary condition

$$\chi(0,s)=\frac{1}{[1+(sl_0/v_0)]^2}$$

(cf. (37)), is:

$$\chi(u, s)=\frac{1}{[1+(sl_0/v_0)][1+(sl(u)/v)]}\exp\left[-s\int_0^u \frac{du'}{s+(v'/l(u'))}\right]. \tag{39}$$

To find $\Psi_0(u, t)$, we must take the Laplace inverse of (36); we obtain:

$$\Psi_0(u, t)=\frac{v_0}{l_0}\exp[-v_0t/l_0]\delta(u)+\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{ds\exp\left[s\left\{t-\int_0^u \frac{du'}{[s+(v'/l(u'))]}\right\}\right]}{[1+(sl(u)/v][1+(sl_0/v_0)]}, \tag{40}$$

where the contour of integration is chosen to the right of all the poles of the integrand. Equation (40) was first obtained by Ornstein and Uhlenbeck (O1) by a method essentially equivalent to the one above.

The evaluation of the second term on the right-hand side of (40) is in general laborious; however, if $[v/l(u)]$ is approximated by certain simple expressions, the Laplace inverse can be obtained immediately from tables.[10] Thus, if the scattering mean free path is assumed to vary as v("$1/v$" cross section—this is a good approximation from 50 kev to 2-3 Mev), so that $v/l(u)=v_0/l_0$ (constant), Eq. (40) reduces to:

$$\Psi_0(u, t)=\frac{v_0}{l_0}\delta(u)\exp[-v_0t/l_0]+\frac{v_0}{l_0}\left(\frac{v_0t}{l_0u}\right)^{\frac{1}{2}}I_1\left[2\left(\frac{v_0ut}{l_0}\right)^{\frac{1}{2}}\right]\exp[-u-(v_0t/l_0)], \tag{41a}$$

where $I_1(z)$ is the first-order Bessel function of imaginary argument. Similarly, if we take $l=l_0$ (constant)—valid below 50 kev—we get:

$$\Psi_0(u, t)=\frac{v_0}{l_0}\delta(u)\exp[-v_0t/l_0]+\frac{1}{2}\left(\frac{v}{l_0}\right)^2t^2\left[1-\frac{v}{v_0}+\frac{2l_0}{tv_0}\right]\exp[-vt/l_0]. \tag{41b}$$

2. Constant Mean Free Path for Heavy Elements

If the mean free path is taken as constant, it is possible to obtain a fairly accurate solution of Eq. (33) for arbitrary mass (M7). If we write $l(u)\equiv l_0$ (constant) and take the Laplace transform of Eq. (33) with respect to t (the notation is the same as in Eq. (35)), we get:

$$\left(1+\frac{sl_0}{v}\right)\Phi_0(u, s)=\int_0^u du'\Phi_0(u', s)f_0(u-u')+\delta(u). \tag{42}$$

If one is interested in large u (which is generally the case), the influence of the solution in the first interval $0\leqslant u\leqslant q_M$ is unimportant and we can rewrite Eq. (42) as follows:

$$\phi(w, s)=\frac{2}{(1-r^2)w^2}\int_{rw}^{w}dw'\frac{w'\phi(w', s)}{(1+w')}\text{ for }u>q_M, \tag{43}$$

where

$$w=l_0s/v,\quad r=(M-1)/(M+1),\quad \phi(w, s)=(1+w)\Phi_0(u, s).$$

Equation (43) is identical with Eq. (29) except that $\phi(w, s)$ is a function of the parameter s; this implies that the solution of the time-dependent problem reduces to taking the Laplace inverse of the solution of (29) or (43). In other words, the time-dependent neutron distribution without capture is the Laplace inversion of the stationary distribution with "$1/v$" capture.

[10] N. W. McLachlan and P. Humbert, *Tabelles des Laplace Transformes.*

The solution of Eq. (43) has already been given in (30). The solution of the time-dependent equation is then:

$$\Psi_0(u, t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dse^{st}\left[\sum_{j=0}^{\infty} \beta_j w^j/\xi(1+w)\right], \tag{44}$$

where the β_j's are defined by (30a). The rigorous evaluation of (44) is very difficult. We therefore have recourse to the following approximation method; we observe that:

$$\langle t^n\rangle_{\text{Av}} = (-)^n \frac{\partial^n}{\partial s^n}\left[\sum_{j=0}^{\infty} \beta_j w^j/(1+w)\right]_{s=0}, \tag{45}$$

where

$$\langle t^n\rangle_{\text{Av}} = \int_0^{\infty} dt t^n \Psi_0(u, t) \Big/ \int_0^{\infty} dt \Psi_0(u, t).$$

Equation (45) can be rewritten as:

$$\langle x^n\rangle_{\text{Av}} = (-)^n \frac{\partial^n}{\partial w^n}\left[\sum_{j=0}^{\infty} \beta_j w^j/(1+w)\right]_{w=0}, \tag{46}$$

where $x = (vt/l_0)$.

Now Placzek has calculated $\langle x^n\rangle_{\text{Av}}$ (P2) with the result:

$$\langle x^n\rangle_{\text{Av}} = n! \prod_{k=1}^{n} (1-\lambda_k)^{-1}. \tag{47}$$

where λ_k is defined by Eq. (30b). In principle, these moments determine a generating function $F(x)$ which is directly related to $\Psi_0(u, t)$. The expression for the nth moment is so complicated, however, that we can at best hope to find an approximate expression for $F(x)$. This we proceed to do. We first notice from the moments that for large x, $F(x)$ behaves like $e^{-x}x^{2/(1-r^2)}$. This follows from the fact that:

$$\begin{aligned}\log \prod_{k=1}^{n} (1-\lambda_k)^{-1} &= \sum_{k=1}^{n} \log (1-\lambda_k)^{-1}\\ &\approx \sum_{k=1}^{n} \log\left[1-\frac{2}{(k+2)(1-r^2)}\right]^{-1}\\ &\approx \sum_{k=1}^{n} \frac{2}{(k+2)(1-r^2)} \underset{\text{asym}}{\approx} \frac{2}{(1-r^2)} \log n = \log n^{2/(1-r^2)},\end{aligned} \tag{48}$$

and

$$\langle x^n\rangle_{\text{Av}} \approx \int_0^{\infty} dx e^{-x} x^{2/(1-r^2)} x^n = \Gamma\left(\frac{2}{1-r^2}+n+1\right) \approx n! n^{2/(1-r^2)}. \tag{49}$$

Secondly, we notice that the integral equation (43) is equivalent to a differential equation with an essential singularity. The simplest essential singularity (at least for integration purposes) is exhibited by $\exp(-b/x)$ (b a constant). We therefore try:

$$F_0(x) = A \exp\left[-\left(\frac{b}{x}+x\right)\right] x^{2/(1-r^2)}, \tag{50}$$

where A is a normalization constant. The constant b is determined by maximizing $F_0(x)$ at the

TABLE I. Constants for time-dependent case.

M	$2/1-r^2$	x_{Av}	a	b
2	2.250	13.6	0.190	4.86
9	5.556	10.38	0.0730	50.15
15	8.533	16.36	0.0526	128.13

point $x=x_{Av}$. The advantage of (50) is that integrals of the form

$$\int_0^\infty dx \exp\left[-\left(\frac{b}{x}+x\right)\right]x^{n+\frac{1}{2}}$$

(n an integer) are readily evaluable. $F_0(x)$ is only a first approximation; after some experimentation, it is found that the relation

$$\int_0^\infty dx x^n F_0(x) = [1+a(n+1)]\langle x^n \rangle_{Av}, \tag{51}$$

where a is a constant and $\langle x^n\rangle_{Av}$ is given by (47), is closely fulfilled. We therefore improve our result by writing:

$$\int_0^\infty dx x^n F_0(x) = [1+a(n+1)]\int_0^\infty dx x^n F_1(x), \tag{52}$$

and solving for $F_1(x)$. It can be shown that (52) is equivalent to the following differential equation for $F_1(x)$:

$$F_1(x) - ax\frac{dF_1(x)}{dx} = F_0(x), \tag{53}$$

with the boundary condition $F_1(\infty)=0$. It is clear that the value of $F_1(x)$ at $x=x_{Av}$ is equal to that of $F_0(x)$. The solution of (53) is:

$$F_1(x) = \frac{1}{a}x^{1/a}\int_x^\infty \frac{dx' F_0(x')}{(x')^{1+1/a}}. \tag{54}$$

If further accuracy is desired, one can express:

$$\int_0^\infty dx x^n F_0(x) = [1+a_1(n+1)+a_2(n+1)(n+2)+\cdots]\langle x^n\rangle_{Av} \tag{55}$$

and obtain higher order differential equations.

The above procedure has been applied to three cases: $M=2$, $M=9$, and $M=15$. Table I gives the corresponding values for $[2/(1-r^2)]$, x_{Av}, a, and b. The quantity a was found from (51) by taking the nearest half-integer for $[2/(1-r^2)]$, i.e., 2.5, 5.5, 8.5 for $M=2$, 9, 15, respectively, and performing the fit at the fifteenth moment. With the value of a thus determined, the ratio r_n of the nth moment, defined in terms of $F_1(x)$, to the nth moment, defined by Eq. (47), is given in Table II for the first thirty moments.

TABLE II. Ratio of approximate nth moment to exact nth moment.

n	$M=2$	$M=9$	$M=15$	n	$M=2$	$M=9$	$M=15$
1	0.951	0.945	0.953	16	0.946	1.001	1.001
2	1.027	0.956	0.956	18	0.912	0.997	1.009
4	1.102	0.974	0.962	20	0.878	0.991	1.017
6	1.118	0.986	0.967	22	0.848	0.984	1.024
8	1.101	0.996	0.973	24	0.818	0.975	1.031
10	1.064	1.000	0.979	26	0.791	0.965	1.038
12	1.028	1.003	0.986	28	0.764	0.954	1.045
14	0.986	1.003	0.994	30	0.737	0.942	1.051

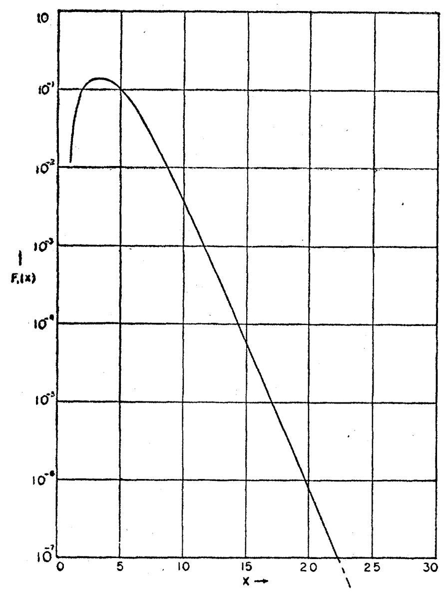

FIG. 3a. Time dependence for $M=2$.

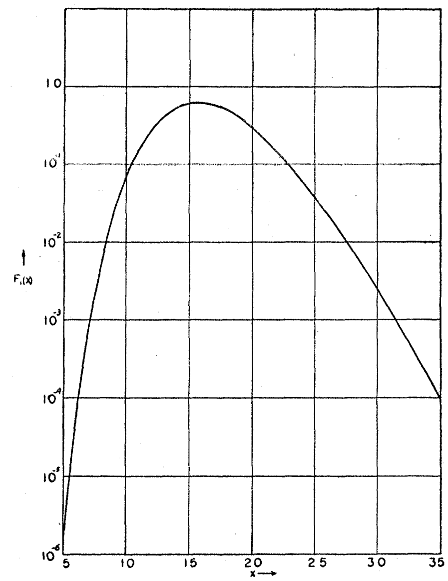

FIG. 3c. Time dependence for $M=15$.

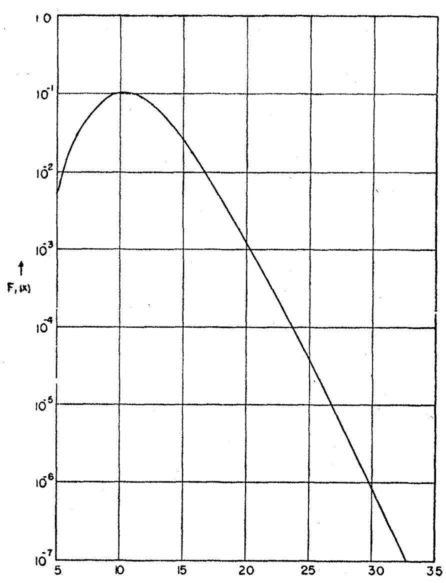

FIG. 3b. Time dependence for $M=9$.

It is seen that the deviation is not more than 5 percent for $M=9$ and $M=15$. However, for $M=2$, the results are not as good. This is to be expected since the derivation of (48) is the less accurate, the smaller the M. Hence, for $M=9$ and $M=15$, the distribution function $F_1(x)$ (given by (54)) should be quite accurate up to rather large values of x. For $M=2$, the distribution function $F_1(x)$ will not hold as far out. Figures 3a–3c contain plots of $\log F_1(x)$ as a function of x up to $x=35$ for $M=2, 9, 15$.

C. Energy Distribution of Neutrons after Given Number of Collisions

For some purposes, it is of interest to know the energy distribution of neutrons which have suffered just n collisions. This problem has been solved by Wick (W5) for hydrogen, and for an element of arbitrary mass by Condon and Breit (C1), by Langevin (L1), and by Dancoff (D1). If $N(n, u)du$ represents the number of neutrons between u and $u+du$ after n collisions, then $N(n, u)$ is different from zero for values of u between 0 and nq_M. We have the integral equation:

$$N(n, u)=\alpha\int_{u-q_M}^{u} du' e^{-(u-u')}N(n-1, u'). \tag{56}$$

The solution of (56) is:

$$N(n, u)=\frac{\alpha^n e^{-u}}{(n-1)!}\sum_{k=0}^{\bar{k}}\binom{n}{k}(-)^k[u-kq_M]^{n-1} \tag{57}$$

where $\bar{k}$ is defined as the largest value of k for which the bracket is negative.

Equation (57) has been generalized to a mixture of m elements, including capture, on the assumption that all scattering cross sections and the capture cross section vary in an identical fashion with the energy (W2). The result is:

$$N(n, u)=\frac{\kappa e^{-u}}{(n-1)!}\sum_{k=0}^{\bar{k}}(-)^k\binom{n}{k}k!\gamma^{n-k}\sum_{k_1+k_2+\cdots+k_m=k}\frac{\gamma_1^{k_1}\gamma_2^{k_2}\cdots\gamma_m^{k_m}}{k_1!k_2!\cdots k_m!}\left(u-\sum_{j=1}^{m}k_jq_{Mj}\right), \tag{57a}$$

where $l^{-1}=l_s^{-1}+l_c^{-1}$, $\kappa=l/l_s$, $\gamma_j=l\alpha_j/l_j$, $\gamma=\sum_{j=1}^{m}k_j$.

The asymptotic behavior of $N(n, u)$ for a single element has also been investigated by Dancoff (D1). He finds:

$$N(n, u)\approx\alpha e^{-u}(\alpha q_M)^{n-1}\left(\frac{6}{\pi n}\right)^{\frac{1}{2}}\exp\left[-12n\left(\frac{u}{nq_M}-\frac{1}{2}\right)^2\right]\left[\sinh 6\left(\frac{u}{nq_M}-\frac{1}{2}\right)\Big/6\left(\frac{u}{nq_M}-\frac{1}{2}\right)\right]. \tag{58}$$

$N(n, u)$ reaches a maximum, and is symmetrical about the point $u=\frac{1}{2}nq_M$; this maximum becomes sharper as n increases. For very large n, (58) may, in turn, be approximated by a Gaussian:

$$N(n, u)\approx\alpha e^{-u}(\alpha q_M)^{n-1}\left(\frac{6}{\pi n}\right)^{\frac{1}{2}}\exp\left[-6n\left(\frac{u}{nq_M}-\frac{1}{2}\right)^2\right]. \tag{59}$$

Equation (59) is valid provided $n\gg 1$ and $|(u/nq_M)-\frac{1}{2}|\ll 1$.

It is clear that the total number of neutrons per unit logarithmic energy interval, $N_0(u)$, is obtained by adding the contributions from all collisions, that is:

$$N_0(u)=\sum_{n=0}^{\infty}N(n, u). \tag{60}$$

The relation between $N_0(u)$ and $\Psi_0(u)$ is simply: $N_0(u)=l(u)/v\Psi_0(u)$ (cf. above).

III. SPATIAL DISTRIBUTION OF SLOWED-DOWN NEUTRONS

In Part II we have treated some characteristic problems associated with the energy distribution of neutrons slowed down by elastic collision with nuclei. However, for many purposes, one requires a knowledge not only of the energy distribution of the slowed-down neutrons, but also of their spatial distribution. In some instances—such as the design of a slow neutron pile—the complete spatial distribution of the neutron density need not be known but only the second moment. In other cases, an accurate knowledge of the neutron density itself is desirable. We, therefore, turn—in this part—to a discussion of the spatial distribution of slowed-down neutrons. Section A is devoted to a fairly complete presentation of the results on the second spatial moment of the neutron density. In Section B, we consider more briefly the higher spatial moments of the neutron density. In Section C, we give a direct derivation of the age approximation to the neutron density and examine the conditions of its validity. Finally, Section D contains an account of improvements on age theory and of the asymptotic neutron density.

A. Second Spatial Moment of the Neutron Density

We work with the stationary, non-capturing form of Eq. (12). Capture and time dependence are complications which can be incorporated into the theory if necessary (cf. below). The discussion of the spatial behavior of the neutron density in an infinite slowing-down medium can be carried out on the basis of a plane δ-source. This follows from the fact that (1) the neutron density, $N_0(\mathbf{r}, u)$,[11] due to an arbitrary distribution of (isotropic) sources $S(\mathbf{r})$ is given by:

$$N_0(\mathbf{r}, u) = \int d\mathbf{r}' S(\mathbf{r}') G_{\mathrm{pt}}(|\mathbf{r}-\mathbf{r}'|, u), \tag{61}$$

where G_{pt} is the neutron density caused by a point source of unit strength; and (2) $G_{\mathrm{pt}}(\mathbf{r}, u)$ is related to the neutron density $G_{\mathrm{pl}}(z, u)$, caused by a plane δ-source of unit strength by:

$$G_{\mathrm{pt}}(r, u) = -\frac{1}{2\pi r}\frac{\partial G_{\mathrm{pl}}(z, u)}{\partial z}\bigg|_{z=\mathbf{r}}. \tag{62}$$

The stationary, non-capturing form of Eq. (12) in one dimension—corresponding to a source $\delta(z)$ —becomes:

$$l(u)\mu(\partial\psi/\partial z) + \psi(z, \mu, u) = \int_0^u du' \int d\mathbf{\Omega}' \psi(z, \mu', u') f(\mu_0, u-u') + \delta(z)\delta(u)/4\pi. \tag{63}$$

Equation (63) is the starting point for our discussion of the spatial moments of the neutron density. We keep the treatment general at first and then specialize to the second spatial moment.

Spatial moments of a distribution function are most easily found by studying the Fourier transform of the distribution function. Let us, therefore, take the Fourier transform of Eq. (63) with respect to z; we get:

$$[1 - iyl(u)]\phi(y, \mu, u) = \int_0^u du' \int d\mathbf{\Omega}' \phi(y, \mu', u') f(\mu_0, u-u') + \delta(u)/4\pi, \tag{64}$$

where

$$\phi(y, \mu, u) \equiv \mathfrak{F}[\psi(z, \mu, u)] = \int_{-\infty}^{\infty} dz e^{iyz}\psi(z, \mu, u).$$

Suppose we expand ϕ, f in spherical harmonics:

$$\phi(y, \mu, u) = \sum_{l=0}^{\infty} [(2l+1)/4\pi]\phi_l(y, u) P_l(\mu), \tag{65a}$$

$$f(\mu; u) = \sum_{l=0}^{\infty} [(2l+1)/4\pi] f_l(u) P_l(\mu), \tag{65b}$$

where

$$\phi_l(y, u) = \int d\mathbf{\Omega} P_l(\mu)\phi(y, \mu, u), \tag{65c}$$

$$f_l(u) = \int d\mathbf{\Omega} P_l(\mu) f(\mu, u). \tag{65d}$$

If we now multiply Eq. (64) by $P_l(\mu)$ and integrate over $d\mathbf{\Omega}$, we arrive at an infinite set of integral

[11] The neutron density, $N_0(\mathbf{r}, u)$ is related to the number of collisions per unit volume per unit time integrated over all angles, $\psi_0(\mathbf{r}, u)$ by $N_0(\mathbf{r}, u) = l(u)/v\, \psi_0(\mathbf{r}, u)$. In the remainder of this article, we shall refer to $\psi_0(\mathbf{r}, u)$ as the neutron density.

equations, thus (the subscript l is not to be confused with the mean free path $l(u)$):

$$\phi_0(y,u)-iyl(u)\phi_1(y,u)=\int_0^u du'\phi_0(y,u')f_0(u-u')+\delta(u), \tag{66a}$$

$$\phi_l(y,u)-\frac{iyl(u)}{(2l+1)}[l\phi_{l-1}(y,u)+(l+1)\phi_{l+1}(y,u)]=\int_0^u du'\phi_l(y,u)'f_l(u-u'). \tag{66b}$$

Each $\phi_l(y,u)$ may be expanded in a power series in y, and from the structure of Eqs. (66a) and (66b), it is seen that $\phi_l(y,u)$ contains only even powers of y when l is even, and odd powers when l is odd. Furthermore, it is easily seen that the expansion for $\phi_l(y,u)$ starts with y^l. Hence we may write:

$$\phi_l(y,u)=\sum_{k=l,\,l+2,\,l+4,\,\cdots}^{\infty} i^k\phi_l^{(k)}(u)y^k/k!. \tag{67}$$

We wish to know the spatial moments of the neutron density; the $2m$th spatial moment is defined as:

$$[z^{(2m)}(u)]_{\mathrm{Av}}=\int_{-\infty}^{\infty}dzz^{2m}\int d\Omega\psi(z,\mu,u)\Big/\int_{-\infty}^{\infty}dz\int d\Omega\psi(z,\mu,u). \tag{68}$$

It is easy to see that (68) reduces to:

$$[z^{(2m)}(u)]_{\mathrm{Av}}=\phi^{(2m)}(u)/\phi_0^{(0)}(u). \tag{69}$$

The procedure for evaluating the various $\phi_0^{(2m)}(u)$ when the mean free path is permitted to vary arbitrarily with energy is clear in principle, but laborious to carry out in practice. Since—in lieu of the actual neutron density itself—the second spatial moment is of greatest interest, we shall in this section derive some fairly rigorous expressions for it. The higher spatial moments will be calculated under more restrictive assumptions in Section B.

