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The Elastic Constants of Anisotropic Materials

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A LIST OF THE PRINCIPAL SYMBOLS USED

Symbol	*Definition*
b	Breadth of specimen of rectangular cross section.
c_{ik}, c'_{ik} $(i, k=1, 2, \cdots 6)$	Elastic parameters in the generalized Hooke's law.
C	Bending moment.
d	Depth of specimen of rectangular cross section.
E	Young's modulus.
F	Orientation function (p. 426).
I	Moment of inertia.
G	Rigidity modulus.
J	Moment of inertia of cross section.
l	Length of specimen.
M	Mass attached to specimen.
M_0	Mass of specimen.
N	Twisting couple.
P	Applied load.
r	Radius of specimen of circular cross section.
R	Radius of curvature in bending.
s_{ik}, s'_{ik} $(i, k=1, 2, \cdots 6)$	Elastic coefficients in the generalized Hooke's law.
s'_g	Inverse free rigidity modulus of an anisotropic specimen of circular cross section.
W	Potential energy of strain.
x, y, z, x', y', z'	Coordinate axes.
x_x, y_y, z_z	Extensional strains.
x_y, y_z, z_x	Shear strains.
X_X, Y_Y, Z_Z	Extensional stresses.
X_Y, Y_Z, Z_X	Shear stresses.
$\alpha_i, \beta_i, \gamma_i$ $(i=1, 2, 3)$	Direction cosines.
ϵ	$=s'^2_{34}/2s'_{33}s'_g$ for a specimen of circular cross section (p. 417).
ϕ	Total angle of twist in torsion.
ν	Frequency.
ρ	Density.

INTRODUCTION AND SUMMARY

THE present paper attempts to review the theoretical and experimental points of interest connected with the elastic constants of anisotropic materials (mainly crystals). Attention is confined exclusively to the "elastic" range, and such topics as ultimate strength and plastic slip and glide are automatically excluded. The aim has been to make the account complete to the end of 1944, but, owing to obvious difficulties imposed by the non-availability of certain periodicals, it is possible that the period 1939–1944 is not fully covered. The originals of the most important papers have been seen, but others are quoted from "Science Abstracts" (since 1941—"Physics Abstracts") of which journal free use has been made throughout.

The elastic behavior of a completely asymmetric material is specified by 21 independent elastic constants, while for an isotropic material, the number is 2. In between these limits the necessary number is determined by the symmetry of the material, and is fundamental to the study of the elastic constants of crystals. Hardly less important is the effect of rotating the axes on the elastic constants; the main mathematical results relating to these two problems are summarized in Section 1.

Sections 2–6 deal with the methods used to measure the elastic constants. They contain a discussion both of the static methods used by Voigt and others, and of the more recently developed dynamic methods, which may involve measurements at sub-sonic, sonic, and supersonic frequencies.

The experimental methods commonly used to measure the elastic constants of anisotropic materials do not, in general, differ in principle from those applicable to isotropic materials. These methods are well known, and details of experimental technique are therefore only mentioned, often incidentally, when they are of special interest.

Although the experimental variables it is necessary to measure are as a rule the same, whether the material is isotropic or anisotropic, the method of calculating the elastic constants from these variables often depends on the nature of the material. In particular, when working on specimens of arbitrary orientation, it is necessary to allow for the coupling which may exist between shear and extension; this effect and the associated corrections are discussed in Sections 3–5.

Sections 7 and 8 give the results of determinations at ordinary temperatures and discuss the main points of interest shown by them. The table of numerical values (Table V) is believed to be the most complete so far assembled.

The effect of temperature is dealt with in Section 9. This work has yielded results of considerable importance and interest, notably in the connection which has been shown to exist between the temperature variation of elastic constants and the order-disorder transition in alloys.

The investigation of the elastic constants of crystals has proved of significance in other branches of physics, for example in the study and development of quartz crystal oscillators, in the interpretation of diffuse x-ray scattering, and in the development of the quantum theory of the crystal lattice. These and other applications are very briefly summarized in Section 10.

1. Resumé of Mathematical Theory

In its most general form, Hooke's law, referred to a Cartesian coordinate system *Ox*, *Oy*, *Oz*, may be written [Love (73) pp. 99 etc.; Voigt (115) p. 568]:

$$\left.\begin{aligned}
X_X &= c_{11}x_x + c_{12}y_y + c_{13}z_z + c_{14}y_z + c_{15}z_x + c_{16}x_y;\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
Y_Z &= c_{41}x_x + c_{42}y_y + c_{43}z_z + c_{44}y_z + c_{45}z_x + c_{46}x_y;\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\tag{1.1}$$

where $X_X \cdots Y_Z \cdots$ are stress components, $x_x \cdots y_z$ are strain components, and the c_{ik} are elastic parameters.[1]

The six components of strain may similarly be written in terms of the six components of stress:

$$\left.\begin{aligned}
x_x &= s_{11}X_X + s_{12}Y_Y + s_{13}Z_Z + s_{14}Y_Z\\
&\qquad + s_{15}Z_X + s_{16}X_Y;\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
y_z &= s_{41}X_X + s_{42}Y_Y + s_{43}Z_Z + s_{44}Y_Z\\
&\qquad + s_{45}Z_X + s_{46}X_Y;\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\tag{1.2}$$

where the s_{ik} are the elastic coefficients. In these equations, a suffix identical with the principal symbol (e.g., x_x, X_X) implies extension (or compression) while a suffix different from the principal symbol (e.g., y_z, Y_Z), implies shear. The quantities c_{ik} and s_{ik} are constants characteristic of the material; the difference between them may be seen by considering, for example, c_{11} and s_{11}.

If, in (1.1), all the strains except x_x are made zero, then, in general, all six stresses exist. Thus

$$c_{11} = X_X/x_x, \tag{1.3}$$

when the stress system is such that the only strain is x_x. Similarly, if in (1.2) all the stresses except X_X are made zero, then all six strains exist and

$$s_{11} = x_x/X_X, \tag{1.4}$$

when the only stress is X_X.

It would be difficult, if not impossible, to realize experimentally the conditions under which (1.3) applies, but it is comparatively simple to realize the conditions of (1.4). In fact, for the particular case considered,

$$s_{11} = 1/E_X, \tag{1.5}$$

where E_X is the Young's modulus in the x direction.

In view of this direct connection between the usual experimental tests and the elastic coefficients s_{ik}, numerical values quoted below refer, as a rule, to s_{ik} and not to the elastic parameters

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[1] The terminology of the quantities specifying the elastic properties of anisotropic materials is very confused and ill defined, in spite of various attempts which have been made to rationalize it [see Love (73), Note A, p. 614]. The most frequent usage appears to be that originated by Voigt (115) according to which the quantities s_{ik} are known as "moduli" (*moduln*) and the quantities c_{ik} as "constants" (*konstanten*). In English, at least, both of these words are open to objection since they have a usage which is well accepted, and does not convey the special meaning attached to s_{ik} and c_{ik}; cf. Young's "modulus," rigidity "modulus," both of which are elastic "constants." In the present paper, the quantities s_{ik} are called "coefficients," and the quantities c_{ik} "parameters." The word "modulus" is reserved for such use as Young's and rigidity modulus, while "constant" is used as a general term covering any of the quantities defining the elasticity of solid bodies and includes moduli, coefficients and parameters. *Editor's Note*: An I.R.E. committee has recently defined the c_{ik} constants as moduli of elasticity and the s_{ik} constants as moduli of compliance.

c_{ik}, although in some cases (e.g., in the elastic theory of plates) the latter are more directly involved.

It is always possible to obtain one set of quantities from the other by the relations [Love (73) p. 106].

$$c_{ik}=\Delta s_{ki}/\Delta s;\quad s_{ik}=\Delta c_{ki}/\Delta c;\quad \Delta c\Delta s=1,$$

where

$$\Delta s=\begin{vmatrix} s_{11}\cdots s_{14}\cdots \\ \vdots \\ s_{41} \\ \vdots \end{vmatrix};\quad \Delta c=\begin{vmatrix} c_{11}\cdots c_{14}\cdots \\ \vdots \\ c_{41} \\ \vdots \end{vmatrix};\quad (1.6)$$

$\Delta s_{ki}=$ minor determinant corresponding with s_{ki};
$\Delta c_{ki}=$ minor determinant corresponding with c_{ki}.
The general properties of s_{ik} are:

(a) $$s_{ik}=s_{ki}. \quad (1.7)$$

These equations reduce the maximum number of independent coefficients from 36 to 21.
(b) $i=k$ (neither >3); s_{ik} relates an extensional stress to a collinear extensional strain,

$$\therefore\quad s_{ik}=1/E_k. \quad (1.8)$$

(c) $i\neq k$ (neither >3); s_{ik} relates an extensional strain to a perpendicular compressional stress (or *vice versa*),

$$\therefore\quad s_{ik}=-\sigma_{ik}/E_i=-\sigma_{ki}/E_k, \quad (1.9)$$

where σ_{ik} is the Poisson's ratio

$$\frac{\text{extensional strain in } k \text{ direction}}{\text{compressive strain in } i \text{ direction}} \text{ for compressive stress in } i \text{ direction.}$$

(d) $i\ngtr 3$; $k>3$; s_{ik} relates an extensional strain to a shear stress.
(e) $i=k$ (each >3), s_{ik} relates a shear strain to a shear stress in the same plane

$$\therefore\quad s_{ik}=1/G, \quad (1.10)$$

where G is a rigidity modulus.
(f) $i\neq k$ (each >3); s_{ik} relates a shear strain to a shear stress in a perpendicular plane.
Suppose now the axes x, y, z are rotated to x', y', z' where the rotation is governed by the scheme of direction cosines:

$$\left.\begin{array}{cccc} & x' & y' & z' \\ x & \alpha_1 & \beta_1 & \gamma_1 \\ y & \alpha_2 & \beta_2 & \gamma_2 \\ z & \alpha_3 & \beta_3 & \gamma_3 \end{array}\right\}. \quad (1.11)$$

Since each set of axes is rectangular, 12 relations of the types:

$$\left.\begin{array}{c} \alpha_1^2+\alpha_2^2+\alpha_3^2=1; \\ \cdots\cdots \\ \alpha_1^2+\beta_1^2+\gamma_1^2=1; \\ \cdots\cdots \\ \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=0; \\ \cdots\cdots \\ \alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3=0, \end{array}\right\} \quad (1.12)$$

must hold among the direction cosines.

The potential energy of strain W is given by

$$\begin{aligned} 2W=s_{11}X_X^2+2s_{12}X_XY_Y+\cdots \\ +2s_{14}X_XY_Z+\cdots+s_{44}Y_Z^2+\cdots \\ +2s_{45}Y_ZZ_X+s_{66}X_Y^2, \quad (1.13) \end{aligned}$$

and the effect of rotation of axes on the elastic constants is obtained from the principle that W is independent of the rotation, i.e.,

$$\begin{aligned} s_{11}X_X^2+2s_{12}X_XY_Y+\cdots \\ =s_{11}'X'^2_{X'}+2s_{12}'X'_{X'}Y'_{Y'}+\cdots \end{aligned}$$

It is convenient to deal with the problem using tensor notation (15, 115, 121),[2] but this method will not be described here. Some typical results may, however, be quoted:

$$\begin{aligned} s'_{33}=s_{11}\gamma_1^4+\cdots+(2s_{23}+s_{44})\gamma_2^2\gamma_3^2+\cdots \\ +2(s_{56}+s_{14})\gamma_1^2\gamma_2\gamma_3+\cdots \\ +2\gamma_1^3(s_{15}\gamma_3+s_{16}\gamma_2)+\cdots, \quad (1.14) \end{aligned}$$

$$\begin{aligned} s'_{44}=4(s_{11}\beta_1^2\gamma_1^2+\cdots)+8(s_{23}\beta_2\gamma_2\beta_3\gamma_3+\cdots) \\ +s_{44}(\beta_2\gamma_3+\gamma_2\beta_3)^2+\cdots \\ +2s_{56}(\beta_1\gamma_2+\gamma_1\beta_2)(\beta_1\gamma_3+\gamma_1\beta_3)+\cdots \\ +4s_{14}\beta_1\gamma_1(\beta_2\gamma_3+\beta_3\gamma_2)+\cdots \\ +4(s_{15}[\beta_1\gamma_3+\gamma_1\beta_3] \\ +s_{16}[\beta_1\gamma_2+\gamma_1\beta_2])\beta_1\gamma_1. \quad (1.15) \end{aligned}$$

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[2] Wooster (121) uses a slightly modified notation for certain of the coefficients as compared with the present one, which is the same as that of Voigt (115). Wooster's notation is such that if $s_{ik(v)}$ denotes the coefficients as defined by Voigt and $s_{ik(w)}$ those defined by Wooster, then

$$\begin{aligned} s_{ik(v)}&=s_{ik(w)} \text{ when both } i \text{ and } k \text{ are 1, 2, or 3} \\ &=2s_{ik(w)} \text{ when either } i \text{ or } k \text{ is 4, 5, or 6} \\ &=4s_{ik(w)} \text{ when both } i \text{ and } k \text{ are 4, 5, or 6.} \end{aligned}$$

The coefficients c_{ik} are defined in the same way by both writers.

(s'_{55} is obtained by writing α for β throughout (1.15).)

$$
\begin{aligned}
\tfrac{1}{2}s'_{34}=s_{11}\beta_1\gamma_1{}^3+\cdots&\\
&+(s_{23}+\tfrac{1}{2}s_{44})(\beta_2\gamma_2\gamma_3{}^2+\beta_3\gamma_3\gamma_2{}^2)+\cdots\\
&+(s_{14}+s_{56})(\beta_1\gamma_1\gamma_2\gamma_3\\
&\qquad+\tfrac{1}{2}\beta_2\gamma_1{}^2\gamma_3+\tfrac{1}{2}\beta_3\gamma_1{}^2\gamma_2)+\cdots\\
&+\tfrac{1}{2}s_{15}(3\beta_1\gamma_1{}^2\gamma_3+\beta_3\gamma_1{}^3)+\cdots\\
&\qquad+\tfrac{1}{2}s_{16}(3\beta_1\gamma_1{}^2\gamma_2+\beta_2\gamma_1{}^3)+\cdots. \qquad (1.16)
\end{aligned}
$$

($\tfrac{1}{2}s'_{35}$ is obtained by writing α for β throughout (1.16).)

In each of the above equations, α_i, β_i, γ_i are the direction cosines defined by (1.11); the missing terms are obtained by cyclic interchange of the suffixes (1, 2, 3) and (4, 5, 6). Equations (1.14) and (1.15) are quoted from Voigt [(115), p. 734] and Eq. (1.16) from Goens (41).

In the particular case of a rotation through an angle θ from x towards y about the axis of z, the direction cosine scheme (1.11) becomes

$$
\left.\begin{array}{cccc}
 & x' & y' & z' \\
x & m & -n & 0 \\
y & n & m & 0 \\
z & 0 & 0 & 1
\end{array}\right\} \qquad (1.17)
$$

where $m=\cos\theta$, $n=\sin\theta$, and the set of equations of which (1.14)–(1.16) are examples becomes:

$$
\left.\begin{aligned}
s'_{11}&=s_{11}m^4+(2s_{12}+s_{66})m^2n^2+s_{22}n^4\\
&\qquad+2s_{16}m^3n+2s_{26}mn^3;\\
s'_{12}&=(s_{11}+s_{22})m^2n^2+s_{12}(m^4+n^4)\\
&\qquad+(s_{16}-s_{26})mn(m^2-n^2)-s_{66}m^2n^2;\\
s'_{13}&=s_{23}n^2+s_{36}mn+s_{13}m^2;\\
s'_{14}&=s_{14}m^3-(s_{15}-s_{46})m^2n\\
&\qquad+(s_{24}-s_{56})n^2m-s_{25}n^3;\\
s'_{15}&=s_{24}n^3+(s_{25}+s_{46})n^2m\\
&\qquad+(s_{14}+s_{56})m^2n+s_{15}m^3;\\
s'_{16}&=-2(s_{11}m^2-s_{22}n^2)mn\\
&\qquad+(2s_{12}+s_{66})mn(m^2-n^2)\\
&\qquad+s_{16}m^2(m^2-3n^2)+s_{26}n^2(3m^2-n^2);\\
s'_{22}&=s_{11}n^4+(2s_{12}+s_{66})m^2n^2+s_{22}m^4\\
&\qquad-2s_{16}n^3m-2s_{26}m^3n;
\end{aligned}\right\}(1.18)
$$

$$
\left.\begin{aligned}
s'_{23}&=s_{23}m^2-s_{36}mn+s_{13}n^2;\\
s'_{24}&=s_{24}m^3-(s_{25}+s_{46})m^2n\\
&\qquad+(s_{14}+s_{56})n^2m-s_{15}n^3;\\
s'_{25}&=s_{14}n^3+(s_{15}-s_{46})n^2m\\
&\qquad+(s_{24}-s_{56})m^2n+s_{25}m^3;\\
s'_{26}&=-2(s_{11}n^2-s_{22}m^2)mn\\
&\qquad-(2s_{12}+s_{66})mn(m^2-n^2)\\
&\qquad+s_{16}n^2(3m^2-n^2)+s_{26}m^2(m^2-3n^2);\\
s'_{33}&=s_{33};\\
s'_{34}&=s_{34}m-s_{35}n;\\
s'_{35}&=s_{34}n+s_{35}m;\\
s'_{36}&=(s_{23}-s_{13})mn+s_{36}(m^2-n^2);\\
s'_{44}&=s_{44}m^2-2s_{45}mn+s_{55}n^2;\\
s'_{45}&=(s_{44}-s_{55})mn+s_{45}(m^2-n^2);\\
s'_{46}&=-(s_{25}-s_{15})2n^2m+(s_{24}-s_{14})\ 2m^2n\\
&\qquad+(s_{46}m-s_{56}n)(m^2-n^2);\\
s'_{55}&=s_{44}n^2+2s_{45}mn+s_{55}m^2;\\
s'_{56}&=(s_{25}-s_{15})2m^2n+(s_{24}-s_{14})2n^2m\\
&\qquad+(s_{56}m+s_{46}n)(m^2-n^2);\\
s'_{66}&=4(s_{11}+s_{22}-2s_{12})m^2n^2\\
&\qquad-4(s_{26}-s_{16})mn(m^2-n^2)+s_{66}(m^2-n^2)^2.
\end{aligned}\right\}\begin{array}{c}(1.18)\\ \textit{contd.}\end{array}
$$

The above equations can now be used to find the effect of the symmetry elements characteristic of the various crystal systems on the number of independent elastic constants which must exist in that system. Thus, suppose a crystal possesses a q-fold rotation axis of symmetry, which is taken as lying in the z direction. The substitution $\theta=2\pi/q$ [i.e., $m=\cos(2\pi/q)$, $n=\sin(2\pi/q)$] is then made in (1.18) and the resulting expressions equated to those obtained by substituting $\theta=0$. If, for example $q=4$, then $\theta=\pi/2$ and $m=0$, $n=1$. The first Eq. (1.18) therefore gives:

$$s'_{11}(\theta=\pi/2)=s_{22};\quad s'_{11}(\theta=0)=s_{11} \text{ and } \therefore\quad s_{11}=s_{22}.$$

Similarly the remaining equations give:

$$s_{13}=s_{23};\quad s_{44}=s_{55};\quad s_{26}=-s_{16};$$

$$s_{14}=s_{15}=s_{24}=s_{25}=s_{34}=s_{35}=s_{36}=s_{45}=s_{46}=s_{56}=0.$$

These equations among the coefficients must therefore hold for a material having a fourfold

TABLE I. The coefficients s_{ik} in the various symmetry systems ($s_1=2[s_{11}-s_{12}]$).

