On a New Theory of Weizsäcker on the Origin of the Solar System

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1. INTRODUCTION

In a recent paper[1] C. F. v. Weizsäcker has proposed a new theory of the origin of the solar system which appears to merit consideration. Weizsäcker's principal idea is to regard the formation of a planetary system around a star as a possible last stage in the formation of the star itself. Before we describe in detail the manner in which Weizsäcker expects this to happen it may be of advantage to state briefly the picture he has in mind.

Weizsäcker imagines that stars are formed by a process of condensation from an interstellar medium in turbulent motion and that during the last stages the newly formed star finds itself surrounded by a cloud of gas with a definite angular momentum. Under certain circumstances this rotating mass of gas will be in the form of a disk with a definite structure (see Fig. 1). Such a disk is, however, expected to be unstable and in consequence dissipate into space. But, and this is the central part of Weizsäcker's theory, certain states of motion in the rotating disk of gas are relatively more stable than others and consist of what may be described as a "quantized arrangement of vortices"[2] (see Fig. 2). Finally it is made plausible that this arrangement of vortices may lead to the formation of planets along certain concentric circles the successive radii of which fall into a sequence of the form

$$r_n = r_0 \epsilon^n, \tag{1}$$

where r_0 and ϵ are constants. In other words Weizsäcker believes to have obtained a theoretical interpretation of Bode's law.

With this brief outline of the new theory we may pass on to a more detailed consideration of the various steps which are involved.

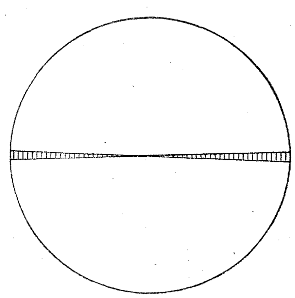

FIG. 1.

[1] C. F. v. Weizsäcker, Zeits. f. Astrophys. **22**, 319 (1944).
[2] To avoid possible misunderstanding, it may be stated that the arrangement arises from considerations which have nothing whatever to do with the quantum theory.

2. THE STRUCTURE OF A ROTATING MASS OF GAS AROUND A STAR

The equation governing the equilibrium of an atmosphere in rotation and under the gravitational influence of a central star of mass, M, can be written in the form

$$\text{grad}\left(\frac{GM}{r}\right)+\frac{1}{\rho}\,\text{grad}\left(\frac{k}{\mu H}\rho T\right)=\omega^2\mathbf{s}, \tag{2}$$

where ρ and T denote the density and temperature at a point distant r from the center, ω the angular velocity at the point under consideration, and $\mathbf{s}$ the position vector at right angles to the axis of rotation, and the rest of the symbols having their usual meanings. It may be noted that in writing Eq. (2) we have supposed that it is permissible to neglect the mass of the atmosphere in comparison with the mass of the central star. In integrating Eq. (2) Weizsäcker supposes that

$$T = ar^{-\frac{1}{2}} \quad (a = \text{constant}), \tag{3}$$

corresponding to the assumption of local thermodynamical equilibrium of the material with the radiation incident on it and that the reduction in the density of radiation by absorption can be neglected.

From Eqs. (2) and (3) it follows as in Poincare's theorem for rotating compressible masses that

$$\omega^2 = f(\Pi) r^{-\frac{1}{2}}, \tag{4}$$

where $f(\Pi)$ is a function of the distance Π from the axis of rotation only. Writing the foregoing equation in the form

$$\omega^2 = q(\Pi)(GM/\Pi^3)(\Pi/r)^{\frac{1}{2}}, \tag{5}$$

Weizsäcker assumes that

$$\left.\begin{aligned} q(\Pi) &= 1 \quad \Pi < \Pi_{max} \\ &= 0 \quad \Pi > \Pi_{max} \end{aligned}\right\}. \tag{6}$$

This assumption corresponds to supposing that, in the equatorial plane and for $\Pi < \Pi_{max}$, the material is "fully supported" by the combined action of the gravitational and centrifugal forces, while for $\Pi > \Pi_{max}$ we have an atmosphere under the sole gravitational influence of the central star. With these assumptions Eq. (2) can be integrated to give the law of density distribution

$$\left.\begin{aligned} \rho &= \rho_0\left(\frac{r}{\Pi_{max}}\right)^{\frac{1}{2}} \\ &\quad\times\exp\left[-\frac{2GM\mu H}{ak}\left(\frac{1}{\Pi^{\frac{1}{2}}}-\frac{1}{r^{\frac{1}{2}}}\right)\right] \quad \Pi < \Pi_{max}, \\ &= \rho_0\left(\frac{r}{\Pi_{max}}\right)^{\frac{1}{2}} \\ &\quad\times\exp\left[-\frac{2GM\mu H}{ak}\left(\frac{1}{\Pi_{max}^{\frac{1}{2}}}-\frac{1}{r^{\frac{1}{2}}}\right)\right] \quad \Pi > \Pi_{max}. \end{aligned}\right\} \tag{7}$$