The slowing down length L_s, which is a term widely used in the literature, is defined in terms of the second spatial moment by:

$$L_s(u)=\left(\frac{[z^2(u)]_{\mathrm{Av}}}{2}\right)^{\frac{1}{2}}\equiv\left(\frac{[r^2(u)]_{\mathrm{Av}}}{6}\right)^{\frac{1}{2}}. \tag{70}$$

In (70), the equivalence of $[z^2(u)]_{\mathrm{Av}}$ and $\frac{1}{3}[r^2(u)]_{\mathrm{Av}}$ (the average r^2 from a point source) follows immediately from (62). Formulae for L_s have been obtained for the following cases[12]:

1. Single element: arbitrary variation of mean free path with energy.
2. Single element: representation of mean free path by sum of decreasing exponentials in u.
3. Mixture containing hydrogen or deuterium: representation of mean free paths by sum of decreasing exponentials in u.
4. Mixture of heavy elements: representation of mean free paths by series of step functions.
5. Hydrogenous mixture: rigorous formula assuming infinite mass for the heavy component.

1. Single Element: Arbitrary Variation of Mean Free Path with Energy

To find the slowing-down length, we must find $\phi_0^{(0)}(u)$ and $\phi_0^{(2)}(u)$ (cf. Eq. (69)). Inspection of the system of Eqs. (66a) and (66b) reveals that the determination of $\phi_0^{(0)}(u)$ and $\phi_0^{(2)}(u)$ only requires (66a) and the $l=1$ equation of (66b), with ϕ_2 omitted. Applying a Laplace transformation

[12] Numerical applications of the formulae for L_s are given in the appendix.

with respect to u to these two equations, we find:

$$\gamma_0(\eta)\Phi_0(y,\eta)-iy\bar{\Phi}_1(y,\eta)=1, \tag{71a}$$

$$\gamma_1(\eta)\Phi_1(y,\eta)-(iy/3)\bar{\Phi}_0(y,\eta)=0, \tag{71b}$$

where

$$\Phi_{0,1}(y,\eta)=\mathfrak{L}[\phi_{0,1}(y,u)],$$

$$\bar{\Phi}_{0,1}(y,\eta)=\mathfrak{L}[l(u)\phi_{0,1}(y,u)],$$

$$\gamma_{0,1}(\eta)=1-g_{0,1}(\eta),$$

with

$$g_0(\eta)\equiv\mathfrak{L}[f_0(u)]=1-\frac{\alpha}{\eta+1}\{1-\exp[-q_M(\eta+1)]\}, \tag{72a}$$

$$g_1(\eta)\equiv\mathfrak{L}[f_1(u)]=1-\frac{\alpha(M+1)}{(2\eta+3)}\{1-\exp[-q_M(\eta+\tfrac{3}{2})]\}+\frac{\alpha(M-1)}{2\eta+1}\{1-\exp[-q_M(\eta+\tfrac{1}{2})]\}. \tag{72b}$$

The subscripts 0 and 1 refer to the zeroth and first spherical harmonics of the various quantities in question.

The expansion (67) applies equally well to the Laplace transform of both sides. Making use of an obvious notation, we have:

$$2L_s^2(u)=\frac{\mathfrak{L}^{-1}[\Phi_0^{(2)}(\eta)]}{\mathfrak{L}^{-1}[\Phi_0^{(0)}(\eta)]}. \tag{73}$$

Substituting the expansion (67) into (71a) and (71b), and equating the coefficients of y^0, y^1, y^2, we obtain:

$$\Phi_0^{(0)}(\eta)=1/\gamma_0(\eta), \tag{74a}$$

$$\Phi_1^{(1)}(\eta)=[1/3\gamma_1(\eta)]\bar{\Phi}_0^{(0)}(\eta), \tag{74b}$$

$$\Phi_0^{(2)}(\eta)=[2/\gamma_0(\eta)]\bar{\Phi}_1^{(1)}(\eta). \tag{74c}$$

Let us denote the Laplace inverse of $1/\gamma_{0,1}(\eta)$ by $\bar{\gamma}_{0,1}(u)$; then:

$$\mathfrak{L}^{-1}[\Phi_0^{(0)}(\eta)]=\bar{\gamma}_0(u). \tag{75}$$

To obtain the Laplace inverse of $\Phi_0^{(2)}(\eta)$, we make use of the inverse of the convolution theorem, i.e.:

$$\mathfrak{L}^{-1}[G_1(\eta)G_2(\eta)]=\int_0^u du' F_1(u-u')F_2(u'),$$

where

$$F_{1,2}(u)=\mathfrak{L}^{-1}[G_{1,2}(\eta)],$$

and also the fact that the Laplace inverse of $\bar{\Phi}$ is $l(u)$ times the Laplace inverse of Φ. Thus, (74c) yields

$$\mathfrak{L}^{-1}[\Phi_0^{(2)}(\eta)]=2\int_0^u du'\bar{\gamma}_0(u-u')l(u')\Psi_1^{(1)}(u'), \tag{76}$$

where

$$\Psi_1^{(1)}(u)=\mathfrak{L}^{-1}[\Phi_1^{(1)}(\eta)].$$

But

$$\mathfrak{L}^{-1}[\Phi_1^{(1)}(\eta)]=\tfrac{1}{3}\int_0^u du'\bar{\gamma}_1(u-u')l(u')\Psi_0(u')du', \tag{77}$$

where

$$\Psi_0^{(0)}(u)=\mathfrak{L}^{-1}[\Phi_0^{(0)}(\eta)].$$

Combining (75)–(77), we arrive at the final formula:

$$L_s^2(u)=\frac{1}{3\bar{\gamma}_0(u)}\int_0^u du'\bar{\gamma}_0(u-u')l(u')\int_0^{u'} du''\bar{\gamma}_1(u'-u'')l(u'')\bar{\gamma}_c(u''). \tag{78}$$

Equation (78) has been derived by Marshak (M4) and Schwinger (S1).

Equation (78) is a rigorous result, and in principle can be used to calculate L_s for an arbitrary variation of mean free path with energy since the γ's are completely well-defined functions. For example, in the case of hydrogen (since $M=1$), we have $\gamma_0(\eta)=\eta/(\eta+1)$ and $\gamma_1(\eta)=(\eta+\frac{1}{2})/(\eta+\frac{3}{2})$, so that $\bar{\gamma}_0(u)=\delta(u)+1$ and $\bar{\gamma}_1(u)=\delta(u)+e^{-u/2}$. A straightforward substitution for $\bar{\gamma}_0(u)$ and $\bar{\gamma}_1(u)$ then leads to the well-known result:

$$L_s^2(u)=\tfrac{1}{3}\Big\{l^2(0)+l^2(u)+l(0)l(u)e^{-u/2}+l(u)\int_0^u du'l(u')e^{-(u-u')/2}+\int_0^u du'l^2(u')$$
$$+l(0)\int_0^u du'l(u')e^{-u'/2}+\int_0^u du'l(u')\int_0^{u'} du''l(u'')e^{-(u'-u'')/2}\Big\}. \tag{79}$$

Equation (79) has been derived by alternative methods (F1 and O1).

Ashkin (A4) has generalized Eq. (79) to the case of a mixture consisting of hydrogen and an absorbing material. He neglects the scattering properties of the absorber but takes account of the variation of capture cross section with energy. He finds for the slowing-down length:

$$L_s^2(u)=\tfrac{1}{3}\Big\{l_t^2(0)+l_t^2(u)+l_t(0)l_t(u)e^{-u/2}+l_t(u)\int_0^u du'\frac{l_t^2(u')}{l_H(u')}e^{-(u-u')/2}+\int_0^u du'\frac{l_t^3(u')}{l_H(u')}$$
$$+l_t(0)\int_0^u du'\frac{l_t^2(u')}{l_H(u')}e^{-u'/2}+\int_0^u du'\frac{l_t^2(u')}{l_H(u')}\int_0^{u'} du''\frac{l_t^2(u'')}{l_H(u'')}e^{-(u'-u'')/2}, \tag{79a}$$

where $1/[l_t(u)]=1/[l_H(u)]+1/[l_c(u)]$ with l_H the scattering mean free path of hydrogen, and l_c the capture mean free path of the absorber.

For elements heavier than hydrogen, the calculation of $\bar{\gamma}_0(u)$ and $\bar{\gamma}_1(u)$ is more complicated, and for large u and not too irregular variation of the mean free path with energy, it is easier to apply the formula derived below. If the mean free path changes irregularly, i.e., there are resonance effects, it is usually preferable to use Eq. (78) (cf. M4).

2. Single Element: Representation of Mean Free Path by Sum of Exponentials

As remarked above, formula (78) is, in general, difficult to apply to nuclei with masses greater than that of hydrogen. If the variation of mean free path with energy is not too irregular, then the following artifice can be resorted to (M2); we write $l(u)$ in the form:

$$l(u)=\sum_{j=0}^{\infty} A_j e^{-a_j u} \tag{80}$$

where A_j and a_j are arbitrary constants with[13] $a_j\geqslant 0$. This representation is quite general since a large number of terms may be taken; it becomes laborious when the variation of $l(u)$ with u is extremely irregular. Using (80) for $l(u)$, we find Eqs. (71a) and (71b) reduce to:

$$\gamma_0(\eta)\Phi_0(y,\eta)-iy\sum_{j=0}^{\infty} A_j\Phi_1(y,\eta+a_j)=1, \tag{81a}$$

$$\gamma_1(\eta)\Phi_1(y,\eta)-\frac{iy}{3}\sum_{j=0}^{\infty} A_j\Phi_0(y,\eta+a_j)=0. \tag{81b}$$

[13] It is necessary that $a_j\geqslant 0$ in order for the Laplace transform to exist (cf. below).

Equations (81a) and (81b) follow from the fact that

$$\mathfrak{L}[e^{-au}F(u)]=G(\eta+a), \quad \text{where} \quad G(\eta)=\mathfrak{L}[F(u)].$$

Following the procedure which led to the derivation of Eqs. (74a)–(74c), we obtain:

$$\Phi_0^{(0)}(\eta)=1/\gamma_0(\eta), \tag{82a}$$

$$\Phi_1^{(1)}(\eta)=\frac{1}{3\gamma_1(\eta)}\sum_{j=0}^{\infty}A_j\Phi_0^{(0)}(\eta+a_j), \tag{82b}$$

$$\Phi_0^{(2)}(\eta)=\frac{2}{\gamma_0(\eta)}\sum_{j=0}^{\infty}A_j\Phi_1^{(1)}(\eta+a_j). \tag{82c}$$

From Eqs. (82a)–(82c), we can immediately write down the solutions for $\Phi_0^{(0)}(\eta)$ and $\Phi_0^{(2)}(\eta)$, the two quantities which are needed for the determination of L_s. These are:

$$\Phi_0^{(0)}(\eta)=1/\gamma_0(\eta), \tag{83a}$$

$$\Phi_0^{(2)}(\eta)=\frac{2}{3\gamma_0(\eta)}\sum_{k=0}^{\infty}\frac{A_k}{\gamma_1(\eta+a_k)}\sum_{j=0}^{\infty}\frac{A_j}{\gamma_0(\eta+a_j+a_k)}. \tag{83b}$$

Since

$$2L_s^2(u)=\frac{\mathfrak{L}^{-1}[\Phi_0^{(2)}(\eta)]}{\mathfrak{L}^{-1}[\Phi_0^{(0)}(\eta)]},$$

we must still take the Laplace inverses of (83a) and (83b). This cannot be done easily; however, for large u, recourse can be had to the Tauberian theorem[14] so that it is only necessary to study the behavior of $\Phi_0^{(0)}(\eta)$ and $\Phi_0^{(2)}(\eta)$ for small η. We find:

$$\mathfrak{L}^{-1}[\Phi_0^{(0)}(\eta)]\approx 1/\gamma_0'(0), \tag{84a}$$

$$\mathfrak{L}^{-1}[\Phi_0^{(2)}(\eta)]\approx\frac{2A_0^2u}{3[\gamma_0'(0)]^2\gamma_1(0)}+\frac{2}{3\gamma_0'(0)}\left\{\sum_{\substack{j=k=0\\(j=k=0\text{ excluded})}}^{\infty}\frac{A_kA_j}{\gamma_1(a_k)\gamma_0(a_j+a_k)}-\frac{A_0^2}{\gamma_0'(0)\gamma_1(0)}\left(\frac{\gamma_0''(0)}{\gamma_0'(0)}+\frac{\gamma_1'(0)}{\gamma_1(0)}\right)\right\}. \tag{84b}$$

In deriving (84a) and (84b), it is assumed that $a_0=0$ and all the other a_j's$\neq 0$. The quantity $2L_s^2$ is the ratio of (84b) to (84a).

For reference purposes, we write down expressions for the γ's and their derivatives[15] which enter into Eqs. (84a) and (84b) (the quantities $\gamma_0(a)$ and $\gamma_1(a)$ are defined by (72a) and (72b)):

$$\gamma_0'(0)=1-\frac{(M-1)^2}{2M}q_M\equiv\xi, \tag{85a}$$

$$\gamma_0''(0)=-2+\frac{(M-1)^2}{2M}q_M(1+q_M), \tag{85b}$$

[14] The Tauberian theorem states that since

$$F(u)=\mathfrak{L}^{-1}[G(\eta)]=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}d\eta e^{\eta u}G(\eta)$$

(where the integration is to the right of all the poles), the behavior of $F(u)$ for large u is found by studying the pole of $G(\eta)$ which has the largest real part (cf. G. Doetsch, *Laplace Transformation*). In the case of $\Phi_0^{(0)}(\eta)$ and $\Phi_0^{(2)}(\eta)$ these poles are at $\eta=0$.

[15] The quantities ξ and $\langle\cos\Theta\rangle_{\text{Av}}$ are the average logarithmic energy loss and average cosine of the angle of deflection, respectively (cf. Eqs. (11a) and (11b).

$$\gamma_1(0) = 1 - \frac{2}{3M} \equiv 1 - \langle\cos\Theta\rangle_{\text{Av}}, \tag{85c}$$

$$\gamma_1'(0) = \frac{1}{3}\left[-2M + \frac{10}{3M} + \frac{1}{2}\left(M^2 - 3 + \frac{2}{M}\right)q_M\right]. \tag{85d}$$

For constant mean free path (independent of energy), Eqs. (84a) and (84b) lead to the following simple formula for $L_s(A_j = 0$ for $j \neq 0)$:

$$L_s^2(u) = \frac{A_0^2}{3\gamma_0'(0)\gamma_1(0)}\left[u - \frac{\gamma_0''(0)}{\gamma_0'(0)} - \frac{\gamma_1'(0)}{\gamma_1(0)}\right]. \tag{86}$$

Equation (86) was first derived by Placzek (P2).

3. *Mixture Containing Hydrogen or Deuterium*

Equations (78) and (84a)–(84b) can be extended to a mixture of elements provided the scattering mean free paths of the component elements vary in exactly the same way over the entire energy interval. If this is the case, the integrals in Eqs. (66a) and (66b) are of convolution type and the method of Laplace transform can be applied as before. If the energy dependence of the mean free paths is not the same, the method of Laplace transform is not immediately applicable and the problem must be examined more closely.

Thus, let us suppose we have a mixture of two elements of masses M and N $(M<N)$, with the variation of the total mean free path, $l(u)$, with energy given by (cf. (80)):

$$l(u) = \sum_{j=0}^{\infty} A_j e^{-a_j u},$$

where A_j is an arbitrary constant, $a_0 = 0$, and $a_j (j>0)$ is an arbitrary positive constant. Furthermore, let us assume that the ratio $c(u) \equiv l(u)/l_M(u)$ has a variation of the form:

$$c(u) = B_0 + \sum_{k=1}^{\infty} B_k \exp[-b_k u],$$

where B_0 and b_k are arbitrary positive constants, and B_k is an arbitrary constant whose absolute value is small compared to B_0. The latter condition is equivalent to the requirement that $c(u)$ is a more slowly varying function of u than $l(u)$. Experimentally, this is usually fulfilled and is true as long as the mean free paths of the component elements do not vary (with u) markedly in opposite directions.

Substituting the expressions for $l(u)$ and $c(u)$ into Eqs. (66a)–(66b) (generalized for a mixture), we obtain the analogs of Eqs. (82a)–(82c), namely:

$$\Phi_0^{(0)}(\eta) = \frac{1}{\gamma_0(\eta)} + \frac{g_0(\eta)}{\gamma_0(n)} \sum_{k=1}^{\infty} B_k \Phi_0^{(0)}(\eta + b_k), \tag{87a}$$

$$\Phi_1^{(1)}(\eta) = \frac{1}{3\gamma_1(\eta)} \sum_{j=0}^{\infty} A_j \Phi_0^{(0)}(\eta + a_j) + \frac{g_1(\eta)}{\gamma_1(\eta)} \sum_{k=1}^{\infty} B_k \Phi_1^{(1)}(\eta + b_k), \tag{87b}$$

$$\Phi_0^{(2)}(\eta) = \frac{2}{\gamma_0(\eta)} \sum_{j=0}^{\infty} A_j \Phi_1^{(1)}(\eta + a_j) + \frac{g_0(\eta)}{\gamma_0(\eta)} \sum_{k=1}^{\infty} B_k \Phi_0^{(2)}(\eta + b_k). \tag{87c}$$

In Eqs. (87a)–(87c), we have set $\gamma_l(\eta) = 1 - [B_0 g_{lM}(\eta) + (1 - B_0) g_{lN}(\eta)]$, and $g_l(\eta) = g_{lM}(\eta) - g_{lN}(\eta)$; the rest of the notation follows closely the previous definitions.

