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The Interaction of Electrons and an Electromagnetic Field*

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CHAPTER I. GENERAL INTRODUCTION

1. Particles in the Maxwell-Lorentz Theory

THE understanding of the ultimate nature of matter has been for a long time a fundamental pursuit of physics and philosophy. In recent years this problem has acquired added interest owing to the discovery of a number of new particles. Besides the electron and the proton, which have been known for quite some time, the existence of the positron, neutron, and positive and negative mesons has been well established by experiment. The existence of a few other particles (such as the neutrino, negative proton, and the neutral meson) has been postulated from various theoretical considerations, although there has as yet been no experimental evidence. It is thought that all matter could be regarded as being built out of a few such elementary particles; consequently, a knowledge of the properties of these particles and of their interactions with each other is of profound importance.

The existence of particles cannot be explained on the basis of the Maxwell-Lorentz theory of classical electromagnetism. Maxwell's equations of classical electrodynamics, which have met with remarkable success in certain of their applications, have given rise to many difficulties when generalized to consider the presence of charged particles.

2. Finite Electron or Point Electron?

In any theory of a particle it is desirable at the outset to decide on the model to be used to describe the particle, whether it is to be considered as having a finite size or as being a point singularity. Lorentz[1] regarded the electron as a small charged sphere, and by using Maxwell's theory he calculated the radiation reaction, obtaining for this force a series in ascending powers of the radius, R, of the electron; thus,

$$F=\tfrac{2}{3}(e^2/c^3)(d^2\mathbf{V}/dt^2)+(\ )R+(\ )R^2+\cdots. \quad (1)$$

When R is made small, only the first term of this series remains; and therefore an approximate form of the equations of motion is

$$md\mathbf{V}/dt-\tfrac{2}{3}(e^2/c^3)(d^2\mathbf{V}/dt^2)=e[\mathbf{E}+(\mathbf{V}/c)\times\mathbf{H}], \quad (2)$$

where $\mathbf{V}$ denotes the velocity of the electron, and $\mathbf{E}$ and $\mathbf{H}$ the electric and magnetic vectors of the incident field.

The Lorentz theory, however, has only a limited applicability. It is a non-relativistic theory, and cannot be made relativistic in a straightforward manner, since "size" is not a relativistically invariant concept, and a sphere in one frame of reference will not be a sphere in another Lorentz frame. Furthermore, the theory does not give a stable model of the electron, since any finite charge distribution would explode if acted upon by purely electromagnetic forces, the different parts of the electron repelling each other according to the Coulomb law. The theory, therefore, has to be supplemented by introducing mechanical forces of some other type to hold the charge on the electron together. No satisfactory way of introducing such forces has yet been discovered.

For these and other reasons it has now become clear that it is better to abandon the extended model of the electron and regard the electron as a point charge, and then proceed to remove the characteristic difficulties that arise in theories of point particles. The chief difficulty is that on the basis of the Maxwell-Lorentz theory, a point electron is supposed to have an infinite self-energy. This difficulty also exists in the quantum theory, where it manifests itself as a divergence in the solution of the wave equation that describes the interaction of an electron and an electromagnetic field. One cannot, therefore, hope to remove these difficulties by merely passing over to the quantum theory; that is, by taking into account the disturbances accompanying measurements. Hence, to build a theory free from the occurrence of infinities, the proper approach is to make the necessary refinements in the classical theory and then to proceed to the quantum theory.

Two ways of avoiding the difficulty of infinities in the classical theory have been suggested. One method was developed by Born and Infeld[2] and

[1] H. A. Lorentz, *The Theory of Electrons* (Leipzig, 1916), second edition.

[2] M. Born, Proc. Roy. Soc. **A143**, 410 (1934); M. Born and L. Infeld, Proc. Roy. Soc. **A144**, 425 (1934).

consists in modifying Maxwell's theory, so that the energy of the field round the point singularity is finite. This method, however, has been only partially successful and has encountered difficulties in the process of second quantization. The other method is due to Dirac[3] and will be discussed in detail later.

One should exspect that this difficulty of infinities would arise as long as a point electron is considered as the limit of a finite charge distribution. The infinite self-energy is the work that has to be done against electromagnetic forces in concentrating the finite charge to a point. If, however, we give up considering the electron as the limit of a finite charge but regard it as an "elementary particle," meaning that, while all matter may be thought of as being built out of certain elementary particles, these particles themselves cannot be divided further into simpler entities, then it becomes clear that an electron cannot have any self-energy, finite or otherwise, because the concept of energy of a system is derived from that of the mechanical work that can be obtained from the system by suitably displacing it, and no work can be extracted from an electron when it is by itself, it being understood that all elementary particles are permanent and immutable in the classical theory.[4, 5] In this way one sees that it should be possible to have a classical theory of point particles, free from the occurrence of infinities.

3. Dirac's Theory

Dirac retains Maxwell's theory to describe the field right up to the point singularity that represents the electron and shows that, in the mathematical formulation of this theory, the terms which give rise to infinities can be subtracted out in a Lorentz invariant way. Such a subtraction process becomes possible for the reasons given in the previous section. The theory is in agreement with well established principles, such as the principle of relativity and the principles of conservation of energy and momentum. The reaction of the radiation field on the motion of the electron is effectively taken into account.

[3] P. A. M. Dirac, Proc. Roy. Soc. **A167**, 148 (1938).
[4] H. J. Bhabha, Proc. Ind. Acad. Sci. **A11**, 347 (1938).
[5] E. A. Milne, Phil. Mag. **34**, 73 (1943).

Pryce[6] has shown that this method is equivalent to replacing the usual energy-momentum tensor, $T_{\mu\nu}$, of the Maxwell field by a tensor of the form

$$T_{\mu\nu}-\partial K_{\sigma\mu\nu}/\partial t,$$

where $K_{\sigma\mu\nu}$ is a tensor which is antisymmetrical in σ and μ, and which depends explicitly on the coordinates of the point charges present in the field. This new energy-momentum tensor gives a finite value to the field energy of a given system of charges. It agrees with the usual tensor, for field points at large distances from the charges.

The equations of motion obtained by Dirac's method are, in their non-relativistic approximations, found to be the same as the Lorentz equations. In this way the equations of Lorentz were rederived by Dirac. But whereas Lorentz' method of derivation makes these equations necessarily approximate, Dirac has suggested that his method of derivation gives room to hope that these equations are exact within the limits of the classical theory.

This method introduced by Dirac has been successfully applied by Bhabha to derive the equations of motion of spinning particles in electromagnetic and meson fields.

4. Difficulties in Dirac's Theory

The present author, in a series of recent papers, has applied Dirac's theory to various problems. The results of these applications reveal several features of the theory which appear to be at variance with familiar ideas of physics. The self-accelerating motions of a free electron, the artificial nature of the only physically allowable solution of the problem of an electron that is disturbed by a pulse of electromagnetic radiation, the inability of the electron in the hydrogen atom to spiral inwards and fall into the nucleus, and the absence of a physically allowable solution of the problem of an electron moving in the field of a thin infinite-charged plate, all appear to suggest that in some respects the Lorentz-Dirac equations are unsatisfactory. However, the correspondence between the classical and the quantum theories is not close enough for one to conclude from the above results that the Dirac

[6] M. Pryce, Proc. Roy. Soc. **A168**, 389 (1938).

classical theory is inadequate owing to its contradiction with experimental results. The theory has to be translated into the quantum theory and the consequences investigated and compared with experimental results. Nevertheless, it seems also desirable to look for possible modifications within the classical theory. In Chapter II we shall consider a generalized classical electrodynamics which includes the Dirac theory as a particular case.

5. Quantum Electrodynamics

Once the classical theory is formulated in a satisfactory way, free from the occurrence of infinities, one has then to take it over into the quantum theory. This may be done by first expressing the classical equations of motion in Hamiltonian form, and then passing over to the quantum theory according to the usual rules, by replacing momenta by operators satisfying certain commutation relations. One has then to examine the physical interpretation of the resulting theory.

It has been shown by Dirac[7] that in order to avoid singularities when expressing the equations of motion in Hamiltonian form, a certain limiting process becomes necessary. This process, called the "λ limiting process," was first introduced by Wentzel[8] and subsequently developed by Dirac, and consists in expressing the equations of motion in terms of a small time-like vector, λ. The equations are then exact only in the limit of λ tending to zero. This limiting process is found to help in the subsequent developments by securing the elimination of certain divergent integrals in quantum electrodynamics.

The next stage of the development concerns the physical interpretation of the quantum electrodynamical equations. In the early attempts to extend the non-relativistic quantum mechanics to make it conform to the special theory of relativity, the mathematical scheme was easily formulated, but there arose several difficulties over an adequate physical interpretation of the theory. It was found that according to these equations a particle has negative energy states in addition to the usual ones of positive energy, and further, if the particle has integral spin, the negative energy states occur with probability values which are negative. The methods which were introduced to meet these difficulties, namely, the Dirac hole theory for particles of half-odd integral spin, and the method of second quantization according to the Pauli-Weisskopf[9] scheme, have met with some success. But a further difficulty arises. When applied to photons, the corresponding wave equation is found to have no valid solution, owing to the occurrence of divergent integrals.

Various attempts have been made to eliminate these divergent integrals. Among the notable contributions to this problem are: (i) the non-linear field theory of Born and Infeld,[2] where the classical Maxwell equations are modified to eliminate divergences, but the quantization of the theory has proved difficult; (ii) the theory of Heitler and Peng,[10] which is a heuristic attempt to demonstrate that the divergent terms may be consistently omitted; (iii) the quantum theory of vacuum fields of Born and Peng;[11] (iv) the quantum electrodynamics of Dirac.[7,12]

In this paper we shall follow the lines of Dirac's quantum electrodynamics. We first express the generalized form of the Lorentz-Dirac equations in Hamiltonian form, and translate them into the quantum theory. We then investigate the interaction of an electron and a radiation field on the basis of these new equations. We shall show that the interaction is entirely free from divergent integrals to any order of approximation in the perturbation theory.

CHAPTER II. GENERALIZED CLASSICAL ELECTRODYNAMICS

6. Fields associated with an electron

Let (x_0, x_1, x_2, x_3) be the time and space coordinates of any point, and let the metric tensor

[7] P. A. M. Dirac, Ann. d. l'Inst. Poincaré **9**, 13 (1939); Proc. Roy. Soc. **A180**, 1 (1942).

[8] G. Wentzel, Zeits. f. Physik **86**, 479, 635 (1933); **87**, 726 (1934).

[9] W. Pauli and V. Weisskopf, Helv. Phys. Acta **7**, 709 (1934).

[10] W. Heitler and H. W. Peng, Proc. Camb. Phil. Soc. **38**, 296 (1942).

[11] M. Born and H. W. Peng, Proc. Roy. Soc. Edinburgh **62**, 40 (1944).

[12] P. A. M. Dirac, Communications of the Dublin Inst, **A**, 1 (1943),

$g_{\mu\nu}$ be given by

$$g_{00}=1, \quad g_{11}=g_{22}=g_{33}=-1,$$

all the other components vanishing. The velocity of light is taken as unity. Assuming that the world-line of the electron in space-time is given by the equations $z_\mu=z_\mu(s)$, where s is the proper time, the electromagnetic potentials A_μ of its field at the point x_μ satisfy the equation

$$\partial A_\mu/\partial x_\mu=0, \quad \Box A_\mu=4\pi j_\mu, \tag{1}$$

where

$$\Box\equiv\frac{\partial^2}{\partial x_0^2}-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}-\frac{\partial^2}{\partial x_3^2},$$

and j_μ is the charge-current density vector, given by

$$j_\mu=e\int\frac{dz_\mu}{ds}\delta(x_0-z_0)\delta(x_1-z_1)\times\delta(x_2-z_2)\delta(x_3-z_3)ds. \tag{2}$$

The field quantities, $F^{\mu\nu}$, are connected with the potentials by the relations

$$F^{\mu\nu}=\partial A^\nu/\partial x_\mu-\partial A^\mu/\partial x_\nu. \tag{3}$$

Particular solutions of Eqs. (1) are the retarded and advanced potentials, first considered by Lienard and Wiechert, and which shall be denoted by A_{ret} and A_{adv}, respectively. A general solution is obtained by adding to either of these particular solutions any solution of

$$\partial A_\mu/\partial x_\mu=0, \quad \Box A_\mu=0. \tag{4}$$

It has been usual to assume that only the retarded field is of relevance to physical problems, the advanced potentials being looked upon as mathematical solutions of a non-physical type. Thus, denoting by F_{el} the field due to the electron, the usual procedure has been to take

$$F_{\text{el}}=F_{\text{ret}}.$$

This assumption is a corner stone in the Maxwell-Lorentz theory. It determines the existence of a radiation field, because when the field F_{ret} is calculated, it is found that in addition to the Coulomb field there is also a transverse field, which is taken as the radiation field that is emitted by the electron. All the properties of the radiation field are thus fixed by the assumption that the field of a moving electron is its retarded field. This assumption is also used in the Dirac theory, and it is, therefore, not surprising that the Lorentz theory and the Dirac theory lead to the same equations of motion.

It seems, however, of some interest to see if any improvement can be brought about by dispensing with the assumption that the field of a moving electron is given by its retarded field, and considering a more general solution of Eqs. (1). We therefore take

$$F_{\text{el}}=F_{\text{ret}}+G, \tag{5}$$

where G describes a radiation field which is derivable from a potential function satisfying (4). G should be finite on the world-line of the electron. Furthermore, when the electron is at rest, the field F_{el} should reduce to the Coulomb field. Hence G must vanish when the electron is stationary. Subject to all the above conditions, G is still arbitrary. By choosing various forms of G we may obtain different radiation fields. We should then examine which of these lead to adequate equations of motion.

An obvious choice for the function G is

$$G=k(F_{\text{ret}}-F_{\text{adv}}), \tag{6}$$

where k is an arbitrary constant. It has been shown by Dirac[3] that on the world-line

$$F_{\text{ret}}{}^{\mu\sigma}-F_{\text{adv}}{}^{\mu\sigma}=\frac{4e}{3}\left(v^\sigma\frac{d^2v^\mu}{ds^2}-v^\mu\frac{d^2v^\sigma}{ds^2}\right), \tag{7}$$

thus showing that G is free from singularities on the world-line. It is easy to verify that (6) satisfies Eq. (4), and also that it vanishes when the electron is at rest. Hence (6) satisfies all the necessary conditions. We note that we have introduced advanced potentials into the expression (6) for G. The use of advanced potentials[13] should not be deprecated, provided all the mathematical requirements are consistently met and the resulting equations of motion lead to physically understandable results. The advanced

[13] After this work was completed, I saw a paper by J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945), who have also introduced advanced potentials but are led to the same results as Dirac, in attempting to provide a radiative mechanism.

potentials may play an important part when the theory is quantized, as for example they may correspond in some way to photons of negative energy which are now used in the physical interpretation of quantum electrodynamics. We do not attempt to give a physical mechanism for the phenomenon of radiation reaction, but shall chiefly be concerned with obtaining an adequate set of equations. In atomic processes it may not be possible to give a description in terms of cause and effect, or past and future. Only a complicated mathematical connection between different events may be feasible.

7. The Equations of Motion

We follow Dirac's method of derivation of the equations of motion of an electron in an electromagnetic field. The world-line of the electron is assumed to be known between the points s_1 and s_2, and is supposed to be surrounded by a thin world-tube; the flow of energy, momentum, and of angular momentum across a three-dimensional section of this tube is calculated. The principles of conservation of energy, of momentum, and of angular momentum[14] require that these rates of flow should be perfect differentials. This requirement gives the relations satisfied by the coordinates of the points on the world line of the electron and hence gives the equations of motion of the electron.

The field being described by the field quantities $F_{\mu\nu}$, its energy-momentum tensor is given by

$$4\pi T_{\mu\nu}=F_{\mu\rho}F^{\rho}{}_{\nu}+\tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \tag{8}$$

which satisfies the equation of conservation in free space

$$\partial T_{\mu\nu}/\partial x_{\nu}=0.$$

The angular momentum density of the field is given by a tensor $M_{\lambda\mu\nu}$, defined by

$$M_{\lambda\mu\nu}=x_{\lambda}T_{\mu\nu}-x_{\mu}T_{\lambda\nu}, \tag{9}$$

which satisfies the equation of conservation in free space

$$\partial M_{\lambda\mu\nu}/\partial x_{\nu}=0.$$

The flow of energy and momentum out of the tube is given by $\int T_{\mu\nu}dS^{\nu}$, the integration being over the surface of the tube, dS^{ν} representing a three-dimensional surface element of the tube, and the flow of angular momentum out of the tube is $\int M_{\lambda\mu\nu}dS^{\nu}$.