According to Eq. (7), for $\Pi < \Pi_{max}$, we have an atmosphere in which the density remains constant on the equatorial plane and falls off exponentially at right angles to it, while for $\Pi > \Pi_{max}$, the density distribution is spherically symmetrical and falls off exponentially with increasing r.

At a distance $\Pi(<\Pi_{max})$ from the center the height, z, of the "homogeneous atmosphere" above the equatorial plane is to a sufficient approximation given by

$$z \simeq \Pi\left[2\Pi^{\frac{1}{2}}\left(\frac{ak}{GM\mu H}\right)\right]^{\frac{1}{2}}. \tag{8}$$

If we choose a in Eq. (2) by the condition that at $r = 10^{13}$ cm (i.e., at the distance of Venus), $T = 300°$K, and assume further that the mean molecular weight $\mu \simeq 4$ (corresponding to a mixture of hydrogen, helium, and oxygen in the proportion 5:4:1 by weight), we find that

$$z/\Pi \simeq 1/30, \tag{9}$$

at planetary distances.

In his further estimates Weizsäcker assumes that

$$\Pi_{max} = 4\times10^{14}\text{ cm} \quad \text{and} \quad \rho_0 = 10^{-9}\text{ g/cm}^3. \tag{10}$$

With these values the mass of the entire gaseous envelope amounts to about a tenth of the solar mass. The general picture of the rotating gaseous envelope is therefore as illustrated in Fig. 1.

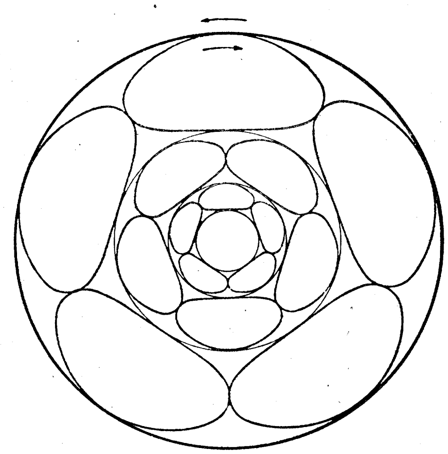

FIG. 2.

3. THE FORMATION AND DISSIPATION OF THE GASEOUS ENVELOPE

The question now arises as to whether the sun could have had surrounding it at any time a rotating disk of gas of the kind discussed in the preceding section. It does not exist today and it almost certainly would be unstable under the present interstellar conditions. On the other hand the present interstellar conditions do not suffice either for the formation of the sun itself by initial condensation and subsequent accretion in a time of the order of 10^9 years. We infer that the sun must have originally been formed under conditions of greater mean densities than those prevailing at present in interstellar space. If we suppose then that the sun was formed out of such a "primeval" nebula, it does not seem improbable that under those conditions the newly formed sun would find itself surrounded by a cloud of gas with a definite angular momentum.[3]

Regarding the nature of this primeval nebula during these early stages it would be difficult to say very much except that the material was probably in a state of turbulent motion with mean eddy velocities of the order of 25 km/sec. as indicated by the peculiar motions of stars.

Once a gaseous envelope of the kind described in Section 2 is formed, it cannot be expected to be a permanent configuration. For, according to the assumptions of Section 2, the angular velocity in the equatorial plane varies, and indeed according to Kepler's third law (cf. Eqs. (5) and (6)). And in consequence of this variation of ω, viscous stresses will come into operation and this, as is well known, will have a tendency to bring about a state of rotation with uniform angular velocity. In other words, the viscous stresses which would be set up will tend to decelerate the inner parts of the rotating disk of gas, and at the same time accelerate the outer parts. This in turn would entail mass motions in the envelope resulting in the outer parts of it dissipating into the space outside, and the inner parts of it falling on to the sun. The energy available for this process will of course be the gravitational energy released by the matter falling on the sun. Thus, if we suppose that a fraction α of the total mass of the enveloping material falls on the sun while the remaining fraction escapes to infinity, the available energy will clearly be (see Fig. 1)