We solve Eqs. (87a)–(87c) by means of a perturbation method. Before we apply perturbation theory, however, let us write down expressions (these are still rigorous) for the Φ's in the neighborhood of $\eta=0$; these will lead to the asymptotic expressions for the Laplace inverses. Thus, neglecting all terms beyond $1/\eta$, we get:

$$\Phi_0^{(0)}(\eta)\approx\frac{1}{\gamma_0'(0)}\cdot\frac{1}{\eta},\tag{88a}$$

$$\Phi_1^{(1)}(\eta)\approx\frac{A_0}{3\gamma_1(0)\gamma_0'(0)}\cdot\frac{1}{\eta},\tag{88b}$$

$$\Phi_0^{(2)}(\eta)\approx\frac{2A_0^2}{3[\gamma_0'(0)]^2\gamma_1(0)}\cdot\frac{1}{\eta^2}+\frac{2}{3\gamma_0'(0)}\left[\sum_{j=1}^{\infty}A_j\Phi_1^{(1)}(a_j)-\frac{A_0^2}{\gamma_0'(0)\gamma_1(0)}\left(\frac{\gamma_0''(0)}{\gamma_0'(0)}+\frac{\gamma_1'(0)}{\gamma_1(0)}\right)\right.$$
$$\left.+\frac{A_0}{\gamma_1(0)}\sum_{j=1}^{\infty}A_j\Phi_0^{(0)}(a_j)+\frac{g_0'(0)A_0^2}{\gamma_0'(0)\gamma_1(0)}\sum_{k=1}^{\infty}B_k\Phi_0^{(0)}(b_k)\right]\cdot\frac{1}{\eta}.\tag{88c}$$

In Eq. (88c), we must insert values for $\Phi_0^{(0)}(d)$ and $\Phi_1^{(1)}(d)$ where $d>0$. Since $|B_k|\ll B_0$, these may be obtained from Eqs. (87a) and (87b) by regarding the terms involving B_k as a perturbation. We therefore first neglect the B_k terms in Eq. (87a) and get:

$$\Phi_0^{(0)}(d)=1/\gamma_0(d).\tag{89}$$

Substituting $[1/\gamma_0(d+b_k)]$ for $\Phi_0^{(0)}(d+b_k)$ into the B_k terms, we find in the next approximation:

$$\bar{\Phi}_0^{(0)}(d)=\Phi_0^{(0)}(d)+\frac{g_0(d)}{\gamma_0(d)}\sum_{k=1}^{\infty}\frac{B_k}{\gamma_0(d+b_k)}.\tag{89a}$$

In a similar fashion, the first approximation for $\Phi_1^{(1)}(d)$ (cf. Eq. (87b)) is:

$$\Phi_1^{(1)}(d)=\frac{1}{3\gamma_1(d)}\sum_{j=0}^{\infty}A_j\Phi_0^{(0)}(d+a_j).\tag{90}$$

In the next approximation we get:

$$\bar{\Phi}_1^{(1)}(d)=\Phi_1^{(1)}(d)+\frac{1}{3\gamma_1(d)}\left\{\sum_{j=0}^{\infty}\frac{A_j}{\gamma_0(d+a_j)}g_0(d+a_j)\sum_{k=1}^{\infty}\frac{B_k}{\gamma_0(d+a_j+b_k)}\right.$$
$$\left.+g_1(d)\sum_{k=1}^{\infty}\frac{B_k}{\gamma_1(d+b_k)}\sum_{j=0}^{\infty}\frac{A_j}{\gamma_0(d+a_j+b_k)}\right\}.\tag{90a}$$

Inserting (89a) and (90a) into Eqs. (88a)–(88c), we obtain the asymptotic expressions for L_s:

$$L_s^2(u)=\frac{1}{3}\frac{A_0^2u}{\gamma_0'(0)\gamma_1(0)}+\frac{1}{3}\left\{\sum_{j=1}^{\infty}\frac{A_j}{\gamma_1(a_j)}\left[\sum_{l=0}^{\infty}\frac{A_l}{\gamma_0(a_j+a_l)}\cdot\left[1+g_0(a_j+a_l)\cdot\sum_{k=1}^{\infty}\frac{B_k}{\gamma_0(a_j+a_l+b_k)}\right]\right.\right.$$
$$\left.+g_1(a_j)\sum_{l=0}^{\infty}A_l\sum_{k=1}^{\infty}\frac{B_k}{\gamma_1(a_j+b_k)\gamma_0(a_j+a_l+b_k)}\right]-\frac{A_0^2}{\gamma_0'(0)\gamma_1(0)}\left(\frac{\gamma_0''(0)}{\gamma_0'(0)}+\frac{\gamma_1'(0)}{\gamma_1(0)}\right)$$
$$\left.+\frac{A_0}{\gamma_1(0)}\cdot\sum_{j=1}^{\infty}\frac{A_j}{\gamma_0(a_j)}\left[1+g_0(a_j)\sum_{k=1}^{\infty}\frac{B_k}{\gamma_0(a_j+b_k)}\right]+\frac{g_0'(0)A_0^2}{\gamma_0'(0)\gamma_1(0)}\cdot\sum_{k=1}^{\infty}\frac{B_k}{\gamma_0(b_k)}\right\}.\tag{91}$$

Of course, it is possible to go on to the next approximation and arrive at more accurate values of $\Phi_0^{(0)}(d)$ and $\Phi_1^{(1)}(d)$ and, therefore, of L_s. However, if the B_k's are really small, Eq. (91) will yield fairly accurate results.

Formula (91) can be generalized to mixtures of more than two elements and is especially useful for mixtures containing hydrogen or deuterium.

4. *Mixtures of Heavy Elements*

If the mixture of elements does not contain hydrogen or deuterium and there are, moreover, rapid variations in the mean free paths (i.e., resonances) of the component elements, a fairly accurate expression for the second moment can be derived if the total logarithmic energy interval, u, can be divided into a finite number of sub-intervals, each large compared with q_{M1} (the maximum logarithmic energy loss associated with the element of the mixture having the smallest M) and over each of which the mean free paths of the different components are sensibly constant. Thus, consider a mixture of m nuclei whose masses are arranged in order of increasing magnitude:

$$M_1 < M_2 < \cdots < M_m;$$

further, let the jth nucleus have a scattering mean free path $l_j(u)$. If $l(u)$ is the total mean free path, i.e.,

$$1/l(u) = \sum_{j=1}^{m} 1/l_j(u),$$

then we can use the transport Eq. (63) provided we understand that:

$$f(\mu_0, u-u') \rightarrow \sum_{j=1}^{m} c_j(u') f_j(\mu_0, u-u'), \tag{92}$$

where

$$c_j(u) = \frac{l(u)}{l_j(u)} \quad (j=1, \cdots m) \quad \text{and} \quad \sum_{j=1}^{m} c_j(u) = 1 \text{ (for all } u).$$

Making use of our basic assumption, we now divide the interval $(0, u)$ into sub-intervals $(0, u_1)$, (u_1, u_2), $\cdots (u_{n-2}, u_{n-1})$, (u_{n-1}, u) such that each sub-interval is large compared with q_{M_1}, and the $l_j(u)$ $(j=1, \cdots m)$ are step-functions, and equal to constants l_{jk} $(k=1, \cdots n)$ in the kth sub-interval. It follows that the value of the total mean free path $l(u)$ in the kth sub-interval—which we denote by l_k—is given by:

$$\frac{1}{l_k} = \sum_{j=1}^{m} \frac{1}{l_{jk}} \quad (k=1, \cdots n), \tag{93}$$

and that the effective concentration, $c_j(u)$ of the jth component in the kth sub-interval, which we denote by c_{jk}, is given by:

$$c_{jk} = (l_k/l_{jk}) \quad (j=1, \cdots m;\ k=1, \cdots n). \tag{94}$$

Finally, to complete our notation, we write $f(\mu_0, u-u)$ in the kth sub-interval as $f^{(k)}(\mu_0, u-u')$ so that:

$$f^{(k)}(\mu_0, u-u') = \sum_{j=1}^{m} c_{jk} f_j(\mu_0, u-u') \quad (k=1, \cdots n). \tag{95}$$

In order to make clear the essential points of our procedure, we work out the case of two sub-intervals. We then write down the general formula for n sub-intervals. If we take $n=2$, Eq. (63)

reduces to:

$$l_1\mu\frac{\partial\psi^{(1)}}{\partial z}+\psi^{(1)}(z,\mu,u)=\int_0^u du'\int d\mathbf{\Omega}'\psi^{(1)}(z,\mu',u')f^{(1)}(\mu_0,u-u')+\frac{\delta(z)\delta(u)}{4\pi}\quad(u<u_1),\tag{96a}$$

$$l_2\mu\frac{\partial\psi^{(2)}}{\partial z}+\psi^{(2)}(z,\mu,u)=\int_0^{u_1} du'\int d\mathbf{\Omega}'\psi^{(1)}(z,\mu',u')f^{(1)}(\mu_0,u-u')+\int_{u_1}^u du'\int d\mathbf{\Omega}'\psi^{(2)}(z,\mu',u')\times f^{(2)}(\mu_0,u-u')\quad(u>u_1).\tag{96b}$$

In Eqs. (96a) and (96b), $\psi^{(1)}$ is defined in the first interval $(0, u_1)$, and $\psi^{(2)}$ is defined in the second interval (u_1, u). Since $u_1 \gg q_{M1}$, we can solve (96a) by the usual asymptotic considerations for constant mean free path. In other words, if we write:

$$\phi_0^{(1)}(y,u)=\phi_{00}^{(1)}(u)-\frac{y^2}{2}\phi_{02}^{(1)}(u)+\cdots,\tag{97a}$$

$$\phi_1^{(1)}(y,u)=iy\phi_{11}^{(1)}(u)+\cdots,\tag{97b}$$

then (cf. Eqs. (74a)–(74c)):

$$\phi_{00}^{(1)}(u)=1/a_1,\tag{98a}$$

$$\phi_{11}^{(1)}(u)=b_1/3a_1\alpha_1,\tag{98b}$$

$$\phi_{02}^{(1)}(u)=\frac{2l_1^2u}{3a_1^2\alpha_1}-\frac{2l_1^2}{3a_1^2\alpha_1}\left(\frac{b_1}{a_1}+\frac{\beta_1}{\alpha_1}\right),\tag{98c}$$

where

$$a_1=\sum_{j=1}^m c_{j1}\gamma_{0j}'(0),\quad b_1=\sum_{j=1}^m c_{j1}\gamma_{0j}''(0),\quad \alpha_1=\sum_{j=1}^m c_{j1}\gamma_{1j}(0),\quad \beta_1=\sum_{j=1}^m c_{j1}\gamma_{1j}'(0),$$

with the γ's defined by Eqs. (85a)–(85d) for the jth nucleus.

With a knowledge of $\phi_{00}^{(1)}(u)$, $\phi_{11}^{(1)}(u)$, and $\phi_{02}^{(1)}(u)$ we can proceed to solve Eq. (96b). If we change the variable in (96b) from u to $u=\bar{u}-u_1$, write $\bar{\psi}^{(2)}(z,\mu,\bar{u})$ for $\psi^{(2)}(z,\mu,u)$, and take the customary Fourier transform with respect to z, and Laplace transform with respect to $\bar{u}$, we find:

$$(1-il_2y\mu)\Phi^{(2)}(y,\mu,\bar{\eta})=B^{(1)}(y,\mu,\bar{\eta})+\int d\mathbf{\Omega}'\Phi^{(2)}(y,\mu',\bar{\eta})g^{(2)}(\mu_0,\eta),\tag{99}$$

where

$$\Phi^{(2)}(y,\mu,\bar{\eta})=\mathfrak{L}_{(\bar{\eta})}\left[\int_{-\infty}^{\infty}dze^{iyz}\bar{\psi}^{(2)}(z,\mu,\bar{u})\right],$$

$$g^{(2)}(\mu_0,\bar{\eta})=\mathfrak{L}_{(\bar{\eta})}f^{(2)}(\mu_0,\bar{u}),$$

$$B^{(1)}(y,\mu,\bar{\eta})=\mathfrak{L}_{(\bar{\eta})}\left[\int_{-\infty}^{\infty}dze^{iyz}\int_0^{u_1}du'\int d\mathbf{\Omega}'\psi^{(1)}(z,\mu',u')f^{(1)}(\mu_0,\bar{u}-u'+u_1).\right.$$

If we then take the zero and first moments of (99) with respect to μ, expand in powers of y, etc., we finally obtain:

$$\gamma_0^{(2)}(\bar{\eta})\Phi_{00}^{(2)}(\bar{\eta})=B_{00}^{(1)}(\bar{\eta}),\tag{100a}$$

$$\gamma_1^{(2)}(\bar{\eta})\Phi_{11}^{(2)}(\bar{\eta})-\frac{l_2}{3}\Phi_{00}^{(2)}(\bar{\eta})=B_{11}^{(1)}(\bar{\eta}),\tag{100b}$$

$$\gamma_0^{(2)}(\bar{\eta})\Phi_{02}^{(2)}(\bar{\eta})-2l_2\Phi_{11}^{(2)}(\bar{\eta})=B_{02}^{(1)}(\bar{\eta}).\tag{100c}$$

In Eqs. (100a)–(100c), the first subscript refers to the moment with respect to μ, the second to the expansion in powers of (iy).

Now

$$2L_s^2(u)=\mathfrak{L}_{(\bar{u})}^{-1}[\Phi_{02}^{(2)}(\bar{\eta})]/\mathfrak{L}_{(\bar{u})}^{-1}[\Phi_{00}^{(2)}(\bar{\eta})],$$

and since it is assumed that $\bar{u}\gg q_{M1}$, we can obtain an expression for L_s valid up to terms of order $\exp[-\bar{u}/q_{M1}]$ by studying the behavior of $\Phi_{00}^{(2)}(\bar{\eta})$ and $\Phi_{02}^{(2)}(\bar{\eta})$ as $\bar{\eta}\rightarrow 0$. The B's have no poles or zeros at $\bar{\eta}=0$; hence, we get:

$$\mathfrak{L}_{(\bar{u})}^{-1}[\Phi_{00}^{(2)}(\bar{\eta})]\approx B_{00}^{(1)}(0)/a_2, \tag{101a}$$

$$\mathfrak{L}_{(\bar{u})}^{-1}[\Phi_{02}^{(2)}(\bar{\eta})]\approx\frac{2l_2^2B_{00}^{(1)}(0)}{3a_2^2\alpha_2}\left[\bar{u}-\left(\frac{b_2}{a_2}+\frac{\beta_2}{\alpha_2}\right)\right]+\frac{2l_2^2}{3a_2^2\alpha_2}\frac{dB_{00}^{(1)}(\bar{\eta})}{d\bar{\eta}}\bigg|_{\bar{\eta}=0}+\frac{2l_2B_{11}^{(1)}(0)}{a_2\alpha_2}+\frac{B_{02}^{(1)}(0)}{a_2}. \tag{101b}$$

Substituting (98a)–(98c) into the definitions of the B's, we find for the quantities which appear in (101a) and (101b) the following:

$$B_{00}^{(1)}(0)=1,\quad \frac{dB_{00}^{(1)}(\bar{\eta})}{d\bar{\eta}}\bigg|_{\bar{\eta}=0}=\frac{b_1}{2a_1},\quad B_{11}^{(1)}(0)=\frac{l_1\beta_1}{3a_1\alpha_1},\quad B_{02}^{(1)}(0)=\frac{2l_1^2}{3a_1\alpha_1}\left[u_1-\left(\frac{b_1}{2a_1}+\frac{\beta_1}{\alpha_1}\right)\right].$$

Inserting these into (101a) and (101b), we obtain as the final two-interval formula for L_s (M3):

$$L_s^2(u)=\frac{1}{3}\left\{\frac{l_2^2}{a_2\alpha_2}\left[u-u_1-\frac{b_2}{a_2}-\frac{\beta_2}{\alpha_2}+\frac{b_1}{2a_1}\right]+\frac{l_1l_2\beta_1}{a_1\alpha_1\alpha_2}+\frac{l_1^2}{a_1\alpha_1}\left[u_1-\frac{b_1}{2a_1}-\frac{\beta_1}{\alpha_1}\right]\right\}. \tag{102}$$

The generalization to n intervals is easy; we find (the terms have been rearranged and we have set $u_0=0$, $u_n=u$):

$$L_s^2(u)=\frac{1}{3}\left\{\sum_{k=1}^{n}\frac{l_k^2(u_k-u_{k-1})}{a_k\alpha_k}+\sum_{k=1}^{n-1}\left\{\left(\frac{b}{2a}\right)_k\left[\left(\frac{l^2}{a\alpha}\right)_{k+1}-\left(\frac{l^2}{a\alpha}\right)_k\right]+\left(\frac{l\beta}{a\alpha}\right)_k\left[\left(\frac{l}{\alpha}\right)_{k+1}-\left(\frac{l}{\alpha}\right)_k\right]\right\}-\left[\frac{l^2}{a\alpha}\left(\frac{b}{a}+\frac{\beta}{\alpha}\right)\right]_n\right\}. \tag{103}$$

Examination of (103) reveals that the first term on the right-hand side is simply the result one would obtain by an application of the age theory under the specified assumptions (cf. Section C). The remaining terms represent the correction to age theory which is a consequence of starting with the rigorous transport equation. Of course, (103) does not constitute a rigorous solution since terms of order $\exp[-(u_k-u_{k-1})q_{M1}]$ have been neglected. However, so long as $(u_k-u_{k-1})\gg q_{M1}$ (for all k), then the error introduced by using (103) is small. Since l_k is never strictly constant, some average value must be used; the most reasonable choice is

$$l_k=\left[\int_{u_{k-1}}^{u_k}l^2(u)du\bigg/(u_k-u_{k-1})\right]^{\frac{1}{2}}$$

(cf. Appendix).

5. *Hydrogenous Mixtures: Rigorous Formula Assuming Infinite Mass for Heavy Element*

Substances containing appreciable amounts of hydrogen—e.g., water or paraffin—slow down neutrons so effectively that it is desirable to have more accurate values of L_s than can be obtained from the approximate formula (91). It is possible to derive a rigorous formula for L_s in a hydrogenous mixture if the heavy element in the mixture[16] is assumed to be infinitely heavy. The latter assumption implies that the scattering, but not the slowing-down property of the heavy element, is taken into account. This assumption introduces only a slight error[17] so long as the scattering cross section of hydrogen is considerably greater than the scattering cross section of the heavy element.

If the heavy element of the hydrogenous mixture is assigned infinite mass, the scattering function can be written:

$$f(\mu_0;\, u,\, u') = \frac{c(u')e^{-(u-u')}}{2\pi}\delta[\mu_0 - e^{-(u-u')/2}] + \frac{[1-c(u')]}{4\pi}\delta(u-u'), \tag{104}$$

where $c(u)$ is the ratio of the total mean free path to the hydrogen mean free path at the energy corresponding to u. Substitution of (104) into Eq. (63) does not lead to an integral equation of convolution type so that a solution for L_s through the use of Laplace transform is not feasible. Instead, a direct solution can be obtained as follows. We take the Fourier transform of Eq. (63) and then the zero and first moments with respect to μ to find:

$$\phi_0(y,\, u) - iyl(u)\phi_1(y,\, u) = \int_0^u du'\phi_0(y,\, u')f_0(u,\, u') + \delta(u), \tag{105a}$$

$$\phi_1(y,\, u) - \frac{iyl(u)}{3}\phi_0(y,\, u) = \int_0^u du'\phi_1(y,\, u')f_1(u,\, u'), \tag{105b}$$

where

$$f_0(u,\, u') = c(u')e^{-(u-u')} + [1-c(u')]\delta(u-u'),$$

$$f_1(u,\, u') = c(u')e^{-\frac{3}{2}(u-u')}.$$

If we expand ϕ_0 and ϕ_1 in powers of (iy), i.e.:

$$\phi_0(y,\, u) = \phi_{00}(u) - \frac{y^2}{2}\phi_{02}(u) + \cdots,$$

$$\phi_1(y,\, u) = iy\phi_{11}(u) + \cdots,$$

separate out the δ-function part, $\phi_{pq}(u) = \chi_{pq}(u) + K_{pq}\delta(u)$ (the K_{pq}'s are constants), and substitute into Eqs. (105a), (105b), we get:

$$K_{00} = \frac{1}{c(0)}, \quad K_{11} = \frac{l(0)}{3c(0)}, \quad K_{02} = \frac{2l^2(0)}{3c^2(0)}, \tag{106}$$

and the integral equations for the χ's:

$$c(u)\chi_{00}(u) = e^{-u} + \int_0^u du'\chi_{00}(u')c(u')e^{-(u-u')}, \tag{107a}$$

$$\chi_{11}(u) - \frac{l(u)}{3}\chi_{00}(u) = \frac{l(0)e^{-3u/2}}{3} + \int_0^u du'\chi_{11}(u')c(u')e^{-\frac{3}{2}(u-u')}, \tag{107b}$$

$$c(u)\chi_{c2}(u) - 2l(u)\chi_{11}(u) = \frac{2l^2(0)e^{-u}}{3c(0)} + \int_0^u du'\chi_{02}(u')c(u')e^{-(u-u')}. \tag{107c}$$

The solutions of (107a) are readily obtained by differentiation, and we get as our final formula for

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[16] We assume that the mixture consists of hydrogen and one heavy element; the presence of several heavy elements can be taken into account if all are assumed to be infinitely heavy.

[17] The error can be estimated (cf. Appendix).

$L_s(u\neq 0)$:

$$L_s^2(u)=\frac{1}{3}\left\{\frac{l^2(0)}{c(0)}+\frac{l^2(u)}{c(u)}+l(0)l(u)\exp\left[-\int_0^u[\tfrac{3}{2}-c(u')]du'\right]\right.$$
$$+\int_0^u\frac{l^2(u')}{c(u')}du'+l(u)\int_0^u l(u')du'\exp\left[-\int_{u'}^u[\tfrac{3}{2}-c(u'')]du''\right]$$
$$+l(0)\int_0^u l(u')du'\exp\left[-\int_0^{u'}[\tfrac{3}{2}-c(u'')]du''\right]$$
$$\left.+\int_0^u l(u')du'\int_0^{u'} l(u'')du''\exp\left[-\int_{u''}^{u'}[\tfrac{3}{2}-c(u''')]du'''\right]\right\}. \quad (108)$$

Fermi first derived (108) using "kinetic-theory" methods; however, Fermi's formula as given in his original article (F1) is incorrect, presumably because of typographical errors. Horway's numbers are also wrong (H1), since he used Fermi's incorrect version. The correct formula has been derived independently by Frankel and Nelson (F4), Marshak (M1), and by Nordheim, Nordheim, and Soodak (N1).

B. Higher Spatial Moments of the Neutron Density

Most of the methods used in the previous section to derive expressions for the second spatial moment can be extended to the calculation of the higher spatial moments of the neutron density. The formulae become more involved and the algebra increasingly laborious. However, in principle the extension is possible and some work has actually been done along these lines.