Full designation	x_x X_X	x_x Y_Y	x_x Z_Z	x_x Y_Z	x_x Z_X	x_x X_Y	y_y Y_Y	y_y Z_Z	y_y Y_Z	y_y Z_X	y_y X_Y	z_z Z_Z	z_z Y_Z	z_z Z_X	z_z X_Y	y_z Y_Z	y_z Z_X	y_z X_Y	z_x Z_X	z_x X_Y	x_y X_Y
Shortened designation	11	12	13	14	15	16	22	23	24	25	26	33	34	35	36	44	45	46	55	56	66
Triclinic system (21 coefficients)	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}	s_{22}	s_{23}	s_{24}	s_{25}	s_{26}	s_{33}	s_{34}	s_{35}	s_{36}	s_{44}	s_{45}	s_{46}	s_{55}	s_{56}	s_{66}
Monoclinic system* (13 coefficients)	s_{11}	s_{12}	s_{13}	0	0	s_{16}	s_{22}	s_{23}	0	0	s_{26}	s_{33}	0	0	s_{36}	s_{44}	s_{45}	0	s_{55}	0	s_{66}
Orthorhombic system (9 coefficients)	s_{11}	s_{12}	s_{13}	0	0	0	s_{22}	s_{23}	0	0	0	s_{33}	0	0	0	s_{44}	0	0	s_{55}	0	s_{66}
Trigonal and hexagonal systems																					
(a) 7 coefficients	s_{11}	s_{12}	s_{13}	s_{14}	$-s_{25}$	0	s_{11}	s_{13}	$-s_{14}$	s_{25}	0	s_{33}	0	0	0	s_{44}	0	$2s_{25}$	s_{44}	$2s_{14}$	s_1
(b) 6 coefficients	s_{11}	s_{12}	s_{13}	s_{14}	0	0	s_{11}	s_{13}	$-s_{14}$	0	0	s_{33}	0	0	0	s_{44}	0	0	s_{44}	$2s_{14}$	s_1
(c) 5 coefficients	s_{11}	s_{12}	s_{13}	0	0	0	s_{11}	s_{13}	0	0	0	s_{33}	0	0	0	s_{44}	0	0	s_{44}	0	s_1
Tetragonal system																					
(a) 7 coefficients	s_{11}	s_{12}	s_{13}	0	0	s_{16}	s_{11}	s_{13}	0	0	$-s_{16}$	s_{33}	0	0	0	s_{44}	0	0	s_{44}	0	s_{66}
(b) 6 coefficients	s_{11}	s_{12}	s_{13}	0	0	0	s_{11}	s_{13}	0	0	0	s_{33}	0	0	0	s_{44}	0	0	s_{44}	0	s_{66}
Cubic system (3 coefficients)	s_{11}	s_{12}	s_{12}	0	0	0	s_{11}	s_{12}	0	0	0	s_{11}	0	0	0	s_{44}	0	0	s_{44}	0	s_{44}
Isotropic system (2 coefficients)	s_{11}	s_{12}	s_{12}	0	0	0	s_{11}	s_{12}	0	0	0	s_{11}	0	0	0	s_1	0	0	s_1	0	s_1

Editor's Note: If the *Y* axis of a monoclinic crystal is taken as the binary axis *b* to agree with the notation of the crystallography, the elastic constants would be s_{11}, s_{12}, s_{13}, s_{15}, s_{22}, s_{23}, s_{25}, s_{33}, s_{35}, s_{44}, s_{46}, s_{55}, s_{66} rather than those in the table.

rotation axis, i.e., one belonging to the tetragonal system. The other systems can be dealt with by similar methods [(37a), (73), (115)]; the results are given in Table I.

The trigonal and hexagonal systems are closely interrelated, and are classed together in Table I; the majority of trigonal crystals, however, have either seven or six independent coefficients, while the majority of hexagonal crystals have five. In these two systems, as in the tetragonal system, the smaller numbers of independent coefficients are appropriate to crystals which possess certain symmetry elements over and above the minimum required to place the crystal in a particular system.

The scheme for the parameters c_{ik} is obtained simply by writing c for s throughout Table I, with the following exceptions [Voigt (115), p. 588]. If $s_{66}=2(s_{11}-s_{12})$, then $c_{66}=\frac{1}{2}(c_{11}-c_{12})$. If $s_{46}=2s_{25}$; $s_{56}=2s_{14}$, then $c_{46}=c_{25}$; $c_{56}=c_{14}$.

The existence of the relations embodied in Table I produces a considerable simplification of all the equations hitherto quoted in connection with the elasticity of anisotropic systems. For example, in the hexagonal system, Eqs. (1.6) become [Voigt (115), p. 747]:

$$\left.\begin{aligned} &c_{11}+c_{12}=s_{33}/s'; \qquad c_{11}-c_{12}=1/(s_{11}-s_{12});\\ &c_{13}=-s_{13}/s'; \qquad c_{44}=1/s_{44};\\ &c_{33}=(s_{11}+s_{12})/s', \end{aligned}\right\} \tag{1.19}$$

where $s'=s_{33}(s_{11}+s_{12})-2s_{13}^2$.

In the cubic system [Voigt (115), p. 741]:

$$\left.\begin{aligned} c_{11}&=\frac{s_{11}+s_{12}}{(s_{11}-s_{12})(s_{11}+2s_{12})};\\ c_{12}&=\frac{-s_{12}}{(s_{11}-s_{12})(s_{11}+2s_{12})};\\ c_{44}&=1/s_{44}. \end{aligned}\right\} \tag{1.20}$$

Equations (1.14)–(1.16) are similarly greatly simplified; for examples see Eqs. (2.3), (2.4), (3.6), (3.7), (7.3), (7.4), (7.9), and (7.10).

2. General Discussion of Methods

In theory, the elastic coefficients of crystals can be obtained by suitable measurements involving the four constants familiar in the elastic theory of isotropic solids—compressibility, Poisson's ratio, Young's modulus, and rigidity modulus.

Thus, the linear compressibilities in the directions of the coordinate axes are:

$$\left.\begin{aligned} x_x/P&=s_{11}+s_{12}+s_{13};\\ y_y/P&=s_{22}+s_{23}+s_{21};\\ z_z/P&=s_{33}+s_{31}+s_{32}; \end{aligned}\right\} \tag{2.1}$$

and the cubic compressibility is therefore:

$$(x_x+y_y+z_z)/P=s_{11}+s_{22}+s_{33}+2(s_{12}+s_{23}+s_{31}), \tag{2.2}$$

where P is a uniform hydrostatic pressure.

If, therefore, the compressibilities and the Young's moduli corresponding to s_{11}, s_{22}, s_{33} are known, s_{12}, s_{23}, and s_{31} can be calculated from the above equations [reference (18)].

The three coefficients s_{12}, s_{23}, and s_{31} can also be obtained from the results of the familiar methods normally used to measure Poisson's ratio; for example, if the direct stress X_X produces a lateral strain y_y, then $s_{12}=-y_y/X_X$. Similarly, if in a material of Young's modulus E, a principal curvature R is accompanied by an anticlastic curvature R_A, then $s_{12}=-R_A/RE$.

The Young's and rigidity moduli may be measured by any of the accepted methods for the determination of these constants (see Table II). By far the most popular methods of any have been those involving static or dynamic bending for Young's modulus, and static or dynamic torsion for rigidity modulus. Young's modulus is also measured fairly often from the frequency of longitudinal vibration, and from static compression or tension experiments. More recently, the tendency has been to apply supersonic methods to measuring the elastic constants; these and other methods are discussed in Section 6.

Equations (1.14) and (1.15) contain between them all of the coefficients and it is therefore possible, by measuring Young's modulus and rigidity modulus on a sufficient number of specimens, each of different but known orientation, to obtain a set of simultaneous equations, the solutions of which yield values for each of the coefficients. For example, in the cubic system, Eqs. (1.14) and (1.15) reduce to

$$s'_{33}=s_{11}-2(s_{11}-s_{12}-\tfrac{1}{2}s_{44})\times(\gamma_1^2\gamma_2^2+\gamma_2^2\gamma_3^2+\gamma_3^2\gamma_1^2); \quad (2.3)$$

$$s'_{44}=s_{44}+4(s_{11}-s_{12}-\tfrac{1}{2}s_{44})\times(\beta_1^2\gamma_1^2+\beta_2^2\gamma_2^2+\beta_3^2\gamma_3^2). \quad (2.4)$$

These equations show that s_{11} and s_{44} can be obtained directly from specimens for which the direction cosine brackets are zero. A measurement of s'_{33} or s'_{44} on a specimen for which the direction cosine bracket is known, but differs from zero, then enables s_{12}, the only unknown principal coefficient, to be calculated. In practice, some modifications of the above method are usually necessary, but the principle remains the same, as will appear later. It is also usual to make measurements of s'_{33} and s'_{44} on the same specimen, so that in the cubic system, a minimum of two specimens is necessary to determine all three coefficients. Naturally, by the use of a greater number of specimens, the results can be cross checked, and hence established with greater certainty.

As the complexity of the symmetry system increases, so does the number of specimens required to determine all the principal coefficients. It appears that no complete measurements have been made in the triclinic or monoclinic systems,[3] but, for example, in the orthorhombic system, at least six specimens are needed. These would most conveniently consist of three specimens cut with their lengths in the three perpendicular directions characterizing the system and three with their lengths in the principal planes, at an angle of 45° with the two principal directions contained in the plane. Bending and torsion measurements on the first three specimens yield s_{11}, s_{22}, s_{33}, s_{44}, s_{55}, s_{66} directly, while bending or torsion measurements on the remaining specimens enable s_{12}, s_{23}, s_{13} to be calculated from the appropriate equations in (1.18).

In some cases, the equations for calculating the Young's and rigidity moduli from the experimental observations are the same whether the specimen is isotropic or not. In other cases, the equations differ, but are usually given in such a form that the modulus for a crystalline material is obtained by combining the usual isotropic equation with a factor which expresses the effect of the anisotropic nature of the material. It is therefore convenient, before dealing with anisotropic materials, to summarize the equations for calculating the constants of isotropic materials from the experimental observations (Table II).

In addition to the quantities defined in the body of the above table, the following notation is used:

A = cross-sectional area.
b = breadth of specimen of rectangular cross section.
d = depth of specimen of rectangular cross section.
$(b>d)$

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[3] Values of s_{11}, s_{22}, $2s_{12}+s_{66}$, s_{16} and s_{26} have been given for the monoclinic crystal gypsum by Reimers (94).

E = Young's modulus.
G = rigidity modulus.
I = moment of inertia of load about long axis of specimen.
J = moment of inertia of cross section.
(Rectangular cross section $J=bd^3/12$; circular cross section $J=\pi r^4/4$.)
l = length of specimen ($l\gg b$; $l\gg r$).
m_i^4 is given by the following table [Rayleigh (93)]

$$\left.\begin{array}{llll} i= & 1 & 2 & 3\\ m_i^4 & 12.36 & 485.3 & 3804\end{array}\right\}\text{after which } m_i\approx\tfrac{1}{2}(2i-1)\pi.$$

m_j^4 is given by the following table [Rayleigh (93)]

$$\left.\begin{array}{llll} j= & 1 & 2 & 3\\ m_j^4 & 500.4 & 3803 & 14590\end{array}\right\}\text{after which } m_j\approx\tfrac{1}{2}(2j+1)\pi.$$

i and j are both integers; i (or j) $=1$ is the fundamental and the successive values give the overtones.
M = mass attached to free end of specimen.
M_0 = mass of specimen.
$n=1, 3, 5\cdots$; $n=1$ is the fundamental
N = twisting couple about long axis of specimen.
r = radius of cross section.
α is a function of M/M_0, often taken as

$$\alpha=3\bigg/\left(\frac{M}{M_0}+\frac{33}{140}\right),\text{ see below.}$$

$\beta=1+(M_0r^2/12I)$.
μ is a function of the dimensions and elastic constants of the specimen; see Table III below.
ν = resonant (natural) frequency of vibration.
ρ = density of specimen.
ϕ = angle of twist at free end of specimen.

The derivations of the majority of the above equations can be found in most standard text books. Equations (2.7) and (2.8) assume that M_0 is negligible in comparison with P_0/g and P_m/g, where g is the acceleration due to gravity. Equation (2.9) is subject to a correction for the effect of lateral vibration (93) and for the fact that the lateral dimensions may not be negligible in comparison with the length (26, 28, 91).

The value of α (Eq. (2.10)) given in the above list of definitions is a first approximation due to Rayleigh. It is usually sufficiently accurate if M/M_0 and l are both large, but it takes no account of shearing in the bar or of rotatory inertia in the system. Exact expressions for α, in which these effects are allowed for and the limitations on M/M_0 and l are removed, have been given by Davies (27).

Equations (2.11) and (2.12) are subject to correction for rotatory inertia and shear effects; these corrections as applied to the latter equation have been discussed by Goens (40, 41).

The quantity μ in Eqs. (2.13) and (2.14) is, according to St. Venant (99), given by Table III for a specimen in which the x', y', z' axes are taken as coinciding, respectively, with the directions of d, b, and l. For an isotropic solid $s'_{55}=s'_{44}$ and μ is therefore a function of d/b only. It is given above as a function of $(d/b)(s'_{55}/s'_{44})^{\frac{1}{2}}$ for convenience of reference when dealing later with anisotropic materials.

A general discussion of torsional vibrations may be found in Davies' paper (28). The quantity β in Eq. (2.17) is due to Goens (39); it is usually negligibly different from unity, and for this reason its effect in Eq. (2.18) is ignored.

3. Effects Introduced by the Anisotropic Nature of the Experimental Material; Free and Pure Elastic Constants

The measurement of the elastic coefficients of anisotropic materials is, in general, complicated by the existence of the coefficients s_{lm} (where $l \ngtr 3$, $m>3$), which relate an extensional stress to a shear strain and *vice versa*. Their existence implies not only that an anisotropic specimen subjected to a pure extension undergoes both extensional and shear strain, but also that the bending of such a specimen is normally accompanied by torsion, and *vice versa*. The existence of this effect was predicted by Voigt (115) and confirmed experimentally by Reimers (94) in 1913. In dealing with crystals, it is often possible to choose orientations for which such effects vanish, a procedure which should be followed if at all possible. This cannot always be done, however, particularly when the crystalline specimen is artificially grown, and it is therefore necessary to take the above effects into account.

For various reasons, specimens of circular cross section are almost invariably used, and attention will therefore be concentrated mainly on such specimens, although for completeness some reference to rectangular cross sections will be made.

Owing to the existence of the coefficients s_{lm}, a distinction has to be drawn between "free" and "pure" elastic constants (22, 41, 53, 123). As already mentioned, anisotropic specimens in general both twist and bend under the influence of either a bending moment or a twisting couple. If a twisting couple is applied, the free rigidity modulus (G_F) corresponds to conditions such that the associated bending can take place, while

TABLE II. Summary of methods for measuring Young's modulus and rigidity modulus of isotropic materials.