$$\left.\begin{aligned} E \backsimeq & -\int_0^{\Pi_{\max}} \rho_0 \frac{GM}{2\Pi}\left(\frac{2z}{\Pi}\right)\Pi(2\pi\Pi)d\Pi \\ & +(1-\alpha)\int_{r_\odot}^{\Pi_{\max}} \rho_0 \frac{GM}{r_\odot}\left(\frac{2z}{\Pi}\right)\Pi(2\pi\Pi)d\Pi, \end{aligned}\right\} \quad (11)$$

where as in Eq. (8) z denotes the height of the "homogenous atmosphere" above the equatorial plane at Π. Further, in Eq. (11) $r_\odot$ denotes the radius of the sun. In Eq. (11) we can substitute for z/Π according to Eq. (9) and obtain after some elementary reductions that

$$E \simeq (2\pi/45)(1-\alpha)\rho_0 GM(\Pi_{\max}{}^3/r_\odot), \quad (12)$$

where use has been made of the fact that $\Pi_{\max}/r_\odot \gg 1$.

Now if the instability of the gaseous disk brings about a state of turbulence, energy will be transported by eddies and the resulting convective flux of energy must be expressible in the form

$$\text{convective flux} = \frac{\partial}{\partial\Pi}\left(-\rho_0\frac{GM}{2\Pi}\right)wl, \quad (13)$$

where w denotes the mean velocity and l the mean free path of the eddies. The transport of energy, S, per unit time across Π is, therefore

$$\left.\begin{aligned} S &= 2\pi\Pi \times 2z \times \frac{\partial}{\partial\Pi}\left(-\rho_0\frac{GM}{2\Pi}\right)wl, \\ &\backsimeq \frac{\pi}{15}\rho_0 GMwl, \end{aligned}\right\} \quad (14)$$

where use has again been made of the approximate relation (9).

Equations (12) and (14) imply a mean life, τ, for the gaseous disk of the order of

$$\tau \backsimeq \frac{E}{S} = \frac{2\Pi_{\max}{}^3(1-\alpha)}{3r_\odot wl}. \quad (15)$$

The fraction α in the foregoing equation can be estimated by the following considerations: At Π the material has an energy (kinetic+potential) $-GM/2\Pi$ and an angular momentum $\Pi\omega^2$ per unit mass where it might be recalled that ω denotes the angular velocity for a circular orbit through Π. If we suppose that a fraction α of the mass at Π escapes to infinity then it must acquire an angular velocity of at least $\sqrt{2}\omega$ corresponding to its receding along a parabolic orbit with its vertex at Π. Accordingly, if we suppose that the matter which escapes to infinity carries with it the entire angular momentum, it is clear that $\alpha \sim 1/\sqrt{2} = 0.71$.[4]

Returning to Eq. (15) we still need estimates

[3] Similar ideas have been expressed by B. M. Peek J. Brit. Astronom. Assn.

[4] In order that this process be energetically possible, it is necessary that

$$-\frac{GM}{2\Pi} + (1-\alpha)\frac{GM}{r_\odot} > 0,$$

or, alternatively

$$2(1-\alpha)\Pi > r_\odot.$$

With $\alpha = 1/\sqrt{2}$ the foregoing condition requires that $\Pi > 1.7r_\odot$; in other words what we have considered in the text can take place for all material outside $\Pi = 1.7r_\odot$.

for l and w. From general considerations we may expect the mean free path l to be of the order of a tenth of the linear dimension of the system in turbulent motion while the considerations to be set forth in Section 5 indicate values for w of the order of 1 km/sec. Using then the values

$$\left.\begin{array}{l}\alpha=0.7, \quad \Pi_{\max}=4\times 10^{14}\text{ cm}, \\ l=10^{13}\text{ cm}, \quad \text{and} \quad w=10^{5}\text{ cm/sec.}\end{array}\right\} \tag{16}$$

we find

$$\tau \simeq 10^{7}\text{ years.} \tag{17}$$

Now, so long as matter from interstellar space falls into the sun, the gaseous envelope around it will not disappear in spite of its instability. However when the addition of further mass ceases, the star will have attached to it a gaseous cloud and it is during these last stages that Weizsäcker anticipates the formation of a planetary system.