Now,

$$F=F_{\mathrm{el}}+F_{\mathrm{ext}}, \tag{10}$$

where F_{ext} denotes the external electromagnetic field. We define a new field $f_{\mu\nu}$ by writing

$$F=\tfrac{1}{2}(F_{\mathrm{ret}}+F_{\mathrm{adv}})+f,$$

so that

$$f=(k+\tfrac{1}{2})(F_{\mathrm{ret}}-F_{\mathrm{adv}})+F_{\mathrm{ext}}. \tag{11}$$

On the world-line

$$f^{\mu\sigma}=\tfrac{2}{3}(2k+1)e\left(v^{\sigma}\frac{d^2v^{\mu}}{dx^2}-v^{\mu}\frac{d^2v^{\sigma}}{ds^2}\right)+F_{\mathrm{ext}}. \tag{12}$$

The integrals $\int T_{\mu\nu}dS^{\nu}$ and $\int M_{\lambda\mu\nu}dS^{\nu}$ are evaluated by taking a tube which is spherical, and of constant radius ϵ, for each value of the proper time, in the particular frame of reference in which the electron is then at rest. We take advantage of calculations already made by Dirac,[3] by following his notation and writing

$$x_{\mu}=z_{\mu}(s_0)+\gamma_{\mu},$$

where the γ's are small, and s_0 is chosen so that

$$(\gamma,v)\equiv\gamma_0v_0-\gamma_1v_1-\gamma_2v_2-\gamma_3c_3=0. \tag{13}$$

Then

$$F_{\mathrm{ret}}{}^{\mu\sigma}=e[1-(\gamma,\dot{v})]^{-\frac{1}{2}}\left\{\left(\frac{\dot{v}^2}{8\epsilon}-\frac{1}{\epsilon^3}\right)(v^{\mu}\gamma^{\sigma}-v^{\sigma}\gamma^{\mu})-\frac{1}{2\epsilon}[1+(\gamma,\dot{v})](\dot{v}^{\mu}\gamma^{\sigma}-\dot{v}^{\sigma}\gamma^{\mu})\right.$$
$$\left.+\frac{1}{2\epsilon}\left(\gamma^{\sigma}\frac{d\dot{v}^{\mu}}{ds}-\gamma^{\mu}\frac{d\dot{v}^{\sigma}}{ds}\right)+\frac{2}{3}\left(v^{\sigma}\frac{d\dot{v}^{\mu}}{ds}-v^{\mu}\frac{d\dot{v}^{\sigma}}{ds}\right)\right\}, \tag{14}$$

$$F_{\mathrm{adv}}{}^{\mu\sigma}=F_{\mathrm{ret}}{}^{\mu\sigma}-\frac{4e}{3}\left(v^{\sigma}\frac{d\dot{v}^{\mu}}{ds}-v^{\mu}\frac{d\dot{v}^{\sigma}}{ds}\right), \tag{15}$$

[14] The conservation of angular momentum was first introduced into this method by H. J. Bhabha, Proc. Ind. Acad. Sci. **A10**, 324 (1939).

to the required degree of approximation in ϵ; dots denote differentiation with respect to the proper time, s. Hence

$$4\pi T_{\mu\rho}=e^2[1-(\gamma,\dot{v})]^{-1}\Bigg\{\left(\frac{1}{\epsilon^6}-\frac{\dot{v}^2}{4\epsilon^4}\right)(\epsilon^2 v_\mu v_\rho-\gamma_\mu\gamma_\rho)-\frac{1}{4\epsilon^2}(\dot{v}_\mu\dot{v}_\rho+\dot{v}^2 v_\mu v_\rho)$$
$$+\frac{1}{2\epsilon^3}[1+(\gamma,\dot{v})][\gamma_\mu\dot{v}_\rho+\gamma_\rho\dot{v}_\mu+2(\gamma,\dot{v})v_\mu v_\rho]-\frac{1}{2\epsilon^4}\bigg[2\dot{v}^2\gamma_\mu\gamma_\rho$$
$$+\left(\gamma,\frac{d^2v}{ds^2}\right)v_\mu v_\rho+\epsilon^2\left(v_\mu\frac{d\dot{v}_\rho}{ds}+v_\rho\frac{d\dot{v}_\mu}{ds}\right)\bigg]-\tfrac{1}{4}g_{\mu\rho}\left[\frac{2}{\epsilon^4}\{1+(\gamma,\dot{v})+(\gamma,\dot{v})^2\}+\frac{\dot{v}^2}{\epsilon^2}\right]\Bigg\}$$
$$+\frac{e}{\epsilon^3}\{(v_\rho\gamma^\sigma-v^\sigma\gamma_\rho)f_{\mu\sigma}+(v_\sigma\gamma_\mu-v_\mu\gamma_\sigma)f_\rho{}^\sigma-g_{\mu\rho}v_\nu\gamma_\sigma f^{\nu\sigma}\},\quad(16)$$

where we have made use of the relations

$$\gamma^2=-\epsilon^2,\quad(\gamma,v)=0,\quad v^2=1,\quad(v,\dot{v})=0,\quad\left(v,\frac{d^2v}{ds^2}\right)+\dot{v}^2=0.\quad(17)$$

An integral over the three-dimensional surface, dS^ν, may be split up into an integral over the two-dimensional spherical section of the tube for a particular proper time, and then an integral along the world-line. If $d\omega$ stands for the elemental solid angle of this two-dimensional spherical surface,

$$dS^\nu=-\gamma^\nu[1-(\gamma,\dot{v})]\epsilon d\omega ds.\quad(18)$$

Hence,

$$\int T_{\mu\nu}dS^\nu=-(4\pi)^{-1}\int\int\left[e^2\left\{\left(\frac{1}{\epsilon^4}+\frac{\dot{v}^2}{\epsilon^2}\right)\gamma_\mu-\frac{1}{2\epsilon^3}[1+\tfrac{3}{2}(\gamma,\dot{v})]\dot{v}_\mu\right\}+\frac{e}{\epsilon}v^\sigma f_{\mu\sigma}\right]d\omega ds$$
$$=\int_{S_1}^{S_2}(\tfrac{1}{2}e^2\epsilon^{-1}\dot{v}_\mu-ev^\sigma f_{\mu\sigma})ds,\quad(19)$$

neglecting terms which vanish with ϵ. Also

$$\int M_{\lambda\mu\nu}dS^\nu=\int(z_\lambda T_{\mu\nu}-z_\mu T_{\lambda\nu})dS^\nu+\int(\gamma_\lambda T_{\mu\nu}-\gamma_\mu T_{\lambda\nu})dS^\nu.\quad(20)$$

The first integral in (20) is

$$-(4\pi)^{-1}\int\int\bigg[z_\lambda\left\{e^2\left(\frac{1}{2\epsilon^3}+\frac{\dot{v}^2}{2\epsilon}\right)\gamma_\mu-\frac{3}{4}\frac{e^2}{\epsilon}(\gamma,\dot{v})\dot{v}_\mu-\left(\frac{e^2}{2\epsilon}\dot{v}_\mu-ev^\sigma f_{\mu\sigma}\right)\right\}$$
$$-z_\mu\left\{e^2\left(\frac{1}{2\epsilon^3}+\frac{\dot{v}^2}{2\epsilon}\right)\gamma_\lambda-\frac{3}{4}\frac{e^2}{\epsilon}(\gamma,\dot{v})\dot{v}_\lambda-\left(\frac{e^2}{2\epsilon}\dot{v}_\lambda-ev^\sigma f_{\lambda\sigma}\right)\right\}\bigg]d\omega ds$$
$$=\int\{z_\lambda(\tfrac{1}{2}e^2\epsilon^{-1}\dot{v}_\mu-ev^\sigma f_{\mu\sigma})-z_\mu(\tfrac{1}{2}e^2\epsilon^{-1}\dot{v}_\lambda-ev^\sigma f_{\lambda\sigma})\}ds.$$

The second integral in (20) is

$$-(4\pi)^{-1}\int\int\gamma_\lambda(\tfrac{1}{2}e^2\epsilon^{-1}\dot{v}_\mu-ev^\sigma f_{\mu\sigma})-\gamma_\mu(\tfrac{1}{2}e^2\epsilon^{-1}\dot{v}_\lambda-ev^\sigma f_{\lambda\sigma})\frac{3e^2}{4\epsilon}(\gamma,\dot{v})(\gamma_\lambda\dot{v}_\mu-\gamma_\mu\dot{v}_\lambda)d\omega ds=0,$$

where we omit terms which vanish with ϵ. Hence

$$\int M_{\lambda\mu\nu}dS^{\nu}=\int_{S_1}^{S_2}\{z_\lambda(\tfrac{1}{2}e^2\epsilon^{-1}\dot{v}_\mu-ev^\sigma f_{\mu\sigma})-z_\mu(\tfrac{1}{2}e^2\epsilon^{-1}\dot{v}_\lambda-ev^\sigma f_{\lambda\sigma})\}ds. \tag{21}$$

From the requirement of conservation of energy and momentum (19) must depend only on conditions at the two ends of the tube, and therefore the integrand must be a perfect differential. Thus

$$\tfrac{1}{2}e^2\epsilon^{-1}\dot{v}_\mu-ev_\sigma f_\mu{}^\sigma=\dot{B}_\mu. \tag{22}$$

From the form of the left-hand side of (22) we see that B_μ is not altogether arbitrary, but must satisfy the condition

$$(v,\,\dot{B})=0. \tag{23}$$

By using (22), (21) becomes

$$\int M_{\lambda\mu\nu}dS^{\nu}=\int_{S_1}^{S_2}(z_\lambda\dot{B}_\mu-z_\mu\dot{B}_\lambda)ds=[z_\lambda B_\mu-z_\mu B_\lambda]_{S_1}^{S_2}-\int_{S_1}^{S_2}(v_\lambda B_\mu-v_\mu B_\lambda)ds, \tag{24}$$

where terms which vanish with ϵ are omitted. Hence for conservation of angular momentum

$$v_\lambda B_\mu-v_\mu B_\lambda \tag{25}$$

must be a perfect differential. Thus the equations of motion are given by (22), where B_μ is any function which is subject to the conditions (23) and (25) but is otherwise arbitrary.

We thus see that the method of derivation of the equations of motion from the conservation principles does not give the equations uniquely. By choosing various possible forms of B_μ satisfying (23) and (25), we can obtain various sets of equations.

A simple form of B_μ which satisfies the conditions (23) and (25) is

$$B_\mu=av_\mu, \tag{26}$$

where a is a constant. By taking

$$a=\tfrac{1}{2}e^2\epsilon^{-1}-m, \tag{27}$$

where m is another constant, Eqs. (22) become

$$m\dot{v}_\mu=ev_\sigma f_\mu{}^\sigma. \tag{28}$$

Thus the effective field that acts on the electron is the $f_{\mu\sigma}$ field. By substituting for $f_\mu{}^\sigma$ from (12), we obtain as the equations of motion of an electron in an electromagnetic field

$$m\dot{v}_\mu-\tfrac{2}{3}e^2(2k+1)[(d\dot{v}_\mu/ds)+\dot{v}^2v_\mu]=ev_\sigma F_\mu{}^\sigma{}_{\text{ext}},\quad v^2\equiv v_\mu v^\mu=1. \tag{29}$$

Equations (29) give a simple generalization of the Lorentz-Dirac equations. They contain an arbitrary constant k. By choosing various values of k we obtain different sets of equations of motion, corresponding to the different radiation fields defined in the theory. We have to apply these equations to various familiar problems and see which value of k gives the most satisfactory equations.

We note that we have obtained the equations of motion (29) by taking a particularly simple form of B_μ. Other possible alternative, but more complicated, forms of B_μ have been investigated by the present author and will be discussed later.

8. Many Electrons

The above theory of a single electron in an electromagnetic field may be readily extended to apply to any number of electrons, interacting with each other and with an external electromagnetic field.

The equations of motion of the nth electron are, in an obvious notation,

$$m\dot{v}_{\mu n}=ev_{\sigma n}f_{\mu n}{}^{\sigma}, \tag{30}$$

where

$$f_n{}^{\mu\sigma}=F_{\text{ext}}{}^{\mu\sigma}+(2e/3)(2k+1)(v_n{}^{\sigma}d\dot{v}_n{}^{\mu}/ds_n-v_n{}^{\mu}d\dot{v}_n{}^{\sigma}/ds_n)+\sum_{m\neq n}(F^{\mu\sigma}{}_{m,\,\text{ret}}+G_m{}^{\mu\sigma}). \tag{31}$$

Hence, (30) becomes

$$m\dot{v}_{\mu n}-\tfrac{2}{3}e^2(2k+1)(d\dot{v}_{\mu n}/ds+\dot{v}_n{}^2v_{\mu n})=ev_{\sigma n}\{F_{\mu}{}^{\sigma}{}_{\text{ext}}+\sum_{m\neq n}(F^{\sigma}{}_{\mu m\,\text{ret}}+G_{\mu m}{}^{\sigma})\}. \tag{32}$$

9. Discussion

The above equations of motion show that the effective field on an electron is the $f_{\mu\sigma}$ field. We may therefore suppose that the field of a moving electron consists of two parts:

$$\tfrac{1}{2}(F_{\text{ret}}+F_{\text{adv}}), \tag{a}$$

$$\tfrac{1}{2}(F_{\text{ret}}-F_{\text{adv}})+G=(k+\tfrac{1}{2})(F_{\text{ret}}-F_{\text{adv}}), \tag{b}$$

and that as far as the force on the electron which gives rise to this field is concerned, only the part (b) gives any contribution. That is, the part (a) of the field emitted by an electron does not exert a force on *that* electron whereas both parts (a) and (b) influence *other* electrons. We note that the part (a) contains all the singular terms in the expression for the field, the part (b) remaining finite on the world-line of the corresponding electron.

The sign of the radiation reaction terms depends on that of $2k+1$. Consequently, by considering the various cases $2k+1>0$, $2k+1=0$, $2k+1<0$ separately, we would have different types of radiation fields. In the first case the radiation field is of the same type as in the Dirac case but with the numerical value altered by a factor $2k+1$, in the second case the usual radiation damping would be absent, and in the third case the radiation field is opposite to that in the first case.

In the next chapter we shall consider various applications of these different equations.

CHAPTER III. APPLICATIONS OF THE CLASSICAL EQUATIONS OF MOTION

(i) The Lorentz-Dirac Equations

10. Physical and Non-Physical Solutions

For the purpose of discussing the Lorentz-Dirac equations, it is convenient if we express them in the form

$$a\frac{d^2\mathbf{r}}{ds^2}-\frac{d^3\mathbf{r}}{ds^3}-\frac{d\mathbf{r}}{ds}\left[\left(\frac{d^2t}{ds^2}\right)^2-\left(\frac{d^2\mathbf{r}}{ds^2}\right)^2\right]=(3/2e)\left[\frac{dt}{ds}\mathbf{E}+\frac{d\mathbf{r}}{ds}\times\mathbf{H}\right],$$

$$t^2-d^2\mathbf{r}/ds^2=1, \tag{1}$$

where $\mathbf{r}$ is the position vector (x, y, z), $\mathbf{E}$ and $\mathbf{H}$ are the external electromagnetic field vectors, and $a=(3/2)(m/e^2)$. The equation involving d^3t/ds^3 is not independent of the others and is therefore ignored.

Unlike the usual Newtonian equations of motion, these involve third-order differential coefficients. Therefore, in order to determine the motion we must be given not only the position and velocity of the electron at one instant of time but also the acceleration. This is a departure from familiar Newtonian ideas, where it is only the initial position and velocity that are prescribed, and the acceleration is automatically determined by the force acting on the particle.

One can think of two ways of meeting this new situation. One is to regard the acceleration also as being entirely arbitrary, so that the solution which describes the motion of the electron will have more arbitrary constants of integration than in the Newtonian case. There is, however, no evidence to support an assumption that the actual motion possesses these arbitrary features when the effects of radiation damping are taken into account. The other alternative is to assume that all the mathematically possible solutions of the equations of motion need not correspond to motions that are observable in nature. We thus see that there are two types of solutions of the equations of motion, namely, the physical and the non-physical solutions. The criterion which distinguishes these two types of solution will be considered later.

11. Free Electron

The motion of an electron in the absence of an external field has been considered by Dirac,[15] who obtained the solution in a special frame of reference in which the motion is rectilinear. It is, however, desirable to know the solution in a general three-dimensional motion, because when one solves, by approximate methods, problems where the electron is subject to forces and where the motion is three-dimensional, then the motion may be representable to a first approximation by the solution corresponding to motion under no forces; and this approximate solution can subsequently be used to carry the solution to a higher approximation.

The equations of motion of the free electron are

$$a\dot{v}_\mu - \frac{d\dot{v}_\mu}{ds} - \dot{v}^2 v_\mu = 0, \tag{2}$$

which has a general solution of the form

$$v_\mu = A_\mu \exp(Ce^{as}) + B_\mu \exp(-Ce^{as}), \tag{3}$$

where A_μ, B_μ, and C are arbitrary except that they are subject to the conditions

$$A^2 = B^2 = 0, \quad (A, B) = \tfrac{1}{2}. \tag{4}$$

This solution will correspond to motion in a plane with the velocity of the electron rapidly increasing and ultimately tending to the velocity of light. Further, the electron is losing energy by radiation at a rapid rate. Such a motion of a free electron has not been observed in nature, and we therefore conclude that this motion is non-physical.

A particular solution is $v_\mu = C_\mu$, where $C_\mu C^\mu = 1$, thus giving a motion in which the electron is moving in a straight line with uniform velocity. There is no loss of energy by radiation. This motion is the one observed in nature and is thus the physical motion.

12. Energy Considerations

In the non-physical motion above, the electron increases its kinetic energy and at the same time loses energy rapidly by the emission of radiation. This is contrary to the usual energy changes. This discrepancy has been examined,[16] and it has been shown, by considering the equations of motion, that for a proper interpretation of energy changes the intrinsic energy of the electron should be regarded as being composed of three parts—kinetic, potential, and acceleration energies. The acceleration energy is $-\frac{2}{3}e^2\dot{v}_0$, which is negative when the acceleration is positive; and hence as the electron acquires this negative acceleration energy, it releases an equal amount of positive energy, which goes towards increasing the kinetic energy of the electron and also contributes to the energy lost by radiation.

13. An Electron in a Uniform Electric Field

(a) *Motion in a Straight Line*

For this simple case, the relativistic equations of motion are easily integrable. If $\mathcal{E}$ is the strength of the field and v is the velocity along the line of motion, the equations of motion give

$$a\dot{v} - \frac{d\dot{v}}{ds} + \frac{v\dot{v}^2}{1+v^2} = \frac{3\mathcal{E}}{2e}(1+v^2)^{\frac{1}{2}}, \tag{5}$$

which has the solution

$$v = \sinh\phi, \quad \dot{t} = \cosh\phi, \tag{6}$$

where

$$\phi = Ae^{as} + B + e\mathcal{E}s/m. \tag{7}$$

The motion is non-physical when A is non-zero. The physical solution, with A zero, is such that

$$d\dot{v}_\mu/ds + \dot{v}^2 v_\mu = 0. \tag{8}$$

That is, there is no force on the electron caused by the radiation reaction. The mathematical solution is exactly the same as when the effect of radiation damping is completely ignored. But the physical interpretation is different. Energy is being lost by radiation; the electron has, besides the usual kinetic and potential energies, also an acceleration energy. The sum of the kinetic and potential energies remains constant, while the energy lost by radiation comes from the acceleration energy.

[15] Dirac, reference 3, p. 156.

[16] C. J. Eliezer, Proc. Ind. Acad. Sci. **A21**, 31 (1945).