Method	Equation for elastic constant	Equation number	Notes
A. Young's modulus—any cross section			
Static tension or compression	$\frac{pl}{A\Delta l}$	(2.5)	p = applied tensile or compressive force. Δl = change in length.
Static uniform bending	$\frac{CR}{J}$	(2.6)	C = applied bending moment. R = radius of curvature.
Static non-uniform bending	$\frac{P_0 l^3}{3Jx_0}$	(2.7)	Specimen forms a cantilever, fixed at one end. P_0 = load applied to free end. x_0 = deflection of free end.
Static non-uniform bending	$\frac{P_m l^3}{48Jx_m}$	(2.8)	Specimen supported on knife edges distance l apart and loaded at middle of span. P_m = load applied to center of span. x_m = deflection of center of span.
Longitudinal vibration	$\frac{4l^2\nu^2\rho}{n^2}$	(2.9)	Specimen clamped at middle point of length and free at both ends.
Flexural vibration	$\frac{4\pi^2 Al^4\nu^2\rho}{J\alpha}$	(2.10)	Specimen clamped at one end and loaded with mass M at the other.
Flexural vibration	$\frac{4\pi^2 M_0 l^3\nu^2}{Jm_i^4}$	(2.11)	Unloaded specimen, clamped at one end, free at other.
Flexural vibration	$\frac{4\pi^2 M_0 l^3\nu^2}{Jm_j^4}$	(2.12)	Unloaded specimen, free at both ends.
B. Rigidity modulus—rectangular cross section			
Static torsion	$\frac{Nl}{\mu bd^3\phi}$	(2.13)	Specimen clamped at one end and twisted through angle ϕ at other.
Torsional vibration	$\frac{4\pi^2 Il\nu^2}{\mu bd^3}$	(2.14)	Specimen clamped at one end and loaded with moment of inertia I at other.
C. Rigidity modulus—circular cross section			
Static torsion	$\frac{2Nl}{\pi r^4\phi}$	(2.15)	Specimen clamped at one end and twisted through angle ϕ at other.
Torsional vibration	$\frac{4l^2\nu^2\rho}{n^2}$	(2.16)	Specimen unloaded and clamped at middle point.
Torsional vibration	$\frac{4\pi lI\nu^2\beta}{r^4}$	(2.17)	Specimen clamped at middle point and loaded at each end with moment of inertia I.
Torsional vibration	$\frac{8\pi lI\nu^2}{r^4}$	(2.18)	Specimen clamped at one end and loaded at other.

the pure rigidity modulus (G_P) corresponds to the application of a bending moment as well as the twisting couple, such that the associated bending is entirely suppressed, and the distortion is therefore purely torsional.

A similar discussion applies to Young's modulus; in bending, E_F corresponds to the application of a bending moment only, so that the distortion is a combination of flexure and torsion, while E_P corresponds to the application of a stress system, such that the distortion is purely flexural.

The free rigidity modulus of a specimen of circular cross section with its length in the z' direction is

$$G_F = 2/(s'_{44}+s'_{55}) = 1/s'_g, \qquad (3.1)$$

while G_P is given by

$$G_P = G_F/(1-\epsilon), \qquad (3.2)$$

where

$$\epsilon = s'^2_{34} + s'^2_{35}/s'_{33}(s'_{44}+s'_{55}). \qquad (3.3)$$

[(22), (41)].

The free Young's modulus is

$$E_F = 1/s'_{33}, \qquad (3.4)$$

while the pure Young's modulus

$$E_P = E_F/(1-\epsilon). \qquad (3.5)$$

The quantity ϵ is necessarily positive, and therefore $E_P > E_F$, $G_P > G_F$.

It follows from (1.16) and (1.12) that in the cubic system,

$$s'^2_{34} + s'^2_{35} = 4(s_{11} - s_{12} - \tfrac{1}{2}s_{44})^2[(\gamma_1^6+\gamma_2^6+\gamma_3^6) - (\gamma_1^4+\gamma_2^4+\gamma_3^4)^2], \qquad (3.6)$$

while in the hexagonal system,

$$s'^2_{34} + s'^2_{35} = 4\gamma_3^2(1-\gamma_3^2)[s_{11}(1-\gamma_3^2) - (s_{13}+\tfrac{1}{2}s_{44})(1-2\gamma_3^2) - s_{33}\gamma_3^2]^2. \qquad (3.7)$$

Goens (41) has shown that $s'^2_{34}+s'^2_{35}$ is invariant for rotation about the z' axis, and, further, that an orientation exists for which s'_{35} vanishes. s'^2_{34} is then a maximum, and, for a specimen of circular cross section, the x' and y' axes are chosen so that these conditions are fulfilled (i.e., the plane of the torsion coupled bending is taken as the $x'z'$ plane). ϵ for a circular cross section Eq. (3.3) can then be written

$$\epsilon = s'^2_{34}/2s'_{33}s'_g. \qquad (3.8)$$

For a specimen of rectangular cross section, z' is taken as coinciding with l, y' with b, and x' with d; it is further assumed that $b \gg d$. Then (53)

$$E_F/E_P = G_F/G_P = 1-\epsilon; \qquad (3.9)$$

$$E_F = 1/s'_{33}; \qquad (3.10)$$

$$G_F = 1/s'_{44}; \qquad (3.11)$$

$$\epsilon = s'^2_{34}/s'_{33}s'_{44}. \qquad (3.12)$$

4. Young's and Rigidity Moduli of Anisotropic Materials by Static Methods

The equations given in Table II for calculating Young's modulus from the results of static experiments (Eqs. (2.5)–(2.8)) are equally applicable whether the material is isotropic or anisotropic. The calculation of s'_{33} from the Young's modulus will, however, obviously depend on whether the experimental conditions correspond to E_F or E_P, and it is thus necessary to be perfectly clear on this point. If E_F is involved, then $s'_{33} = 1/E$, while if E_P is involved, $s'_{33} = 1/E(1-\epsilon)$, E being the Young's modulus calculated from the appropriate equation in Table II.

Table III. Values of μ (Eqs. (2.13) and (2.14)).

$(d/b)(s'_{55}/s'_{44})^{\frac{1}{2}}$	1.0	0.909	0.800	0.714	0.625	0.556	0.500
μ	0.141	0.154	0.172	0.187	0.203	0.217	0.229
$(d/b)(s'_{55}/s'_{44})^{\frac{1}{2}}$	0.400	0.333	0.250	0.200	0.100	0.050	0
μ	0.249	0.263	0.281	0.291	0.312	0.323	0.333

Similar remarks apply to rigidity modulus for a specimen with circular cross section; if G_F is involved, s'_g (Eq. (3.1)) $=1/G$, while if G_P is involved, $s'_g = 1/G(1-\epsilon)$, G being the rigidity modulus obtained directly from Eq. (2.15) (Table II).

For rectangular cross sections, a general equation expressing the angle of twist in terms of the applied couple, the dimensions of the specimen and the elastic coefficients has been given by Voigt [(115), p. 644]. This equation, if valid, would be applicable to any orientation of the specimen, but its validity does not seem to be definitely established, and, moreover, it is somewhat complicated. A full discussion of the problem will be found in Voigt, Sections 320–323, 369, but with one or two exceptions, an accurate measurement of rigidity modulus is only possible on anisotropic specimens of rectangular cross section if (*a*) the specimen length lies in a four- or a sixfold symmetry axis, or (*b*) if two of the specimen edges lie in symmetry axes. These restrictions on orientation are very stringent, and are, in fact, one of the reasons why specimens of rectangular cross section are so seldom used. If the specimen fulfils the conditions, then torsion flexure coupling is absent, and G_F is given by Eq. (2.13) Table II with μ found from Table III.

When the length lies in a four- or sixfold axis, $s_{44} = s_{55}$ and μ is a function of b/d only, just as for isotropic materials.

In the remaining case (2 edges in symmetry axes) s'_{44} and s'_{55} may not be equal and μ must then be found from the value of $(d/b)(s'_{55}/s'_{44})^{\frac{1}{2}}$.

This involves a knowledge both of s'_{44} the quantity it is desired to measure, and s'_{55}, but a procedure of successive approximation can be used which leads to a final satisfactory value for s'_{44} [(52), (56), (57)].

5. Young's and Rigidity Moduli of Anisotropic Materials by Dynamic Methods

The general equations of motion of a cylinder of circular cross section of radius r and density ρ have been given by Voigt [(115), p. 673], Goens (41), and Brown (22). According to Brown, they are:

$$\left.\begin{aligned} \rho\frac{\partial^2\eta}{\partial t^2} &= -\frac{1}{4}r^2E_P\frac{\partial^4\eta}{\partial^4 z}+2C\frac{\partial^2\psi}{\partial z^2}; \\ \rho\frac{\partial^2\psi}{\partial t^2} &= -\frac{1}{4}r^2C\frac{\partial^4\eta}{\partial z^4}+G_P\frac{\partial^2\psi}{\partial z^2}; \end{aligned}\right\} \quad (5.1)$$

where

$$C=(\epsilon E_PG_P/2)^{\frac{1}{2}}; \quad \epsilon=s'^2_{34}/2s'_{33}s'_g;$$

$\eta=$ (bending moment over a cross section)$/\frac{1}{4}\pi r^4$; $\psi=$ (twisting moment over a cross section)$/\frac{1}{2}\pi r^4$. If $s'_{34}=0$, $\epsilon=0$ and we obtain two independent equations

$$\rho\frac{\partial^2\eta}{\partial t^2}=-\frac{1}{4}r^2E\frac{\partial^4\eta}{\partial z^4}, \quad \text{and} \quad \rho\frac{\partial^2\psi}{\partial t^2}=G\frac{\partial^2\psi}{\partial z^2}, \quad (5.2)$$

the usual equations for bending and torsion, respectively. Solutions of (5.1) have been given (*a*) by Goens (41) for the torsional vibrations of a loaded rod and the flexural vibrations of an unloaded rod, and (*b*) by Brown (22) for the torsional vibrations of an unloaded rod.

The derivation of rigorous solutions is a complicated matter, and approximations have to be introduced in order to apply the results to experimental cases. It appears justifiable, therefore, to attempt to derive results on the basis of approximate differential equations (53). Such a procedure is not without its dangers, and, in the last resort, can only be justified by comparison with rigorous solutions. It has the advantage, however, that it not only allows the final equations to be developed more simply and in a more straightforward form, but it also allows the effects of the torsion-flexure coupling in vibration experiments to be more clearly seen.

A treatment of this type has been applied to specimens of rectangular cross section, clamped at one end and loaded at the other (53). The following is a modification of the treatment, applicable to specimens of circular cross section.

Since the specimen will both twist and bend under the action of a simple end force, P, or a simple twisting couple, N, assume that

$$P=\alpha_{11}x+\alpha_{12}\phi; \quad N=\alpha_{12}x+\alpha_{22}\phi; \quad (5.3)$$

$$x=\beta_{11}P+\beta_{12}N; \quad \phi=\beta_{12}P+\beta_{22}N, \quad (5.4)$$

where x is the end displacement, ϕ is the end twist. If Eqs. (5.3) are solved to express x and ϕ in terms of P and N, and the resulting coefficients equated to those of (5.4) we obtain

$$\left.\begin{aligned} &\beta_{11}=\frac{\alpha_{22}}{\alpha_{11}\alpha_{22}-\alpha_{12}^2}; \quad \beta_{12}=\frac{\alpha_{12}}{\alpha_{12}^2-\alpha_{11}\alpha_{22}}; \\ &\beta_{22}=\frac{\alpha_{11}}{\alpha_{11}\alpha_{22}-\alpha_{12}^2}. \end{aligned}\right. \quad (5.5)$$

But by Eqs. (2.7) and (2.15)

$$\left.\begin{aligned} &\alpha_{11}=\frac{3E_P\pi r^4}{4l^3}; \quad \alpha_{22}=\frac{G_P\pi r^4}{2l}; \\ &\beta_{11}=\frac{4l^3}{3E_F\pi r^4}; \quad \beta_{22}=\frac{2l}{G_F\pi r^4}. \end{aligned}\right. \quad (5.6)$$

Now, from the first Eq. (5.5)

$$\alpha_{12}^2=\frac{\alpha_{22}(\beta_{11}\alpha_{11}-1)}{\beta_{11}} \quad (5.7)$$

and therefore, from (5.6), (3.2), and (3.8)

$$\alpha_{12}^2=\frac{3\pi^2r^8}{8l^4}G_P(E_P-E_F)=\frac{3\pi^2r^8s'^2_{34}}{4l^4(s'^2_{34}-2s'_{33}s'_g)^2}. \quad (5.8)$$

Inserting this value of α_{12}^2 into the second Eq. (5.5):

$$\beta_{12}^2=\frac{8l^4(E_P-E_F)}{3\pi^2r^8G_PE_F^2}=\frac{4l^4s'^2_{34}}{3\pi^2r^8}. \quad (5.9)$$

Assuming that the mass of the specimen and rotatory inertia effects in the $z'x'$ plane are negligible, the equations of motion of the attached mass are:

$$\frac{d^2x}{dt^2}+\frac{\alpha_{11}x}{M}+\frac{\alpha_{12}\phi}{M}=0; \tag{5.10}$$

$$\frac{d^2\phi}{dt^2}+\frac{\alpha_{12}x}{I}+\frac{\alpha_{22}\phi}{I}=0, \tag{5.11}$$

or, solving the first equation for ϕ and substituting in the second:

$$\frac{d^4x}{dt^4}+\frac{I\alpha_{11}+M\alpha_{22}}{IM}\frac{d^2x}{dt^2}+\frac{\alpha_{11}\alpha_{22}-\alpha_{12}^2}{IM}x=0. \tag{5.12}$$

Writing $x=e^{-kt}$, we obtain

$$x=A\sin(2\pi\nu_1t+\delta_1)+B\sin(2\pi\nu_2t+\delta_2), \tag{5.13}$$

where A, B, δ_1, and δ_2 are arbitrary constants governed by the boundary conditions, and ν_1 and ν_2 are frequencies.

Putting for brevity $\Delta=\alpha_{12}^2/(I\alpha_{11}-M\alpha_{22})$, then approximately

$$\nu_1=\frac{1}{2\pi}\left(\frac{\alpha_{11}}{M}+\Delta\right)^{\frac{1}{2}},\quad \nu_2=\frac{1}{2\pi}\left(\frac{\alpha_{22}}{I}-\Delta\right)^{\frac{1}{2}}, \tag{5.14}$$

as can be verified by substituting in (5.12). These expressions give the perturbation of the frequencies due to coupling. If the mass of the specimen, and rotatory inertia effects in the $z'x'$ plane are not negligible, then M must be replaced by M_E where

$$M_E\approx\frac{M}{1-9K^2/4l^2}+\frac{33}{140}M_0, \tag{5.15}$$

(53), where K is the radius of gyration in the $z'x'$ plane of M about its point of attachment to the specimen.

If $s'_{34}=0$, then $\alpha_{12}=\Delta=0$, and we obtain two independent frequencies,

$$\frac{1}{2\pi}\left(\frac{\alpha_{11}}{M_E}\right)^{\frac{1}{2}}\quad\text{and}\quad\frac{1}{2\pi}\left(\frac{\alpha_{22}}{I}\right)^{\frac{1}{2}}, \tag{5.16}$$

appropriate, respectively, to flexural and torsional vibrations, and in agreement with Eqs. (2.10) and (2.18).

Squaring the first Eq. (5.14) and inserting the values of α_{11} and Δ, in terms of the elastic coefficients, we obtain

$$\frac{16\pi M_E\nu_1^2l^3}{3r^4}=E$$

$$=\frac{E_F}{1-\epsilon}\left[1+\frac{M_El^2s'^2_{34}}{(3Is'_g-2M_El^2s'_{33})s'_g}\right], \tag{5.17}$$

where E is the Young's modulus calculated directly by the usual isotropic equation from the experimental results, no account being taken of the effects of the torsion-flexure coupling.

Similarly, the second Eq. (5.14) gives

$$\frac{8\pi\nu_2^2Il}{r^4}=G$$

$$=\frac{G_F}{1-\epsilon}\left[1-\frac{3Is'^2_{34}}{(3Is'_g-2M_El^2s'_{33})2s'_{33}}\right], \tag{5.18}$$

where G is the rigidity modulus corresponding to E.

These equations show that by making l sufficiently large, the constants measured directly are E_F and G_P, whereas by making l sufficiently small, the constants measured directly are E_P and G_F. Some of these facts have been utilized in the measurement, by vibration methods, of the elastic constants of strips of wood cut at an angle to the grain (53).

These findings are qualitatively in agreement with those of Goens (41) and Brown (22). Goens gives an exact solution for the case of a loaded specimen executing torsional vibrations and quantitative comparison is therefore possible.

Goens' equation is:

$$s'_g=s_I(1-\gamma)/(1-\epsilon), \tag{5.19}$$

where s_I is the coefficient calculated directly from the equation appropriate to isotropic materials ((2.17) for Goens' experiments).

γ is a complicated third-order determinantal function involving, among other quantities, the elastic coefficients and dimensions of the specimen, and the mass and moment of inertia of the end loads.

Since $s_I=1/G$, and $s'_g=1/G_F$, Eq. (5.18) is of the same form as (5.19) and both can be written

$$s'_g=s_I\eta, \tag{5.20}$$

TABLE IV. Correction factors in torsional vibration.

	(1) Copper								
Crystal number	7	10	6	5	9	4	2	3	8
$1/(1-\epsilon)$	1.040	1.140	1.078	1.076	1.032	1.024	1.030	1.019	1.008
η (Goens and Weerts)	1.032	1.118	1.078	1.079	1.036	1.028	1.036	1.031	1.014
η (Eq. (5.18))	1.041	1.148	1.084	1.083	1.035	1.028	1.034	1.056	1.009

	(2) Gold					(3) Lead				
Crystal number	8	9	14	1	6	4	6	5	9a	9b
$1/(1-\epsilon)$	1.105	1.112	1.045	1.013	1.004	1.031	1.189	1.154	1.030	1.030
η (Goens and Weerts)	1.078	1.090	1.041	1.015	1.005	1.045 1.023	1.118 1.086	1.116	1.033	1.064
η (Eq. (5.18))	1.145	1.117	1.048	1.015	1.005	1.032	1.193	1.160	1.033	1.038

	(4) Aluminum											
Crystal number	*S*11	*S*11	*S*5	*S*21	*S*25	*S*16	*S*16	*W*4	*W*4	*W*12	*W*6	*W*6
$1/(1-\epsilon)$	1.002	1.002	1.003	1.003	1.004	1.001	1.000	1.001	1.001	1.002	1.001	1.007
η (Goens)	1.004	0.996 1.013	1.004	1.013 1.008	1.001	1.002 1.000	0.999 0.995	1.011 1.007	0.997 0.93	1.039 0.921	1.017 1.008	0.998 0.985
η (Eq. (5.18))	1.003	1.003	1.003	1.005	1.001	1.001	1.000	1.004	1.004	1.006	1.003	1.010

where η is a correction factor to be applied to the isotropic equation.