4. A QUASI-STABLE CONFIGURATION OF VORTICES

We have seen in the preceding section that a rotating disk of gas of the kind considered in Section 2 is unstable and that in consequence a state of turbulent motion is likely to be set up. But the question arises, whether in conformity with Kepler's laws we cannot construct a state of motions in which the viscous stresses would operate the least. If such a state could be found, we may expect the dissipation to be less for this state of motion than for others. We shall now show following Weizsäcker how such a state of quasi-stability can be found.

Consider first two particles (elements of gas in the present context) one of which is describing a circular orbit with radius Π_0 and the other a nearly circular orbit, both with the same constant of areas. If we arrange that at time $t=0$, the two particles are along the same radius vector with a separation Π_{10}, and further refer the particle describing the nearly circular orbit to the particle describing the circular orbit (i.e., refer the nearly circular orbit to a frame of reference rotating with an angular velocity ω_c), then it is well known that second particle describes an ellipse centered on the first.[5] In fact if X and Y denote the Cartesian coordinates in the rotating frame of reference and along the radial and the transverse directions, respectively, then (Chandrasekhar, reference 5, Eqs. (4.333)–(4.337))

$$X=\Pi_{10}\cos n_1 t; \quad Y=-2\frac{\Theta_c}{\Pi_0 n_1}\sin n_1 t, \tag{18}$$

where Θ_c denotes the rotational velocity in the circular orbit through Π_0 and

$$n_1^2=2\frac{\Theta_c}{\Pi_0}\left(\frac{\partial\Theta_c}{\partial\Pi}+\frac{\Theta_c}{\Pi}\right)_{\Pi=\Pi_0}. \tag{19}$$

Accordingly

$$X^2+\frac{\Pi_0}{2\Theta_c}\left(\frac{\partial\Theta_c}{\partial\Pi}+\frac{\Theta_c}{\Pi}\right)_{\Pi=\Pi_0}Y^2=\Pi_{10}^2. \tag{20}$$

For an inverse square field of force,

$$\Theta_0=\frac{GM}{\Pi_0} \quad \text{and} \quad \frac{\Pi_0}{2\Theta_c}\left(\frac{\partial\Theta_c}{\partial\Pi}+\frac{\Theta_c}{\Pi}\right)_{\Pi=\Pi_0}=\frac{1}{4}. \tag{21}$$

The ellipse (20) is therefore one with a ratio of the axes 2, the major axes being in the transverse direction. Moreover, according to Eq. (18) the ellipse (20) is described in a sense contrary to that of Θ_c.

It is now apparent that if we consider a family of nearly circular orbits, all with the same constant of areas[6] (and therefore with slightly differing eccentricities) then in the rotating frame of reference, we have a set of *similar* elliptical orbits all described with the same period and about a common center. For such a state of motion no viscous stresses can operate. However this will cease to be true when it should become necessary to take into account second-order terms to describe the nearly circular orbits, i.e., when the deviations from a circular orbit become appreciable. Under these circumstances the orbits in the rotating frame of reference will also cease to be strict ellipses though they will continue to be closed curves so long as we consider particles with the same energy. Actually, orbits which deviate substantially from circular orbits describe in the rotating frame of reference orbits such as those shown in Fig. 2. In

[5] Cf. S. Chandrasekhar, *Principles of Stellar Dynamics* Chicago, 1942), pp. 154–160.

[6] Weizsäcker considers particles with the same energy. But in the first order the two assumptions are equivalent.

any case it is clear that so long as we consider nearly circular orbits, we can construct motions of the nature of a *vortex cell* in which the viscous dissipation will be appreciable—if at all—only in the outer most parts of the cell. Consequently an arrangement of vortex cells such as that shown in Fig. 2 has a certain stability. For, with the arrangement as shown, a particle belonging to any one vortex never encounters directly a particle belonging to any other vortex. Each vortex continues its motion independent of the others and interfering with none.