(b) *The Non-Relativistic Equations*

For the three-dimensional motion the relativistic equations cannot be solved exactly; we therefore consider first the solution of the non-relativistic equations of motion. These equations become

$$a d\mathbf{v}/dt - d^2\mathbf{v}/dt^2 = \tfrac{3}{2}\mathbf{E}/e. \tag{9}$$

The complete solution is

$$\mathbf{v} = \mathbf{A}e^{at} + \mathbf{B} + e\mathbf{E}t/m. \tag{10}$$

The physical solution occurs for **A** equal to zero. The electron then describes a parabola, the axis of which is parallel to the direction of the field.

(c) *The Relativistic Equations, Solved Approximately*

The equations of motion are

$$a\dot{v}_\mu - d\dot{v}_\mu/ds - \dot{v}^2 v_\mu = f_\mu, \quad v^2 = 1, \tag{11}$$

where

$$f_\mu = \tfrac{3}{2}(\mathcal{E}/e)(\mathbf{v}\cdot\mathbf{l},\, v_0\mathbf{l}), \tag{12}$$

and the three-dimensional vector, **l**, gives the direction cosines of the electric field vector, $\mathcal{E}$ is the strength of the electric field, and **V** is the three-dimensional spatial part of the velocity four-vector v_μ. To avoid the frequent occurrence of a in the solution, we change the independent variable from s to a new variable $\tau = as$. If we use dashes to denote differentiation with respect to τ, we obtain

$$v_\mu' - v_\mu'' - v'^2 v_\mu = \lambda(\mathbf{v}\cdot\mathbf{l},\, v_0\mathbf{l}), \tag{13}$$

where

$$\lambda = 3\mathcal{E}/2ea^2. \tag{14}$$

We solve these equations approximately by expressing the velocity as a series in ascending powers of λ, λ being considered small enough for the expansion to be valid. We therefore try the solution

$$v_\mu = u_\mu + \lambda u_\mu^{(1)} + \lambda^2 u_\mu^{(2)} + \cdots. \tag{15}$$

By substituting in (13) and equating the terms involving the same powers of λ, we obtain differential equations to determine successively u_μ, $u_\mu^{(1)}$, $u_\mu^{(2)}\cdots$. Without giving details, we merely quote the result that the physical solution is of the form

$$\begin{aligned} v_\mu = A_\mu &+ \lambda(\mathbf{A}\cdot\mathbf{l},\, A_0\mathbf{l})as + \lambda^2[-\{A_0^2 - (\mathbf{A}\cdot\mathbf{l})^2\}A_\mu as \\ &+ \{A_0,\, (\mathbf{A}\cdot\mathbf{l})\mathbf{l}\}(as + \tfrac{1}{2}a^2s^2)] \\ &+ \lambda^3(\mathbf{A}\cdot\mathbf{l},\, A_0\mathbf{l})[-2\{A_0^2 - (\mathbf{A}\cdot\mathbf{l})^2\} \\ &\times(as + \tfrac{1}{2}a^2s^2) + as + \tfrac{1}{2}a^2s^2 + \tfrac{1}{6}a^3s^3] + \cdots, \end{aligned} \tag{16}$$

where $A_\mu^2 = 1$. The part of the solution that is independent of λ corresponds to uniform motion of a free electron. Correct to the first approximation, the motion is in a parabola whose axis is parallel to the direction of the field. In the higher approximations the effect of radiation damping becomes operative, and the path deviates from the parabola, but is still confined to a plane. The direction of motion tends more and more to be parallel to the direction of the field.

14. An Electron in a Uniform Magnetic Field

(a) *The Non-Relativistic Equations*

The relativistic equations of motion cannot be solved exactly, and we therefore consider first the non-relativistic case. This will be sufficient for the purpose of knowing the general nature of the physical and non-physical solutions.

The non-relativistic equations

$$m\frac{d\mathbf{v}}{dt} - \frac{2}{3}e^2\frac{d^2\mathbf{v}}{dt^2} = e(\mathbf{v}\times\mathbf{H}) \tag{17}$$

have the solution[17]

$$\left.\begin{aligned} x &= e^{\alpha\tau}(A\cos\beta\tau + B\sin\beta\tau) \\ &\qquad + e^{\alpha_1\tau}(A_1\cos\beta\tau + B_1\sin\beta\tau), \\ y &= e^{\alpha\tau}(C\cos\beta\tau + D\sin\beta\tau) \\ &\qquad + e^{\alpha_1\tau}(C_1\cos\beta\tau + D_1\sin\beta\tau), \\ z &= Ee^{\tau} + F\tau, \end{aligned}\right\} \tag{18}$$

where $\tau = at$, and where all the letters on the right-hand side, except τ, denote constants, some of which are arbitrary. Further $\alpha < 0$, $\alpha_1 > 0$, and $\exp(\alpha\tau)\to 0$, $\exp(\alpha_1\tau)\to\infty$ at $\tau\to\infty$.

When E and F vanish, the motion is confined to a plane. When the terms in $\exp(\alpha_1\tau)$ are non-zero, the motion is non-physical; the electron then spirals outwards with steadily increasing

[17] C. J. Eliezer, Proc. Camb. Phil. Soc. 42, 40 (1946).

velocity and ultimately goes off to infinity. In the physical motion the electron describes an equiangular spiral, with steadily diminishing velocity, ultimately coming to rest at the pole of the spiral.

The non-planar physical motion is such that on the motion in the equiangular spiral considered above, there is superimposed a uniform velocity normal to the plane, that is, parallel to the direction of the magnetic field. The resulting path of the electron is in the shape of a corkscrew which has its axis parallel to the field. After some time the motion tails off to one of uniform velocity along the axis of the corkscrew.

(*b*) *The Relativistic Equations, Solved Approximately*

The equations of motion are of the form

$$a\dot{v}_\mu - d\dot{v}_\mu/ds - \dot{v}^2 v_\mu = f_\mu, \qquad (19)$$

where $f_\mu = (3/2)e^{-1}(0, \mathbf{V}\times\mathbf{H})$. The physical solution is obtained as a series in ascending powers of λ, where $\lambda = 3\mathcal{H}/(2ea^2)$, and is of the form

$$\begin{aligned} v_\mu = A_\mu &+ \lambda(0, \mathbf{A}\times\mathbf{1})as + \lambda^2[-(\mathbf{A}\times\mathbf{1})^2 A_\mu as \\ &+ \{0, (\mathbf{A}\times\mathbf{1})\times\mathbf{1}\}(as+\tfrac{1}{2}a^2s^2)] \\ &- \lambda^3(0, \mathbf{A}\times\mathbf{1})\{(\mathbf{A}\times\mathbf{1})^2(2as+a^2s^2) \\ &+ as + \tfrac{1}{2}a^2s^2 + \tfrac{1}{6}a^3s^3\} + \cdots, \end{aligned} \qquad (20)$$

where A_μ is an arbitrary constant vector such that $A_\mu A^\mu = 1$.

To see if this motion differs in any appreciable way from that in the non-relativistic case, we consider the magnitude, V, of the velocity of the electron, which is seen to be

$$V = (v_0^2-1)^{\frac{1}{2}} = (A_0^2-1)^{\frac{1}{2}}\{1 - \lambda^2(\mathbf{A}\times\mathbf{1})^2 a^2 s^2 / \times (A_0^2-1) + 0(\lambda^4)\}. \qquad (21)$$

Hence the velocity begins to decrease as s increases, and after a time, V will be small enough for the non-relativistic solution to be valid.

15. An Electron Disturbed by a Pulse

Dirac[18] has considered the behavior of an electron which is initially at rest and which is disturbed for a moment by a pulse of electromagnetic radiation passing over it. The equation of motion is then of the form

$$adv/dt - d^2v/dt^2 = k\delta(t). \qquad (22)$$

The solution which corresponds to the electron being at rest at $t = -\infty$, and which satisfies the boundary conditions at $t=0$, is

$$\begin{aligned} v &= (k/a + C)e^{at}, \quad t<0; \\ v &= k/a + Ce^{at}, \quad t>0. \end{aligned} \qquad (23)$$

The constant, C, is still arbitrary. A solution which is such that the electron is at rest till it comes directly under the influence of the pulse is obtained by taking $C = -k/a$. Then

$$v=0, \quad t<0; \quad v=(k/a)(1-e^{at}), \quad t>0, \qquad (24)$$

But in this solution the motion after t is zero appears to be non-physical. We can however have a motion, which after t is zero is physical, by choosing $C=0$. Then

$$v=(k/a)e^{at}, \quad t<0; \quad v=k/a, \quad t>0. \qquad (25)$$

This is the solution given by Dirac, and corresponds to a motion in which the electron is gradually building up an acceleration till it meets the pulse. The difference between this motion and a uniform motion is too small to be observable; and hence (25) may be regarded as a physical solution. But this solution introduces certain contradictions with some elementary ideas of causality. As Dirac says, "the electron seems to know about the pulse before it arrives and to get up an acceleration (as the equations of motion will allow it to do) just sufficient to balance the effect of the pulse when it does arrive." Dirac tried to surmount the difficulty of accepting this situation by interpreting the result in a more natural way by supposing that the electron behaved as though it had a finite size. "There is then no need for the pulse to reach the center of the electron before it starts to accelerate. It starts to accelerate and radiate as soon as the pulse meets its outside." A close examination of this explanation shows that it implies "that it is possible for a signal to be transmitted faster than light through the interior of the electron. The finite size of the electron now reappears in a new sense, the interior of the electron being a region of failure,

[18] Dirac, reference 3, p. 198.

not of the field equations of electromagnetic theory, but of some of the elementary properties of space-time."

In this way one is led to many departures from familiar ideas of Newtonian mechanics.

16. The Principle of Field-Balance[19]

In Section (10), and in the subsequent applications of the equations of motion, we have seen that the equations of motion have physical and non-physical solutions. A prominent defect in the above theory of the electron is that there has been no adequate criterion to discriminate between the physical and the non-physical solutions. In each application that we have considered, we had to guess which of the mathematically possible solutions appeared to be physically permissible.

On examination of the solutions of all the problems we have considered above, we note that one point of difference between the physical and the non-physical solutions is that in the physical motion, after a sufficient lapse of time, a state of equilibrium is reached between the emitted and the absorbed radiation fields. The radiation field consists of two parts: (i) $\frac{2}{3}e^2 d\dot{v}_\mu/ds$, which corresponds to a reversible form of emission or absorption of radiation, and (ii), $\frac{2}{3}e^2\dot{v}^2 v_\mu$, which corresponds to an irreversible emission of radiation, $\dot{v}^2$ being always negative as $\dot{v}_\mu$ is a space-like vector. For the physical solutions above, the changes in these two fields tend to cancel each other out, in process of time. That is,

$$d\dot{v}_\mu/ds+\dot{v}^2 v_\mu \rightarrow 0 \quad \text{as} \quad s\rightarrow\infty. \tag{26}$$

We may hope that this principle is of general validity. It implies that the physical solution should satisfy not only the conservation laws, and therefore the equations of motion which are derived from them, but also a further principle, namely, this principle of field-balance. This principle is similar to that of detailed balancing widely used in thermodynamics and in theories of radiation. But whereas in these theories the balancing is a statistical effect, where a large number of particles are concerned, in our case the principle is assumed to apply to a single electron.

[19] C. J. Eliezer and A. W. Mailvaganam, Proc. Camb. Phil. Soc. 41, 184 (1945).

17. An Integral for Central Fields

When the field on the electron is directed towards a fixed center of force, and the strength of the field depends only on the radial distance, then the equations of motion are of the form

$$a\frac{d^2\mathbf{r}}{ds^2}-\frac{d^3\mathbf{r}}{ds^3}-\frac{d\mathbf{r}}{ds}\left[\left(\frac{d^2t}{ds^2}\right)^2-\left(\frac{d^2\mathbf{r}}{ds^2}\right)^2\right] = -\tfrac{3}{2}(er)^{-1}f(r)\mathbf{r}, \tag{27}$$

where $\dot{t}^2=1+\dot{\mathbf{r}}^2$. By taking scalar product of this equation with $\mathbf{r}\times\dot{\mathbf{r}}$, we obtain

$$a\left[\mathbf{r},\frac{d\mathbf{r}}{ds},\frac{d^2\mathbf{r}}{ds^2}\right]-\left[\mathbf{r},\frac{d\mathbf{r}}{ds},\frac{d^3\mathbf{r}}{ds^3}\right]=0,$$

where the square bracket indicates a triple scalar product. This equation is directly integrable, thus giving

$$\left[\mathbf{r},\frac{d\mathbf{r}}{ds},\frac{d^2\mathbf{r}}{ds^2}\right]=Ce^{as}, \tag{28}$$

where C is an arbitrary constant. If the initial conditions are such that C is zero, that is, if initially the position, velocity, and acceleration vectors are in a plane, then $[\mathbf{r}, d\mathbf{r}/ds, d^2\mathbf{r}/ds^2]=0$ always, and the subsequent motion is also confined to this plane. If, however, the initial position, velocity, and acceleration vector are not co-planar, then the fact that the right-hand side of (28) increases as s increases indicates that the motion is likely to deviate more and more from the motion in a plane.

18. The Hydrogen Atom

With the classical picture of the hydrogen atom, that is, an electron revolving in an orbit round a fixed proton, if one takes into account the effect of radiation damping, then one would expect that, owing to loss of energy by radiation, the electron would approach the nucleus closer and closer and ultimately fall into it. The Lorentz-Dirac equations have been applied to this problem,[20] and evidence has been obtained that there may be no solutions of these equations which correspond to the electron spiralling

[20] C. J. Eliezer, Proc. Camb. Phil. Soc. 39, 173 (1943).

round and finally falling into the nucleus. A rigorous proof of this result is possible for the case when the electron is moving in a straight line, directly towards the proton. The equation that describes the motion is

$$a\ddot{x}\frac{d^3x}{ds^3}+\frac{\dot{x}\ddot{x}^2}{1+\dot{x}^2}=-\frac{3}{2}\frac{(1+\dot{x}^2)^{\frac{1}{2}}}{x^2}, \qquad (29)$$

where $\dot{x}$ denotes the velocity. The behavior of the solution of this differential equation has been examined in detail in the paper referred to, and it has been shown that with whatever initial velocity the electron may be projected towards the proton, the electron would be brought to rest before it could reach the proton. The electron then turns back and moves away from the proton with velocity and acceleration which keep on increasing. Ultimately, the electron escapes to infinity.

The types of possible two- and three-dimensional motions of an electron in the field of a fixed proton have also been considered. While a rigorous treatment of all the mathematically possible solutions of the equations of motion appears to be very complicated, the probable solutions have been examined, and the weight of the evidence that has been gathered points to the result that there is no solution with the electron falling into the nucleus.

One may think that in the above work we are not justified in taking the proton as fixed,even by considering its mass to be infinite, because the proton is subjected to a force which becomes infinitely large as the electron approaches the proton, and hence the proton may tend to have a motion which cannot be neglected. This likelihood has been examined in the paper referred to, and it has been shown to be not possible, by considering the straight line motion of two electrons of unlike charges, in which case it is also found that the two particles do not come into collision.

19. Like Charges

Having found that two unlike charges do not collide with each other, it is of interest to see what happens when the two charges are of a like nature. We shall consider the motion of a particle of charge e in the presence of a fixed particle, also of charge e. The charge e may be positive or negative. The equations of straight line motion reduce to

$$a\ddot{x}\frac{d^3x}{ds^3}+\frac{\dot{x}\ddot{x}^2}{1+\dot{x}^2}=\frac{3}{2}\frac{(1+\dot{x}^2)^{\frac{1}{2}}}{x^2}, \qquad (30)$$

which can be shown to have an approximate solution,[21] for small x, of the form

$$\dot{x}=-(3\log x^{-1})^{\frac{1}{2}}. \qquad (31)$$

Thus the moving particle approaches the fixed particle with increasing velocity and ultimately falls into it, with velocity that tends to that of light.

In the two-dimensional motion there is also a collision possible. The equations of motion are

$$a\frac{d^2\mathbf{r}}{ds^2}-\frac{d^3\mathbf{r}}{ds^3}-\frac{d\mathbf{r}}{ds}\left[\left(\frac{d^2t}{ds^2}\right)^2-\left(\frac{d^2\mathbf{r}}{ds^2}\right)^2\right]=\tfrac{3}{2}\dot{t}r^{-3}\mathbf{r}, \qquad (32)$$

where $\dot{t}^2=1+\dot{\mathbf{r}}^2$. We can show that a possible solution is one in which the particle spirals inwards into the nucleus with velocity which for small r is approximately of the value $(3\log r^{-1})^{\frac{1}{2}}$.

Hence we see that when the two charges are alike, a collision can take place, whereas when the two charges are unlike a collision cannot take place. In both cases, the result is the opposite of what we should expect from elementary physical considerations.

20. Discussion

Several interesting points arise out of the results of the last two sections.

(a) As a result of certain experiments on the scattering of electrons, as for example those done by G. P. Thomson, it has been suggested that, when an electron is close to a nucleus, the force on the electron is given by the Coulomb inverse square law. It appears that in these experiments the electron behaves as though at points near the nucleus, the nucleus exerts a repulsive force on the electron, instead of the usual attractive force. It is now seen that this behavior of the electron may be interpreted as being caused not by any alterations in the field of the nucleus but by the effect of radiation damping on the motion

[21] C. J. Eliezer, Ph.D. thesis, submitted at Cambridge (1945).

of the electron. We saw above that when an electron is projected towards a nucleus, whatever the initial conditions may be, as the electron approaches the nucleus it starts acquiring an acceleration which is away from the nucleus. This acceleration is what has been interpreted as being caused by a repulsive force that is exerted by the nucleus, but it is now seen that this may be interpreted as being caused by radiation damping.

(b) Attempts to study the structure of an atomic nucleus, on the basis of the quantum theory, have led to the belief that the electron is not a nuclear constituent. If an electron is packed into any region as small as that which the nucleus is believed to be, then Dirac's relativistic quantum-mechanical equations of an electron require that the electron should escape from this region by jumping into a state of negative energy. There is a strong similarity between this result obtained on the basis of the quantum theory and the classical result we obtained above. It appears that in the classical theory too the electron cannot be a nuclear constituent, because an electron cannot be contained within a small region in in which the potential energy varies rather steeply, since the electron then escapes, as we saw earlier. According to the classical solution above, the total energy of the electron, which consists of kinetic energy, potential energy, and acceleration energy, becomes eventually negative.