Thus, Placzek (P2) has obtained asymptotic formulae for all moments for the case of constant mean free path, i.e., $l(u)\equiv l_0$(const). Assuming that $u\gg q_M$, he has worked out an expression going as far as the two highest powers of u. He finds (for a single element):

$$[z^{2m}(u)]_{\mathrm{Av}}=\frac{(2m)!}{m!}\left(\frac{l_0^2u}{K}\right)^m\left[1+\frac{m(mB_{11}+B_{10})}{u}+\cdots\right], \quad (109)$$

where

$$K=3\xi(1-\langle\cos\Theta\rangle_{\mathrm{Av}}), \quad (110a)$$

$$B_{11}=\delta_1+\delta_2+\delta_3, \quad (110b)$$

$$B_{10}=\delta_1-\delta_3; \quad (110c)$$

with

$$\delta_1=\frac{8M-(M-1)^2q_M(q_M+2)}{8M-2(M-1)^2q_M},$$

$$\delta_2=\frac{1}{6}\left[\frac{12M^2-20-3(M-1)^2(M+2)q_M}{3M-2}\right]$$

$$\delta_3=\frac{32}{15}\left[\frac{4M-(M-1)^2q_M}{4M(M^2+1)-(M^2-1)^2q_M}\right].$$

Equation (109) can be applied with obvious modifications to mixtures (W3) provided the scattering mean free paths of all the component elements are constant (independent of energy). For $M=1$, Eqs. (110a)–(110c) reduce to

$$K=1,\quad B_{11}=11/15,\quad B_{10}=-1/15. \quad (111)$$

For $M \gg 1$, Eqs. (110a)–(110c) yield:

$$K = \frac{6}{M}\left(1 - \frac{4}{3M} + \cdots\right), \tag{112a}$$

$$B_{11} = \frac{18}{5M}\left(1 - \frac{2}{3M} + \cdots\right), \tag{112b}$$

$$B_{10} = -\frac{4}{15M}\left(1 - \frac{7}{3M} + \cdots\right). \tag{112c}$$

For $m=1$, Eq. (109) contains a term proportional to u and a constant term, reducing to Eq. (86) as it should. The terms omitted in (109) are of exponentially decreasing order in (u/q_M). For $m>1$, the terms omitted in (109) are powers of u. Hence, as m increases (the higher the moment), Eq. (109) yields increasingly poorer values for the higher moments. It must be remarked, however, that (109) is a considerable improvement over age theory (cf. Section C below) which leads—under the assumption of constant mean free path—to the formula:

$$[z^{2m}(u)]_{\text{Av}} = [(2m)!/m!](l_0^2 u/K)^m. \tag{113}$$

Equation (109) was derived assuming constant mean free path. A formula for variable mean free path, with the mean free path represented by a sum of exponentials, i.e.,

$$l(u) = \sum_{j=0}^{\infty} A_j \exp[-a_j u]$$

(cf. Section A above), has been derived by Marshak (M2) to the order of approximation represented by (109). He finds:

$$[z^{(2m)}(u)]_{\text{Av}} = \frac{(2m)!}{m!}\left(\frac{A_0^2 u}{K}\right)^m \left\{1 + \frac{m(mB_{11}+B_{10})}{u} + \frac{m}{3}\left(\frac{K}{A_0^2 u}\right) \sum_{\substack{j=k=0 \\ (j=k=0 \\ \text{excluded})}}^{\infty} \frac{A_j A_k}{\gamma_0(a_j+a_k)\gamma_1(a_j)}\right\}. \tag{114}$$

Marshak (M2) has also calculated $[z^4(u)]_{\text{Av}}$ for the same variation of mean free path with energy, but the formula is so long that we refrain from citing it here. However, it is worth quoting Table III for carbon (M2), showing the percentage deviations of $[z^2(u)]_{\text{Av}}$ and $[z^4(u)]_{\text{Av}}$ from age theory as a function of energy for both constant and variable mean free path. Column 1 of Table III gives the initial energy of the neutrons; the final energy is always taken as 1.44 ev (the indium resonance energy). Columns 2 and 4 are derived by assuming a linear variation of scattering mean free path with energy which fits the energy range in question quite well for carbon (cf. Appendix). The variation in mean free path is about a factor three from $E_0=0.1$ to $E_0=2.7$ Mev, and decreases only about 10 percent from 0.1 Mev to 1.44 ev. Columns 3 and 5 are obtained from Eq. (109). Table III makes manifest the relative importance of the variation of the mean free path and the order of the moment in causing deviations from age theory. A rough measure of the deviation from age theory is the quantity $\{[z^4]_{\text{Av}}/3[z^2]_{\text{Av}}^2\}$ which is identically one in age theory. Examination of Table III shows that this quantity decreases from 1.07 to 1.05 for variable mean free path and increases from 1.04 to 1.05 for constant mean free path.

C. Age Theory

If all the moments of a physical distribution function are known, then the distribution function is uniquely determined except for a normalization constant. Thus, the distribution function associated

TABLE III. Percent deviations from age theory.

E (Mev)	$\Delta[z^2]_{Av}$ (variable m.f.p.)	$\Delta[z^2]_{Av}$ (constant m.f.p.)	$\Delta[z^4]_{Av}$ (variable m.f.p.)	$\Delta[z^4]_{Av}$ (constant m.f.p.)
2.7	6.8	1.8	21.2	7.6
1.0	3.4	1.9	12.0	8.0
0.5	2.6	2.0	10.3	8.4
0.1	2.3	2.3	9.8	9.6

with the sequence of moments (109) is easily found to be:

$$\psi_0(z,u)=C\exp[-z^2K/4l_0^2u]\left\{1+\frac{1}{4u}\left[(B_{11}-2B_{10})-(2B_{11}-B_{10})\frac{z^2K}{l_0^2u}+\frac{B_{11}}{4}\left(\frac{Kz^2}{l^2u}\right)^2\right]\right\}, \tag{115}$$

where the normalization constant[18] $C=1/\xi l_0(4\pi u/K)^{\frac{1}{2}}$. The age approximation to the neutron density is obtained by retaining only the first term on the right-hand side of (115), i.e.:

$$\psi_0^{(\text{age})}(z)=C\exp[-z^2K/4l_0^2u] \tag{115a}$$

where C is the same as in (115). Equation (115a) is the distribution function whose moments are given by (113). It is also possible to derive the distribution function corresponding to the sequence of moments (114); we get:

$$\psi_0(z,u)=C\exp[-z^2K/4A_0^2u]\left\{1+\frac{1}{4u}\left[\left(B_{11}-2B_{10}-\frac{2K}{3A_0^2}\sum_{\substack{j=k=0\\(j=k=0\\\text{excluded})}}^{\infty}\frac{A_jA_k}{\gamma_0(a_j+a_k)\gamma_1(a_j)}\right)\right.\right.$$
$$\left.\left.-\left(2B_{11}-B_{10}-\frac{K}{3A_0^2}\sum_{\substack{j=k=0\\(j=k=0\\\text{excluded})}}^{\infty}\frac{A_jA_k}{\gamma_0(a_j+a_k)\gamma_1(a_j)}\right)\cdot\frac{z^2K}{A_0^2u}+\frac{B_{11}}{4}\left(\frac{z^2K}{A_0^2u}\right)^2\right]\right\}, \tag{116}$$

where $C=1/\xi A_0(4\pi u/K)^{\frac{1}{2}}$.

Equations (115)–(116) represent approximate expressions for the neutron density, valid for sufficiently large u and sufficiently small z. The limitation to large u follows from the use of the Tauberian theorem in deriving these equations. The limitation to small z follows from the increasing inaccuracy of the expressions for the higher moments. The particular range of applicability of each distribution function is obscured by its indirect deviation. We, therefore, start *de novo* with the transport equation (63) and derive directly from it successive approximations to the neutron density. The first approximation to the neutron density will turn out to be the widely used age approximation. The use of age theory seems to have started with the investigation of Bethe, Korff, and Placzek (B4) into the slowing down of cosmic-ray neutrons in the atmosphere. The age approximation can be derived in many ways: from the time-dependent diffusion equation (P2), from simple arguments about the conservation of neutrons (F3), or from the rigorous transport equation (M8). We prefer to derive age theory from the rigorous transport equation in order to make evident the nature of the restrictions underlying the approximation, and to facilitate improvements upon it.

The first condition on which the age approximation is based is that the neutron distribution function $\psi(z,\mu,u)$ is almost isotropic; we retain the first two terms of an expansion in Legendre

[18] We normalize $\psi_0(z,u)$ so that $\int_{-\infty}^{\infty}\psi_0(z,u)dz=\frac{1}{\xi}$; this corresponds to normalizing the slowing down density to unity (cf. Eq. (122a)). It can easily be shown that the correction term to the Gaussian contributes nothing to the integral (as follows immediately from Eq. (142) since $\int_{-\infty}^{\infty}\psi_0(z,u)dz=\mathfrak{L}^{-1}\Phi_0(0,\eta)$); this explains the values of C specified in Eqs. (115a) and (116).

polynomials,[19] namely:

$$\psi(z, \mu, u) = \frac{1}{4\pi}[\psi_0(z, u) + 3\mu\psi_1(z, u)], \tag{117}$$

where

$$\psi_{0,1}(z, u) = \int d\Omega P_{0,1}(\mu)\psi(z, \mu, u). \tag{117a}$$

To take advantage of the approximation (117) to the distribution function, we take the zero and first moments (with respect to μ) of Eq. (63); we get:

$$l(u)\frac{\partial\psi_1(z, u)}{\partial z} + \psi_0(z, u) = \int_0^u du'\psi_0(z, u')f_0(u-u') + \delta(z)\delta(u), \tag{118a}$$

$$\tfrac{1}{3}l(u)\frac{\partial\psi_0(z, u)}{\partial z} + \psi_1(z, u) = \int_0^u du'\psi_1(z, u')f_1(u-u'); \tag{118b}$$

where

$$f_0(u) = \alpha e^{-u}, \quad f_1 = \alpha\left[\frac{(M+1)}{2}e^{-3u/2} - \frac{(M-1)}{2}e^{-u/2}\right], \quad \left(\alpha = \frac{(M+1)^2}{4M}\right).$$

It is understood in (118a) and (118b) that f_0 and f_1 are zero when $(u-u') > q_M$, so that the integrals are taken over an interval q_M (except for $u < q_M$, which does not affect the results since it will be assumed that $u \gg q_M$).

The second condition underlying the derivation of age theory is that the ψ_0 and ψ_1 under the integral signs in (118a) and (118b) vary slowly with u'. This condition will be the more closely satisfied, the smaller the collision interval[20] q_M (q_M is smaller, the larger M), and the more slowly does the mean free path vary within it. Expanding ψ_0 and ψ_1 about $u' = u$, we get:

$$\psi_0(z, u') = \psi_0(z, u) - (u-u')\frac{\partial\psi_0}{\partial u}(z, u) + \cdots, \tag{119a}$$

$$\psi_1(z, u') = \psi_1(z, u) + \cdots \tag{119b}$$

One more term is carried in ψ_0 because it is assumed that $\psi_1 \ll \psi_0$. Insertion of (119a) and (119b) into (118a) and (118b) yields:[21]

$$l(u)\frac{\partial\psi_1}{\partial z} + \psi_0 = \psi_0\int_{u-q_M}^u du' f_0(u-u') - \frac{\partial\psi_0}{\partial u}\int_{u-q_M}^u du'(u-u')f_0(u-u') + \delta(z)\delta(u), \tag{120a}$$

$$\tfrac{1}{3}l(u)\frac{\partial\psi_0}{\partial z} + \psi_1 = \psi_1\int_{u-q_M}^u du' f_1(u-u'). \tag{120b}$$

[19] If only the first term were kept, the approximation would be too crude since there would be no net flow of neutrons (zero current).

[20] We assume that $M > 1$; in hydrogen, if the energy of the source neutrons is greater than 100 kev, age theory is very poor because the scattering mean free path varies so rapidly. If the source energy is less 100 kev, age theory gives a fair approximation; in this case, the arguments given below can be extended to hydrogen with similar results.

[21] This derivation is correct only when $u > q_M$ since it involves replacing the integral $\int_0^u$ by $\int_{u-q_M}^u$. However, it can be shown by Laplace transform methods (cf. below) that it is also correct for $u < q_M$, and that the source term is correct as given in (121a).

But

$$\int_{u-q_M}^{u} du' f_0(u-u') = 1 \qquad \text{(from Eq. (10))}$$

$$\int_{u-q_M}^{u} du'(u-u') f_0(u-u') = \xi \qquad \text{(from Eq. (11a))}$$

$$\int_{u-q_M}^{u} du' f_1(u-u') = \langle \cos\Theta \rangle_{\text{Av}}. \qquad \text{(from Eq. (11b))}$$

Hence, Eqs. (120a) and (120b) become:

$$l(u)\partial\psi_1/\partial z = -\xi\partial\psi_0/\partial u + \delta(z)\delta(u), \tag{121a}$$

$$\psi_1 = -\frac{l(u)}{3(1-\langle\cos\Theta\rangle_{\text{Av}})}\frac{\partial\psi_0}{\partial z}. \tag{121b}$$

Substituting (121b) into (121a) leads to the age equation:

$$\frac{\partial\chi(z,\tau)}{\partial\tau} = \frac{\partial^2\chi(z,\tau)}{\partial z^2} + \delta(z)\delta(\tau), \tag{122}$$

where

$$\tau = \int_0^u du' l^2(u')/3\xi(1-\langle\cos\Theta\rangle_{\text{Av}})$$

is called the age[22] of the neutrons and has the dimensions of an area and $\chi(z, \tau) = \xi\psi_0(z, u)$ is called the slowing-down density and represents the number of neutron per unit volume per unit time which reach the age τ.

The solution of (122) for an infinite slowing-down medium is:

$$\chi(z,\tau) = (4\pi\tau)^{-\frac{1}{2}} \exp[-z^2/4\tau], \tag{122a}$$

where χ is normalized to unity over all space. If we calculate $\langle z^2(u)\rangle_{\text{Av}}$ using (122a), we find:

$$\langle z^2(u)\rangle_{\text{Av}} \equiv 2L_s^2(u) = 2\tau. \tag{122b}$$

In other words, in the age approximation, the square of the slowing-down length is precisely the age of the neutrons.

It is consistent to use (122a) to determine the magnitudes of the various approximations made in deriving (122). Thus, in writing (117) we assumed that $\psi_1 \ll \psi_0$; using (121b) and (122a), this condition is equivalent to:

$$z \ll \left[\frac{6[1-\langle\cos\Theta\rangle_{\text{Av}}]L_s}{[l(u)]_{\max}}\right] L_s. \tag{123}$$

Next, in writing (119a), we assumed that $q_M\partial\psi_0/\partial u \ll \psi_0$; substituting (122a) into this condition leads to one inequality which is essentially identical with (123), while the other is:

$$[l^2(u)_{\max}] \ll \frac{1}{\xi}\int_0^u l^2(u')du'. \tag{124}$$

[22] The term "age"—sometimes "symbolic age"—has its origin in the formal analogy of Eq. (122) to the heat conduction equation where τ corresponds to the time. Moreover, τ is related to the average distance a neutron travels from the time it is born with an energy E_0 until it reaches the energy E.

Inequality (124) has been derived by using the approximate equality (for $M>1$) $q_M \approx 2\xi$. Finally, the expansion (119b) was based on the assumption $q_M(\partial\psi_0/du) \ll \psi_1$; this condition yields three inequalities of which two are essentially (123) and (124), while the third is:

$$q_M\left[\frac{d \log l(u)}{du}\right]_{\max} \ll 1. \tag{125}$$

The inequalities (123)–(125) delimit the range of validity of age theory; they state, respectively, that:

(1) Age theory can only be used up to distances of the order L_s^2/l; at greater distances, the age approximation breaks down and (and as will be shown in Section D) the Gaussian solution (122a) goes over into an exponential solution.

(2) Age theory can only be used provided the average number of slowing-down collisions is large; this follows from the fact that the average number of collisions, ΔN, which degrade the energy of a neutron by an amount, Δu, is $(\Delta u/\xi)$. For small (u/ξ), age theory is a poor approximation.

(3) Age theory can only be used if the fractional rate of change of mean free path in one collision interval (i.e., q_M) is small; if the mean free path changes very rapidly, age theory will break down.

While the above conditions may appear somewhat restrictive, they are sufficiently well satisfied for many practical problems to make age theory extremely useful. The great virtue of age theory is that the differential equation for the slowing-down density is formally identical with the time-dependent heat-conduction equation, for which a large number of solutions are known for different geometries and boundary conditions. As a matter of fact, Eq. (122) can be generalized to three dimensions and a general source distribution, namely:

$$\frac{\partial\chi(\mathbf{r}, \tau)}{\partial\tau} = \nabla^2\chi(\mathbf{r}, \tau) + S(\mathbf{r})\delta(\tau), \tag{126}$$

where $S(\mathbf{r})$ represents the general source distribution.

Wallace and Le Caine (W1) have worked out a great variety of solutions of Eq. (126). They consider chiefly cases in which the slowing-down media and source distributions are either plane, or spherically symmetrical. Furthermore, they always apply the initial condition $\chi(\mathbf{r}, 0) = S(\mathbf{r})$ and the boundary condition $\chi(\mathbf{r}, \tau) = 0$ (for all τ) at an outer bounding surface of the medium under consideration. They do not solve problems involving media with different slowing-down properties.

The two problems we now proceed to solve supplement Wallace and Le Caine's work. The solution of the first problem clarifies the extent to which the boundary condition $\chi(\mathbf{r}, \tau) = 0$ (for all τ) at an outer bounding surface is correct. The solution to the second problem, apart from having a certain intrinsic interest, exhibits the boundary conditions which obtain at the interface between two different slowing-down media. The second problem also illustrates the power of the Laplace transform method for solving problems of neutron aging.

Problem 1: Semi-Infinite Slowing-Down Medium Bounded by Vacuum

The first problem is: given a plane δ-source, $\delta(z-z')$ $(z'>0)$ in a semi-infinite slowing-down medium extending from $z=0$ to $z=\infty$, what is the extrapolated end-point?[23] The age equation valid for this problem is (cf. Fig. 4):

$$\partial\chi(z, \tau)/\partial\tau = \partial^2\chi(z, \tau)/\partial z^2 + \delta(z-z')\delta(\tau). \tag{127}$$

The boundary conditions are:

[23] I.e., the point where the slowing-down density vanishes.

(a) $$\chi_{II}(z,\tau)\to 0 \quad \text{as} \quad z\to\infty, \quad \text{all } \tau,$$

(b) $$\chi_I(z,\tau)=\chi_{II}(z,\tau)$$

(c) $$\frac{\partial\chi_I}{\partial z}(z,\tau)-\frac{\partial\chi_{II}}{\partial z}(z,\tau)=1$$

(b), (c) at $z=z'$, all τ,

(d) $$\chi_I(z,\tau)-\frac{2}{3}\frac{l(u)}{(1-\langle\cos\Theta\rangle_{Av})}\frac{\partial\chi_I(z,\tau)}{\partial z}=0 \quad \text{at } z=0, \quad \text{all } \tau.$$

Boundary conditions (a)–(c) are the usual ones at infinity and across a δ-source for an equation of the heat-conduction type. Boundary condition (d) follows from the fact that the total incoming current at $z=0$ is zero, and that this incoming current is

$$\int_0^1 d\mu\mu\psi(z,\mu,u)$$

where (in the age approximation—cf. Eq. (117)):

$$\psi(z,\mu,u)=\frac{1}{4\pi\xi}\left[\chi(z,u)-\mu\frac{l(u)}{[1-\langle\cos\Theta\rangle_{Av}]}\frac{\partial\chi(z,\tau)}{\partial z}\right].$$

The solution of (127), subject to the above boundary conditions, is difficult to obtain for arbitrary variation of $l(u)$. We therefore resort to a perturbation calculation (M6) which gives good results even for rapidly (though smoothly) varying $l(u)$. We write:

$$l(u)=l_0+\epsilon(u), \tag{128}$$

where l_0 is a constant, and $\epsilon(u)$ is an arbitrary function having the property $\epsilon(u)\to 0$ as $u\to\infty$; physically, l_0 represents the mean free path for slow neutrons (e.g., indium resonance neutrons). Substituting (128) into boundary condition (d) yields:

$$\chi_I(0,\tau)-\frac{2l_0}{3(1-\langle\cos\Theta\rangle_{Av})}\frac{\partial\chi_I(0,\tau)}{\partial z}=\frac{2\epsilon(u)}{3(1-\langle\cos\Theta\rangle_{Av})}\frac{\partial\chi_I(0,\tau)}{\partial z}. \tag{129}$$

We now set the right-hand side of (129) equal to zero and solve the "zero-order" problem.