The following quantities are given in Table IV.

(1) $1/(1-\epsilon)$.

(2) η as calculated by Goens (42) for certain aluminum crystals, and by Goens and Weerts (44) for certain copper, gold, and lead crystals, using Goens' accurate Eq. (5.19). An entry of two values in a single space indicates the limits between which η lies.

(3) η as calculated from Eq. (5.18) using Goens' and Goens and Weerts' primary data.

These figures show that, as a rule, the value of η calculated from (5.18) does not differ from the value given by Goens' equation by more than a few parts in a thousand. Unfortunately, however, this is not always the case, and some of the discrepancies are of the order of a few percent. Thus, in work of high accuracy, it appears necessary to use Goens' equation rather than Eq. (5.18).

A very good approximation to the accurate value of η is given for many of the crystals by $1/(1-\epsilon)$. This implies that the torsional vibrations are approximately pure, a result which appears to be generally true for specimens of the length normally used (22). On the other hand, the flexural vibrations of such specimens are approximately free, and, with an error less than that of the experiments, $s'_{33}=1/E$ where E is the Young's modulus calculated directly from the appropriate isotropic equation (41), (43). It will be seen from (5.18) that in order to calculate s'_g, the values of s'_{33}, s'_g, and s'_{34} must be known. This paradox is more apparent than real, since the three coefficients enter into the correction terms γ and ϵ (Eq. (5.19)) which are usually small compared with unity. The coefficients need not therefore be known with great accuracy. At the same time, it is desirable to have available a set of provisional fundamental constants for the material involved when measuring s'_g at an arbitrary orientation.

If a torsional frequency occurs near a flexural frequency, a state of affairs termed "inner resonance" by Goens (41), then the corrections may become very large, as pointed out both by Goens (41) and Brown (22). According to Brown, this state of affairs would at once be obvious, and the specimen rejected as useless.

6. Discussion of Some Experimental Points

Throughout his work, Voigt (115) used exclusively static methods, and in this he has been followed by Bridgman (18, 19), Mandell (77, 78), Hanson (51), Hinz (54), Swift and Tyndall (112), and others.

The majority of these workers used bending and torsion experiments to obtain the elastic coefficients. Hinz' experiments are, however, an interesting exception, since he obtained the complete set of 9 coefficients for Rochelle Salt from direct compression measurements only. This he did by observing the longitudinal and lateral strains in compression of specimens cut in the directions of the principal axes and also the longitudinal strains on specimens cut in the principal planes, at angles of 45° to the principal directions. In connection with static experiments, the use by Kantola (64) of an electrical capacitance ultramicrometer to measure compressional strain should be mentioned.

Direct application of vibration methods to crystals have been made by Wright (123), Davies (26), Goens (42, 43), Balamuth (3), Rose (98), Mason (81, 82), Hunter and Siegel (59), and others. The usual vibration methods nearly always depend on the following principles. An alternating driving force is applied to the specimen so as to excite the appropriate vibrations (longitudinal, flexural, or torsional) in it. The response, or some function of response, is measured over a range of driving frequencies; the elastic constant is then calculated from the dimensions of the specimen, and the frequency at which the response is a maximum (the resonant frequency). In practice, the situation is complicated by the existence of harmonic modes of vibration; when a resonant frequency has been observed it is often not easy to decide to which harmonic it belongs [see e.g., Kammer and Atanasoff (63)]. A further complication is the tendency of some systems to exhibit "subharmonics" and "multiple resonance," i.e., a tendency for the specimen to execute a subintegral or integral number (>1) of vibrations during one forcing cycle [Ludeke (74)]. Whatever the reason for the effect (e.g. non-linear response, or the existence of harmonics in the forcing vibration), its occasional occurrence appears to be established and may have to be borne in mind in interpreting the spectrum of resonant frequencies observed on any specimen.

The methods used to excite the vibrations and to detect the response are described in the papers cited above. There is, however, one method of comparatively recent development which is specially applicable to crystal specimens, over a wide range of temperature—the method of the piezoelectric oscillator [Balamuth (3), Rose (98), and numerous later papers by other workers in *The Physical Review*].

This method depends on having a suitably cut quartz rod, provided with gold leaf electrodes. The crystal specimen on which measurements are to be made is of the same cross section as the quartz, and is cemented to it end to end. The electrodes are disposed relative to the quartz in such a way that when an alternating potential of constant amplitude is applied to them the piezoelectric effect in the quartz tends to set up the required vibrations in the composite oscillator.

Balamuth's arrangement for longitudinal vibration is shown in Fig. 1. The resonant frequency is found by observing the variation with frequency of the electrical impedance between the electrodes. The elastic constant involved in the vibrations can then be found from the resonant frequency, the dimensions of the crystal specimen, and certain constants of the quartz rod.

Other methods which have been applied to crystals include the use of sonic (32) and supersonic waves (4, 6a, 8, 9, 10, 32, 75, 102) and the calculation of the constants from the frequencies of vibration of plates (1, 63, 84 see p. 433).

The supersonic method described by Schaefer and Bergmann (6a, 102) is noteworthy for the fact that all the constants can be determined on a single sample of the material. A beam of light is passed through the specimen, which is set into vibration by a piezoelectric oscillator. The light is diffracted by the ultrasonic "grating" in the crystal, and the diffraction pattern is photographed. The elastic constants are determined from certain dimensions of the diffraction pattern, and a factor containing the exciting frequency, the wave-length of the light, and the distance of the crystal from the plane of the picture. Although the practice of the method is simple, the theory is somewhat complicated; for details, reference may be made to Bergmann's book (6a). As described above, the method is obviously only suitable for transparent solids, but in a later development (6a) the light is

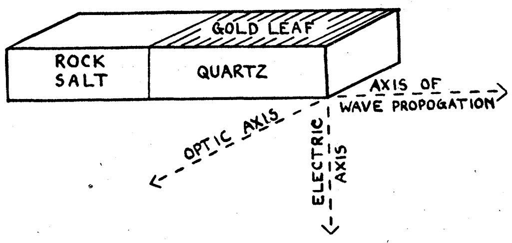

FIG. 1. Piezoelectric oscillator (Balamuth).

reflected from the surface of the vibrating specimen instead of passing through it, and in this way, the method can be used for opaque bodies.

Another supersonic method has been applied by Bhagavantam and co-workers (8, 9, 10). The crystalline specimen in the form of a plate is excited by a quartz wedge driven by alternating current of variable frequency, thus providing a continuous supersonic spectrum. The specimen is in contact with a liquid through which a beam of light is passed. As the exciting frequency is varied, the resonances of the specimen are detected by the maxima in the diffraction effects on the light produced by the supersonic "grating" in the liquid. It has so far proved possible to work only with longitudinal vibrations in the thickness direction of the crystal, and the supersonic measurements have therefore to be supplemented by others in order to determine all of the constants. In the work under discussion, static torsion experiments were made to measure the shear constant of the materials, all of which belonged to the cubic system.[4]

The preparation of the crystalline specimens is of obvious importance, but forms almost a subject on its own.[5] Voigt cut his specimens from large natural crystals and this method is still in use. Recently, however, the practice has become common of making monocrystalline rods by cooling the melted poly-crystalline material under suitable conditions [see e.g., Bridgman (18, 19); Hanson (51)]. The orientation of the resulting single crystals can sometimes be controlled by "seeding" with a small crystal of the required orientation during the formation of the main crystal [see e.g., Grüneisen and Goens (50)]. Metal crystals prepared in this way (whether with or without seeding) require careful annealing before the measurements are taken (see Fig. 4 and p. 425).

Another matter of obvious importance is the determination of orientation, but this again is too wide a subject to be dealt with adequately in the present paper. Methods which have been used include measurement of cleavage or fracture angle (18, 50); etching the crystal and observing the reflection of light from the etched surface (18, 112, 119); the use of x-rays (44, 45, 50); and measurement of the linear coefficient of thermal expansion (50).

7. Results at Ordinary Temperatures

Voigt's work covered the period roughly from 1875–1910. Before the end of this time, the theory of the elastic coefficients and their measurement had been established, mainly by Voigt himself, and was in a form susceptible only of refinement (7, 61) and extended application (22, 41). Nevertheless, apart from Voigt's own measurements (115, 116, 117) no experiments appear to have been reported until 1924, with the exception of isolated and incomplete determinations such as those of Reimers (94), Sieg and Miller (105), and Perrier and Mandrot (89).

In 1924, complete results for a number of metals were published by Grüneisen and Goens (50) and by Bridgman (17) [see also Bridgman (18)]. A little later, the measurements of Mandell (77, 78) on Rochelle salt, of Masima and Sachs (80) and Karnop and Sachs (65) on copper alloys, and of Bridgman (19) on alkali halides were published. In this latter work, the coefficients were obtained by measuring statically the Young's and rigidity moduli of artificially grown specimens and combining the results with Slater's measurements (108) of compressibility.

The specimens were of circular cross section, and the methods used correspond to Eqs. (2.8) and (2.15), Table II. Although the coupling between torsion and flexure had previously been pointed out by Voigt (115) its importance had not been emphasized at the time these measurements were made, and Bridgman, in effect, assumed that the conditions of his experiments produced free bending and free torsion.

Thus (Eq. (2.8))

$$s'_{33}=12\pi r^4 x_m/P_m l^3, \qquad (7.1)$$

and (Eqs. (3.1) and (2.15))

$$2s'_g=s'_{44}+s'_{55}=\pi r^4\phi/Nl. \qquad (7.2)$$

Further, for cubic crystals, to which group the alkali halides belong, Eqs. (1.14), (1.15), and (2.2) give, respectively,

$$s'_{33}=s_{11}-2(s_{11}-s_{12}-\tfrac{1}{2}s_{44})\\ \times(\gamma_1^2\gamma_2^2+\gamma_2^2\gamma_3^2+\gamma_3^2\gamma_1^2); \qquad (7.3)$$

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[4] More recently (1945) the shear constants have been measured by the high frequency method.

[5] See several papers in *The Review of Scientific Instruments* (1933–9).

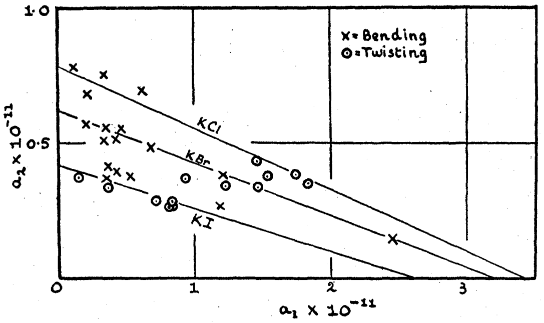

FIG. 2. Bridgman's results for potassium halides. Each point belongs to line nearest to it.

$$s'_{44}+s'_{55}=4(s_{11}-s_{12})$$

$$-4(s_{11}-s_{12}-\tfrac{1}{2}s_{44})(\gamma_1{}^4+\gamma_2{}^4+\gamma_3{}^4); \quad (7.4)$$

$$K=3(s_{11}-2s_{12}), \quad (7.5)$$

where K is the cubic compressibility.

Now, s'_{33} can be eliminated between (7.1) and (7.3), and $(s'_{44}+s'_{55})$ between (7.2) and (7.4). Equation (7.5) can be used to eliminate s_{12}; the expressions resulting from these operations can be written

$$a_{11}s_{11}+a_{12}s_{44}=1, \quad (7.6)$$

$$a_{21}s_{11}+a_{12}s_{44}=1. \quad (7.7)$$

The quantities a are functions of orientation, specimen dimensions, and compressibility; in addition, a_{11} and a_{12} are functions of P_m and x_m, while a_{21} and a_{22} are functions of N and ϕ. Equations (7.6) and (7.7) must obviously be identical, since they refer to s_{11} and s_{44} which are constants of the material. They can therefore be written

$$a_1s_{11}+a_2s_{44}=1, \quad (7.8)$$

and, for any one substance, the points obtained by plotting a_1 against a_2 should lie on a single straight line irrespective of whether a_1 and a_2 were obtained from bending or torsion.

Bridgman's results reduced in this way are plotted in Fig. 2; the intercepts of the straight lines on the axes give, respectively, $1/s_{11}$ and $1/s_{44}$. It appears that Bridgman's results in themselves would have been sufficient to determine the principal coefficients, and, in view of the method adopted, it is difficult to decide to what extent the final accuracy depends on Bridgman's measurements, and to what extent on Slater's. Figure 2, however, shows that all the points fall reasonably close to the appropriate straight lines, thus justifying to this extent the reliability of the results.

In 1930, some measurements on tungsten were reported by Wright (123). These are interesting, not only in themselves, but also as being apparently the first for which all the coefficients in a system were measured by vibration methods.

The possibility of using these methods was realized by Voigt, who set up the differential equations of motion of thin crystalline rods, but did not pursue the matter any further. Sieg and Miller (105) in 1921 obtained the rigidity modulus of selenium single crystals and Grüneisen and Goens (50) in 1924 the Young's modulus of zinc and cadmium crystals, by vibration methods. At that time, however, there was no exact theory in terms of which the vibration results could be interpreted, except in particular cases; this deficiency was later remedied by Goens (41), Kimura (68), and Brown (22). It so happens that tungsten, in spite of belonging to the cubic system, cannot be distinguished experimentally from an isotropic material as far as elastic properties are concerned (see p. 430). Thus torsion-flexure coupling is absent at all orientations, and the vibration results are not complicated by this effect, although Wright realized the possibility of such complication, and was in a position to apply the necessary corrections, had they been required.

Wright's paper was followed by others from Bridgman (20); Röhl (97); Kimura (66); Goens (42, 43); and Davies (26), some of whom used static and others dynamic methods. From this time onwards, dynamic methods began to come more into favor, a process which was furthered, at least in America, by Balamuth's description (3) in 1934 of the piezoelectric oscillator (see p. 421). This device was originally applied to the measurement of Young's modulus; Balamuth carried out a series of tests establishing the validity of the method, which was later extended to the measurement of rigidity modulus by Rose (98) and Durand (30). Since this time many measurements have been made by vibration methods in America, and in the great majority of them the piezoelectric oscillator has been used. Other vibration methods have also been developed and

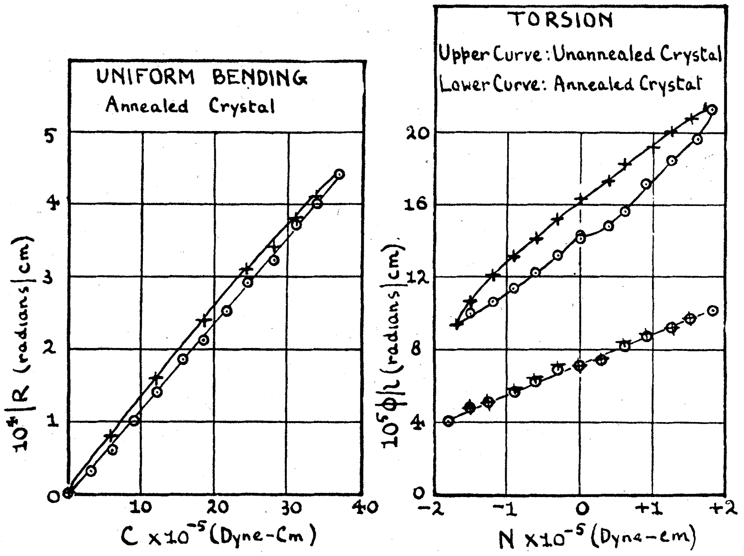

FIG. 3. Stress-strain curves of zinc single crystals (Hanson). ⊙ = loading + = unloading.

used, mainly outside America; one reason for the recent popularity of such methods is undoubtedly the variety of reliable and accurate apparatus, such as valve oscillators, cathode-ray oscillographs, etc., which are now available. Another reason for the popularity of vibration methods is their convenience and adaptability; they have proved particularly useful in studies of the effect of temperature on the elastic coefficients.

It may perhaps be pointed out here that surprisingly little fundamental information has been published regarding vibration methods as applied to crystal specimens. For example, little, if any, data are available concerning the effect of such quantities as amplitude and frequency of vibration on the measured constants. Further, direct comparisons of dynamic with static methods are very rare; a few such comparisons have, however, been made by Goens and Weerts (44) on gold and copper crystals in torsion and by Kimura (68) on iron crystals without obtaining any significant difference between the two methods.

Although the majority of measurements made since, say 1936, have employed dynamic methods, static measurements are still made, and in fact, the bulk of the information regarding the fundamental elastic behavior of crystals has been gained from static experiments. The utility of static methods is particularly well shown in some work by Hanson (51) on zinc crystals. In the hexagonal system, to which zinc belongs, the equations for reciprocal Young's and rigidity modulus of a specimen of circular cross section are:

$$s'_{33}=s_{11}(1-\gamma_3^2)^2+s_{33}\gamma_3^4+(2s_{13}+s_{44})(1-\gamma_3^2)\gamma_3^2; \quad (7.9)$$

$$s'_g=\tfrac{1}{2}(s'_{44}+s'_{55})=s_{44}+(s_{11}-s_{12}-\tfrac{1}{2}s_{44})(1-\gamma_3^2)+2(s_{11}+s_{33}-2s_{13}-s_{44})(1-\gamma_3^2)\gamma_3^2, \quad (7.10)$$

where γ_3 is the cosine of the angle between the specimen axis and the hexagonal axis.