(c) In the problem of the hydrogen atom it appears that the radiation field which is emitted by the electron has a strong repulsive effect which exceeds the Coulomb attraction of the nucleus and so prevents a collision. In the final motion the electron escapes to infinity with increasing velocity and acceleration. The radiation field has thus the effect of accelerating the motion instead of damping it. Furthermore, the presence of the external force has the effect of increasing the velocity and acceleration in the opposite direction! One would be inclined to consider that such a motion, which is so strikingly different from known ones, should be looked upon as a non-physical motion, which arises only because of certain non-physical initial conditions. One may assert that in the above motion the acceleration is opposite to the force, a non-physical characteristic which is responsible for the non-physical nature of the solution. But in the case of the electron that is moving in the field of a proton this circumstance cannot be avoided, because even when initially the electron is projected towards the proton with an acceleration which is in the same direction as the force, then it is found from the equations of motion that the acceleration eventually changes sign and becomes opposite to the force. It may be that such a reversal in the direction of the acceleration, and the subsequent escape of the electron to infinity, occurs only after the electron has approached very close to the nucleus, and that such electrons are not observed but only those which have not been too near the nucleus.

It appears that we have two possible ways of viewing these results. One is to consider that the above motion, which is the only one allowed by the equations of motion, is the appropriate physical motion. The other is to conclude that the Lorentz-Dirac equations of motion are not exact, and do not give an adequate description of the radiation forces in regions within atomic distances from particles.

21. An Electron in the Field of a Thin Infinite Charged Plate[22]

As a final application of the Lorentz-Dirac equations, we consider an electron moving in a straight line along the x-axis in the field of a thin infinite-charged plate in the yz-plane. The velocity, v, is then given by equations of the form

$$\left.\begin{aligned} a\dot{v}-(d^2v/ds^2)+v\dot{v}^2/(1+v^2)&=\alpha(1+v^2)^{\frac{1}{2}}, \quad x<0\\ a\dot{v}-(d^2v/ds^2)+v\dot{v}^2/(1+v^2)&\\ &=-\alpha(1+v^2)^{\frac{1}{2}}, \quad x>0\end{aligned}\right\}\quad(33)$$

where we shall take α to be positive. We solve these equations by transforming to ϕ where $v=\sinh\phi$. The equations then become

$$a\dot{\phi}-(d^2\phi/ds^2)=\pm\alpha, \qquad (34)$$

and the solution is of the form

$$v=\sinh(Ae^{as}+B\pm\alpha s/a). \qquad (35)$$

Suppose that initially the electron is moving towards the origin, 0, from the left, and reaches

[22] C. J. Eliezer, Bull. Calcutta Math. Soc. **37**, 125 (1945).

0 when $s=0$ with velocity v_0. Till $s=0$ the velocity will be of the form

$$v=\sinh(A_0e^{\alpha s}+B_0+\alpha s/a). \qquad (36)$$

As s increases from $s=0$, the electron will begin to move into the region $x>0$, and thereafter the velocity will have the form

$$v=\sinh(A_1e^{\alpha s}+B_1-\alpha s/a), \quad s>0, \qquad (37)$$

as long as $x>0$. The velocity and acceleration should be continuous at $x=0$, and, therefore, $A_0+B_0=A_1+B_1$, $\alpha A_0+\alpha=aA_1-\alpha$. If A_1 is non-zero, the motion after $s=0$ is non-physical. Hence following the method of Section (13), we take $A_1=0$, thus obtaining a solution which corresponds to a physical motion after $s=0$. Then we have

$$v=\sinh(\sinh^{-1}v_0-\alpha s) \qquad (38)$$

and, therefore

$$x=\alpha^{-1}\{\cosh(\sinh^{-1}v_0)-\cosh(\sinh^{-1}v_0-\alpha s)\}. \qquad (39)$$

Hence as s increases from $s=0$, v gradually decreases and vanishes and then becomes negative. That is, the electron moves towards the right with decreasing velocity, comes to rest, and commences to move backwards towards 0 with gradually increasing speed. It reaches 0 again at $s=2\alpha^{-1}\sinh^{-1}v_0=s_1$, say, and then goes over into the region of space $x<0$. The motion in this region will be given by a solution of the form

$$v=\sinh(A_2e^{\alpha s}+B_2+\alpha s/a), \quad s>s_1. \qquad (40)$$

where A_2 is non-zero, and will, therefore, be a non-physical motion.

22. Conclusion

Much of the evidence that has accumulated from the work of the last few sections suggests that the signs of certain terms in the Lorentz-Dirac equations do not fit in appropriately. This same point was arrived at by Dirac when he first derived these equations and applied them to consider the motion of a free electron. The solution he obtained for this problem is of the form

$$\dot{x}=\sinh(e^{\alpha s}+b), \quad \dot{t}=\cosh(e^{\alpha s}+b), \qquad (41)$$

where the notation is obvious. Dirac's comments on this solution are relevant to our discussion and are quoted below. "One would be inclined to say that there is a mistake in sign in our equations and that we ought to have $e^{-\alpha s}$ instead of $e^{\alpha s}$ in (41). With this alteration we should have a theory in which, if an electron is disturbed in any way and then left alone, it would rapidly settle down into a state of constant velocity, with emission of radiation while it is settling down. This would be a reasonable behavior for an electron according to our present-day physical ideas. However, it is not possible to tamper with the signs in our theory in any relativistic way to obtain this result, without getting equations of motion which would make the electron in the hydrogen atom spiral outwards, instead of spiralling inwards and ultimately falling into the nucleus, as it should in the classical theory. We are, therefore, forced to keep the signs in (41) as they are and to see what interpretation we can give to the equations as they stand."

It is now seen from our results outlined in Section (18) that the part of the above statement of Dirac that concerns the hydrogen atom needs revision. The fact that there is now every indication that the Lorentz-Dirac equations of motion do not allow the electron in the hydrogen atom to spiral inwards and fall into the nucleus gives added reason to suppose that a change in sign is necessary.

While the Lorentz-Dirac equations can be given up as inadequate only after they have been taken over into the quantum theory and then found to fail, nevertheless, it appears desirable to investigate possible modifications within the frameworks of the classical theory.

(ii) Alternative Equations I

23. Introduction of Higher Differential Coefficients

When we look out for modifications of the Lorentz-Dirac equations there are two possible methods of approach. One is to keep to the assumption that the field of a moving electron is given by its retarded field and to obtain equations of motion different from the Lorentz-Dirac equations by introducing higher differential coefficients of velocity than the second. The other method is to abandon the assumption that the field of an electron is its retarded field, and to consider a combination of retarded and advanced fields. In this section we shall follow the first method of approach.

The equations of motion obtained from the conservation principles are of the form

$$\dot{A}_\mu - \tfrac{2}{3}e^2(d\dot{v}_\mu/ds + \dot{v}^2 v_\mu) = ev_\sigma F_{\mu\ \mathrm{ext}}{}^\sigma, \quad (42)$$

where A_μ is not entirely arbitrary, but is restricted by the relation $(v, \dot{A}) = 0$ and the condition that $v_\lambda A_\mu - v_\mu A_\lambda$ should be a perfect differential. The choice $A_\mu = 0$ corresponds to particles of zero rest mass which are not of interest in the classical theory, and the choice $A_\mu = mv_\mu$ leads to the Lorentz-Dirac equations. I have examined[23] various forms of A_μ, and shown that A_μ cannot have the form

$$A_\mu = Pv_\mu + Q\dot{v}_\mu, \quad (43)$$

but can be of the form

$$A_\mu = Pv_\mu + Q\dot{v}_\mu + Rd\dot{v}_\mu/ds, \quad (44)$$

where, from the conditions to be satisfied by A_μ, we obtain

$$P = \text{constant} - \tfrac{3}{2}R\dot{v}^2, \quad Q = 0, \quad R = \text{constant}. \quad (45)$$

Thus we take

$$A_\mu = mv_\mu + k(\tfrac{2}{3}d\dot{v}_\mu/ds + \dot{v}^2 v_\mu), \quad (46)$$

where m and k are constants.

Hence we obtain as a possible set of equations of motion

$$m\dot{v}_\mu - \frac{2}{3}e^2\left(\frac{d\dot{v}_\mu}{ds} + \dot{v}^2 v_\mu\right) + k\frac{d}{ds}\left(\frac{2}{3}\frac{d\dot{v}_\mu}{ds} + \dot{v}^2 v_\mu\right) = ev_\nu F_\mu{}^\nu. \quad (47)$$

The first question that arises concerns the physical interpretation of the two constants in the equations, namely, m and k. Comparison with the Lorentz equations shows that m is the rest mass of the electron. In the classical theory, a point electron has associated with it two constants, the rest mass and the charge, and no other. The extra constant, k, has either to be interpreted as describing an additional physical property of the electron, or has to be regarded as expressible in terms of the constants e and m.

[23] C. J. Eliezer, Proc. Camb. Phil. Soc. 42, 278 (1946).

The author's hope, when developing these ideas, that the first alternative may be possible, thus giving scope for the introduction of Planck's constant, h, into the framework of the classical theory, has not proved to be promising. It is the second alternative, that is, that k can be expressed in terms of e and m, that has to be followed. It is easily seen that k has the dimensions of (mass)×(length)^2, or in terms of e and m has the dimensions of e^4/m. Hence we can take

$$k = \sigma e^4 m^{-1}, \quad (48)$$

where σ is a dimensionless constant whose value can be determined by comparison with experimental results.

24. Applications

(a) A Free Electron

For the purpose of determining in a general way the nature of the solutions, it is sufficient to consider the non-relativistic equations. Then the equations giving the three Cartesian components of velocity, **V**, are

$$m\frac{d\mathbf{V}}{dt} - \frac{2}{3}C^2\frac{d^2\mathbf{V}}{dt^2} + \frac{2}{3}k\frac{d^3\mathbf{V}}{dt^3} = 0, \quad (49)$$

where the (x, y, z, t) notation is now being used. These equations may be written as

$$\frac{d^3\mathbf{V}}{dt^3} - 2\alpha\frac{d^2\mathbf{V}}{dt^2} + \beta\frac{d\mathbf{V}}{dt} = 0, \quad (50)$$

where

$$\alpha = \tfrac{1}{2}e^2k^{-1}, \quad \beta = \tfrac{3}{2}mk^{-1}.$$

The nature of the solution depends on the value of σ in Eq. (48). If $\sigma < 0$, then

$$\alpha < 0 \qquad \beta < 0$$
$$\alpha + (\alpha^2 - \beta)^{\frac{1}{2}} > 0, \quad \alpha - (\alpha^2 - \beta)^{\frac{1}{2}} < 0,$$

and the solution is

$$\mathbf{V} = \mathbf{A} + e^{\alpha t}\{\mathbf{B}\exp[(\alpha^2 - \beta)^{\frac{1}{2}}t] + \mathbf{C}\exp[-(\alpha^2 - \beta)^{\frac{1}{2}}t]\}. \quad (51)$$

When **B** is non-zero the motion is self-accelerating, and when **B** is zero and **C** non-zero, the motion is self-retarding. This latter motion is such that any initial acceleration the electron may possess gradually diminishes and dies out, with the velocity tending to a finite constant value. Such a motion would be in keeping with our present-day physical ideas. When **B** and **C** are both zero, the motion is one of uniform velocity.

If $0<\sigma<\frac{1}{6}$, then the solution is of the same form as with

$$\alpha+(\alpha^2-\beta)^{\frac{1}{2}}>0, \quad \alpha-(\alpha^2-\beta)^{\frac{1}{2}}>0.$$

The physical solution should have both **B** and **C** zero, all other motions being self-accelerating.

If $\sigma=\frac{1}{6}$, then

$$\mathbf{V}=\mathbf{A}+e^{\alpha t}(\mathbf{B}t+\mathbf{C}), \tag{52}$$

and the physical solution should have both **B** and **C** zero.

If $\sigma>\frac{1}{6}$, then

$$\mathbf{V}=\mathbf{A}+e^{\alpha t}\{\mathbf{B}\cos[(\beta-\alpha^2)^{\frac{1}{2}}t] + \mathbf{C}\sin[(\beta-\alpha^2)^{\frac{1}{2}}t]\}, \tag{53}$$

giving for non-zero **B** or **C**, an oscillatory motion, the amplitude of which increases with time. Such a motion is quite foreign to the classical theory and also cannot be compared to the wave packets of the quantum theory.

Thus we see that when σ is negative there exists, apart from the motion of uniform velocity, another physically understandable motion in which the velocity rapidly becomes uniform. For other values of σ, the physical motion is only of one kind, namely, the uniform motion.

(*b*) *An Electron Disturbed by a Pulse*

We consider the problem which was examined in Section (15) on the basis of the Lorentz-Dirac equations, using now this new set of equations. The equation of motion is then of the form

$$\frac{d^3v}{dt^3}-2\alpha\frac{d^2v}{dt^2}+\beta\frac{dv}{dt}=\gamma\delta(t), \tag{54}$$

where α and β are as before, and $\gamma=\frac{3}{2}e\epsilon k^{-1}$, where ϵ is such that $\epsilon\delta(t-y)$ is the x component of the electric field of the pulse.

The nature of the solution of the Eq. (54), like that of the Eq. (50), depends on the value of σ. When σ is negative the solution that remains finite always is

$$\left.\begin{aligned} v&=\frac{1}{2}\frac{\gamma}{\beta}\{1-\alpha(\alpha^2-\beta)^{-\frac{1}{2}}\}\exp\{\alpha+(\alpha^2-\beta)^{\frac{1}{2}}\}t, \quad t<0\\ &=\frac{\gamma}{\beta}[1-\tfrac{1}{2}\{1+\alpha(\alpha^2-\beta)^{-\frac{1}{2}}\}\}\\ &\qquad\times\exp\{\alpha-(\alpha^2-\beta)^{\frac{1}{2}}\}t, \quad t>0.\end{aligned}\right\} \tag{55}$$

This solution corresponds to a motion in which for large negative values of t the electron is at rest, to a high approximation, but as t approaches zero the electron gradually acquires a velocity, when t is zero the electron is acted upon by a pulse, and thereafter the velocity gradually increases, and for large positive values of t the electron has uniform motion with the velocity γ/β.

We note that this solution is more symmetrical than the corresponding solution given by the Lorentz-Dirac equations, in which case the electron is supposed to accelerate until t is zero and then to move with uniform velocity.

It is of interest to obtain an expression for the energy emitted by the electron per unit frequency range and compare with the result given by the Lorentz-Dirac equations. The rate of emission of energy is $\frac{2}{3}e^2(dv/dt)^2$ and its spectral distribution is obtained from the Fourier resolution of dv/dt. We readily obtain that the energy emitted per unit frequency range is

$$E_\nu=6e^4\epsilon^2/\{9(m-4\pi^2k\nu^2)^2+16\pi^2e^4\nu^2\}. \tag{56}$$

For small frequencies ν, E_ν is approximately $\frac{2}{3}e^4\epsilon^2/m^2$, which is in agreement with the classical formula of Thomson. For large values of ν

$$E_\nu\sim\tfrac{2}{3}e^4\epsilon^2k^{-2}(2\pi\nu)^{-4}. \tag{57}$$

Thus it is seen that the cross section for scattering of light is in agreement with Thomson's formula when the frequency is small, and decreases as ν^{-4} for large ν.

With the Lorentz-Dirac equations of motion the expression for E_ν is[24]

$$E_\nu=6e^4\epsilon^2/\{9m^2+16\pi^2e^4\nu^2\}, \tag{58}$$

which agrees with (56) for small values of ν, but

[24] Dirac, reference 3, p. 160.

differs from it for large frequencies. The value of (58) for large ν is approximately $\frac{3}{2}\epsilon^2(2\pi\nu)^{-2}$.

We thus obtain a point of difference between the equations of motion (47) and the Lorentz-Dirac equations. Cross sections which vary as ν^{-4} for large ν, though not so common as those which vary as ν^{-2} do, however, occur in prevailing classical theories for certain scattering processes, as, for example, in the scattering of neutral, longitudinally polarized meson waves by heavy particles.[25]

The above solution was obtained for the case when σ is negative. For other values of σ, the method of solution proceeds in the same way. The solution obtained, however, is not symmetrical, but is such that after the action of the pulse the electron moves with uniform velocity. In these cases also, E_ν is found to be given by the same expression (56).

(c) *The Hydrogen Atom*

It appeared in Section (15) that the motion of the electron in the hydrogen atom, as given by the Lorentz-Dirac equations, had the characteristics of a non-physical motion. It is, therefore, very desirable to examine the motion as given by Eqs. (47).

For the sake of simplicity we shall consider the straight line motion of an electron towards a fixed proton. The equations to be solved are

$$m\dot{v}-\frac{2}{3}e^2\frac{d^2v}{dS^2}-\frac{2}{3}e^2(\dot{v}_0{}^2-\dot{v}^2)v+k\frac{d}{dS}\left\{\frac{2}{3}\frac{d^2v}{dS^2}+(\dot{v}_0{}^2-\dot{v}^2)v\right\}=e^2(1+v^2)^{\frac{1}{2}}x^{-2}, \quad (57)$$

$$v_0{}^2-v^2=1.$$

We transform the dependent variable to y, where $y=(1+\dot{x}^2)^{\frac{1}{2}}$, and the independent variable to x. If dashes now denote differentiation with respect to x, the equations become

$$(y^2-1)y'''+yy'y''-\tfrac{1}{2}y'^3+2\alpha(y^2-1)^{\frac{1}{2}}y''+\beta y'+3\alpha x^{-2}=0, \quad (58)$$

where α and β are as before.

The author has examined the solution of these equations in the paper referred to above, and shown that for small values of x the following approximate solutions are possible:

$$\text{If } \sigma<0, \quad y=1+2(-\alpha x)^{\frac{1}{2}}, \quad (59)$$

$$\text{if } \sigma>0, \quad y=C+3(C^2-1)^{-1}\alpha x\log x. \quad (60)$$

In both cases the electron falls into the proton. The nature of the motion is in conformity with our present day physical ideas. It appears then that in some respects these equations of motion, though mathematically much more complicated than the Lorentz-Dirac equations, are preferable from the physical point of view.