We may next ask as to what governs the size of a vortex cell. Weizsäcker argues in the following manner: If we consider a very small vortex cell, the particles with the maximum eccentricity which occurs in it and which will be describing the outermost orbit in the rotating frame of reference will be characterized by a small eccentricity. If such a vortex should encounter another particle (with, say, the same energy) the chances are that its eccentricity will be greater than the maximum present in the vortex. It is therefore possible for the intruder to be captured: for, it can "attach" itself to the vortex without dearranging the already existing arrangement since on account of its greater eccentricity it will describe an ellipse outside all the others. A small vortex will accordingly have a tendency to grow. But it cannot grow indefinitely, for once it has grown to a certain size it would more often encounter particles with eccentricities less than the maximum already present, than otherwise, and the capture of such particles would only tend to a disruption of the existing structure. Weizsäcker therefore concludes that the vortices of the kind we have been considering will grow to an optimum size characterized by a certain maximum eccentricity $e_{\max}$ which will occur in it. (Later considerations (Section 5 below) suggest that $e_{\max}\sim\frac{1}{3}$.) It would further appear that, along any given circle, we will have an integral number of vortices in near contact. The condition that there be an integral number of vortices along a circle would further limit the possible values of $e_{\max}$ to a certain discrete sequence. Such an arrangement is shown in Fig. 2 where there are five cells in each ring. For this arrangement $e_{\max}=0.31$.

It would appear then that an arrangement of cells which may be expected to materialize is one which would consist of an integral number of cells in each ring and such that the common aphelion distance for the particles in the inner ring is very nearly equal to perihelion distance for the particles of the outer ring. If each ring be assumed to contain the same number of cells, then it is not difficult to find the ratio of the radii, r_n/r_{n-1}, of the series of concentric circles defining the various rings. For,

$$\left.\begin{aligned}\frac{r_n}{r_{n-1}}&=\frac{\text{Aphelion distance for a particle with } e=e_{\max}}{\text{Perihelion distance for the same particle}},\\&=\frac{1+e_{\max}}{1-e_{\max}}.\end{aligned}\right\}\tag{22}$$

Hence,

$$r_n=r_0\left(\frac{1+e_{\max}}{1-e_{\max}}\right)^n.\tag{23}$$

For the case illustrated in Fig. 2 (five cells in a ring)

$$e_{\max}=0.307,\tag{24}$$

and the law (23) becomes

$$r_n=r_0(1.894)^n.\tag{25}$$

5. ON A THEORETICAL INTERPRETATION OF BODE'S LAW

While the arrangement of the vortices described in the preceding section has been designed primarily with a view to minimizing the dissipation due to viscous forces, it appears that, whatever dissipation there is, will occur principally along the circles r_n where the vortices in one ring pass those in another. At r_n for example, the particles belonging to the vortices of the inner ring are at their aphelion while the particles belonging to the vortices of the outer ring are at perihelion. Accordingly, in the fixed frame of reference, the velocities (tangential to the circle of radius r_n) of the particles belonging, respectively, to the inner and the outer rings are

$$\left.\begin{aligned} v_{\text{inner}} &= \left(\frac{GM}{r_n^3}\right)^{\frac{1}{2}}(1-e_{\max})^{\frac{1}{2}}, \\ v_{\text{outer}} &= \left(\frac{GM}{r_n^3}\right)^{\frac{1}{2}}(1+e_{\max})^{\frac{1}{2}}, \end{aligned}\right\}. \tag{26}$$

For $e_{\max}=\frac{1}{3}$ and at the distance of the earth, the difference between v_{inner} and v_{outer} amounts to as much as 10 km/sec.; while for the same value of $e_{\max}$ the difference amounts to 3 km/sec. at the distance of Jupiter. This difference of velocities along the circles of contact r_n is therefore quite considerable and turbulence is to be expected. *Secondary eddies* will therefore form along these circles and since $v_{\text{outer}}>v_{\text{inner}}$, the sense of rotation of these secondary eddies will be contrary to that of the principal vortex cells and accordingly in the same sense as that of Θ_c. These secondary eddies (in contrast to the small vortices considered in Section 4) will have no tendency to grow since the direction of rotation of these eddies is in the same sense as Θ_c.

It is thus seen that with the arrangement of vortices as shown in Fig. 2, dissipation is localized along the circles r_n and Weizsäcker anticipates that here, if anywhere, are the chances most favorable for the formation of planetary bodies (see Section 6 below). If this be accepted, we have a means of interpreting the law of planetary distances. We now turn our attention to this matter.

As is well known, Bode's law of planetary distances can be expressed in the form

$$r_n=a+2^n b, \tag{27}$$

with $a=0.4$ and $b=0.3$ when r_n is measured in units of the distance of the earth from the sun. For the outer planets a can be ignored and we have

$$r_n \simeq b\cdot 2^n, \tag{28}$$

which is the form Eq. (23) will have for $e_{\max}=\frac{1}{3}$. It is therefore remarkable that this value of $e_{\max}$ should be so near to a value, namely 0.31, required for an integral number of vortices in a given ring. However, as Weizsäcker goes on to show, the formula (25) actually provides a better representation of the distances of Mars, the average of the asteroids, Jupiter, Saturn, and Uranus than does even Bode's law. But the point at issue here is clearly, not that Eq. (25) provides a better interpolation formula than Bode's law; it is, rather, that a physical interpretation of Bode's law has become possible. Indeed Weizsäcker regards this as the greatest argument, *a priori*, for his theory.