To solve the "zero-order" problem,[24] we take the Laplace transform of (127) and of the boundary conditions. We get:

$$\frac{d^2\phi(z,\zeta)}{dz^2}-\zeta\phi(z,\zeta)=-\delta(z-z'), \tag{130}$$

and

(a) $$\phi_{II}(z,\zeta)\to 0 \quad \text{as } z\to\infty, \quad \text{all } \zeta,$$

(b) $$\phi_I(z,\zeta)=\phi_{II}(z,\zeta)$$

(c) $$\frac{\partial\phi_I}{\partial z}(z,\zeta)-\frac{\partial\phi_{II}}{\partial z}(z,\zeta)=1$$

(b), (c) $z=z'$, all ζ,

(d) $$\phi_I(z,\zeta)-\tfrac{2}{3}l_0\frac{\partial\phi_I}{\partial z}(z,\zeta)=0 \quad \text{at } z=0, \quad \text{all } \zeta,$$

[24] The "zero-order" problem is equivalent to a heat-conduction problem wherein the δ-source is contained in a semi-infinite medium which radiates into vacuum at zero temperature. The heat problem is solved in H. Carlsaw, *Theory of Conduction of Heat* (Dover Publishers), §82, through the use of Green's function. We present the Laplace transform method of solution because it can so easily be extended to other problems.

with $\bar{l}_0 = l_0/(1-\langle\cos\Theta\rangle_{\text{Av}})$. Equation (130) is a diffusion equation with ζ appearing as a parameter; its solution is:

$$\phi_{\text{I}}(z,\zeta)=\frac{1}{2(\zeta)^{\frac{1}{2}}}\exp[-(z'-z)(\zeta)^{\frac{1}{2}}]-\frac{1}{2(\zeta)^{\frac{1}{2}}}\left(\frac{1-\frac{2\bar{l}_0}{3}(\zeta)^{\frac{1}{2}}}{1+\frac{2\bar{l}_0}{3}(\zeta)^{\frac{1}{2}}}\right)\exp[-(z+z')(\zeta)^{\frac{1}{2}}], \tag{131a}$$

$$\phi_{\text{II}}(z,\zeta)=-\frac{1}{2(\zeta)^{\frac{1}{2}}}\left\{\left(\frac{1-\frac{2}{3}\bar{l}_0(\zeta)^{\frac{1}{2}}}{1+\frac{2}{3}\bar{l}_0(\zeta)^{\frac{1}{2}}}\right)\exp[-2z'(\zeta)^{\frac{1}{2}}]-1\right\}\exp[-(z-z')(\zeta)^{\frac{1}{2}}]. \tag{131b}$$

The Laplace inverses of (131a) and (131b) can be obtained from tables of Laplace transforms; thus, the result for (131a) is:

$$\chi(z,\tau)=\frac{1}{(4\pi\tau)^{\frac{1}{2}}}\Bigg\{\exp[-(z-z')^2/4\tau]+\exp[-(z+z')^2/4\tau] -\frac{3(\pi\tau)^{\frac{1}{2}}}{\bar{l}_0}\exp\left[\left(\frac{9\tau}{4\bar{l}_0^2}+\frac{3(z+z')}{2\bar{l}_0}\right)\right]\cdot\left[1-\text{erf}\left(\frac{3(\tau)^{\frac{1}{2}}}{2\bar{l}_0}+\frac{(z+z')}{2(\tau)^{\frac{1}{2}}}\right)\right]\Bigg\}. \tag{132}$$

Equation (132) is the "zero-order" solution for variable mean free path, but is the exact solution (in the age approximation) for constant mean free path. In the latter case, the extrapolated end-point z_0 may be found by setting $\chi(z,\tau)$, as given by (132), equal to zero. It turns out that for values of z and τ for which age theory is valid (cf. (123)–(125)) that[25] $z_0 \approx -0.70\bar{l}_0$.

For variable mean free path, i.e., when $\epsilon(u)\neq 0$ in (128), we substitute (132) into the right-hand side of (129) and then solve for the "first-order" χ subject to the boundary conditions (a)–(c) and (129) (which is now correct to "first-order"). This "first-order" problem is equivalent to a heat-conduction problem radiating into a medium whose temperature is a specified function of the time. This problem is solved in §83 for Carslaw (cf. reference 24). If we use that solution (which we refrain from writing out because of its length), we find that even for as much as a fivefold linear variation of mean free path with energy (from several Mev to 1 ev), the extrapolated end point z_0 is within a few percent still[26] $-0.70\bar{l}_0$. Thus, for variable mean free path the boundary condition (d) can be replaced by (129) with the right-hand side $=0$, provided we are interested in the density of slow neutrons. In other words, the extrapolated end point corresponding to a particular age is of the order of the transport mean free path at that age. Since the transport mean free path usually decreases at the low energy end (large ages), the effective extrapolated end point for large ages becomes very small.

Problem 2: Two Different Slowing-Down Media[27]

The second problem is: given a point source of fast neutrons in one of two adjacent semi-infinite media of different slowing-down properties, what is the slowing-down density everywhere in space? To simplify the calculation, we assume that the point source of mono-energetic fast neutrons is situated at the interface between the two semi-infinite media of different slowing-down properties. The more general problem of the source located in one medium has also been solved (B2).

The equations for the slowing-down densities $\chi_1(\rho,z,\tau_1)$ and $\chi_2(\rho,z,\tau_2)$ in the two media are (it is

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[25] This can be seen by expanding the error function for large values of the argument (since $(\tau)^{\frac{1}{2}}\gg\bar{l}_0$); we get: $e^{z'z_0/4\tau}-1+\frac{2}{3}(z'+z_0)\bar{l}_0/\tau=0$. Furthermore, it is implicit that $z'\gg z_0$; making use of this fact and the hypothesis that $z_0\ll(\tau)^{\frac{1}{2}}$ leads to the result $z_0\approx-\frac{2}{3}\bar{l}_0$. More accurate evaluation changes the $\frac{2}{3}$ to 0.70.

[26] Computations were carried out with $u=15$, $z'=3(\tau)^{\frac{1}{2}}$. Larger values of z' (i.e., greater distances of source from boundary) and smaller values of u (i.e., higher source energies) increase z_0 somewhat. A calculation was also performed for a twofold linear variation of mean free path (over the same range of energy) with similar results (M6).

[27] The two-medium problem with a plane distribution of sources has been solved by Friedman (F5).

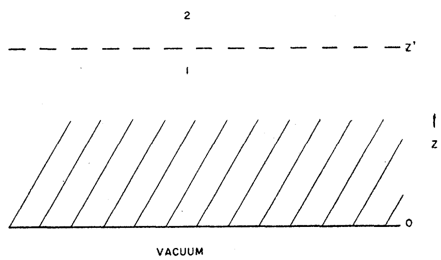

FIG. 4. Semi-infinite medium with plane δ-source.

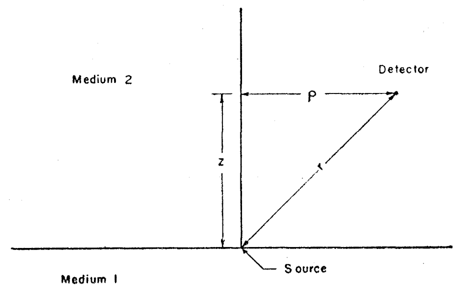

FIG. 5. Two-medium slowing-down problem.

convenient to use cylindrical coordinates—cf. Fig. 5):

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\chi_1}{\partial\rho}\right)+\frac{\partial^2\chi_1}{\partial z^2}=\frac{\partial\chi_1}{\partial\tau_1}-\frac{\delta(r)\delta(\tau_1)}{4\pi r^2}, \tag{133a}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\chi_2}{\partial\rho}\right)+\frac{\partial^2\chi_2}{\partial z^2}=\frac{\partial\chi_2}{\partial\tau_2}-\frac{\delta(r)\delta(\tau_2)}{4\pi r^2}, \tag{133b}$$

where

$$\tau_{1,2}=\int_0^u\frac{du' l_{1,2}^2(u')}{3\xi_{1,2}[1-\langle\cos\Theta_{1,2}\rangle_{\text{Av}}]}.$$

Equations (133a) and (133b) are to be solved subject to two boundary conditions at the interface between the two media, namely[28]:

$$(l_1(u)/\xi_1)\chi_1(\rho, 0, u(\tau_1))=(l_2(u)/\xi_2)\chi_2(\rho, 0, u(\tau_2)) \tag{A}$$

(continuity of neutron density for all ρ and u),

$$\frac{l_1^2(u)}{\xi_1[1-\langle\cos\Theta_1\rangle_{\text{Av}}]}\frac{\partial\chi_1}{\partial z}(\rho, 0, u(\tau_1))=\frac{l_2^2(u)}{\xi_2[1-\langle\cos\Theta_2\rangle_{\text{Av}}]}\frac{\partial\chi_2}{\partial z}(\rho, 0, u(\tau_2)) \tag{B}$$

(continuity of normal neutron current for all ρ and u).

We now assume that $l_1(u)$ and $l_2(u)$ vary arbitrarily, but in such a manner that their ratio is always constant. Then (A) and (B) can be rewritten as:

$$\chi_1(\rho, 0, u(\tau_1))=(D\delta)^{\frac{1}{2}}\chi_2(\rho, 0, u(\tau_2)), \tag{A'}$$

$$(\partial\chi_1/\partial z)(\rho, 0, u(\tau_1))=D(\partial\chi_2/\partial z)(\rho, 0, u(\tau_2)), \tag{B'}$$

where D and δ are constants[29] defined by:

$$D=\frac{l_2^2(u)}{\xi_2(1-\langle\cos\Theta_2\rangle_{\text{Av}})}\bigg/\frac{l_1^2(u)}{\xi_1(1-\langle\cos\Theta_1\rangle_{\text{Av}})},\quad \delta=\frac{(1-\langle\cos\Theta_2\rangle_{\text{Av}})}{\xi_2}\bigg/\frac{(1-\langle\cos\Theta_1\rangle_{\text{Av}})}{\xi_1}.$$

Hence $\tau_2=D\tau_1$ so that Eq. (133a) becomes:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\chi_1}{\partial\rho}\right)+\frac{\partial^2\chi_1}{\partial z^2}=D\frac{\partial\chi_1}{\partial\tau_2}-\frac{D\delta(r)\delta(\tau_2)}{4\pi r^2}. \tag{134}$$

[28] Use is made of the relation between the neutron density N_0 and the slowing-down density, i.e., $\chi=\xi vN_0/l$. The neutron current is $-lv/3(1-\langle\cos\Theta\rangle_{\text{Av}})\,\text{grad}N_0$.

[29] The subscripts 1 and 2 are chosen so that $D>1$; δ may be greater or less than unity.

Applying a Laplace transformation with respect to τ_2 to Eqs. (133b) and (134) and to the boundary conditions (A′) and (B′), we get:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi_1}{\partial\rho}\right)+\frac{\partial^2\phi_1}{\partial z^2}=D\zeta\phi_1-D\frac{\delta(r)}{4\pi r^2}, \tag{134a}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi_2}{\partial\rho}\right)+\frac{\partial^2\phi_2}{\partial z^2}=\zeta\phi_2-\frac{\delta(r)}{4\pi r^2}, \tag{134b}$$

$$\phi_1(\rho,0,\zeta)=(D\delta)^{\frac{1}{2}}\phi_2(\rho,0,\zeta), \tag{A''}$$

$$\frac{\partial\phi_1}{\partial z}(\rho,0,\zeta)=D\frac{\partial\phi_2}{\partial z}(\rho,0,\zeta), \tag{B''}$$

where

$$\phi_{1,2}(\rho,z,\zeta)\equiv\mathfrak{L}\chi_{1,2}(\rho,z,\tau_2)=\int_0^\infty d\tau_2 e^{-\zeta\tau_2}\chi_{1,2}(\rho,z,\tau_2).$$

The solutions of (134a) and (134b), subject to the boundary conditions (A″) and (B″), are:

$$\phi_1(\rho,z,\zeta)=\frac{D(\delta)^{\frac{1}{2}}}{2\pi}\int_0^\infty\frac{\lambda d\lambda J_0(\lambda\rho)\exp[(\lambda^2+D\zeta)^{\frac{1}{2}}z]}{\{[\delta(\lambda^2+D\zeta)]^{\frac{1}{2}}+[D(\lambda^2+\zeta)]^{\frac{1}{2}}\}}, \tag{135a}$$

$$\phi_2(\rho,z,\zeta)=\frac{(D)^{\frac{1}{2}}}{2\pi}\int_0^\infty\frac{\lambda d\lambda J_0(\lambda\rho)\exp[-(\lambda^2+\zeta)^{\frac{1}{2}}z]}{\{[\delta(\lambda^2+D\zeta)]^{\frac{1}{2}}+[D(\lambda^2+\zeta)]^{\frac{1}{2}}\}}, \tag{135b}$$

where $J_0(x)$ is the Bessel function of zero order.

We must now take the Laplace inverses of Eqs. (135a) and (135b). This can be done for arbitrary D, δ, ρ and z. However, to illustrate the method, we work out a limiting case, namely, $\delta=1$, $z=0$, D and ρ arbitrary. Consider ϕ_2 when $\delta=1$, $z=0$; we have:

$$\phi_2(\rho,0,\zeta)=\frac{(D)^{\frac{1}{2}}}{2\pi}\int_0^\infty\frac{\lambda d\lambda J_0(\lambda\rho)}{\{[\lambda^2+D\zeta]^{\frac{1}{2}}+[D(\lambda^2+\rho)]^{\frac{1}{2}}\}}. \tag{136}$$

The Laplace inverse of (136) can be easily obtained by rationalizing the denominator and making use of the identity:

$$\int_0^\infty\lambda d\lambda\exp[-\lambda^2\tau_2]J_0(\lambda\rho)=\frac{1}{2\tau_2}\exp[-\rho^2/4\tau_2];$$

we get:

$$\chi_2(\rho,0,\tau)\equiv\mathfrak{L}^{-1}\phi_2=\frac{1}{(4\pi\tau_2)^{\frac{3}{2}}(1-1/D)}\left\{Ei\left(-\frac{\rho^2D}{4\tau_2}\right)-Ei\left(-\frac{\rho^2}{4\tau_2}\right)\right\}, \tag{136a}$$

where $Ei(x)$ is the exponential integral function. In the limit $D\rightarrow1$, (136a) goes over into the solution for a point source in an infinite medium, namely:[30]

$$\chi_2(\rho,\tau_2)=\frac{1}{(4\pi\tau_2)^{\frac{3}{2}}}\exp[-\rho^2/4\tau_2]. \tag{136b}$$

Equation (136a) represents the slowing-down density in the less dense of two media for which the ratio of densities is $(D)^{\frac{1}{2}}$. For small ρ, χ_2, as given by (136a), is greater than χ_2, as given by (136b);

[30] Equation (136b) agrees with the point-source solution obtained by substituting (122a) into Eq. (62) as it should.

this is to be expected qualitatively since the dense medium 1 enhances the slow neutron density in medium 2 through more efficient "aging." For large ρ, (136a) gives smaller values than (136b) for a given τ_2 since the greater "aging" in medium 1 carries a greater percentage of neutrons beyond τ_2 as compared with the situation when only medium 2 is present. This completes our discussion of problems 1 and 2.

The age equation (122) was derived on the assumption of stationarity and absence of capture. If we wish to take account of time dependence and capture, we must start with Eq. (10). If we then make the assumptions which led to Eq. (122), and furthermore assume that the capture is weak and that only the time variation of ψ_0 (and not ψ_1) need be considered, we get:

$$\frac{l(u)}{v}\frac{\partial\psi_0}{\partial t}+l(u)\frac{\partial\psi_1}{\partial z}+[1-h(u)]\psi_0=-\xi\frac{\partial\psi_0}{\partial u}+\delta(z)\delta(u)\delta(t), \tag{137a}$$

$$\tfrac{1}{3}l_t(u)\frac{\partial\psi_0}{\partial z}+\psi_1=0, \tag{137b}$$

where $l_t(u)=l(u)/(1-\langle\cos\Theta\rangle_{\mathrm{Av}})$ with $l(u)$ the scattering mean free path. Combining Eqs. (137a) and (137b) into one equation, and introducing the age τ and the slowing-down density χ, one obtains the equation:

$$\frac{\partial\chi}{\partial\tau}(z,\tau,t)+\frac{3}{vl_t(u)}\frac{\partial\chi}{\partial t}+\frac{3[1-h(u)]\chi}{l(u)l_t(u)}=\frac{\partial^2\chi}{\partial z^2}+\delta(z)\delta(\tau)\delta(t). \tag{138}$$

Examination of Eq. (138) reveals that both the time-dependent and capture terms lead to multiplicative factors in the final solution; thus, if we write

$$\chi(z,\tau,t)=\chi(z,\tau)F_1(h)F_2(t),$$

we find that:

$$F_1(h)=\exp\left\{-\int_0^u\frac{du'}{\xi}[1-h(u')]\right\}, \tag{139a}$$

$$F_2(t)=\delta\left\{t-\int_0^u\frac{l(u')du'}{v\xi}\right\}, \tag{139b}$$

and that $\chi(z,\tau)$ satisfies the stationary, non-capturing age equation (122). In other words, within the limits of validity of age theory, time dependence and capture can be included by merely multiplying the solution of (122), subject to the customary boundary conditions, by (139a) and by (139b).

D. Neutron Density Throughout Space

In Section C we derived the age approximation in slowing-down theory, and specified the three conditions which must be fulfilled in order for this approximation to be valid. If any one of the three conditions is not fufilled, then age theory becomes a poor approximation. In particular, when z becomes comparable to, and larger than (L_s^2/l), age theory becomes increasingly poorer and for very large z the Gaussian distribution of slow neutrons due to a plane δ-source of fast neutrons must go over into an exponential with a decay length equal to the mean free path of the fast neutrons.

We divide our discussion in this section into three parts. In the first two parts, we discuss those improvements upon age theory which lead to a more accurate spatial distribution of the slow neutron density as z becomes comparable to, or somewhat larger than (L_s^2/l). In this discussion, two alternative methods are presented which start with age theory as the first step in a series of successive approximations. In the third part of this section, we consider the behavior of the neutron density for very large z, i.e., $z \gg (L_s^2/l)$. Most of the work on the asymptotic neutron density is still unpublished and has been carried out by Wick.[31] Unless otherwise stated, it is assumed that the scattering mean free path is constant for all energies (taken equal to unity) and that [32] $u/\xi \gg 1$.

1. Improvements on Age Theory: Spherical Harmonic Method

Equation (115) already contains a first correction to age theory—a correction which is of order (ξ/u) for $z \approx L_s$. It was derived by obtaining more accurate expressions for the moments than those following from simple age theory (cf. Eqs. (109) and (113)). Equation (115) can also be derived in another way—by starting directly with the transport Eq. (63). A direct derivation has the virtue that it is capable of obvious generalization and has actually been carried out to order $(\xi/u)^2$. The direct approach to the neutron density through the transport equation has already been applied in the derivation of the age approximation (cf. Section C). When one attempts to extend the same treatment to the higher approximations, one gets a series of complicated partial differential equations. However, if one works in transform space—Fourier transform with respect to z, and Laplace transform with respect to u—improvements on age theory can be obtained much more readily. Let us then start with the Laplace-Fourier transform of (63), namely:

$$(1-iy\mu)\Phi(y,\mu,\eta)=\int d\Omega'\Phi(y,\mu,\eta)g(\mu_0,\eta)+\frac{1}{4\pi}, \tag{140}$$

where

$$\Phi(y,\mu,\eta)\equiv\mathfrak{F}\mathfrak{L}\psi(z,\mu,u)=\int_{-\infty}^{\infty}dze^{iyz}\int_0^{\infty}due^{-\eta u}\psi(z,\mu,u), \tag{140a}$$

$$g(\mu_0,\eta)\equiv\mathfrak{L}f(\mu_0,\eta)=\frac{\alpha}{\pi}\frac{1}{(\mu_0^2+M^2-1)^{\frac{1}{2}}}\left(\frac{\mu_0+(\mu_0^2+M^2-1)^{\frac{1}{2}}}{M+1}\right)^{2(\eta+1)}. \tag{140b}$$

If we take the moments of (140) with respect to μ, we get the infinite sequence of equations:

$$\gamma_0(\eta)\Phi_0(y,\eta)-iy\Phi_1(y,\eta)=1, \tag{141a}$$

$$\gamma_l(\eta)\Phi_l(y,\eta)-\frac{iy}{(2l+1)}[l\Phi_{l-1}(y,\eta)+(l+1)\Phi_{l+1}(y,\eta)]=0,\quad (l>0), \tag{141b}$$

where

$$\Phi_l(y,\eta)=\int d\Omega P_l(\mu)\Phi(y,\mu,\eta),$$

$$\gamma_l(\eta)\equiv 1-g_l(\eta)\quad\text{with}\quad g_l(\eta)=\int d\Omega P_l(\mu)g(\mu,\eta).$$

[31] Dr. G. C. Wick very kindly communicated his results to the author prior to publication.
[32] That is, only the first of the three conditions underlying the derivation of age theory is lifted.

The solution for $\Phi_0(y, \eta)$ can be written down at once, namely:

$$\Phi_0(y, \eta) = \frac{\begin{vmatrix} 1 & -iy & 0 & 0 & \cdots \\ 0 & \gamma_1(\eta) & -\frac{2}{3}iy & 0 & \cdots \\ 0 & -\frac{2}{5}iy & \gamma_2(\eta) & -\frac{3}{5}iy & \cdots \\ 0 & 0 & -\frac{3}{7}iy & \gamma_3(\eta) & \cdots \\ \vdots & \cdots & & & \end{vmatrix}}{\begin{vmatrix} \gamma_0(\eta) & -iy & 0 & 0 & \cdots \\ \frac{-iy}{3} & \gamma_1(\eta) & -\frac{2}{3}iy & 0 & \cdots \\ 0 & -\frac{2}{5}iy & \gamma_2(\eta) & -\frac{3}{5}iy & \cdots \\ 0 & 0 & -\frac{3}{7}iy & \gamma_3(\eta) & \cdots \\ \vdots & \cdots & & & \end{vmatrix}}. \tag{142}$$

The rigorous evaluation of the infinite determinants seems hopeless. Nor does the problem become any more tractable when it is realized that the neutron density is the Laplace-Fourier inverse of Eq. (142). However, for all practical purposes, one is interested in the density of slow neutrons from a fast neutron source (i.e., in large u). The assumption of large u (and not too large z—cf. below) permits the problem to be solved by a method of successive approximations in which age theory turns out to be the lowest approximation.