(iii) Alternative Equations II

25. Discussion of the Equations

In Chapter II it was shown that a simple generalization of the Lorentz-Dirac equations could be obtained by taking the field of an electron to be a combination of retarded and advanced fields. The equations of motion corresponding to the assumption

$$F_{\text{el}}=F_{\text{ret}}+k(F_{\text{ret}}-A_{\text{adv}})$$

were shown to be

$$m\dot{v}_\mu-\tfrac{2}{3}e^2(2k+1)[(d\dot{v}_\mu/ds)+\dot{v}^2v_\mu]=ev_\sigma F_{\mu,\,\text{ext}}{}^\sigma. \quad (61)$$

If $2k+1$ is negative, then the value of the radiation reaction would be opposite in sign to that as given by Lorentz-Dirac equations. Consequently we may expect that the behavior of the solutions of these equations would differ substantially from the corresponding solutions of the Lorentz-Dirac equations. We shall therefore consider a few applications of these equations with $2k+1$ negative. What corresponds to self-acceleration in the Lorentz-Dirac case would now correspond to self-retardation, and what corresponds to emission of radiation would now be absorption of radiation.

26. Applications

(a) *A Free Electron*

If we choose a Lorentz frame of reference in which the initial velocity and acceleration are in the same direction, that of the x axis say, then the subsequent motion will be always confined

[25] H. J. Bhabha, Proc. Roy. Soc. **A178**, 314 (1941).

to this axis, and the equations of motion will give

$$d\dot{v}/ds+(\dot{v}_0^2-\dot{v}^2)v+\alpha\dot{v}=0, \quad v_0^2=1+v^2, \quad (62)$$

where $\alpha=-3m/2e^2(2k+1)>0$. Eliminating v_0 we obtain

$$d\dot{v}/ds+\alpha\dot{v}-v\dot{v}^2/(1+v^2)=0,$$

the solution of which is easily seen to be

$$v=\sinh(Ae^{-\alpha s}+B). \quad (63)$$

This solution is such that with whatever initial conditions the electron may be projected it rapidly settles down to a motion of uniform velocity. Such a motion is in conformity with familiar notions in mechanics, in contrast to the self-accelerating motions of the Dirac theory. Further, in the Dirac theory, the data that are necessary to solve the equations of motion and the appropriate physical solution are the initial position, velocity, and the final acceleration. With the Alternative Eqs. II, however, the initial position, velocity, and acceleration suffice.

(*b*) *An Electron Disturbed by a Pulse*

A problem for which the Lorentz-Dirac equations do not give an altogether satisfactory solution is that of an electron disturbed for a moment by a pulse of electromagnetic radiation. We shall therefore consider this problem using Eqs. (61). It is easily seen that the motion is given by an equation of the form

$$d\dot{v}/ds+\alpha\dot{v}=\kappa\delta(t), \quad (64)$$

and hence the solution is of the form

$$\begin{aligned} v&=Ae^{-\alpha t}+B, \quad t<0,\\ v&=Ce^{-\alpha t}+D, \quad t>0, \end{aligned} \quad (65)$$

where A, B, C, and D are constants which have to be determined from the initial conditions and the boundary conditions at time $t=0$. If we suppose that the electron is at rest till the instant $t=0$, when it comes under the influence of the pulse, then the solution is

$$v=0, \quad t<0; \quad v=(\kappa/\alpha)(1-e^{-\alpha t}), \quad t>0. \quad (66)$$

This solution is such that the pulse imparts an acceleration κ to the electron, the velocity of the electron thereby gradually increasing from zero to κ/α while the acceleration gradually diminishes and eventually dies out. Ultimately, the motion is one of uniform velocity. Hence when we use Eqs. (61) the solution is entirely satisfactory. The total momentum acquired by the electron agrees with the result of Dirac's theory, as also with that of the elementary theory which neglects radiation damping.

(*c*) *The Hydrogen Atom*

I have examined[26] the motion of an electron in the hydrogen atom on the basis of these equations. It is seen that it is possible for the electron to fall into the nucleus. For the case of rectilinear motion of the electron towards the proton, the velocity of the electron at points close to the proton is, approximately

$$v=-\left(\frac{3}{2k+1}\log x\right)^{\frac{1}{3}}.$$

If the motion is confined to a plane, then the electron can fall into the nucleus by describing a spiral whose equation at points close to the nucleus is given, approximately, by

$$\theta=Ar[\log(1/r)]^{-\frac{1}{3}}.$$

Thus these applications show that there are many respects in which these equations give results in harmony with our usual notions of physics. It is somewhat surprising that in a theory which uses advanced potentials, and where, therefore, the physical mechanism is by no means clear, the equations derived from the theory lead to physically understandable results, whereas when the theory uses retarded potentials alone, the corresponding equations lead to many unexpected results.

(iv) ALTERNATIVE EQUATIONS III

27. Discussion of the Equations

An interesting particular case of the generalized classical equations of Chapter II occurs when $k=-\frac{1}{2}$. Then

$$F_{\text{el}}=\tfrac{1}{2}(F_{\text{ret}}+F_{\text{adv}}), \quad (67)$$

and the corresponding equation of motion for a

[26] C. J. Eliezer, Phys. Rev. 71, 49 (1947).

single particle is

$$m\dot{v}_{\mu}=ev_{\sigma}F_{\mu,\,\mathrm{ext}}{}^{\sigma}, \tag{68}$$

and for a system of n particles is

$$m\dot{v}_{\mu i}=ev_{\sigma i}\{F_{\mu,\,\mathrm{ext}}{}^{\sigma}+\sum_{i\neq j}(F_{\mu j,\,\mathrm{ret}}{}^{\sigma}+F_{\mu j,\,\mathrm{adv}}{}^{\sigma})\}. \tag{69}$$

These equations have been considered by Fokker[27] who chose this particular form in order to express the equations of motion in variational form. We have here obtained it by deducing the equations from the conservation laws by following Dirac's method of subtracting infinities in a Lorentz invariant way.

We observe that Eq. (68) does not contain the usual radiation reaction terms. Thus for a single-electron problem these equations are the same as those in a theory which neglects radiation damping altogether. Hence this theory would allow the existence of stationary states unlike the other theories in which it is impossible for an electron in the presence of an electromagnetic field to describe the same orbit continually. The fact that Bohr's semiclassical methods of describing the motion of the electron in the hydrogen atom has met with such remarkable success might be taken to lend support to the view that Eq. (68) has a certain range of applicability.

These equations for the single-electron case have been extensively applied to all problems of interest; and the equations for the many-body problem, while they may be formally written down easily, are difficult to solve, except in the form of very rough approximations. We do not, therefore, consider here any applications of these equations.

28. Discussion of the Results

The evidence gathered from the above work shows that if we keep to the assumption that the field of a moving electron is given by its retarded field, the corresponding equations of motion, namely, the Lorentz-Dirac equations or their modifications containing higher differential coefficients of velocity such as the Alternative Eqs. I, do not always have physically understandable solutions. But the correspondence between the classical and the quantum theories is not always very close, and therefore one cannot conclude that this theory is unsatisfactory without first translating it into the quantum theory and investigating its consequences there.

In the Alternative Eqs. II, the advanced field contributes more than the retarded field but the resulting equations seem to be satisfactory in many respects. In III there is a perfect symmetry between the retarded and the advanced fields, and for a single electron there is no radiation damping. This may be regarded as a contradiction with experiment, as it is generally accepted that the phenomenon of radiation reaction is well borne out by experimental results. The Lorentz equations have been supposed to explain satisfactorily, within the limits of the classical theory, certain phenomena such as the line breadth of spectral lines, dispersion, the potential needed to drive a wireless antenna, and so on. But the view may be taken that because certain of these phenomena (as for example dispersion or line breadth) are explainable in the quantum theory by taking over classical equations which do not contain the radiation damping terms, similar procedure may be possible for all the other phenomena.

Furthermore, it may be that in microscopic regions the notion of a single electron in an external field considered as "given" may be an unjustifiable abstraction, and that only the many-body problem has physical significance. Unfortunately, Eq. (69) for the many-body problem are very involved, with the past and future of all particles interconnected in a complicated manner, so that it is difficult to extract any information from them. It may be that the effect of these interconnections is the experimentally observed damping.

From the various results obtained above the author is inclined to the view that for those problems where the classical theory is applicable the Alternative Eqs. III appear to be the most satisfactory. As far as the passage to the quantum theory is concerned, it is likely that whatever form of the classical theory one starts with would not substantially affect the mathematical formulation in the quantum theory, provided the classical theory is consistent with the conservation laws and with the special theory

[27] A. D. Fokker, Zeits. f. Physik 58, 386 (1929).

of relativity. In the remaining chapters of this paper we shall discuss the method of taking over the generalized equations of motion into the quantum theory, and then apply the resulting theory to a few important problems.

CHAPTER IV. QUANTUM ELECTRODYNAMICS

29. Difficulties in the Quantum Theory of Radiation

The equations which describe the behavior of electrons in the quantum theory correspond to classical equations based on the point charge model of the electron. Hence the difficulties in classical electrodynamics arising from the occurrence of infinities and which were removed by the methods given in Chapter II also exist in quantum electrodynamics. In addition to the infinities of classical origin there may also be infinities which are purely quantum effects. In the quantum theory the infinities appear when we solve the wave equation that describes the interaction of particles and an electromagnetic field. The Fourier components of the field which correspond to very short wave-lengths give rise to divergent integrals in the solution of the wave equation. Methods which avoid these infinities by arbitrarily cutting off these short wavelengths have met with some success when applied to certain elementary problems, such as the emission and absorption of radiation whose wavelength is not too short. Nevertheless the limitations of the theory are of a fundamental nature, and many attempts have been made to formulate a quantum theory of radiation free from these divergent integrals.

The infinities of classical origin having been satisfactorily removed as in Chapter II, the next step is to take over this modified form of classical electrodynamics into the quantum theory. This may be accomplished by expressing the classical equations of motion in Hamiltonian form, and replacing, in the usual way, the momenta by certain Hermitian operators which satisfy appropriate commutation relations. The wave equation thus obtained is then solved by means of the perturbation method. One has then to interpret physically the mathematical solution.

30. The Hamiltonian Form of the Classical Equations of Motion

A convenient method of expressing the equations of motion in Hamiltonian form is by making use of the action principle of classical electrodynamics in the form given by Fokker[27] and Dirac.[7] The action integral is of the form

$$S=\sum_i m_i\int (v_i^2)^{\frac{1}{2}}ds_i+\sum_i\sum_{j\neq i} e_ie_j\int\int \delta(z_i-z_j)^2 ds_i ds_j+\sum_i e_i\int M_\mu(z_i)v_i^\mu ds_i, \quad (1)$$

where $M_\mu(x)$ is the vector potential which for the present shall be taken as an arbitrary function of the field point x_μ, and the rest of the notation is as before. The condition $v_i^2=1$ should not be used before the variation. Suppose that the limits of integration for each s_i are s_i^0 and s_i', and let the corresponding z_i and v_i be z_i^0, z_i' and v_i^0, v_i'. We restrict s_i^0 and s_i' so that the points z_i^0, z_i' all lie outside each other's light cones. By making variations in the world-lines of the particles, we obtain the equation of motion of the ith particle in the form

$$m_i\frac{dv_i^\mu}{ds_i}=e_i\left[\frac{\partial A_i^\sigma}{\partial x_\mu}-\frac{\partial A_i^\mu}{\partial x_\sigma}\right]_{z_i} v_{\sigma i}, \quad (2)$$

where

$$A_i^\sigma(x)=M^\sigma(x)+\sum_{j\neq i} e_j\int_{s_j^0}^{s_j'}\delta(x-z_j)^2 v_j^\sigma ds_j. \quad (3)$$

We should take A_i to be the potential function corresponding to the effective field f_i in the classical electrodynamics of Chapter II. If F_{act} denotes the actual field at any point, then

$$f_i=F_{\text{act}}-\tfrac{1}{2}(F_{i,\text{ret}}+F_{i,\text{adv}}).$$

Defining the fields F_{in} and F_{out} (as Dirac does but with certain modifications here as we are considering a combination of retarded and advanced fields) by the relations

$$\begin{aligned}F_{\text{act}}&=F_{\text{in}}+\sum_i F_{i,\text{ret}}+\sum_i(k+\tfrac{1}{2})(F_{i,\text{ret}}-F_{i,\text{adv}})\\&=F_{\text{out}}+\sum_i F_{i,\text{adv}}+\sum_i(k+\tfrac{1}{2})(F_{i,\text{adv}}-F_{i,\text{ret}}),\end{aligned} \quad (4)$$

we obtain

$$f_i=\tfrac{1}{2}(F_{\text{in}}+F_{\text{out}})+\tfrac{1}{2}\sum_{j\neq i}(F_{j,\,\text{ret}}+F_{j,\,\text{adv}}).$$

Hence for $A_{i,\text{ in}}$ we take

$$A_i=\tfrac{1}{2}(A_{\text{in}}+A_{\text{out}})+\tfrac{1}{2}\sum_{j\neq i}(A_{j,\,\text{ret}}+A_{j,\,\text{adv}}), \quad (5)$$

which is the same as in the Dirac case except that A_{in} and A_{out} are defined differently here.

A_{in} and A_{out} given by (4) both satisfy $\Box A=0$, and much of the mathematical formalism to follow will be independent of what particular value we assign to k. For the sake of definiteness, however, we shall take k equal to zero in the rest of this paper. It seems appropriate that we first consider in detail the particular theory with the assumption that the field of a moving electron is given by its retarded field and investigate its chief consequences, and then proceed to extend the theory to include other values of k. Such an extension would be straightforward as seen from the above work and will not be considered here. In the remaining part of this paper we shall therefore restrict the discussion to Dirac's quantum electrodynamics.

Equation (5) may be written

$$A_i^\sigma=\tfrac{1}{2}(A_{\text{in}}^\sigma+A_{\text{out}}^\sigma)+\sum_{j\neq i}e_j\int_{-\infty}^{\infty}\delta(x-z_j)^2v_j^\sigma ds_j, \quad (6)$$

and hence

$$M^\sigma(x)=\tfrac{1}{2}(A_{\text{in}}^\sigma+A_{\text{out}}^\sigma)+\sum_{j\neq i}e_j\left[\int_{-\infty}^{s_j^0}+\int_{s_j'}^{\infty}\right]\delta(x-z_j)^2v_j^\sigma ds_j. \quad (7)$$

It is however preferable to take

$$M^\sigma(x)=\tfrac{1}{2}(A_{\text{in}}^\sigma+A_{\text{out}}^\sigma)+\tfrac{1}{2}\sum_{j}e_j\left[\int_{-\infty}^{s_j^0}-\int_{s_j'}^{\infty}\right]\Delta(x-z_j)v_j^\sigma ds_j, \quad (8)$$

which is independent of i and therefore makes it possible to derive the equations of motion of all the particles from the same action integral; further, (8) satisfies the equation

$$\Box M^\sigma(x)=0, \quad (9)$$

since the Jordan-Pauli Δ-function satisfies

$$\Box\Delta(x)=0.$$

The expressions (7) and (8) do not differ in the region inside the future light cones from the points z_j^0 and past light cones from the points z_j', and it is the field in this region which contributes to the equations of motion, the points being subject to the conditions

$$(z_i^0-z_j^0)^2<0, \quad (z_i'-z_j')^2<0.$$

The λ-Limiting Process.

When using the action principle in the above work, it was assumed that $M(x)$ is a continuous function in the neighborhood of the world-lines, but (8) has discontinuities. Dirac has shown that this difficulty of discontinuity may be eliminated by employing a certain limiting process known as the λ-limiting process which was first introduced by Wentzel.

A small four vector, λ_μ, is taken which is such that $\lambda^2>0$, $\lambda_0>0$. We replace $\delta(z_i-z_j)^2$ of (1) by $\delta(z_i-z_j+\lambda)^2$ if $(z_i-z_j)_0>0$, and by $\delta(z_i-z_j-\lambda)^2$ if $(z_i-z_j)_0<0$. Then interaction between pairs of particles will occur at points on the world-lines that lie one just outside the light cone from the other. The expression for $M(x)$ is correspondingly modified and is

$$M^\sigma(x)=\tfrac{1}{2}(A^\sigma{}_{\text{in}}+A^\sigma{}_{\text{out}})+\tfrac{1}{2}\sum_j e_j\left[\int_{-\infty}^{s_j^0}\Delta(x-z_i+\lambda)v_j^\sigma ds_j-\int_{s_j'}^{\infty}\Delta(x-z_i-\lambda)v_j^\sigma ds_j, \quad (10)\right.$$

where

$$(z_{i^0}-z_{j^0}\pm\lambda)<0, \quad (z_i'-z_j'\pm\lambda)<0. \quad (11)$$

The result of this alteration is that the singularities in $M(x)$ are now displaced a little from the end points on the world-lines. The variation principle can then be applied provided the variations in the world-lines are smaller than λ. The resulting equations of motion will then be exact only in the limit λ tending to zero. In practical applications, one is usually interested in all the times z_{0i} being equal, and the condition (11) is then equivalent to saying that the distance between the particles should not be smaller than the order of λ.