6. ON THE FORMATION OF THE PLANETS

We next turn to the question of how probable it is for planets to be formed along the circles where secondary eddies are being created by the viscous shear between the successive rings of vortices.

It is not difficult to grant that at the low temperatures at which most of the gaseous envelope finds itself, we would already have fine dust particles or tiny droplets of colloidal or sub-colloidal sizes. But it is more difficult to visualize how such particles can coalesce to form larger particles and eventually particles of planetary dimensions. A full discussion of this problem is obviously a matter of considerable complexity but Weizsäcker puts forward the following tentative considerations to estimate the orders of magnitude involved.

When dust particles of similar or comparable sizes collide, the chances are that they fragment each other into still finer pieces. But when two particles of widely different sizes collide, it is likely that the larger particle increases its mass since on account of its larger presentation area the fragmentation pieces of the smaller particle will adhere to it. In other words, we may expect that it is only the largest particles that have a chance to grow and since in an assembly of n particles the largest may have sizes $\log n$ times the average, it is not improbable that particles of the size necessary for growth are already present. We assume, then, the existence of such particles and find that they will grow, in the first instance, by straightforward capture of the much smaller average sized particles. The cross section for such captures will simply be the average presentation area of the particle. However, once the particle has grown to an extent that its *gravitational radius*

$$R=2Gm/v_g^2 \tag{29}$$

(where m denotes the mass of the particle, and v_g the root mean square velocity of the gaseous *elements* relative to it) becomes comparable to its linear dimensions, the particle will begin to increase its mass more rapidly on account of gravitational attraction. For such gravitational accretion, we may expect a radius of capture $R\delta$ where δ is a factor of order $\frac{1}{10}$.[7] It appears, then, that we can distinguish two stages in the growth of a particle: the stage of direct capture and the stage of gravitational capture. Assuming for the sake of simplicity that the particles are spherical, capture by gravitational attraction will set in when (cf., Eq. (29))

$$r=(2G/v_g^2)((4/3)\pi r^3\rho_p)\delta, \tag{30}$$

where ρ_p denotes the density of the particle and r its radius. Accordingly the radius r_* at which direct capture gives place to gravitational capture is given by

$$r_*=\left(\frac{3}{8\pi\delta G\rho_p}\right)^{\frac{1}{2}} v_g. \tag{31}$$

On the ideas outlined in the preceding paragraph it is not difficult to follow the growth of a particle. Again, restricting ourselves to spherical particles, the equation of growth during the stage of direct capture is

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\rho_p\right)=\pi r^2 v\rho_d, \tag{32}$$

where ρ_d is the mass per unit volume of the *medium* which is in the form of fine dust particles, and v is the mean relative speed between the larger particle and the fine dust particles. From Eq. (32) we obtain

$$r-r_0=\tfrac{1}{4}(\rho_d/\rho_p)vt, \tag{33}$$

where r_0 is the initial radius of the particle at time $t=0$. From Eq. (33) we conclude that the time, t_*, required for the particle to grow to the size r_* is

$$t_*\simeq\frac{4\rho_p}{\rho_d}\frac{r_*}{v}=\left(\frac{6\rho_p}{\pi\delta G}\right)^{\frac{1}{2}}\frac{v_g}{\rho_d v}. \tag{34}$$

After this time the particle will begin to grow by gravitational capture and the corresponding equation of growth can be written as (cf., Eq. (29))

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{4}{3}\pi r^3\rho_p\right)&=\pi\delta^2R^2\rho_d v,\\ &=\frac{64\pi^3\delta^2}{9v_g^4}G^2\rho_p^2\rho_d r^6 v,\end{aligned}\right\} \tag{35}$$

or,

$$\frac{1}{r^4}\frac{dr}{dt}=\frac{16\pi^2\delta^2}{9}\frac{v}{v_g^4}G^2\rho_p\rho_d. \tag{36}$$

