Random Distribution of Lines in a Plane

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INTRODUCTION

In many of the early cloud-chamber experiments the possibility existed of erroneous interpretation because several tracks might accidentally seem to originate from the same point. Professor Bohr asked me once to study the chance that several independent tracks intersected at almost the same point. The need for the answer to this problem had fortunately vanished before any progress was made towards its solution. Lately, however, the same problem arose in other connections and in the following we discuss the first steps so far obtained towards the solution.

THE IDEALIZED PROBLEM

We consider a plane covered with straight lines distributed at random in position and direction. These lines cut the plane into triangles and polygons. What is wanted is the probability distribution of the areas of the fragments, namely what fraction of the fragments has an area lying between given limits.

In order to avoid difficulties with infinities it seems advantageous to consider the problem on a sphere instead of in a plane. In that case the straight lines are replaced by great circles on the sphere. These great circles will intersect an arbitrarily chosen equator at randomly distributed points, in fact one needs to consider only a half sphere as the two halves are identical.

SIMPLIFIED PROBLEM

The following simplified version of the problem can be solved completely. We assume that the plane is covered by straight lines which are not arbitrary in direction but parallel to the x axis and the y axis. The plane is then cut into rectangular fragments by two perpendicular sets of lines. The lines in each set are distributed at random.

Let the density of the lines be chosen such that on the average the x axis is intersected by one line per unit length and similarly the y axis. The problem is now reduced to the well-known distribution of random points on a line. Considering one rectangular fragment the probability that its horizontal dimension is between η and $\eta+d\eta$ and its vertical dimension between ξ and $\xi+d\xi$ is given by

$$f(\xi,\eta)d\xi d\eta = e^{-(\xi+\eta)}d\xi d\eta.$$

We are, however only interested in the area $\sigma=\xi\eta$. We introduce new variables, namely

$$\sigma=\xi\eta, \quad \text{area}$$
$$u=\xi+\eta, \quad \text{half circumference}$$
$$f(\xi,\eta)d\xi d\eta = e^{-u}du d\sigma/(u^2-4\sigma)^{\frac{1}{2}}.$$

Note that

$$u^2 \geq 4\sigma.$$

For the probability that the area of a fragment is σ the simplified problem gives

$$F(\sigma)d\sigma = d\sigma\int_{\sqrt{4\sigma}}^{\infty} du e^{-u}/(u^2-4\sigma)^{\frac{1}{2}},$$

$$F(\sigma)=\tfrac{1}{2}\pi i H_0^{(1)}(i\sqrt{4\sigma}). \tag{1}$$

These functions are tabulated in Jahnke-Emde.

It is very easy to obtain the averages of powers of σ,

$$\langle\sigma^k\rangle_{\mathrm{Av}} = \int_0^{\infty}\int \xi^k\eta^k e^{-(\xi+\eta)}d\xi d\eta = (k!)^2. \tag{2}$$

Note that our choice of one line per unit length on the average has normalized these averages so that

$$\langle\sigma\rangle_{\mathrm{Av}} = 1. \tag{3}$$

THE GENERAL PROBLEM

For the general problem it is better to consider the distribution on a sphere. We assume that there are N halves of great circles distributed at random on a half sphere, not counting the "equator." We can then derive the following properties, the N lines divide the half sphere into $\frac{1}{2}N(N+1)+1$ fragments; and on the average

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each fragment has four sides when N increases indefinitely.

The first statement can easily be proved by induction. The Nth line intersects all $(N-1)$ previous lines and in doing so cuts N fragments into two, thus adding just N fragments to the total. As we started with one fragment, the whole half sphere, the number of fragments is given by

$$\text{number of fragments} = 1+\Sigma N = \tfrac{1}{2}N(N+1)+1. \quad (4)$$

The second statement is also derived very simply. Each one of the N great circle halves is cut by the others into N segments. Each one of these segments serves as side to two adjacent fragments. In addition the "equator" is cut up into $2N$ segments bordering one fragment each. The total number of sides to the fragments is therefore

$$\text{total sides} = 2N^2+2N.$$

The average number of sides per fragment is

$$\text{average number of sides} = 4 \text{ for } N\rightarrow\infty. \quad (5)$$

When going over to the plane case we have to let N become infinite. In that case we can also use an exponential distribution for the lengths of the segments into which the lines are cut, which is not strictly correct in the final case.

We must watch the normalization this time. If, like above, we wish to adjust it so that the average size of the fragments is unity, the area of the half sphere has to be $\frac{1}{2}N^2$ for large N. The radius of the half sphere is then $\frac{1}{2}N/\sqrt{\pi}$, the length of the great circle half is $\frac{1}{2}N\sqrt{\pi}$. The mean length of the segments into which the great circles are cut is thus $\frac{1}{2}\sqrt{\pi}$ and not unity, as it was in the simplified rectangular case.

THE MEAN SQUARE AREA

We use an artifice to compute the mean square area of the fragments in the general case. We consider two arbitrarily chosen points and ask for the probability that they happen to lie in the same fragment. This probability can be expressed in terms of the mean square area of the fragments. We next consider the line which is determined by the two points and ask for the probability that the two points fall both in one of the segments into which the line is divided by all the other lines. This latter probability can easily be computed and thus the mean square area obtained.

The probability that the first arbitrarily chosen point lies in a fragment of size between σ and $\sigma+d\sigma$ is given by the fraction of the total area which is covered by such fragments, namely

$$\sigma SG(\sigma)d\sigma \Big/ \int \sigma SG(\sigma)d\sigma,$$

where S is the number of fragments considered and $G(\sigma)$ the distribution in size.

The probability that a second arbitrary point falls in the same fragment is given by the area of that fragment divided by the total area,

$$\sigma \Big/ \int \sigma SG(\sigma)d\sigma.$$

The probability P_2 that the two points lie in the same fragment irrespective of its size is equal to the product of these two expressions integrated over all sizes,

$$P_2 = \int \sigma^2 SG(\sigma)d\sigma \Big/ \left\{ \int \sigma SG(\sigma)d\sigma \right\}^2 = \langle\sigma^2\rangle_{\text{Av}}/S\langle\sigma\rangle_{\text{Av}}^2. \quad (6)$$

We next consider the probability that the second point lies at a distance l to $l+dl$ from the first. This is given by the area of a ring, namely

$$2\pi l dl/\text{total area}.$$

Through the two points we pass a line, which will be cut into segments by all the lines already present. The chance that no intersection will occur between the two chosen points is

$$e^{-l/\frac{1}{2}\sqrt{\pi}}.$$

The factor $\frac{1}{2}\sqrt{\pi}$ arises from the normalization discussed above, the mean distance between intersections is $\frac{1}{2}\sqrt{\pi}$.

The probability that irrespective of their distance the two points are not separated by one of the lines is thus given by

$$P_2 = \int dl\cdot 2\pi l e^{-l/\frac{1}{2}\sqrt{\pi}}/S\langle\sigma\rangle_{\text{Av}} = \tfrac{1}{2}\pi^2/S,$$

where we made use of the fact that $\langle\sigma\rangle_{\text{Av}}=1$ in the chosen normalization.

Comparison with Eq. (6) gives at once the final result

$$\langle\sigma^2\rangle_{\text{Av}}/\langle\sigma\rangle_{\text{Av}}^2 = \tfrac{1}{2}\pi^2. \quad (7)$$

Note that this differs from the simplified case with rectangular fragments which gives 4 instead.