We treat $M^\sigma(x)$ for each point in space-time as a coordinate depending on s_i'. If $\kappa_\mu(x)$ is its conjugate momentum,

$$\kappa_\mu(x)=\sum_i e_i\int_{s_i^0}^{s_i'}\delta(x_0-z_{i0})\delta(x_1-z_{1i})\times\delta(x_2-z_{2i})\delta(x_3-z_{3i})v_{\mu i}ds_i. \quad (12)$$

The field function

$$N_\mu(x)=\tfrac{1}{2}\int\int\int\int_{-\infty}^{\infty}\Delta(x-x')\kappa_\mu(x')\times dx_0'dx_1'dx_2'dx_3' \quad (13)$$

$$=\tfrac{1}{2}\sum e_i\int_{s_i^0}^{s_i'}\Delta(x-z_i)v_{\mu i}ds_i \quad (14)$$

satisfies the equation

$$\Box N_\mu(x)=0. \quad (15)$$

Also

$$[N_\mu(x),\ M_\sigma(x')]=\tfrac{1}{2}g_{\mu\sigma}\Delta(x-x'),$$
$$[N_\mu(x),\ N_\sigma(x')]=0,\quad [M_\mu(x),\ M_\sigma(x')]=0. \quad (16)$$

The particle momenta p_i^μ are given by

$$p_i^\mu=m_iv_i'^\mu+e_i[M^\mu(z_i')+\tfrac{1}{2}\sum_j e_j\int_{s_j^0}^{s_j'}\Delta(z_i'-z_j+\lambda)v_j^\mu ds_j$$
$$=m_iv_i'^\mu+eA^\mu(z_i'), \quad (17)$$

where

$$A^\mu(x)=M^\mu(x)+N^\mu(x+\lambda). \quad (18)$$

$A_\mu(x)$ then satisfies the Poisson bracket relations

$$[A_\mu(x),\ A_\sigma(x)]=\tfrac{1}{2}g_{\mu\sigma}[\Delta(x-x'+\lambda)+\Delta(x-x'-\lambda)] \quad (19)$$

and also the equations

$$\Box A_\mu=0, \quad (20)$$

$$\frac{\partial A_\mu(x)}{\partial x_\mu}=-\tfrac{1}{2}\sum_i e_i\{\Delta(x-z_i+\lambda)+\Delta(x-z_i-\lambda)\}. \quad (21)$$

Since $v_i^2=1$, we have

$$F_i\equiv-\frac{1}{2m_i}[\{p_i-eA(z_i)\}^2-m_i^2]=0, \quad (22)$$

there being one such equation for each particle. The F's may then be used as Hamiltonians in determining the law of variation of a dynamical variable according to the equation

$$d\xi/ds_i=[\xi,\ F_i], \quad (23)$$

where the condition F_i equal to zero should not be used before evaluating a Poisson bracket.

31. The Passage to the Quantum Theory

The Hamiltonian formulation given above makes it possible to pass over from the classical to the quantum theory, in the usual way, by replacing the momenta by operators satisfying certain commutation relations. The wave equations will be

$$[\{p_i-eA(z_i)\}^2-m_i^2]\psi=0, \quad (24)$$

the wave function, ψ, being a function of the coordinates z_i of all the particles and of the field variables, $M(x)$. Corresponding to the Poisson bracket relation (19) we now have the commutation relations to be satisfied by the potentials

$$[A_\mu(x),\ A_\sigma(x)]=\tfrac{1}{2}g_{\mu\sigma}[\Delta(x-x'+\lambda)+\Delta(x-x'-\lambda)]. \quad (25)$$

The particle variables, z_i, p_i, satisfy the usual commutation relations

$$[z_{\mu i},\ z_{\sigma j}]=[p_{\mu i},\ p_{\sigma j}]=0;\quad [p_{\mu i},\ z_{\mu j}]=g_{\mu\sigma}\delta_{ij}, \quad (26)$$

and the particle variables commute with the field variables. Equations (26) can be taken over unchanged into the quantum theory as they are consistent with the commutation relations. This shows that the field, A, may be resolved into waves travelling with the velocity of light. Equations (21) must be taken over as supplementary conditions

$$\left[\frac{\partial A_\mu(x)}{\partial x_\mu}+\tfrac{1}{2}\sum_i e_i\{\Delta(x-z_i+\lambda)+\Delta(x-z_i-\lambda)\}\right]\psi=0. \quad (27)$$

It is easily verified that the wave equations and the supplementary conditions are all consistent with one another.

The above Hamiltonian as it stands applies to spinless electrons. The corresponding equations for particles of spin $\frac{1}{2}\hbar$ will be of the form

$$[p_{0i}-eA_0(z_i) + \sum \alpha_{ri}\{p_{ri}-eA_r(z_i)\}+\beta_i m_i]\psi=0, \quad (28)$$

the α_{ri}, β_i being the usual Dirac spin variables.

Since

$$A_\mu(x)=M_\mu(x)+N_\mu(x+\lambda),$$

where

$$\begin{gathered}\Box M_\mu=0, \quad \Box N_\mu=0,\\ [M_\mu(x), M_\sigma(x')]=[N_\mu(x), N_\sigma(x')]=0, \\ [M_\mu(x), N_\sigma(x')]=\tfrac{1}{2}g_{\mu\sigma}\Delta(x-x'),\end{gathered} \quad (29)$$

there are now eight functions of position instead of the four A_μ. This shows that some redundant variables have been introduced, and it is easy to see that

$$B_\mu(x)=M_\mu(x)-N_\mu(x-\lambda) \quad (30)$$

gives the redundant variables, as these commute with all the variables that occur in the wave equations and the supplementary conditions. Hence the wave function represents a Gibbs ensemble, and appropriate weight function should be chosen for the purpose of physical interpretation.

32. Difficulties in Physical Interpretation—Negative Energy and Negative Probability

The methods of the previous sections show how a theory of quantum electrodynamics may be formulated. The next stage of the development concerns the physical interpretation of the solution of the quantum electrodynamical equations.

In the early attempts to extend the non-relativistic quantum mechanics to make it conform to the requirements of the special theory of relativity, the necessary mathematical methods were readily found; but there arose several obstacles in interpreting adequately the physical implications of the theory. The chief difficulty was that if one follows the same method of physical interpretation as in non-relativistic quantum mechanics, particles are found to have, in addition to the usual states of positive energy, also states of negative energy. These negative energy states in the quantum theory cannot be excluded as in the classical theory, because even if initially the particle is in a state of positive energy, there is the possibility of a quantum jump into a state of negative energy. Under the action of suitable perturbing forces such quantum transitions do, in fact, take place.

An additional difficulty concerns the sign of the expression for the probability of the particle being in a certain state. In non-relativistic quantum mechanics if $\psi(x_0, x_1, x_2, x_3)$ denotes the wave function, then $|\psi|^2$ gives the probability of the particle being in the neighborhood of the point (x_1, x_2, x_3) at the time x_0. This expression for the probability is always positive. For a relativistic theory this scalar expression for probability will not be valid, since probability should transform under Lorentz transformations like the time components of a four vector. The proper relativistic expression for probability was suggested by Gordon and Klein and is

$$P=\frac{1}{4\pi i}\left\{\frac{\partial\bar{\psi}}{\partial x_0}\psi-\bar{\psi}\frac{\partial\psi}{\partial x_0}\right\}, \quad (31)$$

which is easily seen to transform like the time component of a four vector.

This expression, however, unlike $|\psi|^2$, is not definitely positive; and thereby arises the difficulty that in some cases the probability value may be negative. It is found that particles whose spin is an integral number of quanta have states of negative energy with probability values which are negative. In this way there arise the two chief difficulties—negative energy states and negative probability.

33. Particles of Half-Odd Integral Spin

Most of the well-known particles that occur in nature come under this group of particles, the chief among them being the electron, positron, proton, and neutron. It has also been speculated whether some type of meson that has been identified in cosmic rays also belongs to this group. This group of particles is believed to obey the Fermi-Dirac statistics. All their states occur with positive probability and the difficulty is only over the existence of negative energy states. In order to resolve this difficulty Dirac introduced the idea of unoccupied states.

The Dirac "hole theory," which uses this idea, has met with substantial agreement with experiment. In the original form of this theory all the negative energy states were supposed to be occupied, and then an unoccupied negative energy state would appear as a hole which can be identified with the positron. The theory removes the difficulty of negative energy states but meets with other difficulties. A recent modification of the theory is to make it apply to a hypothetical world in which nearly all the states of negative energy are unoccupied, that is, to a world which is almost saturated with positrons. This makes it possible to calculate the probability of any collision process taking place in this hypothetical world, whereas with the previous form of the theory the calculations required unreliable approximations, even for the simplest type of collision processes. We then assume that the probability coefficients for the actual world are the same as for the hypothetical world. This assumption, equating the probability coefficients of the actual and hypothetical worlds, now replaces the old assumption in the former theory concerning the non-observability of the vacuum distribution of negative energy electrons.

34. Particles of Integral Spin

Particles which have spin whose magnitude is an integral number of quanta are believed to satisfy the Einstein-Bose statistics, and cannot be considered by the method of the previous section. For these particles both the difficulties, namely, that of negative energy states and of negative probability, occur. Among this group of particles is the light-quantum, the photon. It is believed that the cosmic-ray particle, the meson, also comes in this group.

The method of second quantization may be applied to this group of particles. This method uses certain operators of emission into, and of absorption from, states; and it is found that these operators have the same transformation equations and the same equations of motion as certain wave functions and their conjugate functions. The difficulty of negative energy states and of negative probability may then be got over by following the method which was introduced by Pauli and Weisskopf to deal with spinless particles.[9] They allowed only positive energy states and removed the difficulty of negative energy states by replacing the operators of emission into, and absorption from, negative energy states (which occur in the application of second quantization) by the operators of absorption from, and emission into, positive energy states, respectively, for particles having the opposite charge, Such a replacement does not conflict with the principles of conservation of energy, momentum, and charge.

The method of Pauli and Weisskopf may be applied in a degenerate form to photons. Since photons have no charge, one has to start with a one-particle theory in which the wave functions are real. The part of the wave function referring to positive energy states is then replaced by absorption operators from positive energy states, and the part referring to negative energy states is replaced by emission operators into positive energy states. These operators are then put into correspondence with the classical theory of electrodynamics by the usual rules. The resulting quantum electrodynamics thus involves only positive energy states, and is therefore free from the difficulties of negative energy and negative probability. It is the quantum electrodynamics of Heisenberg and Pauli.[28]

Although the theory of Heisenberg and Pauli removes the difficulties of negative energy and negative probability, it encounters a difficulty when we proceed to solve the wave equation. The difficulty is caused by the occurrence of divergent integrals in the solution of the wave equations. If we consider the motion of an electron and an electromagnetic field, and follow the method of the perturbation theory and try for the wave function a solution of the form

$$\psi=\psi_0+e\psi_1+e^2\psi_2+\cdots,$$

a series in ascending powers of the charge, e, of the electron, then it is found that ψ_0 and ψ_1 are finite, but ψ_2 contains divergent integrals. This makes it impossible to pass on to the higher approximations of the solution, and imposes severe restrictions on the range of application of the theory.

[28] W. Heisenberg and W. Pauli, Zeits. f. Physik **56**, 1; *ibid.* **59**, 168 (1929).

The divergent integrals that appear in the quantum electrodynamics of Heisenberg and Pauli are of the form

$$\int_0^\infty f(k_0)dk_0, \tag{32}$$

where $f(k_0) \sim k_0^n$ for large k_0, and the values $n=-1$, 0, and 1 occur frequently. When the theory of Heisenberg and Pauli is modified by the use of the λ-limiting process, then instead of (32) one obtains

$$\int_0^\infty f(k_0)\cos\lambda_0 k_0 dk_0. \tag{33}$$

Dirac has examined these integrals and shown that when n has even values the factor $\cos\lambda_0 k_0$ secures the elimination of divergence. When n is odd the divergence still remains. This may be because of the unsymmetrical treatment of the photon states of positive energy and of negative energy. Dirac's form of quantum electrodynamics treats the photon states of positive and negative energy symmetrically, and one then has instead of (33)

$$\tfrac{1}{2}\int_{-\infty}^{\infty}\{f(k_0)\exp(ik_0\lambda_0) + f(-k_0)\exp(-ik_0\lambda_0)\}dk_0. \tag{34}$$

When $f(k_0)$ is a rational algebraic function and when the upper and lower limits of integration are approached at the same rate, which is justifiable on physical grounds, the integrals become convergent. There is, however, no guarantee that $f(k_0)$ will be a rational algebraic function for all the integrals that occur in quantum electrodynamics. For the interaction of an electron and an electromagnetic field it is found that up to the second order of approximation in the perturbation method $f(k_0)$ is an algebraic function and the divergences are eliminated to this order. It is also found that the higher approximation terms of the general solution do involve nonrational functions, but the divergence in this case may be eliminated by taking an appropriate solution, as will be shown in the next chapter.

Two special features in which Dirac's form of quantum electrodynamics differs from that of Heisenberg and Pauli are:

(i) It involves a limiting process corresponding to an analogous limiting process necessary in classical electrodynamics to express the equations of motion in Hamiltonian form.

(ii) The representation used is different in that the wave function involves certain field functions as coordinates. This new representation requires for its physical interpretation both positive energy and negative energy photons, whereas in the representation of Heisenberg and Pauli, positive energy photons alone suffice. The wave function is expressed as a power series in certain variables ξ_+ and ξ_-, which correspond to positive energy photons and negative energy photons, respectively. The coefficients of terms such as $\xi_+^m\xi_-^n$ will enable us to calculate the probability of a state with $m+n$ photons, m photons with positive energy, and n with negative energy. This probability is then reinterpreted as the probability of m photons having been emitted and n photons having been absorbed. In this way the negative energy difficulty is removed without conflicting with the principles of conservation of energy and momentum.

The negative probability difficulty still remains and has then to be circumvented by a rather artificial device. The probability of a process in which an odd number of photons are absorbed is found to be negative. This is caused by the fact that according to the above scheme the probability of there being r photons in the initial distribution is

$$P_r = 2(-1)^r. \tag{35}$$

This probability distribution has no physical meaning but has to be used as a mathematical weight function for interpreting the Gibbs ensemble which as mentioned earlier arises from the use of the redundant variables. This weight function may be instrumental in securing convergence. We have to suppose that this distribution applies to a hypothetical mathematical world and calculate the probability coefficients for radiative transition processes in this hypothetical world by using Einstein's laws of radiation. The probability coefficients obtained by this method are then seen to be positive. We have then to assume that these coefficients also hold for the actual world.

CHAPTER V. THE INTERACTION OF AN ELECTRON AND A RADIATION FIELD—THE HIGHER APPROXIMATIONS

35. The Wave Equation and Its Solution. First- and Second-Order Terms

Let us consider the motion of an electron in an electromagnetic field.[29] Suppose that initially the electron is at rest. The problem in which the electron is initially not at rest can be reduced to this case by means of an appropriate Lorentz transformation. The wave equation is

$$[p_0 - eA_0 - \boldsymbol{\alpha}\cdot(\mathbf{p} - e\mathbf{A}) - m\beta]\psi = 0, \tag{1}$$

where the energy and momentum operators are given by

$$p_0 = i\partial/\partial x_0, \quad \mathbf{p} = -i\partial/\partial\mathbf{x},$$

and α_1, α_2, α_3, β are the usual Dirac matrices. The wave function, ψ, is taken as a matrix with four rows and four columns and not as a column matrix, because this representation is more convenient for averaging over the initial states. The units are so chosen that the velocity of light, and also h, Planck's constant divided by 2π, is unity. The four-vector potential A_μ is expressed as a Fourier expansion by

$$A_\mu(x) = 2^{-\frac{1}{2}}(2\pi)^{-1}\sum\int\{\xi_{k\mu}\exp[i(k, x)] + \xi_{k\mu}^*\exp[-i(k, x)]\}\partial k, \tag{2}$$

where ξ_k and ξ_k^* are, respectively, the operators of emission and of absorption of a photon with energy and momentum given by the four vector, k_μ, these operators being defined only for $k^2 = k_0^2 - \mathbf{k}^2 = 0$; $\sum$ means a summation over both values $\pm(k_1^2 + k_2^2 + k_3^2)^{\frac{1}{2}}$ for k_0, and

$$\partial k = k_0^{-1}dk_1dk_2dk_3.$$

The potentials satisfy the commutation relations

$$[A_\mu(x), A_\sigma(x')] = \tfrac{1}{2}g_{\mu\sigma}\{\Delta(x - x' + \lambda) + \Delta(x - x' - \lambda)\}. \tag{3}$$

These commutation relations for the potentials lead to the following commutation relations for ξ and ξ^*:

$$\begin{gathered}\xi_{k\mu}\xi_{k'\sigma} - \xi_{k'\sigma}\xi_{k\mu} = 0, \quad \xi_{k\mu}^*\xi_{k'\sigma}^* - \xi_{k'\sigma}^*\xi_{k\mu}^* = 0,\\ \xi_{k\mu}^*\xi_{k'\sigma} - \xi_{k'\sigma}\xi_{k\mu}^* = -\tfrac{1}{2}g_{\mu\sigma}(k_0 + k_0')\delta(\mathbf{k} - \mathbf{k}')\exp[-i(k, \lambda)],\end{gathered} \tag{4}$$

where $\delta(\mathbf{k} - \mathbf{k}')$ stands for $\delta(k_1 - k_1')\delta(k_2 - k_2')\delta(k_3 - k_3')$. The vector, λ_μ, lies within the future light cone, so that

$$\lambda_0 > 0, \quad \lambda^2 = \lambda_0^2 - \boldsymbol{\lambda}^2 > 0, \tag{5}$$

and is eventually made to tend to zero.

We may divide the field, A_μ, into longitudinal and transverse parts, and then eliminate the longitudinal waves from the Hamiltonian formalism by means of an appropriate transformation. The supplementary conditions are then automatically satisfied. In the rest of this work we shall, therefore, take A_0 to be zero. The commutation relations[30] for ξ and ξ^* will, after elimination of the longitudinal waves, read

$$\begin{gathered}\xi_{kr}\xi_{k's} - \xi_{k's}\xi_{kr} = 0, \quad \xi_{kr}^*\xi_{k'\sigma}^* - \xi_{k'\sigma}^*\xi_{kr}^* = 0,\\ \xi_{kr}^*\xi_{k's} - \xi_{k's}\xi_{kr}^* = -\tfrac{1}{2}(g_{rs} + k_rk_s/k_0^2)(k_0 + k_0')\delta(\mathbf{k} - \mathbf{k}')\exp[-i(k, \lambda)].\end{gathered} \tag{6}$$

[29] C. J. Eliezer, Proc. Roy. Soc. **A187**, 197 (1946).

[30] By an oversight the term k_rk_s/k_0^2 in Eq. (6) had been omitted in reference 29. A few expressions there have to be modified but the final conclusion is the same.