Integrating this equation we have

$$r^3=\frac{3v_g^4}{16\pi^2\delta^2G^2\rho_p\rho_d v}\frac{1}{(T-t)}, \tag{37}$$

where T is a constant of integration to be determined by the condition that at time $t=0$, $r=r_*$. Accordingly,

$$T=\frac{3v_g^4}{16\pi^2\delta^2G^2\rho_p\rho_d v}\frac{1}{r_*^3}, \tag{38}$$

or, substituting for r_* from Eq. (31), we have (cf., Eq. (34))

$$T=\left(\frac{2\rho_p}{3\pi\delta G}\right)^{\frac{1}{2}}\frac{v_g}{\rho_d v}=\frac{1}{3}t_*. \tag{39}$$

But according to Eq. (37) $r\to\infty$ as $t\to T$. We may therefore expect, on the basis of these ideas, that in a time of the order of

$$t_\infty=\frac{4}{3}t_*=\left(\frac{32\rho_p}{3\pi\delta G}\right)^{\frac{1}{2}}\frac{v_g}{\rho_d v}, \tag{40}$$

bodies of planetary dimensions could be formed.[8] With the values

$$\rho_p=3\text{ g/cm}^3,\quad\text{and}\quad \rho_d=10^{-11}\text{ g/cm}^3, \tag{41}$$

[7] For a comprehensive theory of capture by gravitational attraction see W. C. H. Eakin and W. H. McCrea, Proc. Roy. Irish. Acad. **46**, 91 (1940).

[8] It may be noted that our formula (40) for t_∞ does not agree with the formula given by Weizsäcker. For $\delta=0.1$ the difference amounts to a factor 5 with Weizsäcker over-estimating the time.

Eq. (40) becomes

$$t_\infty = 4.2\times10^7 \frac{1}{(\delta)^{\frac{1}{2}}}\frac{v_g}{v} \text{ years.} \tag{42}$$

In order that we may use the foregoing formula to estimate t_∞, we still require to know the relevant values for the relative motions v_g and v. If these velocities are simply those to be expected in consequence of thermal motions only, the values of t_∞ which would follow are so large as in fact to rule out the present theory. But if instead we suppose that the gaseous medium in which the process of planetary formation is believed to be taking place is in a state of turbulence and that, further, the fine dust particles are being carried with eddies, we may expect values for v_g and v of the order of kilometers per second, and the resulting values for t_∞ will not be incompatible with the estimated mean life of the order of 10^7 years (Eq. (17)) for the rotating disk of gas. But before we can make these assumptions, we must verify if, under the conditions envisaged, the dust particles will in fact be carried with the eddies and thus share in the turbulence. In other words, we have to determine whether or not the mean free path, λ, of the dust particles is comparable to the size of the eddies as indicated by *their* mean free path l.

We can estimate the mean free path, λ, by the following elementary considerations: During a time t, a dust particle of radius r will sweep out an amount of gas of mass $\pi r^2\rho_g wt$, where ρ_g denotes the density of the gas in the medium and w the mean relative velocity between the dust particle and the gas. This mass of gas, $\pi r^2\rho_g wt$, will on the average be endowed with an additional velocity of the order w so that the work done by the particle during a time t will be of the order $\pi r^2\rho_g w^2 t$. And if this should become comparable to the initial momentum of the dust particle, namely $(4/3)\pi r^3\rho w$ (where ρ is the density of the particle), we may expect that the particle has suffered substantial changes in the direction and magnitude of its motion. The "time of relaxation" is therefore

$$\text{time of relaxation} \simeq \frac{\frac{4}{3}\pi r^3\rho w}{\pi r^2\rho_g w^2} = \frac{4}{3}\frac{\rho r}{\rho_g w}. \tag{43}$$

Accordingly for the mean free path, λ, we may write

$$\lambda = \text{time of relaxation} \times w = \frac{4}{3}\frac{\rho}{\rho_g} r. \tag{44}$$

With

$$\rho = 3\text{ g/cm}^3 \quad\text{and}\quad \rho_g = 10^{-9}\text{ g/cm}^3, \tag{45}$$

we have

$$\lambda \simeq 4\times10^9 r. \tag{46}$$

In order then that particles of radius r be carried with eddies of size l, and share in the turbulence, it is necessary that

$$l \geq 4\times10^9 r. \tag{47}$$

For fine dust particles of radii $10^{-6}-10^{-5}$ cm, $l\sim10^4-10^5$ cm. But particles which have grown into bodies of any reasonable size ($r>10^4$ cm), cannot be carried along with even the largest of the vortices present.