Hanson was one of the first workers to take account of Goens' work (41) on the coupling between flexure and torsion [see, however, Kimura (66)], and his experiments were carried out in general so as to minimize the errors introduced by this coupling.

Stress-strain curves, obtained by Hanson, are shown in Fig. 3. It will be seen that the curves for uniform bending is not reversible, but the hysteresis shown in the figure was found to be perfectly reproducible and was not removed by annealing.[6] The ascending part of the curve is

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[6] Later evidence shows that the hysteresis is probably caused by the apparatus used, and is not an inherent property of the zinc crystals. See Webb (119) footnote 26, p. 303.

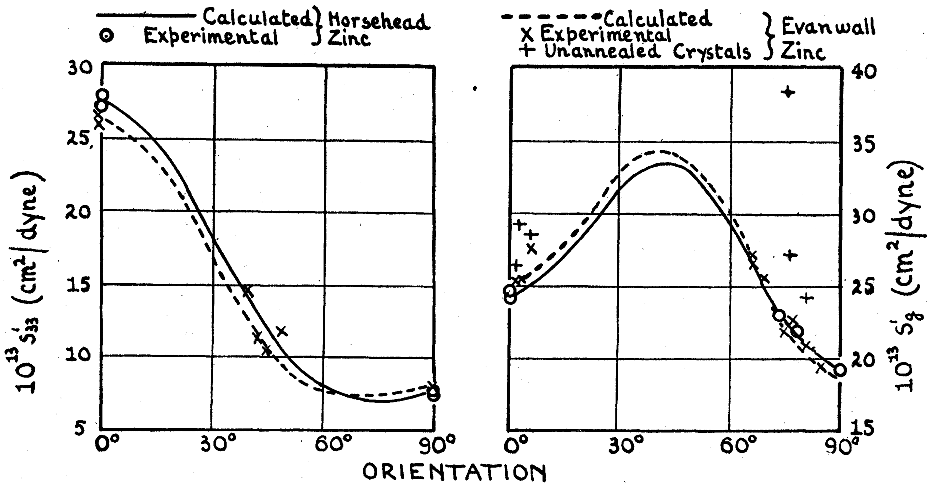

FIG. 4. Effect of orientation on reciprocal Young's and rigidity modulus of zinc single crystals (Hanson).

straight, and was used by Hanson in calculating the Young's moduli. The torsion curves show the importance of proper annealing. The upper curve, for the unannealed crystal, is neither straight nor reversible. Moreover, it has a considerably different average slope from that of the lower curve. This latter refers to the annealed crystal and is both straight and reversible. The necessity for annealing is again brought out by Fig. 4 on which the experimental results are compared with the theoretical curves. The values for the unannealed crystals, obtained from the average slope of the stress-strain curve show that errors of considerable magnitude may be introduced by failure properly to anneal the specimens.

Hanson used two varieties of zinc, each 99.99+ percent pure, yet, as Fig. 4 shows, still obtained considerable differences between them (see also Table V, 4). According to Hanson, the differences are larger than can be accounted for by experimental error, or by differences in treatment, and he concludes that the elastic coefficients of zinc are quite sensitive to small amounts of impurity (see, however, p. 430). Hanson points out that the coefficients of the Evanwall zinc are closer to Goens' values (42), while those of the Horsehead zinc agree better with Bridgman's determinations (17). He therefore suggests that the differences between the results of these two workers are genuine, and are caused by the material.

The compressibility of the Evanwall zinc, computed by Hanson from his coefficients, is 14.18×10^{-13} and that of the Horsehead 18.40 $\times10^{-13}$ cm²/dyne. The compressibilities of these two materials have been measured directly by Bridgman (21), who could detect no difference, and says that the computed differences probably originate in small internal strains. A further attempt to account for the differences has been made by Tyndall (114), who claims that a satisfactory set of coefficients can be based on Bridgman's compressibility measurements, and Hanson's coefficients for the Horsehead zinc.

Following this work, further results for other materials were published by Goens and Weerts (44), Goens and Schmid (45), Michaelow (84), and Mason (81), while Kimura (68) has made a comprehensive study of the coefficients of iron, a substance which is of particular interest in view of its ferromagnetic nature.

Before 1938, measurements of the elastic coefficients made in America by vibration methods had been carried out on specimens for which the torsion flexure coupling was zero. This was not so for the sodium specimens used by Quimby and Siegel (92), nor for the Cu_3Au alloy specimens used by Siegel (106, 107).

The longitudinal frequencies were interpreted on the assumption that the modulus involved was the free Young's modulus (E_F) and the torsional frequencies on the assumption that the

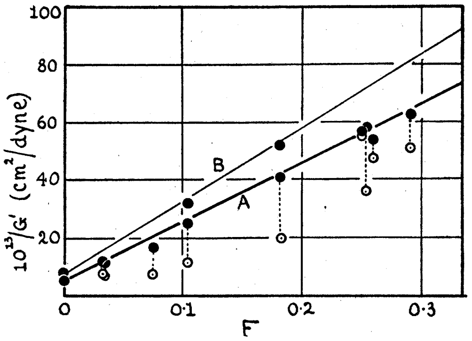

FIG. 5. Effect of orientation on reciprocal rigidity modulus of β-brass (Good).

modulus involved was the pure rigidity modulus (G_P). These assumptions had virtually been justified by Goens (41), but it was not until Brown (22) derived and extended the results by a more compact method that the full significance of the torsion flexure coupling was realized in America.

The effect of this coupling may be seen from Fig. 5, taken from a paper by Good (47). The results were obtained on β-brass, using the composite piezoelectric oscillator. The open circles represent reciprocal rigidity modulus uncorrected for effects of torsion flexure coupling, while the solid circles represent the same quantities after correction, i.e., they are values of s'_g. The figure shows that large errors may be introduced by failure to take the effects of coupling into account.

β-brass is a cubic crystal, and Eqs. (7.3), (7.4), and (3.8), which apply to the cubic system, can be rewritten

$$s'_{33}=s_{11}-2sF; \tag{7.11}$$

$$s'_g=s_{44}+4sF; \tag{7.12}$$

$$\epsilon=2s^2(F-4F^2+3\chi)/s'_{33}s'_g, \tag{7.13}$$

where the following abbreviations have been used

$$s=(s_{11}-s_{12}-\tfrac{1}{2}s_{44}); \tag{7.14}$$

$$F=(\gamma_1{}^2\gamma_2{}^2+\gamma_2{}^2\gamma_3{}^2+\gamma_3{}^2\gamma_1{}^2); \tag{7.15}$$

$$\chi=\gamma_1{}^2\gamma_2{}^2\gamma_3{}^2. \tag{7.16}$$

Thus, if s'_{33} is plotted against F, the slope of the line should be $-2s$, while if s'_g is plotted against F, the slope should be $+4s$. The slope of the heavy line A, through the corrected points was determined not from any consideration of best fit, but from the value of s obtained by Rinehart (95, 96), who measured s'_{33} of β-brass as a function of orientation also using the piezo electric oscillator. The fit between the solid circles and the straight line A in Fig. 5 not only substantiates this method of handling the data, but also provides some indirect proof of the accuracy of the torsion flexure correction. The experimental points and the line A refer to room temperature; the line B refers to a temperature of about 460°C.

In 1942, Swift and Tyndall (112) carried out a study by static methods of the elastic properties of lead single crystals. They obtained some particularly interesting results in their torsion experiments [see Goens (42), for an earlier discussion of this subject]. The fixed ends of their specimens were embedded in Wood's metal, while the movable ends were held in jaws on a torsion head, which was free to rotate in ball bearings. The stress-strain relations of specimens with no torsion-flexure coupling ($\epsilon=0$) were straight lines and the slopes were consistent to 1 percent. For crystals with appreciable torsion flexure coupling ($0<\epsilon<1$), the results were more complicated. The ball bearings in the torsion head had just about sufficient play to permit free torsion. Thus, a slope corresponding to the free rigidity modulus (G_F) might be expected. If, however, the bearing reaction produced a partial restraint, the slope should correspond to something between G_F and G_P. In the extreme case, where the torsion head was initially definitely in contact with one side of the bearing, torque in one sense would produce no bending (i.e., the torsion would be pure), while torque in the opposite sense would produce the bending requisite for free torsion. The stress-strain relation would then be two straight lines, one with slope corresponding to G_P and the other to G_F.

Broadly speaking, if ϵ was greater than 0, the following results were obtained. Either single straight lines, with slope corresponding to G_F, were observed, or else the curve showed two distinct slopes. Some typical results are shown in Fig. 6 in which ϕ is the angle of twist, and N the applied torque. A is a single slope curve and B a double slope curve, each obtained on the same specimen. Curve C is for a different specimen and shows the double slope more markedly than does *B*.

From the various stress/strain slopes obtained in the course of from 5 to 20 runs on a given specimen, the extreme values were picked out. The minimum slope was assumed to correspond with G_F and the maximum with G_P. G_F was then calculated from the latter values by means of the equation

$$G_F = 1/s'_g = G_P(1-\epsilon). \qquad (7.17)$$

In the Young's modulus measurements, Swift and Tyndall used a static tension method, and obtained straight-line load—extension curves, with no appreciable hysteresis or permanent set. They also obtained results which further emphasize the need for careful annealing. Thus, two of their specimens gave $1/E = s'_{33} = 97.1$ and 94.7×10^{-13} cm²/dyne before annealing, while after annealing, for 3 hours at 200°C, the respective values were 94.7 and 93.1 The differences due to annealing are considerably smaller than those observed by Hanson (51) for the rigidity modulus of zinc crystals (see p. 425), but they are still appreciable.

The procedure adopted by Swift and Tyndall was to measure the Young's modulus on the unannealed specimen, and if the values differed noticeably from that indicated by the results of Goens and Weerts (44), the specimen was annealed, and Young's modulus redetermined. On this basis, it was only found necessary to anneal the two specimens quoted above out of a total of five.

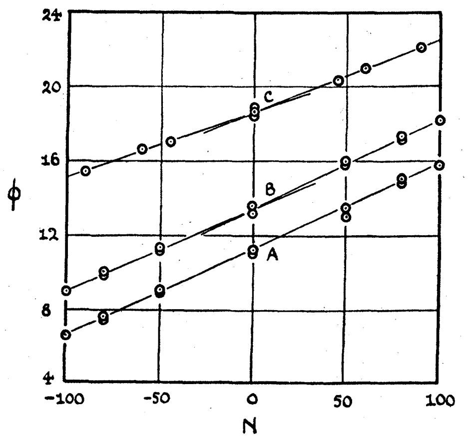

FIG. 6. Stress-strain curves of lead single crystals in torsion (Swift and Tyndall). Unit of torque = 1960 dyne-cm; unit of twist = 5.6×10^{-5} radians.

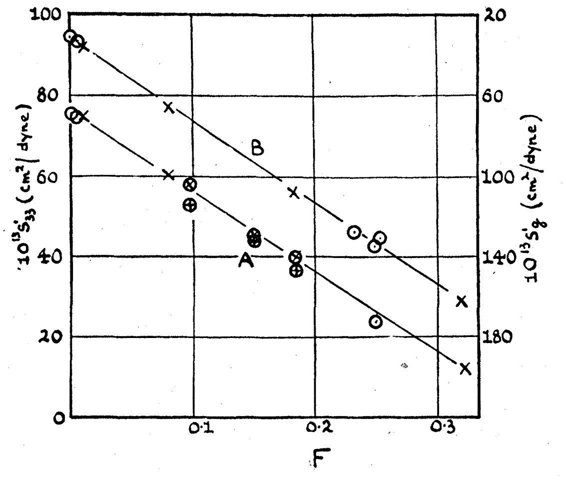

FIG. 7. Effect of orientation on reciprocal rigidity modulus (*A*) and reciprocal Young's modulus (*B*) of lead single crystals. *X*, Goens and Weerts; ⊙, Swift and Tyndall; ⊕, Swift and Tyndall (min. slope); ⊗, Swift and Tyndall (max. slope and $1-\epsilon$).

The final results are plotted in Fig. 7, in which the notation is the same as that of Eqs. (7.11) and (7.12). Goens and Weerts' results are also plotted; the agreement throughout is very close, a fact which is reflected in the excellent agreement obtained by the two pairs of workers for the principal coefficients (see Table V, 2).

It will be seen that slope of curve $A = -2 \times$ slope of curve *B*, as predicted by Eqs. (7.11) and (7.12).

Numerical values of the independent principal coefficients for at least the majority of systems on which complete determinations have been made are given in Table V; the results quoted are all at room temperature (20°C). A similar, but less exhaustive, table has been published by Seitz and Read (104). One or two sets of results, for which it has not been possible to identify the original authority, are quoted in Table V direct from Seitz and Read's paper.

8. Discussion of Results at Ordinary Temperatures

Theoretically, static experiments yield the isothermal elastic coefficients and dynamic experiments the adiabatic coefficients, provided the

TABLE V. Principal elastic coefficients at room temperature (unit $=10^{-13}$ cm^2/dyne).

(1) Cubic system (3 coefficients) alloys

Material	s_{11}	s_{44}	s_{12}	Authority
Cu_3Au	13.4	15.1	−5.65	Siegel (106, 107)
72% Cu, 28% Zn	19.4	13.9	−8.4	Masima and Sachs (80)
β-brass (CuZn)	38.8	5.78	−15.2	Rinehart (96); Good (47)
5% Cu, 95% Al	15	37	−6.9	Karnop and Sachs (65)
75% Ag, 25% Au	20.7	20.5	−8.91	Röhl (97)
50% Ag, 50% Au	19.7	19.7	−8.52	Röhl (97)
25% Ag, 75% Au	20.5	20.6	−9.09	Röhl (97)

(2) Cubic system (3 coefficients) metals

Material	s_{11}	s_{44}	s_{12}	Authority
Aluminum	15.9	35.2	−5.8	Goens (43)
Copper	12.9	16.4	−5.2	Kimura (66)
Copper	14.9	13.3	−6.3	Goens and Weerts (44)
Gold	22.7	22.9	−10.3	Röhl (97)
Gold	23.3	23.8	−10.7	Goens and Weerts (44)
Gold	22.9	23.4	−10.4	Seitz and Read (104)
Iron	7.65	8.95	−2.89	Kimura and Ohno (67)
Iron[a]	7.72	9.02	−2.85	Kimura (68)
Iron[b]	7.73	8.98	−2.72	Kimura (68)
Iron	7.57	8.62	−2.82	Seitz and Read (104)
Lead	93.0	69.4	−42.6	Goens and Weerts (44)
Lead	92.8	69.4	−42.4	Swift and Tyndall (112)
Potassium	833	380	−370	Seitz and Read (104)
Silver	23.2	22.9	−9.9	Röhl (97)
Sodium[c]	571	235	−248	Quimby and Siegel (92)
Tungsten	2.534	6.55	−0.726	Bridgman (17)
Tungsten	2.573	6.55	−0.729	Wright (123)

(3) Cubic system (3 coefficients) minerals and artificial crystals

Material	s_{11}	s_{44}	s_{12}	Authority
Diamond	1.47	2.4	−0.45	Bhagavantam and Bhimasenachar (8)
Fluorspar	6.91	29.6	−1.49	Voigt (115)
Fluorspar	6.94	27.7	−1.53	Bergmann (6a)
Lithium fluoride	10.59	15.92	−2.85	Bergmann (6a)
Magnesium oxide	4.01	6.46	−0.93	Durand (30)
Potassium alum	54	117	−16	Voigt (117)
Potassium bromide	31.7	161	−4.7	Bridgman (19)
Potassium chloride (sylvine)	27.4	156	−1.37	Voigt (115)
Potassium chloride	29.4	127	−5.3	Bridgman (19)
Potassium chloride (sylvine)	25.9	158	−3.5	Durand (30)
Potassium iodide	39.2	238	−5.4	Bridgman (19)
Pyrites	2.88	9.48	+0.44	Voigt (115)
Pyrites	2.86	9.61	+0.39	Bhagavantam and Bhimasenachar (9)
Sodium bromide	40.0	75.4	−11.5	Bridgman (19)
Sodium chlorate	24.6	83.6	+12.5	Voigt (115)
Sodium chloride (rocksalt)	24.4	78.22	−5.23	Voigt (115)
Sodium chloride (rocksalt)	23	78	−5	Bridgman (19)
Sodium chloride (rocksalt)	22.5	78.7	−4.67	Durand (30)
Sodium chloride (rocksalt)	22.8	78.1	−4.5	Hunter and Siegel (59)
Sodium chloride	24.0	84.0	−5.0	Bergmann (6a)
Zinc blende	19.4	22.9	−7.3	Voigt (116)
Zinc blende	20.0	24.3	−8.0	Bhagavantam and Suryanarayana (10)

frequency of vibration is sufficiently high. The difference between the two has been dealt with by Voigt [(115) Sections 392–393], whose equation can be written:

$$\Delta_{ik}=-Ta_i a_k/\rho C_P, \qquad (8.1)$$

where

$\Delta_{ik}=$ (adiabatic $s_{ik}-$ isothermal s_{ik}),
$T=$ absolute temperature,
a_i, $a_k=$ thermal expansion coefficients,
$\rho=$ density,

TABLE V.—*Continued*

(4) Hexagonal system (5 coefficients)