To solve the wave equation we follow the method of the perturbation theory and try a solution of the form

$$\psi=\psi_0+e\psi_1+e^2\psi_2+\cdots, \tag{7}$$

assuming that e is small. Substituting in (1) and equating coefficients we see that the successive terms, ψ_0, ψ_1, $\cdots$ are connected by the equation

$$(p_0-\boldsymbol{\alpha}\cdot\mathbf{p}-m\beta)\psi_n=-(\boldsymbol{\alpha}\cdot\mathbf{A})\psi_{n-1}. \tag{8}$$

When $n=0$, we have

$$(p_0-\boldsymbol{\alpha}\cdot\mathbf{p}-m\beta)\psi_0=0. \tag{9}$$

This is the wave equation of a free electron; and since the electron is supposed to be at rest initially, we take as the solution

$$\psi_0=u\exp[-imx_0], \tag{10}$$

where u is a matrix with four rows and columns and satisfies the equation

$$(1-\beta)u=0. \tag{11}$$

ψ_1 is given by

$$(p_0-\boldsymbol{\alpha}\cdot\mathbf{p}-m\beta)\psi_1=-(\boldsymbol{\alpha}\cdot\mathbf{A})u\exp[-imx_0]$$

which, when we substitute for $\mathbf{A}$, gives

$$(p_0-\boldsymbol{\alpha}\cdot\mathbf{p}-m\beta)\psi_1=-2^{-\frac{1}{2}}(2\pi)^{-1}\sum\int\alpha_r\xi_{kr}\exp[i(k,x)-imx_0]u\partial k, \tag{12}$$

where we have used the relation

$$\xi^*u=0, \tag{13}$$

the initial state being one in which no photons are present. Hence

$$\psi_1=-2^{-\frac{1}{2}}(2\pi)^{-1}\sum\int\{(m-k_0)^2-\mathbf{k}^2-m^2\}^{-1}(m-k_0-\boldsymbol{\alpha}\cdot\mathbf{k}+m\beta)\alpha_r\xi_{kr}\exp[i(k,x)-imx_0]u\partial k$$

$$=-2^{-\frac{1}{2}}(4\pi m)^{-1}\sum\int(1+l_s\alpha_s)\alpha_r\xi_{kr}\exp[i(k,x)-imx_0]u\partial k, \tag{14}$$

where (l_1, l_2, l_3) are the direction cosines of $\mathbf{k}$. We have used the condition (11).

ψ_2 is given by

$$(p_0-\boldsymbol{\alpha}\cdot\mathbf{p}-m\beta)\psi_2=-(\boldsymbol{\alpha}\cdot\mathbf{A})\psi_1.$$

When the expressions for $\mathbf{A}$ and ψ_1 are substituted in the right-hand side, and the result simplified the application of the commutation relations (6) and Eq. (13), we obtain two terms, one of the second degree and the other of zero degree in the ξ's. Let us denote by $\psi_{n,m}$ that part of ψ_n which is of the mth degree in the ξ's, that is, the part that refers to m photons. When n is even, m is even; and when n is odd, m also is odd.

$\psi_{2,0}$ satisfies

$$(p_0-\boldsymbol{\alpha}\cdot\mathbf{p}-m\beta)\psi_{2,0}=-(32m\pi^2)^{-1}\sum\sum\int\int\alpha_t(1+l_s\alpha_s)\alpha_r(g_{rt}+l_rl_t)(k_0+k_0')\delta(\mathbf{k}-\mathbf{k}')$$

$$\times\exp[i(k-k',x)-imx_0-i(k,\lambda)]u\partial k\partial k',$$

$$=-(16m\pi^2)^{-1}\sum\int\alpha_t(1+l_s\alpha_s)\alpha_r(g_{rt}+l_rl_t)\exp[-imx_0-i(k,\lambda)]u\partial k. \tag{15}$$

In evaluating an integral of the form

$$\sum \int f(k_0, \mathbf{k}) dk_1 dk_2 dk_3,$$

we take

$$\int \{f(k_0, \mathbf{k}) + f(-k_0, -\mathbf{k})\} dk_1 dk_2 d_{k3}$$

and integrate over the **k**-space by using polar coordinates so that

$$dk_1 dk_2 dk_3 = k_0^2 dk_0 d\Omega = k_0^2 \sin\theta dk_0 d\theta d\phi.$$

The right-hand side of (15) is then

$$i(16m\pi^2)^{-1} \int \alpha_t(1+l_s\alpha_s)\alpha_r(g_{rt}+l_r l_t) e^{-imx_0} d\Omega \int_{-\infty}^{\infty} k_0 \sin k_0(\lambda_0 - 1\cdot\lambda) dk_0$$

and

$$\int_{-\infty}^{\infty} k_0 \sin k_0(\lambda_0-1\cdot\lambda) dk_0 = \left[-\frac{k_0 \cos k_0(\lambda_0-1\cdot\lambda)}{(\lambda_0-1\cdot\lambda)^2} + \frac{\sin k_0(\lambda_0-1\cdot\lambda)}{(\lambda_0^2-1\cdot\lambda)^2} \right]_{-\infty}^{\infty}$$

which can be taken as zero, according to the usual procedure in quantum mechanics where one takes for the value of an oscillating function its mean value. Hence

$$\psi_{2,0} = 0. \tag{16}$$

$\psi_{2,2}$ is given by

$$(p_0 - \boldsymbol{\alpha}\cdot\mathbf{p} - m\beta)\psi_{2,2} = (32m\pi^2)^{-1} \sum\sum \int\int \{\alpha_t(1+l_s\alpha_s)\alpha_r + \alpha_r(1+l_s', \alpha_s)\alpha_t\} \times \xi_{k't}\xi_{kr} \exp\{i(k+k', x) - imx_0\} u\partial k\partial k', \tag{17}$$

where the right-hand side has been made symmetrical. Operating on (17) from the left by $p_0 + \boldsymbol{\alpha}\cdot\mathbf{p} + m\beta$, we obtain

$$(p_0^2 - \mathbf{p}^2 - m^2)\psi_{2,2} = (32m\pi^2)^{-1} \sum\sum \int\int \{m(1+\beta) - k_0 - k_0' - \boldsymbol{\alpha}\cdot(\mathbf{k}+\mathbf{k}')\}$$

$$\times \{\alpha_t(1+l_s\alpha_s)\alpha_r + \alpha_r(1+l_s'\alpha_s)\alpha_t\} \xi_{k't}\xi_{kr} \exp[i(k+k', x) - imx_0] u\partial k\partial k'. \tag{18}$$

In solving an equation of the form

$$(p_0^2 - \mathbf{p}^2 - m^2)\psi = F \exp[i(a_0 x_0 - \mathbf{a}\cdot\mathbf{x})], \tag{19}$$

we note that apart from the solution

$$(a_0^2 - \mathbf{a}^2 - m^2)^{-1} F \exp[i(a_0 x_0 - \mathbf{a}\cdot\mathbf{x})]$$

there also exist solutions of the form

$$\delta(a_0^2 - \mathbf{a}^2 - m^2) F \exp[i(a_0 x_0 - \mathbf{a}\cdot\mathbf{x})]$$

because

$$(a_0^2 - \mathbf{a}^2 - m^2)\delta(a_0^2 - \mathbf{a}^2 - m^2) = 0.$$

Hence the complete solution of (19) is

$$\psi = \left\{ \frac{1}{a_0^2 - \mathbf{a}^2 - m^2} + \Phi\delta(a_0^2 - \mathbf{a}^2 - m^2) \right\} F \exp[i(a_0 x_0 - \mathbf{a}\cdot\mathbf{x})], \tag{20}$$

where Φ is an arbitrary function of integration, which must be chosen to suit the special physical conditions of each problem under consideration.

Equation (18) is of the form (19) with $a_0 = k_0 + k_0' - m$, $\mathbf{a} = \mathbf{k} + \mathbf{k}'$. Therefore,

$$a_0^2 - \mathbf{a}^2 - m^2 = -2m(k_0 + k_0') + 2k_0 k_0'(1 - \mathbf{1}\cdot\mathbf{1}')$$

and

$$\delta(a_0^2 - \mathbf{a}^2 - m^2) = \frac{1}{2|m - k_0(1 - \mathbf{1}\cdot\mathbf{1}')|}\delta\left\{k_0' + \frac{mk_0}{m - k_0(1 - \mathbf{1}\cdot\mathbf{1}')}\right\}, \tag{21}$$

the result being expressed in this form because we are going to integrate first with respect to k_0'. The simplest possible choice for Φ, namely, Φ is zero, leads to divergences as will be shown later. The next simple choice is to take Φ equal to a constant. But when we proceed to evaluate ψ_3 we obtain integrals of the form

$$\int |w|^{-1} \exp(-i\lambda_0 w^{-1}) dw,$$

where the range of integration includes the point w equal to zero. This integral does not tend to a finite value as λ_0 tends to zero. If, however, we take $\Phi = \pm C$ according as $m - k_0(1 - \mathbf{1}\cdot\mathbf{1}') \gtrless 0$, then the corresponding integrals in ψ_3 are of the form

$$\int w^{-1} \exp(-i\lambda_0 w^{-1}) dw,$$

which tends to a finite value as λ_0 tends to zero. This choice has the physical interpretation that $\Phi = +C$ for positive energies of the electron and $\Phi = -C$ for negative energies of the electron, because the electron's energy $m - k_0 - k_0'$ has the same sign as $m - k_0(1 - \mathbf{1}\cdot\mathbf{1}')$ when $\mathbf{k}$, $\mathbf{k}'$ are such that the δ function in (21) does not vanish, that is, when

$$m(k_0 + k_0') = k_0 k_0'(1 - \mathbf{1}\cdot\mathbf{1}').$$

The particular case when C is equal to $i\pi$ turns out to be an important one, as the divergent integrals then disappear, to any order of approximation in the perturbation theory. The solution then corresponds to outgoing waves of the electron, as will be shown later.

It is convenient to employ the following notation: If b_μ denotes a four vector $(b_0, \mathbf{b})$, let

$$\gamma_b \equiv (m - b_0)^2 - \mathbf{b}^2 - m^2, \quad R_b \equiv m(1 + \beta) - b_0 - \boldsymbol{\alpha}\cdot\mathbf{b}. \tag{22}$$

The solution of Eq. (18) is then taken to be

$$\psi_{2,2} = (32m\pi^2)^{-1} \sum\sum \int\int \{\gamma_{k+k'}^{-1} \pm C\delta(\gamma_{k+k'})\} R_{k+k'} \{\alpha_t(1 + l_s\alpha_s)\alpha_r + \alpha_r(1 + l_s'\alpha_s)\alpha_t\}$$
$$\times \xi_{k't}\xi_{kr} \exp\{i(k + k', x) - imx_0\} u\partial k\partial k', \tag{23}$$

where the sign in $\pm$ is determined by

$$\pm\delta(\gamma_{k+k'}) = \frac{-1}{2\{m - k_0(1 - \mathbf{1}\cdot\mathbf{1}')\}}\partial\left\{k_0' + \frac{mk_0}{m - k_0(1 - \mathbf{1}\cdot\mathbf{1}')}\right\}. \tag{24}$$

36. The Higher Approximations

ψ_3 consists of two terms: $\psi_{3,3}$ and $\psi_{3,1}$. The term $\psi_{3,3}$ can be readily evaluated. We obtain

$$\psi_{3,3} = -2^{-\frac{1}{2}}(64m\pi^3)^{-1} \sum\sum\sum \int\int\int \{\gamma_{k+k'+k''}^{-1} \pm C\delta(\gamma_{k+k'+k''})\}\{\gamma_{k+k'} \pm C\delta(\gamma_{k+k'})\} R_{k+k'+k''}\alpha_q R_{k+k'}$$
$$\times \{\alpha_t(1 + l_s\alpha_s)\alpha_r + \alpha_r(1 + l_s'\alpha_s)\alpha_t\} \xi_{k''q}\xi_{k't}\xi_{kr} \exp\{i(k + k' + k'', x) - imx_0\} u\partial k\partial k'\partial k''. \tag{25}$$

$\psi_{3,1}$ is given by

$$(p_0-\boldsymbol{\alpha}\cdot\mathbf{p}-m\beta)\psi_{3,1}=2^{-\frac{1}{2}}(128m\pi^3)^{-1}\sum\sum\sum\int\int\int\{\gamma_{k+k'}{}^{-1}\pm C\delta(\gamma_{k+k'})\}\alpha_q R_{k+k'}$$
$$\times\{\alpha_t(1+l_s\alpha_s)\alpha_r+\alpha_r(1+l_s'\alpha_s)\alpha_t\}\{(g_{qr}+l_ql_r)(k_0''+k_0)\delta(\mathbf{k}''-\mathbf{k})e^{-i(k,\lambda)}\xi_{k't}$$
$$+(g_{qt}+l_q'l_t')(k_0''+k_0')\delta(\mathbf{k}''-\mathbf{k}')e^{-i(k',\lambda)}\xi_{kr}\}\exp[i(k+k'-k'',x)-imx_0]u\partial k\partial k'\partial k'',$$

where the right-hand side has been reduced by the use of the commutation relations (6) and Eq. (13). Integrating with respect to k'' and using the symmetry in k and k', we obtain

$$(p_0-\boldsymbol{\alpha}\cdot\mathbf{p}-m\beta)\psi_{3,1}=2^{-\frac{1}{2}}(32m\pi^3)^{-1}\sum\sum\int\int\{\gamma_{k+k'}{}^{-1}\pm C\delta(\gamma_{k+k'})\}\alpha_q R_{k+k'}\{\alpha_t(1+l_s\alpha_s)\alpha_r+\alpha_r(1+l_s'\alpha_s)\alpha_t\}$$
$$\times(g_{qt}+l_q'l_t')\xi_{kr}\exp\{i(k,x)-imx_0-i(k',\lambda)\}u\partial k\partial k'. \quad (26)$$

Hence,

$$\psi_{3,1}=2^{-\frac{1}{2}}(32m\pi^3)^{-1}\sum\int\gamma_k{}^{-1}R_kI\xi_{kr}\exp\{i(k,x)-imx_0\}u\partial k, \quad (27)$$

where

$$I\equiv\sum\int\{\gamma_{k+k'}{}^{-1}\pm C\delta(\gamma_{k+k'})\}\alpha_qR_{k+k'}\{\alpha_t(1+l_s\alpha_s)\alpha_r+\alpha_r(1+l_s'\alpha_s)\alpha_t\}(g_{qt}+l_q'l_t')e^{-ik_0'\lambda_0}\partial k'. \quad (28)$$

We have taken $\lambda_\mu=(\lambda_0, 0)$ which is permissible from the inequalities (5). To evaluate I we note that if $f(k_0')$ is any polynomial and a is independent of k_0', then

$$\sum\int_{-\infty}^{\infty}\frac{f(k_0')}{k_0'-a}e^{-ik_0'\lambda_0}dk_0'=f(a)\int_{-\infty}^{\infty}\left\{\frac{e^{-ik_0'\lambda_0}}{k_0'-a}-\frac{e^{ik_0'\lambda_0}}{k_0'+a}\right\}dk_0'$$
$$=2f(a)\left\{a\int_{-\infty}^{\infty}\frac{\cos k_0'\lambda_0}{k_0'^2-a^2}dk_0'-i\int_{-\infty}^{\infty}\frac{k_0'\sin k_0'\lambda_0}{k_0'^2-a^2}dk_0'\right\}=-2i\pi f(a)e^{-ia\lambda_0}. \quad (29)$$

We have here used the result

$$\int_{-\infty}^{\infty}x^ne^{ix}dx=0,$$

where n is any positive integer. Again

$$\sum\int f(k_0')\delta(k_0'-a)e^{-ik_0'\lambda_0}dk_0'=2f(a)e^{-ia\lambda_0}. \quad (30)$$

Hence

$$\sum\int\left\{\frac{1}{k_0'-a}+C\delta(k_0'-a)\right\}f(k_0')e^{-ik_0'\lambda_0}dk_0'=2(C-i\pi)f(a)e^{-ia\lambda_0}. \quad (31)$$

Thus on integrating (28) with respect to k_0', we obtain

$$I=\tfrac{1}{2}(C-i\pi)\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\frac{mk_0(l_q'l_t'\alpha_q-\alpha_t)}{\{m-k_0(1-\mathbf{1}\cdot\mathbf{1}')\}^2}\left\{m(1+\beta)-k_0(1+l_s\alpha_s)+\frac{mk_0(1+l_s'\alpha_s)}{m-k_0(1-\mathbf{1}\cdot\mathbf{1}')}\right\}$$
$$\times\{\alpha_t(1+l_s\alpha_s)\alpha_r+\alpha_r(1+l_s'\alpha_s)\alpha_t\}\exp\left\{\frac{imk_0\lambda_0}{m-k_0(1-\mathbf{1}\cdot\mathbf{1}')}\right\}\sin\theta'd\theta'd\phi'. \quad (32)$$

When integrating over ϕ' it is convenient to take the direction of $\mathbf{1}$ for the polar axis, and then to express the result in tensor form. Thus

$$\int_0^{2\pi} l_t'd\phi' = 2\pi\omega l_t, \quad \int_0^{2\pi} l_t'l_s'd\phi' = \pi(1-\omega^2)\delta_{ts} + \pi(3\omega^2-1)l_tl_s. \tag{33}$$

Writing

$$\mathbf{1}\cdot\mathbf{1}' = \cos\theta' = \omega, \quad \omega_0 = 1 - m/k_0,$$

we have

$$I = \tfrac{1}{2}(C-i\pi)\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\frac{m}{k_0(\omega-\omega_0)^2}\Big[4m(l_r'l_t'\alpha_t - \alpha_r) - k_0\{2(\omega l_t'\alpha_t - 1)(1+l_s\alpha_s)\alpha_r$$

$$+\alpha_t(1+l_s\alpha_s)\alpha_r(1+l_s'\alpha_s)(l_t'-\alpha_t)\} - \frac{2m}{\omega-\omega_0}(1-l_s'\alpha_s)(\alpha_r + l_s\alpha_s\alpha_r + 2l_r')\Big]\exp\Big[\frac{im\lambda_0}{\omega-\omega_0}\Big]\sin\theta' d\theta' d\phi'$$

$$= \tfrac{1}{2}(C-i\pi)\int_{-1}^{1}\frac{m\pi}{k_0(\omega-\omega_0)^2}\Big[-4m(1+\omega^2) + 2k_0(1-\omega^2)(1+l_s\alpha_s)$$

$$-\frac{4m(1-\omega)}{\omega-\omega_0}(l_s\alpha_s - \omega)\Big]\alpha_r\exp\Big(\frac{im\lambda_0}{\omega-\omega_0}\Big)d\omega, \tag{34}$$

where we have omitted certain terms in l_r and $(1-\beta)$ owing to the relations $l_r\xi_{kr}=0$, $(1-\beta)u=0$. Hence we obtain

$$I = -(C-i\pi)m\pi\int_{-1}^{1}\Big[1 + l_s\alpha_s + \frac{2m}{k_0} + \frac{2}{\omega-\omega_0}\Big\{1 + \frac{2m}{k_0} - \frac{2m^2}{k_0^2} + \Big(1-\frac{m}{k_0}\Big)l_s\alpha_s\Big\}$$

$$+\frac{m}{k_0(\omega-\omega_0)^2}\Big\{4 - \frac{7m}{k_0} + \frac{2m^2}{k_0^2} + \Big(\frac{m}{k_0} - 4\Big)l_s\alpha_s\Big\} + \frac{2m^2}{k_0^2(\omega-\omega_0)^3}\Big(\frac{m}{k_0} - 1 + l_s\alpha_s\Big)\Big]\alpha_r\exp\Big[\frac{im\lambda_0}{\omega-\omega_0}\Big]d\omega. \tag{35}$$

We may integrate with respect to ω and substitute for I in (27) and show that $\psi_{3,1}$ is free from divergent integrals. But in using its value to determine $\psi_{4,2}$ and $\psi_{4,0}$ it seems more convenient to postpone the integration.