Now, as we have already stated in Sections 4 and 5, in the regions where the vortex cells in the successive rings pass each other, we may anticipate the formation of a large number of smaller secondary eddies—"lubrication eddies" as Weizsäcker calls them. It is hardly likely that the secondary eddies will have mean free paths greater than 10^4–10^5 cm. Thus, while the secondary eddies may be large enough to carry along with them the fine dust particles they will be unable even to "move" any of the particles which have grown appreciably beyond the colloidal size. And moreover, since in the regions where the secondary eddies are being formed, shearing velocities of the order of 3–10 km/sec. are present (cf. Eq. (26)) we may suppose that under these circumstances

$$v_g \sim v \sim 3\text{ km/sec.} \tag{48}$$

With these values and with $\delta = 0.1$, Eq. (42) yields

$$t_\infty = 10^8 \text{ years.} \tag{49}$$

It would appear, then, that the mechanism suggested is adequate for the building up of bodies of planetary dimensions though it might seem at first sight that t_∞ is rather too large with a mean life of the order of 10^7 years estimated for the gaseous envelope (Eq. (17)). But as Weizsäcker points out, one might adduce empirical evidence for believing that when the formation of the planets is approaching the stage of completion, most of the material in the envelope has already been dissipated. The reasoning is this: The elements which constitute more than half the weight of the earth, for example, are higher up in the periodic table than oxygen, while in the sun the elements hydrogen, helium, and oxygen alone constitute probably as much as 99 percent of its total weight. This difference in the constitution of the sun and the planets generally can perhaps be understood if, when the planetary bodies have grown to a size when they can retain an atmosphere of their own, most of the gases have already escaped. In other words, the difference of a factor 10 in t_∞ and τ may just exactly be what is needed to account for the relative poverty of the planets in the lighter gaseous elements.

Since our discussion of the formation of the planets depends so much on the "quantized" arrangement of the vortices described in Section 4, we might ask the following questions: How stable really is the arrangement of vortices described in Section 4? How long can we expect such a structure to continue its existence? And what is it which determines the scale of the arrangement (i.e., r_0 in Eq. (23))? It is difficult to answer these questions, but as Weizsäcker points out, for our purposes it is strictly not necessary that the arrangement of the vortices (essential as it is for the interpretation of the law of planetary distances) be stable for a period of the order of 10^6 or 10^7 years. For, once such a system of vortices comes into existence, in the regions where the secondary eddies are being formed particles of sizes sufficient not to be carried along with the largest of the eddies present will be formed in a period of a ten or a hundred years. And once such large bodies have been formed the system of the vortices can gradually disperse without any substantial changes in the picture which has been presented. But it is essential for Weizsäcker's theory that the quasi-stable arrangement of the vortices come once into being and continue to exist for a period of the order of a hundred years at least in order that we may account for Bode's law. Conversely, the interpretation of Bode's law which Weizsäcker's theory makes possible may be taken as an argument *a posteriori* for the coming into existence at one time of such a system of vortices.

7. CONCLUDING REMARKS

We may conclude by referring to certain related matters.

First, regarding the sense of rotation of the planets: As has been pointed out, the sense of rotation of the secondary eddies is contrary to that of the larger quasi-stable vortex cells and accordingly is in the same sense as the rotation of the gaseous envelope. And if, as has been suggested, the planets are formed in the regions of the secondary eddies, it would follow that the sense of rotation of the planets will be in the same sense as that of the secondary eddies and therefore in the sense of the orbital rotation. This is in agreement with observation.

Second, as regards the formation of the satellite systems, it is not difficult to imagine that the whole complex of events which we have considered in connection with the formation of the planetary system around the sun could take place, on a naturally reduced scale, with respect to each of the planets. For the planets are also formed in a turbulent medium and during the last stages of their formation they may be also surrounded by clouds of gas in rotation about them. And the dynamics of such clouds of gas around a planet may be expected to have all the qualitative features which we have encountered in connection with a cloud surrounding the sun. However, we may expect that the quasi-stable arrangement of vortices in a gaseous disk rotating about the planets may not have the same "perfection" we may expect around the sun. This may account for the "irregularities" in the satellite systems such as occasionally retrograde orbits, high inclination of the orbits, and the not too strict a form of Bode's law which is valid for the satellite distances.

Finally, it is clear that, if the broad outlines of Weizsäcker's picture of planetary formation should be substantiated, the underlying ideas will have applications to a large number of other problems such as the origin of the meteorites, double star formations, and the like.