Material	s_{11}	s_{33}	s_{44}	s_{12}	s_{13}	Authority
Beryl	4.42	4.70	15.3	−1.37	−0.86	Voigt (115)
Cadmium	12.3	35.5	54.0	−1.5	−9.3	Grüneisen and Goens (50)
Cadmium	12.9	36.9	64.0	−1.5	−9.3	Bridgman (17)
Magnesium	22.3	19.8	59.5	−7.7	−4.5	Seitz and Read (104)
Magnesium	22.1	19.7	60.3	−7.7	−4.9	Goens and Schmid (45)
Zinc	8.23	26.4	25.0	+0.34	−6.6	Bridgman (17)
Zinc	8.0	28.2	25.0	−0.5	−6.1	Grüneisen and Goens (50)
Zinc	8.4	28.7	26.4	+1.1	−7.7	Goens (42)
Zinc	8.08	26.3	25.1	+1.6	−7.9	Hanson (51)
Zinc	7.70	27.7	24.4	+0.45	−6.4	Hanson (51)

(5) Tetragonal system (b) (6 coefficients)

Material	s_{11}	s_{33}	s_{44}	s_{66}	s_{12}	s_{13}	Authority
Tin	18.5	11.8	57.0	135	−9.9	−2.5	Bridgman (17)

(6) Trigonal and hexagonal systems (b) (6 coefficients)

Material	s_{11}	s_{33}	s_{44}	s_{12}	s_{13}	s_{14}	Authority
Antimony	17.7	33.8	41.0	−3.8	−8.5	−8.0	Bridgman (17)
Bismuth	26.9	28.7	104.8	−14.0	−6.2	+16.0	Bridgman (17)
Calcspar	11.3	17.5	40.3	−3.7	−4.3	+9.1	Voigt (115)
Haematite	4.41	4.43	11.9	−1.02	−0.23	+0.79	Voigt (115)
Quartz	13.0	9.9	20.0	−1.66	−1.52	−4.30	Voigt (115)
Quartz	12.79	9.56	19.78	−1.53	−1.10	−4.46	Mason (82)
Tellurium	48.7	23.4	58.1	−6.9	−13.8	[d]	Bridgman (17)
Tourmaline	3.98	6.24	15.1	−1.03	−0.16	+0.58	Voigt (115)

(7) Orthorhombic system (9 coefficients)

Material	s_{11}	s_{22}	s_{33}	s_{44}	s_{55}	s_{66}	s_{12}	s_{13}	s_{23}	Authority
Aragonite	6.95	13.2	12.2	24.2	39.0	23.4	−3.0	+0.4	−2.4	Voigt (115)
Barytes	16.4	18.9	10.6	84.0	34.8	36.0	−9.0	−1.9	−2.5	Voigt (115)
Barytes	17.2	19.9	10.9	85.5	35.8	39.2	−9.9	−1.7	−3.0	Bergmann (6a)
Rochelle salt	47	32	28	61	305	80	−8.0	−22	−17	Mandell (77)
Rochelle salt[e]	56	38	37	88	360	118	−9	−34	−5	Mandell (78)
Rochelle salt	52.3	34.3	32.4	96.3	338	118	−21.8	−16.9	−13.3	Hinz (54)
Rochelle salt	51.8	34.9	33.4	79.8	328	101	−15.3	−21.1	−10.3	Mason (81)
Topaz	4.43	3.53	3.84	9.25	7.52	7.64	−1.38	−0.86	−0.66	Voigt (115)
Wood (Beech)	878	72.6	447	640	2250	965	−37.9	−325	−32.7	Hörig (55, 56)
Wood (Oak)	1040	175	468	828	2740	1320	−88.4	−303	−59.4	Hörig (55, 56)
Wood (Pine)	1110	61.0	770	1110	13400	1190	−24.6	−461	−24.0	Hörig (55, 56)
Wood (Spruce)	1580	59.9	1230	1600	28400	1170	−34.0	−527	−22.0	Hörig (55, 56)

[a] Static measurements.
[b] Dynamic measurements.
[c] Extrapolated values at 20°C.
[d] Not measured.
[e] Potassium ion replaced by ammonium.

C_P = specific heat at constant pressure in mechanical units.

Table VI is based on Voigt's data, with the exception of the results for lead [Swift and Tyndall (112)] and aluminum [Goens (43)]. The quantity entered in the table is 100 Δ_{ik}/s_{ik}, and represents the difference between the adiabatic and isothermal s_{ik} expressed as a percentage of the isothermal s_{ik}.

The figures in Table VI are almost invariably smaller (some of them considerably smaller) than the error involved in measuring the associated s_{ik}, even by modern methods and, for this reason, the difference between isothermal and adiabatic coefficients is ignored in Table V; in special applications, however, it is necessary to take the difference into account.

A comparison of the coefficients measured on a single chemical substance can be made in a limited number of cases. For the three materials lead, magnesium, and tungsten, the results obtained by different workers are so close as to be indistinguishable; further, for at least two of

TABLE VI. Percentage difference between adiabatic and isothermal coefficients.

Material	System	s_{11}	s_{12}	s_{13}	s_{33}	Other relations
Fluorspar	Cubic	-0.48	2.2	(2.2)	(-0.48)	$\Delta_{44}=0$
Rocksalt (NaCl)	Cubic	-0.82	3.8	(3.8)	(-0.82)	
Sylvine (KCl)	Cubic	-0.93	18	(18)	(-0.93)	
Lead	Cubic	-0.17	3.8	(3.8)	(-0.17)	
Aluminum	Cubic	-0.43	1.2	(1.2)	(-0.43)	
Beryl	Hexagonal		←Negligible→			
Quartz	Trigonal (b)	-0.23	1.8	1.0	-0.08	$\Delta_{44}=\Delta_{66}=\Delta_{14}=0.$
Calcspar	Trigonal (b)	-0.04	0.11	-0.45	-0.49	
Tourmaline	Trigonal (b)	-0.03	0.10	1.2	-0.08	

these substances (lead and tungsten), one set of coefficients was obtained by static and the other by dynamic methods.

For other substances, as pointed out by Zwicky (125) differences of the order of 5–10 percent are not uncommon, even when the measurements are made by a single worker, using a single method.

Bridgman (19), for example, obtained the following results on specimens cut from two crystals of natural rocksalt;

	CRYSTAL *A*	CRYSTAL *B*
$s_{11}\times 10^{13}$ cm^2/dyne	21.6, 20.9, 20.2	23.0, 23.2
$s_{44}\times 10^{13}$ cm^2/dyne	78.2	74.2

The differences observed by Hanson (51) between two samples of zinc, each 99.99+ percent pure have already been discussed (p. 425; see also Table V, 4), but they show differences of the same order as the above. The elastic coefficients of crystals cannot therefore always be regarded as invariable quantities, but are affected to some extent by such factors as mode of origin or preparation of the specimen, and, possibly, by the presence of impurities.

Little information is available, however, regarding the effect of impurities, but the indications are that this effect may be smaller than has sometimes been supposed. The following results, due to Goens and Schmid (45) refer to magnesium contaminated with known amounts of zinc.

% Zinc by weight	ϕ	$s'_{33}\cdot 10^{13}$	$\delta\cdot 10^{13}$
1	23°45′	21.4	$+0.1$
1	46°30′	23.1	-0.1
1	74°45′	22.3	-0.3
1.6	78°15′	22.3	-0.1
2.3	85°30′	22.1	-0.0

ϕ is the angle between the hexagonal axis and the axis of the specimen; s'_{33} is the value as measured; δ is the difference between s'_{33} and the value calculated for a pure magnesium crystal of the same orientation [Eq. (7.9); $\gamma_3=\cos\phi$]. The entries under δ above do not exceed experimental error.

A further factor which may affect the values of the elastic constants is the presence of imperfections in the crystal. Zwicky (125) has shown that if the crystal is ideal and the forces purely electrical, the elastic parameters should be numerically of the order of 10^{12} dynes/cm^2. He has also given a short qualitative discussion of the effect of such imperfections as holes and fissures, "knots" and twins.

Some of the materials in Table V, although belonging to a non-isotropic system, are approximately isotropic as far as elastic properties are concerned, e.g., haematite, magnesium

$$(s_{44}\approx 2[s_{11}-s_{12}];\quad s_{11}\approx s_{33})$$

and tungsten

$$(s_{44}\approx 2[s_{11}-s_{12}]).$$

A further confirmation of the elastic isotropy of tungsten is provided by the work of Wright (123), who measured s'_{44} of this element at a number of orientations and found that none of the differences exceeded 0.5 percent. There is some evidence that a material may be isotropic at one temperature and anisotropic at all others (see p. 432 and Fig. 8).

The occasional occurrence of positive values of the coefficients s_{12}, s_{23}, s_{31} is of interest, since it implies that the Poisson's ratio associated with these values is itself negative. In the case of pyrites, Voigt (115) obtained $s_{12}=+0.44\times 10^{-13}$

cm^2/dyne. It has been suggested (73) that this result may have been caused by twinning in the specimen used by Voigt, but Bhagavantam and Bhimasenachar (9) using a combination of supersonic and static methods (see p. 422), obtained $+0.39\times 10^{-13}$ and conclude that the possibility of twinning is ruled out by this agreement.

The constants measured on galena by Bhagavantam and Bhimasenachar (9) have been omitted from Table V. The values quoted in the original paper are $c_{11}=2.65$, $c_{12}=6.99$, $c_{44}=4.47$ all$\times 10^{11}$ dynes/sq. cm, and these values check with independent measurements made by the authors, who suggest that the low value of c_{11} may be caused by the easy cleavage of galena along the (100) plane. If, however, the above parameters are put into the conversion equations analogous to (1.20) the following values are obtained for the coefficients: $s_{11}=-13.4$, $s_{44}=22.4$, $s_{12}=+9.7$, all $\times 10^{-13}$ cm^2/dyne. The coefficients thus lead to a negative principal Young's modulus, which is impossible, and to a principal Poisson's ratio of 0.72. It would seem that the results for galena are caused to some peculiarity of the material rather than by the method, since the measurements on pyrites (9) and zinc blende (10) gave good agreement with Voigt's values (see Table V, 3). Whatever the explanation, the results for galena should be accepted with caution until they are either confirmed or superseded by better values.

The only non-crystalline material included in Table V is wood. There is now a considerable support for the identification of wood with a rhombic system, at least as a first approximation (52, 53, 55, 56, 62). In spite of its defects, such as imperfect structure and elasticity, some important aspects of the elasticity of anisotropic materials may be brought out by a study of wood quite apart from its technical importance (48, 49, 53).

A question which at one time caused considerable controversy is that of the so-called "Cauchy relations" [Love (73), pp. 14, 99–100, 616–628; Voigt (115), Sections 292–303, 369, etc.]. It is now accepted that these relations are not valid, but a consideration of them is still instructive.

According to Eqs. (1.7), $s_{ik}=s_{ki}$, and similarly $c_{ik}=c_{ki}$. These equations reduce the maximum possible number of independent coefficients or parameters from 36 to 21 and are necessary if the strain energy function W (Eq. (1.13)) is to exist [Love (73), p. 99]. They are, therefore, independent of any molecular hypothesis. If, however, it is assumed that the crystal is made up of a number of material points which act on each other at a distance, and that the field of force of each point possesses spherical symmetry, then certain new relations appear among the elastic constants. Expressed in terms of the parameters, these "Cauchy" relations are

$$\begin{gathered} c_{23}=c_{44};\quad c_{31}=c_{55};\quad c_{12}=c_{66}; \\ c_{14}=c_{56};\quad c_{25}=c_{46};\quad c_{45}=c_{36}. \end{gathered} \tag{8.2}$$

If these relations were true they would reduce the maximum possible number of independent coefficients from 21 to 15. Table I shows that in designating the coefficients s_{ik} with two suffixes only, a contraction has been made, and the same is true of the parameters c_{ik}. Thus, in full, c_{12} is $c(xxyy)$, c_{21} is $c(yyxx)$, and c_{66} is $c(xyxy)$. The equations $c_{ik}=c_{ki}$ imply that if the parameters are written in full, the order of suffixes, taken in pairs, is immaterial. The Cauchy relations go a step further and imply that the order of suffixes, however taken, is unimportant, so that, for example, $c(xxyy)$ and $c(xyxy)$ are equivalent, i.e., $c_{12}=c_{66}$.

When applied to particular systems, the relations become:

In the cubic system: $c_{12}=c_{44}$.

In the hexagonal and trigonal (b) systems:

$$\begin{gathered} c_{13}=c_{44};\quad c_{11}=3c_{12} \\ (\text{since } c_{66}=\tfrac{1}{2}(c_{11}-c_{12})). \end{gathered} \tag{8.3}$$

In the rhombic system:

$$c_{23}=c_{44};\quad c_{31}=c_{55};\quad c_{12}=c_{66}.$$

Some results are given in Table VII which show that the relations are only obeyed in isolated cases. Bridgman (19) however, considers that the relations, though not generally true, may nevertheless hold for the alkali halides. Voigt's values for sylvine appear to contradict this view, but these values had been previously

TABLE VII. Test of the Cauchy relations. Unit $=10^{11}$ dynes/cm^2.

A. Cubic system

Material	c_{12}	c_{44}	Authority
Fluorspar	4.48	3.38	Voigt (115)
Rocksalt	1.29	1.27	Voigt (115)
Sylvine	0.194	0.642	Voigt (115)
Sodium chlorate	−2.10	1.19	Voigt (115)
Pyrites	−4.74	10.52	Voigt (115)
Copper	12.3	7.53	Goens and Weerts (44)
Gold	15.7	4.20	Goens and Weerts (44)
Lead	4.09	1.44	Goens and Weerts (44)
Aluminum	6.2	2.84	Goens (43)

B. Hexagonal and trigonal (b) systems

Material	c_{13}	c_{44}	c_{11}	$3c_{12}$	Authority
Beryl	6.60	6.53	27.0	28.8	Voigt (115)
Zinc	4.37	4.00	16.1	13.0	Grüneisen and Goens (50)
Cadmium	4.42	1.85	12.1	14.4	Grüneisen and Goens (50)
Calcspar	4.51	3.42	13.7	13.7	Voigt (115)
Quartz	1.41	5.70	8.5	2.1	Voigt (115)
Tourmaline	0.88	6.67	27.0	20.7	Voigt (115)
Haematite	1.57	8.52	24.2	16.5	Voigt (115)

C. Rhombic system

Material	c_{23}	c_{44}	c_{31}	c_{55}	c_{12}	c_{66}	Authority
Topaz	8.8	10.8	8.4	13.2	12.5	13.1	Voigt (115)
Barytes	2.68	1.20	2.69	2.87	4.59	2.78	Voigt (115)
Aragonite	1.57	4.27	0.17	2.56	3.72	4.12	Voigt (115)

re-examined by Försterling, (33), who concluded that one of the specimens used by Voigt was faulty, and that the relation $c_{12}=c_{44}$ does actually hold for this material. More recent measurements on sylvine (30) and rocksalt (6a, 30, 59) have confirmed Bridgman's suggestion.

9. The Effect of Temperature on the Elastic Coefficients

A number of papers have appeared within recent years, dealing with aspects of the effect of temperature on the coefficients. Durand (30) working with natural crystals of MgO, NaCl, and KCl found that his results for the isothermal parameters were represented with an error probably less than that of the experiments by the equations

$$c_{12}=\text{constant};\quad c_{11}=K_1\exp(-b_1f_1);$$

$$c_{44}=K_2\exp(-b_2f_2),$$

where K_1, b_1, K_2, and b_2 are constants different for each substance; f_1 and f_2 are functions of T/θ ($T=$absolute temperature, $\theta=$Debye critical temperature), and are the same for all three substances.

The data necessary for the calculation of the parameters at any temperature are shown in Tables VIII and IX. Since $f_1=f_2=0$ at $T=0$, K_1 and K_2 are, respectively, the values of the isothermal c_{11} and c_{44} at absolute zero.

Brjukhanov (23) found the anisotropy of rocksalt to be affected by temperature, and stated that at 713°K this crystal becomes elastically isotropic. In Fig. 8, part of the data of Hunter and Siegel (59) for rocksalt is plotted against temperature. For isotropic materials, $s_{44}=2(s_{11}-s_{12})$, and Fig. 8 shows that, according to the data used, this relation is true at 680°K. Brjukhanov's main finding is thus confirmed, but there is a difference of some 30° between the temperature as found by him, and the one derived from Hunter and Siegel's results.

The elastic constants of quartz as affected by temperature have been discussed by several

TABLE VIII.

Substance	$K_1\times10^{-11}$	$K_2\times10^{-11}$	b_1	b_2	θ(°K)	$C_{12}\times10^{-11}$
MgO	28.89	15.679	0.230	0.0920	946	8.57
Na	5.85	1.339	0.210	0.0601	320	1.17
KCl	4.95	0.669	0.168	0.0440	246	0.60

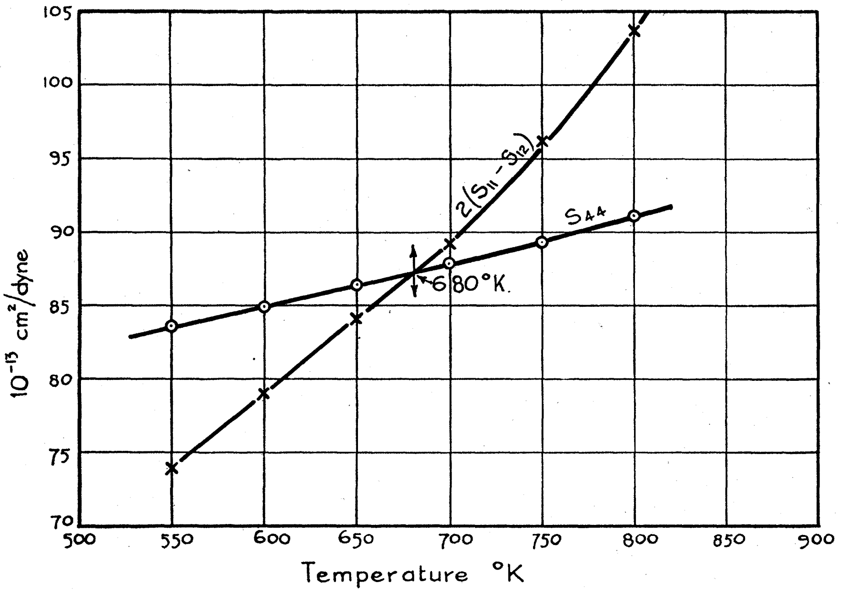

FIG. 8. Hunter and Siegel's data for NaCl.

workers. At ordinary temperatures, this material belongs to the trigonal (b) system and has, therefore, 6 independent coefficients, s_{11}, s_{12}, s_{13}, s_{14}, s_{33}, s_{44}. This form is stable up to 846°K, at which temperature a transition to β-quartz takes place; the coefficient s_{14} disappears, and the symmetry becomes hexagonal.