Thus, consider the expression for $\psi_0(z, u)$:

$$\psi_0(z, u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iyz} dy \cdot \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\eta u} d\eta \Phi_0(y, \eta). \tag{142a}$$

Since u is large, the dominant term arising from the Laplace inverse integral in (142a) is contributed by the pole of $\Phi_0(y, \eta)$ with the largest real part. But the poles of $\Phi_0(y, \eta)$ are defined by the zeros of the denominator of (142), and the zero of this denominator with the largest real part can be written in the form:

$$\eta = -ay^2 + by^4 + \cdots, \tag{142b}$$

where a and b are constants ($a > 0$). Consequently, the largest contribution to the Fourier inverse integral arises from small y. Further study of (142) on the basis of these remarks leads to the conclusion that one can arrive at increasingly improved results for the neutron density by taking larger sub-determinants of the infinite determinants.

If we choose a sub-determinant of order $(n+1)$ in (142), we say that we are working in the nth spherical harmonic approximation or the P_n-approximation. Thus, the lowest approximation which gives a spatial dependence of the neutron density is the P_1-approximation; in this approximation we have:

$$\Phi_0(y,\eta)=\frac{\begin{vmatrix}1 & -iy\\ 0 & \gamma_1(\eta)\end{vmatrix}}{\begin{vmatrix}\gamma_0(\eta) & -iy\\ \frac{-iy}{3} & \gamma_1(\eta)\end{vmatrix}}. \tag{143}$$

The pole of $\Phi_0(y, \eta)$ is obtained in the form (142b) by expanding $\gamma_0(\eta)$ and $\gamma_1(\eta)$ about $\eta=0$; we have

$$\gamma_0(\eta)=\eta\gamma_0'(0)+\cdots;\quad \gamma_1(\eta)=\gamma_1(0)+\cdots,$$

where $\gamma_0'(0)=\xi$ and $\gamma_1(0)=1-\langle\cos\Theta\rangle_{Av}$ (cf. (85a) and (85c)). Hence, the pole is at

$$\eta=-y^2/3\xi(1-\langle\cos\Theta\rangle_{Av}).$$

Calculating the residue at this point, we find:

$$\psi_0(z,u)=\frac{1}{2\pi\xi}\int_{-\infty}^{\infty}\exp[-iyz-y^2u/3\xi(1-\langle\cos\Theta\rangle_{Av})]dy \tag{144}$$

or

$$\psi_0(z,\tau)=\exp[-z^2/4\tau]/\xi(4\pi\tau)^{\frac{1}{2}}, \tag{145}$$

with

$$\tau=u/3\xi(1-\langle\cos\Theta\rangle_{Av}).$$

In other words, the P_1-approximation is identical with age theory[33] when one retains the first non-vanishing terms in $\gamma_0(\eta)$ and $\gamma_1(\eta)$.

The above procedure can be continued: in the P_2-approximation it turns out that one must retain terms in γ_0 up to η^2, in γ_1 up to η, and in γ_2 up to the constant. Furthermore, terms up to and including y^4 are present in the pole-condition (142b). As might be expected, the P_2-approximation leads to a result identical with Eq. (115).

The expression for ψ_0 becomes more and more involved in still higher approximations. However, it seems worth while to quote the results of a P_3-calculation (B1) in which γ_0 is expanded up to η^3, γ_1 to η^2, γ_2 to η, and γ_3 to the constant and where the pole-condition involves the retention of all terms up to and including y^6. The result is:

$$\psi_0(z,\tau)=\frac{\exp[-x^2]}{(4\pi\tau)^{\frac{1}{2}}}\left\{A_0+\frac{1}{\tau}\left[A_1(\tfrac{1}{2}-x^2)+\frac{D_1A_3}{E_0}(x^4-3x^2+\tfrac{3}{4})\right]+\frac{1}{\tau^2}\left[A_2(x^4-3x^2+\tfrac{3}{4})\right.\right.$$
$$\left.\left.+\frac{D_1A_4}{E_0}\left(-x^6+\frac{15}{2}x^4-\frac{45}{4}x^2+\frac{15}{8}\right)+\frac{D_1^2A_5}{E_0^2}\left(x^8-14x^6+\frac{105}{2}x^4-\frac{105}{2}x^2+\frac{105}{16}\right)\right]\right\}, \tag{146}$$

where

$$x^2=z^2/4\tau,\quad \tau=u/3\gamma_0'(0)\gamma_1(0),$$

$$A_0=\frac{F_0}{D_1},\quad A_1=\frac{G_0}{D_1}+\frac{1}{D_1^2}(2\delta_2F_0E_0-F_0E_1-F_1E_0),$$

$$A_2=\frac{1}{D_1^2}\left(\frac{6}{35}\delta_2F_0+2\delta_0G_0E_0-G_0E_1-\frac{3}{35}F_1-G_1E_0\right)+\frac{1}{D_1^3}(F_2E_0^2+2E_0E_1F_1-3\delta_2F_1E_0^2$$
$$+2F_0E_0E_2+F_0E_1^2-6\delta_2F_0E_0E_1+6\delta_2^2F_0E_0^2-3\delta_3F_0E_0^2),$$

[33] The Laplace—transform derivation of the age solution for variable mean free path (corresponding to (145)) can be obtained by a method analogous to the one which led to the result (78).

$$A_3 = -\frac{3F_0}{35D_1^2} + \frac{1}{D_1^3}(F_0E_1E_0 - \delta_2 F_0 E_0^2),$$

$$A_4 = -\frac{3G_0}{35D_1^2} + \frac{1}{D_1^3}\left(\frac{3}{35}F_1E_0 + G_0E_1E_0 + \frac{6}{35}F_0E_1 - \delta_2 G_0 E_0^2 - \frac{12\delta_2 E_0 F_0}{35}\right) + \frac{1}{D_1^4}(\delta_2 E_0^3 F_1 - F_1 E_0^2 E_1$$
$$- F_0E_0^2E_2 - 2F_0E_0E_1^2 + 6\delta_2 F_0 E_0^2 E_1 - 4\delta_2^2 F_0 E_0^3 + \delta_3 F_0 E_0^3),$$

$$A_5 = \frac{D_1}{2F_0}A_3^2$$

with

$$F_0 = \gamma_1(0)\gamma_2(0)\gamma_3(0), \quad F_1 = \gamma_3(0)[\gamma_2'(0)\gamma_1(0) + \gamma_1'(0)\gamma_2(0)], \quad F_2 = \gamma_3(0)\left[\gamma_1'(0)\gamma_2'(0) + \frac{\gamma_1''(0)}{2}\gamma_2(0)\right],$$

$$G_0 = \frac{9}{35}\gamma_1(0) + \frac{4}{15}\gamma_3(0), \quad G_1 = \frac{9}{35}\gamma_1'(0), \quad G_2 = \frac{9}{70}\gamma_1''(0),$$

$$D_1 = \gamma_0'(0)F_0, \quad D_2 = \frac{\gamma_0''(0)}{2}F_0 + \gamma_0'(0)F_1, \quad D_3 = \gamma_0'(0)F_2 + \frac{\gamma_0''(0)}{2}F_1 + \frac{\gamma_0'''(0)}{6}F_0,$$

$$E_0 = \tfrac{1}{3}\gamma_2(0)\gamma_3(0), \quad E_1 = \gamma_0'(0)G_0 + \frac{\gamma_2'(0)}{3}\gamma_3(0), \quad E_2 = \frac{9}{35}\left[\gamma_0'(0)\gamma_1'(0) + \frac{\gamma_0''(0)}{2}\gamma_1(0)\right] + \frac{2}{15}\gamma_0''(0)\gamma_3(0),$$

$$\delta_2 = D_2/D_1, \quad \delta_3 = D_3/D_1.$$

The quantities $\gamma_0'(0)$, $\gamma_0''(0)$, $\gamma_1(0)$, and $\gamma_1'(0)$ have already been tabulated (Eqs. (85a)–(85d)); the remaining γ's which enter above are:

$$\gamma_0'''(0) = 6 - \frac{3(M-1)^2}{2M}q_M\left(1 + \frac{q_M}{2} + \frac{q_M^2}{6}\right), \tag{147a}$$

$$\gamma_1''(0) = -\frac{32M}{9}\left(\frac{7}{6M^2} - 1\right) - \frac{2(4M+5)(M-1)^2}{9M}q_M - \frac{(M+2)^2(M-1)^2}{6M}q_M^2, \tag{147b}$$

$$\gamma_2(0) = 1 - \frac{(5-3M^2)}{8} - \frac{3}{32M}(M^2-1)^2 q_M, \tag{147c}$$

$$\gamma_2'(0) = -\frac{(9M^2-7)}{16} + \frac{(M-1)^2}{64M}q_M(9M^2+6M-7) + \frac{3}{64M}(M^2-1)^2 q_M, \tag{147d}$$

$$\gamma_3(0) = 1. \tag{147e}$$

We have derived the first three spherical harmonic approximations to the neutron density. Higher approximations can be obtained if necessary although the numerical work becomes increasingly more troublesome. The question now arises as to the range of validity of the different approximations. Examination of Eq. (146) (the P_2-approximation is obtained by dropping the $(1/\tau^2)$ term, the P_1-approximation by dropping both the $(1/\tau)$ and $(1/\tau^2)$ terms) reveals that the spherical harmonic method is essentially a development in powers of[34] $\epsilon = D_1 A_3 x^4 / E_0 A_0 \tau$. If ϵ is small compared to unity, the convergence is rapid and the P_3-approximation is adequate. For very small ϵ, the P_1-approximation (age theory) of course suffices. If ϵ is not small compared to unity, it would appear necessary to go to even higher approximations than the P_3[35].

[34] In a previous notation $\epsilon = B_{11}x^4/u$ (cf. Eq. (115)).
[35] The convergence condition turns out to be much less stringent due to considerable cancellation of terms (cf. Tables IV and V).

For large M (mass of the scattering nucleus), the convergence criterion for the spherical harmonic method, i.e. $\epsilon \ll 1$, can be written as $z^4 \ll (u/\xi)^3$. For given M and u, this condition very soon breaks down with increasing distance from the source. Each higher approximation extends the region of applicability of the theory to larger distances from the source. However, in principle it is not expected that any finite P-approximation will be accurate beyond distances of the order $z \sim (u/\xi)^{3/4}$. In order to push the theory to greater distances, it is necessary to somehow work in the P_∞-approximation. Placzek has partially succeeded in accomplishing this end (P2) by retaining only the highest power of z in the polynomial correction contributed by each successive P-approximation. This leads him to the following expression in the P_∞-approximation:

$$\psi_0(z, u) = \frac{A_0 \exp[-x^2]}{(4\pi\tau)^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!}$$

$$= \frac{A_0}{(4\pi\tau)^{\frac{1}{2}}} \exp[-x^2+\epsilon]. \tag{146a}$$

Equation (146a) constitutes a correction to age theory which does not require that $\epsilon \ll 1$. Indeed $\epsilon > 1$ is also possible; however, there are two restrictions on Eq. (146a), namely: $x^2 \gg 1$ and $z \ll (u/\xi)^{5/6}$. The first inequality justifies the sole retention of the highest power of z in each polynomial, the second follows from the fact that the term neglected in the exponential of Eq. (146a) is of the form[36] (ϵ^2/x^2) and this must be small compared to unity. For very large (u/ξ) (very large M and u), Eq. (146a) provides a significant extension of the theory.

We present some numerical results[37] in order to throw light on the extent of the deviations from age theory and the nature of the improvements upon it. We have chosen $M=12$ (carbon) and $\tau = 183$ cm^2; the value of the age corresponds to neutrons emitted at about 150 kev (Raγ+Be source) and detected at 1.44 ev (indium resonance energy). Table IV gives the slow neutron density as a function of the distance from a *plane* source of fast neutrons while Table V gives the slow neutron density as a function of distance from a *point* source of fast neutrons. The point-source values are obtained from the plane-source values through the use of Eq. (62). Column 1 of each table lists the square of the distance from the source in units of 4τ. Columns 2–4 list the P_1, P_2, and P_3 results, respectively, the latter two given in units of the P_1 values. Finally, Column 5 lists the ratio of the neutron density predicted by formula (146a) (or its appropriate modification) to the P_1 value.

It is seen from Tables IV and V that for carbon the spherical harmonic corrections to age theory do not become appreciable until $z \approx 7L_s$. According to the criterion discussed above, the correction should become appreciable at distances $z \approx (u/\xi)^{3/4} \approx 5L_s$. The greater range of validity of these results is due to the strong cancellation of terms in the P_2 and P_3 corrections. In virtue of the same cancellation of terms, the correction predicted by formula (146a) becomes suspect. Instead of giving

TABLE IV. Neutron density from plane source ($\psi_0(z, \tau)$).

$z^2/4\tau$	P_1 (in cm^{-1})	P_2/P_1	P_3/P_1	Exponential correction
0	$1.32\cdot10^{-1}$	1.00	1.00	1.00
1	$4.86\cdot10^{-2}$	1.00	1.00	1.02
2	$1.79\cdot10^{-2}$	1.00	1.00	1.10
3	$6.57\cdot10^{-3}$	1.01	1.01	1.25
4	$2.42\cdot10^{-3}$	1.03	1.03	1.48
5	$8.89\cdot10^{-4}$	1.05	1.05	1.85
10	$5.99\cdot10^{-6}$	1.28	1.29	11.59
15	$4.04\cdot10^{-8}$	1.67	1.82	247.7

[36] This can be guessed from the structure of the P_3-approximation; it is also confirmed by the one-dimensional model (cf. ref. P2).

[37] The computations were performed by the Los Alamos Computing Group under the direction of Mr. B. Carlson.

TABLE V. Neutron density from point source ($\psi_0(r, \tau)$).

$r^2/4\tau$	P_1 (in cm^{-3})	P_2/P_1	P_3/P_1	Exponential correction
0	$5.74\cdot10^{-5}$	1.01	1.01	1.00
1	$2.11\cdot10^{-5}$	1.00	1.00	0.97
2	$7.77\cdot10^{-6}$	0.99	0.99	0.99
3	$2.86\cdot10^{-6}$	1.00	1.00	1.06
4	$1.05\cdot10^{-6}$	1.01	1.01	1.19
5	$3.87\cdot10^{-7}$	1.02	1.02	1.39
10	$2.61\cdot10^{-9}$	1.21	1.22	5.91
15	$1.76\cdot10^{-11}$	1.58	1.68	65.64

reasonable corrections to age theory up to distances of order $z\approx(u/\xi)^{5/6}\approx 7L_s$, formula (146a) should overestimate them. This is confirmed by the fact that whereas for $(z^2/4\tau)=10$, the exact P_3-correction, for example, is only 1.29, the "exponential" P_3-correction (i.e. $(1+\epsilon+\epsilon^2/2)$) is 6.45. Similarly, for $(z^2/4\tau)=15$, the corresponding numbers are 1.82 and 21.53. We conclude from this calculation that the P_3-approximation is fairly reliable up to distances of the order $z\sim(u/\xi)^{3/4}$ and that formula (146a) should only be used in the region $z=(u/\xi)^{3/4}-(u/\xi)^{5/6}$ for *very* large M (and u).

2. *Improvements on Age Theory: One-Velocity Method*

The spherical harmonic method has furnished improvements on age theory for an infinite slowing-down medium. It can also be applied to finite media. However, a study of the limitations of age theory for finite media can be carried out more readily by means of an alternative method which we call the one-velocity method. Infinite media can also be treated by the one-velocity method.

The idea of the one-velocity method is to perform a Laplace transform (with respect to u) on the many-velocity transport equation (63); one obtains:[38]

$$\mu\partial\phi/\partial z+\phi(z,\mu,\eta)=\int d\Omega'\phi(z,\mu',\eta)g(\mu_0,\eta)+\delta(z)/4\pi, \tag{148}$$

where $\phi(z,\mu,\eta)=\mathfrak{L}\psi(z,\mu,u)$. Since η can be regarded as a parameter independent of z and μ, Eq. (148) is in Laplace space a one-velocity transport equation with anisotropic scattering and capture. The scattering function is

$$g(\mu_0,\eta)=\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi}g_l(\eta)P_l(\mu_0)$$

where all the g_l's are different from zero. Furthermore, an equivalent capture is present since $g_0(\eta)=1$ only for $\eta=0$; for all other values of $\eta(>0)$, $g_0(\eta)<1$.

In principle, once Eq. (148) has been solved subject to the appropriate boundary conditions for an infinite or finite medium, the Laplace inverse of the solution is the solution of the corresponding slowing-down problem. In practice, there are difficulties arising from the fact that: (1) the one-velocity problem with general anisotropic scattering and capture has not been solved, (2) even if the problem were solved, it would be impossible to take the Laplace inverse. While a rigorous solution of the problem is therefore excluded, it is possible to obtain approximate solutions by setting some of the g_l's equal to zero and by taking advantage of the fact that the slowing-down density is desired for large u. The latter point implies (cf. above) that the one-velocity solution need only be examined in the neighborhood of $\eta=0$, i.e., weak capture.

To illustrate the method, let us consider the case of isotropic scattering, i.e., we set $g_l(\eta)\equiv0$ for

[38] The mean free path is assumed constant and set equal to unity, as before.

$l>0$. Then, for an infinite medium, (148) leads to the following solution for $\phi_0(z,\eta)$ (P3):

$$\phi_0(z,\eta)=\frac{(1-\nu_0^2)\nu_0 e^{-\nu_0|z|}}{g_0(\eta)[\nu_0^2+g_0(\eta)-1]}+\frac{1}{2}\int_1^\infty \frac{dse^{-s|z|}}{s\left\{\left[1-\frac{g_0(\eta)}{s}\tanh^{-1}\left(\frac{1}{s}\right)\right]^2+\frac{\pi^2 g_0^2(\eta)}{4s^2}\right\}}, \tag{149}$$

where

$$\tanh^{-1}\nu_0/\nu_0=1/g_0(\eta).$$

Now, it is well known that the integral on the right-hand side of (149) is only important within one mean free path of the source. At larger distances, the contribution of the integral decreases rapidly compared to the first term on the right-hand side of (149). We therefore drop the integral and try to take the Laplace inverse of the first term. The exact Laplace inverse cannot be written down but, as mentioned before, we may study the behavior of ϕ_0 for small η. We expand $g_0(\eta)=1-\xi\eta+\cdots$ and $(\tanh^{-1}\nu_0)/\nu_0=1+\nu_0^2/3+\cdots$; substituting these expressions into the first term on the right-hand side of (149), we get to order η:

$$\phi_0(z,\eta)\approx\left(\frac{3}{4\xi}\right)^{\frac{1}{2}}\frac{\exp[-(3\xi\eta)^{\frac{1}{2}}|z|]}{(\eta)^{\frac{1}{2}}}. \tag{150}$$

The Laplace inverse of (150) is the Gaussian (145) with the age[39] $\tau=u/3\xi$. To obtain the analog of the P_2-approximation for isotropic scattering one would expand $g_0(\eta)$ up to η^2, and $\tanh^{-1}\nu_0/\nu_0$ up to ν_0^4. One can proceed in this way to get arbitrarily high approximations. However, as we have seen in our discussion of the spherical harmonic method, a consistent P_1-approximation includes $\gamma_1(0)$ (cf. fn. 39), a consistent P_2-approximation includes $\gamma_0''(0)$, $\gamma_1'(0)$, and $\gamma_2(0)$ (in addition to $\gamma_1(0)$) and so on. In other words, simultaneous with the retention of higher powers of η in the expression for ϕ_0, more g_l's in the scattering function $g(\mu_0,\eta)$ must be taken into account. Just as in the isotropic case, the "branch-point"[40] contribution, represented by an integral of the sort appearing in (149), can be neglected compared to the "pole"[40] contribution of the sort appearing as the first term on the right-hand side of (149). Thus, for linear scattering (i.e., $g_l(\eta)\equiv 0$ for $l>1$), we get as the "pole" contribution:

$$\phi_0(z,\eta)(=)\frac{\left[1-\frac{3g_1(\eta)}{\nu_1^2}\left(\frac{\tanh^{-1}\nu_1}{\nu_1}-1\right)\right]\exp[-\nu_1|z|]}{g_0(\eta)\left\{\frac{g_0(\eta)\tanh^{-1}\nu_1}{\nu_1^2}-\frac{g_0(\eta)}{\nu_1(1-\nu_1^2)}+3g_1(\eta)[g_0(\eta)-1]\left[\frac{3-2\nu_1^2}{\nu_1^3(1-\nu_1^2)}-\frac{3\tanh^{-1}\nu_1}{\nu_1^4}\right]\right\}}, \tag{151}$$

where ν_1 is defined by:

$$1+\frac{3}{\nu_1^2}g_1(\eta)\left(1-\frac{\tanh^{-1}\nu_1}{\nu_1}\right)-g_0(\eta)\frac{\tanh^{-1}\nu_1}{\nu_1}-\frac{3}{\nu_1^2}g_0(\eta)g_1(\eta)\left(1-\frac{\tanh^{-1}\nu_1}{\nu_1}\right)=0.$$

If Eq. (151) is now treated in accordance with the approximations $g_0=1-\xi\eta$, and $g_1(\eta)=\langle\cos\Theta\rangle_{\text{Av}}$, etc., one obtains the Gaussian (145) for $\psi_0(z,u)$. The derivation of the analog to the P_2-approximation in the spherical harmonic method requires a knowledge of the one-velocity "pole" solution for quadratic scattering, and so on for the analogs to the higher P-approximations.