To proceed to terms of the fourth order, we note that ψ_4 consists of $\psi_{4,4}$, $\psi_{4,2}$, and $\psi_{4,0}$. $\psi_{4,4}$ can be written down easily, the expression for it being similar to that of $\psi_{3,3}$ in (25). The term $\psi_{4,2}$ is given by

$$(p_0 - \boldsymbol{\alpha}\cdot\mathbf{p} - m\beta)\psi_{4,2} = -2^{-\frac{1}{2}}(2\pi)^{-1}\sum\int\alpha_p\{\xi_{k'''p}e^{i(k''',x)}\psi_{3,1} + \xi_{k'''p}^*e^{-i(k''',x)}\psi_{3,3}\}\partial k''',$$

and it may be seen without much difficulty that the right-hand side is free from divergence. The term $\psi_{4,0}$ is given by

$$(p_0 - \boldsymbol{\alpha}\cdot\mathbf{p} - m\beta)\psi_{4,0} = -2^{-\frac{1}{2}}(2\pi)^{-1}\sum\int\alpha_p\xi_{k'p}^*e^{-i(k',x)}\psi_{3,1}\partial k'$$

$$= (2^4m\pi^2)^{-2}\sum\int(\alpha_r - l_rl_p\alpha_p)\{m(1+\beta) - k_0(1+l_s\alpha_s)\}Iu\exp(-imx_0 - ik_0\lambda_0)k_0^{-1}\partial k, \tag{36}$$

where we have used the commutation relations (6) and Eq. (13) and integrated over $\mathbf{k}'$-space.

Operating on this equation by $p_0+\boldsymbol{\alpha}\cdot\mathbf{p}+m\beta$ from the left we have

$$(p_0^2-\mathbf{p}^2-m^2)\psi_{4,0}=-m(1+\beta)(2^4m\pi^2)^{-2}\sum\int(\alpha_r-l_rl_p\alpha_p)(1+l_s\alpha_s)Iu\exp(-imx_0-ik_0\lambda_0)\partial k.$$

Substituting for I from (35), using the commutation relations satisfied by $\boldsymbol{\alpha}$ and β, and the relation $\beta u=u$, we obtain

$$(p_0^2-\mathbf{p}^2-m^2)\psi_{4,0}=(C-i\pi)(4\pi)^{-2}\sum\int\int\int\left\{1+\frac{m}{k_0}+\frac{1}{\omega-\omega_0}\left(2+\frac{m}{k_0}-\frac{2m^2}{k_0^2}\right)+\frac{m^2}{k_0^2(\omega-\omega_0)^2}\left(\frac{m}{k_0}-3\right)\right.$$
$$\left.+\frac{m^3}{k_0^3(\omega-\omega_0)^3}\right\}u\exp\left[\frac{im\lambda_0}{\omega-\omega_0}-imx_0-ik_0\lambda_0\right]k_0dk_0d\Omega d\omega$$
$$=(C-i\pi)m^2(8\pi)^{-1}ue^{-imx_0}\int_{-\infty}^{\infty}\int_{x^{-1}-1}^{x^{-1}}\left[x+2+\frac{x^2+x-4}{xy}+\frac{2-3x}{x^2y^2}+\frac{1}{x^2y^3}\right]$$
$$\times\exp\left[\tfrac{1}{2}im\lambda_0\left(\frac{1}{y}-x\right)\right]dxdy, \quad (37)$$

after integrating with respect to Ω and substituting $k_0=\frac{1}{2}mx$, $\omega-\omega_0=2y$. This integral has been examined by the author. It is divergent. The discussion is rather long and involved and is not reproduced here. It is clear that as we proceed to the higher approximations, the integrand becomes non-rational and therefore there is no reason to expect convergence. The integrand becomes more and more complicated as we continue the solution step by step to higher orders of approximations, especially due to the factor $\exp\{-i(k,\lambda)\}$ in the commutation relations (6), and one would expect that divergence would naturally arise at some stage of the calculations. The hope expressed by some authors[31] that the integrand would always remain a rational algebraic function is therefore not justified.

If, however, we take the particular case when C has the value $i\pi$, then we see that owing to the factor $C-i\pi$ in I, $\psi_{3,1}$ and $\psi_{4,0}$ vanish. It is easy to see that the integral in any term such as $\psi_{n,m}$ with m not equal to n, is of the type (31) and, therefore, the factor $C-i\pi$ will occur in each such term. Hence by taking C equal to $i\pi$, we obtain a particular solution with

$$\psi_{n,m}=0, \quad \text{all } m\neq n. \quad (38)$$

The value of $\psi_{n,n}$ can be written down easily as

$$\psi_{n,n}=2^{-n/2}(-2\pi)^{-n}\sum\sum\cdots\sum\int\int\cdots\int R_{k+k'+\cdots k^{(n-1)}}\alpha_{r^{(n-1)}}R_{k+k'+\cdots k^{(n-2)}}\alpha_{r^{(n-2)}}\cdots$$
$$\times R_k\alpha_r\{\gamma_{k+k'+\cdots k^{(n-2)}}\pm i\pi\delta(\gamma_{k+k'+\cdots k^{(n-2)}})\}\cdots\{\gamma_{k+k'}{}^{-1}\pm i\pi\delta(\gamma_{k+k'})\}\gamma_k{}^{-1}\xi_{k^{(n-1)}r^{(n-1)}}\cdots$$
$$\times\xi_{k'r'}\xi_{kr}u\exp[i(k+k'+\cdots k^{(n-1)},x)-imx_0]\partial k^{(n-1)}\cdots\partial k'\partial k. \quad (39)$$

We should now consider the physical significance of the condition C equal to $i\pi$. It will be seen that this condition is equivalent to taking only those solutions which correspond to outgoing waves of the electron. To show this we follow a method given by Dirac[32] for the treatment of collision problems,

[31] W. Pauli, Rev. Mod. Phys. **15**, 175 (1943).
[32] P. A. M. Dirac, *Quantum Mechanics* (Oxford University Press, New York, 1935), second edition, p. 198.

where he shows that for outward moving particles the solution has a factor

$$1/(W'-W)-i\pi\delta(W'-W). \tag{40}$$

In the case considered by Dirac, the proof applied only to positive energies and we must verify that it also applies to negative energies. When W' is negative, some alterations in the equations given by Dirac will have to be made. On the right-hand side of his Eq. (27), instead of W' we would have $-W'$ which is equal to $|W'|$, while in his Eq. (28) there would still be W'. Then the condition (40) above means that the coefficient of $\exp[iP'r]$ vanishes and the term with $\exp[-iP'r]$ remains, P' being positive. The phase factor thus will be $\exp[i(-P'r-W't)]$ which is $\exp i[-P'r+|W'|t]$. Hence the solution corresponds to outgoing waves of the electron for both positive and negative W'. It remains to show that the condition (40) leads to the value $i\pi$ for C. If we consider the solution of Eq. (18) and write

$$W'=m-k_0-k_0', \quad |W|=\{(\mathbf{k}+\mathbf{k}')^2+m^2\}^{\frac{1}{2}}, \tag{41}$$

then we see that

$$\gamma_{K+K'}{}^{-1}\pm i\pi\delta(\gamma_{k+k'})=\frac{1}{W'+W}\left\{\frac{1}{W'-W}-i\pi\delta(W'-W)\right\}. \tag{42}$$

The result may then be extended to any equation of the form (18) involving any number of photons.

37. Discussion

The above work shows that whereas the general solution involves divergent integrals, there exists a particular solution corresponding to outgoing waves of the electron and which is free from divergence to all orders of approximation in the perturbation theory. The wave function in this particular solution may be written down explicitly, to any required order of approximation. One may therefore hope that this solution may be applied to calculate exactly the probability coefficients of transition processes involving any number of photons, whereas previously this was not possible owing to the divergence of the higher order terms.

CHAPTER VI. FURTHER APPLICATIONS

38. Hydrogen-Like Atoms

The results of the last chapter on the interaction of a free electron may be extended to apply to any one-electron problem. In particular, we may consider the interaction of hydrogen-like atoms with a radiation field. The quantum electrodynamics of Heisenberg and Pauli gives an infinite displacement of the spectrum lines of hydrogen-like atoms, owing to the divergent terms in the wave function. It is therefore of interest to investigate this question on the basis of Dirac's quantum electrodynamics.

We take the nucleus as fixed and obtain the wave function of an electron of charge e bound to a nucleus of charge $-Ze$. The wave equation is

$$(p_0-\boldsymbol{\alpha}\cdot(\mathbf{p}-e\mathbf{A})-m\beta+Ze^2/r)\psi=0, \tag{1}$$

in the same notation as before. We try a solution of the form

$$\psi=\psi_0+\psi_1+\psi_2+\cdots, \tag{2}$$

where ψ_0, ψ_1, $\cdots$ are in decreasing order of magnitude. The Born approximation which treats the Coulomb field as a perturbation is not valid for our purposes here, and we must therefore retain the term Ze^2/r in the Hamiltonian of the unperturbed system, and treat only the radiation field as a perturbation. The term $\psi_0, \psi_1, \cdots$ are connected by the equation

$$(p_0-\boldsymbol{\alpha}\cdot\mathbf{p}-m\beta+Ze^2/r)\psi_n=-e(\boldsymbol{\alpha}\cdot\mathbf{A})\psi_{n-1}. \quad (3)$$

Taking $n=0$, we have

$$(p_0-\boldsymbol{\alpha}\cdot\mathbf{p}-m\beta+Ze^2/r)\psi_0=0, \quad (4)$$

which is the relativistic wave equation giving the stationary states of the atom. Supposing that initially there are no photons present and that the atom is in a stationary state of energy E_m, we take

$$\psi_0=u_m\exp(-iE_mx_0). \quad (5)$$

ψ_1 is given by

$$(p_0-\boldsymbol{\alpha}\cdot\mathbf{p}-m\beta+Ze^2/r)\psi_1=-2^{-\frac{1}{2}}(2\pi)^{-1}e\sum\int\alpha_r\xi_{kr}u_m\exp(-iE_mx_0)\partial k, \quad (6)$$

where we have used the relation $\xi^*u_m=0$, there being no photons in the initial state. We solve[33] for ψ_1 by expanding in terms of the eigenfunctions of the unperturbed system, obtaining

$$\psi_1=-2^{-\frac{1}{2}}(2\pi)^{-1}e\sum\sum_p\int\{(E_m-E_p-k_0)^{-1}-i\pi\delta(E_m-E_p-k_0)\}$$

$$\times\left\{\int\int\int u_p^*\alpha_ru_me^{-i\mathbf{k}\cdot\mathbf{x}}dx_1dx_2dx_3\right\}\xi_{kr}u_p\exp[i(k_0-E_p)x_0]\partial k, \quad (7)$$

we take the solution which corresponds to outgoing waves. The factor

$$(E_m-E_p-k_0)^{-1}-i\pi\delta(E_m-E_p-k_0)$$

ensures that when we proceed to evaluate $\psi_{2,0}$, (after using the commutation relations for ξ and ξ^*), we obtain integrals of the form

$$\int_{-\infty}^{\infty}[(a-k_0)^{-1}-i\pi\delta(a-k_0)]k_0e^{-ibk_0}dk_0, \quad (8)$$

where a and b are independent of k_0, and this integral has the value zero. Hence $\psi_{2,0}$ is zero. It is easy to see that the integrals that arise in the higher approximation terms have also the same form as (8), and hence we have the general result

$$\psi_{n,m}=0 \quad \text{for } m\neq n. \quad (9)$$

Thus all the divergent terms are eliminated.

39. A Note on the Interaction of Two Moving Particles

The methods of the previous section cannot be applied if both the particles are moving. It would be very desirable to evaluate the interaction of moving particles, without having to use Born's approximation, but so far this has not been found possible. Although the Born approximation is sufficient for certain purposes, it is, however, unreliable for heavy particles and for low energies.

For the present we have to restrict the calculation to this approximation and so consider an expansion in ascending powers of e. The Coulomb interaction then appears as a perturbation of the second order. On this basis of approximation, the interaction has been evaluated by Moller,[34] correct to the terms of order e^4. From the work

[33] C. J. Eliezer, Bull. Calcutta Math. Soc. (to be published).

[34] W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, New York, 1936).

of Chapter V where all the terms of the type $\psi_{n,m}$, where m is not equal to n, were shown to vanish, one may hope that Moller's formula has a wide range of applicability.

40. The Compton Effect and Radiation Damping

The scattering cross section in the Compton effect, where an incident photon is scattered by a free electron with a change in the frequency of the photon, is given by the Klein-Nishina formula.[34] This formula is in excellent agreement with experiment even when the incident photon is of fairly high energy. The derivation of this formula does not take into account the effect of radiation damping. It is therefore of much interest to investigate the effect of radiation damping on this process, on the basis of Dirac's quantum electrodynamics.

In terms of the notation of Chapter V, the wave function from which the scattering cross section is calculated is

$$\psi = e^2\psi_{2,2} + e^4\psi_{4,2} + \cdots + e^{2n}\psi_{2n,2} + \cdots,$$

where the terms after the first give the effect of radiation damping. In the derivation of the Klein-Nishina formula, only the part $e^2\psi_{2,2}$ is used. According to the theory we have been considering, $\psi_{n,m}$ is zero when m is different from n, and it therefore follows that on the basis of this theory radiation damping has no effect on this process, indicating that the Klein-Nishina formula gives the exact cross section,

There is, however, the possibility of certain other terms contributing to the cross section and which should not be ignored. The wave function $e^4\psi_{4,4}$ will apply to a process in which four photons take part. If two of these have the same frequency and direction, and one of them is absorbed and the other is emitted, then one would have a process which is physically indistinguishable from one in which only two photons take part. Similar contributions would arise from terms of higher order also. Calculations of such contributions are in progress, but they are rather long and involved and not yet completed.

41. Multiple Processes

The wave functions obtained in Chapter V may be used to calculate the probability coefficients of certain multiple processes. These multiple processes are of interest for the following reasons. They are typical quantum effects and their probability cannot be estimated from the classical theory by using the correspondence principle. Also, the existence of showers in cosmic radiation makes it desirable to know if in photon showers the quanta are emitted simultaneously or one after another in a short range.

If we consider a process in which an incident photon is scattered into two photons in the presence of a free electron, the wave function that is applicable to such a process is

$$\psi = e^3\psi_{3,3} + e^5\psi_{5,3} + \cdots + e^{2n+1}\psi_{2n+1,3} + \cdots,$$

which in our theory becomes

$$\psi = e^3\psi_{3,3}.$$

We have to choose an appropriate form of ψ in order to have a probability with a physical meaning. The expression for probability so obtained will then be proportional to the time, and we may therefore define the probability per unit time. The diagonal sum of the matrix $\psi\phi$, where ϕ is the Hermitian conjugate of ψ, gives the density of electrons, which when divided by the initial density gives the transition probability. This has been calculated by the author.[35] It is found that if the emitted protons have frequencies of the same order of magnitude, the probability of the double scattering is at least 1/137 times smaller than that for the single scattering. But if one of the emitted photons is allowed to have low energy then the probability of the process could become large, the transition probability varying inversely as the frequency of the low energy photon. This result could be generalized to apply to any number of photons, and one would infer that showers in which the photons are emitted simultaneously are not disallowed by quantum electrodynamics.

The probability which is of physical significance in processes involving low energy photons would be obtained by integrating the probability expression over the low energy photons. Preliminary calculations done by the author seem to suggest that the integrated probability is not finite. If this be the case it would

[35] C. J. Eliezer, Proc. Roy. Soc. **A187**, 210 (1946).

imply that the theory given above is not free from the "infra-red catastrophe" of the previous theories. To deal with such problems one would, therefore, have to give up the expansion in powers of $e^2/\hbar c$, and for the present make use of the Bloch-Nordsieck transformation.[36]

42. Conclusion

This paper gives an account of some methods which have been developed recently in formulating a scheme of electrodynamics which is consistent with well-established principles of physics, such as the conservation laws and the special theory of relativity, and which is free from the divergence difficulties which are characteristic of theories of point particles. It is too premature to suggest that the above theory is satisfactory in all essential respects. It seems reasonable to suppose that those infinities which are of classical origin would be satisfactorily eliminated by the methods outlined above. Some of the purely quantum infinities would also be eliminated, and one may hope that the other difficulties which still remain may be solved by an appropriate extension of the above formalism. It appears to the author that an important line of advance would be to obtain solutions of the wave equation without making use of expansions in powers of the fine structure constant. Such solutions are likely to show closer correspondence between the classical and the quantum theories, and may pave the way for future development.

[36] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).