Early contributions to the problem of the effect of temperature near 846°K on the coefficients were made by Perrier and Mandrot (89), Freederickz and Michaelow (34), and Mandrot (79), but a complete set of coefficients for β-quartz was not available until the work of Kammer and Atanasoff (63) at a temperature of 873°K, was published.

TABLE IX.

T/θ	f_1	f_2
0.1	0.007	0.007
0.2	0.062	0.052
0.3	0.148	0.123
0.4	0.247	0.210
0.5	0.362	0.311
0.6	0.480	0.424
0.7	0.605	0.553
0.8	0.735	0.695
0.9	0.865	0.845
1.0	1.000	1.000
1.1	1.136	1.146
1.2	1.280	1.296
1.3	1.427	1.447
1.4	1.577	1.597
1.5	1.730	1.750

The method used was essentially that originally described by Atanasoff and Hart (1) and applied by Atanasoff and Kammer (2) to the measurement of the c_{44} parameter of β-quartz. It depends on the frequency of vibration of plates of the material, use being made of high harmonics in order to eliminate effects due to the finite dimensions of the plates. Kammer and Atanasoff's results are given in Table X under K and A, and compared with the previous results of Perrier and Mandrot [P and M (89)], and Lawson [L (70)]. Perrier and Mandrot's measurements were made statically, while the remainder were made dynamically. This introduces some uncertainty into the comparisons; a further uncertainty arises from the fact that the values given as Perrier and Mandrot's are actually their results interpolated to 873°K.

The quantities $s(45°)$ and $s(50°)$ are inverse Young's moduli of specimens having angles of 45° and 50°, respectively, between the specimen axis and the hexagonal axis. Kammer and Atanasoff's values at these angles were not measured directly, but were calculated from their principal coefficients and the transformation Eq. (7.9). Lawson's $s(45°)$ was measured directly from the frequency of longitudinal vibration of a specimen cut in this direction, using the piezoelectric oscillator; Lawson calculated s_{13} from $s(45°)$ using as supplementary data the results of Perrier and Mandrot (89), and Atanasoff and Kammer (2).

The value of c_{44} corresponding to Kammer and Atanasoff's s_{44} is $10^{13}/27.89=30.85\times10^{10}$; Osterberg and Cookson (87) give 19.36×10^{10} dynes/cm^2 for this parameter, a value which is regarded as untenable both by Kammer and Atanasoff and by Lawson.

Some extremely interesting work has been carried out in America on the temperature variation of the elastic coefficients as affected by the order-disorder transition in alloys. For a complete account of this transition, reference may be made to a paper by Nix and Shockley (86), but it has been briefly described by Good (47), as it occurs in β-brass, as follows: "Whether ordered or disordered, β-brass (approx. $\frac{1}{2}$Cu$\frac{1}{2}$Zn) is body centered cubic in structure. In the ordered state, the Cu atoms occupy the corners of the cubes, and the Zn atoms the centers. In the disordered state, the Cu and Zn atoms have equal probabilities of appearing either at the corner or the center lattice positions, as is shown by x-ray experiments."

Figure 9, which is taken from Good's paper, is based on Good's own measurements of s'_{44} and Rinehart's measurements of s'_{11} (95, 96). The change in the slopes of the curves at 468°C, the critical temperature for the order-disorder transition, will easily be seen. Similar but perhaps even more pronounced breaks have been obtained by Siegel (106, 107) at the critical temperature of Cu_3Au alloy.

The effect of temperature on the elastic coefficients of the piezoelectric substances Rochelle salt and potassium di-hydrogen phosphate has been investigated by Lüdy (76). Specimens of these substances, cut at various orientations, were provided with suitable electrodes to which a frequency modulated alternating voltage was applied. If a resonant frequency lies within the range of the frequency modulations, vibrations are induced in the specimen, and are detected by the beats they produce with the driving frequency. Lüdy's main results are summarized in Table XI. The unit is 10^{-13} cm^2/dyne. Rochelle salt, of course, belongs to the orthorhombic system, but potassium phosphate belongs to the tetragonal (b) system, in which $s_{11}=s_{22}$, $s_{23}=s_{13}$, $s_{44}=s_{55}$. The figures in Table XI have been read off from small scale graphs, and are therefore not very accurate; the change in all the quantities with temperature is approximately linear. The numerical values are in satisfactory agreement with those obtained in static tests.

Lüdy discusses at some length the anomalies observed in the particular case when the exciting field is parallel with the electric axis, and the

TABLE X. Elastic coefficients of β-quartz at 873°K, unit = 10^{-13} cm^2/dyne.

Coefficient	K and A	
s_{11}	9.257	9.345 (P and M)
s_{12}	−0.802	
s_{13}	−2.52	−2.26 (L)
s_{33}	10.85	10.50 (P and M)
s_{44}	27.89	
$s(45°)$	10.73	10.67 (L)
$s(50°)$	10.57	10.75 (P and (M)

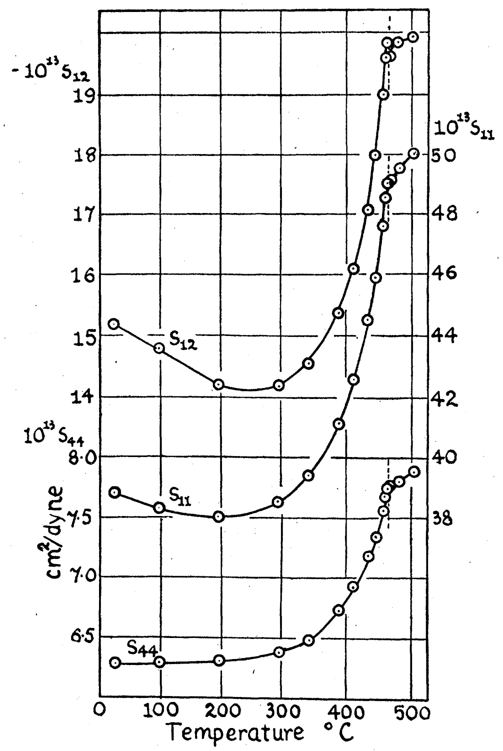

FIG. 9. Effect of temperature on elastic coefficients of β-brass (Good).

electrodes are in close contact with the specimen. These anomalies had been observed previously [see e.g., Müller (85)] and consist in a broadening of the resonance curve, a departure of the frequencies of the harmonics from the usual integral relationships, and an extremely irregular variation of resonant frequency with temperature. The coefficients calculated from the frequencies observed under these conditions differ materially from those obtained in static tests, but the anomalies disappear, and the agreement is restored if an air gap is introduced between each electrode and the face of the specimen with which it was formerly in contact.

Lüdy explains the effect theoretically in terms of the interaction between elastic, piezoelectric, and electric properties. For details of this explanation, and for a bibliography, Lüdy's original paper should be consulted.

Papers not specifically referred to in the above discussion, but which deal with the effect of temperature are contained in the following list: (3, 6, 29, 39, 68, 92, 110, 119, 125).

10. Miscellaneous Applications of Experimental Results

A full discussion of the actual and potential uses of experimental values of the elastic coefficients would form too much of a digression, but, for completeness, a brief summary of some applications, additional to those already referred to will be made.

The velocities of propagation of elastic waves (46, 100) have been calculated as a function of orientation for the hexagonal zinc and cadmium crystals by Grüneisen and Goens (50), and for certain cubic gold-silver crystals by Röhl (97). These velocities are connected with the x-ray diffuse reflections (16, 72) and also with the Debye theory of specific heat. They therefore furnish a link with the quantum theory of crystals; attempts have been made by Fuchs (35, 36, 37) to calculate the elastic constants of some of the simpler crystals from quantum mechanical considerations. In his original papers, Fuchs concentrated on the monovalent cubic metals, Cu, Li, Na, and K. The calculations were supported by the results for copper, which, at that time, were the only ones available. Later determinations on sodium (6, 29, 92) have further confirmed Fuchs' calculations. The problem for hexagonal crystals is far more complicated, but Huntington (60) has endeavored to evaluate some of the factors involved, while Saksena (101) has succeeded in estimating the constants of α-quartz from Raman effect data; the attempts of Wasastjerna (118), Bruggeman (24), and Zdanow (124) to calculate certain constants of cubic metals may also be mentioned. For further details, reference may be made to a book by Seitz (103).

The single crystals on which the measurements are made are the exception, rather than the rule. Industrial materials are usually polycrystalline, though some "preferred" orientation may be present [see e.g., Wood (120)]. For this reason, the connection between elastic constants as determined on single crystals and those of the polycrystalline material has been much discussed. General contributions to this question have been made by Kuntze (69), Huber and Schmid (58), Boas and Schmid (12), and Boas (14). The three latter papers have developed a method of finding values of the constants for a multiple cubic or hexagonal crystal by forming the average value of the single crystal constant over the whole range of orientation. This method has been applied to the special case of tin, and the results compared with experiment by Boas (13). Other discussions dealing with similar topics have been made by Srinavasan (109), and Birch and Bancroft (11).

The elastic constants of crystals are obviously of major importance in determining the frequencies of vibration of crystalline prisms. Mention has already been made (pp. 422, 433) of some investigations in which these frequencies have been studied in order to measure the elastic constants. Sometimes, however, the emphasis is

TABLE XI. Effect of temperature.

Substance	Rochelle salt			Potassium phosphate	
Temperature °C	−40	+20	+40	−180	+20
s_{11}	47	54	56	16	19
s_{22}	32	37	38	(16)	(19)
s_{33}	33	34	34	20	22
$2s_{23}+s_{44}$	41	44	45	80	80
$2s_{13}+s_{55}$	300	330	340	(80)	(80)
$2s_{12}+s_{66}$	54	55	56	136	153

shifted, and the elastic constants are used in order to interpret the observed frequencies. Early theoretical and experimental contributions to the subject included those of Lissütin (71), Wright and Stuart (122), Giebe and Scheibe (38), Colwell (25), Straubel (111), Petrzilka (90), and Osterberg and Cookson (88). This interest was stimulated partly by the use of quartz and to a lesser extent tourmaline, for the production of frequency control crystals. In recent years, the use of quartz for this purpose has increased enormously, and has been accompanied by a considerable increase in the volume of research, some of which has been reviewed by Bergmann (6a), Mason (82), and Sykes (113).

Five main types of vibration are recognized:

1. Extensional vibration in the length direction (longitudinal vibration).
2. Extensional vibration in the thickness direction.
3. Shear vibration in the width-length plane (face shear vibration).
4. Shear vibration in the thickness-length plane (thickness shear vibration).
5. Flexural vibration.

Most crystals vibrate in complex modes, with one mode predominating; the frequency of the crystal is determined by its dimensions, elastic constants, and mode of vibration. It is important for some purposes that the coupling between certain modes shall be a minimum, and a crystal orientation is therefore chosen for which the elastic coefficient governing the coupling is a minimum. If, for example, the length of the crystal lies in the x' direction, the width in the y' direction, and the thickness in the z' direction, and it is desired to minimize the coupling between face and thickness shear modes, then an orientation will be chosen for which s'_{56} is a minimum. Similarly, if it is desired to minimize the coupling between longitudinal and face shear modes, an orientation will be chosen for which s'_{16} is a minimum.

Another important factor governing choice of orientation is the temperature coefficient of frequency, which, for highly accurate frequency control, must be as small as possible. One way of achieving this end is to choose an orientation for which the temperature variation of the elastic coefficient principally involved in the vibration is a minimum or zero. For instance, if a low temperature coefficient crystal vibrating in a longitudinal mode is required, an orientation will be chosen for which the temperature coefficient of s'_{11} is a minimum or zero. In some cases, an orientation may be deliberately chosen for which the vibration consists of two coupled modes, having temperature coefficients of opposite signs, thus leading to a low over-all temperature coefficient.

The above discussion does no more than indicate some of the factors governing choice of orientation; for further details, reference should be made to the papers of Mason and Sykes, already quoted, and to other papers in the same series. It is apparent that the various considerations must sometimes conflict, and under these circumstances, a compromise orientation must be adopted.

On the theoretical side, the problem is exceedingly complicated, although some progress has been made by Eckstein (31), McSkimin (83), and others in dealing with special cases. Eckstein has applied a modification of the perturbation method used in quantum mechanics, and has obtained good agreement with earlier experimental results, notably those of Bechmann [(5) and other papers]. At the moment, however, the practical side of the work is considerably in advance of theory, as is perhaps only to be expected in view of the complexity of the problem. The approach, in fact, appears to be largely empirical, supplemented by semiquantitative appeal to such theory as is available.

11. Bibliography and References

(Note: The figures in italics refer to page numbers in the present paper.)

(1) J. V. Atanasoff and P. J. Hart, "Dynamical determination of the elastic constants and their temperature coefficients for quartz," Phys. Rev. **59**, 85 (1941) (*421*, *433*).

(2) J. V. Atanasoff and E. Kammer, "Determination of the c_{44} elastic constant of β quartz," Phys. Rev. **59**, 97 (1941) (*433*, *434*).

(3) L. Balamuth, "Elastic moduli and temperature variation of the principal Young's modulus of rock salt," Phys. Rev. **45**, 715 (1934) (*421*, *423*, *435*).

(4) E. Baumgardt, "Supersonic measurements of the elastic constants of crystals," Comptes rendus, **208**, 1280 (1939) (*421*).

(5) R. Bechmann, "Natural elastic vibrations of a rectangular quartz parallelopiped," Zeits. f. Physik **122**, 510 (1944) (*436*).

(6) O. Bender, "Elasticity of monocrystalline alkali metals at low temperature," Ann. d. Physik **34**, 359 (1939) (*435*).

(6*a*) L. Bergmann, *Ultrasonics*, translated by H. Stafford Hatfield (G. Bell and Sons, London, 1938) (*421*, *428*, *429*, *432*, *436*).

(7) S. Bhagavantam, "Photoelastic effect in crystals," Proc. Ind. Acad. Sci. **16**, 359 (1942) (*422*).

(8) S. Bhagavantam and J. Bhimasenachar, "Elastic constants of diamond," Nature **154**, 546 (1944) (*421*, *422*).

(9) S. Bhagavantam and J. Bhimasenachar, "Elastic constants of crystals: A new method and its application to pyrites and galena," Proc. Ind. Acad. Sci. **20**, 298 (1944) (*421*, *422*, *428*, *431*).

(10) S. Bhagavantam and D. Suryanarayana, "Elastic constants of piezoelectric crystals," Proc. Ind. Acad. Sci. **20**, 304 (1944) (*421*, *422*, *428*, *431*).

(11) F. Birch and D. Bancroft, "Elasticity of rocks and massive minerals," Am. J. Sci. **37**, 7 (1939) (*435*).

(12) W. Boas and E. Schmid, "Calculation of the physical constants of quasi-isotropic multiple crystals," Helv. Phys. Acta **7**, 628 (1934) (*435*).

(13) W. Boas, "Elastic behavior of single and multiple crystals of tin," Helv. Phys. Acta, **7**, 878 (1935) (*435*).

(14) W. Boas, "Torsion modulus of quasi-isotropic multiple crystals from single crystal constants," Helv. Phys. Acta **8**, 674 (1935) (*435*).

(15) W. L. Bond, "The mathematics of the physical properties of crystals," Bell Sys. Tech. J. **22**, 1 (1943) (*411*).

(16) M. Born, "Theoretical investigations on the relation between crystal dynamics and x-ray scattering," *Reports on Progress in Physics* (Phys. Soc., London) **9**, 294 (1943) (*435*).

(17) P. W. Bridgman, "Some properties of single metal crystals," Proc. Nat. Acad. Sci. Wash. **10**, 411 (1924) (*422*, *425*, *428*, *429*).

(18) P. W. Bridgman, "Physical properties of single crystals of antimony, bismuth, tellurium, cadmium, zinc, and tin," Proc. Am. Acad. **60**, 305 (1925) (*414*, *420*, *422*).

(19) P. W. Bridgman, "The elastic moduli of five alkali halides," Proc. Am. Acad. **64**, 19 (1929) (*420*, *422*, *428*, *430*, *431*).

(20) P. W. Bridgman, "Physical properties of single crystal magnesium," Proc. Am. Acad. **67**, 29 (1932) (*423*).

(21) P. W. Bridgman, "The effect of slight impurities on the elastic constants, particularly the compressibility, of zinc," Phys. Rev. **47**, 393 (1935) (*425*).

(22) W. F. Brown, Jr., "The interpretation of the torsional frequencies of crystal specimens," Phys. Rev. **58**, 998 (1940) (*415*, *417*, *418*, *419*, *420*, *422*, *423*, *426*).

(23) A. Brjukhanov, "Point of elastic isotropy in rock salt," J. Exp. and Theor. Phys., U.S.S.R. **9**, 722 (1939) (*432*).