While the one-velocity method does not seem to have any special advantage over the spherical harmonic method as far as the infinite-medium problem is concerned, it would appear to be useful for the study of certain aspects of the finite-medium problem,—e.g., the extrapolated end point,

[39] The age should be $u/3\xi(1-\langle\cos\Theta\rangle_{\text{Av}})$; however, since we have assumed isotropy, $\langle\cos\Theta\rangle_{\text{Av}}=0$.

[40] This refers to the origin of the term when the one-velocity problem is solved by the Fourier transform method.

albedo, etc. In particular, some light is thrown on these matters by working in the approximation of isotropic scattering in the laboratory system[41] (i.e., $g_l(\eta)\equiv 0$ for $l>0$). Thus, one may investigate the extrapolated end point in the "isotropic" approximation by considering the problem of a plane δ-source of mono-energetic fast neutrons situated in a semi-infinite slowing-down medium at a distance z' from the boundary (cf. Fig. 4). The solution ("pole" contribution) valid more than a mean free path from the boundary and from the source is (for isotropic scattering):[42]

$$\phi_0(z,\eta)(=)\frac{2\nu_0(1-\nu_0^2)}{g_0(\eta)[\nu_0^2+g_0(\eta)-1]}\exp[-\nu_0\{z'+z_0(\eta)\}]\sinh\{\nu_0[z+z_0(\eta)]\}, \tag{152}$$

where $\tanh^{-1}\nu_0/\nu_0=1/g_0(\eta)$ and $z_0(\eta)$ is the extrapolated end-point defined in terms of η by

$$z_0(\eta)=\bar{z}_0\left[1+\frac{\nu_0^2}{3}+O(\nu_0^4)\right], \tag{153}$$

with[43] $\bar{z}_0=0.7104$. Equation (152) is a solution of the "diffusion-like" equation:

$$\partial^2\phi_0/\partial z^2-\nu_0^2\phi_0(z,\eta)=-Q(\nu_0)\delta(z-z') \tag{154}$$

subject to the boundary condition:

$$\phi_0(z,\eta)-\frac{\tanh[\nu_0 z_0(\eta)]}{\nu_0}\frac{\partial\phi_0(z,\eta)}{\partial z}=0 \quad \text{at } z=0. \tag{154a}$$

In Eq. (154), $Q(\nu_0)$, the effective source strength, is the coefficient of $e^{-\nu_0|z|}$ in the infinite medium solution (cf. (149)).

To obtain information about the neutron density in a finite medium, one expands all functions of η in (152) about $\eta=0$. In the first approximation, one keeps the first non-vanishing term; Eq. (152) becomes:

$$\phi_0(z,\eta)=(3/4\xi\eta)^{\frac{1}{2}}\{\exp[-(3\xi\eta)^{\frac{1}{2}}(z'-z)]-\exp[-(3\xi\eta)^{\frac{1}{2}}(z'+z+2\bar{z}_0)]\}, \quad (0\le z<z'), \tag{155}$$

of which the Laplace inverse is ($\tau=u/3\xi$):

$$\psi_0(z,u)=\frac{1}{\xi(4\pi\tau)^{\frac{1}{2}}}\{\exp[-(z'-z)^2/4\tau]-\exp[-(z'+z+2\bar{z}_0)^2/4\tau]\}, \quad (0\le z<z'). \tag{156}$$

Equation (156) is the age solution with the extrapolated end point at $z=-\bar{z}_0$. In the next approximation, one expands all functions of η in (152) about $\eta=0$, and retains the first two non-vanishing terms, and so on. In this fashion one can study the dependence of the extrapolated end point on z' and u by starting with the Wiener-Hopf solution for the one-velocity problem.

3. *Asymptotic Neutron Density*

Neither the spherical harmonic method nor the one-velocity method is suitable for finding the neutron density for very large z. A method of successive approximations which starts with age theory as a first approximation cannot easily yield the asymptotic neutron density. The problem must be tackled in some other way.

Physically, it is to be expected that most slow neutrons at great distances from the source arrive at these distances after a large number of small-angle collisions, each associated with a small energy

[41] The assumption of isotropic scattering in the laboratory system becomes better, the larger M.
[42] Equation (152) is derived using Wiener-Hopf techniques (cf. N. Wiener and E. Hopf, Berliner Ber. Math. Phys. Klasse (1931), p. 696).
[43] It is to be remembered that all lengths are measured in units of the mean free path.

loss. Consequently, an upper bound on the asymptotic neutron density can be obtained by assuming that the neutrons which reach large z all travel in a straight line. If this assumption is made, the probability $P_n(z)$ that a neutron will reach a distance z after suffering just n collisions, is (the mean free path is set equal to unity):

$$P_n(z) = z^n e^{-z}/n!. \tag{157}$$

The number of neutrons per unit logarithmic energy interval after just n collisions is $N(n, u)$ where $N(n, u)$ is given by Eq. (57). Hence, the total number of neutrons per unit logarithmic energy interval at distance z is obtained by summing over-all collisions and we get:

$$N(z, u) = \sum_{n=1}^{\infty} P_n(z) N(n, u). \tag{158}$$

Since the greatest contribution to $N(z, u)$ comes from terms corresponding to a large number of collisions, we can replace $N(n, u)$ by its asymptotic value (for large n, and for fixed but large u—cf. Eq. (57)), namely:

$$N(n, u) \approx \frac{\alpha e^{-u}(\alpha u)^{n-1}}{(n-1)!}, \quad \left(\alpha = \frac{(M+1)^2}{4M}\right). \tag{159}$$

Substituting (159) into (158), we find:

$$N(z, u) \approx \frac{e^{-u-z}}{u} \sum_{n=1}^{\infty} \frac{(\alpha uz)^n}{n!(n-1)!} = \left(\frac{\alpha z}{u}\right)^{\frac{1}{2}} \exp\left[-u-z-\frac{\pi i}{2}\right] J_1(2i(\alpha uz)^{\frac{1}{2}}). \tag{160}$$

For large values of z, the asymptotic behavior of $N(z, u)$ is easily obtained from the asymptotic behavior of the Bessel function; we get:

$$N(z, u) \approx \frac{1}{(4\pi)^{\frac{1}{2}}} \left(\frac{\alpha z}{u^3}\right)^{\frac{1}{4}} \exp[-u-z+2(\alpha uz)^{\frac{1}{2}} + O(1/z^{\epsilon})], \tag{161}$$

where $\epsilon > 0$. Equation (161) was first derived by Wigner (W8) for the case of hydrogen ($M=1$).

It is difficult to judge the extent to which the upper bound for the asymptotic neutron density given by (161) is in error. It is therefore desirable to obtain a rigorous expression for the asymptotic neutron density, not merely an upper bound. The correct asymptotic neutron density could then be fitted on to formula (146), say, and a fairly complete curve for the neutron density as a function of distance could be drawn (at least for constant mean free path).

Breit (B6) and Placzek (P2) were the first to investigate the asymptotic neutron density in some detail. Breit assumed that the scattering is spherically symmetric in the laboratory system (i.e., the mass of the scattering nucleus is infinite) and studied the asymptotic behavior of the Laplace transform of the neutron density. Placzek looked into the asymptotic neutron density on the basis of Fermi's "one-dimensional" model of slowing down and assuming constant mean free path. In the one-dimensional model, it is supposed that the neutrons move in a straight line and that the effect of a collision is to leave the direction of the neutron velocity unaltered, or to simply reverse its direction. The energy loss in a collision can be chosen arbitrarily. Assuming that the energy loss in a collision is a constant (equal, for example, to the average logarithmic energy loss in the actual case), Placzek found for the asymptotic neutron density:

$$\psi_0(z, u) \approx \frac{1}{(4\pi z)^{\frac{1}{2}}} \left(\frac{|z|}{2p}\right)^p \exp[-|z|+p], \tag{162}$$

where $p = (u/\xi)+1$.

Bothe (B5) has also looked into the question of the asymptotic neutron density, even attempting to take account of the variation of the mean free path with energy. However, his underlying assumptions are so stringent—isotropic scattering in the laboratory system and a single average energy loss per collision—that his results are not very satisfactory.

Recently, Wick (W6) re-examined the entire problem and deduced the correct asymptotic formula for very large distances and for constant mean free path. Wick's procedure is quite ingenious and we shall present it briefly here; we start with Eq. (140) rewritten as:

$$(1-iy\cos\theta)\Phi(y,\theta,\eta)=\int d\mathbf{\Omega}'\Phi(y,\theta',\eta)g(\Theta,\eta)+(1/4\pi), \tag{163}$$

where $\cos\theta$ replaces μ, and

$$g(\Theta,\eta)=\frac{\alpha}{\pi}\frac{1}{(M^2-\sin^2\Theta)^{\frac{1}{2}}}\left\{\frac{\cos\Theta+(M^2-\sin^2\Theta)^{\frac{1}{2}}}{M+1}\right\}^{2(\eta+1)}. \tag{163a}$$

It is to be recalled that the neutron density, $\psi_0(z, u)$ is defined in terms of $\Phi(y, \theta, \eta)$ by:

$$\psi_0(z,u)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}d\eta e^{\eta u}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}dye^{-iyz}\int d\mathbf{\Omega}\Phi(y,\theta,\eta), \tag{164}$$

where σ is to the right of all the poles (and can be taken as large as desired). A rigorous evaluation of (164) is impossible. However, an asymptotic formula can be derived by taking advantage of the fact that for large z, only the discrete pole of the Fourier inverse integral need be evaluated, and that the Laplace inverse integral can be calculated by means of the "saddle-point" method.

We first observe that $g(\Theta, \eta)$ has a strong maximum for small collision angles, i.e., $\Theta\approx 0$ (which give the largest contribution to the neutron density at large distances—cf. above) for large values of η. This leads to the expectation (which is fulfilled—cf. below) that an expression for the asymptotic neutron density can be obtained by studying Eq. (163) in the region of large η. We are, therefore, interested in the behavior of $g(\Theta, \eta)$ for large values of η (i.e., small values of Θ); we find that $g(\Theta, \eta)$ can be expanded in the form:

$$g(\Theta,\eta)=\frac{\alpha}{\pi M}\exp[-\Theta^2\eta'+\Theta^4\eta''+\cdots],$$

where

$$\eta'=\frac{1}{M}\left[\eta+1-\frac{1}{2M}\right],\quad \eta''=\frac{1}{12M}\left(1-\frac{3}{M^2}\right)\left\{\eta+1-\frac{1}{M}\frac{[2-(3/M^2)]}{[1-(3/M^2)]}\right\}. \tag{165}$$

It will appear below that the values of θ, θ', and Θ which have to be considered, are of order $1/(\eta')^{\frac{1}{2}}$; hence, the term in Θ^4 in the exponent in Eq. (165) is of order $(1/\eta')$ compared with the quadratic term. In the first approximation, we therefore take:[44]

$$g(\Theta,\eta)=(\alpha/\pi M)\exp[-\Theta^2\eta']. \tag{165a}$$

[44] A check on the accuracy of the first-order expression (165a) for large η can be obtained by calculating

$$g_l(\eta)=\int d\mathbf{\Omega}P_l(\cos\theta)g(\theta,\eta)\approx 2\pi\int_0^\infty\left(\frac{\alpha}{\pi M}\right)\exp[-\eta'\theta^2]J_0[(l+\tfrac{1}{2})\theta]\theta d\theta=\frac{\alpha}{\eta}\exp\left[-\frac{M}{4\eta}(l+\tfrac{1}{2})^2\right],$$

and comparing the result with the exact expression for $g_l(\eta)$. We have replaced $P_l(\cos\theta)$ by $J_0[(l+\frac{1}{2})\theta]$ (since large η implies small θ), and η' by (η/M) (valid for large M). The following numbers were obtained for $M=12$ (carbon) and $\eta=100$:

l	g_l (exact)	g_l (approximate)
0	0.0349	0.0350
1	0.0328	0.0329
2	0.0291	0.0292
3	0.0242	0.0243
4	0.0189	0.0192
5	0.0140	0.0142

The agreement is excellent.

The second approximation would consist in writing:

$$g(\Theta, \eta) = (\alpha/\pi M)(1+\Theta^4\eta'') \exp[-\Theta^2\eta'], \tag{165b}$$

and so on.

The next step is to solve Eq. (163) for $\Phi_0(y, \eta)$, and to take its Fourier inverse. As we have already remarked, the evaluation of the Fourier inverse integral in (164) requires a knowledge of the discrete pole of $\Phi_0(y, \eta)$. It can be shown that this pole occurs on the imaginary axis, i.e., that the pole is at a point $y_0(\eta)$, where $|y_0(\eta)|<1$ for finite $\eta(|y_0(\eta)|\to 1$ as $\eta\to\infty)$. Furthermore, it can be shown that the residue at the pole of $\Phi_0(y, \eta)$ is given in terms of the homogeneous solution, $\bar{\Phi}_0(y, \eta)$ of Eq. (163) (i.e., the solution with the source term omitted) by means of the relation:

$$\lim_{y\to y_0(\eta)} (y-y_0)\Phi_0(y, \eta) = iC\bar{\Phi}_0(y_0, \eta), \tag{166}$$

where C is a normalization factor.

On the basis of Eq. (166), the residue of $\Phi_0(y, \eta)$ at $y=y_0(\eta)$ can be found by studying the homogeneous equation. Substituting (165a) into the homogeneous part of (163), and expanding $\cos\theta\approx 1-\frac{1}{2}\theta^2$, $d\mathbf{\Omega}'\approx\theta' d\theta' d\phi'$ leads to the first-order equation for $\bar{\Phi}(y, \theta, \eta)$:

$$(1-k)\bar{\Phi}(y, \theta, \eta) = -\frac{\theta^2}{2}\bar{\Phi}(y, \theta, \eta) + \frac{\alpha}{\pi M}\int \theta' d\theta' d\phi' \exp[-\Theta^2\eta']\bar{\Phi}(y, \theta', \eta), \tag{167}$$

where we can write, to the same order:

$$\Theta^2 = \theta^2+\theta'^2-2\theta\theta' \cos(\phi-\phi'). \tag{167a}$$

The quantity k takes the place of iy in the inhomogeneous equation and, as we shall see, plays the role of an "eigenvalue"; that is to say, a bounded solution of the homogeneous equation (167) only exists for certain specified values of k (really functions of η) which are called "eigenvalues." The limiting value of k for $\eta\to\infty$ is always unity; for large but finite η, k is close to unity. Use has been made of the latter fact in deriving Eq. (167).

The largest "eigenvalue" k will determine the asymptotic behavior of the neutron density. To obtain this "eigenvalue," it is convenient to take the Fourier-Bessel transform of Eq. (167). This corresponds to expanding $\bar{\Phi}(y, \theta, \eta)$ in the analogon to Legendre polynomials for very small[45] θ, namely:

$$\bar{\Phi}(y, \theta, \eta) = \frac{1}{2\pi}\int_0^\infty s ds \bar{\phi}(y, s, \eta) J_0(sr), \tag{168}$$

with

$$\bar{\phi}(y, s, \eta) = \int_0^\infty r dr \bar{\Phi}(y, \theta, \eta) J_0(sr). \tag{168a}$$

In Eq. (168) we have used the variable $r=(\eta')^{\frac{1}{2}}\theta$ instead of θ. If we write $k=1-k_1/\eta'$, and insert the Fourier-Bessel integral (168) into (167), we get the following eigenvalue equation for k (independent[46] of η' and therefore of η):

$$\frac{d^2\bar{\phi}}{ds^2}+\frac{1}{s}\frac{d\bar{\phi}}{ds}+\left\{-2k_1+\frac{2\alpha}{M}\exp[-s^2/4]\right\}\bar{\phi}=0. \tag{169}$$

We are interested in the lowest "eigenvalue" of Eq. (169) where $\bar{\phi}$ satisfies the boundary conditions: $d\bar{\phi}/ds$ at $s=0$ and $\bar{\phi}=0$ at $s=\infty$. Equation (169) has the form of a Schrödinger equation with a

[45] This follows from the fact that $P_l(\cos\theta)\approx J_0(l\theta)$ for small θ. We have used $J_0(l\theta)$ instead of $J_0[(l+\frac{1}{2})\theta]$ for the sake of simplicity; the difference is not significant for most of the l's.

[46] This is the justification for writing $k=1-(k_1/\eta')$, and shows that $k\to 1$ as $\eta'\to\infty$.

Gaussian potential. As is well known, an analytic solution of this problem does not exist; however, the variational method easily yields accurate results for the lowest eigenvalue. Thus, if we choose $\bar{\phi}$ of the form $A\exp[-as^2/4]$, and carry out the variational calculation, we find $a=(2\alpha/M)^{\frac{1}{2}}-\frac{1}{2}$, and $k_1=a^2/2$. The two limiting cases, $M=1$ and $M\gg1$, give values for k_1 of 0.418 and 0.0215, respectively.[47] These values are lower than the correct values, and additional calculations show they are not in error by more than about 10 percent.

Knowledge of the quantity k is not sufficient for the derivation of the asymptotic neutron density. The calculation of the lowest eigenvalue must be pushed to second order. One writes $k=1-k_1/(\eta'-c)+O(1/\eta'^3)$, where c is determined by substituting (165b) for $g(\Theta, \eta)$ into (163), and all expansions are carried out one step further in powers of $(1/\eta')$. The precedure is straightforward and we shall not write down the result. Terms of order $(1/\eta'^3)$ and higher can be neglected completely.

Once k_1 and c are known, the residue of $\Phi_0(y, \eta)$ at the pole $iy_0=k$ can be determined from (166). To get the correct asymptotic formula, it suffices to evaluate the residue in the first-order approximation. To do this, we write down the analog of (167) for $\Phi(y, \theta, \eta)$, namely:

$$(1-iy)\Phi(y, \theta, \eta)=-\frac{\theta^2}{2}\Phi(y, \theta, \eta)+\frac{\alpha}{\pi M}\int\theta' d\theta' d\phi'\Phi(y, \theta', \eta)\exp[-\Theta^2\eta']+\frac{1}{4\pi}. \tag{170}$$

We multiply Eq. (167) by $\Phi(y, \theta, \eta)$ and Eq. (170) by $\bar{\Phi}(y, \theta, \eta)$, subtract and integrate over $\theta d\theta$; we get:

$$(k-iy)\int\theta d\theta\Phi(y, \theta, \eta)\bar{\Phi}(y, \theta, \eta)=\frac{1}{4\pi}\int\theta d\theta\bar{\Phi}(y, \theta, \eta). \tag{171}$$

Now,

$$\lim_{y\to -ik}(y+ik)\Phi(y, \theta, \eta)=iC\bar{\Phi}(-ik, \theta, \eta)$$

(cf. (166)); moreover,

$$\int\theta d\theta\bar{\Phi}(y, \theta, \eta)=\frac{1}{\eta'}\bar{\phi}(y, 0, \eta),$$

and in accordance with the convolution theorem for the Fourier-Bessel transform,

$$\int\theta d\theta\bar{\Phi}^2(y, \theta, \eta)=\frac{1}{\eta'}\int sds\bar{\phi}^2(y, s, \eta).$$

Hence, the residue of $\Phi_0(y, \eta)$ at $y=y_0(\eta)$ is given by:

$$iC\bar{\Phi}_0(-ik, \eta)=bM/2\eta'; \tag{172}$$

where

$$b=\bar{\phi}^2(-ik, 0, \eta)\Big/\int_0^\infty sds\bar{\phi}^2(-ik, s, \eta), \tag{172a}$$

and

$$k=1-(k_1/\eta'). \tag{172b}$$

Collecting our results and inserting them into (164) yields the final contour integral which enables us to find the asymptotic neutron density; we get:

$$\psi_0(z, u)\approx\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}d\eta\left(\frac{bM}{2\eta'}\right)\exp\left[\eta u-z\left(1-\frac{k_1}{\eta'-c}\right)\right], \tag{173}$$

where the procedure for finding the constants k_1, b, and c (which depend on the mass of the scattering

[47] It is interesting to note that the numbers 0.418 and 0.0215 are much smaller than the corresponding numbers 8 and 2 associated with the upper limit for the neutron density defined by Eq. (161) (cf. Eq. (175) below).

nucleus) has been completely specified above. The evaluation of (173) can be carried out by means of the saddle-point method. The exponential term in (173) has a saddle point at $\eta=\eta_s$, where η_s is obtained from the equation:

$$u-zk_1M/(\eta_s'-c)^2=0, \tag{174}$$

with

$$\eta_s'=\frac{1}{M}\left[\eta_s+1-\frac{1}{2M}\right].$$

Proceeding in the usual fashion (i.e., expanding the various functions about $\eta=\eta_s$ and retaining up to the quadratic terms in the exponential, etc.) leads to the final asymptotic formula:

$$\psi_0(z,u)\approx\frac{bM}{4(\pi)^{\frac{1}{2}}c}\left(\frac{k_1}{Mu^3z}\right)^{\frac{1}{4}}\exp[-z+(k_1Muz)^{\frac{1}{2}}+cu]. \tag{175}$$

Equation (175) is the correct asymptotic formula for the neutron density—correct up to and including the constant term—provided the mean free path for scattering is constant (independent of the energy).

While Eq. (175) is correct up to and including the constant term, the leading term neglected in the exponential is of order $(u/\xi)^{\frac{3}{2}}z^{-\frac{1}{2}}$. This implies that Eq. (175) is valid at distances (measured in units of the mean free path) large compared to $(u/\xi)^3$. These are very large distances indeed! However, Wick points out in his paper (W6) how the neutron density can be found at smaller distances from the source at the expense of greater numerical work.

Wick (W6) has also examined the asymptotic neutron density for variable mean free path. He assumes that the mean free path for scattering decreases monotonically with the energy. He finds:

$$\psi_0(z,u)\sim f(u)z^\delta e^{-z}, \tag{176}$$

where $\delta>-1$, $f(u)$ is a function of u alone behaving like be^{cu} (b and c are constants) for very large u, and where the distance is measured in units of the mean free path at the initial energy. Comparison of Eqs. (175) and (176) reveals that the term proportional to $z^{\frac{1}{2}}$ in the exponential is absent in the variable mean free path case whereas it is present in the case of constant mean free path. In other words, the neutron density at large distances decreases more rapidly for variable mean free path than for constant mean free path. This is to be expected in view of the fact that the slowing down process (for variable mean free path) becomes increasingly more efficient as the energy decreases. The absence of the $z^{\frac{1}{2}}$ term for variable mean free path has the additional consequence that the energy spectrum of the slow neutrons tends to a limiting form at large distances in contrast to the constant mean free path case where the ratio of slow to fast neutrons increases continuously with distance. For further details, the reader should consult Wick (W6) and also Wick and Verde (W7) where some numerical results for the asymptotic neutron density on the assumption of a "$1/v$" scattering cross section are given for the particular case of hydrogen.

It is a pleasure to thank Dr. George Placzek for helpful discussion of some points in this article.

IV. APPENDIX: NUMERICAL RESULTS FOR THE SECOND SPATIAL MOMENT[48]

The formulae derived in Section A of Part II permit fairly accurate predictions of the second spatial moment of the density of neutrons in a slowing-down medium, provided the scattering mean free path is known as a function of the energy. In this appendix, we apply those formulae to the more common slowing-down substances (C, O, H, D_2O, H_2O) for which measurements of the scattering cross section are available. For the purpose of reference, the relevant scattering cross sections are plotted in Figs. 6–9 as a function of energy.[49]

[48] The author is indebted to the Los Alamos Computing Group under Mr. Bengt Carlson for the computations on C, O, H, and D_2O. The computations on H_2O were carried out by the Montreal Computing Group, again under Mr. Carlson's direction.

[49] These curves were taken from C. L. Bailey *et al*, Phys. Rev. **70**, 580 (1946); D. H. Frisch, Phys. Rev. **70**, 589 (1946) and C. L. Bailey *et al*, Phys. Rev. **70**, 805 (1946).

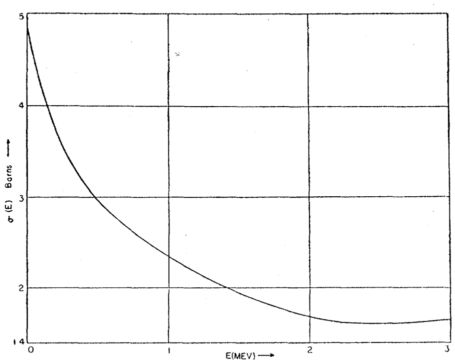

FIG. 6. Scattering cross section of carbon.

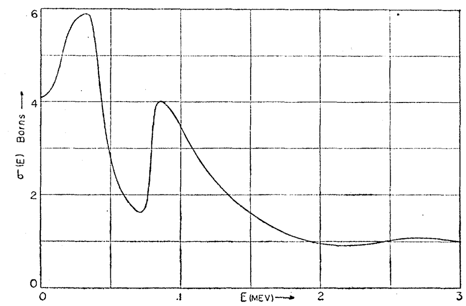

FIG. 7. Scattering cross section of oxygen.

The results for the slowing-down length[50] of neutrons in C, O, H, D_2O are given in Tables VI–IX, respectively. In each of the four tables, column 1 gives the initial energy E_i of the fast neutrons, and column 2 the corresponding value of $u(\equiv\log(E_i/E_f))$, where E_f is the energy at which the slow neutrons are detected. The detection energy has in all cases been chosen as the indium resonance energy, i.e., 1.44 ev. Column 3 in all four tables lists the slowing-down lengths obtained by inserting the measured scattering cross sections (as a function of energy) into Eq. (122b). In the case of D_2O, proper account has been taken of the fact that we are dealing with a mixture. Column 3, therefore, represents the age approximation for the slowing-down length.

The remaining two columns in Tables VI–IX have to be discussed separately. Column 4 of Tables VI and VII lists the slowing-down lengths derived from the "interval" formula (103). Since the collision interval

$$q_M\left(\equiv\log\left(\frac{M+1}{M-1}\right)^2\right)$$

is quite small for both carbon and oxygen (cf. Table X),[51] the sub-intervals, introduced for the purpose of applying (103), can be chosen sufficiently small so that the variation of the mean free path (with energy) over each sub-interval is not more than 10 percent (cf. remarks preceding Eq. (92)). The value of the mean free path in each sub-interval is chosen in accordance with the remarks following Eq. (103). While the "interval" formula was developed primarily for its application to *mixtures* of heavy elements, it can of course be used for the determination of the slowing-down length of neutrons in single heavy elements. In particular, for oxygen, formula (103) is especially convenient because of the existence of resonances in the scattering cross section; by virtue of this fact, the slowing-down lengths in oxygen as given by the "interval" formula are probably more accurate than those given by the "exponential" equations (84a) and (84b) (cf. column 5 of Table VII). In the case of carbon, column 4 is included for the sake of comparison[52] with the certainly more accurate results[53] obtained on the basis of the "exponential" equations (84a) and (84b).

Column 5 of Tables VI and VII lists the slowing-down lengths in carbon and oxygen predicted by Eqs. (84a) and (84b) when the scattering mean free paths are fitted (in their variation with u) by sums of exponentials. Rather than choose a large number of exponentials and fit the coefficients at a series of points, we have written the mean

TABLE VI. Slowing down length of neutrons in carbon ($\rho=1.6$).

E_i(Mev)	u	Age theory	Interval theory	Exponential theory
3.0	14.55	19.2 cm	19.7 cm	19.8 cm
2.0	14.14	17.7	18.2	18.2
1.0	13.45	15.9	16.2	16.2
0.50	12.76	14.7	15.0	15.0
0.25	12.06	13.9	14.1	14.1
0.10	11.15	13.2	13.3	13.3

TABLE VII. Slowing-down length of neutrons in oxygen ($\rho=1.0$).

E_i(Mev)	u	Age theory	Interval theory	Exponential theory
3.0	14.55	56.8 cm	58.8 cm	62.6 cm
2.0	14.14	48.8	51.2	50.0
1.0	13.45	42.2	42.5	42.6
0.50	12.76	38.6	39.2	39.1
0.25	12.06	37.9	38.0	38.2
0.10	11.15	36.8	37.1	37.1

[50] The slowing-down length is related to the second spatial moment of the neutron density by Eq. (70).

[51] Table X contains the values of the constants characteristic of H, D, C, and O which were used in computing the various slowing-down lengths.

[52] It actually turns out that the "interval" and "exponential" values of the slowing-down length in carbon are almost identical (cf. column 5 of Table VI).

[53] L. Nordheim, G. Nordheim, and H. Soodak (N1) have derived values for the slowing-down length of neutrons in carbon from the formula:

$$L_s^2=\frac{1}{3}\left\{\int_0^u\frac{l^2(u')}{\alpha a}du'+l^2(0)+l^2(u)\right\}.$$

This formula is not as accurate as the "interval" or "exponential" formula; however, their results agree with the ones given in Table VI to one unit in the last place.

TABLE VIII. Slowing-down length of neutrons in hydrogen ($\rho=1.0$).

E_i(Mev)	u	Age theory	Interval theory	Exponential theory
3.0	14.55	0.728 cm	0.837 cm	0.865 cm
2.0	14.14	0.608	0.693	0.707
1.0	13.45	0.463	0.516	0.520
0.50	12.76	0.375	0.407	0.411
0.25	12.06	0.328	0.348	0.352
0.10	11.15	0.293	0.306	0.309

TABLE XI. Slowing-down length of neutrons in heavy water ($\rho=1.1$).

E_i(Mev)	u	Age theory	Interval theory	Exponential theory
3.0	14.55	10.5 cm	11.4 cm	11.9 cm
2.0	14.14	10.1	10.8	10.9
1.0	13.45	9.7	10.1	10.2
0.50	12.76	9.4	9.9	9.8
0.25	12.06	9.1	9.5	9.5
0.10	11.15	8.8	9.2	9.2

free path as a constant plus one exponential, i.e., $l(u)=A_0+A_1e^{-a_1u}$, and on this representation are placed the requirements that the mean free path of the slow neutrons (e.g., at the indium resonance energy of 1.44 ev) be correctly given, and further that:

$$\int_0^{u_0}(A_0+A_1e^{-a_1u})^2du=\int_0^{u_0}l^2(u)du, \quad \text{(A1)}$$

$$\int_0^{u_0}u(A_0+A_1e^{-a_1u})du=\int_0^{u_0}ul(u)du, \quad \text{(A2)}$$

where the mean free path under the integral signs denotes the actual mean free path as a function of energy, and u_0 depends on the initial energy. Condition (A1) is chosen since it expresses the fact that age theory is a good first approximation to the final result. Condition (A2) is chosen since it weighs the low energy region more than the high energy region; this is necessary because, while the mean free path varies most rapidly in the high energy region, the number of slowing-down collisions is greater in the low energy region (where the mean free path is usually fairly constant). For carbon, conditions (A1) and (A2) lead to the expression: $l(u)=2.69+6.20e^{-0.743u}$, where $l(u)$ is measured

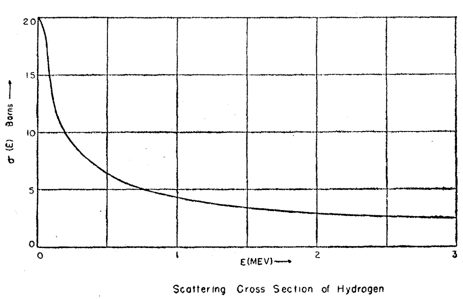

FIG. 8. Scattering cross section of hydrogen.

in cm, and the density has been taken as 1.6 g/cm^3; the fit is made for $u_0=14.55$ (i.e., $E_i=3$ Mev). For oxygen, we get $l(u)=6.50+40.3e^{-2.68u}$, corresponding to a density of 1.0 g/cm^3.

Column 4 of Tables VIII and IX lists the slowing-down lengths in hydrogen and heavy water derived from the "integral" form of the "interval" formula (103). That is, Eq. (103) is replaced by the limiting form assumed when the number of sub-intervals is permitted to become infinite, namely:

$$L_s^2(u)=\frac{1}{3}\left\{\int_0^u\frac{l^2(u')du'}{a(u')\alpha(u')}+\int_0^u du'\left[\frac{b(u')}{2a(u')}\frac{d}{du'}\left(\frac{l^2(u')}{a(u')\alpha(u')}\right)+\frac{l(u')\beta(u')}{a(u')\alpha(u')}\frac{d}{du'}\left(\frac{l(u')}{\alpha(u')}\right)\right]-\frac{l^2(u)}{a(u)\alpha(u)}\left(\frac{b(u)}{a(u)}+\frac{\beta(u)}{\alpha(u)}\right)\right\}. \quad \text{(A3)}$$

Equation (A3) can be written more simply for a single element:[54]

$$L_s^2(u)=\frac{1}{3}\left\{\frac{1}{a\alpha}\int_0^u l^2(u')du'-\frac{1}{2a\alpha}\left(\frac{b}{a}+\frac{\beta}{\alpha}\right)[l^2(0)+l^2(u)]\right\}. \quad \text{(A4)}$$

We call Eq. (A3) (and (A4)) the integral form of Eq. (103). The application of the "interval" formula (103) (and therefore of (A3) or (A4)) to hydrogen and heavy water should, in principle, lead to poor results since the scattering mean free paths of neither hydrogen nor heavy water are sensibly constant over properly chosen sub-intervals. However, it is a relatively easy matter to apply Eqs. (A3) and (A4) and this has been done for hydrogen and heavy water to judge the accuracy of the interval-type approximation (by comparison with the more accurate "exponential" values for the slowing-down length listed in column 5 of Tables VIII and IX). It is seen that the "interval" values for the slowing down length are not very

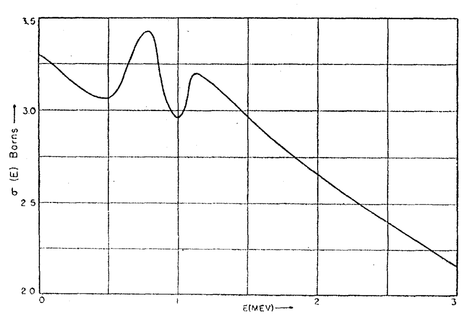

FIG. 9. Scattering cross section of deuterium.

[54] Equation (A4) could also be used for carbon and oxygen in place of (103); however, both formulae lead to essentially identical results (which is to be expected). Presumably, Eq. (A4) (and (A3)) could be derived directly from the transport equation.

TABLE X. Constants for H, D, C, O.

Element	M	$q_M\left(\equiv\log\left(\frac{M+1}{M-1}\right)^2\right)$	$\gamma_0'(0)(\equiv\xi)$	$\gamma_0''(0)$	$\gamma_1(0)(\equiv 1-\langle\cos\Theta\rangle_{\frac{1}{2}})$	$\gamma_1'(0)$
H	1	∞	1.0000	−2.0000	0.3333	0.4444
D	2	2.1972	0.7254	−0.8472	0.6667	−0.0454
C	12	0.3341	0.1578	−0.0341	0.9444	−0.0466
O	16	0.2503	0.1200	−0.0196	0.9583	−0.0366

different from the "exponential" values and are a considerable improvement over the "age" values.

Column 5 of Table VIII lists the slowing-down lengths obtained from Eqs. (84a) and (84b) by choosing a representation of the scattering mean free path of neutrons in hydrogen[55] in accordance with (A1) and (A2). This choice leads to the following expression for $l(u)$ in cm (for a density of 1.0 g/cm^3): $l(u)=0.083+0.677e^{-0.793u}$. Since heavy water is a mixture, Eqs. (84a) and (84b) cannot be used; however, the "exponential" approximation has been extended to mixtures in the form of Eq. (91). Column 5 of Table IX lists the values obtained for the slowing-down length of neutrons in heavy water by an application of Eq. (91). Both the total scattering mean free path, and its ratio to the scattering mean free path in deuterium have been chosen in accordance with relations of the type (A1) and (A2). We find (for a density of 1.1 g/cm^3):

$$l(u)=2.83+3.85e^{-2.04u},\quad c(u)=0.617+0.183e^{-1.05u}.$$

The slowing-down length of neutrons in ordinary water can be calculated along the lines of the slowing-down length in heavy water. However, if the mass of the oxygen nucleus is assumed to be infinite, a rigorous formula for the slowing-down length of neutrons in water is available, namely Eq. (108). Computations based on Eq. (108) have been carried out[56] (M5) and the results for the slowing-down length are plotted in Fig. 10 as a function of the initial energy (in Mev). Just as in the previous computations, the final energy is chosen as 1.44 ev (the indium resonance energy). In contrast to the previous computations, results for the slowing-down length are included for values of the initial energy as high as 5 Mev. For convenience, the total scattering mean free path, and the ratio of the total mean free path to the mean free path in hydrogen, as obtained from Figs. 7–8 augmented by the measured values in the range 3–5 Mev, are plotted up as functions of the energy in Fig. 11.

The results for the slowing-down length given in Fig. 10 are completely rigorous provided one accepts the assumption of an infinitely heavy oxygen nucleus.[57] Because of the rather large mass of the oxygen nucleus, and because the scattering cross section of hydrogen is much greater than that of oxygen, the error introduced by the infinite-

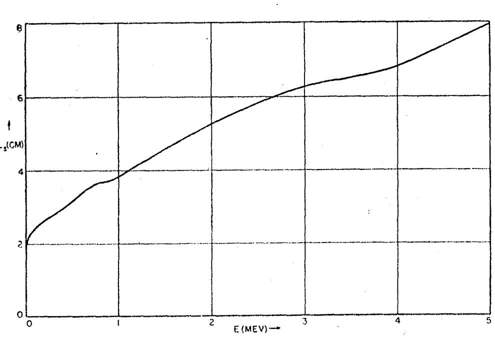

FIG. 10. Slowing-down length of neutrons in water.

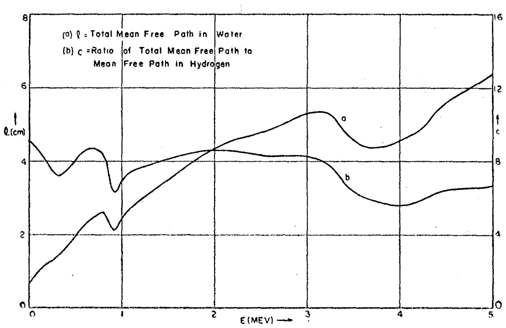

FIG. 11. Scattering mean free path of neutrons in water as function of energy.

[55] In the case of hydrogen, we have available a rigorous formula for the slowing-down length as a function of the energy (i.e., Eq. (79)). However, since hydrogen by itself is not used as a slowing-down medium, the tedious numerical calculations connected with the rigorous formula have not been carried out. Instead, the rigorous formula for the slowing-down length of neutrons in a hydrogen-containing mixture has been applied to ordinary water (cf. below). The approximate calculations have been performed for hydrogen, the lightest element and the one for which the scattering mean free path varies most rapidly, to provide a severe test of the various methods for calculating the slowing-down length.

[56] L. Nordheim, G. Nordheim, and H. Soodak (N1) have also calculated the slowing-down length of neutrons in water on the basis of formula (108). However, they used earlier (and less accurate) measurements of the cross sections and hence their computations have been superseded by the present ones.

[57] Implicit in this statement is the assumption that the scattering cross section is isotropic in the center of mass system for both hydrogen and oxygen. This is very closely true for hydrogen, but may not be true for oxygen. As a matter of fact, the existence of resonances in the scattering cross section of oxygen makes it unlikely that the scattering is isotropic throughout the energy range considered. However, the deviations from isotropy are unknown and the net effect on the slowing-down length is probably small.

TABLE XI. Slowing-down length of neutrons in water ($\rho=1.0$).

E_i(Mev)	u	Interval theory
3.0	14.55	6.4 cm
2.0	14.14	5.3
1.0	13.45	3.8
0.50	12.76	3.1
0.25	12.06	2.7
0.10	11.15	2.4

mass assumption should be small. To estimate this error, we resort to the following device: we represent $l(u)$ (the total mean free path), and $c(u)$ (the ratio of the total mean free path to the mean free path in hydrogen) by expressions of the exponential type, namely ($u=0$ corresponds to $E_i=5$ Mev):

$$l(u)=0.655+10.25e^{-0.64u},\quad c(u)=0.91-0.665e^{-0.115u}.$$

These representations of course smear out the oxygen resonances but otherwise reproduce the curves in Fig. 11 fairly well. Once the exponential expressions are adopted, it is possible to apply Eq. (91). Two calculations for the slowing-down length have been performed using Eq. (91): one assuming that the mass of the oxygen nucleus is infinite, and the second assuming the correct mass. The percentage deviation between the slowing-down length obtained on the infinite-mass hypothesis and that obtained on the finite-mass hypothesis turns out to be 1.5 percent at 5 Mev and is a monotonically decreasing function of the energy.

As a final remark, it may be noted that the approximate "interval" formula (A3) has been applied to water for several initial energies (the final energy is again taken as 1.44 ev) with the results for the slowing-down length given in Table XI. The purpose of this calculation was to ascertain the degree of accuracy obtainable with a relatively simple formula like (A3) as contrasted with the laborious calculations required by the use of the rigorous formula (108). A comparison of Table XI with Fig. 10 shows that the agreement is excellent. In view of this, it seems justified to employ (A3) for future calculations on the slowing-down length of neutrons in water (and other hydrogenous mixtures), necessitated, say, by improved measurements of the scattering cross sections of hydrogen and oxygen.

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