(24) D. A. Bruggeman, "Elastic moduli of cubic metals," Zeits. f. Physik **92**, 561 (1934) (*435*).

(25) R. C. Colwell, "Vibration of quartz plates," Proc. I. R. E. **20**, 808 (1932) (*436*).

(26) R. M. Davies, "Elastic constants of Rochelle salt by a dynamic method," Phil. Mag. **16**, 97 (1933) (*415*, *421*, *423*).

(27) R. M. Davies, I—"The frequency of vibration of a loaded fixed-free bar," Phil. Mag. **22**, 892 (1936); II—"The effect of rotatory inertia of the load," Phil. Mag. **23**, 464 (1937); III—"The effect of rotatory inertia of the bar," Phil. Mag. **23**, 563 (1937); and IV—"The effect of shearing in the bar," Phil. Mag. **23**, 1129 (1937) (*415*).

(28) R. M. Davies, "The frequency of longitudinal and torsional vibration of loaded and unloaded bars," Phil. Mag. **25**, 364 (1938) (*415*).

(29) R. H. V. M. Dawton, "Elastic properties of sodium wires between −183°C and 90°C," Proc. Phys. Soc. London, **50**, 483 (1938) (*435*).

(30) M. Durand, "Temperature variation of the elastic moduli of NaCl, KCl, and MgO," Phys. Rev. **50**, 449 (1936) (*423*, *428*, *432*).

(31) H. Eckstein, "Free vibration of aelotropic bodies," Phys. Rev. **66**, 108 (1944) (*436*).

(32) A. B. Focke, R. B. Lindsay, and C. R. Wilks, "Velocity of sound in single crystals of bismuth," J. Acous. Soc. Am. **9**, 348 (1938) (*421*).

(33) K. Försterling, "Elastic constants of sylvine," Zeits. f. Physik **2**, 172 (1920) (*432*).

(34) V. Freedericks and G. Michaelow, "Variation with temperature of the piezoelectric constants of quartz," Zeits. f. Physik **76**, 328 (1932) (*433*).

(35) K. Fuchs, "Quantum mechanical calculation of the elastic constants of monovalent metals," Proc. Roy. Soc. **A153**, 622 (1936) (*435*).

(36) K. Fuchs, "Elastic constants and specific heats of the alkali metals," Proc. Roy. Soc. **A157**, 444 (1936) (*435*).

(37) K. Fuchs and H. W. Peng, "Crystal theory of metals: Calculation of elastic constants," Proc. Roy. Soc. **A180**, 451 (1942) (*435*).

(37*a*) J. W. Geckeler, "Elastic theory of anisotropic bodies," in *Handbuch der Physik* (Vol. VI), edited by H. Geiger and K. Scheel (Verlagsbuchhandlung Julius Springer, Berlin, 1928) (*413*).

(38) E. Giebe and A. Scheibe, "Series law for elastic proper frequencies of quartz rods," Ann. d. Physik **9**, 93 (1931) (*436*).

(39) E. Goens, "Dynamic method for determining the relation between temperature and elastic constants of rods," Ann. d. Physik **4**, 733 (1930) (*415*, *435*).

(40) E. Goens, "Determination of Young's modulus from flexural oscillations," Ann. d. Physik **11**, 649 (1931) (*415*).

(41) E. Goens, "Torsional and flexural oscillations of a monocrystalline rod," Ann. d. Physik **15**, 455, 902 (1932) (*412, 415, 418, 419, 420, 422, 423, 424, 426*).

(42) E. Goens, "Improved apparatus for the measurement of torsional modulus of crystal rods, and application to single crystals of zinc," Ann. d. Physik **16**, 793 (1933) (*420, 421, 423, 426, 429*).

(43) E. Goens, "Elastic constants of aluminum single crystals," Ann. d. Physik **17**, 233 (1933) (*420, 421, 428, 429, 432*).

(44) E. Goens and J. Weerts, "Elastic constants of single crystals of copper, gold, and lead," Physik. Zeits. **37**, 321 (1936) (*420, 422, 424, 425, 427, 428, 432*).

(45) E. Goens and E. Schmid, "Elastic constants, resistivity and thermal expansion of single magnesium crystals," Physik. Zeits. **37**, 385 (1936) (*422, 425, 429, 430*).

(46) E. Goens, "Velocities of propagation of elastic waves in crystals," Ann. d. Physik **29**, 279 (1937) (*435*).

(47) W. A. Good, "Rigidity modulus of β-brass single crystals," Phys. Rev. **60**, 605 (1941) (*426, 428, 434*).

(48) A. E. Green and G. I. Taylor, "Stress systems in aelotropic plates—I," Proc. Roy. Soc. **A173**, 162 (1939) (*431*).

(49) A. E. Green, "Stress Systems in Aelotropic Plates" —II. Proc. Roy. Soc. **A173**, 173 (1939); —IV, Proc. Roy. Soc. **A180**, 173 (1942) (*431*).

(50) E. Grüneisen and E. Goens, "Researches on metal crystals—I. Elastic constants of zinc and cadmium," Zeits. f. Physik **26**, 235 (1924) (*422, 423, 429, 432, 435*).

(51) A. W. Hanson, "Elastic behavior and elastic constants of zinc single crystals," Phys. Rev. **45**, 324 (1934) (*420, 422, 424, 427, 429, 430*).

(52) R. F. S. Hearmon and W. W. Barkas, "The effect of grain direction on the Young's moduli and rigidity moduli of beech and Sitka spruce," Proc. Phys. Soc. London **53**, 674 (1941) (*418, 431*).

(53) R. F. S. Hearmon, "The significance of the coupling between shear and extension in the elastic behavior of wood and plywood," Proc. Phys. Soc. London **55**, 67 (1943) (*415, 418, 419, 431*).

(54) H. Hinz, "Elastic deformation in Rochelle salt," Zeits. f. Physik **111**, 617 (1939) (*420, 429*).

(55) H. Hörig, "The elasticity of spruce—I," Zeits. f. tech. Physik **12**, 369 (1931) (*429, 431*).

(56) H. Hörig, "The numerical evaluation of torsion measurements on wood," Ingen.-Arch. **4**, 570 (1933) (*418, 429, 431*).

(57) H. Hörig, "The calculation of the coefficients s_{44}, s_{55}, s_{66} from the torsion of rods with rectangular cross section and rhombic symmetry," Ingen.-Arch. **7**, 165 (1936) (*418, 429*).

(58) A. Huber and E. Schmid, "Elastic properties of quasi-isotropic multiple crystals," Helv. Phys. Acta **7**, 620 (1934) (*435*).

(59) L. Hunter and S. Siegel, "Variation with temperature of the principal elastic moduli of NaCl near the melting point," Phys. Rev. **61**, 84 (1942) (*421, 428, 432*).

(60) H. B. Huntington, "Contributions to the shear constants of hexagonal crystals," Phys. Rev. **57**, 60 (1940) (*435*).

(61) H. A. Jahn, "Elastic constants of anisotropic solids," Zeits. f. Krist. **98**, 191 (1937) (*422*).

(62) C. F. Jenkin, "Report on materials used in the construction of aircraft and aero engines," (H.M. Stationery Office, London 1920) (*431*).

(63) E. Kammer and J. V. Atanasoff, "A determination of the elastic constants of β-quartz," Phys. Rev. **62**, 395 (1942) (*421, 433*).

(64) M. Kantola, "Determination of elastic constants by a pressure method and its application to NaCl crystals," Acta Soc. Sci. Fenn. **3**, 1 (1939) (*421*).

(65) R. Karnop and G. Sachs, "The flow of metallic crystals under torsion," Zeits. f. Physik **53**, 605 (1929) (*422*).

(66) R. Kimura, "Elastic constants of single crystals of copper," Tohoku Univ. Sci. Rep. **22**, 553 (1933) (*423, 424, 428*).

(67) R. Kimura and K. Ohno, "Elastic constants of single crystals of iron," Tohoku Univ. Sci. Rep. **23**, 359 (1934) (*428*).

(68) R. Kimura, "Elastic moduli of ferromagnetic materials," Proc. Phys. Math. Soc. Japan: I—"Iron crystals," **21**, 686 (1939); II—"Change in Young's modulus due to magnetization and temperature," **21**, 786 (1939); III—"The ΔE effect of single iron crystals," **22**, 45 (1940); IV—"The ΔE effect at high temperatures, and the ΔK effect," **22**, 219 (1940) (*423, 424, 425, 428, 435*).

(69) W. Kuntze, "Rigidity of the multiple crystal," Zeits. f. Physik **81**, 80 (1933) (*435*).

(70) A. W. Lawson, "Determination of the elastic modulus s_{13} of quartz," Phys. Rev. **59**, 608 (1941) (*433*).

(71) A. Lissütin, "Oscillations of quartz plates," Zeits. f. Physik **69**, 850 (1931) (*436*).

(72) K. Lonsdale, "Experimental study of x-ray scattering in relation to crystal dynamics," *Reports on Progress in Physics* (Phys. Soc., London) **9**, 257 (1943) (*435*).

(73) A. E. H. Love, *The Mathematical Theory of Elasticity* (Cambridge University Press, Cambridge, England, 1927), fourth edition (*410, 413, 431*).

(74) C. Ludeke, "Resonance," J. App. Phys. **13**, 418 (1942) (*421*).

(75) H. F. Ludloff, "Ultrasonics and elasticity," J. Acous. Soc. Am. **12**, 193 (1940) (*421*).

(76) W. Lüdy, "The influence of temperature on the dynamic-elastic behavior of piezoelectric substances," Helv. Phys. Acta **15**, 527 (1942) (*434*).

(77) W. Mandell, "Determination of the elastic moduli of the piezoelectric crystal Rochelle salt by a static method," Proc. Roy. Soc. **A116**, 623 (1927) (*421, 422, 429*).

(78) W. Mandell, "Change in elastic properties on replacing the potassium atom of Rochelle salt by the ammonium atom," Proc. Roy. Soc. **A121**, 122 (1928) (*421*, *422*, *429*).

(79) R. de Mandrot, "Isothermal and adiabatic moduli of elasticity of crystallized quartz," Helv. Phys. Acta **5**, 362 (1932) (*433*).

(80) M. Masima and G. Sachs, "Mechanical properties of brass crystals," Zeits. f. Physik **50**, 161 (1928) (*422*, *428*).

(81) W. P. Mason, "A dynamic measurement of the elastic, electric and piezoelectric constants of Rochelle salt," Phys. Rev. **55**, 775 (1939) (*421*, *425*, *429*).

(82) W. P. Mason, "Quartz crystal applications," Bell Sys. Tech. J. **22**, 178 (1943) (*421*, *429*, *436*).

(83) H. J. McSkimin, "Theoretical analysis of modes of vibration for isotropic plates having all surfaces free," Bell Sys. Tech. J. **23**, 151 (1944) (*436*).

(84) G. Michaelow, "Elastic vibration in a piezoelectric crystal of Rochelle salt," Tech. Phys., U.S.S.R. **3**, 652 (1936) (*421*, *425*).

(85) H. Müller, "The properties of Rochelle salt.—IV," Phys. Rev. **58**, 805 (1940) (*435*).

(86) F. C. Nix and W. Shockley, "Order-disorder transformations in alloys," Rev. Mod. Phys. **10**, 1 (1938) (*434*).

(87) H. Osterberg and J. W. Cookson, "Some piezoelectric and elastic properties of β-quartz," J. Frank. Inst. **220**, 361 (1935) (*434*).

(88) H. Osterberg and J. W. Cookson, "Longitudinal, shear and transverse modes of vibration in quartz and tourmaline," Physics **6**, 246 (1935) (*436*).

(89) A. Perrier and R. de Mandrot, (quoted by Kammer and Atanasoff (631)) "Elasticity and symmetry of quartz at high temperatures," Memoires de la Soc. Vaudoise des sci. Naturelles: No. 7, **1**, 333 (1923). Imprimeries Rennies S. A. Lausanne (*422*, *433*, *434*).

(90) V. Petrzilka, "Longitudinal and flexural vibrations of tourmaline plates," Zeits. f. Physik **97**, 436 (1935) (*436*).

(91) J. Prescott, "Elastic waves and vibrations of thin rods," Phil. Mag. **33**, 703 (1942) (*415*).

(92) S. L. Quimby and S. Siegel, "Variation of elastic constants of crystalline sodium between 80°K and 210°K," Phys. Rev. **54**, 293 (1938) (*425*, *428*, *435*).

(93) Lord Rayleigh, *The Theory of Sound* (Macmillan and Company, London, 1877), Vol. 1 (*415*).

(94) T. Reimers, "Secondary torsion accompanying primary bending of crystalline cylinders," Physik. Zeits. **14**, 276 (1913) (*414*, *415*).

(95) J. S. Rinehart, "Temperature dependence of Young's modulus of β-brass single crystals," Phys. Rev. **58**, 365 (1940) (*426*, *434*).

(96) J. S. Rinehart, "Behavior of Young's modulus of β-brass single crystals at low temperatures," Phys. Rev. **59**, 308 (1941) (*426*, *428*, *434*).

(97) H. Röhl, "Elastic properties of gold-silver single crystals," Ann. d. Physik **16**, 887 (1933) (*423*, *428*, *435*).

(98) F. C. Rose, "Variation of adiabatic elastic moduli of rock salt with temperature between 80°K and 270°K," Phys. Rev. **49**, 50 (1936) (*421*, *423*).

(99) M. B. de Saint Venant, Note to Section 156, Art. 5 of *De la Résistance des Corps Solides* by L. M. H. Navier (Dunod, Paris, 1864) (*415*).

(100) Z. Sakadi, "Elastic waves in crystals," Proc. Phys. Math. Soc. Japan **23**, 539 (1941) (*435*).

(101) B. D. Saksena, "Calculation of the elastic constants of α-quartz at room temperature from Raman effect data," Proc. Ind. Acad. Sci. **A19**, 357 (1944) (*435*).

(102) C. Schaefer and L. Bergmann, "Optical determination of elastic constants of crystals," Atti. Accad. Lincei **21**, 701 (1935) (*421*).

(103) F. Seitz, *The Modern Theory of Solids* (McGraw Hill Book Company, Inc., London and New York, 1940), first edition (*435*).

(104) F. Seitz and T. A. Read, "Theory of the plastic properties of solids," J. App. Phys. **12**, 100, 170, 470, 538 (1941) (*427*, *428*).

(105) L. P. Sieg and R. F. Miller, "Coefficients of rigidity and Young's modulus for selenium crystals," Proc. Iowa Acad. Sci. **28**, 103 (1921) (*422*, *423*).

(106) S. Siegel, "Variation of the principal elastic moduli of Cu_3Au with temperature," Phys. Rev. **57**, 537 (1940) (*425*, *428*, *434*).

(107) S. Siegel, "Kinetics of order-disorder transformation in Cu_3Au," J. Chem. Phys. **8**, 860 (1940) (*425*, *428*, *434*).

(108) J. C. Slater, "Measurement of the compressibility of the alkali halides," Proc. Am. Acad. **61**, 135 (1926) (*422*).

(109) P. S. Srinavasan, "Elastic properties of mother-of-pearl," Proc. Ind. Acad. Sci. **A5**, 463 (1937) (*435*).

(110) T. Steinebach, "Determination of the elasticity and torsion coefficients of rock salt at low temperatures," Zeits. f. Physik **33**, 664 (1925) (*435*).

(111) H. Straubel, "Temperature coefficient of quartz oscillators," Physik. Zeits. **35**, 657 (1934) (*436*).

(112) I. H. Swift and E. P. T. Tyndall, "Elasticity and creep of lead single crystals," Phys. Rev. **61**, 359 (1942) (*420*, *422*, *426*, *428*, *429*).

(113) R. A. Sykes, "Modes of motion in quartz crystals, the effects of coupling and methods of design," Bell Sys. Tech. J. **23**, 52 (1944) (*436*).

(114) E. P. T. Tyndall, "Probable values of the elastic constants of the zinc crystal," Phys. Rev. **47**, 398 (1935) (*425*).

(115) W. Voigt, *Lehrbuch der Kristallphysik* (B. G. Teubner, Leipzig, 1910, reprinted 1928) (*410*, *411*, *413*, *415*, *417*, *422*, *428*, *429*, *430*, *431*).

(116) W. Voigt, "The elastic constants of zinc blende," Nach. Ges. Wiss. Göttingen, p. 424 (1918) (*422*, *428*).

(117) W. Voigt, "The elastic constants of crystallized potassium alum," Nach. Ges. Wiss. Göttingen, p. 85 (1918) (*422*, *428*).

(118) J. A. Wasastjerna, "Forces acting between atoms and ions and the physical properties of matter in bulk," Phil. Trans. Roy. Soc. **A237**, 105 (1938) (*435*).

(119) W. Webb, "Study of β-brass in single crystal form," Phys. Rev. **55**, 297 (1939) (*422*, *424*, *435*).

(120) W. A. Wood, "X-ray diffraction and the deformation of metals," J. Sci. Inst. **18**, 153 (1941) (*435*).

(121) W. A. Wooster, *Crystal Physics* (Cambridge University Press, Cambridge, England, 1938) first edition (*411*).

(122) R. B. Wright and D. M. Stuart, "Vibrations of quartz plates," Bur. Stand. J. Research **7**, 519 (1931) (*436*).

(123) S. J. Wright, "Elasticity of Pintsch crystals of tungsten," Proc. Roy. Soc. **A126**, 613 (1930) (*415*, *423*, *428*).

(124) V. Zdanow, "Calculation of the compressibility coefficients of crystals," Zeits. f. Physik **101**, 86 (1936) (*435*).

(125) F. Zwicky, "On the physics of crystals—I," Rev. Mod. Phys. **6**, 193 (1934) (*430*, *435*).